Outcrossing rates of marked Poisson cluster processes in structural reliability

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A marked Poisson cluster process (PCP) is defined as a model for live loads in buildings. The outcrossing rate for this PCP and the superposition of such processes are derived for the determination of structural failure probabilities. For the equilibrium process a Poisson limit theorem for the failure probability and an asymptotic approximation for the outcrossing rate are given. Numerical results for normally distributed marks are presented, and comparisons with two other similar load models are made.

Keywords: cluster processes, outcrossing rate, structural reliability, asymptotic analysis

Introduction

Various attempts have been made to realistically describe structural loads by appropriate stochastic models and to solve the corresponding combination problem when calculating structural reliability. The class of marked jump processes has been found useful when describing certain loading phenomena (e.g., live loads in buildings). Most of these models rest on the assumption that occurrence, intensity, and duration of the load pulses are independent random variables. Only under this assumption and, in part, in special forms of the load pulses has the combination problem found practical solutions.[1–8] However, real loading phenomena often exhibit pronounced dependencies of various kinds. There may be amplitude dependencies (Rackwitz[9]), dependencies between pulse durations, or dependencies between pulse amplitude and pulse duration (Madsen[10]). One of the more important dependencies is occurrence clustering. Wen[6] and Wen and Pearce[11] (the latter reference should also be consulted for a general discussion of the matter of dependence for structural loads) give examples where such clustering can be observed in reality. Wen and Pearce used a kind of Bartlett-Lewis[12] cluster process to handle the problem. By the method of load coincidence they constructed an approximation for the distribution of the lifetime maximum value for the sum of such load processes.

In this paper a more general clustering phenomenon is described by a marked point process. It may be used when modelling loads due to vehicle traffic, but other applications in nonstructural areas can also be visualized. The maximum lifetime distribution of sums or more complicated functions of such processes is approximated by the well-known outcrossing rate method.

Denote by $P_F(t) = P(T \leq t)$, $t \geq 0$, the distribution function of the time T to first failure (i.e., entrance of the marked point process $X(t)$ into the failure domain $F \subseteq R^1$). We obtain an upper bound to the failure probability[13]

$$P_F(t) \leq P_F(0) + E[M(t)] = P_F(0) + \int_0^t \kappa(\sigma)\,d\sigma \quad (1)$$

where $P_F(0)$ is the initial failure probability, $E[M(t)]$ is the mean value of the point process of exits of $X(t)$ into F, and $\kappa(\sigma)$ is the instantaneous crossing rate of $M(t)$ into F. A lower bound to $P_F(t)$ involving higher moments of $M(t)$ can also be given, but the well-known asymptotic approximation[13,14]

$$P_F(t) \sim 1 - \exp[-E[M(t)]] \quad (2)$$

valid under certain conditions, generally is of more practical interest.

In the following, a formally strict description and several properties of the unmarked Poisson cluster point

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process are presented. Then a special marked Poisson cluster process (PCP) is defined. Its crossing rate out of safe domains of structural states is given. The results are extended to sums of such processes. The Poisson convergence theorem leading to equation (2) is given together with an asymptotic formula for the crossing rate out of arbitrary domains. The results are illustrated by some examples.

Unmarked cluster point process

In the context of point process theory, clustering means that points which occur along a time scale can be separated into main points, the cluster centers, and subsidiary points, the points within a cluster.

For our purposes the following point process is introduced:

$$N(t) = \int_0^t N_p(\sigma;t)\,dN_c(\sigma) \qquad (3)$$
$$= \begin{cases} 0 & \text{for } N_c(t) = 0 \\ \sum_{i=1}^{N_c(t)} N_p(\tau_i;t) & \text{for } N_c(t) \in N \end{cases}$$

where

(i) $N_c(\sigma)$ is a homogeneous Poisson counting process with intensity parameter $\lambda > 0$ and jump times τ_i, $i \in$ N ($\tau_0 = 0$), and

(ii)

$$N_p(\sigma;t) = \sum_{j=1}^{Z(\sigma)} I(\{\rho_j^\sigma \in (\sigma,t]\})$$

is a process counting the number of events of a renewal process in the interval $(\sigma,t]$ with renewal times ρ_j^σ, $j \in N_0$ ($\rho_0^\sigma = \sigma$) and continuous waiting time distribution function $F(u)$, $u \geqslant 0$,

$$P(\Delta \leqslant u) = P(\Delta_j \leqslant u)$$
$$= P(\rho_{j+1}^\sigma - \rho_j^\sigma \leqslant u) = F(u) \qquad j \in N_0$$

which has expectation $E[\Delta] = \mu$.

The renewal process is assumed to be finite; i.e., there exists a probability distribution with

$$P(Z(\sigma) = k) = p_k \qquad k \in N_0$$
$$\sum_{k \in N_0} p_k = 1 \quad \text{and} \quad E[Z] < \infty$$

Process (3) is similar to the Bartlett-Lewis point process.[12] Cluster centers are the starting points τ_i of the subsidiary processes $N_p(\tau_i;\infty) = Z(\tau_i)$. The clusters are the points in each subsidiary process. For the derivation of the important stochastic properties of the subsidiary process $N_p(\sigma;t)$ and the cluster process $N(t)$ it is assumed that the random variables τ_i, ρ_j^σ, and $Z(\tau_i)$ are mutually independent for $i,j \in$ N.

By some well-known results of point process theory[15] it can be shown that the cluster point process (3) is nonstationary, regular, and evolves with aftereffects. Its expected value is given by[12]

$$E[N(t)] = \int_0^t E[N_p(\sigma;t)]E[dN_c(\sigma)]$$
$$= \lambda \int_0^t H(t-\sigma)\,d\sigma \qquad (4)$$

with

$$H(\sigma) = \sum_{i=1}^{\infty} F^{i*}(\sigma)P(Z \geqslant i) = E[N_p(0;\sigma)]$$

the renewal function of $N_p(0;\sigma)$ (F^{i*} denoting the ith convolution of F).

We also shall need the distribution of the number of subsidiary processes D_σ, which are active at time instant σ, $\sigma \in (0,\infty)$. Using the fact that $N_c(\sigma)$ is a homogeneous Poisson process and some standard results in probability theory, it can be shown[12] that the distribution $V(D_\sigma)$ is a Poisson distribution; i.e.,

$$q_k(\sigma) = P(D_\sigma = k) \qquad (5)$$
$$= \exp(-\lambda(\sigma))\frac{(\lambda(\sigma))^k}{k!} \qquad k \in N_0$$

with parameter

$$\lambda(\sigma) = \lambda \int_0^\sigma R(u)\,du \quad \text{and}$$

$$R(u) = \sum_{i=1}^{\infty} p_i(1 - F^{i*}(u))$$

$R(\sigma) = P(\gamma_\sigma \leqslant \infty)$ is the probability that the subsidiary process is still active at time σ.

The distribution of the duration $L = L(\sigma) = \sum_{i=0}^{D_\sigma} \Delta_i$, $\Delta_0 \equiv 0$, of the subsidiary process starting in σ is easily derived by conditioning on the event $\{Z(\sigma) = i\}$:

$$G(l) = P(L(\sigma) \leqslant l) = \sum_{i=0}^{\infty} p_i F^{i*}(l) \qquad (6)$$

For the subsidiary process we further determine the distribution of the forward recurrence time γ_σ which is the time to the next event in $N_p(\tau_i;\infty)$ from an arbitrary time instant σ, $\sigma \in (\tau_i,\infty)$. This distribution is again derived by using results in Lewis[12] and by remembering that the jump times τ_i under the condition $\{D_\tau = k\}$ have the following distribution:[15]

$$P(\tau_i \leqslant \tau) = \frac{1}{\bar{\sigma}} \int_0^\tau \lambda(u)\,du \qquad \tau \leqslant \sigma \qquad (7)$$

with $\bar{\sigma} = \int_0^\sigma \lambda(u)\,du$ and $\lambda(u)$ as in equation (5).

Thus, we obtain for the forward recurrence time

$$P(\gamma_\sigma \leqslant h | Z_\sigma = k)$$
$$= \frac{1}{\bar{\sigma}} \int_0^\sigma \lambda(\tau) \int_0^{\sigma-\tau} (F(u+h) - F(u))\,dH^*(u)\,d\tau \qquad (8)$$

where $H^*(u) = \sum_{i=0}^{\infty} P(Z > i)F^{i*}(u)$

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where $H^*(u) = \sum_{i=0}^{\infty} P(Z > i) F^{i*}(u)$

The inner integral in equation (8) represents the probability of a renewal in the interval $(u, u + h]$ given that there are more than i renewals and the lifetime up to the ith renewal is not greater than u.

As mentioned before, the cluster process (3) is nonstationary and evolves with aftereffects. Stationarity for the cluster process can be regained by using a limit operation common in renewal theory. Instead of using the process (3) which starts at time instant zero, another point process is considered which starts in the infinite past. The resulting point process which is observed at time zero, the so-called equilibrium point process, is stationary. Further information and details about this limit operation can be found in Westcott.[16]

Further derivations are mainly restricted to this equilibrium point process. Equation (4), for example, reads[17]

$$E[N(t)] = \lambda t E[Z] \tag{9}$$

Some other characteristics of this equilibrium point process which, to a large extent, can be found in Lewis[17] are repeated here for easy reference. The number of subsidiary processes D which are active at time instant σ is given by a Poisson distribution with parameter $\lambda_e = \lambda\mu E[Z] = \lambda E[\Delta]E[Z]$; i.e.,

$$P(D = k) = q_k = \exp(-\lambda_e)\frac{\lambda_e^k}{k!} \qquad k \in \mathrm{N}_0 \tag{10}$$

Equation (10) reveals some insight into the structure of the cluster process $N(t)$. The expected length of a subsidiary process $Z(\sigma) = N_p(\sigma;\infty)$ is

$$E\left(\sum_{j=0}^{Z(\sigma)} \Delta_j\right) = E[Z]E[\Delta] \tag{11}$$

The expected length of a renewal interval of $N_c(t)$ is

$$E[\tau_{i+1} - \tau_i] = \frac{1}{\lambda} \tag{12}$$

Now, if $E[\Delta]E[Z] < 1/\lambda$, the expected length of a cluster does not exceed the expected waiting time for the next cluster and, thus, the clusters usually are separated. The probability that clusters overlap is small. For $E[\Delta]E[Z] = 1/\lambda$, the expected number of Z_σ is 1; i.e., there is always one subsidiary process active at time instant σ and the clusters change with the same rate as the cluster centers. This resembles very much the well-known case of a Poisson square wave process. For $E[\Delta]E[Z] > 1/\lambda$ the clusters overlap, but in the equilibrium case the expected number of Z_σ is finite, provided that $E[\Delta]E[Z] < \infty$. This is exactly the condition ensuring the existence of the stationary process.

The distribution function of the lifetime of a cluster is given by

$$P(L \leq l) = \frac{1}{\mu E[Z]}\int_0^l R(u)\,\mathrm{d}u \qquad l \geq 0 \tag{13}$$

and the distribution function of the forward recurrence time in the equilibrium process can be shown to be

$$P(\gamma \leq h) = \frac{1}{\mu}\int_0^h (1 - F(u))\,\mathrm{d}u \qquad h \geq 0 \tag{14}$$

Marked Poisson cluster process

The previous point process model can be generalized by assigning a mark to each renewal. The marks are independent, identically distributed (i.i.d.) random vectors which assume values in a mark space which is a part of R^l. Formally, we have the following definition for the PCP:

$$X(t) = \sum_{i=1}^{N_c(t)} \sum_{j=1}^{\infty} A_i^j I(\{N_p(\tau_i;t) = j\}) I(\{L(\tau_i) \geq t\}) \tag{15}$$

with A_i^j i.i.d. random vectors with distribution $V(A) = V(A_i^j)$. It is further assumed that A_i^j, τ_i, ρ_i^n, and $Z(\sigma)$ are mutually independent. In structural reliability the interpretation of equation (15) is as follows: Load changes occur according to the point process equation (3). To each change there is associated a random vector which characterizes the load, for example, by attributes such as amplitude, oscillator frequency, and pulse shape. If a failure domain $F \subseteq \mathrm{R}^l$ is assumed, the structure fails if the process (15) enters this domain. A typical realization of the PCP is shown in *Figure 1*.

In order to bound the probability of failure according to equation (1), let $M_F(t)$ denote the counting process which counts the number of exits of $X(\sigma)$ from $\bar{F}$ into F during the time interval $(0,t]$. As shown in Schrupp[18] the expected value of $M_F(t)$ is

$$E[M_F(t)] = \int_0^t \kappa(\sigma)\,\mathrm{d}\sigma \tag{16}$$

with outcrossing rate

$$\kappa(\sigma) = \lim_{h \downarrow 0}\frac{1}{h}P(M_F(\sigma,\sigma + h) = 1)$$

and where $\{M_F(\sigma,\sigma + h) = 1\}$ denotes the event that the counting process M_F has an event in $(\sigma, \sigma + h]$.

For the calculation of the instantaneous crossing rate $\kappa(\sigma)$ in equation (16), we observe that the PCP (15) can have three different types of jumps in a small interval $(\sigma, \sigma + h]$ (compare *Figure 1*): A jump occurs if

(i) a new subsidiary process is generated in $(\sigma, \sigma + h]$, denoted by $\{N_1(\sigma,\sigma + h) = 1\}$,
(ii) an active subsidiary process dies out in $(\sigma, \sigma + h]$, denoted by $\{N_2(\sigma,\sigma + h) = 1\}$,
(iii) an active subsidiary process has a renewal in $(\sigma, \sigma + h]$, denoted by $\{N_3(\sigma,\sigma + h) = 1\}$.

For small h the regularity of the cluster point process ensures that these three cases exhaust all possibilities for a jump in the PCP.

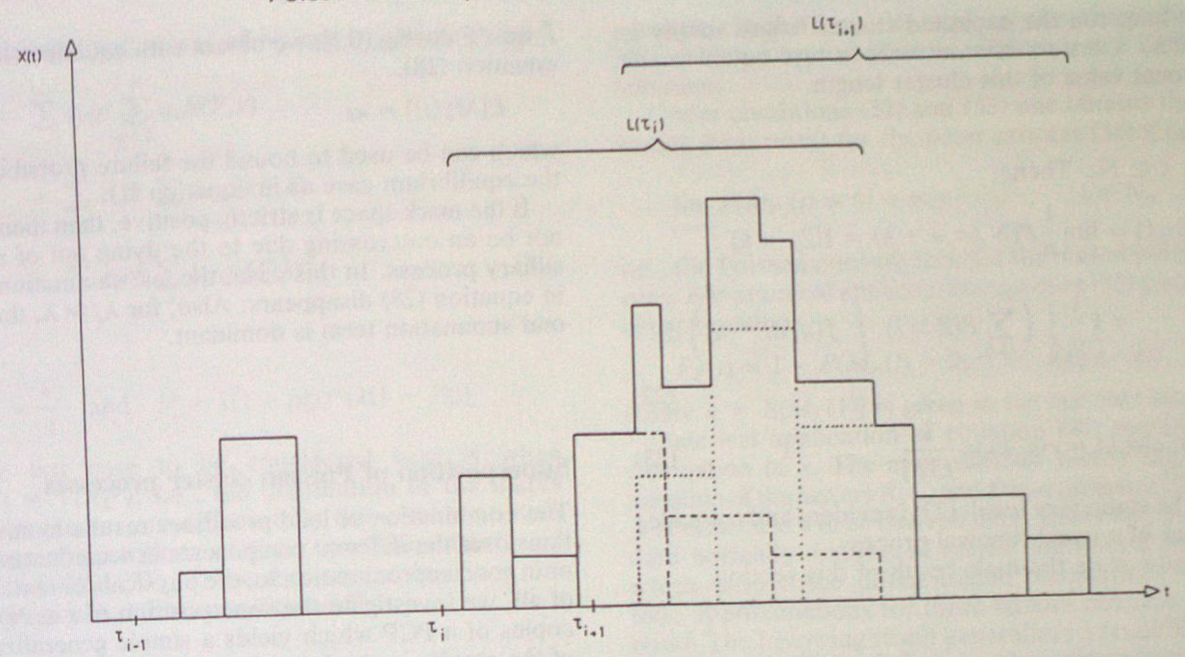

Figure 1 The Poisson cluster process (PCP)

The conditional intensity rates

$$\kappa_i(\sigma,\kappa) = \lim_{h \downarrow 0}\frac{1}{h}P(N_i(\sigma,\sigma + h) = 1 | Z_\sigma = k) \qquad k \in \mathrm{N}_0 \quad i = 1, 2, 3 \tag{17}$$

and their limits for the steady state $\sigma \to \infty$ are given below. The proofs can be found in Schrupp.[18]

Lemma 1

Let $k \in \mathrm{N}_0$ and $F' = f$. Then

$$\begin{aligned} b = b(\sigma,k) &= \lim_{h \downarrow 0}\frac{1}{h}P(N_1(\sigma,\sigma + h) = 1 | Z_\sigma = k) \\ &= \lambda(1 - p_0)f(\lambda(1 - p_0)) \end{aligned} \tag{18}$$

with

$$\begin{aligned} f(\tau(\sigma)) &= \lim_{h \downarrow 0}\frac{1}{h}P(N_1(\sigma,\sigma + h) \\ &= 1 | \bar{N}_c(0,\sigma), N_1(0,\sigma)) \end{aligned}$$

being the conditional intensity of N_1 given the entire history of $\bar{N}_c$ and N_1 up to time instant σ, $\bar{N}_c$ a Poisson process with rate $\lambda(1 - p_0)$,

$$f(s) = \int_0^\infty e^{-su} f(u)\,\mathrm{d}u$$

the Laplace transform of f, and

$$\tau(\sigma) = \sigma - \min\{u \leq \sigma | \bar{N}_c(0,u) = \bar{N}_c(0,\sigma)\}$$

the backward recurrence time of $\bar{N}_c$ at instant σ.

To make this result more transparent, let us consider the following example with $V(\Delta_1)$ an exponential distribution with parameter $\beta > 0$. We obtain

$$\begin{aligned} b &= \lambda(1 - p_0)f(\lambda(1 - p_0)) \\ &= \lambda(1 - p_0)\int_0^\infty e^{-\lambda(1-p_0)u}\beta e^{-\beta u}\,\mathrm{d}u \\ &= \frac{\beta\lambda(1 - p_0)}{\lambda(1 - p_0) + \beta} = \frac{1}{1/\beta + 1/\lambda(1 - p_0)} \\ &= \frac{1}{E[\Delta_1] + 1/E[\bar{N}_c(0,1)]} \end{aligned} \tag{19}$$

Observing that $E[\Delta_1] > 0$, we obtain $b < \lambda(1 - p_0)$ and equation (19) is plausible. The quantity $\lambda(1 - p_0)$ is the rate of the Poisson process $\bar{N}_c$. The process N_1 starts with a delay measured from the last event of $\bar{N}_c$. Thus, the intensity has to be smaller and depends on the first waiting time distribution.

Lemma 2

Let $k \in \mathrm{N}$. Then

$$\begin{aligned} d(\sigma,k) &= \lim_{h \downarrow 0}\frac{1}{h}P(N_2(\sigma,\sigma + h) = 1 | Z_\sigma = k) \\ &= \frac{k}{\bar{\sigma}}\int_0^\sigma \lambda(\tau)\sum_{k=0}^\infty p_k f^{k*}(\sigma - \tau)\,\mathrm{d}\tau \end{aligned} \tag{20}$$

with $\bar{\sigma}$ from equation (7).

$$\lim_{\sigma \to \infty} d(\sigma,k) = \frac{k}{E[Z]E[\Delta]} \tag{21}$$

Result (21) for the stationary case is again plausible.

In the long run the expected cluster length simply is $E[Z]E[\Delta]$. Clusters die out with a rate equal to the reciprocal value of this cluster length.

Lemma 3

Let $k \in \mathrm{N}_0$. Then

$$c(\sigma,k) = \lim_{h \downarrow 0} \frac{1}{h} P(N_3(\sigma,\sigma + h) = 1 | Z_\sigma = k)$$
$$= k\frac{1}{\sigma}\int_0^{\sigma} \left(\sum_{i=0}^{\infty} P(Z > i) \int_0^{\sigma-\tau} f(u)\,\mathrm{d}F^{i*}(u) \right) \mathrm{d}\tau \tag{22}$$

$$\lim_{\sigma\to\infty} c(\sigma,k) = c(\infty,k) = \frac{k}{E[\Delta]} \tag{23}$$

Again the stationary result (23) is evident and coincides with that of a usual renewal process.

We now state the main result of this section.

Theorem

The outcrossing rate $\kappa(\sigma)$ of the PCP (15) is given by

$$\kappa(\sigma) = b \sum_{k=0}^{\infty} q_k(\sigma)B(F,k) + \sum_{k=1}^{\infty} d(\sigma,k)q_k(\sigma)D(F,k) + \sum_{k=1}^{\infty} c(\sigma,k)q_k(\sigma)C(F,k) \tag{24}$$

with b, $d(\sigma,k)$, and $c(\sigma,k)$ given in equations (18), (20), and (22), respectively, and $q_k(\sigma)$ given in equation (5):

$$B(F,k) = \int_F P(A + x \in F) f^{k*}(x)\,\mathrm{d}x \qquad k \in \mathrm{N}$$
$$B(F,0) = P(A \in F) I(\{0 \in \overline{F}\}) \tag{25}$$

$$D(F,k) = \int_F (A + x \in \overline{F}) f^{(k-1)*}(x)\,\mathrm{d}x \qquad k \geq 2$$
$$D(F,1) = P(A \in \overline{F}) I(\{0 \in F\}) \tag{26}$$

$$C(F,k) = \int_{\mathbb{R}^l} P(A + x \in F) P(A + x \in \overline{F})\, f^{(k-1)*} \times (x)\,\mathrm{d}x \qquad k \geq 2 \tag{27}$$
$$C(F,1) = P(A \in F) P(A \in \overline{F}).$$

The outcrossing rate κ of the stationary equilibrium PCP simplifies to

$$\kappa = \lim_{\sigma\to\infty} \kappa(\sigma)$$
$$= b \sum_{k=0}^{\infty} q_k B(F,k) + \frac{1}{E[Z]E[\Delta]} \sum_{k=1}^{\infty} k q_k D(F,k) + \frac{1}{E[\Delta]} \sum_{k=1}^{\infty} k q_k C(F,k) \tag{28}$$

with $q_k = \exp(-\lambda_e)\lambda_e^k/k!$, b given in equation (18) and $\lambda_e = \lambda E[\Delta]E[Z]$ given in equation (9). Denoting by $M_F^e(t)$ the equilibrium number of crossings of $X(\sigma)$ from $\overline{F}$ into F during $(0,t]$, we obtain with equation (16) and equation (28),

$$E[M_F^e(t)] = \kappa t \tag{29}$$

which can be used to bound the failure probability in the equilibrium case as in equation (1).

If the mark space is strictly positive, then there cannot be an outcrossing due to the dying out of a subsidiary process. In this case, the last summation term in equation (28) disappears. Also, for $\lambda_e \gg \lambda$, the second summation term is dominant.

Superposition of Poisson cluster processes

The combination of load processes results in summations over the different components in accordance with, or in good approximation to, the physical context. First of all, we investigate the superposition of $s \in N$ i.i.d. copies of a PCP which yields a simple generalization of the results in the foregoing section. Only the equilibrium results are given.

Let

$$X(t) = \sum_{\nu=1}^{s} X_\nu(t)$$
$$= \sum_{\nu=1}^{s} \sum_{i=1}^{N_c(t)} \sum_{j=1}^{\infty} A_j^{i,\nu} I(\{N_p^\nu(\tau_i;t)\}) I(\{L^\nu(\tau_i) \geq t\}) \tag{30}$$

be the superposition of s i.i.d. PCPs $X_\nu(t)$. As will be seen, the calculation of the crossing rate of equation (30) is rather straightforward. The only difference to the univariate case is that the number of active processes at instant σ needs to be considered. Observing that

$$D = \sum_{\nu=1}^{s} D^\nu$$

with D^ν the number of active clusters in the component ν, and using the i.i.d. condition, it follows that $V(D)$ is Poissonian with parameter $s\lambda_e$ with λ_e from equation (5). Thus, the results of the last section can be used with this slight modification.

More generally, the process $X(t)$ is defined as in equation (3), all variables are independent, but now the cluster processes N_p^ν have different distributions. This can be described by different waiting time distributions $V(\Delta^\nu)$ or different occurrence distributions $V(Z^\nu)$ in the component processes $\nu = 1, \ldots, s$. Both results in a modification of $V(D) = V(\Sigma_{\nu=1}^{s} D^\nu)$ and a change in the conditional intensity functions given in Lemmas 1–3. The first type of modification is simple. $V(D)$ is still Poissonian, but now with parameter $\lambda = \Sigma_{\nu=1}^{s}\lambda_\nu$ and $\lambda_\nu = E[Z^\nu]$ as in equation (5) with the obvious modifications. More subtle is the change of the intensity functions when calculating $\kappa(\sigma)$.

The crossing rate κ of the superposition process (30) for different distributions of N_p^ν can be shown to be

$$\kappa = \sum_{\nu=1}^{s} r_\nu b^\nu \sum_{k=0}^{\infty} q_k B(F,k) + \sum_{\nu=1}^{s} r_\nu d^\nu \sum_{k=1}^{\infty} k q_k D(F,k) + \sum_{\nu=1}^{s} r_\nu c^\nu \sum_{k=1}^{\infty} k q_k C(F,k) \tag{31}$$

with

$$r_\nu = \frac{\lambda_\nu}{\lambda} \quad \text{and} \quad b^\nu = \lambda(1 - p_0^\nu) f^\nu(\lambda(1 - P_0^\nu))$$

The last case to be considered here is when $V(A_j^{i,\nu}) \neq V(A_k^{l,\mu})$; i.e., the distribution of the marks in the processes are different. In this case we obtain different compositions of the convolution factors in equations (25)–(27). These factors can be given explicitly but are rather complicated.[18]

Poisson convergence for the equilibrium process and an asymptotic approximation for the outcrossing rate

Using equation (1) and the results of the last two sections, we can give an upper bound for the failure probability. We now discuss another approach by using the fact that under some specific conditions the equilibrium point process of crossings $M_F(t)$ has an asymptotic Poisson distribution. This limit distribution can be used to approximate the failure probability $P_F(t)$, a procedure which is common in deriving limit distributions for the extreme values of stochastic processes.[14]

The first condition for the process $M_F(t)$ to approach the Poisson process is the mixing condition; i.e.,

$$\sup_A \sup_B |P(A \cap B) - P(A)P(B)| \leq g(t) \tag{32}$$

with $\lim_{t\to\infty} g(t) = 0$ for all events A and B, where A depends only on the behavior of the process M_F until time instant t_1, and B depends only on the behavior of $M_F(\cdot)$ after time instant $t_2 > t_1$, with $t_2 - t_1 > t$.

Condition (32) states that the probabilistic behavior of events which are separated in time becomes asymptotically independent. The equilibrium process M_F satisfies condition (32) as shown in Schrupp.[18]

The second condition is achieved by a scaling of the time axis. Let $(F_n)_{n\in\mathrm{N}}$ be a sequence of failure domains with

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} E[M_{F_n}(1)] = 0 \tag{33}$$

We consider the sequence of scaled processes

$$\tilde{M}_{F_n}(t) = M_{F_n}\left(\frac{t}{m_n}\right) \tag{34}$$

The meaning of (34) is as follows: Under condition (33) the point process $M_{F_n}(t)$ counts rare events and degenerates in the limit. By scaling, the distribution of M_{F_n} is blown up to obtain a nondegenerate limit distribution.

Under conditions (32) and (33) one obtains the following limit result for the point process (34):[18]

$$\lim_{n\to\infty} P(\tilde{M}_{F_n}(t) = k) = \exp\frac{(-t)t^k}{k!} \qquad k \in \mathrm{N}_0 \tag{35}$$

i.e., the Poisson convergence for the scaled point process. For practical applications equation (35) yields the approximation

$$P_F(t) = 1 - P(M_F(t) = 0) \approx 1 - \exp(-\kappa t) \tag{36}$$

where $\kappa = E[M_F(1)]$ is given in the last two sections.

Practical application of equation (36) requires the calculation of κ. The most difficult part is the determination of the factors B, C, and D in equations (25)–(27). A special case will be derived later. However, for standard normally distributed marks and failure domains which have twice differentiable boundaries, asymptotic approximations for these factors can also be derived. The following result generalizes a result of Breitung,[19] where an asymptotic approximation for the outcrossing rate of the classical renewal square wave process is given. Since, for independent marks, standard normality can always be achieved by an appropriate probability distribution transformation, the result is of quite general nature.[20]

Let the failure domain F for k active processes be given by

$$F_k = \{\mathbf{x} \in \mathrm{R}^k \mid g_k(\mathbf{x}) < 0\} \tag{37}$$

and the safe domain S by

$$S_k = \{\mathbf{x} \in \mathrm{R}^k \mid g_k(\mathbf{x}) > 0\} \tag{38}$$

with $g_k\colon \mathrm{R}^k \to \mathrm{R}$ a twice differentiable function. The factors in equations (25)–(27) can also be written as

$$B(F,k) = P(\{X(t-) \in S_k\} \cap \{X(t) \in F_{k+1}\}) \tag{39}$$
$$B(F,0) = I(\{0 \in \overline{F}\}) P(X(t) \in F_1)$$
$$C(F,k) = P(\{X(t-) \in S_k\} \cap \{X(t) \in F_k\}) \tag{40}$$
$$D(F,k) = P(\{X(t-) \in S_k\} \cap \{X(t) \in F_{k-1}\}) \tag{41}$$
$$D(F,1) = P(X(t) \in \overline{F}) I(\{0 \in F\})$$

where $X(t-)$ and $X(t)$ are the left and right limits at time instant t of the PCP, respectively.

Asymptotic approximations for equations (39)–(41) can then be developed by closely following the arguments in Breitung.[19] Making use of $P(A \cap B) = P(B) - P(\overline{A} \cap B)$ and neglecting the last summand asymptotically, one obtains

$$B(F,k) \approx \varphi(-\beta_{k+1}) \prod_{j=1}^{k} (1 - \rho_j^{k+1})^{-1/2} \qquad k \in \mathrm{N}$$
$$B(F,0) \approx \varphi(-\beta_1) \tag{42}$$

$$C(F,k) \approx \varphi(-\beta_k) \prod_{j=1}^{k-1} (1 - \rho_j^{k})^{-1/2} \qquad k \in \mathrm{N} \tag{43}$$

$$D(F,k) \approx \varphi(-\beta_{k-1}) \prod_{j=1}^{k-2} (1-\rho_j^{k-1})^{-1/2} \quad k>2 \qquad (44)$$

$$D(F,1) \approx 0$$

Here, β_k is the minimal distance of $g_k(\mathbf{x}) = 0$ to the origin:

$$\beta_k = \|\mathbf{x}_k^*\| = \min\{\|\mathbf{x}\| \mid \mathbf{X} \in \mathrm{R}^k, g_k(\mathbf{x}) = 0\} \qquad (45)$$

and ρ_j^k are the main curvatures of $g_k(\mathbf{x}) = 0$ at $\mathbf{x} = \mathbf{x}_k^*$. For large values of β_k these approximations can be used in equation (28). As shown by Breitung,[19] the limit results (35) and (36) remain valid when these approximations are used.

A final remark appears suitable. An obvious difficulty in practical computations is the infinite summation required for the calculation of exact, or even asymptotic, outcrossing rates. Observing that the factors B, C, and D are not greater than unity and using the bounds

$$\sum_{k \in N_0} q_k < \sum_{k=K+1}^{\infty} q_k \equiv \epsilon_1$$

and

$$\sum_{k \in N} k q_k < \sum_{k=K+1}^{\infty} k q_k = \lambda_e \sum_{k=K}^{\infty} q_k \equiv \epsilon_2$$

the error of a finite summation, up to K, say, can be bounded by

$$r(K) \leq b\epsilon_1 + \left(\frac{1}{E[Z]E[\Delta]} + \frac{1}{E[\Delta]}\right)\epsilon_2 \equiv \epsilon$$

Thus, for given $\epsilon > 0$ we can find a $K = K(\epsilon,\lambda_e)$ such that the rest of the series is smaller than ϵ.

Example

We now discuss a special but practical important example in which κ can be given explicitly. Let $N_c(t)$ be a homogeneous Poisson process with parameter $\lambda = 1$ generating the cluster centers τ_i, $i \in N$. The equilibrium crossing rate for the superposition of $s \in N$ i.i.d. PCPs is given by $\lambda_e = sE[\Delta]E[Z]$, according to the fourth section.

The mark space for the point process is assumed to be the real line. The distribution of the marks is $V(A) = N(\mu,\sigma^2)$ and the failure domain is $F = (a,\infty)$, $a > 0$. Then, according to equations (25)–(28), we have the exact formulae by using a table for normal integrals given by Owen:[21]

$$B(F,k) = \int_{-\infty}^{a} \phi\left(\frac{x+\mu-a}{\sigma}\right)\frac{1}{\sqrt{k}\sigma}\varphi\left(\frac{x-k\mu}{\sqrt{k}\sigma}\right)\mathrm{d}x$$

$$= \phi_2\left[\frac{((k+1)\mu-a)/\sigma}{\sqrt{k+1}}, \frac{a-k\mu}{\sqrt{k}\sigma}; -\frac{\sqrt{k}}{\sqrt{k+1}}\right] \qquad k \in \mathrm{N}$$

$$B(F,0) = \phi\left(\frac{\mu-a}{\sigma}\right)$$

$$D(F,k+1) = \int_{a}^{\infty} \phi\left(\frac{a-x-\mu}{\sigma}\right)\frac{1}{\sqrt{k}\sigma}\varphi\left(\frac{x-k\mu}{\sqrt{k}\sigma}\right)\mathrm{d}x$$

$$= \phi\left[\frac{(a-(k+1)\mu)/\sigma}{\sqrt{k+1}}\right] - \phi_2\left[\frac{(a-(k+1)\mu)/\sigma}{\sqrt{k+1}}, \frac{(a-k\mu)/\sigma}{\sqrt{k}}; \frac{\sqrt{k}}{\sqrt{k+1}}\right] \qquad k \in \mathrm{N}$$

$$D(F,1) = 0$$

$$C(F,k+1) = \int_{\mathrm{R}} \phi\left(\frac{x+\mu-a}{\sigma}\right)\phi\left(\frac{a-x-\mu}{\sigma}\right)\frac{1}{\sqrt{k}\sigma}\varphi\left(\frac{x-k\mu}{\sqrt{k}\sigma}\right)\mathrm{d}x$$

$$= \phi_2\left[\frac{(a-(k+1)\mu)/\sigma}{\sqrt{k+1}}, \frac{((k+1)\mu-a)/\sigma}{\sqrt{k+1}}; -\frac{k}{\sqrt{k+1}}\right] \qquad k \in \mathrm{N}$$

$$C(F,1) = \phi\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{\mu-a}{\sigma}\right)$$

where $\phi_2[x,y;\rho] = \phi(x)\phi(y) + \int_0^\rho \varphi_2(x,y;\rho)\,\mathrm{d}\rho$ is the standardized bivariate normal distribution function with correlation coefficient ρ, and $\varphi_2(x,y;\rho)$ is its density.

The factors B, C, and D could also be given analytically for marks having nonvanishing auto- and cross-correlations, following the approach in Rackwitz.[9]

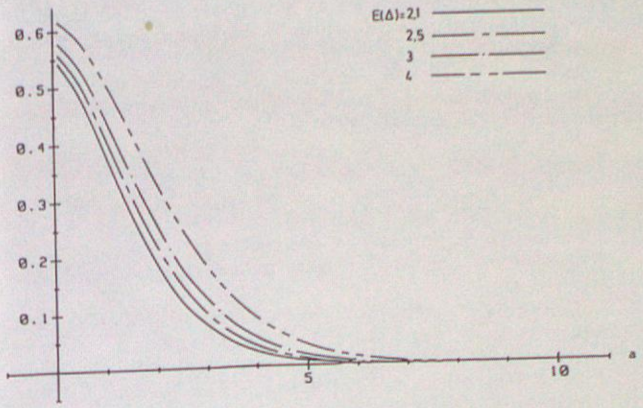

Figure 2 Variation of $E(\Delta)$

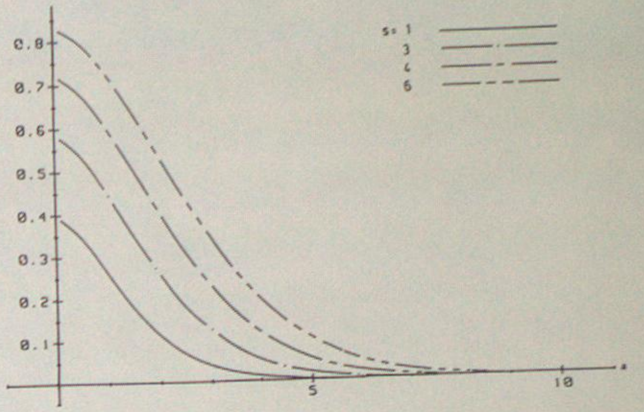

Figure 4 Variation of S

However, it appears difficult to assess a proper correlation matrix among the marks of different subsidiary processes, so this possibility is not pursued further.

In *Figures 2–4* the outcrossing rates for different values of the parameters $E[\Delta]$, $E[Z]$, and s are plotted against the threshold $a \in [0,10]$. In each figure one parameter varies while standard values are kept for the two other parameters. The standard values are $E[\Delta] = 1$, $E[Z] = 1$, and $s = 1$. The parameters of the normal distribution are $\mu = 0$ and $\sigma = 1$. The summation procedure in equation (30) has been truncated at $K = 30$, implying an error smaller than 10^{-9} in all cases. As expected, the outcrossing rate in equation (28) is dominated by the last term, which refers to the load changes.

In *Figure 5* lower and upper bounds for the failure probability are given for the standard values of the parameters. The upper bound is given in equation (1); the lower bound can be found in Bolotin,[13] page 377. As can be seen, the estimation of the failure probability by equation (1) becomes fairly accurate even for moderate values of the threshold a.

In *Figure 6* the outcrossing rate equation (28) is plotted against the outcrossing rate of a Poisson square wave process with the same parameters for λ and $E[\Delta]$. This rate can be found in Breitung and Rackwitz.[8] The figure shows that clustering increases the outcrossing rate and, thus, cannot be ignored when modelling stochastic load processes for clustered phenomena.

In *Figure 7* the crossing rate equation (28) was calculated with the parameters as in Wen and Pearce.[11] They used a normal distribution with $\mu = 1$ and $\sigma = 0{,}3$. Their expected cluster length is $E[\Delta]E[Z] = E[L] = 0.001$, the Poisson parameter is $\lambda = 6$, and the number of superposed processes is $s = 2$. For different values of $E[\Delta]$, keeping the product $E[\Delta]E[Z]$ at the constant value 0.001, the outcrossing rate equation (28) is plotted against the threshold $a \in [1,4]$. For smaller values of $E[\Delta]$ the outcrossing rate increases in accordance with equation (28). The change rate of the loads $1/E[\Delta]$ increases and, simultaneously, with it the contribution of the last summand in equation (28). In their model Wen and Pearce obtained smaller values for κ, which is due to the strong filtering effect; i.e., the probability that clusters in a single process overlap is small.

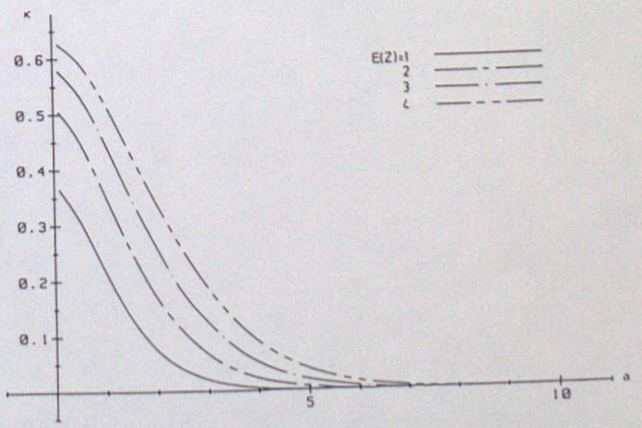

Figure 3 Variation of $E(z)$

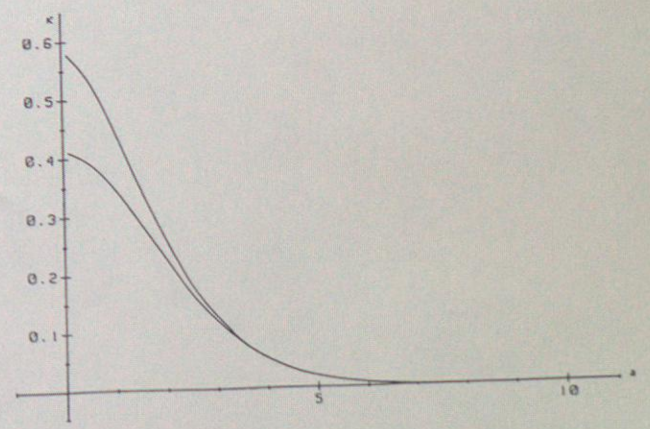

Figure 5 Upper and lower bound for the failure probability

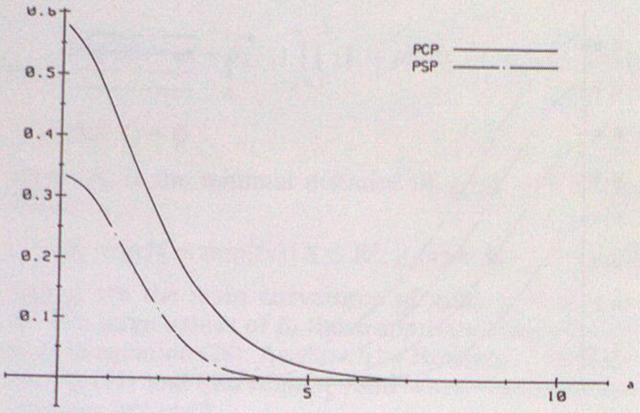

Figure 6 Comparison PCP and Poisson square wave process (PSP)

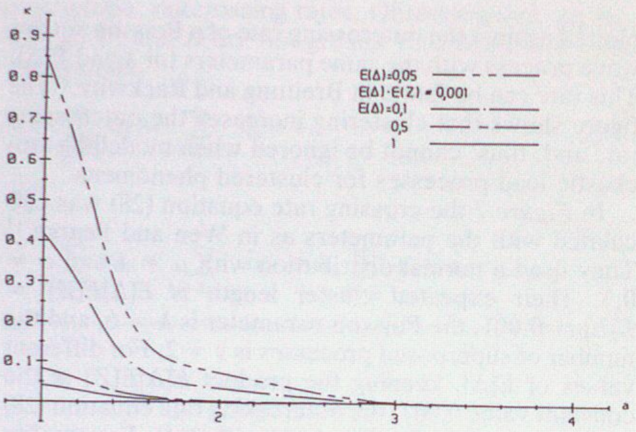

Figure 7 Outcrossing rate of PCP for the parameters in Wen and Pearce

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