NON-NORMAL DEPENDENT VECTORS IN STRUCTURAL SAFETY

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INTRODUCTION

In the last few years, essential contributions have been made toward the formulation of the problem of calculating reliability measures for structures. Originating from second moment reliability concepts (1), the method given in Ref. 8 for example, has been shown to be a simple and efficient tool for obtaining reliability measures in terms of either operational failure probabilities or reliability indices. If required, a one-to-one correspondence between these two measures can be established (3). The word "operational" merely refers to the fact that such measures are conditioned to the stochastic model adopted for the uncertainties, rather than to the techniques with which uncertainties are mathematically manipulated.

In essence, the method in Ref. 8 requires the formulation of the domain in which the structure behaves safely. This is done in the space of basic uncertainty variables, $\mathbf{X} = (X_1, \ldots, X_n)^T$, such as actions, geometrical parameters, and strengths of materials which, for the moment, are assumed independent and normally distributed. The surface separating the "safe" from the "failure" domain is called "failure" or "limit state surface." If the vector $\mathbf{X}$ is standardized by $\mathbf{U} = [\mathbf{X} - E(\mathbf{X})]/D(\mathbf{X})$ in which $E(\mathbf{X}) = (\mu_1, \ldots, \mu_n)^T$ = the mean value vector and $D(\mathbf{X}) = (\sigma_1, \ldots, \sigma_n)^T$ = the vector of standard deviations, the point $\mathbf{u}^*$ on the boundary of the safe domain nearest to the coordinate origin and the "β-point," with distance $\beta = \|\mathbf{u}^*\|$ to the origin, must be found (5). Then, an approximation of the failure surface at the "β-point" by tangent hyperplanes or quadratics, for which the probability content of the safe domain can be easily computed, produces accurate estimates of, or close bounds to, the exact failure probability (4). The reason such estimates are usually good is that the rotational symmetry of the standard multinormal density (3), and its rapid decay with increasing distance from the origin, together with the fact that failure

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surfaces are generally rather smooth. Thus, the original task of integrating probabilities over a given domain is reduced to defining a search algorithm for the "β-point," plus the evaluation of the standard normal integral or the distribution function of a quadratic form in normal variates. If, on the other hand, a single point on the failure surface is not sufficiently informative about the shape of the safe domain, approximation of its boundary in several points (multiple-point-checking) by suitably chosen quadratics or polyhedra, yields accurate results as well. Then, the selection strategy for the approximation points is most important. This study concentrates on the first mentioned single-point-checking method, since generalization to multiple point-checking methods is straightforward.

Elementary Transformations

Solutions to less idealized problems, with respect to the stochastic model, can be obtained by suitable probability distribution transformations. If the original uncertainty vector has independent but non-normal components with cumulative distribution functions

$$F_i(x_i) = P(X_i \leqq x_i) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

the necessary transformation, T, simply is

$$(u_1, \ldots, u_n)^T = \mathbf{T}(x_1, \ldots, x_n) = [T_1(x_1), \ldots, T_n(x_n)]^T \quad \ldots\ldots\ldots\ldots\ldots (2)$$

in which $u_i = T_i(x_i) = \Phi^{-1} F_i(x_i)$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

in which $i = 1, \ldots, n$; and $\Phi(\cdot)$ = the standard normal integral. Then, the transformed vector, $\mathbf{U} = \mathbf{T}(\mathbf{X})$, is a standard normal and the usual techniques can be applied. In passing we note that the transformation (Eq. 2) is equivalent to matching the probability density and the cumulative distribution function of X_i at $x_i^* = T_i^{-1}(u_i^*) = F_i^{-1}[\Phi(u_i^*)]$ by an approximating normal distribution (8). In the sequel, we shall denote the space of the arbitrarily distributed vector **X** by x-space, and the space of the standard-normal, transformed vector **U** by **u**-space.

If **X** is a dependent vector but normal, log-normal, or mixed normal-log-normal, a simple transformation (logarithm, shift, and rotation of the coordinate system) generates the required space of uncorrelated standard normal variates.

However, to the writers knowledge, general stochastic dependence has not been dealt with so far, even though such dependent vectors frequently are encountered in practical applications. Stochastic dependence may, in fact, have a significant influence on reliability, particularly when many dependent variables are involved. Subsequently, a general transformation method is developed. It is based on an idea of Rosenblatt (10) and appears to have wide applicability.

Definition of General Transformation

Assume that the uncertainty vector, $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$, possesses a probability density f, such that

$$P(\mathbf{X} \in S) = \int_S f(\mathbf{x})\, d\mathbf{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which S may be viewed as an arbitrary domain, e.g., the safe domain. Define

$$f_i(x_1, ..., x_i) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} f(x_1, x_2, ..., x_i, s_{i+1}, ..., s_n)\, ds_{i+1} ... ds_n \quad ..... (5)$$

to be the density of the partial vector $(X_1, ..., X_i)^T$. Further, introduce the conditional distribution function H_i of X_i conditioned on $x_1, ..., x_{i-1}$:

$$H_i(x_i|x_1, ..., x_{i-1}) = \frac{1}{N_i} \int_{-\infty}^{x_i} f_i(x_1, ..., x_{i-1}, s_i)\, ds_i \quad \text{for} \quad i \geqq 2 \quad ..... (6)$$

in which N_i = a normalizing factor given by

$$N_i = \int_{-\infty}^{+\infty} f_i(x_1, ..., x_{i-1}, s_i)\, ds_i \quad ..... (7)$$

and $H_1(x_1) = F_1(x_1)$. According to Rosenblatt (10), the transformation, $\mathbf{T}_R$, then, is

$$\mathbf{u} = \mathbf{T}_R(\mathbf{x}) = [T_1(x_1), T_2(x_1, x_2), ..., T_n(x_1, ..., x_n)]^T \quad ..... (8)$$

in which $u_1 = T_1(x_1) = \Phi^{-1}[H_1(x_1)]$ ..... (9)

$$u_2 = T_2(x_1, x_2) = \Phi^{-1}[H_2(x_2|x_1)] \quad ..... (10)$$

$$u_n = T_n(x_1, ..., x_n) = \Phi^{-1}[H_n(x_n|x_1, ..., x_{n-1})] \quad ..... (11)$$

In summary, in the first step, X_1, is transformed into a standard normal variate. In the second step, all conditional distributions of $X_2|X_1 = x_1$ are transformed into a standard normal variate, and so forth. Thus, the vector $\mathbf{U} = \mathbf{T}_R(\mathbf{X})$ is standard normally distributed with independent components. Obviously, some difficulties may exist in the evaluation of $H_i(\cdot|\cdot)$. In some cases, these must be determined numerically, while in others they can be derived theoretically, as shown in the upcoming examples.

In some applications, the inverse of Eq. 8, $\mathbf{x} = \mathbf{T}_R^{-1}(\mathbf{u})$, at least in some points, is required. From Eq. 8 it immediately follows that the inversion of $\mathbf{T}_R$ can be made stepwise in inverting sequentially n one-dimensional transformations:

$$x_1 = H_1^{-1}[\Phi(u_1)] \quad ..... (12)$$

$$x_2 = H_2^{-1}[\Phi(u_2)|x_1] \quad ..... (13)$$

$$x_n = H_n^{-1}[\Phi(u_n)|x_1, ..., x_{n-1}] \quad ..... (14)$$

It is seen that now the inverse conditional distributions, $H_i^{-1}(\cdot|\cdot)$, inverted with respect to the first variable, must be known.

The Rosenblatt transformation has some useful properties. First, for an independent but non-normal vector, $\mathbf{T}_R$ is identical to the transformation of Eq. 2 with Eq. 3. For a dependent normal vector $\mathbf{X}$, it can be shown that $\mathbf{T}_R$ (and $\mathbf{T}_R^{-1}$) are linear and give the same β-index as the elementary transformation previously described. Further, if $\mathbf{u} = \tau(\mathbf{x}) = [\tau_1(x_1), ..., \tau_n(x_n)]^T$ and $\mathbf{T}_R^u$ of $\mathbf{u}$ is known, and if all τ_i are strictly increasing and differentiable, then $\mathbf{T}_R^u(\tau)$ is the Rosenblatt transformation of $\mathbf{X}$. Note that this enables one to solve for the Rosenblatt transformation of a normal-log-normal vector and of those vectors obtained through translation systems, as proposed in Ref. 6.

APPLICATIONS

A fundamental problem when dealing with dependent random vectors in structural reliability or other areas is the appropriate characterization of its dependence structure. Seldom are sufficient data available to determine the joint cumulative distribution function. Lack of data frequently leads to the selection of elementary normal or log-normal correlation models which then permit a classical solution. In many areas, however, multivariate or conditional distributions are generated through statistical considerations or in the way an uncertain phenomenon is derived from physical facts.

USE OF INVERSE TRANSFORMATION

First the writers shall give two examples in which application of the inverse transformation of Eqs. 12–14 is advantageous. This will be so in all cases where the inverse conditional distributions, Eqs. 12–14, can be given analytically or can be evaluated without too much difficulty. Note that $\mathbf{x} = \mathbf{T}_R^{-1}(\mathbf{u})$ implies a transformation of the safety condition, $\mathbf{X} \in S_X$ or $g(\mathbf{X}) \geqq 0$, respectively, in the x-space into a corresponding condition in the u-space, i.e.

$$\mathbf{U} \in S_U = \{\mathbf{u} : \mathbf{T}_R^{-1}(\mathbf{u}) \in S_X\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

$$g[\mathbf{T}_R^{-1}(\mathbf{U})] \geqq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

In this manner, the particular type of stochastic model for X is completely accounted for by the corresponding safe set in the u-space.

Example 1.—Consider a manufacturing process producing structural elements the resistances of which can be described by an independent normal sequence $(Y_1, Y_2, \ldots)$ and the mean and standard deviation of which are known from a sample of limited size, and for which some additional prior knowledge exists. For a structure where m of these elements are to be used, the limit state function, g, is defined in a space of m such resistance variables and a set of action variables. In a Bayesian context it often can be assumed that the *a posteriori* distribution of the uncertain mean, M, and standard deviation, Σ, is of normal-inverse-gamma-2-type with *a posteriori* parameters $(\bar{y}'', n'', s'', \nu'')$ (9). In particular

$$P(\Sigma \leqq \sigma) = H_1(\sigma | s'', \nu'') = \int_0^{\sigma} \frac{2 \exp\left(-\frac{\nu''}{2} s''^2 \tau^{-2}\right)\left(\frac{\nu''}{2} s''^2 \tau^{-2}\right)^{\nu''/2+1/2}}{\Gamma\left(\frac{\nu''}{2}\right)\left(\frac{\nu''}{2} s''^2\right)^{1/2}} d\tau \quad \ldots\ldots\ldots\ldots \quad (17)$$

is said to be inverse-gamma-2-distributed and

$$P(M \leqq \mu | \Sigma = \sigma) = H_2(\mu | \sigma, \bar{y}'', n'') = \Phi\left(\frac{u - \bar{y}''}{\frac{\sigma}{\sqrt{n''}}}\right) \quad \ldots\ldots\ldots\ldots \quad (18)$$

while, given $M = \mu$ and $\Sigma = \sigma$, the future realizations of structural elements to be built in have the strength distribution

$$P(Y_i \leqq y \,|\, M = \mu, \Sigma = \sigma) = H_i(y_i|\mu, \sigma) = \Phi\left(\frac{y_i - \mu}{\sigma}\right)$$

for $i = 1, 2, \ldots, m$ . . . . . (19)

Integrating out the uncertainty on (M, Σ) in Eq. 19 yields a multivariate Student's-t-distribution for the future strengths (2) which generally is not amenable to a classical reliability solution. However, taking $(X_1, \ldots, X_n) = (\Sigma, M, Y_1, Y_2, \ldots, Y_m)$ as the new basic uncertainty vector, the transformation technique easily can be applied. In this special case, the transformation (Eq. 8) and its inverse (Eqs. 12–14) reduce to

$$u_1 = \Phi^{-1}[H_1(\sigma|s'', \nu'')]; \quad \sigma = H_1^{-1}[\Phi(u_1)|s'', \nu''] \quad . . . . . (20)$$

$$u_2 = \frac{\mu - \bar{y}''}{\frac{\sigma}{\sqrt{n''}}}; \quad \mu = \frac{u_2 \sigma}{\sqrt{n''}} + \bar{y}'' \quad . . . . . (21)$$

$$u_3 = \frac{y_1 - \mu}{\sigma}; \quad y_1 = u_3 \sigma + \mu = \left(\frac{u_3 + u_2}{\sqrt{n''}}\right)\sigma + \bar{y}'' \quad . . . . . (22)$$

$$u_n = \frac{y_m - \mu}{\sigma}; \quad y_m = u_n \sigma + \mu = \left(\frac{u_n + u_2}{\sqrt{n''}}\right)\sigma + \bar{y}'' \quad . . . . . (23)$$

When using Eq. 16 together with the right hand parts of Eqs. 20–23, the only difficulty lies in the inversion of H_1, which must be done numerically. To be more specific, assume that the structural elements under consideration form a perfectly ductile, "parallel" system, the failure surface of which can be given as a linear combination of the uncertain variables:

$$g(\mathbf{X}) \equiv \sum_{i=1}^{m} c_i Y_i - Y = 0 \quad . . . . . (24)$$

in which the Y_i's = the elemental strengths; c_i = certain deterministic constants; and Y = an independent loading variable with exponential distribution function $F_y(y) = 1 - \exp(-\lambda y)$. Application of Eqs. 20–23 and the inverse of Eq. 3 to the variable Y yields the new failure surface

$$\tilde{g}(\mathbf{u}) \equiv \sum_{i=3}^{n} c_{i-2} \left\{ \left(\frac{u_i + u_2}{\sqrt{n''}}\right) H_1^{-1}[\Phi(u_1); \quad s'', \nu''] + \bar{y}'' \right\}$$

$$+ \frac{\ln \Phi(-u_{n+1})}{\lambda} = 0 \quad . . . . . (25)$$

in which the U's now = independent, standard normal variates. Note that this is no longer a plane surface and has $(n + 1) + 2$ dimensions.

It is clear from this example that a similar approach would solve all cases in which stochastic dependence of an originally independent uncertainty vector is introduced by a set of additional conditioning variables. These variables may represent statistically uncertain or, via filtering, somehow modified distribution parameters of the original uncertainty vector. They may also derive from the particular physical model for which the following example is sketched.

Example 2.—Earthquakes are a major threat to buildings in many countries. In earthquake-resistant design, a large number of uncertainties have to be simultaneously considered so that widely simplified reliability methods have been proposed. As an illustration of a more refined integrated approach, assume a linear, viscously damped, single-degree-of-freedom oscillator which is excited by a stationary Gaussian base acceleration, $\ddot{x}_0(t)$, with given frequency spectrum $G_{\ddot{x}_0}(\omega|x_1,x_2,x_3)$, e.g. the spectrum for earthquakes in the strong motion phase, as proposed by Tajimi/Kanai (6), and where x_1 is a parameter describing the total incoming energy, while x_2 and x_3 are site-dependent parameters characterizing the filtering properties of the surrounding soil. The stationary output spectrum (displacements) is given by

$$G_x(\omega|x_1,\ldots,x_6) = |H(\omega|x_4,x_5,x_6)|^2\, G_{\ddot{x}_0}(\omega|x_1,x_2,x_3) \quad \ldots\ldots\ldots\ldots (26)$$

in which the transfer-function $H(\omega|\cdot)$ depends on the uncertain structural damping constant, x_4, stiffness, x_5, and mass, x_6, of the oscillator. The extreme value distribution of displacements, x_8, in a time-interval $(0,x_7)$ (the uncertain, equivalent duration of the strong motion phase) may be approximated by

$$F_8(x_8|x_1,\ldots,x_7) = \exp\left\{-\left(\frac{\lambda_2}{\lambda_0}\right)^{1/2}\cdot\frac{x_7}{\pi}\cdot\exp\left(-\frac{x_8}{2\lambda_0}\right)\cdot\frac{1-\exp\left[-\left(\frac{\pi}{2}\frac{\lambda_0\lambda_2-\lambda_1^2}{\lambda_0\lambda_2}\right)^{1/2}\cdot\frac{x_8}{\lambda_0}\right]}{1-\exp\left[-\frac{x_8}{2\lambda_0}\right]}\right\} \quad \ldots\ldots\ldots\ldots (27)$$

in which $\lambda_i = \lambda_i(x_1,\ldots,x_6)$ = the ith spectral moment $\lambda_i = \int_0^\infty \omega^i\, G_x(\omega|x_1,\ldots,x_6)\, d\omega$ (11). (For analytical expressions, given the Tajimi/Kanai-spectrum, see Ref. 11.) A possible failure criterion of the structure now is

$$g(\mathbf{x}) \equiv x_9 - h(x_8) \leqq 0 \quad \ldots\ldots\ldots\ldots (28)$$

in which x_9 = some cross-sectional strength; and $h(x_8)$ = a known, generally linear function relating displacement to action effects. Assume, for simplicity, independence of $X_1, \ldots, X_7, X_9$. Then the inverse transformation, Eqs. 12–14, lead to

$$g(\mathbf{x}) \equiv x_9 - h(F_8^{-1}[\Phi(u_8)|x_1,\ldots,x_7] \leqq 0 \quad \ldots\ldots\ldots\ldots (29)$$

which requires inversion of Eq. 27 which easily can be carried out numerically. The inverse distribution, F_8^{-1}, is an inverse with respect to the first variable, for fixed $x_1,\ldots,x_7$.

Let us further assume that seismicity in the region considered (one potential earthquake source!) is described by a stationary Poisson process with intensity ν, and the variable x_1 is related to the magnitude, m, of earthquakes by a given attenuation law $x_1 = a(m)$. The common Gutenberg-Richter rule leads to a distribution of magnitudes:

$$F_1(x_1) = \frac{1 - \exp\,[-b\cdot(a^{-1}(x_1) - m_1)]}{1 - \exp\,[-b\,(m_2 - m_1)]} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

in which $m_1 \leqq m \leqq m_2$ = the range of magnitudes of engineering interest; and b = a region-dependent constant. The maximum of the Poisson sequence, (X_1), in a given-time-interval, $(0, T)$, e.g., the anticipated lifetime of the structure, has distribution

$$F_{1,\max}(x_1) = \exp\,\{-\nu T[1 - F_1(x_1)]\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

Therefore, we may use

$$x_1 = F_1^{-1}\left[1 + \frac{\ln \Phi(u_1)}{\nu\, T}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

in Eq. 29 and proceed as usual in further substituting $x_i = F_i^{-1}\,[\phi(u_i)]$ for $i = 2, \ldots, 7, 9$ to finally arrive at formulation Eqs. 15–16. It should be obvious that statistical uncertainties on some of the distribution parameters of the uncertainty vector, **X**, e.g., on ν and b, could, in principle, be included, as outlined in the first example. It might also be obvious that a solution was possible since the mechanical problem has been formulated in the basic variable space (original or standardized). However, repeated nested inversion of distribution functions may cause numerical difficulties. For such cases, the exact inversion might be substituted by approximate, local inversion techniques, as described in the next section.

Direct Transformation

The algorithm proposed in Ref. 8 easily can be generalized for dependent uncertainty vectors. Its basic idea is to linearize $\mathbf{R}_R$ in each iteration point: denoting $\mathbf{L}_{R,\mathbf{x}_0}$, the linearization of $\mathbf{T}_R$ at $\mathbf{x}_0$, the linearization of $\mathbf{T}_R^{-1}$ at $\mathbf{u}_0 = \mathbf{T}_R(\mathbf{x}_0)$ is simply $\mathbf{L}_{R,\mathbf{x}_0}^{-1}$.

A suitable algorithm is as follows:

1. Select a starting point $\mathbf{x}_0$ in the x-space.
2. Calculate $\mathbf{T}_R(\mathbf{x}_0) = \mathbf{u}_0$ (in the **u**-space).
3. Determine the (triangular) Jacobian matrix $\mathbf{D}_0$ of $\mathbf{T}_R$ in $\mathbf{x}_0$ and its inverse $\mathbf{D}_0^{-1}$. It is often useful to note that

$$\frac{\partial T_i}{\partial x_j}(\mathbf{x}) = \frac{\partial\{\Phi^{-1}[H_i(x_i|x_1, \ldots, x_{i-1})]\}}{\partial x_j} = \frac{1}{\phi(u_i)}\,\frac{\partial H_i(x_i|x_1, \ldots, x_{i-1})}{\partial x_j} \quad (33)$$

in which $\phi(\cdot)$ = the standard normal density function. If necessary, $\mathbf{D}_0$ can be evaluated numerically.

4. Evaluate the structural state function, $g(\mathbf{x}_0)$, and its gradient, grad $g(\mathbf{x}_0)$. For the transformed state function $\tilde{g} = g(\mathbf{T}_R^{-1})$

$$\tilde{g}(\mathbf{u}_0) = g(\mathbf{x}_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

$$\text{grad}\ \ \tilde{g}(\mathbf{u}_0) = (\mathbf{D}_0^{-1})^T\ \ \text{grad}\ \ g(\mathbf{x}_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

which determines a linear approximation, $\tilde{\mathcal{Z}}_0$, of $\tilde{g}$ in $\mathbf{u}_0$.

5. The "β-point" of $\tilde{\mathcal{Z}}_0$ is obtained from

$$\tilde{\mathbf{u}}_1 = \frac{1}{[\mathbf{grad}\,\tilde{g}(\mathbf{u}_0)]^2}\,[\mathbf{u}_0 \cdot \mathbf{grad}\,\tilde{g}(\mathbf{u}_0) - \tilde{g}(\mathbf{u}_0)]\,\mathbf{grad}\,\tilde{g}(\mathbf{u}_0) \quad \ldots\ldots\ldots\ldots \quad (36)$$

6. Approximate (linear) backwards transformation of $\tilde{\mathbf{u}}_1$ into the original space by $\mathbf{D}_0^{-1}$ yields

$$\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{D}_0^{-1}(\tilde{\mathbf{u}}_1 - \mathbf{u}_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

as a new starting point.

7. Repeat steps 1–6 until convergence is reached, e.g., $\mathbf{u} = \mathbf{T}_R(\mathbf{x})$ lies on the transformed failure surface and is nearest to the origin in the u-space.

Other search algorithms can be established by similar procedures.

Example 3.—As an illustration, consider the two-dimensional exponential

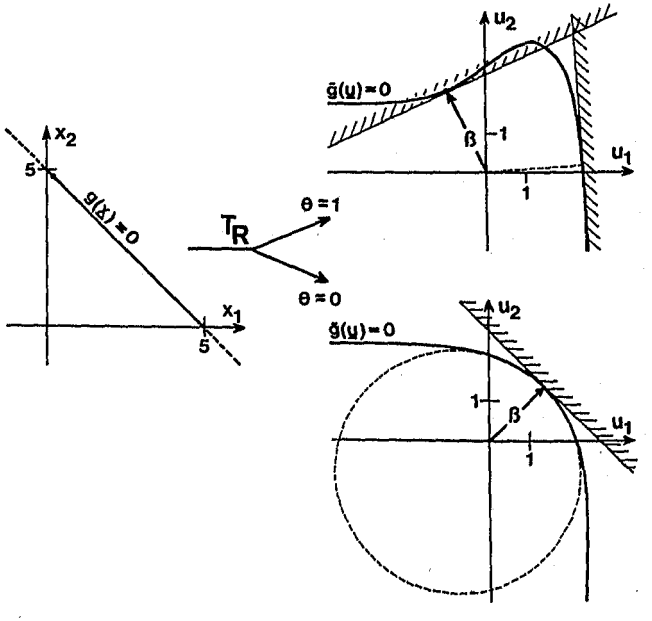

FIG. 1.—Safe Domain in u-Space

distribution with cumulative distribution function

$$F(x_1, x_2) = \begin{cases} 0 & \text{for } x_1 \leqq 0 \text{ or } x_2 \leqq 0 \\ 1 - \exp(-x_1) - \exp(-x_2) + \exp(-x_1 - x_2 - \theta x_1 x_2) & \text{else} \end{cases} \quad (38)$$

for $0 \leqq \theta \leqq 1$. We have, from Eq. 6

$$H_1(x_1) = 1 - \exp(-x_1) \quad \text{for } x_1 > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

$$H_2(x_2|x_1) = 1 - (1 + \theta x_2)\exp(-x_2 - \theta x_1 x_2)$$

$$\text{for } x_1 > 0 \text{ and } x_2 > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

and, therefore, according to Eqs. 8–11:

$$\mathbf{u} = \mathbf{T}_R(x_1, x_2) = \{\Phi^{-1}[1 - \exp(-x_1)],$$

$$\Phi^{-1}[1 - (1 + \theta x_2)\exp(-x_2 - \theta x_1 x_2)]\}^T$$

if $x_1 > 0$ and $x_2 > 0$; and $\mathbf{T}_R(x_1, x_2) = (-\infty, -\infty)^T$, otherwise. (41)

For example, if the safe domain in the original space is given by $S = [(x_1, x_2)^T : 5 - (x_1 + x_2) \geqq 0]$ it changes its form in the u-space as drastically as shown in Fig. 1.

We now illustrate the "Beta-algorithm" in this case. The Jacobian matrix in the point $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is

$$\mathbf{D}_0 = \begin{pmatrix} \dfrac{1}{\phi(u_1)} \exp(-x_1) & , 0 \\ \dfrac{1}{\phi(u_2)} \dfrac{(1 + \theta x_2) x_2}{\exp(x_2 + \theta x_1 x_2)} & , \dfrac{1}{\phi(u_2)} \dfrac{(1 + \theta x_1)(1 + \theta x_2) - \theta}{\exp(x_2 + \theta x_1 x_2)} \end{pmatrix} \quad (42)$$

and its inverse

$$\mathbf{D}_0^{-1} = \begin{pmatrix} \phi(u_1) \exp(x_1) & , 0 \\ \phi(u_1) \dfrac{(1 + \theta x_2)\theta x_2 \exp(x_1)}{(1 + \theta x_1)(1 + \theta x_2) - \theta} & , \dfrac{\phi(u_2)\exp(x_2 + \theta x_1 x_2)}{(1 + \theta x_1)(1 + \theta x_2) - \theta} \end{pmatrix}. \quad (43)$$

Let, for instance, $\theta = 1$, which corresponds to a correlation coefficient

$$\rho(X_1, X_2) \approx -0.40, \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 0.1 \\ 3 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (44)$$

Then, using Eq. 41 (step 2)

$$\mathbf{u}_0 = \begin{pmatrix} -1.31 \\ 1.05 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (45)$$

and from Eqs. 42 and 43 (step 3)

$$\mathbf{D}_0 = \begin{pmatrix} 5.35 & 0 \\ 1.93 & 0.546 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_0^{-1} = \begin{pmatrix} 0.187 & 0 \\ -0.660 & 1.83 \end{pmatrix}. \quad \ldots\ldots\ldots\ldots \quad (46)$$

For the structural state function defined above we obtain (step 4)

$$\text{grad}\ g(\mathbf{x}_0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{and}$$

$$\text{grad}\ \tilde{g}(\mathbf{u}_0) = \begin{pmatrix} 0.187 & -0.660 \\ 0 & 1.83 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.473 \\ -1.83 \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (47)$$

Since $g(\mathbf{x}_0) = \tilde{g}(\mathbf{u}_0) = 1.9$, the linear approximation $\tilde{\mathcal{Z}}_0$ of $\tilde{g}$ at the point $\mathbf{u}_0$ is

$$\tilde{\mathcal{Z}}_0(\mathbf{u}) = \tilde{g}(\mathbf{u}_0) + (\mathbf{u} - \mathbf{u}_0) \cdot \text{grad}\ \tilde{g}(\mathbf{u}_0) = 0.473\, u_1 - 1.83\, u_2 + 4.44 \quad \ldots\ldots \quad (48)$$

This gives a "linearized limit state surface" $[\mathbf{u} : \tilde{\mathcal{Z}}_0(\mathbf{u}) = 0]$ and its "β-point"

with minimum distance to the origin (step 5)

$$\tilde{\mathbf{u}}_1 = \frac{1}{0.473^2 + 1.83^2}\left[\begin{pmatrix}-1.31\\1.05\end{pmatrix}\cdot\begin{pmatrix}0.473\\-1.83\end{pmatrix} - 1.9\right]\begin{pmatrix}0.473\\-1.83\end{pmatrix} = \begin{pmatrix}-0.588\\2.27\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (49)$$

Approximate retransformation by $\mathbf{D}_0^{-1}$ generates a new starting point (step 6)

$$\mathbf{x}_1 = \begin{pmatrix}0.1\\3\end{pmatrix} + \begin{pmatrix}0.187 & 0\\-0.660 & 1.83\end{pmatrix}\begin{pmatrix}-0.588 + 1.31\\2.27 - 1.05\end{pmatrix} - \begin{pmatrix}0.235\\4.76\end{pmatrix} \quad \ldots\ldots \quad (50)$$

Repetition of the sequence of steps yields the final β-point

$$\mathbf{u}^* = \begin{pmatrix}-0.97\\2.06\end{pmatrix} \quad \text{corresponding to} \quad \mathbf{x} = \mathbf{T}_R^{-1}(\mathbf{u}) = \begin{pmatrix}0.18\\4.82\end{pmatrix};$$

the β-distance is $\|\mathbf{u}\| = 2.28$ . . . . . (51)

It may happen that, due to the approximate inverse transformation, a new starting point falls into the "singular domain" of the assumed distribution or into a domain where no structural state function exists. In this case, the algorithm has to be modified suitably. For example, denoting $\mathbf{x}_l$ and $\mathbf{x}_{l+1}$ the old and the new starting point, the algorithm is continued with

$$\mathbf{x}_{l+2} = \mathbf{x}_l + \lambda(\mathbf{x}_{l+1} - \mathbf{x}_l),\ 0 < \lambda < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (52)$$

and λ not too small, i.e., such that $\mathbf{x}_{l+2}$ is not in the singular domain. This appears meaningful since the algorithm depends on a linearization of both $\mathbf{T}_R$ and g.

In setting $\theta = 0$ (corresponding to $\rho(X_1, X_2) = 0$) in Eq. 22 one has the bivariate exponential distribution with independent components. It is also shown in Fig. 1. Applying the same algorithm (or the simplified algorithm working only in the u-space, which is appropriate in this case) yields $\beta = 1.97$.

The possible distortion of the original safe domain by the transformation, as demonstrated for $\theta = 1$, may raise doubts whether the β-point is an informative checking point. The linear approximation of $g(\mathbf{u}) = 0$ at the β-point yields a failure probability $P_f = 1.13 \cdot 10^{-2}$, whereas exact integration yields $1.73 \cdot 10^{-2}$. In the second case, with $\theta = 0$, one has $P_f = \Phi(-1.97) = 2.46 \cdot 10^{-2}$ while exact integration yields $P_f = 4.04 \cdot 10^{-2}$. In fact, it appears that multiple point checking methods, which approximately fit polyhedra or quadratics to the failure surface, are more in order. With a polyhedrical approximation for the dependent case, as shown in Fig. 1, the much closer value of $1.92 \cdot 10^{-2}$ is obtained. For the independent case, an approximating non-central sphere with the same curvature in the β-point has failure probability $4.46 \cdot 10^{-2}$. Details of such methods will be analyzed in a separate paper. The writers, however, wish to emphasize that refined methods are only rarely needed in practical applications.

Summary and Conclusions

A general probability distribution transformation has been developed with

which complex structural reliability problems involving non-normal, dependent uncertainty vectors can be reduced to the standard case of first-order-reliability, i.e., the problem of determining the failure probability or the reliability index in the space of independent, standard normal variates. The method requires the knowledge of the joint cumulative distribution function or a certain set of conditional distribution functions of the original vector.

Some basic properties of the transformation are discussed. Details of the transformation technique are given. A suitable algorithm for computing reliability measures has been proposed. The field of potential applications is indicated by a number of examples, e.g., when statistical uncertainties on distributional parameters are present, or for an integrated approach to structural reliability against earthquakes. The method can, in fact, deal with problems where until now complex stochastic models had to be reduced to simple normal correlation models.

Appendix I.—References

1. Ang, A. H.-S., and Cornell, C. A., "Reliability Bases of Structural Safety and Design," *Journal of the Structural Division*, ASCE, Vol. 100, No. ST9, Sept., 1974, pp. 1755–1769.
2. Chew, V., "Simultaneous Prediction Intervals," *Technometrics*, Vol. 10, 1968, pp. 323–330.
3. Ditlevsen, O., "Generalized Second Moment Reliability Index," *Journal of Structural Mechanics*, Vol. 7, No. 4, 1979, pp 435–451.
4. Fiessler, B., Neumann, H.-J., and Rackwitz, R., "Quadratic Limit States in Structural Reliability," *Journal of the Engineering Mechanics Division*, ASCE, No. EM4, Aug., 1979, pp. 661–676.
5. Hasover, A. M., and Lind, N. C., "An Exact and Invariant First-order Reliability Format," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 100, No. EM1, 1974, pp. 111–121.
6. Kanai, K., "An Empirical Formula for the Spectrum of Strong Earthquake Motions," *Bulletin of Earthquake Research Institute*, Univ. of Tokyo, Japan, Vol. 39, pp. 85–95.
7. Parkinson, D. B., "First-order Reliability Analysis Employing Translation Systems," Engineering Structures, IPC Science and Technologic Press, Guilford, England, Vol. 1, Oct. 1978, pp. 31–40.
8. Rackwitz, R., and Fiessler, B., "Structural Reliability under Combined Random Load Sequences," *Computers & Structures*, Pergamon Press, New York, N.Y., Vol. 9, 1978, pp. 489–494.
9. Raiffa, H., and Schlaifer, R., *Applied Statistical Decision Theory*, Massachusetts Institute of Technology Press, Cambridge, Mass., 1968.
10. Rosenblatt, M., "Remarks on a Multivariate Transformation," *Annals of Mathematical Statistics*, Institute of Mathematical Statistics, Baltimore, Md., Vol. 23, pp. 470–472.
11. Vanmarcke, E. H., "Structural Response to Earthquakes," *Seismic Risk and Engineering Decisions*, C. Lomnitz and E. Rosenblueth, eds., Elsevier Book Co., Amsterdam, The Netherlands, 1976.

Appendix II.—Notation

The following symbols are used in this paper:

$\mathbf{D}_0 = (\partial T_i/\partial x_j)_{i,j}$ = Jacobian matrix of $\mathbf{T}_R$;
$f_I(x_1, \ldots, x_I)$ = probability density of partial vector $(X_1, \ldots, X_I)$;
$g(\mathbf{x})$ = structural state function in x-space;
$\tilde{g}(\mathbf{u}) = g(\mathbf{T}_R^{-1}(\mathbf{u}))$ = transformed structural state function in u-space;
grad g = gradient of g;

$H_i(x_i|x_1,\ldots,x_{i-1})$ = conditional distribution function of X_i, conditioned on $X_1 = x_1,\ldots,X_{i-1} = x_{i-1}$;

$\mathbf{T}_R(\mathbf{x}) = (T_1(x_1), T_2(x_1,x_2),\ldots, T_n(x_1,\ldots,x_n))^T$ = "Rosenblatt" transformation;

$\mathbf{U} = (U_1,\ldots,U_n)^T = \mathbf{T}_R(\mathbf{X})$ = transformed, standard normal uncertainty vector;

u-space = space of standard-normal variates;

$\mathbf{X} = (X_1,\ldots,X_n)^T$ = basic uncertainty vector;

x-space = space of basic uncertain variables;

β = safety index;

Φ = standard normal cumulative distribution function;

ϕ = standard normal density function;

$\rho[X_1,X_2]$ = correlation coefficient of X_1 and X_2; and

$\|\cdot\|$ = norm (length) of vector.