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Nonlinear Combination of Load Processes

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ABSTRACT

An important problem in structural reliability is the combination of time-variant stochastic loadings. The failure probability in the case of nonlinear combinations of stochastic processes can be approximated by the mean crossing rates of time-variant loadings out of the safe domain of structural states. Except for linearly bounded safe domains, these rates are difficult to compute; therefore, the quality of various linearization points is investigated. Comparisons are made of some examples showing that for time-variant problems the Hasofer-Lind point is often suboptimal. A linearization at the point of maximum local outcrossing density might generally produce non-degenerate, sufficiently accurate results.

I. INTRODUCTION

The combination of actions upon a structure is one of the crucial points in safe and economical design. Efficient combination rules must take proper account of the variations in time and space. The time-variant aspects are most important. In the past a variety of exact or approximate methods for the probabilistic treatment of linear combination have been proposed. The set of solutions and types of load processes considered are sufficient to fulfill many practical needs, e.g., in code making or direct probabilistic design. However, examples of nonlinear combination are very few. The only nonlinear, approximate solutions of practical interest appear to be for loads modeled as independent random sequences [13], special independent filtered Poisson processes [14], and for Gaussian processes [15]. While the first two references make use of the fast convolution techniques of first-order reliability methods [6], a different approach has been chosen in Ref. 15. The latter is based on a multivariate extension of the problem of first passages of random processes out of given domains; here, in particular, domains of safe structural states. Typically, the computation of the mean outcrossing rate is required which frequently is rather simple if the boundary of the safe domain (failure surface) is a hyperplane and, therefore, the problem is reduced to a problem of crossings of the sum of random processes above a given threshold (see Refs. 2 and 8).

Tangent linearization of nonlinear failure surfaces has been the key to highly accurate and numerically straightforward solutions to time-invariant reliability problems [6]. Specifically, in the space of independent standard normal variates the so-called Hasofer-Lind point has been shown to be superior to alternative choices of linearization [7]. Therefore, it appears important to investigate whether the same point is best for time-variant problems, too, or, if not, to select another linearization point which then would enable utilization of the various solutions available for mean crossing rates out of linearly bounded safe domains. *A priori*, the Hasofer-Lind point appears suitable only in special cases since its determination does not use any information on the nature of variations in time of the various load processes involved.

In the following some results on mean outcrossing rates for different types of load processes will be collected. These results will then be used to discuss appropriate linearization points.

II. TIME-DEPENDENT FAILURE PROBABILITIES

Let D be the safe domain of structural states in the Euclidean space R^n and denote by $\mathbf{X}(t)$ an n-dimensional stationary stochastic process with known probability law. The exact failure probability of a system in a given time-interval $[0, t]$ can be written as

$$P_f(t) = 1 - P(\mathbf{X}(\tau) \in D \text{ for } 0 \leqq \tau \leqq t) \tag{1}$$

which, generally, is difficult to compute. However, it can be bounded by considering the counting process, $M_D(t)$, describing the number of crossings of $\mathbf{X}(\tau)$ out of D in $[0, t]$. In particular, denoting with $P_f(0)$ the probability that the process starts in the failure domain at $\tau = 0$, one can write:

$$\begin{aligned} P_f(t) &= P_f(0) + [1 - P_f(0)]P[M_D(t) > 0 \mid X(0) \in D] \\ &= P_f(0) + [1 - P_f(0)] \sum_{j=1}^{\infty} P[M_D(t) = j \mid X(0) \in D] \\ &\leqq P_f(0) + [1 - P_f(0)] \sum_{j=1}^{\infty} jP[M_D(t) = j \mid X(0) \in D] \\ &= P_f(0) + [1 - P_f(0)]E[M_D(t) \mid X(0) \in D] \\ &\leqq P_f(0) + E(M_D(t)) \\ &= P_f(0) + \lambda(D)\cdot t \end{aligned} \tag{2}$$

where $\lambda(D)$ is the stationary mean outcrossing rate defined by

$$\lambda(D) = \lim_{\Delta \to 0} \frac{E[M_D(\Delta)]}{\Delta} \tag{3}$$

Equation (2) should be close to the exact result for sufficiently small $P_f(0)$ and $P[M_D(t) = 1] \gg \sum_{j=2}^{\infty} jP[M_D(t) = j]$. Note that the first line of the right-hand side of Eq. (2) simply is the sum of the probability that $\mathbf{X}(\tau)$ starts in the failure domain at $\tau = 0$ plus the probability that it starts in the safe domain times the probability of at least one outcrossing in $[0, t]$.

III. SOME MULTIVARIATE RESULTS ON MEAN OUTCROSSING RATES

A. Renewal Square Wave Load Processes

The univariate process can be defined by

$$X(t) = Y_k \qquad \text{if } \sum_{r=1}^{k-1} T_k \leqq t < \sum_{r=1}^{k} T_k \tag{4}$$

where Y_k ($k \geqq 1$) determines the random amplitude of the loads and the sequence T_k ($k \geqq 1$) determines the random duration of a square wave pulse. The sequence T_k has renewal intensity λ. This process has repeatedly been suggested for the modeling of occupancy loading (see, e.g., Ref. 12). Definition (4) can be generalized to multivariate Poisson processes, e.g., a vector process whose ith component ($i = 1, ..., n$) is determined by the renewal load process with intensity λ_i and amplitude $Y_{k,i}$ ($k = 1, 2, ...$), all processes being independent of each other. Figure 1 illustrates a typical sample path in the plane also indicating schematically the safe and unsafe set of structural states. The independence assumption just made explains why the mean number of outcrossings in an interval $[0, t]$ out of an arbitrary domain D is just the sum of the mean numbers of renewals of a component multiplied by the probability that the process leaves D in the direction of that component. Therefore, for the mean outcrossing intensity, we have [4]:

$$\lambda(D) = \sum_{i=1}^{n} \lambda_i \int_{R^n} P[\mathbf{Y}_{1,i} + \mathbf{x} \in D] P[\mathbf{Y}_{1,i} + \mathbf{x} \notin D] f_i^*(\mathbf{x})\, d\mathbf{x} \qquad (5)$$

where

$$f_i^*(\mathbf{x}) = \delta_0(x_i) \prod_{\substack{j=1 \\ j \neq 1}}^{n} f_j(x_j)$$

with $f_j(x_j)$ denoting the probability density function of $Y_{1,j}$ and $\delta_0(x_i)$ denoting the Dirac-delta-function. Obviously, $\lambda_i = 0$ corresponds to a time-invariant component. The integral in Eq. (5) is understood as an n-dimensional integral over the entire state space of $\mathbf{X}$. $\mathbf{Y}_{1,i} = (0, ..., Y_{1,i}, ..., 0)$ designates the component of $\mathbf{X}(\tau)$ which has a renewal and has the same

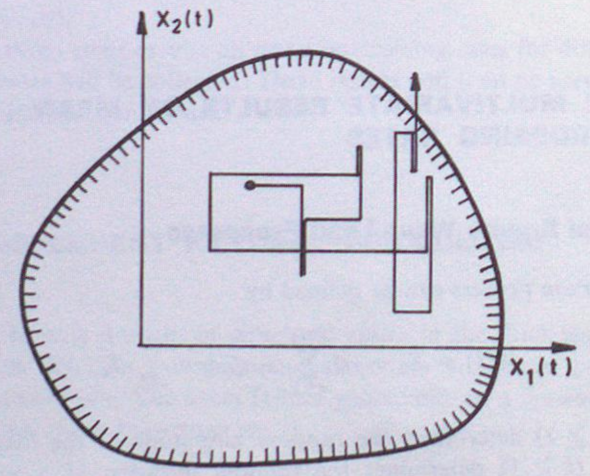

Fig. 1 Sample path of a two-dimensional renewal load process.

distribution as $Y_{k,l}$. In words: The first term in the integral is the probability of remaining inside D, the second term the probability of being outside D given a renewal of the ith component and the other components fixed at $\mathbf{x} = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n)$ while $f_i^*(\mathbf{x})$ is the joint probability density of the state vector $\mathbf{X}_i$ of the components when X_i has a renewal.

It is worth mentioning that Eq. (5) also holds if the renewals occur deterministically, i.e., for independent random sequences, since only information about the mean number of renewals is used. In this case the renewal intensity simply is given by $\lambda_i = r_i/t$ where r_i is the number of independent load changes or repetitions of the ith load during $[0, t]$.

Formula (5) can be specialized for a number of cases. First, assume that the sequence $Y_{k,i}$ is a normal sequence with mean $E[Y_{k,i}] = 0$ and variance $\mathrm{Var}[Y_{k,i}] = 1$. If $Y_{k,i}$ is not a standard normal sequence, a probability distribution transformation such that $F_i(Z_{1,i}) = \phi(Y_{1,i})$ can be applied where $F_i(.)$ is the original marginal distribution function of the ith component amplitude and $\Phi(.)$ is the standard normal integral. Second, the safe domain D has a particular shape (in the transformed space). Subsequent solutions for certain simple shapes will then provide approximations to more complex problems.

B. Rectangular Safe Domain D_{Re}

The safe domain is given by $D = D_{\mathrm{Re}} = \{\mathbf{x};\ \mathbf{x} \in [\mathbf{a}, \mathbf{b}]\}$. Application of Eq. (5) yields (see Appendix A)

$$\lambda(D_{\mathrm{Re}}) = \sum_{i=1}^{n} \lambda_i[\phi(b_i) - \phi(a_i)][1 - \phi(b_i) + \phi(a_i)] \prod_{\substack{j=1 \\ j \neq i}}^{n} [\phi(b_j) - \phi(a_j)] \qquad (6)$$

In setting a_i or b_i equal to $-\infty$ or $+\infty$, respectively, one obtains semi-hypercube-shaped safe domains.

C. Safe Domain Bounded by a Hyperplane

For $D = D_H = \{\mathbf{x};\ \boldsymbol{\alpha}\mathbf{x} - \beta \leq 0\}$ in which $\boldsymbol{\alpha}$ is the vector of direction cosines and β the distance of the hyperplane $\boldsymbol{\alpha}\mathbf{x} - \beta = 0$ to the coordinate origin, we have (see Appendix B):

$$\lambda(D_H) = \sum_{i=1}^{n} \lambda_i[\phi(\beta) - \phi(\beta, \beta; 1 - \alpha_i^2)] \qquad (7)$$

where

$$\phi(x, y; \rho) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-(x^2 - 2\rho xy + y^2)/(2(1-\rho^2))\right] dy\, dx \qquad (8)$$

which is the binormal distribution function. If the direction cosines are equal, e.g., for the simple case $X(\tau) = \sum_{i=1}^{n} X_i(\tau)$, the correlation coefficient reduces to $\rho = \rho_i = (n-1)/n$. Equation (8) can easily be evaluated (see Ref. 1).

D. Safe Domain Bounded by a Central Hypersphere D_R

Finally, if $D = D_R = \{\mathbf{x};\ \mathbf{x}\mathbf{x}^T < R\}$, a central hypersphere with radius $R^{1/2}$, Eq. (5) yields

$$\lambda(D_R) = \sum_{i=1}^{n} \lambda_i \int_0^R f_{n-1}(x) F_1(R-x)\cdot[1 - F_1(R-x)]\, dx \qquad (9)$$

where $f_{n-1}(.)$ is the probability density function of a χ^2-variable with $n-1$ degrees of freedom and F_1 is the corresponding distribution function for one degree of freedom (for the derivation and some computational techniques, see Appendix C).

E. Filtered Poisson Processes with Rectangular Response Function

Filtered Poisson processes form a wide class of processes suitable for modeling loads. For example, the process can be used to model loads due to vehicles on bridges or storage loads in warehouses. The process is formed by the superposition of a sequence of independent, identically distributed rectangular load pulses X_k with distribution function $F_1(x)$. The pulses are generated by the sequence of times T_k, $k = 1, 2, \ldots$, corresponding to a homogeneous Poisson process with intensity λ. The durations of pulses form a sequence of independent exponentially distributed variables with probability density $h(d) = \nu \exp[-\nu\cdot d]$. In Ref. 3 it is shown that the number of pulses present at time τ has a Poisson distribution with mean $(\lambda/\nu)(1 - \exp[-\nu\tau])$ which approaches stationarity as $\tau \to \infty$. This stationary state is studied further on. The probability density of $X(\tau)$ can then be given as

$$g(x) = \exp\left[-\frac{\lambda}{\nu}\right] \sum_{j=1}^{\infty} \frac{\left(\frac{\lambda}{\nu}\right)^j}{j!} \tilde{f}_j(x) \qquad (10)$$

where $\tilde{f}_k(x)$ is the probability density of $\sum_{i=1}^{k} X_i$ for $k \geqq 1$ and $\delta_0(x)$ for $k = 0$. In passing, we note that $X(\tau)$ has stationary mean $\lambda/\nu\, E[X_1]$ and variance $\lambda/\nu\,(\mathrm{Var}[X_1] + E^2[X_1])$. For small λ/ν, $g(x)$ tends to $f_1(x)$ with a spike of magnitude $\exp[-\lambda/\nu]$ at $x = 0$ while $g(x)$ tends to the normal density for large λ/ν with the parameters given just before. Generalization of the univariate process to an independent vector process is straightforward in characterizing each component by the occurrence rate λ_i, the decay rate ν_i, and the corresponding random load pulses $X_{i,k}$ having probability density $f_{i,1}(x)$. In the same manner $g_i(x)$ denotes the stationary probability density of the ith component and $f_{i,k}(x)$ the probability density of $\sum_{l=1}^{k} X_{i,l}$. The mean crossing rate of a filtered Poisson vector process out of a domain D can now be shown to be

$$\begin{aligned}\lambda(D) = \sum_{i=1}^{n} \Bigg(\lambda_i \int_D g(\mathbf{x}) P[\mathbf{x} + \mathbf{X}_{1,i} \notin D]\, d\mathbf{x} \\ + \nu_i \sum_{k=1}^{\infty} k\cdot P_{k,i} \int_D g^*_{i,k-1}(\mathbf{x}) P[\mathbf{x} + \mathbf{X}_{1,i} \in D]\, d\mathbf{x}\Bigg) \qquad (11)\end{aligned}$$

where $\bar{D}$ is the failure domain,

$$g(\mathbf{x}) = \prod_{i=1}^{n} g_i(x_i)$$

$$P_{k,i} = \frac{(\lambda_i/\nu_i)^k}{k!} e^{-\lambda_i/\nu_i}$$

$$g^*_{i,k}(\mathbf{x}) = \prod_{\substack{j=1 \\ j\neq i}}^{n} g_j(x_j) f_{i,k}(x_i)$$

and $\mathbf{Y}_{l,i}$ is an n-dimensional vector with the ith component having the value of the random variable $Y_{l,i}$ and all other components being zero. Here the first term is the crossing probability if a new pulse occurs in the ith component. The second term is the crossing probability if a pulse vanishes which clearly depends on the number of pulses present, i.e., their probability. Note that unless the infinite sum in Eq. (11) converges rather fast, e.g., for small λ_i/ν_i, the numerical part may become quite cumbersome.

F. Gaussian Vector Processes

In order to enrich the spectrum of loads, some of the results given in Ref. 15 are repeated here. Let $\mathbf{X}(\tau)$ be a stationary continuously differentiable n-dimensional Gaussian process with probability density $f_{\mathbf{X}}(\mathbf{x})$ and $\dot{\mathbf{X}}(\tau)$ be the derivative process. Then, in applying the generalized Rice formula, the mean outcrossing rate is given by

$$\lambda(D) = \int_S E_0^\infty[\dot{X}_n | \mathbf{X} = \mathbf{x}] f_{\mathbf{X}}(\mathbf{x})\, ds(\mathbf{x}) \tag{12}$$

where $S = \{x; g(x) = 0\}$ the failure surface with $ds(\mathbf{x})$ denoting the surface integral, $\dot{X}_n = n^T(\mathbf{x}) \cdot \dot{X}(t)$ with $n(\mathbf{x})$ the unit vector normal to S in $\mathbf{x}$ pointing toward the exterior of D and $E_0^\infty[\dot{X}_n | \mathbf{X} = \mathbf{x}]$ the conditional expectation over positive values of $\dot{X}_n$ for all $\mathbf{X} = \mathbf{x}$. If the process $\mathbf{X}(t)$ has stationary standard normal distribution, e.g., $X(\tau) \sim N(O, I)$ and $\dot{X}(\tau)$ follows $N(\mathbf{O}, \text{diag}\,(\sigma_{ii}^2))$ with I the identity matrix and $\text{diag}\,(\sigma_{ii}^2)$ a diagonal matrix with elements σ_{ii}^2 and no cross-correlations between $X(\tau)$ and $\dot{X}(\tau)$, the mean crossing rate out of a hyperplane $S = S_H = \{\mathbf{x}; \alpha\mathbf{x} - \beta = 0\}$ is given by

$$\lambda(S_H) = \frac{1}{2\pi}\left(\sum_{i=1}^{n} \alpha_i^2 \sigma_{ii}^2\right)^{1/2} \exp\left[-\frac{1}{2}\beta^2\right] \tag{13}$$

For the crossing rates out of other safe domains, e.g., out of rectangles, central and noncentral spheres, the reader is referred to Ref. 15.

G. Other Processes

Quite a number of results on upcrossing rates for sums of random processes, either exact or approximate, have been collected in Refs. 9 and 3 including sums of renewal processes with different type. Further formulas appear to be deriveable, e.g., by using the techniques of Laplace or Fourier transforms. Outcrossing results for other shapes of the safe domain, however, appear to be almost inexistent and, in any case, very difficult to compute.

IV. LOCAL OUTCROSSING RATES

For later use the local outcrossing rate for the above-mentioned processes is briefly derived. In general, it can be defined as follows:

$$\lambda'(\mathbf{x}) = \lim_{|\Delta S| \to 0} \frac{1}{|\Delta S|}\left(\begin{array}{l}\text{Expected number of outcrossings out of } D\\ \text{through } \Delta S \text{ during one time unit}\end{array}\right) \tag{14}$$

where ΔS, containing $\mathbf{x}$, is a certain surface element of $S = \{\mathbf{x}; g(\mathbf{x}) = 0\}$

and $|\Delta S|$ is the area of ΔS. For example, for renewal processes it can immediately be seen that Eq. (14) can be written in the form

$$\lambda'(\mathbf{x}) = \lim_{|\Delta S| \to 0} \frac{1}{|\Delta S|} \sum_{i=1}^{n} \left(\begin{array}{l}\text{Expected number of outcrossings through}\\ \Delta S \text{ in the unit time interval caused by}\\ \text{renewals of the } i\text{th component}\end{array}\right) \tag{15}$$

For the moment, let D be a convex safe domain, containing the coordinate origin.

Also, define

$$g_i(\mathbf{x}) = \min_{-\infty < z < +\infty} \{z; (x_1, x_2, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \in D\} \tag{16}$$

and

$$h_i(\mathbf{x}) = \max_{-\infty < z < +\infty} \{z; (x_1, x_2, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \in D\} \tag{17}$$

for a point $\mathbf{x}$ on $g(\mathbf{x}) = 0$ (see Fig. 2).

If, in the vicinity of $\mathbf{x}$, the surface $g(\mathbf{x}) = 0$ can be described parametrically, then each term of Eq. (15) can be computed by the calculus of surface integrals. Passing to the limits reveals that an outcrossing at $\mathbf{x}$ occurs if the ith component has a renewal and jumps from a value in $[g_i(\mathbf{x}), h_i(\mathbf{x})]$ to a value larger than $h_i(\mathbf{x})$ for $x_i = h_i(\mathbf{x})$ or to a value smaller than $g_i(\mathbf{x})$ for

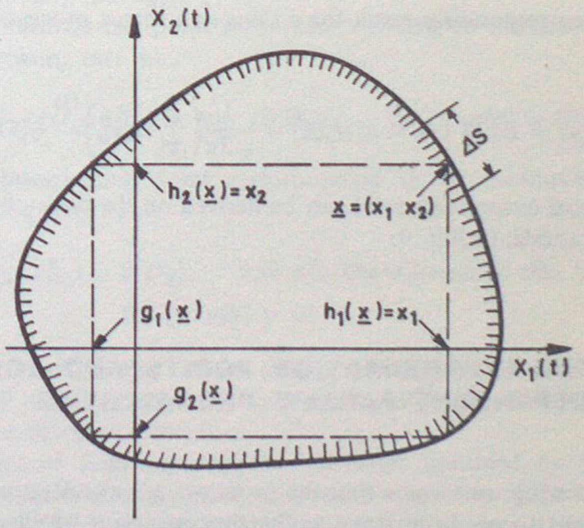

Fig. 2 Two-dimensional example showing $g_i(\mathbf{x})$ and $h_i(\mathbf{x})$.

$x_i = g_i(\mathbf{x})$ provided that the other components take on values at $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Therefore, for normally distributed loads we have

$$\lambda'(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \varphi(x_j)\{\phi[h_i(\mathbf{x})] - \phi[g_i(\mathbf{x})]\}\phi[\mu(x_i)] \cdot |\alpha_i| \tag{18}$$

where $\mu(x_i) = -x_i$ if $x_i = h_i(\mathbf{x})$ and $\mu(x_i) = x_i$ if $x_i = g_i(\mathbf{x})$ and where α_i is the ith component of the unit vector normal to the surface element ΔS in $\mathbf{x}$. In the same manner, local outcrossing rates can be derived for convex safe domains not containing the origin. The same result, of course, is valid for random sequences. The derivation of the local outcrossing rate for filtered Poisson vector processes follows essentially the same line so that only the result is given here:

$$\lambda'(\mathbf{x}) = \sum_{i=1}^{n} g_i^*(\mathbf{x})\left(\lambda_i \sum_{k=0}^{\infty} P_{k,i} P\left(\mathbf{x}_i + \sum_{l=1}^{k} \mathbf{Y}_{l,i} \in D, \mathbf{x}_i + \sum_{l=1}^{k+1} \mathbf{Y}_{l,i} \notin D\right)\right.$$
$$\left. + \nu_i \sum_{k=1}^{\infty} k \cdot P_{k,i} P\left(\mathbf{x}_i + \sum_{l=1}^{k} \mathbf{Y}_{l,i} \in D, \mathbf{x}_i + \sum_{l=1}^{k-1} \mathbf{Y}_{l,i} \notin D\right)\right)|\alpha_i| \tag{19}$$

where

$$g_i^*(\mathbf{x}) = \prod_{\substack{j=1 \\ j \neq i}}^{n} g_j(x_j) \qquad \text{and} \qquad \mathbf{x}_i = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

Finally, the corresponding result for a Gaussian vector process as described above is

$$\lambda'(\mathbf{x}) = E_0^{\infty}[\dot{X} \mid \mathbf{X} = \mathbf{x}] \cdot f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{2\pi}}\left(\sum_{i=1}^{n} \alpha_i^2 \sigma_{ii}^2\right)^{1/2} \cdot f_{\mathbf{x}}(\mathbf{x}) \tag{20}$$

Further local outcrossing rates can be derived on the basis of the material given, for example, in Ref. 9.

V. OPTIMAL APPROXIMATION POINTS AND BOUNDS TO TIME-DEPENDENT FAILURE PROBABILITIES

In the following we assume that the processes are already transformed in the normalized space. In analogy to the time-invariant problem, an upper bound to the failure probability is now obtained by taking an inscribing

central hypersphere as an approximation to the true failure surface (see Ref. 5). Hence Eq. (1) becomes

$$P_f(t) \leqq [1 - \chi_n^2(\beta_{\mathrm{HL}}^2)] + \chi_n^2(\beta_{\mathrm{HL}}^2)\lambda(D_R)t \tag{21}$$

in which $\lambda(D_R)$ is calculated from Eq. (9) or the corresponding formulas for the other processes (see, e.g., Ref. 15) and $\chi_n^2(\cdot)$ is the χ_n^2-distribution with n degrees of freedom. β_{HL} is the safety index according to Hasofer/Lind. However, as pointed out in Ref. 15, this bound can be very conservative both for the time-invariant and time-variant part, particularly for higher dimension of $\mathbf{X}(t)$ so that it is of little practical value. Improved bounds, of course, could be derived for other quadratic approximation surfaces. Their probability content, but more so, their mean outcrossing rate, generally is difficult to compute. While there are some results for the time-invariant part (see, e.g., Ref. 5), very few, precisely only for Gaussian processes, are known for the mean outcrossing rates [15].

Fortunately, practical failure surfaces turn out to be sufficiently flat so that it frequently suffices to use the simpler hyperplane approximations and, hence, $P_f(0) \approx 1 - \phi(\beta_{\mathrm{HL}})$ for the time-invariant part.

As concerns the time-variant part, three linearization points can be proposed (the linearization hyperplane is the one going through the linearization point having the same normal vector as the failure surface at this point):

(*a*) The linearization point $\mathbf{x}_{\mathrm{HL}}^*$ according to Hasofer-Lind with mean outcrossing rate $\lambda_{\mathrm{HL}}(D_H; \mathbf{x}_{\mathrm{HL}}^*)$

(*b*) The linearization point $\mathbf{x}_{\mathrm{MM}}$ corresponding to the maximum mean outcrossing rate, i.e.,

$$\lambda_{\mathrm{MM}}(D_H; \mathbf{x}_{\mathrm{MM}}^*) = \max\{\lambda(D_H)\} \qquad \text{for } \{\mathbf{x}: g(\mathbf{x}) = 0\} \tag{22}$$

(*c*) The linearization point corresponding to the maximum local outcrossing rate, i.e.,

$$\lambda_{\mathrm{ML}}(D_H; x_{\mathrm{ML}}^*) = \lambda(D_H) \qquad \text{with } \mathbf{x}_{\mathrm{ML}}^* \text{ the solution of } \max\{\lambda'(\mathbf{x})\}$$
$$\text{for } \{\mathbf{x}: g(\mathbf{x}) = 0\} \tag{23}$$

For the evaluation of Eq. (22) or (23), one of the well-known methods in mathematical programming may be used [10], although these methods often involve major practical difficulties.

The quality of failure probability estimates obtained by linearization techniques depends strongly on the convexity properties of the safe domain. Obviously, any linearization of the boundary of convex failure domains yields upper bounds on the failure probability while for convex safe domains only lower bounds are obtained. In practice, the convexity properties of the safe domain as well as more detailed descriptions of its shape generally remain unknown and, therefore, results obtained by Eq. (3) together with linearization techniques should be viewed as estimates rather than as probability bounds.

The reason why the Hasofer-Lind point (Alternative *a*) must be rejected as a superior choice simply is the fact that its superiority in the time-invariant case rests on the rotational symmetry of the independent standard multinormal density. This rotational symmetry obviously can, in the nonisotropic time-variant case, be achieved either for states or state changes of the vector process but generally not simultaneously for both characteristics. Only if state changes occur in an isotropic manner, e.g., $\lambda_i = \lambda$, does the Hasofer-Lind point appear suitable.

On the other hand, a linearization at a point where the mean outcrossing rate becomes maximal can produce bounds of uncertain quality (Alternative *b*). For example, if the failure domain is convex, rather conservative probability bounds may be obtained (see Fig. 3a). Moreover, the assumptions made for the validity of Eq. (2) may no longer be met. On the other hand, if the safe domain is convex, the sharpest (linear) lower bound for the second term of Eq. (2) is obtained (Fig. 3b).

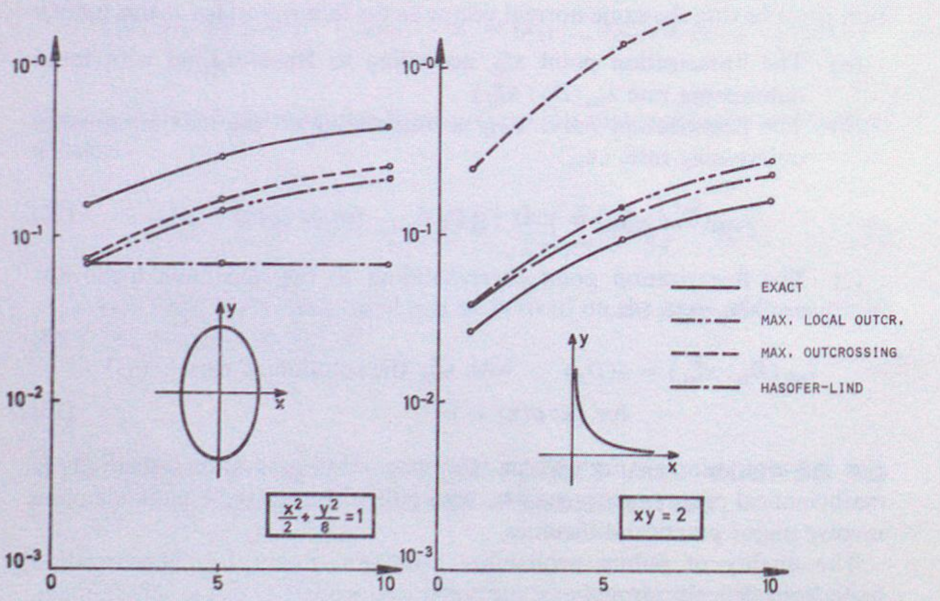

Fig. 3 Approximations to the outcrossing rates for an elliptical (3a) and a hyperbolic domain (3b). (Mean outcrossing rate versus λ_2/λ_1 with $\lambda_1 = 1$.)

The third alternative, a linearization at the point of maximum local outcrossing, appears to be the only stationary point which simultaneously takes proper account of the shape of the safe domain and the "velocity" of state changes as well. At least it does not produce degenerate results for arbitrarily shaped safe domains as they incidentally might be obtained when using the Alternatives *a* or *b*. Clearly, a time-variant analogue to the Hasofer-Lind point viewed as the point of maximum probability density $f(\mathbf{x})$ on $g(\mathbf{x}) = 0$ (in the standardized normal space) complies at best with Alternative *c* which, instead of maximizing the probability density of X on $g(\mathbf{x}) = 0$, maximizes the density of the stream of crossings toward the exterior of the safe domain on $g(\mathbf{x}) = 0$.

Nevertheless, it is concluded that in time-variant problems there is no uniquely best choice of a linearization point, although there may be no significant differences among the three alternatives if the failure surfaces are rather flat.

VI. DISCUSSION AND EXAMPLES

Consider first two simple but extreme two-dimensional cases where either the safe or the failure domain is convex and for which one expects any linearization to produce rather inaccurate results. The two components are renewal load processes X_i with renewal intensity λ_i ($i = 1, 2$) and standard normally distributed amplitudes. Figure 3 shows that for the elliptical region all linearizations yield unconservative mean outcrossing rates. The maximum mean outcrossing rate is, as expected, closest to the exact one. Linearization at the Hasofer-Lind point does not take into account any changes in the ratio of the renewal rates. In the hyperbolic case the contrary is true. The Hasofer-Lind linearization yields the best results but is very close to the maximum local outcrossing rate linearization whereas linearization at the point of maximum mean outcrossing rate produces unrealistically conservative failure probabilities. If the problem dimension is increased, one finds even more drastic differences.

Next, we present a more practical application. Consider the column in a 4-story building in which four random process occupancy loads X_2 to X_5 are acting. In contrast to similar studies for occupancy loadings (see, e.g., Ref. 12), the amplitude distribution is, for illustration purposes, chosen to be of the Weibull type with parameters u_i and k_i, $i = 2, 3, 4$, and 5. The resistance X_1 of the basement column is log-normal with parameters $\hat{\mu}_1$ and δ_1. Dead load is treated as deterministic. Thus the resistance is reduced by that amount. The office loads are independent and change according to the Poisson renewal process with the same intensity λ. The lifetime of the structure is $T = 50$ (years). Then the failure surface can be written as

$$x_1 - \sum_{i=2}^{5} x_i = 0 \tag{24}$$

or, in the normalized space:

$$g(\mathbf{x}) \equiv \hat{\mu}_1 \exp\left[x_1 \cdot \delta_1 - \frac{1}{2}\delta_1^2\right] - \sum_{i=2}^{5} u_i[-\ln\{1 - \phi(x_i)\}]^{1/k_i} = 0 \tag{25}$$

in which the constants $\delta_1 = 0.2$ ($\approx$coefficient of variation), $\hat{\mu}_1 = 10, 15, 20$ ($\approx$mean) and $u_i = 1$ ($\approx$mean), $k_i = 12$ (or coefficient of variation ≈ 0.2) for $i = 2$ to 5. Note that nonlinearity has been introduced into the failure surface solely by the probability distribution transformation which, alternatively or in addition, could have been achieved by a more complex formulation of the mechanical problem. For illustration, Fig. 4(a) demonstrates the nonlinearity of the failure surface due to the probability distribution transformation for the three different means of column resistance.

Figure 4(b) shows the resulting failure probabilities as a function of λt. Due to the slight curvature of the failure surface, the linearization at the point

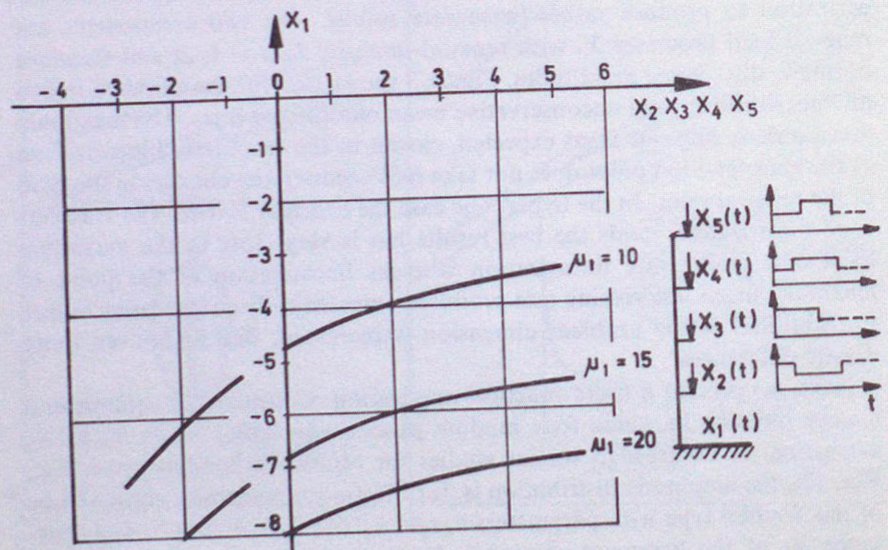

Fig. 4a Failure surface for building example.

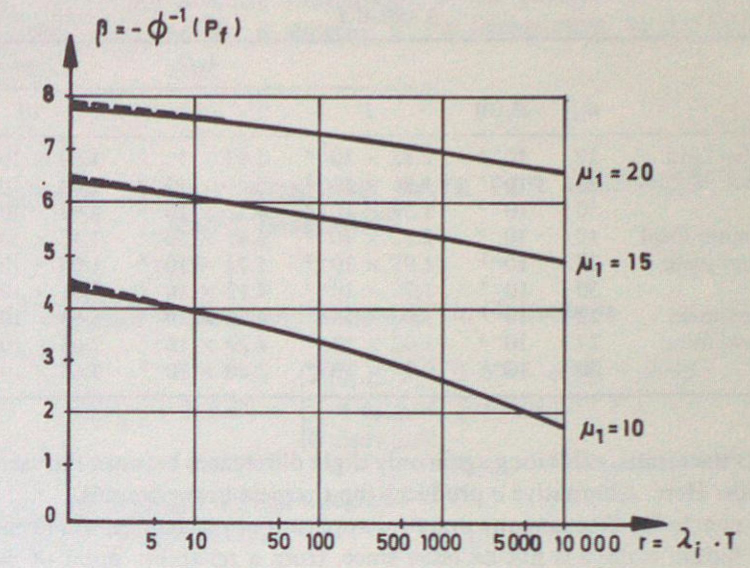

Fig. 4b Failure probability of building example versus renewal intensity λ (the broken line shows the results of the fact convolution technique).

of maximum local outcrossing and at the Hasofer-Lind point as well as application of the fast convolution technique in Ref. 13 yields results which, in Fig. 4(b), are virtually indistinguishable.

Finally, the mechanically highly nonlinear case of a brick wall (no tensile strength) with a normal force X_1, bending moment X_2, and wall strength X_3 is studied (see Refs. 6 and 13 for further details). The normal force X_1 is modeled as a renewal load process with intensity $\lambda_1 = 1$, the bending moment as a renewal load process with varying intensity λ_2. The failure surface is given by

$$g(X_1, X_2, X_3) = \begin{cases} X_3 - (X_1/\delta + 6X_2/\delta^2) = 0 & \text{for } X_2/X_1 < \delta/6 \\ X_3 - (2/3)X_1/(\delta/2 - X_2/X_1) = 0 & \text{for } \delta/6 < X_2/X_1 < \delta/2 \end{cases}$$

where δ is the thickness of the wall. The variables X_1, X_2, X_3 are assumed to have a normal distribution with mean values m_i and variances σ_i^2 ($i = 1, 2, 3$). For $\sigma_1 = 1$, $m_2 = 3$, $\sigma_2 = 0.5$, $m_3 = 150$, $\sigma_3 = 15$ three different values of m_1 and δ are studied, i.e., $(m_1, \delta) = (12, 0.926; 27, 0.610; 50, 0.740)$, where the parameter δ is adjusted such that $P_f(0) = 10^{-3}$. Applying Alternatives a and c, the approximations $P_f(0) + \lambda(D_H) \approx P_f(1)$ are calculated. These are compared with the results of the fast convolution technique. Table 1

TABLE 1

	m_1	$P_f(0)$	λ_2/λ_1 1	5	10
Hasofer-Lind	12	10^{-3}	2.82×10^{-3}	5.93×10^{-3}	1.08×10^{-2}
	27	10^{-3}	1.91×10^{-3}	5.48×10^{-3}	1.04×10^{-2}
	50	10^{-3}	1.59×10^{-3}	4.12×10^{-3}	6.93×10^{-3}
Maximum local outcrossing	12	10^{-3}	2.82×10^{-3}	6.81×10^{-3}	1.17×10^{-2}
	27	10^{-3}	1.97×10^{-3}	5.73×10^{-3}	1.05×10^{-2}
	50	10^{-3}	1.75×10^{-3}	4.12×10^{-3}	7.01×10^{-3}
Approximate convolution	12	10^{-3}	1.00×10^{-3}	4.60×10^{-3}	8.63×10^{-3}
	27	10^{-3}	1.02×10^{-3}	4.29×10^{-3}	7.68×10^{-3}
	50	10^{-3}	9.9×10^{-4}	2.60×10^{-3}	3.65×10^{-3}

collects the results, exhibiting again only slight differences between the various methods. Here, Alternative *c* produces the sharpest lower bounds.

It is concluded that in many practical applications mechanical nonlinearity of the failure surface is not an issue since, from a reliability point of view, failure surfaces remain sufficiently flat locally. Therefore, linearization at the Hasofer-Lind point or, better, at the point of maximum local outcrossing, produces fairly accurate failure probability estimates. The same conclusion holds for the nonlinearity introduced by probability distribution transformations as long as the original distribution functions do not deviate too much from the normal. It might still hold for convex safe domains whose center of gravity, however, is far off the coordinate origin (in the standardized space). The contrary is true for convex safe domains centered around the coordinate origin. Such a situation might, for example, occur in the reliability analysis of slender structures under wind or earthquake excitation or centrally located columns in office buildings. Then other approximating failure surfaces should be used, e.g., hypercubes, hyperspheres, or polyhedra, for which, however, many outcrossing results of practical interest are still missing.

VII. SUMMARY

The nonlinear combination of renewal load processes and of filtered Poisson processes is investigated. The failure probability is approximated by the crossing rates out of the safe domain of structural states. Since these rates are difficult to compute except for domains bounded by hyperplanes, various linearization techniques are tested. Some numerical examples indicate that in the time-variant case the linearization at the point of maximum local outcrossing rate generally is superior to the linearization at the Hasofer-Lind point.

APPENDIX A: OUTCROSSING RATE FOR RECTANGULAR SAFE DOMAINS

The first factor in the second integral in Eq. (5) becomes

$$P[Y_{1,i} + \mathbf{X} \in D] = \begin{cases} P[a_i \leqq Y_{1,i} \leqq b_i] = \phi(b_i) - \phi(a_i) \\ \quad \text{if } a_j \leqq x_j \leqq b_j \text{ for all } j \neq i \\ 0 \text{ elsewhere} \end{cases} \quad \text{(A.1)}$$

with $\phi(.)$ the standard normal integral.

Similarly, the second factor is

$$P[Y_{1,i} + \mathbf{x} \notin D] = P[Y_{1,i} < a_i \cup Y_{1,i} > b_i] = 1 - \phi(b_i) + \phi(a_i) \quad \text{(A.2)}$$

The joint density $f_i^*(\mathbf{x})$ can generally be written as

$$f_i^*(\mathbf{x}) = \delta_0(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \varphi(x_j) \quad \text{(A.3)}$$

in which $\delta_0(.)$ is the Dirac-delta-function and $\varphi(.)$ the standard normal density. Remember the assumed independence assumption between the components of $\mathbf{X}(t)$. Therefore, one obtains

$$\lambda(D_R) = \sum_{i=1}^{n} \lambda_i \int_{a_1}^{b_1} \cdots \int_{a_{i-1}}^{b_{i-1}} \cdots \int_{a_{i+1}}^{b_{i+1}} \cdots \int_{a_n}^{b_n} [\phi(b_i) - \phi(a_i)][1 - \phi(b_i) + \phi(a_i)]$$
$$\times \prod_{\substack{j=1 \\ j \neq i}}^{n} \varphi(x_j)\, dx_n \cdot \cdots \cdot dx_{i+1} \cdot dx_{i-1} \cdot \cdots \cdot dx_1$$
$$= \sum_{i=1}^{n} \lambda_i [\phi(b_i) - \phi(a_i)][1 - \phi(b_i) + \phi(a_i)] \prod_{\substack{j=1 \\ j \neq i}}^{n} (\phi(b_j) - \phi(a_j)) \quad \text{(A.4)}$$

APPENDIX B: CROSSING RATE OUT OF SEMISPACES

With D_H given by $\{\mathbf{x};\, \alpha\mathbf{x} - \beta \leq 0\}$, the first probability in Eq. (5) can be written as

$$
\begin{aligned}
P[Y_{1,i} + \mathbf{x} \in D_H] &= P\left[\alpha_i Y_{1,i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j x_j \leq \beta\right] \\
&= P\left[Y_{1,i} \leq \left(\beta - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j x_j\right)\Big/\alpha_i\right] \\
&= \phi\left[\left(\beta - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j x_j\right)\Big/\alpha_i\right] \qquad \text{(B.1)}
\end{aligned}
$$

whereas the second probability is just the complement with which Eq. (5) becomes on using (A.3):

$$
\begin{aligned}
\lambda(D_H) = \sum_{i=1}^{n} \lambda_i \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi\left(\left(\beta - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j x_j\right)\Big/\alpha_i\right) \cdot \phi\left(\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j x_j - \beta\right)\Big/\alpha_i\right) \\
\times \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi(x_j)\, dx_n \cdot \cdots \cdot dx_{i+1} \cdot dx_{i-1} \cdot \cdots \cdot dx_2 \cdot dx_1 \qquad \text{(B.2)}
\end{aligned}
$$

This $(n-1)$-dimensional integral can be reduced to a one-dimensional integral by applying separately the rotation

$$
(x_1, x_{i-1}, x_{i+1}, \ldots, x_n) \to (y_1 \cdots y_{n-1}) \text{ with:} \quad y_{n-1} = \frac{\sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j x_j}{\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j^2\right)^{1/2}} \qquad \text{(B.3)}
$$

for each coordinate. Substitution of (B.3) into (B.2) yields

$$
\begin{aligned}
\lambda(D_H) = \sum_{i=1}^{n} \lambda_i \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi\left[\left(\beta - \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j^2\right)^{1/2} y_{n-1}\right)\Big/\alpha_i\right] \\
\times \phi\left[\left(\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j^2\right)^{1/2} y_{n-1} - \beta\right)\Big/\alpha_i\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{j=1}^{n-1} \varphi(y_j)\, dy_{n-1} \cdot dy_{n-2} \cdot \cdots \cdot dy_1 \\
&= \sum_{i=1}^{n} \lambda_i \int_{-\infty}^{+\infty} \phi\left[\left(\beta - \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j^2\right)^{1/2} y_{n-1}\right)\Big/\alpha_i\right] \\
&\quad \times \phi\left[\left(\left(\sum_{\substack{j=1\\ j\neq 1}}^{n} \alpha_j^2\right)^{1/2} y_{n-1} - \beta\right)\Big/\alpha_i\right] \cdot \varphi(y_{n-1})\, dy_{n-1}
\end{aligned}
$$

which can be expressed in terms of the binormal distribution (see Ref. 13, p. 407)

$$
\lambda(D_H) = \sum_{i=1}^{n} \lambda_i(\phi(\beta) - \phi(\beta, \beta; \rho_i)) \qquad \text{with } \rho_i = 1 - \alpha_i^2 \qquad \text{(B.4)}
$$

APPENDIX C

In order to compute the outcrossing rate $\lambda(D_R)$ for a central sphere, we first determine the Laplace transform of the outcrossing rate, i.e., if $\lambda(D_R)$ is taken as a function of R, the Laplace transform $g(s)$ is given by

$$
g(s) = \int_0^{\infty} e^{-sR} \lambda(D_R)\, dR
$$

with s denoting a complex variable. Denoting the Laplace transform of a function $f(x)$ by $L[f(x)]$, we have for $\lambda(D_R)$ due to the rules of Laplace transforms:

$$
\begin{aligned}
L[\lambda(D_R)] &= \sum_{i=1}^{n} \lambda_i L[f_{n-1}(R)] L[F_1(R)(1 - F_1(R))] \\
&= \sum_{i=1}^{n} \lambda_i (L[f_{n-1}(R)] L[F_1(R)] - L[f_{n-1}(R)] L[F_1(R)^2]) \qquad \text{(C.1)}
\end{aligned}
$$

For the first term we obtain

$$
\begin{aligned}
&L[f_{n-1}(x)] L[F_1(x)] \\
&\quad = (1/s) L[f_{n-1}(x)] L[f_1(x)] = (1/s) L[f_n(x)] = 1/s(0.5/(0.5 + s))^{n/2} \qquad \text{(C.2)}
\end{aligned}
$$

and for the second term

$$L[f_{n-1}(x)]L[F_1(x)^2]$$
$$= (1/s)L[f_{n-1}(x)]L\left[\frac{d}{dx}F_1(x)^2\right] = (1/s)L[f_{n-1}(x)]2L[f_1(x)F_1(x)] \quad \text{(C.3)}$$

Since $F_1(x) = 2\phi(\sqrt{x}) - 1$, we have

$$L[f_1(x)F_1(x)] = \int_0^\infty e^{-sx}\frac{x^{-1/2}}{\sqrt{2\pi}}e^{-x/2}(2\phi(\sqrt{x}) - 1)\,dx$$
$$= (0.5/(0.5 + s))^{1/2}\frac{2}{\pi}\arctan((0.5/(0.5 + s))^{1/2} \quad \text{(C.4)}$$

where the substitution $x \rightarrow x^{1/2}$ and a table for normal integrals given in Ref. 11 have been used. Therefore, the Laplace transform can be written as:

$$L[\lambda(D_R)] = \sum_{i=1}^{n} \lambda_i(0.5/(0.5 + s))^{n/2}/s\left(1 - \frac{4}{\pi}\arctan((0.5/(0.5 + s))^{1/2})\right) \quad \text{(C.5)}$$

or, in expanding the arctan function:

$$L[\lambda(D_R)] = \left(\sum_{i=1}^{n} \lambda_i\right)(0.5/(0.5 + s))^{n/2}$$
$$\cdot\left(1 - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j + 1}(0.5/(0.5 + s))^{j+1/2}\right)/s \quad \text{(C.6)}$$

Formula (C.6) now can be inverted term by term, yielding

$$\lambda(D_R) = \left(\sum_{i=1}^{n} \lambda_i\right)\left(F_n(R) - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j + 1}\right.$$
$$\left.\times\left(1 - \frac{2^{-(j+(n+1)/2)}}{\Gamma\left(j + \frac{n+1}{2}\right)}\int_R^\infty x^{j+(n-1)/2}e^{-x/2}dx\right)\right) \quad \text{(C.7)}$$

where $F_n(R)$ is the χ^2-distribution with n degrees of freedom. Since

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{2j + 1} = \frac{\pi}{4}$$

we finally have

$$\lambda(D_R)$$
$$= \left(\sum_{i=1}^{n} \lambda_i\right)\left(F_n(R) - 1 + \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{2j + 1}\frac{2^{-(j+(n+1)/2)}}{\Gamma\left(j + \frac{n+1}{2}\right)}\int_R^\infty x^{j+(n-1)/2}e^{-x/2}\,dx\right) \quad \text{(C.8)}$$

The integrals in (C.8) can be computed by using the recurrence relation

$$\int_x^\infty y^k e^{-y/2}\,dy = 2x^k e^{-x/2} + 2k\int_x^\infty y^{k-1}e^{-y/2}\,dy \quad \text{(C.9)}$$

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