Polar Code Design for SCL Decoding: An Information Theoretic Perspective

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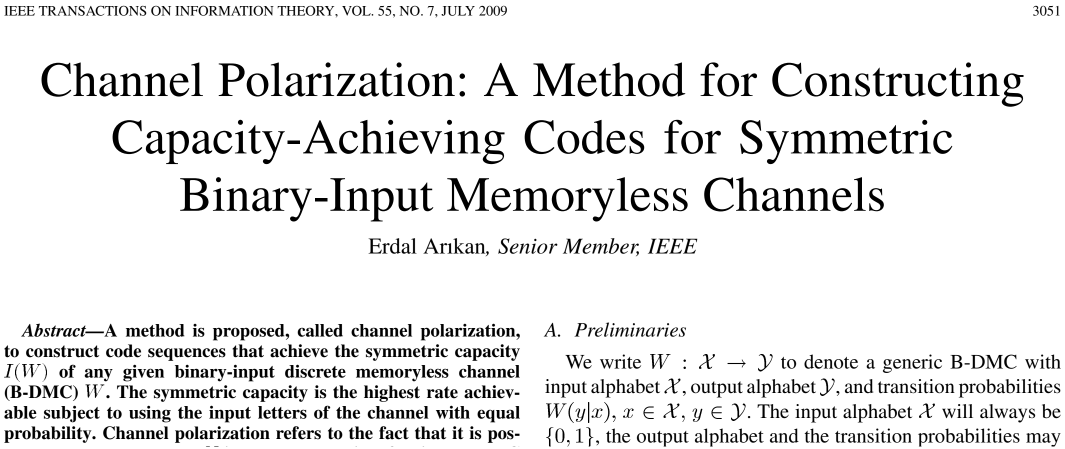
Polar Codes



IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 55, NO. 7, JULY 2009 3051

Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels

Erdal Arıkan, *Senior Member, IEEE*

***Abstract*—A method is proposed, called channel polarization, to construct code sequences that achieve the symmetric capacity $I(W)$ of any given binary-input discrete memoryless channel (B-DMC) W. The symmetric capacity is the highest rate achievable subject to using the input letters of the channel with equal probability. Channel polarization refers to the fact that it is pos-**

A. Preliminaries

We write $W : \mathcal{X} \to \mathcal{Y}$ to denote a generic B-DMC with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and transition probabilities $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. The input alphabet $\mathcal{X}$ will always be $\{0, 1\}$, the output alphabet and the transition probabilities may

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- But successive cancellation (SC) decoding performs poorly for small blocks.

Successive List Cancellation Decoding

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List Decoding of Polar Codes

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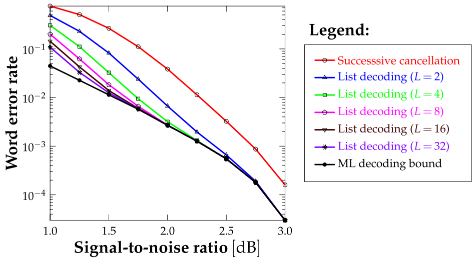

Fig. 1. List-decoding performance for a polar code of length $n = 2048$ and rate $R = 0.5$ on the BPSK-modulated Gaussian channel. The code was constructed using the methods of [15], with optimization for $E_b/N_0 = 2\,\text{dB}$.

- SC list (SCL) decoding with CRC and large list-size performs very well and matches maximum-likelihood (ML) [TV15].

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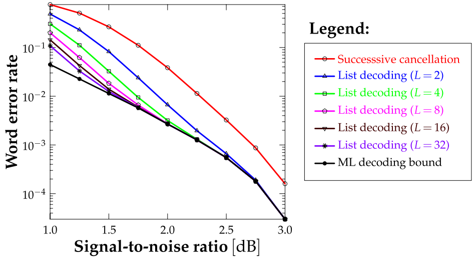

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- SC list (SCL) decoding with CRC and large list-size performs very well and matches maximum-likelihood (ML) [TV15].
- It can also be used to decode other codes (e.g., Reed–Muller codes).

Polar Codes with Dynamic Frozen Bits

254 IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS, VOL. 34, NO. 2, FEBRUARY 2016

Polar Subcodes

Peter Trifonov, *Member, IEEE*, and Vera Miloslavskaya, *Member, IEEE*

***Abstract*—An extension of polar codes is proposed, which allows some of the frozen symbols, called dynamic frozen symbols, to be data-dependent. A construction of polar codes with dynamic frozen symbols, being subcodes of extended BCH codes, is proposed. The proposed codes have higher minimum distance than classical polar codes, but still can be efficiently decoded using the successive cancellation algorithm and its extensions. The codes with Arikan, extended BCH and Reed-Solomon kernel are considered. The proposed codes are shown to outperform LDPC and turbo codes, as well as polar codes with CRC.**

RM codes, and are therefore likely to provide better finite length performance. However, there are still no efficient MAP decoding algorithms for these codes.

It was suggested in [17] to construct subcodes of RM codes, which can be efficiently decoded by a recursive list decoding algorithm. In this paper we generalize this approach, and propose a code construction "in between" polar codes and EBCH codes. The proposed codes can be efficiently decoded using the techniques developed in the area of polar coding, but provide

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- Later, polar codes were extended with the concept of dynamic frozen bits, which enabled state-of-art designs.
- It is also shown that any code can be decoded using SCL decoding, but some require very large complexity for a good performance.

Motivating Question

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Based on joint works with Henry D. Pfister [CP20, CP21]

Outline

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Polar Codes and Density Evolution

$$x_1^N = u_1^N \mathbf{G}_2^{\otimes n} \quad \text{where} \quad \mathbf{G}_2 \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad N = 2^n$$

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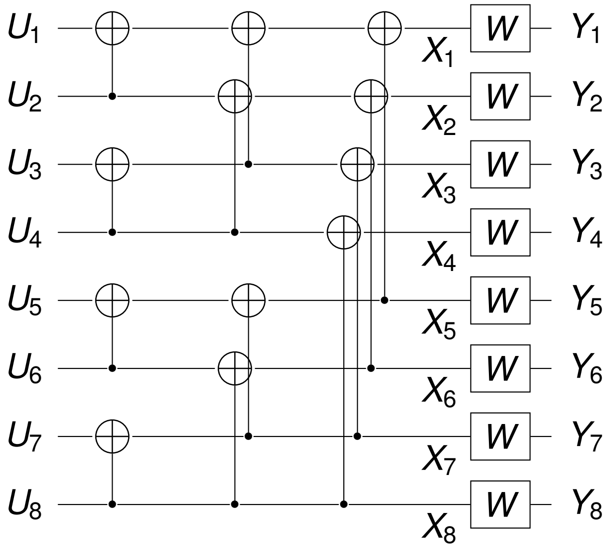

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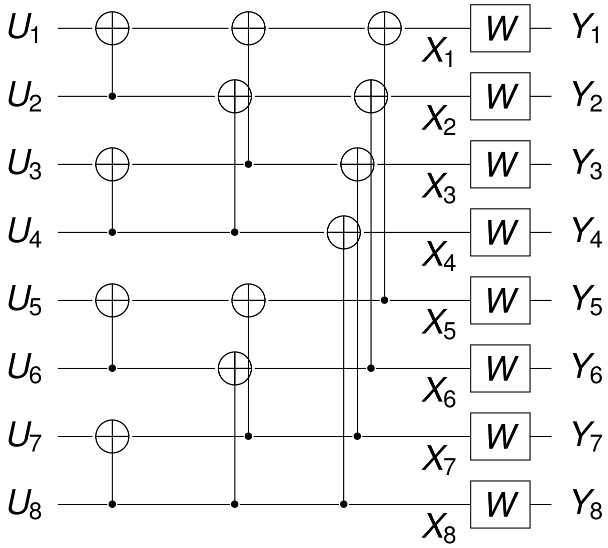
$$W_N^{(i)}\left(y_1^N, u_1^{i-1} | u_i\right)$$



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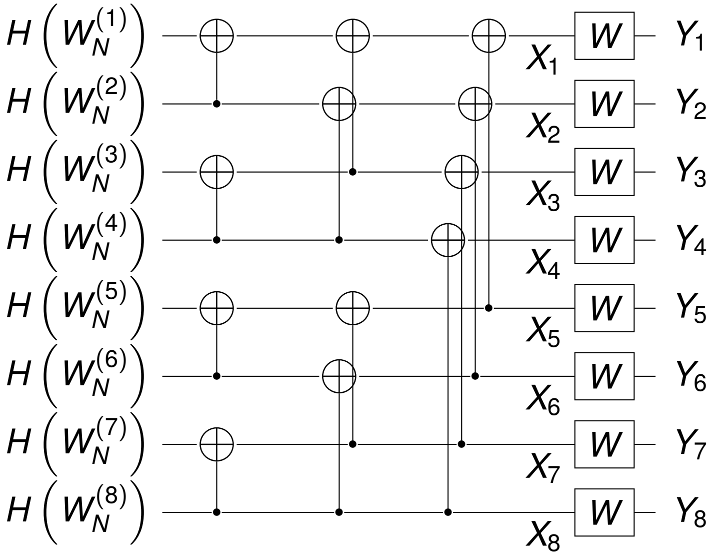

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$$\mathcal{F} = \{1, 2, 3, 5\}$$

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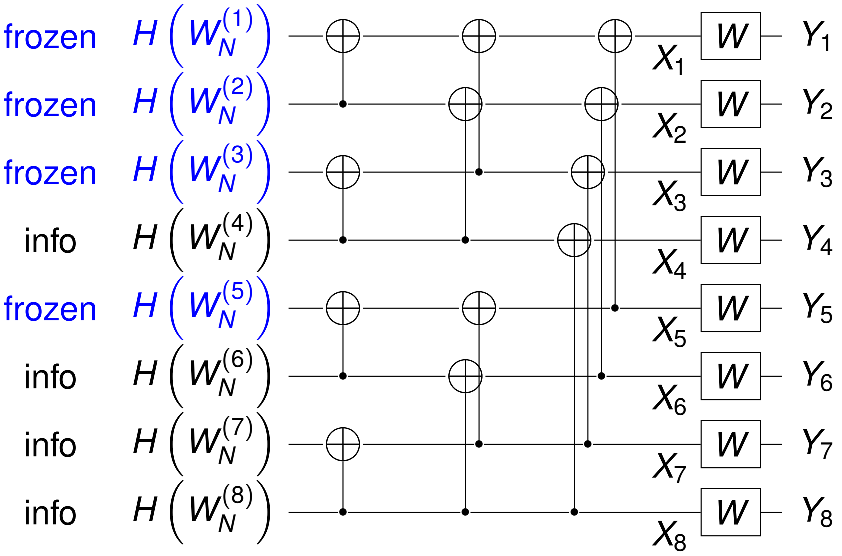

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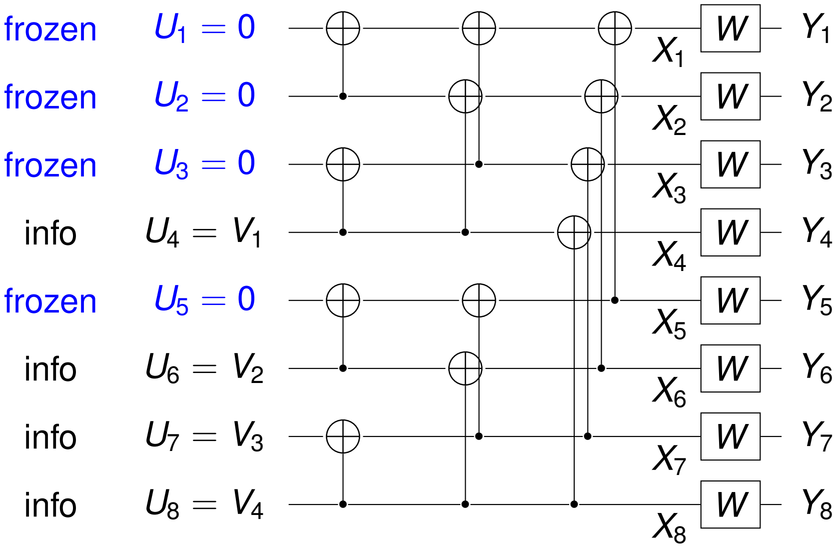

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- A frozen bit whose value depends on past inputs is called dynamic.
- SC/SCL decoding easily modified for polar codes with dynamic frozen bits.
- Any binary linear block code can be represented as a polar code with dynamic frozen bits!

Successive Cancellation List Decoding

- Idea of SCL decoding of y_1^N is to recursively $(i = 1, 2, \ldots)$ compute

$$Q_i(\tilde{u}_1^i) \propto \mathbb{P}\left(U_1^i = \tilde{u}_1^i, Y_1^N = y_1^N\right)$$

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- After N-th stage, estimate $\hat{u}_1^N \in \mathcal{U}_N$ chosen to maximize $Q_N(\tilde{u}_1^N)$.

Outline

1. Preliminaries

2. Information-Theoretic Perspective on the SCL Decoding

    - Binary Erasure Channel

3. Conclusions

An Information-Theoretic Perspective (1)

Basic Idea: After m steps, consider the conditional entropy $H\left(U_1^m \middle| Y_1^N\right)$.

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Note: this ignores frozen bits and will be modified soon!

An Information-Theoretic Perspective (2)

- For the first m input bits, the information/frozen sets are denoted as

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- **Key Idea:** information entropy given frozen bits and difference sequence

$$\bar{D}_m \triangleq H\left(U_{\mathcal{A}^{(m)}} \middle| Y_1^N, U_{\mathcal{F}^{(m)}}\right) \quad \text{and} \quad \Delta_m \triangleq \bar{D}_m - \bar{D}_{m-1}$$

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$$\sum_{i \in \mathcal{A}^{(m)}} H\left(U_i | Y_1^N, U_1^{i-1}\right) - \sum_{i \in \mathcal{F}^{(m)}} \left(1 - H\left(U_i | Y_1^N, U_1^{i-1}\right)\right) \leq \bar{D}_m \leq \sum_{i \in \mathcal{A}^{(m)}} H\left(U_i | Y_1^N, U_1^{i-1}\right)$$

Bounding the List Size

Theorem

Upon observing y_1^N when u_1^N is sent, we define the set (for $\alpha \in (0,1]$)
$\mathcal{S}_\alpha^{(m)}\left(u_1^m, y_1^N\right) \triangleq \left\{\tilde{u}_1^m : \mathbb{P}\left(\tilde{u}_{\mathcal{A}^{(m)}} | y_1^N, \tilde{u}_{\mathcal{F}^{(m)}}\right) \geq \alpha \mathbb{P}\left(u_{\mathcal{A}^{(m)}} | y_1^N, u_{\mathcal{F}^{(m)}}\right)\right\}$. *Then,*

$$\mathrm{E}\left[\log_2 \left|\mathcal{S}_\alpha^{(m)}\right|\right] \leq \bar{D}_m + \log_2 \frac{1}{\alpha} = H\left(U_{\mathcal{A}^{(m)}} \middle| Y_1^N, U_{\mathcal{F}^{(m)}}\right) + \log_2 \frac{1}{\alpha}$$

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Proof.

$$\log_2 \left|\mathcal{S}_\alpha^{(m)}\right| = \log_2 \overbrace{\sum_{\tilde{u}_1^m} \mathbb{1}\left\{\mathbb{P}\left(\tilde{u}_{\mathcal{A}^{(m)}} | y_1^N, \tilde{u}_{\mathcal{F}^{(m)}}\right) \geq \underbrace{\alpha \mathbb{P}\left(u_{\mathcal{A}^{(m)}} \middle| y_1^N, u_{\mathcal{F}^{(m)}}\right)}_{q}\right\}}^{L}$$

$$\leq \log_2 1 / \left(\alpha \mathbb{P}\left(u_{\mathcal{A}^{(m)}} \middle| y_1^N, u_{\mathcal{F}^{(m)}}\right)\right)$$

Valid for all u_1^N and y_1^N; thus, we take expectation over all u_1^m and y_1^N □

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$$\mathrm{E}\left[\log_2 |\mathcal{S}_\alpha^{(m)}|\right] \leq \bar{D}_m + \log_2 \frac{1}{\alpha} = H\big(U_{\mathcal{A}^{(m)}} \big| Y_1^N, U_{\mathcal{F}^{(m)}}\big) + \log_2 \frac{1}{\alpha}$$

- For an SCL decoder with max list size L_m during the m-th decoding step,
  - the decoder needs $L_m \geq |\mathcal{S}_1^{(m)}|$ for the true u_1^m to stay on the list
  - Choosing $\alpha < 1$ (say 0.94) captures near misses and matches entropy better.

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- Significance for code design:
  - A first-order code design criterion can be seen as $\log_2 L_m \geq d_m$.
  - Based on this, the improved code designs will be reported.

Dynamic Reed-Muller Codes

- dRM code ensemble [CNP20, CP21]:
  - Let $\mathcal{A}$ be the information indices of an RM code.
  - u_i is an information bit if $i \in \mathcal{A}$.
  - $u_i = \sum_{j \in \mathcal{A}^{(i)}} A_{ij} u_j$ if $i \in \mathcal{F}$ where A_{ij} iid $\sim$ Bernoulli(0.5).

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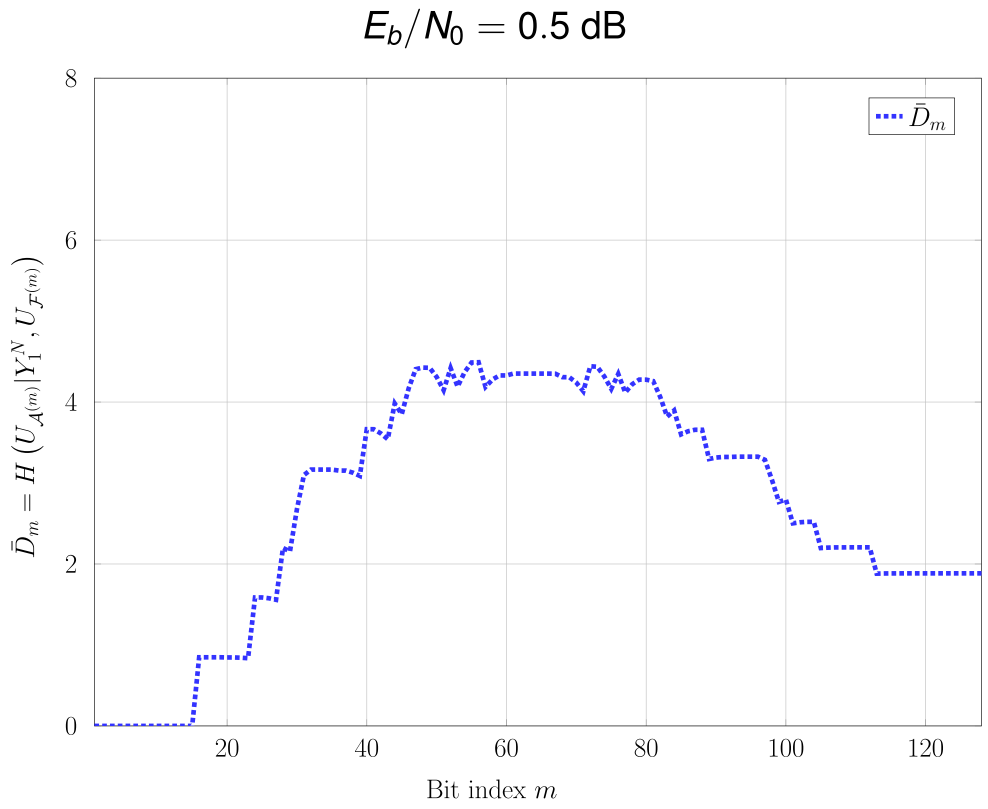
- Closely related to polarization-adjusted convolutional (PAC) codes [Arı19].
- PAC is outperformed slightly by (random instances of) dRM code under SCL decoding with the same list sizes.

$(128, 64)$ dRM Code over the AWGN Channel



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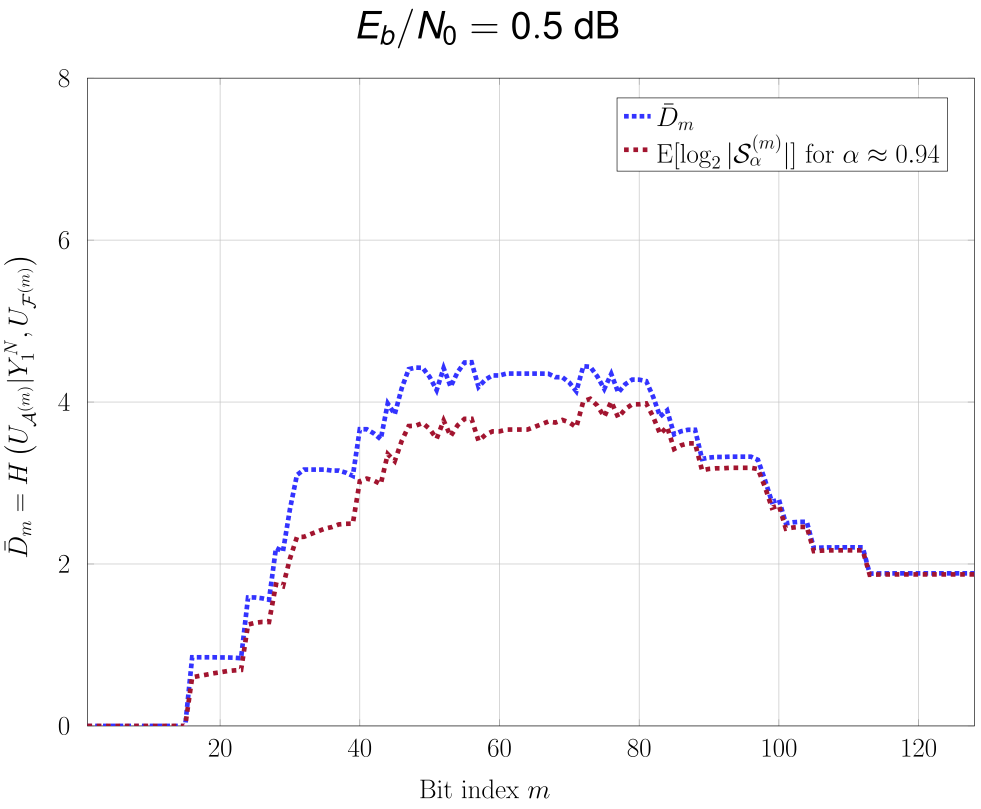

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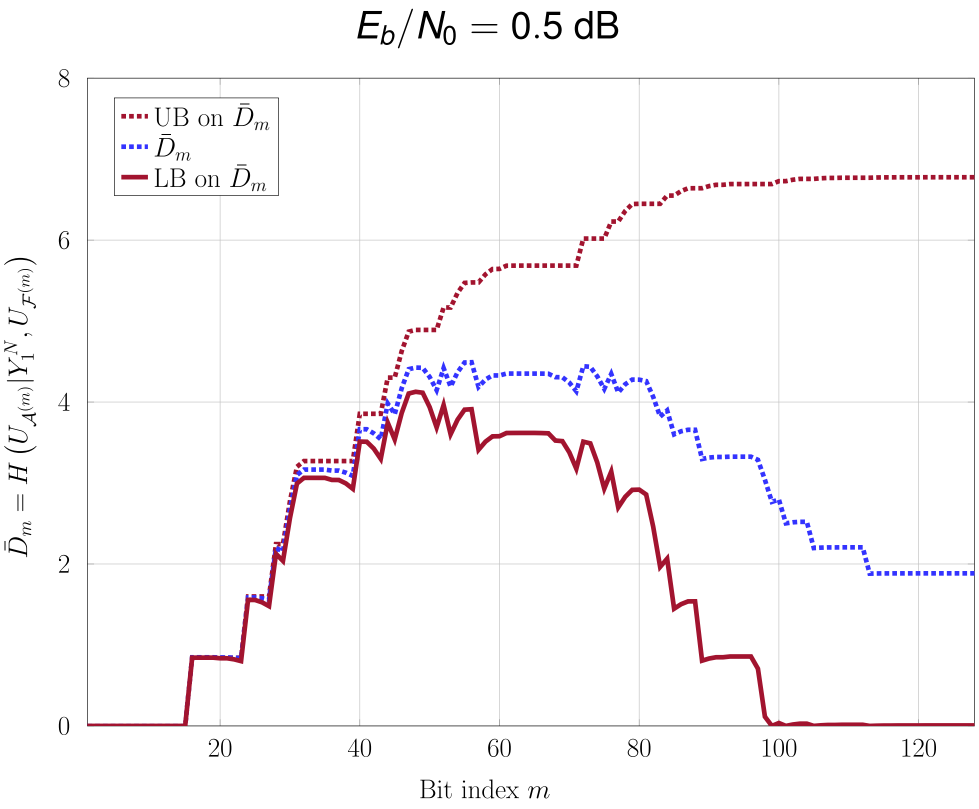
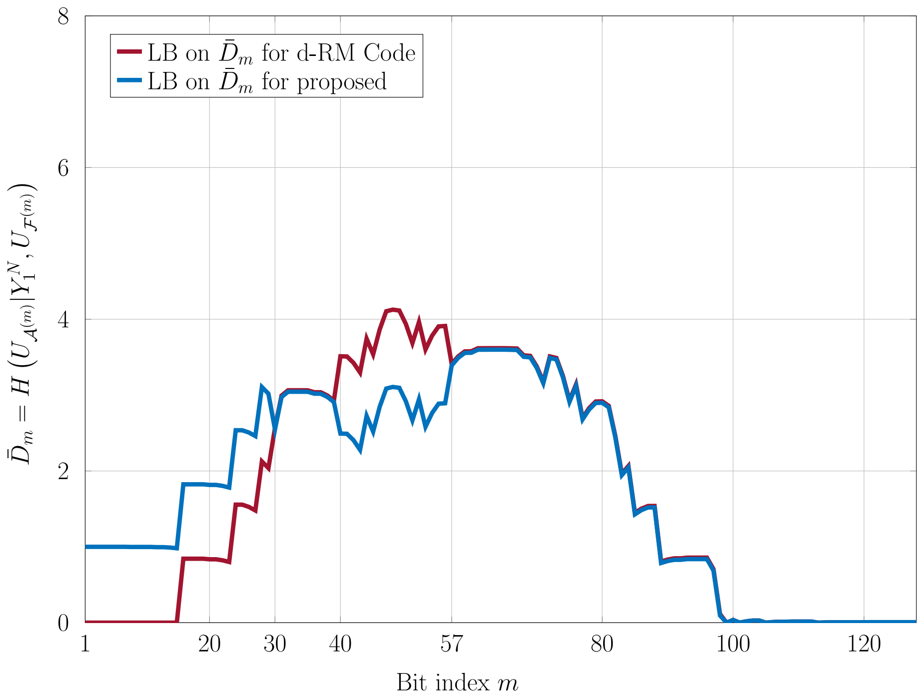


$(128, 64)$ Proposed vs dRM Code over the AWGN Channel

Proposed Code

- $u_{\{30,40\}}$ dynamic frozen bits
- $u_{\{1,57\}}$ info. bits

$E_b/N_0 = 0.5$ dB



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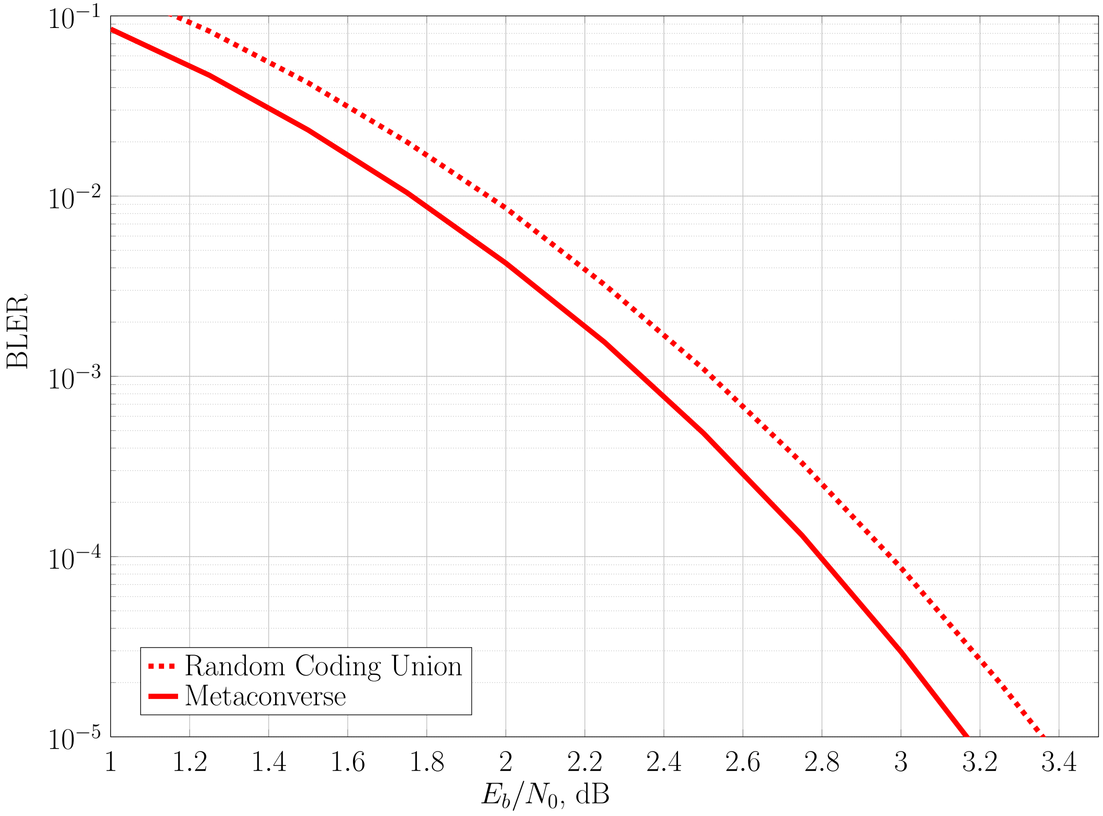

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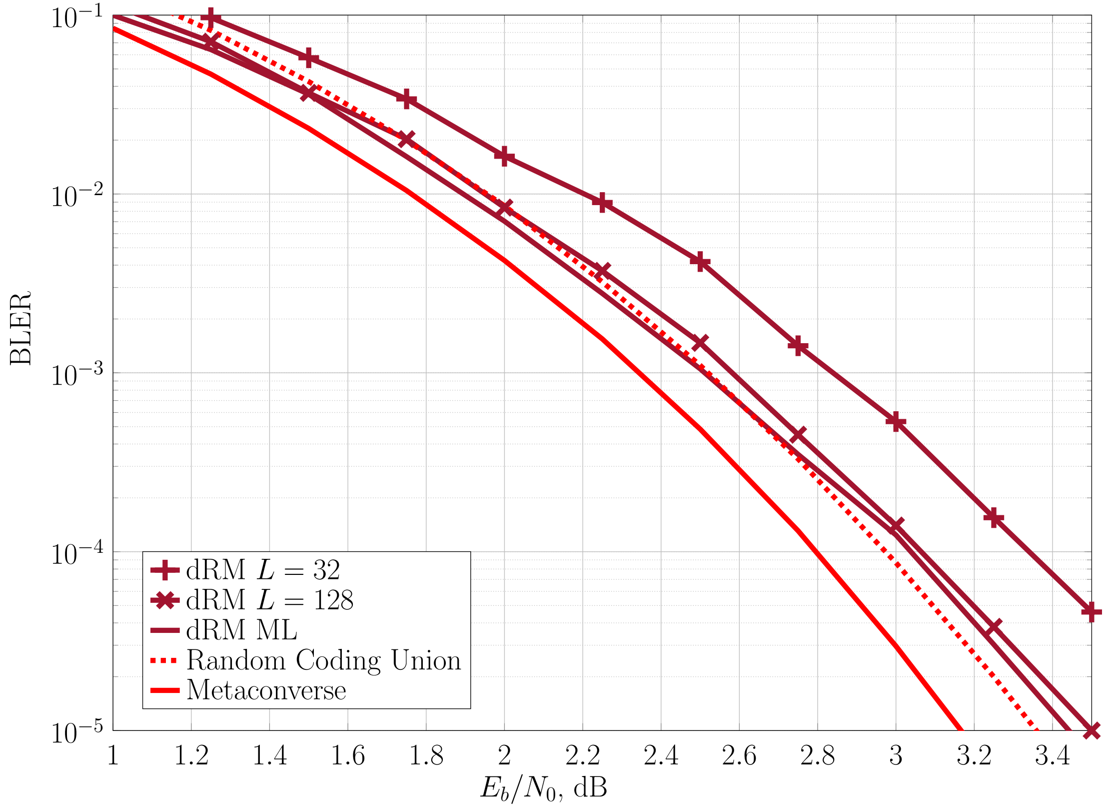

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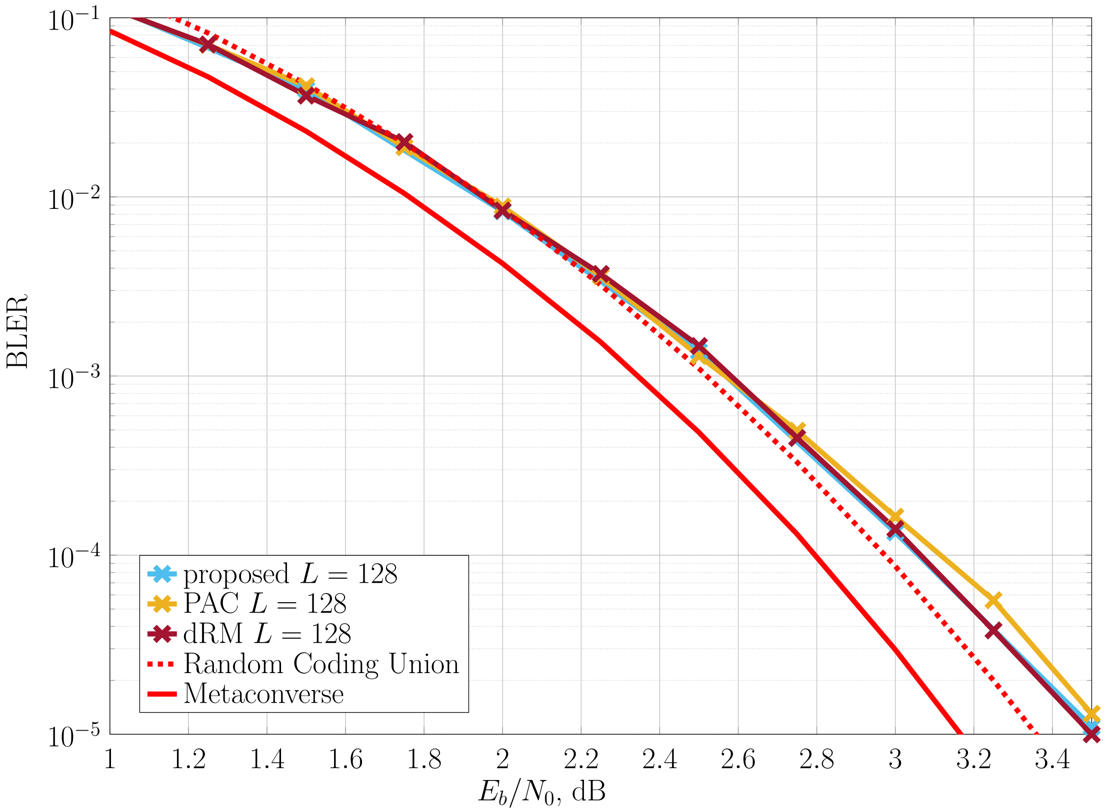

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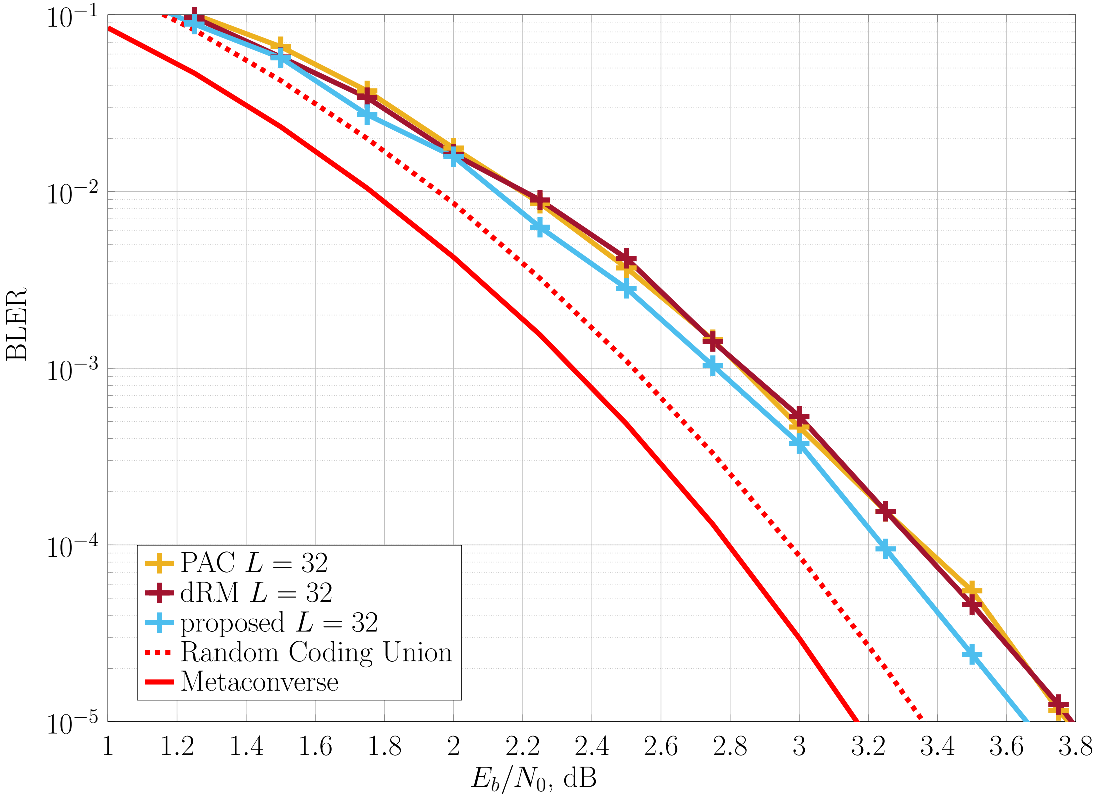
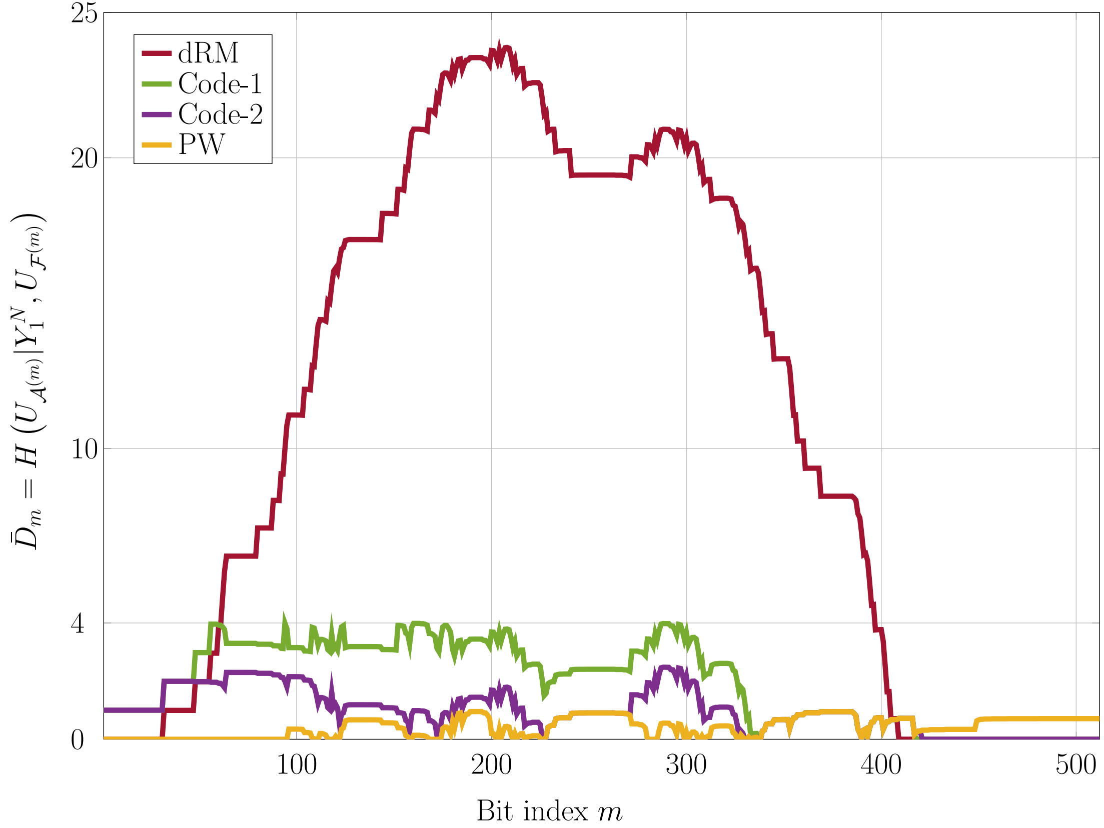


(512, 256) Proposed vs dRM Codes over the AWGN Channel

$E_b/N_0 = 0.5$ dB



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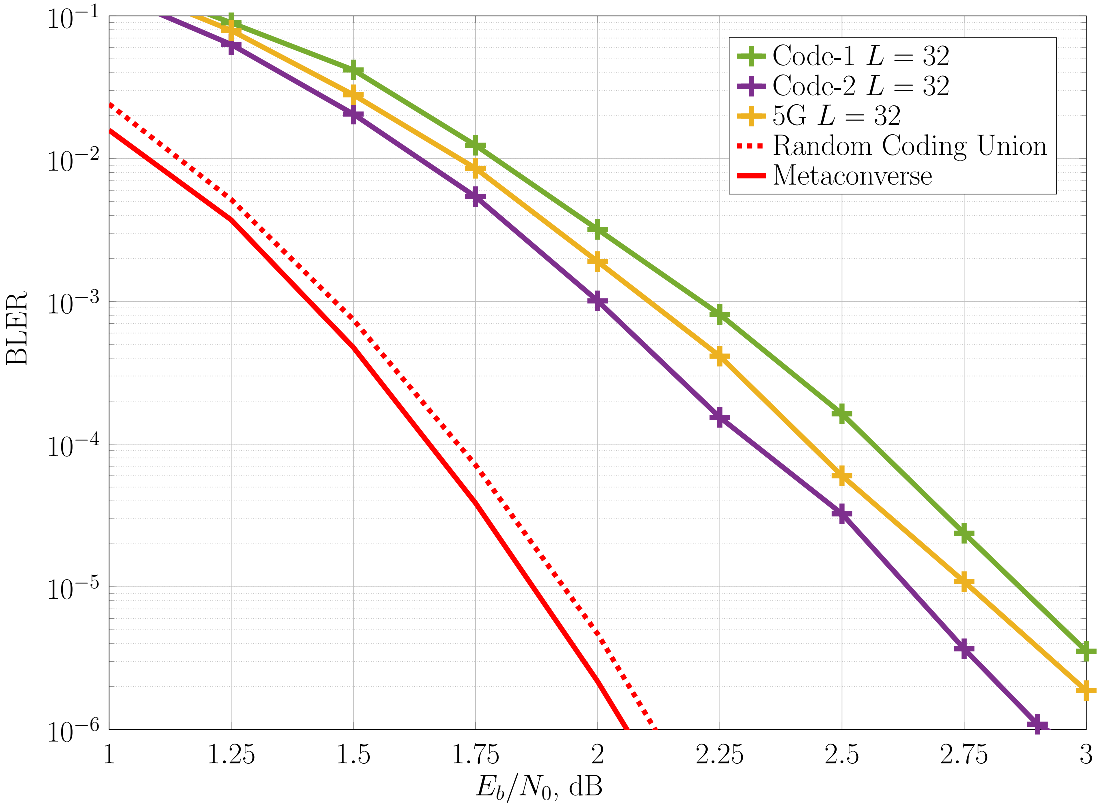

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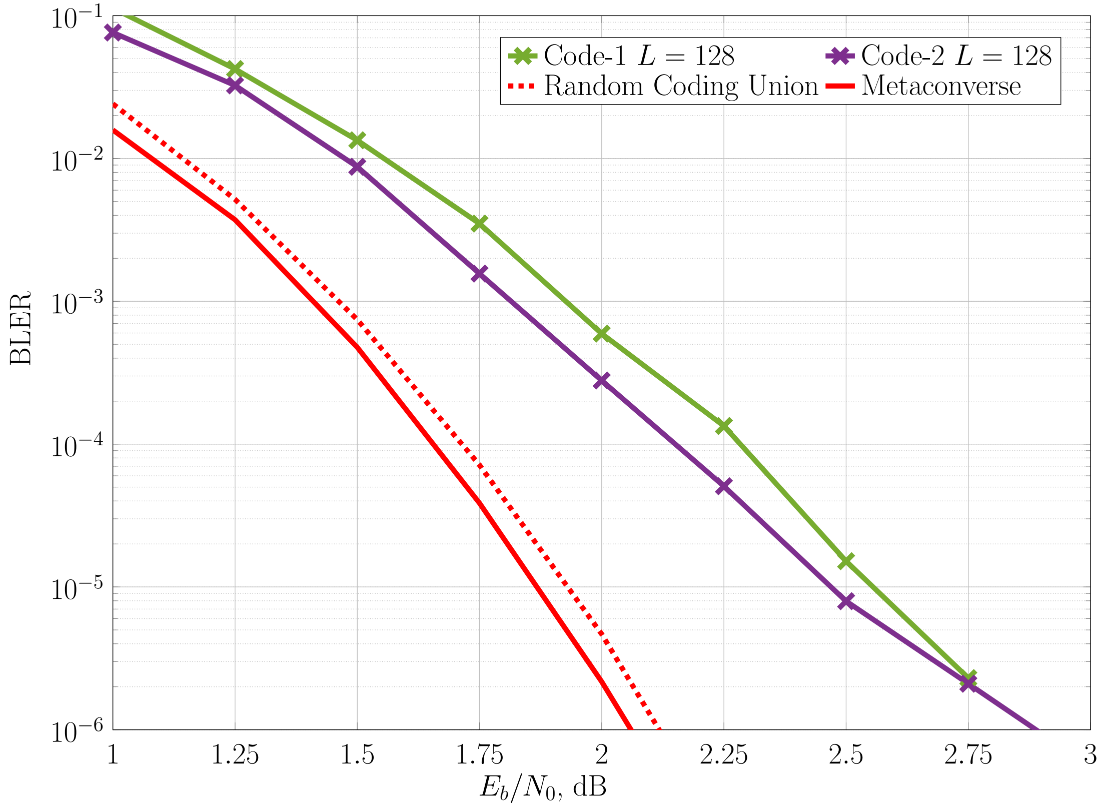

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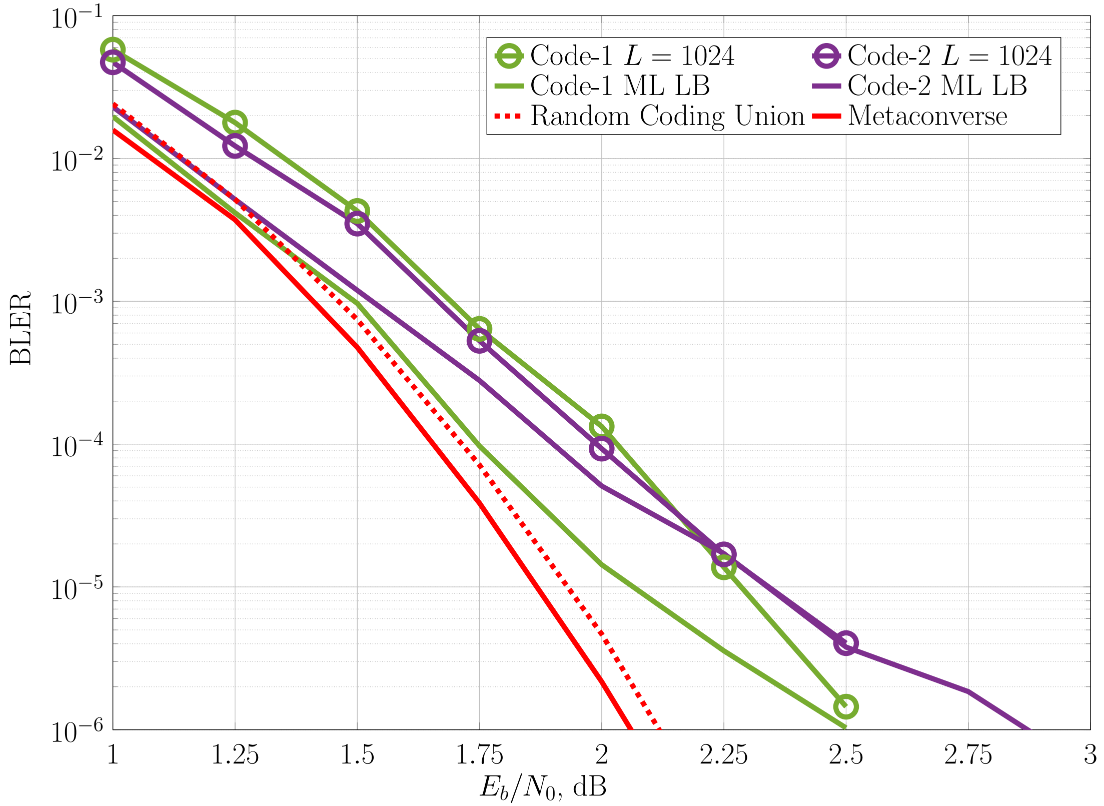

Outline

The Uncertainty Dimension

- For a fixed y_1^N, the subspace dimension is

$$d_m(y_1^N) = H\left(U_{\mathcal{A}^{(m)}} \middle| Y_1^N = y_1^N, U_{\mathcal{F}^{(m)}}\right)$$

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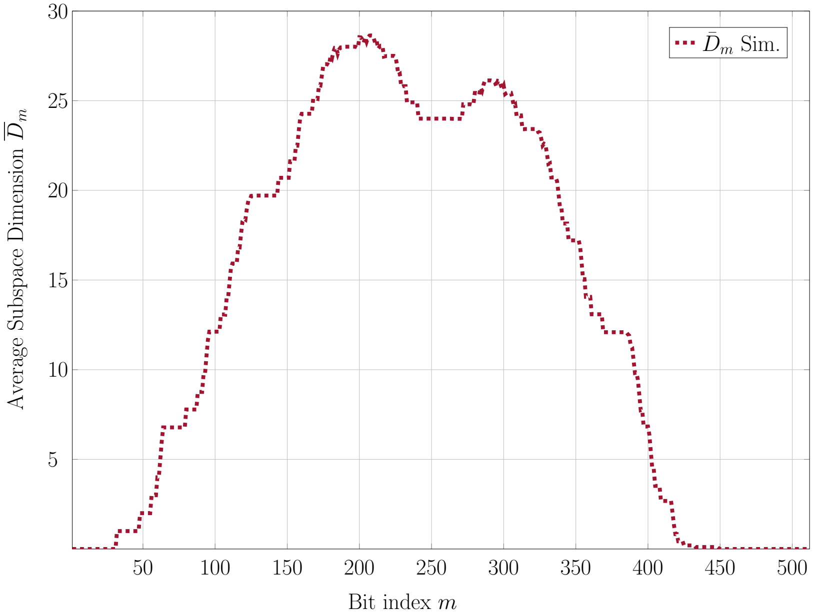
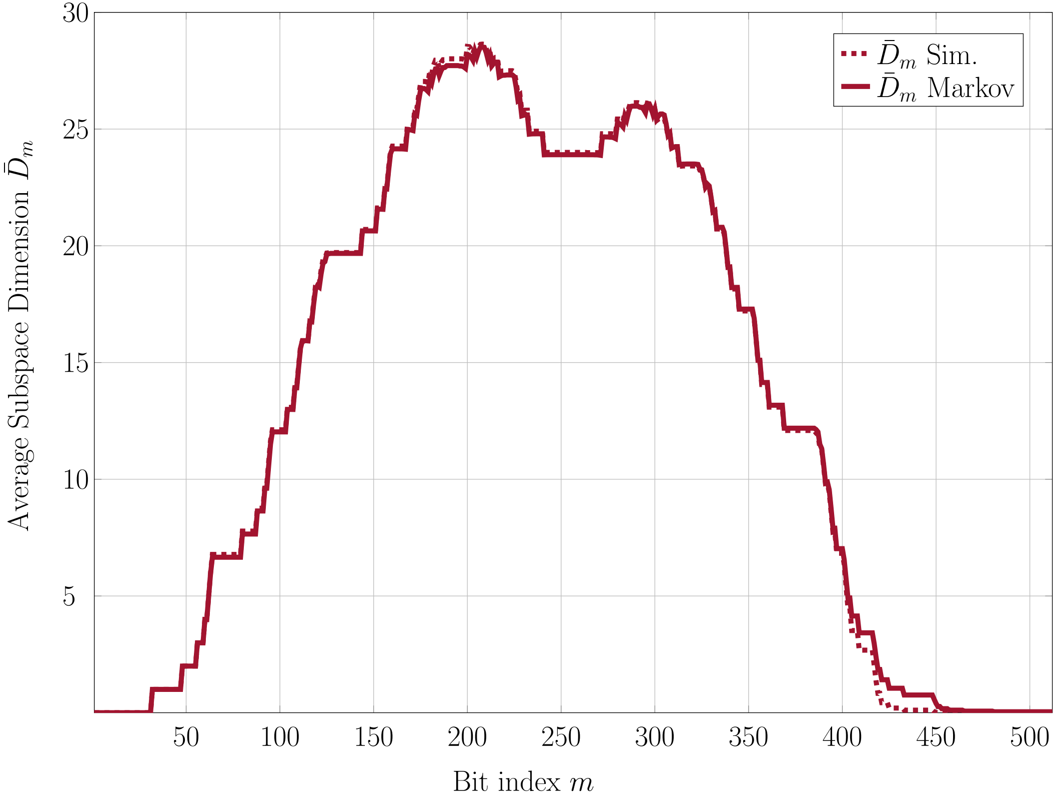


$(512, 256)$ dRM Code

A fixed-weight BEC with exactly $\texttt{round}(512 \times 0.48) = 246$ erasures



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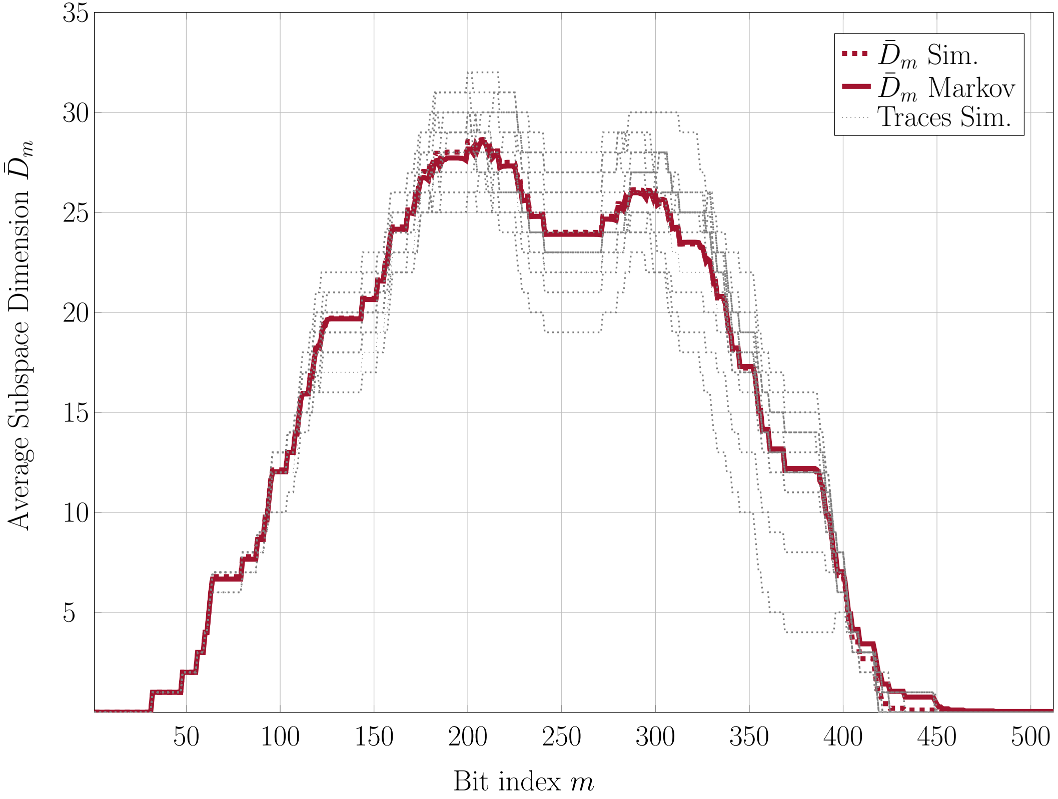

Concentration of the Subspace Dimension

Theorem

The subspace dimension D_m for a particular random realization Y_1^N concentrates around the mean $\bar{D}_m$ for sufficiently large block lengths [CP21], i.e., for any $\beta > 0$, we have

$$\mathbb{P}\left\{\frac{1}{N}|D_m - \bar{D}_m| > \beta\right\} \leq 2\exp\left(-\frac{\beta^2}{2}N\right). \tag{1}$$

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Proof.

Key observation: at any decoding stage, the subspace dimension satisfies Lipschitz$-$1 condition: For all $i \in [N]$ and all values y_1^N and $\tilde{y}_i$, we have

$$|d_m(y_1^N) - d_m(y_1^{i-1}, \tilde{y}_i, y_{i+1}^N)| \leq 1.$$

Then, use Azuma-Hoeffding inequality by forming a Doob's Martingale. □

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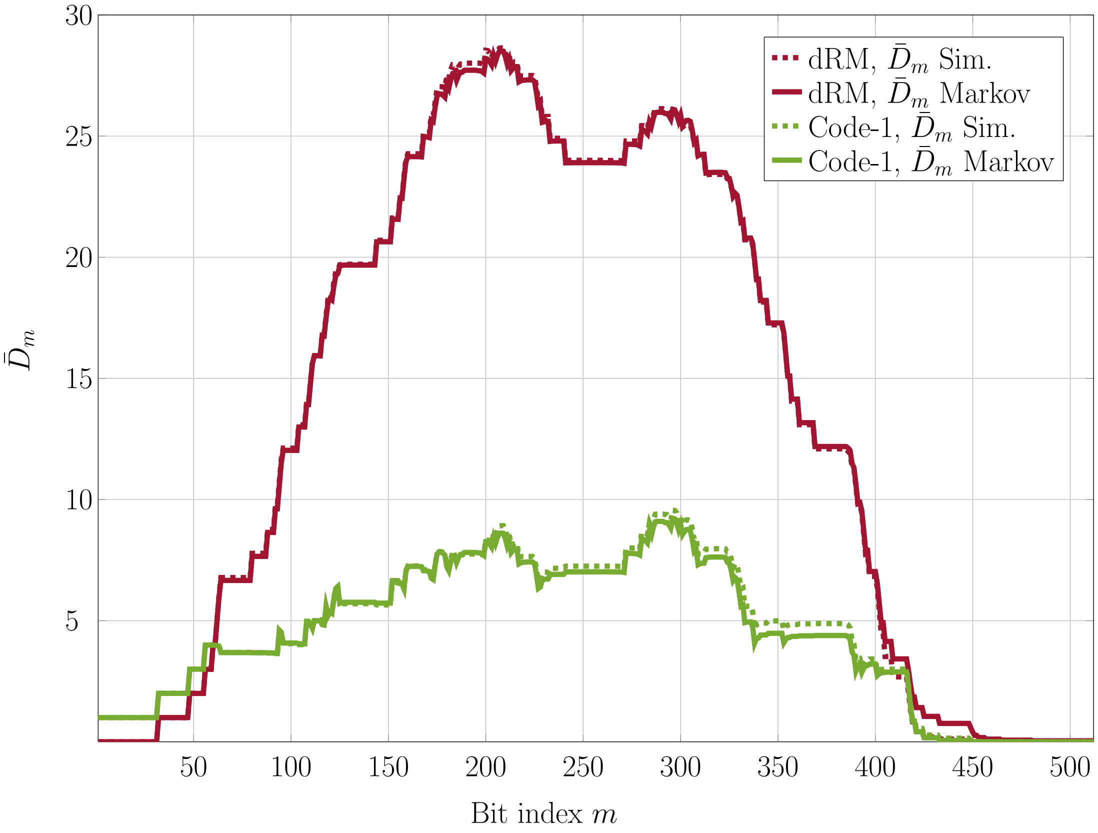
- We may use the theorem above to give bounds on the average complexity of ML decoding of a given code implemented via SCI decoding.
- Extension to more general class of channels follows [CP21].

(512, 256) Codes over the BEC



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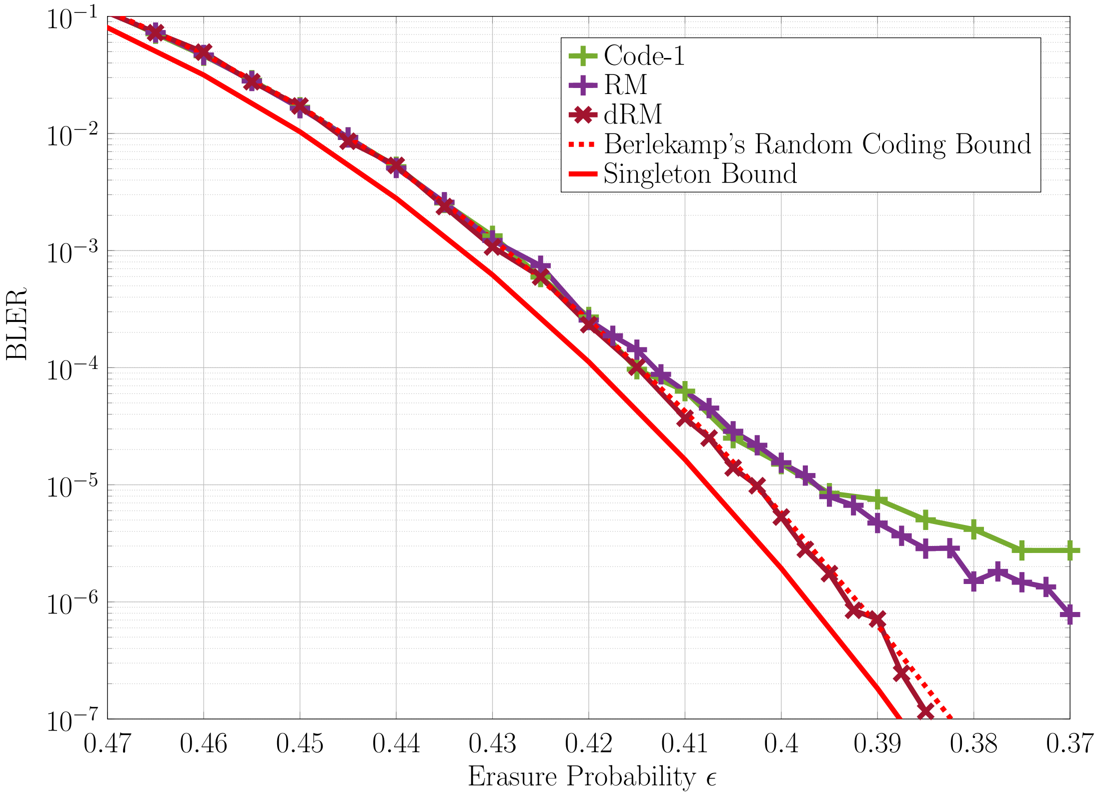

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  - Not only for $N = 128$ but also for $N = 512$.
- Recent advances (dynamic frozen bits + SCL) in polar codes allow performance near random coding union bound for $(128, 64)$ with moderate complexity.
  - Moderate-length regime, e.g., $2048 \geq N \geq 256$, has more margin to improve.

Summary (2)

BEC

- The logarithm of the random required list size is shown to concentrate around the mean and a simple but accurate approximation of this mean is provided for the BEC.
  - Holds for more general class of channels (see [CP21] for details).

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Outlook and Future Work

- The concentration of the random subspace dimension makes the average analysis meaningful; hence, we may upper bound the average complexity of SC inactivation decoding (for the BEC).
- Constructive algorithm to design longer codes with good performance vs. complexity trade-off under SCL decoding.

Thanks

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