Technische Universität München

Fakultät für Mathematik

Lehrstuhl für Wahrscheinlichkeitstheorie

Mixing times and limit theorems for exclusion processes

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigten Dissertation.

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Die Dissertation wurde am 28.04.2021 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 11.06.2021 angenommen.

Acknowledgments

First of all, I am genuinely grateful to my supervisor and mentor, Nina Gantert, for her encouragement, guidance, and advice in the last years. Since my time as a second-year Bachelor's student, she raised my interest in probability theory and introduced me to the world of interacting particle systems. Giving me the chance to develop and pursue my own research interests, giving me the freedom and flexibility to work on different problems and topics, encouraging me to travel to various summer schools, and helping me to establish connections to researchers from all over the world, is outstanding support and by far not standard. I am deeply grateful for these opportunities!

Next, I thank all present, former and honorable members of the M14 group for a delightful time and for creating an excellent working atmosphere. In particular, I am grateful to Noam Berger for sharing his intuition about mathematics with me. In every single conversation, there is at least one new insight that I get from him. In particular, I want to thank him for changing my thinking on how to cut pizza forever. Moreover, special thanks go to Stefan Junk for sharing the office for almost three years and introducing me to coffee and Go, and Felizitas Weidner for always having an open door for discussions and fantastic cake.

Furthermore, I want to thank all my collaborators and other researchers who I met during the last years, for an amazing time on the past (and future) work. Doing math together makes it even more enjoyable, and I am grateful to learn a lot from all of you during our conversations! In particular, I want to thank Nicos Georgiou and Perla Sousi for hosting me during a research visit in the UK, Patrik Ferrari and Caroline Lasser for being members of the Ph.D. committee, and Sam Olesker-Taylor and Sebastian Weiß who helped proof-reading this thesis. Special thanks go to Evita Nestoridi for being a fantastic collaborator and at the same time mentor and advisor for many mathematical and non-mathematical questions, as well as to Alexandros Pardon for his joyful contributions to our discussions.

Moreover, I want to thank the TopMath program and the Studienstiftung des Deutschen Volkes for their great support during my Ph.D. time, which enabled me to carry out large parts of the research with many amazing people from all over the world.

Finally, I want to sincerely thank my parents, my girlfriend Tanja, and my friends for their outstanding support. They listened to endless stories about mathematics and many other things but were always encouraging, helpful, and supportive. Without you, this thesis would not be possible!

To my grandparents

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Part I

Introduction

1 General introduction

Imagine that we have a set of indistinguishable particles and a graph. Initially, we place the particles on different sites of the graph. We then let the particles move independently according to a given transition rule. However, when a particle tries to move to an occupied site, this move is suppressed. For a given graph, a given initial distribution of the particles, and a given transition rule, what can we say about the long-term behavior of the system?

The main goal of this thesis is to investigate this question when the particles perform simple random walks. In this case, the above-described dynamic is called the simple exclusion process. The simple exclusion process is a model with a long history. In the late 1960s, a version of this model was first studied in biology by MacDonald, Gibbs, and Pipkin, where the particles represent ribosomes moving along the RNA [100]. Nowadays, the exclusion process has various applications, such as a model for cars in a traffic jam or molecules in gases [30, 71, 111]. A first study in the mathematical literature is attributed to Spitzer in 1970; see [133]. Since then, over the last decades, the exclusion process equally raises scientists' attention from probability, statistical mechanics, and combinatorics; see [21, 89, 94, 140] for review papers in the respective areas.

The simple exclusion process is a natural example of a Markov process. Informally speaking, for a Markov process, it suffices to know its current state to determine the law of its future evolution. When the underlying graph of the exclusion process is infinite but locally finite, a characterization of the limit behavior is of significant interest. We achieve first insights by investigating the set of invariant measures. For example, it is a classical result that the Bernoulli product measures with constant parameter are extremal invariant measures for the exclusion process when the graph and the rates are translation invariant under spatial shifts; see [94]. In general, studying the behavior of the exclusion process in and out of equilibrium reveals surprising phenomena. These include phase transitions and the formation of shocks, i.e., an abrupt change in the particle density; see [56, 51, 53, 54, 137]. The main observable to describe these phenomena is the current. The current quantifies the number of particles passing through a given location over time. The moments of the current are closely related to the motion of second class particles [10]. Intuitively, second class particles are perturbations of the system as they act as particles with respect to empty sites and as empty sites with respect to particles. In particular, the fluctuations of the current of a simple exclusion process on the integers with drift can be expressed using the mean of the displacement of a single second class particle started from the origin within a Bernoulli product measure. We see either a diffusive or a sub-diffusive behavior of the current depending on the product measure parameter; see [11, 53, 112]. Furthermore, we can identify shocks using second class particles [51, 54, 55]. More precisely, suppose that we consider the simple exclusion process on the integers and start with a shock at the origin, i.e., we associate two product measures to the positive and negative integers with different parameters. We place a second class particle at the transition point. It is a classical result that the second class particle will stay close to the shock location for all times under certain assumptions on the parameters in the product measures [51].

A different way of gaining insights on the long-term behavior of the exclusion process is to study the motion of a single particle within the system, referred to as the tagged particle. In the following, we focus on the speed of the tagged particle and its fluctuations around the expected position over time. Consider the d-dimensional lattice, where the transition probabilities are given by a simple random walk with drift, and where we start from the invariant Bernoulli-ρ-product measures with parameter ρ. In this case, the speed of the tagged particle is $1-\rho$ times the speed of a single particle; see [78, 114]. This formula for the speed is what one expects: since the density of empty sites is $1-\rho$, only a proportion of $1-\rho$ of the steps are carried out. In one dimension, Arratia proved that the symmetric simple exclusion process has a subdiffusive behavior [4]. Kipnis and Varadhan studied the fluctuations of the symmetric simple exclusion process on $\mathbb{Z}^d$ with $d \geq 2$ and proved a diffusive behavior [79]. These articles had a significant impact on further work for exclusion processes and related processes; see [80] for an overview. For a general introduction to exclusion processes and other interacting particle systems, we refer to Liggett [94, 95].

When the infinite system is too complicated to be analyzed directly, an alternative is to study finite particle systems as approximations. We treat the corresponding exclusion processes as continuous-time Markov chains on a finite state space. When the underlying graph is the integer lattice, we consider the simple exclusion process with open boundaries as a natural model to approximate the infinite system. In this process, we allow particles to enter and exit at the segment's endpoints in addition to the particle movement under the exclusion rule. The simple exclusion process with open boundaries is, in general, not reversible and it is one of the simplest examples of a non-equilibrium system. Informally speaking, this means that the mean position of the particles does not stay fixed over time. We quantify this observation by studying currents for the simple exclusion process with open boundaries; see Section 6.3.2 for a more detailed discussion. Currents were investigated for symmetric transition rates within the segment in [87]. For general boundary parameters and drift, Uchiyama et al.

determined the first-order of the current using Askey–Wilson polynomials [137] . This extended previous results of Blythe et al. [22]. Current fluctuations for the asymmetric simple exclusion process with open boundaries were investigated in [69, 88]. Related spectral properties can be found in [42, 41, 43], among others. Furthermore, there are deep connections to the Kardar–Parisi–Zhang universality class; see [35, 37, 39, 108].

Over the last decades, invariant measures of the simple exclusion process with open boundaries were intensively studied; see [94, Part III, Section 3]. An important tool is the matrix product ansatz. It is in an implicit form already given in [92] and was successfully applied for the simple exclusion process with open boundaries in [45] when particles can move only in one direction. Informally speaking, we assign in the matrix product ansatz to every configuration a weight that consists of a product of matrices and vectors. The matrices and vectors must satisfy certain relations, usually called the DEHP algebra; see [45]. The matrix product ansatz allows to study the mean current, the density profile, and correlations within the stationary distribution; see [117, 137, 138]. Representing the weights in the matrix product ansatz is a combinatorial question that gained lots of recent attention. It led to descriptions such as weighted Catalan paths and staircase tableaux; see [23, 33, 101]. These representations are closely related to Askey–Wilson polynomials, building on the works of Sasomoto and Uchiyama et al. [116, 137]. Similar expressions hold for the simple exclusion process with second class particles using Koornwinder polynomials — see [26, 32] — and were recently extended for more than two different kinds of particles [27, 61, 102].

When no particles are allowed to enter or exit the finite system, a main question is to quantify the convergence to the equilibrium from an arbitrary starting configuration. This problem has a particular motivation coming from card shuffling: suppose that we have a deck of cards which we shuffle in the following way. In each step of the shuffle, we pick a pair of adjacent cards uniformly at random and swap their positions. How many steps does it take until a deck of N cards is well mixed? This dynamic of shuffling cards is known as the random adjacent transposition shuffle, and more generally, as the interchange process for arbitrary finite graphs. We reobtain an instance of the simple exclusion process with k particles by projecting the first k cards to particles and all other cards to empty sites. We study (total-variation) mixing times to quantify how much time is required to mix. Informally speaking, for general Markov chains, the mixing time determines when the law of the chain is close to its equilibrium for any possible starting configuration. We are interested in the mixing time when the system's size grows. We say that a family of Markov chains exhibits the cutoff phenomenon, if the transition from unmixed to mixed occurs in a short time interval compared to the total amount of time needed to mix. Diaconis and Shahshahani discovered this phenomenon in [46] in the year 1981 and it obtained its name in a seminal paper by Aldous and Diaconis [2]. It appears for a wide range of families of Markov chains and is still not fully understood today; see [15, 18, 48, 65] for recent progress. However, many techniques for achieving good bounds on the mixing time are now available. An introduction to these tools can be found in [91] and we discuss a selection of them in Section 4.

For the symmetric simple exclusion process (SSEP), i.e., when the particles perform symmetric simple random walks on the segment, the first-order term of the mixing time is determined in [139] and [84]. In particular, we see cutoff when the number of particles and empty sites goes to infinity together with the size of the segment. The proof for the lower bound in [139] uses spectral techniques. The upper bound in [84] follows from a clever combination of various properties of the SSEP. For the asymmetric simple exclusion process (ASEP), i.e., when the particles perform biased simple random walks on the segment, Benjamini et al. showed that the mixing time is linear in the size of the segment [19]. Recently, Labbé and Lacoin proved cutoff for the ASEP [81]. Furthermore, mixing times for exclusion processes were investigated for size-dependent bias as well as on general graphs [72, 82, 90, 107]. All these investigations have in common that the underlying simple exclusion process is reversible. Many techniques for precise bounds on the mixing time require reversibility and do, for example, not apply for the simple exclusion process with open boundaries. To our best knowledge, mixing times for a non-reversible simple exclusion process were previously only investigated for the asymmetric simple exclusion process on the cycle [60].

2 Overview of the results

This thesis is based on the following five publications and preprints:

- [119, Section 5]; published in Electronic Journal of Probability,
- [66]; submitted, joint work with Nina Gantert and Evita Nestoridi,
- [29]; published in Electronic Communications in Probability, joint work with Dayue Chen, Peng Chen and Nina Gantert,
- [67]; published in Electronic Journal of Probability, joint work with Nina Gantert,
- [62]; submitted, joint work with Nina Gantert and Nicos Georgiou.

The content of the material presented in this thesis mainly follows the publications and preprints. The presentation will differ at several points, e.g., we provide additional explanations and figures. We will now give an outline of the thesis.

In the remainder of Part I, we introduce the exclusion process as an interacting particle system. We state preliminary results on the existence of the exclusion process as a Feller process and present a selection of classical properties. This includes the characterization of the extremal invariant measures of the simple exclusion process as well as its graphical representation and canonical coupling.

In Part II, we study the simple exclusion process on segments of the integers. We focus on two cases: The simple exclusion process in a random environment and the simple exclusion process with open boundaries. Following [119, Section 5], we study in Section 5 a site-dependent random environment $\omega = \{\omega_x\}_{x\in\{1,\dots,N\}}$ for the segment of size N, where the elements ω_x are i.i.d. according to a distribution ν on $(0,1]$. Here, a particle at a site x moves to the right at rate ω_x and to the left at rate $1-\omega_x$ whenever the target is a vacant site. We quantify the speed of convergence to the equilibrium using mixing times. Suppose that the environment law is marginally nestling with a bias to the right-hand side, i.e., ν is supported on $[\frac{1}{2},1]$. Then the order of the mixing time is at most $N\log^3(N)$ with probability tending to 1 when N goes to infinity.

We then study mixing times for the simple exclusion process with open boundaries in homogeneous environments. This part is based on joint work with Nina Gantert and Evita Nestoridi [66]. For symmetric rates and general boundary conditions, we prove that the mixing time is at least the mixing time of the simple exclusion process on the closed segment to first-order; see also [84]. Moreover, we establish the cutoff phenomenon when particles can enter and exit only from one side. The analysis for the simple exclusion process with drift is more delicate. For closed segments, the mixing time is always linear in the size of the segment; see [19]. We see different regimes of the mixing time depending on the boundary parameters. In particular, we study the case where particles can enter and exit at both sides of the segment and produce a positive linear current. This is from a physical perspective arguably the most exciting regime. Compared to previous results for exclusion processes, this case is harder to analyze since many essential properties such as a monotone height function representation, a reversible measure, or the conservation of particles are no longer available. We introduce multi-class particle arguments to prove that the mixing time in the high density and in the low density regime is linear in the size of the segment.

In Part III, we start with a brief overview of related work for currents and tagged particles. We then investigate tagged particles in exclusion processes on trees. This part is based on joint work with Dayue Chen, Peng Chen, and Nina Gantert [29, 67]. We consider the exclusion process on rooted d-regular trees. We assume that the particles move according to symmetric transition rates that are translation invariant, irreducible, and of finite range. For $\rho \in (0,1)$ fixed, we start from a Bernoulli-ρ-product measure conditioned on having a particle at the root — our tagged particle. Our goal is to prove limit laws for the statistics of tagged particles, measured in terms of the shortest path distance on the tree. For $d \geq 3$, we show that the tagged particle has a positive linear speed, which we determine explicitly. It turns out that the speed scales linearly in the particle density ρ. Further, we prove a central limit theorem for the tagged particle on the d-regular tree when $d \geq 3$, following the classical results of Kipnis and Varadhan [79]. For $d = 2$ and nearest neighbor transition rates, a subdiffusive behavior was shown in [4].

For simple exclusion processes, i.e., when the particles perform only nearest neighbor moves, we extend the results to non-regular trees. More precisely, we consider an offspring distribution with support in $\mathbb{N}$. We define a Galton–Watson tree by choosing a starting position o as the root. Recursively, starting from o, we then draw for every site a number of descendants independently according to the offspring distribution. Since the root has, on average, one neighbor fewer than all other sites, we add one additional descendant to o, and apply the same recursion to obtain an augmented Galton–Watson tree. We define the simple exclusion process on a fixed augmented Galton–Watson tree in two different ways: In the variable speed model, each particle at a site x has an exponential waiting time with parameter $\deg(x)$ independently of all other particles. In the constant speed model, the particles have exponential waiting times with parameter 1. In both cases, we let the simple exclusion process start from an equilibrium distribution with non-vanishing particle density and consider the tagged particle initially placed at the root. We show in both models that the tagged particle has almost surely a positive linear speed and we give explicit formulas for the speeds. We see a linear scaling in the particle density for the variable speed model, similar to the d-regular tree. In the constant speed model, it turns out that the scaling of the speed is in general smaller than linear in the averaged particle density.

In Sections 10 and 11, we consider the totally asymmetric simple exclusion process (TASEP) on rooted trees with a reservoir. This part is based on joint work with Nina Gantert and Nicos Georgiou [62]. In the TASEP, particles can only jump on an edge along the direction pointing away from the root. In addition, particles are created at the root according to a given rate whenever the root is vacant. Our interests are two-fold. On the one hand, we study invariant measures for the TASEP on trees and provide sufficient conditions for the existence of non-trivial equilibrium distributions; see Section 10. In particular, we are interested in conditions which ensure that the invariant measure has a non-trivial density when starting with all sites being empty. On the other hand, we study the current of the TASEP, i.e., the number of particles that have passed through a specific level of the tree until a particular time; see Section 11. In this case, we let the underlying graph be a Galton–Watson tree with the above assumptions on the offspring distribution. We give upper and lower bounds on the current for general rates. For regular trees with the same rates at each tree level, we provide refined bounds on the current. An essential step in the analysis of the current is to understand how the particles disentangle on the tree over time. This is a question of independent interest.

3 Preliminaries on exclusion processes

We will now give a brief introduction to interacting particle systems. We focus on the simple exclusion process and related processes. The presented material is mainly taken from Chapters 3 and 4 of [96] and Chapter VIII of [95] by Liggett, and we refer the reader to these references for a more detailed discussion.

3.1 Construction and basic definitions

Let $G = (V, E)$ be a locally finite graph with vertex set V and edge set E. Let $S = \{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$ be a set of colors, also called **spins**. For a fixed graph G and a set of spins S, let $\Omega := S^V$ be the space of all vertex-colorings of G using elements of S. We endow Ω with the product topology.

In the following, we focus on the case $S = \{0, 1\}$. We say for a configuration $\eta \in \Omega$ that the site $v \in V$ is **vacant** if $\eta(v) = 0$ holds, and **occupied** otherwise. Intuitively, we think of the spins as indicators whether a certain site $v \in V$ is occupied by a particle ($\eta(v) = 1$) or empty ($\eta(v) = 0$). On the space $\{0, 1\}^V$, we define two basic operations: **flipping** and **swapping**. For $\eta \in \{0, 1\}^V$, we write

$$\eta^x(z) := \begin{cases} 1 - \eta(z) & \text{if } z = x \\ \eta(z) & \text{if } z \neq x \end{cases} \tag{3.1}$$

for the configuration which we obtain from η by flipping the value at $x \in V$ and leaving all other spins unchanged. For $x, y \in V$ with $x \neq y$, we denote by

$$\eta^{x,y}(z) := \begin{cases} \eta(x) & \text{if } z = y \\ \eta(y) & \text{if } z = x \\ \eta(z) & \text{if } z \neq x \end{cases} \tag{3.2}$$

the configuration which we obtain from η by swapping the values at x and y, and again leaving all other spins unchanged.

With these notions at hand, we will now define a stochastic process $(\eta_t)_{t \geq 0}$ on Ω. Consider a pair of functions $c \colon V \times \Omega \mapsto \mathbb{R}_0^+$ and $p \colon V \times V \mapsto \mathbb{R}_0^+$, and let the operator $\mathcal{L}$ be given by

$$\mathcal{L}f(\eta) = \sum_{x \in V} c(x, \eta)\,[f(\eta^x) - f(\eta)] + \sum_{x,y \in V \colon \eta(x)=1, \eta(y)=0} p(x, y)\,[f(\eta^{x,y}) - f(\eta)] \tag{3.3}$$

for all cylinder functions $f\colon \Omega \to \mathbb{R}$. The following theorem gives sufficient conditions such that the closure of $\mathcal{L}$ yields a Feller process; see Chapter 3 in [96] for a definition and a general introduction to Feller processes. It is a direct consequence of Theorem 4.3 and Theorem 4.68 in [96].

Theorem 3.1. *Suppose that the function p in* (3.3) *satisfies*

$$\sup_{x\in V} \sum_{y\in V\colon y\neq x} [p(x,y)+p(y,x)] < \infty \ . \tag{3.4}$$

Moreover, assume that c in (3.3) *is uniformly bounded, continuous in the second component, and satisfies*

$$\sup_{x\in V} \sum_{y\in V\colon y\neq x} \sup_{\eta\in S} |c(x,\eta^y)-c(x,\eta)| < \infty \ . \tag{3.5}$$

Then the closure of the operator $\mathcal{L}$ is the generator of a Feller process.

When Theorem 3.1 holds for some operator $\mathcal{L}$, we call the associated Feller process $(\eta_t)_t \geq 0$ the **exclusion process** on G with **transition rates** p and c and **generator** $\mathcal{L}$. We say that the transition rates are **nearest neighbor** if

$$\begin{aligned} p(x,y) > 0 \quad &\Rightarrow \quad \{x,y\} \in E \\ c(x,\eta) \neq c(x,\eta^y) \quad &\Rightarrow \quad \{x,y\} \in E \ \text{ or } \ x = y \end{aligned}$$

holds for all $x, y \in V$. If the transition rates are nearest neighbor and $c \equiv 0$, we call $(\eta_t)_{t\geq 0}$ a **simple exclusion process** on G. For nearest neighbor transition rates, Theorem 3.1 implies that for any graph G of uniformly bounded degree, i.e., when G has a finite maximum degree, the operator $\mathcal{L}$ always gives raise to a Feller process on Ω, provided that c and p are uniformly bounded. However, when the underlying graph is locally finite, but does not have a uniformly bounded degree, the assumption (3.4) in Theorem 3.1 may fail even for uniformly bounded rates p and c.

The next theorem generalizes Theorem 3.1 for the simple exclusion process on a larger class of graphs. We remark that results of this type are well-known in the interacting particle system community. However, we will give a formal statement and proof in the following as we could not find an appropriate reference. Recall that $G = (V,E)$ is a locally finite, connected graph, and let $o \in V$ be some distinguished, but fixed site of G, called the **root**. Let $p_G \in [0,1]$ denote the critical value for bond percolation on G, i.e., we set

$$p_G := \sup \{p \geq 0\colon P_p(o \text{ is contained in an infinite open cluster}) = 0\} \ ,$$

where P_p denotes the law of Bernoulli bond percolation on G with parameter p.

Theorem 3.2. *Let G be a graph such that $p_G > 0$ holds. Let $(p(x,y))_{x,y\in V}$ be a family of nearest neighbor transition rates which are uniformly bounded from above. Then the associated simple exclusion process on G with transition rates $(p(x,y))_{x,y\in V}$ is a Feller process with respect to the operator given in* (3.3).

The proof of Theorem 3.2 is deferred to Section 3.5. We will see an application of Theorem 3.2 in Sections 9 and 11, where we study the simple exclusion process on Galton–Watson trees. In the following, we will only consider graphs and transition rates which satisfy either the assumptions of Theorem 3.1 or of Theorem 3.2. Hence, the associated exclusion process will indeed be a Feller process.

Remark 3.3. *Theorem 3.2 can be extended to show that interacting particle systems with only nearest neighbor interactions and uniformly bounded transition rates on locally finite graphs with strictly positive critical value for bond percolation are well-defined Feller processes.*

3.2 Invariance and ergodicity

In order to understand the limit behavior of an exclusion process, we are in particular interested in its invariant distributions. We say that a probability measure μ on Ω is **invariant** for the exclusion process $(\eta_t)_{t\geq 0}$ if the process, when initialized with law μ, will have law μ at any fixed time. Moreover, we call an exclusion process $(\eta_t)_{t\geq 0}$ **stationary** or **in equilibrium** if $(\eta_t)_{t\geq 0}$ is started from an invariant measure. An equivalent way of defining invariant measures for Feller processes is the following criterion which involves the generator. It is an immediate consequence of Theorem 3.37 in [96].

Theorem 3.4. *Let $(\eta_t)_{t\geq 0}$ be an exclusion process with generator $\mathcal{L}$. A probability measure μ is invariant for $(\eta_t)_{t\geq 0}$ if and only if*

$$\int \mathcal{L} f \, \mathrm{d}\mu = 0 \tag{3.6}$$

for all cylinder functions f.

For a given exclusion process, we denote by $\mathcal{I}$ the set of all of its invariant measures. Note that $\mathcal{I}$ is convex and denote by $\mathcal{I}_{\mathrm{e}}$ its extreme points. We say that a probability measure μ on Ω is **extremal invariant** for the exclusion process $(\eta_t)_{t\geq 0}$ if $\mu \in \mathcal{I}_{\mathrm{e}}$. Extremal invariant measures play an important role in order to determine the long-term behavior of the exclusion process. This can be see in the next theorem, which follows from Theorem B.50 and Theorem B.52 in [94].

Theorem 3.5. *Let $(\eta_t)_{t\geq 0}$ be a stationary exclusion process with respect to some invariant measure $\mu \in \mathcal{I}$. Then the following two statements are equivalent:*

- *Let A be an event for the exclusion process $(\eta_t)_{t\geq 0}$, which is time shift-invariant, i.e.,*

$$(\eta_t)_{t\geq 0} \in A \quad \Rightarrow \quad (\eta_t)_{t\geq T} \in A \tag{3.7}$$

for all $T \geq 0$. Then either A or A^{c} must hold almost surely.

- *It holds that $\mu \in \mathcal{I}_{\mathrm{e}}$.*

Moreover, if one of the above equivalent statements holds, then

$$\lim_{t\to\infty} \frac{1}{t} \int f(\eta_s)\mathrm{d}s = \int f(\eta)\mathrm{d}\mu \tag{3.8}$$

almost surely for all bounded, measurable functions $f\colon \Omega \to \mathbb{R}$.

In general, when a stationary process satisfies the first statement of Theorem 3.5, we call the process **ergodic**. The consequence (3.8) for stationary and ergodic processes is known as Birkhoff's ergodic theorem.

3.2.1 Construction of invariant measures

In the following, we will construct a family of invariant measures for the simple exclusion process on a graph $G = (V, E)$ under certain assumptions on the transition rates. To do so, we require a bit of setup. For a given function $a\colon V \to [0, 1]$, we denote by ν_a the Bernoulli product measure on Ω with marginals given by a, i.e.,

$$\nu_a(\eta(x) = 1) = 1 - \nu_a(\eta(x) = 0) = a(x) \tag{3.9}$$

holds for all $x \in V$. We will write ν_ρ for $\rho \in [0, 1]$ whenever $a \equiv \rho$. Note that the measures ν_1 and ν_0 are always invariant measures for the simple exclusion process for any choice of transition rates $(p(x, y))_{x,y\in V}$. We will now give a sufficient condition such that ν_ρ is invariant for all $\rho \in [0, 1]$. We say that the transition rates $(p(x, y))_{x,y\in V}$ of a simple exclusion process satisfy a **flow rule** if

$$\sum_{v\in V} p(x, v) = \sum_{w\in V} p(w, x) \tag{3.10}$$

holds for all $x \in V$. The following theorem is an immediate consequence of the proof of Theorem 2.1 in [95, Chapter VIII].

Theorem 3.6. *Let $(\eta_t)_{t\geq 0}$ be a simple exclusion process on Ω with transition rates $(p(x, y))_{x,y\in V}$. If the transition rates satisfy a flow rule, then $\nu_\rho \in \mathcal{I}$ for all $\rho \in [0, 1]$.*

We obtain a one-parameter family of invariant measures for the simple exclusion process in this way. In the particular case where the underlying graph and the transition rates are translation invariant under spatial shifts, one can show that the measures ν_ρ are actually extremal invariant for all $\rho \in [0, 1]$; see Theorem 1.17 of [94, Part III].

3.2.2 Construction of reversible measures

In the following, we will give another way of constructing invariant measures for the simple exclusion process. The resulting measures will satisfy the stronger conditions of reversibility. We say that a probability measure μ is **reversible** for an exclusion process $(\eta_t)_{t\geq 0}$ if its generator $\mathcal{L}$ satisfies

$$\int f\,(\mathcal{L}g)\,\mathrm{d}\mu = \int g\,(\mathcal{L}f)\,\mathrm{d}\mu \tag{3.11}$$

for all cylinder functions $f, g\colon V \to \mathbb{R}$. Note that every reversible measure μ for an exclusion process $(\eta_t)_{t\geq 0}$ satisfies $\mu \in \mathcal{I}$. This can be seen by taking $g \equiv 1$ in (3.11) and using Theorem 3.4. The following theorem provides sufficient conditions for the existence of reversible measures for the simple exclusion process. It is an immediate consequence of the proof of Theorem 2.1 in [95, Chapter VIII].

Theorem 3.7. *Let $(\eta_t)_{t\geq 0}$ denote the simple exclusion process on Ω with transition rates $(p(x,y))_{x,y\in V}$. Assume that there exists a measure π on V such that $(p(x,y))_{x,y\in V}$ satisfies*

$$\pi(x)p(x,y) = \pi(y)p(y,x) \tag{3.12}$$

for all $x, y \in V$. Then ν_a with $a(x) = \frac{\pi(x)}{1+\pi(x)}$ for all $x \in V$ is reversible for $(\eta_t)_{t\geq 0}$.

Similar to Theorem 3.6, note that we obtain a one-parameter family of reversible measures. While both Theorem 3.6 and Theorem 3.7 provide a way of constructing invariant measures for the simple exclusion process, it is in general a difficult task to determine for a simple exclusion process the set of all invariant measures $\mathcal{I}$. In Section 3.4, we will address this question for the simple exclusion process on the integers.

3.3 Graphical representation and canonical coupling

Next, we introduce the graphical representation of the simple exclusion process, which allows us to study the evolution of the process for different initial configurations simultaneously. The **graphical representation** has the following description:

Consider a simple exclusion process $(\eta_t)_{t\geq 0}$ and transition rates $(p(x,y))_{x,y\in V}$. For every pair $(x,y) \in V \times V$ with $p(x,y) > 0$, we assign a sequence of independent Exponential-$p(x,y)$-distributed random variables, which we call **(Poisson) clocks**. Each time a clock associated to a transition rate $p(x,y)$ rings at time t, we check whether the site x is occupied and the site y is empty, i.e.,

$$\eta_t(x) = 1 - \eta_t(y) = 1$$

holds. If this is the case, then move the particle from x to y. Otherwise, the current configuration remains unchanged. Note that there are at most countably many clocks, and hence, there are almost surely no two clocks ringing at the same time.

Consider a pair of simple exclusion processes $(\eta_t)_{t\geq 0}$ and $(\tilde{\eta}_t)_{t\geq 0}$ with the same underlying graph and the same transition rates, but which may have different initial distributions. We say that $(\eta_t)_{t\geq 0}$ and $(\tilde{\eta}_t)_{t\geq 0}$ are within the **canonical coupling**, also called **basic coupling**, if we use the same clocks to determine the evolution of both processes within the graphical representation. Note that the canonical coupling naturally gives raise to a **grand coupling**, in which the evolution of the simple exclusion process can be determined for any possible starting configuration using only the clocks of the graphical representation.

The canonical coupling has the property that it preserves the component-wise order for the simple exclusion process. More precisely, we say for $\eta, \zeta \in \{0,1\}^V$ that η **dominates** ζ with respect to the **component-wise order**, and we write $\eta \succeq_{\mathrm{c}} \zeta$, if

$$\eta(x) \geq \zeta(x) \tag{3.13}$$

for all $x \in V$. The following proposition is an immediate consequence of the construction of the canonical coupling, so we omit the proof.

Proposition 3.8. *Consider two simple exclusion processes $(\eta_t)_{t\geq 0}$ and $(\tilde{\eta}_t)_{t\geq 0}$ within the canonical coupling, and denote by $\mathbf{P}$ their joint law. Then*

$$\mathbf{P}\left(\eta_t \succeq_{\mathrm{c}} \tilde{\eta}_t \text{ for all } t \geq 0 \mid \eta_0 \succeq_{\mathrm{c}} \tilde{\eta}_0\right) = 1\,. \tag{3.14}$$

We will discuss various extensions of the canonical coupling in the course of this thesis; see Section 6.3.1 for the simple exclusion process with open boundaries and Section 10.2 for the TASEP on rooted trees with a particle source at the root. Moreover, we will see that the canonical coupling will preserve a different partial order for the simple exclusion process on the segment, introduced by its height function representation; see Section 3.4.2 and Section 3.4.3 as well as Section 5.2 and Section 6.3.1 for extensions to random environments and open boundaries, respectively.

3.4 The simple exclusion process on the integers

We now study the simple exclusion process (SEP) in the special case where the underlying graph G is a connected subgraph of the one-dimensional integer lattice, i.e., the vertex set is either the full line $\mathbb{Z}$, the half-line $\mathbb{N}$ or a segment $[N] := \{1, \dots, N\}$ for some $N \in \mathbb{N}$, and we place an edge between two sites $x, y \in [N]$ if and only if the Euclidean distance between x and y is equal to 1.

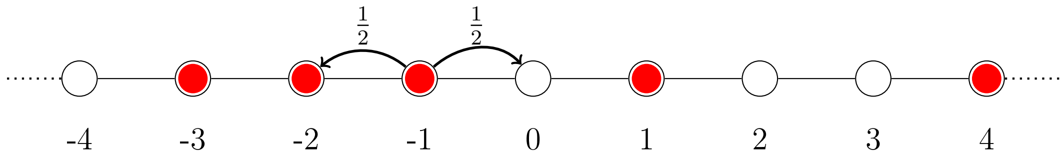

Figure 1: Example of a configuration of the symmetric simple exclusion process on the integers. Particles are depicted as red dots. Note that the particle at position -1 may move to position 0, while the position -2 is blocked.

3.4.1 The symmetric simple exclusion process on the integers

We start by considering the simple exclusion process on $\mathbb{Z}$, where we choose all transition rates to be equal to $\frac{1}{2}$, i.e., we set

$$p(x,y) = \begin{cases} \frac{1}{2} & \text{if } y \in \{x-1, x+1\} \\ 0 & \text{otherwise;} \end{cases} \tag{3.15}$$

see Figure 1. The associated Feller process is called the **symmetric simple exclusion process (SSEP)** on the integers generated by the closure of

$$\mathcal{L}f(\eta) = \sum_{\substack{x \in \mathbb{Z}\colon \eta(x)=1 \\ \eta(x+1)=0}} \frac{1}{2}\left[f(\eta^{x,x+1}) - f(\eta)\right] + \sum_{\substack{x \in \mathbb{Z}\colon \eta(x)=1 \\ \eta(x-1)=0}} \frac{1}{2}\left[f(\eta^{x,x-1}) - f(\eta)\right] . \tag{3.16}$$

Theorem 3.1 ensures that the symmetric simple exclusion process on $\mathbb{Z}$ is indeed a Feller process. Since all particles are indistinguishable in the symmetric simple exclusion process on $\mathbb{Z}$, one can instead of $\mathcal{L}$ also consider the operator

$$\tilde{\mathcal{L}}f(\eta) = \sum_{x \in \mathbb{Z}} \left[f(\eta^{x,x+1}) - f(\eta)\right] \tag{3.17}$$

for all cylinder functions f. We will revisit this construction in Section 4.5 when introducing the interchange process. By Theorem 3.7, the Bernoulli-ρ-product measures ν_ρ with $\rho \in [0,1]$ are reversible for any simple exclusion process with **symmetric rates**, i.e., where $p(x,y) = p(y,x)$ holds for all x, y. In fact, when the rates satisfy in addition a flow rule, these measures are also extremal invariant. Moreover, we can give a complete characterization of the set of extremal invariant measures when the underlying graph is the integer lattice; see Theorem 1.10 and Theorem 1.16 in [94, Part III].

Theorem 3.9. *For a simple exclusion process with symmetric rates which satisfy a flow rule, the set of extremal invariant measures satisfies*

$$\{\nu_\rho \text{ for } \rho \in [0,1]\} \subseteq \mathcal{I}_{\mathrm{e}} \,. \tag{3.18}$$

Moreover, the symmetric simple exclusion process on $\mathbb{Z}$ satisfies (3.18) *with equality.*

3.4.2 The asymmetric simple exclusion process on the integers

Next, we study the **asymmetric simple exclusion process (ASEP)** on the integers, which we obtain by taking the transition probabilities $(p(x,y))_{x,y\in\mathbb{Z}}$ equal to

$$p(x,y) = \begin{cases} p & \text{if } y = x+1 \\ (1-p) & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases} \tag{3.19}$$

for some $p \in [0,1]$ with $p \neq \frac{1}{2}$. Intuitively, all particles perform biased simple random walks under the exclusion rule. Note that the asymmetric simple exclusion process is a Feller process generated by the closure of

$$\mathcal{L}f(\eta) = \sum_{\substack{x\in\mathbb{Z}\colon \eta(x)=1 \\ \eta(x+1)=0}} p\left[f(\eta^{x,x+1}) - f(\eta)\right] + \sum_{\substack{x\in\mathbb{Z}\colon \eta(x)=1 \\ \eta(x-1)=0}} (1-p)\left[f(\eta^{x,x-1}) - f(\eta)\right] \tag{3.20}$$

for all cylinder functions f; see Theorem 3.1. Since the rates satisfy a flow rule, we see by Theorem 3.4 that the measures ν_ρ are invariant for the asymmetric simple exclusion process for all $\rho \in [0,1]$. Furthermore, by Theorem 3.7, we note that the product measures ν_a with

$$a(x) = \frac{cp^x}{(1-p)^x + cp^x} \tag{3.21}$$

for all $x \in V$ are reversible for the ASEP for all $c > 0$ when $p \notin \{0,1\}$. Observe that under this measure ν_a

$$\nu_a\left(\{\eta\colon \exists\, C_\eta > 0 \text{ s.t. } \eta(x) = 1\ \forall x > C_\eta \text{ and } \eta(x) = 0\ \forall x < -C_\eta\}\right) = 1$$

holds when $p > \frac{1}{2}$ and

$$\nu_a\left(\{\eta\colon \exists\, C_\eta > 0 \text{ s.t. } \eta(x) = 0\ \forall x > C_\eta \text{ and } \eta(x) = 1\ \forall x < -C_\eta\}\right) = 1$$

when $p < \frac{1}{2}$. In the following, we will assume without loss of generality that $p > \frac{1}{2}$ as all arguments apply similarly for $p < \frac{1}{2}$ by reflecting the underlying graph along a vertical axis. For all $n \in \mathbb{N}$, we denote by A_n the set of configurations

$$A_n := \left\{\eta \in \{0,1\}^{\mathbb{Z}}\colon \sum_{x>n}(1-\eta(x)) = \sum_{x\le n}\eta(x) < \infty\right\}. \tag{3.22}$$

Note that $\nu_a(A_n) > 0$ for all $p < 1$, and define the **blocking measure** $\nu_{(n)}$ on A_n to be $\nu_{(n)}(\cdot) := \nu_a(\,\cdot\,|A_n)$ for all $n \in \mathbb{N}$. When $p = 1$, we let $\nu_{(n)}$ be the Dirac measure on the configuration $\vartheta_n \in A_n$ given by

$$\vartheta_n(x) = \mathbb{1}_{x>n} \tag{3.23}$$

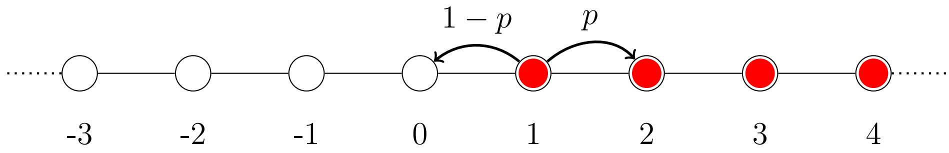

Figure 2: Visualization of the ground state ϑ_0 for the asymmetric simple exclusion process with $p > \frac{1}{2}$ and restricted to the set of configurations A_0.

for all $x \in \mathbb{Z}$. We call ϑ_n the **ground state** on A_n; see Figure 2. Note that A_n consists of countable many configurations and that the asymmetric simple exclusion process restricted to A_n is a positive recurrent continuous-time Markov chain for all $n \in \mathbb{N}$. Its unique reversible measure is the blocking measure $\nu_{(n)}$ on A_n when $p \in (0,1)$.

The following theorem, which can be found as Theorem 1.4 in [93], guarantees that all invariant measures for the asymmetric simple exclusion process must be convex combinations of Bernoulli product measures with constant parameters and blocking measures.

Theorem 3.10. *For the asymmetric simple exclusion process with* $p \in (\frac{1}{2}, 1]$

$$\mathcal{I}_{\mathrm{e}} = \{\nu_\rho \textit{ for some } \rho \in [0,1]\} \cup \{\nu_{(n)} \textit{ for some } n \in \mathbb{Z}\} . \tag{3.24}$$

For the simple exclusion process restricted to the sets A_n for some $n \in \mathbb{Z}$, we can define a partial order $\succeq_{\mathrm{h}}$ as follows. For two configurations $\eta \in A_n$ and $\zeta \in A_m$ with $n, m \in \mathbb{Z}$

$$\eta \succeq_{\mathrm{h}} \zeta \quad \Leftrightarrow \quad \sum_{i=-\infty}^{j} \eta(i) \geq \sum_{i=-\infty}^{j} \zeta(i) \quad \text{for all } j \in \mathbb{Z} . \tag{3.25}$$

Intuitively, $\eta \succeq_{\mathrm{h}} \zeta$ holds whenever η has for any given reference point on $\mathbb{Z}$ at least as many particles to the left-hand side as ζ. Note that on the set of configurations A_n, the ground state ϑ_n is the unique minimal element with respect to the partial order $\succeq_{\mathrm{h}}$. Moreover, we observe that the canonical coupling preserves the partial order $\succeq_{\mathrm{h}}$ on A_n for all $n \in \mathbb{Z}$. This is immediate from its construction, and summarized in the following proposition, where we recall Proposition 3.8 for a similar statement with respect to the partial order $\succeq_{\mathrm{c}}$.

Proposition 3.11. *Consider two simple exclusion processes* $(\eta_t)_{t \geq 0}$ *and* $(\tilde{\eta}_t)_{t \geq 0}$ *on* A_n *for some* $n \in \mathbb{Z}$ *within the canonical coupling and denote by* $\mathbf{P}$ *their joint law. Then*

$$\mathbf{P}\left(\eta_t \succeq_{\mathrm{h}} \tilde{\eta}_t \textit{ for all } t \geq 0 \mid \eta_0 \succeq_{\mathrm{h}} \tilde{\eta}_0\right) = 1 \tag{3.26}$$

holds.

3.4.3 The simple exclusion process on the segment

We now consider the simple exclusion process on a finite segment $[N]$ for some $N \in \mathbb{N}$ and initially k particles for some $k \in [N-1]$. Note that the number of particles k is preserved in the exclusion process. Hence, we can restrict the dynamics to be defined as a continuous-time Markov chain with state space $\Omega_{N,k}$ given by

$$\Omega_{N,k} := \left\{ \eta \in \{0,1\}^N : \sum_{i=1}^{N} \eta(i) = k \right\} . \tag{3.27}$$

When $p(x, x+1) > 0$ holds for all $x \in [N-1]$, the simple exclusion process on $\Omega_{N,k}$ has a unique stationary distribution μ. If in addition $p(x+1, x) > 0$ holds for all $x \in [N-1]$, the simple exclusion process is reversible with respect to μ, and

$$\mu(\eta) = \frac{1}{Z} \prod_{j=1}^{N-1} \left(\frac{p(j, j+1)}{p(j+1, j)} \right)^{\sum_{i=j+1}^{N} \eta(i)} \tag{3.28}$$

holds for all $\eta \in \Omega_{N,k}$ with respect to some normalization constant Z.

Similar to the asymmetric simple exclusion process restricted to the set of configurations A_n, we define the partial order $\succeq_{\mathrm{h}}$ on the state space $\Omega_{N,k}$. More precisely, for a given $N \in \mathbb{N}$ and $k \in [N-1]$

$$\eta \succeq_{\mathrm{h}} \zeta \quad \Leftrightarrow \quad \sum_{i=1}^{j} \eta(i) \geq \sum_{i=1}^{j} \zeta(i) \quad \text{for all } j \in [N] \tag{3.29}$$

for configurations $\eta, \zeta \in \Omega_{N,k}$. In words, $\eta \succeq_{\mathrm{h}} \zeta$ holds if and only if the i^{th} particle in η is further to left than the i^{th} particle in ζ for all $i \in [k]$. As an analogue of the ground state, we get unique minimal and maximal elements $\vartheta_{N,k}$ and $\theta_{N,k}$ on $\Omega_{N,k}$ with

$$\vartheta_{N,k}(i) := \mathbb{1}_{i > N-k} , \tag{3.30}$$

where all particles are on the right-hand side and

$$\theta_{N,k}(i) := \mathbb{1}_{i \leq k} , \tag{3.31}$$

where all particles are on the left-hand side, and $i \in [N]$. Again, it is immediate that the canonical coupling for the simple exclusion process preserves the partial order $\succeq_{\mathrm{h}}$.

Proposition 3.12. *Consider two simple exclusion processes $(\eta_t)_{t \geq 0}$ and $(\tilde{\eta}_t)_{t \geq 0}$ on $\Omega_{N,k}$ for some $n \in \mathbb{N}$ and $k \in [N-1]$ within the canonical coupling, and denote by* $\mathbf{P}$ *their joint law. Then*

$$\mathbf{P} \left(\eta_t \succeq_{\mathrm{h}} \tilde{\eta}_t \text{ for all } t \geq 0 \mid \eta_0 \succeq_{\mathrm{h}} \tilde{\eta}_0 \right) = 1 . \tag{3.32}$$

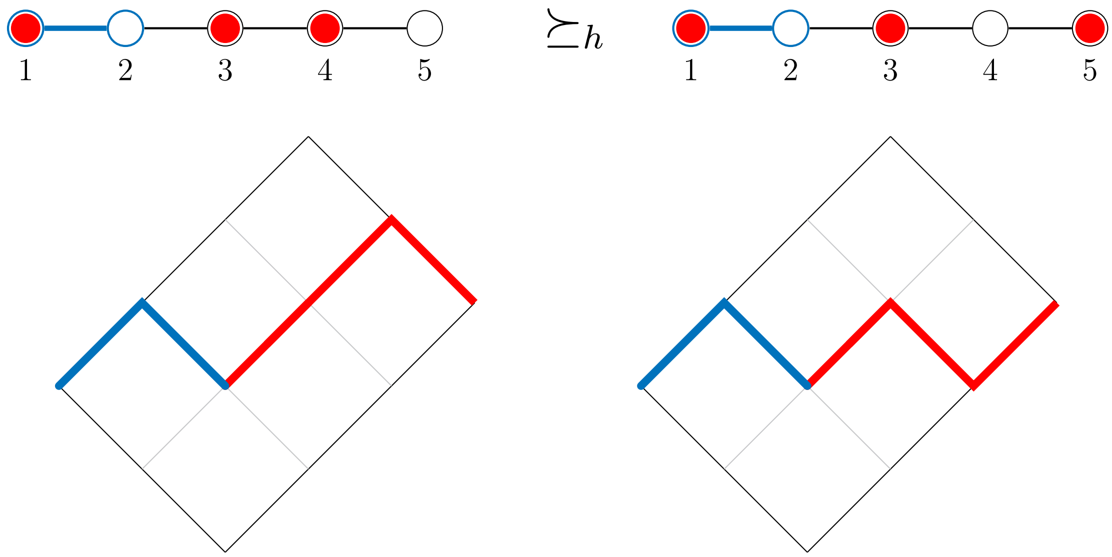

Figure 3: Visualization of two ordered configurations of the simple exclusion process on a segment of size $N = 5$ with $k = 3$ particles, and their corresponding paths which we obtain as linear interpolations of their height functions. Note that two adjacent sites with different spins, as for example in the blue edge, corresponds to a local extremum in the path representation.

The partial order $\succeq_{\mathrm{h}}$ on $\Omega_{N,k}$ has also a graphical interpretation. For $\eta \in \Omega_{N,k}$, let $h\colon \{0, \ldots, N\} \mapsto \mathbb{R}$ be the **height function** of η, given by

$$h_\eta(x) := \sum_{z=1}^{x} \left(\eta(z) - \frac{k}{N} \right) \tag{3.33}$$

for all $x \in [N]$. Note that $\eta \succeq_{\mathrm{h}} \zeta$ if and only if $h_\eta(x) \geq h_\zeta(x)$ for all $x \in [N]$. A visualization of the height function in terms of paths in a rotated coordinate system is shown in Figure 3. Note that configurations of the simple exclusion process and height functions are in a one-to-one correspondence, and we will exploit this connection in Section 6.4 in more detail, where we discuss approximated eigenfunctions for the symmetric simple exclusion process with open boundaries.

We remark that one can define a similar height function representation for the simple exclusion process on $\mathbb{Z}$. When particles can only move in one direction and the rates are homogeneous, the path representation of the height function yields a natural interpretation of the simple exclusion process on the integers as an i.i.d. exponential corner growth model on $\mathbb{Z}^2$. This description is a key to many exact expression for various observables of the exclusion process; see [34, 126, 128] for an introduction and overview. In Section 11.3.2, we will make use of this corner growth representation when studying last passage times for a slowed down exclusion process on trees.

3.5 Feller property for locally finite graphs

In this section, we present the proof of Theorem 3.2, which states a criterion for the existence of Feller processes on locally finite graphs. Again, for graphs with uniformly bounded degree, we refer to Theorem 3.1 due to Liggett; see also [96] for a more comprehensive discussion.

Proof of Theorem 3.2. We start by defining the simple exclusion process as a Markov process $(\eta_t)_{t\geq 0}$ on the state space Ω. Recall the graphical representation of the exclusion process from Section 3.3. To every (directed) edge $(x,y) \in E$ of the underlying graph G, we assign rate $p(x,y)$ Poisson clocks. Whenever a clock rings and the site x is occupied, we move the particle from x to y, provided that y is empty; otherwise nothing happens. Set

$$C := \max_{x,y\in V} p(x,y)$$

and let $\tau = \frac{p_G}{3C}$. Note that $P(X \leq \tau) < p_G$ holds for X being Exponential-$(2C)$-distributed. Observe that the value of $\eta_t(x)$ for $x \in V$ and $t \in [0,\tau]$ does only depend on the sites which are in the percolation cluster containing x spanned by the edges on which at least one clock rings until time τ. By our choice of τ, each such cluster is almost surely finite. Hence, the number of transitions in the evolution of $(\eta_t(x))_{t\in[0,\tau]}$ is almost surely finite for all $x \in V$, and therefore, the evolution of $(\eta_t(x))_{t\in[0,\tau]}$ is well-defined. For $t \geq \tau$, observe that the graphical representation is Markovian and iterate the above argument to conclude.

In order to show that $(\eta_t)_{t\geq 0}$ is a Feller process, it remains to verify the **Feller property**, i.e., to show that for any continuous function $f\colon \{0,1\}^V \to \mathbb{R}$, the mapping

$$\zeta \mapsto \mathbb{E}_\zeta[f(\eta_t)] \tag{3.34}$$

is continuous in the starting configuration ζ for all $t \geq 0$. Using the Markov property of $(\eta_t)_{t\geq 0}$, it suffices to check this for $t \in [0,\tau]$. Moreover, since the state space $\{0,1\}^V$ is equipped with the product topology, and is hence compact, we can assume that f is a bounded function. We denote its ℓ_∞-norm by $\|f\|_\infty$. Let $\varepsilon > 0$ be fixed. For a given bounded continuous function f, we can choose $r = r(G,\varepsilon)$ such that

$$\sup\left\{|f(\eta) - f(\zeta)| : \eta|_{B_r(G,o)} = \zeta|_{B_r(G,o)}\right\} < \frac{\varepsilon}{2} \tag{3.35}$$

holds, where $B_r(G,o)$ denotes the ball of radius r around the root o of G and $\eta|_{B_r(G,o)}$ is the configuration η restricted to $B_r(G,o)$. We claim that there exists a constant $r' = r'(r,\tau)$ such that

$$\mathbb{P}\left(\eta_t|_{B_r(G,o)} \neq \zeta_t|_{B_r(G,o)} \text{ for some } t \in [0,\tau] \;\middle|\; \eta_0|_{B_{r'}(G,o)} = \zeta_0|_{B_{r'}(G,o)}\right) < \frac{\varepsilon}{2\|f\|_\infty} \tag{3.36}$$

where $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ perform simple exclusion processes according to the graphical representation using the same clocks. Note that the event in (3.36) can only occur if there exists a path of edges, which are updated until time τ, connecting the boundaries of the balls $B_r(G,o)$ and $B_{r'}(G,o)$. Since we are in the subcritical regime of percolation on G, the probability of this event goes to 0 for r' tending to infinity. Thus, for $\eta, \zeta \in \{0,1\}^V$ with $\eta|_{B_{r'}(G,o)} = \zeta|_{B_{r'}(G,o)}$, we conclude that

$$|\mathbb{E}_\eta[f(\eta_t)] - \mathbb{E}_\zeta[f(\eta_t)]| < \varepsilon$$

holds for all $t \in [0,\tau]$ by combining (3.35) and (3.36). Hence, the simple exclusion process on G is a Feller process. The fact that the generator of the simple exclusion process has indeed the form stated in (3.3) follows from a comparison with the graphical representation. □

Part II

Mixing times for exclusion processes

4 Preliminaries on mixing times

In this part of the thesis, we focus on exclusion processes on finite graphs. We will assume that the exclusion process has a unique stationary distribution that we approach from any starting configuration. Our goal is to quantify the speed of convergence to the equilibrium when the size of the underlying state space grows towards infinity. We achieve this using the notion of total-variation mixing times. In the following, we give a brief introduction to mixing times and related quantities. Most of the presented material can be found in [91] in the context of discrete-time Markov chains. We continue with an overview of recent related work on mixing times for exclusion processes. In Sections 4.3 to 4.5, we discuss three techniques, which play a crucial role in achieving precise upper and lower bounds on the mixing time of exclusion processes. We study Wilson's lower bound technique [139], the censoring inequality introduced by Peres and Winkler [110], and second class particle arguments, which were first used by Benjamini et al. in [19] in the context of mixing times. We present generalizations of all three mentioned concepts, which we will use throughout this thesis.

4.1 Definition and the cutoff phenomenon

We start by introducing the notion of mixing times for general continuous-time Markov chains. Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on a finite state space $\mathcal{S}$ with generator $\mathcal{A}$. We assume that $(X_t)_{t\geq 0}$ has a unique stationary distribution π to which $(X_t)_{t\geq 0}$ converges for any initial state $s \in \mathcal{S}$. For two probability measures $\nu, \tilde{\nu}$ on $\mathcal{S}$, we define their **total-variation distance** as

$$\|\nu - \tilde{\nu}\|_{\mathrm{TV}} := \max_{A\subseteq \mathcal{S}} \nu(A) - \tilde{\nu}(A) = \frac{1}{2}\sum_{x\in\mathcal{S}} |\nu(x) - \tilde{\nu}(x)| . \tag{4.1}$$

We denote by $\mathbb{P}_\nu(X_t \in \cdot\,) = \mathbb{P}(X_t \in \cdot\, | X_0 \sim \nu)$ the law of the process $(X_t)_{t\geq 0}$ at time $t \geq 0$, where $X_0 \sim \nu$, and write $\mathbb{P}_x(X_t \in \cdot)$ when ν is the Dirac measure δ_x on $x \in \mathcal{S}$. We define

$$d(t) := \sup_{x\in\mathcal{S}} \|\mathbb{P}_x(X_t \in \cdot\,) - \pi\|_{\mathrm{TV}} \tag{4.2}$$

for all times $t \geq 0$ to be the **maximal distance** from equilibrium at time t. Note that $d(t)$ takes only values in $[0, 1]$ and is a monotone decreasing function in t; see for

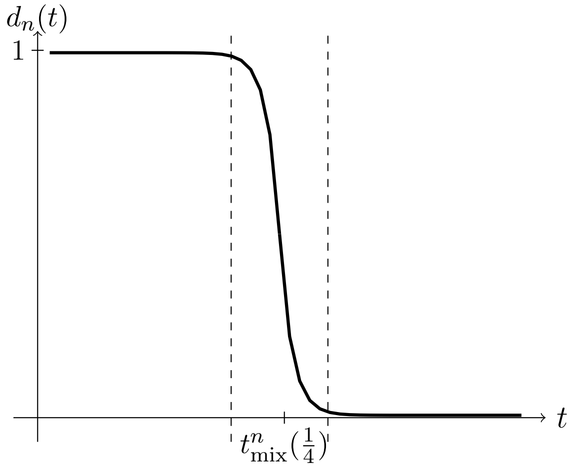

Figure 4: Visualization of the cutoff phenomenon for a family of Markov chains (X^n_t). The maximal distance $d_n(\cdot)$ exhibits a sharp transition from (close to) 1 to (close to) 0 at the order of the mixing time when n is large.

example Exercise 4.2 in [91]. The total-variation **ε-mixing time** $t_{\text{mix}}(\varepsilon)$ of $(X_t)_{t\geq 0}$ is

$$t_{\text{mix}}(\varepsilon) := \inf\{t \geq 0 : d(t) \leq \varepsilon\} \tag{4.3}$$

for all $\varepsilon \in (0,1)$. Note that the ergodic theorem for Markov chains ensures that $t_{\text{mix}}(\varepsilon)$ is almost surely finite under the above assumptions. For $\varepsilon = \frac{1}{4}$, we call $t_{\text{mix}} = t_{\text{mix}}(\frac{1}{4})$ simply the **mixing time** of the Markov chain $(X_t)_{t\geq 0}$.

In the following, our goal is to provide sharp upper and lower bounds on the mixing times of a sequence of Markov chains. In particular, we are interested in the dependence on the parameter ε. Consider a family of Markov chains $((X^n_t)_{t\geq 0})_{n\in\mathbb{N}}$ with ε-mixing times $t^n_{\text{mix}}(\varepsilon)$. We say that this family exhibits **pre-cutoff** if there exist constants $c_1, c_2 > 0$ such that for any $\varepsilon \in (0,1)$

$$c_1 \leq \liminf_{n\to\infty} \frac{t^n_{\text{mix}}(1-\varepsilon)}{t^n_{\text{mix}}(\varepsilon)} \leq \limsup_{n\to\infty} \frac{t^n_{\text{mix}}(1-\varepsilon)}{t^n_{\text{mix}}(\varepsilon)} \leq c_2 \tag{4.4}$$

holds. Moreover, we say that the **cutoff phenomenon** occurs, if we can choose $c_1 = c_2 = 1$ in (4.4). In words, a sequence of Markov chains exhibits pre-cutoff if the ε-mixing time is located in a fixed time-window of the order of the mixing time for any $\varepsilon \in (0,1)$ when n goes to infinity. If the size of this window is of strictly smaller order than the mixing time, we have cutoff; see Figure 4. In general, it is a challenging task to determine whether cutoff occurs. For a more comprehensive discussion of cutoff, we refer to Chapter 18 of [91]. We will now discuss three quantities, the coupling, hitting and the relaxation time, which are closely related to mixing times.

4.1.1 Coupling and hitting times

Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two continuous-time Markov chains on some common state space $\mathcal{S}$, which have the same generator $\mathcal{A}$. For a coupling $(X_t, Y_t)_{t\geq 0}$ with initial configurations $X_0 = \eta$ and $Y_0 = \zeta$, we define the **coupling time** with respect to $\eta, \zeta \in \mathcal{S}$ to be

$$\tau_{\text{couple}}(\eta, \zeta) := \inf\{t \geq 0\colon X_t = Y_t\} \ . \tag{4.5}$$

The following upper bound on the mixing time is well-known; see Corollary 5.5 in [91].

Proposition 4.1. *Let the Markov chains associated with the generator $\mathcal{A}$ on the state space $\mathcal{S}$ be irreducible. Moreover, assume that for every pair $\eta, \zeta \in S$, we have a coupling $(X_t, Y_t)_{t\geq 0}$ of Markov chains according to $\mathcal{A}$ with $X_0 = \eta$ and $Y_0 = \zeta$ almost surely. Then*

$$d(t) \leq \max_{\eta,\zeta\in S} \mathbb{P}\left(\tau_{\text{couple}}(\eta, \zeta) > t\right) \tag{4.6}$$

holds for all $t \geq 0$. In particular, $t_{\text{mix}} \leq 4 \max_{\eta,\zeta\in S} \mathbb{E}\left[\tau_{\text{couple}}(\eta, \zeta)\right]$ holds.

Whenever the state space $\mathcal{S}$ is equipped with a partial order which is respected by the coupling, the right-hand side in (4.6) can be simplified. More precisely, assume that $\mathcal{S}$ is equipped with a partial order $\succeq$. A coupling $(X_t, Y_t)_{t\geq 0}$ with law $\mathbf{P}$ is said to be **monotone** with respect to $\succeq$, if

$$\mathbf{P}\left(X_s \succeq Y_s \ \text{ for all } s \geq 0 \mid X_0 \succeq Y_0\right) = 1 \tag{4.7}$$

holds $\mathbf{P}$-almost surely. We say that a grand coupling is **monotone** if (4.7) holds $\mathbf{P}$-almost surely for any pair of initial configurations. Note that the graphical representation for the simple exclusion process on the segment with respect to the partial order $\succeq_{\text{h}}$ yields a monotone grand coupling by Proposition 3.12. For any pair of configurations $\eta, \zeta \in \mathcal{S}$, we let

$$\tau^{\zeta}_{\text{hit}}(\eta) := \inf\{t \geq 0\colon X_t = \eta, X_0 = \zeta\} \tag{4.8}$$

be the **hitting time** of the configuration η starting from ζ. The following result is an immediate consequence of Proposition 4.1.

Corollary 4.2. *Suppose that the state space $\mathcal{S}$ is equipped with a partial order $\succeq$, and that we have a monotone grand coupling with law $\mathbf{P}$. Further, assume that there exist unique minimal and maximal elements $\vee, \wedge \in \mathcal{S}$ such that*

$$\vee \preceq \eta \preceq \wedge \tag{4.9}$$

for all $\eta \in \mathcal{S}$. Then for all $t \geq 0$

$$d(t) \leq \mathbf{P}\left(\tau_{\text{couple}}(\vee, \wedge) > t\right) \leq \min(\mathbf{P}\left(\tau^{\vee}_{\text{hit}}(\wedge) > t\right), \mathbf{P}\left(\tau^{\wedge}_{\text{hit}}(\vee) > t\right)) \, . \tag{4.10}$$

4.1.2 The relaxation time and a necessary criterion for pre-cutoff

In order to get precise bounds on the mixing time, one possible way is to analyze the spectrum of the process. Historically, studying eigenvalues and the corresponding eigenfunctions allowed Diaconis and Shahshahani in [46] to show cutoff for the first time, although the term "cutoff" was only introduced later by Aldous and Diaconis [2]. For a continuous-time Markov chain $(X_t)_{t\geq 0}$ on a a state space $\mathcal{S}$ with generator $\mathcal{A}$, we consider pairs (λ, f) for $\lambda \in \mathbb{C}$ and $f \colon V \to \mathbb{C}$ such that

$$(\mathcal{A}f)(x) = \lambda f(x) \tag{4.11}$$

holds for all $x \in V$. When (4.11) holds, we say that f is an **eigenfunction** of $(X_t)_{t\geq 0}$ with respect to the **eigenvalue** λ. Note that the function $f \equiv 1$ is always an eigenfunction with respect to the eigenvalue $\lambda = 0$.

When $(X_t)_{t\geq 0}$ is reversible, all eigenvalues λ of $\mathcal{A}$ are real-valued and satisfy $\lambda \leq 0$; see Theorem 12.1 in [91]. In this case, our goal is to investigate the **spectral gap** λ^* of the process, which is the absolute value of the second largest eigenvalue of $\mathcal{A}$. In practice, it will often be convenient to consider the **relaxation time** $t_{\text{rel}} := (\lambda^*)^{-1}$. The following proposition show that the relaxation time can be used to provide a lower bound on the mixing time. The proof is similar to the one of Lemma 20.11 in [91].

Proposition 4.3. *For any eigenvalue $\lambda < 0$ of $(X_t)_{t\geq 0}$, it holds that*

$$d(t) \geq \frac{1}{2}\exp(\lambda t)\,. \tag{4.12}$$

In particular

$$t_{\text{mix}}(\varepsilon) \geq \log\left(\frac{1}{2\varepsilon}\right)(t_{\text{rel}} - 1) \tag{4.13}$$

holds for any $\varepsilon \in (0, \frac{1}{2})$.

Moreover, Proposition 4.3 gives raise to a necessary criterion for a family of Markov chains to exhibit pre-cutoff. The proof is similar to the one of its discrete-time analogue Proposition 18.4 in [91].

Proposition 4.4. *Let $(t^n_{\text{mix}}(\varepsilon))_{n\geq 1}$ and $(t^n_{\text{rel}})_{n\geq 1}$ denote the ε-mixing times and relaxation times of a family of reversible, irreducible continuous-time Markov chains $((X^n_t)_{t\geq 0})_{n\geq 1}$. Assume that $t^n_{\text{rel}} \geq 1$ for all $n \geq 1$. If for some $\varepsilon \in (0,1)$*

$$\limsup_{n\to\infty} \frac{t^n_{\text{mix}}(\varepsilon)}{t^n_{\text{rel}}} < \infty\,, \tag{4.14}$$

then $((X^n_t)_{t\geq 0})_{n\geq 1}$ does not exhibit pre-cutoff.

4.2 Mixing times for the simple exclusion process

Equipped with these tools, we now study the mixing time of the simple exclusion process. For the simple exclusion process $(\eta_t)_{t\geq 0}$ on the segment $[N]$, we say that we have a **homogeneous environment** if

$$p(x,y) = \begin{cases} p & \text{if } y = x+1 \\ (1-p) & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

holds for all $x, y \in [N]$ and some $p \in (0,1)$ fixed. In accordance with the simple exclusion process on the integers, we call $(\eta_t)_{t\geq 0}$ the **symmetric simple exclusion process (SSEP)** on the segment when $p = \frac{1}{2}$, and the **asymmetric simple exclusion process (ASEP)** when $p \neq \frac{1}{2}$.

4.2.1 Mixing times for the symmetric simple exclusion process

We start with the mixing time of the symmetric simple exclusion process on the segment $[N]$ for $N \to \infty$. When the number of particles $k = k(N)$ is bounded from above uniformly in N, it is well-known that the resulting simple exclusion process has a mixing time and a relaxation time of order N^2, and hence by Lemma 4.4 no cutoff; see also Chapter 12 in [91]. When the number of particles $k = k(N)$ and the number of empty sites $N - k(N)$ both grow with N, a lower bound of the correct leading order was shown by Wilson for a discrete-time version of the exclusion process [139]. Over 10 years later, Lacoin proved a matching upper bound [84]. In total, we obtain the following result on the mixing time of the symmetric simple exclusion process.

Theorem 4.5 (c.f. Theorem 2.4 in [84])**.** *Let $t^{N,k}_{\text{mix}}(\varepsilon)$ denote the ε-mixing time of the symmetric simple exclusion process on $[N]$ and with $k = k(N)$ particles. Assume that*

$$\lim_{N\to\infty} \min(k(N), N - k(N)) = \infty \tag{4.16}$$

holds. Then for all $\varepsilon \in (0,1)$

$$\lim_{N\to\infty} \frac{t^{N,k}_{\text{mix}}(\varepsilon)}{N^2 \log(\min(k(N), N-k(N)))} = \frac{1}{\pi^2}\,. \tag{4.17}$$

In particular, the symmetric simple exclusion process exhibits cutoff.

The crucial idea for the lower bound in Theorem 4.5 is to gain detailed knowledge about the spectral gap and the corresponding eigenfunction. More precisely, Wilson derives an explicit formula of the eigenvalue corresponding to the spectral gap and the corresponding eigenfunction by interpreting the evolution of the mean of the height function as a solution to a discretized, one-dimensional heat equation. With this explicit eigenvalue at hand, as well as a bound on the fluctuations of the corresponding eigenfunction, Wilson then identifies the times at which the symmetric simple exclusion process started from the ground state has not reached equilibrium with high probability. This idea will be formalized in Section 4.3 where we discuss Wilson's lower bound technique for general Markov chains.

For the upper bound, Lacoin combines various techniques for the simple exclusion process, including correlation properties, scaling limits of the height function representation, as well as the censoring technique, which will be discussed in Section 4.4. We will revisit parts of the proof in Section 6.5.2, where we give precise bounds on the mixing time for the symmetric simple exclusion process with one open boundary.

At this point, we note that similar results were achieved by Lacoin for the symmetric simple exclusion process on the cycle. It turns out that under the above assumptions of Theorem 4.5, the symmetric simple exclusion process on the cycle exhibits cutoff, where the right-hand side in (4.17) is replaced by $1/(4\pi^2)$. Moreover, the limiting profile of the maximal distance from equilibrium can be described; see [83, 85].

4.2.2 Mixing times for the asymmetric simple exclusion process

We now study the asymmetric simple exclusion process on the segment of size $[N]$. We assume that the number of particles $k = k(N)$ satisfies

$$\lim_{N\to\infty} \frac{k(N)}{N} = \rho \tag{4.18}$$

for some $\rho \in [0, 1]$. The following result on the mixing time of the asymmetric simple exclusion process is due to Labbé and Lacoin [81].

Theorem 4.6 (c.f. Theorem 2 in [81])**.** *Fix some $p \in (\frac{1}{2}, 1]$. Let $t^{N,k}_{\text{mix}}$ denote the mixing time of the asymmetric simple exclusion process with parameter p and $k = k(N)$ particles. Moreover, assume that* (4.18) *holds for some $\rho \in [0, 1]$. Then*

$$\lim_{N\to\infty} \frac{t^{N,k}_{\text{mix}}}{N} = \frac{(\sqrt{\rho} + \sqrt{1-\rho})^2}{2p-1} \, . \tag{4.19}$$

In contrast to the results for the symmetric simple exclusion process, the order of the mixing time of the asymmetric simple exclusion process on the segment does not depend on the number of particles k. We note that the first bound of the correct order on the mixing time was given by Benjamini et al. in [19]. In their proof, they reduce the question of estimating mixing times to bounding the hitting time of the ground state for an asymmetric simple exclusion process on $\mathbb{Z}$ with a certain initial configuration. This is formalized in the following lemma.

Theorem 4.7 (c.f. Lemma 2.8 in [19]). *Let $p \in (\frac{1}{2}, 1]$. We fix some N and $k \in [N-1]$. For $\eta \in \Omega_{N,k}$, let $\Theta^{\eta}_{N,k}$ denote the configuration on $\{0,1\}^{\mathbb{Z}}$ which is given by*

$$\Theta^{\eta}_{N,k}(x) := \begin{cases} 0 & \text{if} \quad x \leq -(N-k) \\ \eta(x+(N-k)) & \text{if} \quad -(N-k) < x \leq k \\ 1 & \text{if} \quad x > k\,, \end{cases} \tag{4.20}$$

and we write $\Theta_{N,k}$ when $\eta = \theta_{N,k}$, i.e., the particles in $\Theta_{N,k}$ are placed on the sites $\{-N+k+1, \dots, -N+2k\} \cup \{k+1, \dots\}$. Recall ϑ_0 from (3.23). Then for all $\varepsilon > 0$,

$$\mathbb{P}(\tau_{\text{hit}}^{\Theta_{N,k}}(\vartheta_0) > t) \leq \varepsilon \tag{4.21}$$

implies that $t^N_{\text{mix}}(\varepsilon) \leq t$ for the ε-mixing time of the asymmetric simple exclusion process with state space $\Omega_{N,k}$.

Intuitively, (4.21) yields an upper bound on the mixing time by considering the projection $(\tilde{\eta}_t)_{t \geq 0}$ of an asymmetric simple exclusion process on $\mathbb{Z}$ started from $\Theta_{N,k}$ onto the segment $[-(N-k)+1, k]$. This argument can be formalized using the graphical representation. We identify edges of the asymmetric simple exclusion process on the segment and the integers, and apply Corollary 4.2; see also Figure 5. We will return to the idea of using the simple exclusion process on $\mathbb{Z}$ for studying mixing times of the simple exclusion process on the segment in Section 5.3.2 and Section 6.6.

We conclude this paragraph by noting that mixing times were also studied when we allow the parameter p of the drift to depend on N. More precisely, when

$$\lim_{N \to \infty} p(N) = \frac{1}{2} \tag{4.22}$$

we say that we are in the **weakly asymmetric regime**. In this setup, mixing times were investigates by Levin and Peres who showed the existence of two phase transitions in the rate of decay of $(p(N))_{N \in \mathbb{N}}$ [90]. Recently, cutoff results were obtained in the weakly asymmetric regime by Labbé and Lacoin [82].

4.2.3 Mixing times for general simple exclusion processes

We now consider mixing times for the simple exclusion process on general graphs $G = (V, E)$ with $k \in [|V|-1]$ particles. We will focus on the case of symmetric transition rates, i.e., where

$$p(x,y) = p(y,x) \tag{4.23}$$

holds for all $\{x, y\} \in E$. In this case, the simple exclusion process has a unique reversible distribution, which is the Uniform distribution on the state space. In general, it is a difficult task to bound the mixing time of exclusion processes on some arbitrary

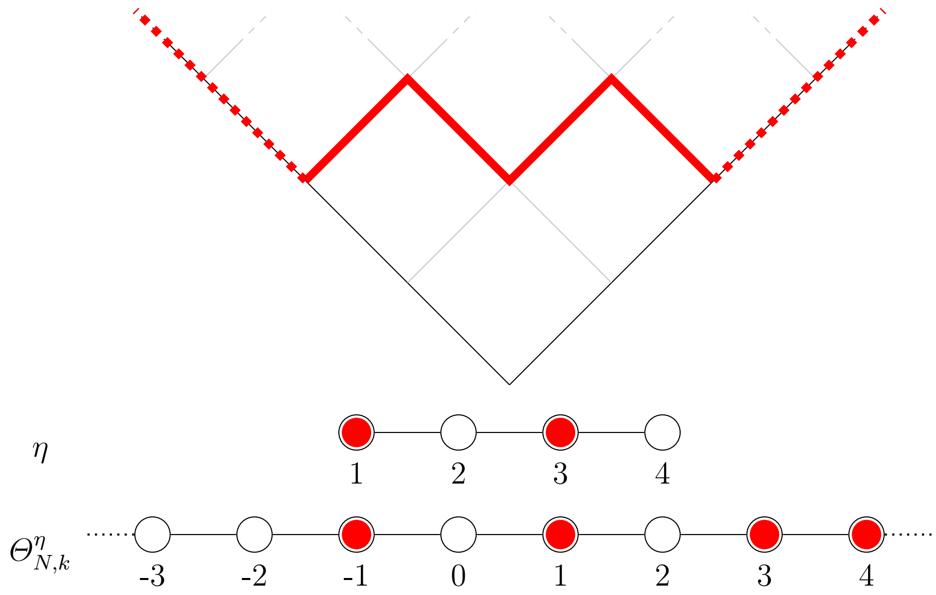

Figure 5: Visualization of a configuration $\eta \in \Omega_{N,k}$ of the simple exclusion process with $N = 4$ and $k = 2$, and its extension to the set A_0 using the embedding $\Theta^{\eta}_{N,k}$ defined in (4.20) of Theorem 4.7. The embedding $\Theta^{\eta}_{N,k}$ provides a natural way of associating a path representation for configurations $\tilde{\eta} \in A_0$ with $\tilde{\eta}(x) = 0$ for all $x \leq -N + k$ and $\tilde{\eta}(x) = 1$ for all $x > k$, respectively.

graph G. Hence, it is a natural question whether the mixing time of the simple exclusion process can be bounded from above using the mixing time of a single particle, i.e., when we choose $k = 1$. The following result due to Oliveira shows that the mixing time of the simple exclusion process with symmetric rates differs only by a factor of order $\log(|V|)$ from the mixing time of a single particle [107].

Theorem 4.8 (c.f. Theorem 1.1 in [107]). *Let $G = (V, E)$ be a graph with symmetric transition rates, such that the corresponding simple exclusion process on G is irreducible. Let $t^{G,k}_{\text{mix}}(\varepsilon)$ denote the ε-mixing time of the simple exclusion process on G when we have $k \in [|V| - 1]$ particles. Then for all $k \in [|V| - 1]$ and $\varepsilon > 0$*

$$t^{G,k}_{\text{mix}}(\varepsilon) \leq C \log\left(\frac{1}{\varepsilon}\right) \log(|V|) t^{G,1}_{\text{mix}}\left(\frac{1}{4}\right) \tag{4.24}$$

holds for some universal constant C.

Recently, the results by Oliveira were extended by Hermon and Pymar in [72]. They derive upper bounds with an explicit dependence on the number of particles k for certain families of graphs. Moreover, they show that the mixing time for a single particle can never exceed the mixing time of a simple exclusion process for any number of particles $k \in [|V| - 1]$ by more than some universal factor; see Proposition 1.6 in [72]. Note that in general, when the transition rates do not satisfy (4.23), Theorem 4.8 may fail; see Section 5 for an example.

4.3 A generalized Wilson's lemma

For lower bounds on the mixing time, recall Proposition 4.3 which states that the mixing time can be bounded from below using the relaxation time. In general, when we have a detailed knowledge about an eigenvalue and the corresponding eigenfunction, this may allow us to refine this lower bound on the mixing time.To state such a refined estimate, we will use the following definition. For a continuous-time Markov chain $(X_t)_{t\geq 0}$ with generator $\mathcal{A}$, let $(M_t)_{t\geq 0}$ be the associated martingale given by

$$M_t := F(X_t) - F(X_0) - \int_0^t (\mathcal{A}F)(X_s)\mathrm{d}s \tag{4.25}$$

for all $t \geq 0$. We denote its quadratic variation by $(\langle M\rangle_t)_{t\geq 0}$. For an introduction to martingales and their quadratic variations, we refer to Chapter 3 and Chapter 5 in [96]. The following bound is due to Wilson in [139] for discrete-time Markov chains. It naturally extends to continuous-time Markov chains as follows; see also [83, 84].

Theorem 4.9 (Wilson's lemma in continuous time)**.** *Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on some finite state space $\mathcal{S}$ with a unique stationary distribution π and generator $\mathcal{A}$. Let $F\colon \mathcal{S} \to \mathbb{R}$ be a function with*

$$\mathcal{A}F(y) = -\lambda F(y) \text{ for all } y \in \mathcal{S}, \tag{4.26}$$

with constant $\lambda > 0$. We assume that the quadratic variation $(\langle M\rangle_t)_{t\geq 0}$ of the associated martingale defined in (4.25) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\langle M\rangle_t\right] \leq R \tag{4.27}$$

for some $R > 0$ and all $t \geq 0$. Then for all $\varepsilon \in (0,1)$, the ε-mixing time $t_{\mathrm{mix}}(\varepsilon)$ of $(X_t)_{t\geq 0}$ satisfies

$$t_{\mathrm{mix}}(1-\varepsilon) \geq \frac{1}{\lambda}\log\left(\|F\|_\infty\right) - \frac{1}{2\lambda}\log\left(\frac{16R}{\lambda\varepsilon}\right). \tag{4.28}$$

Although we may use any eigenvalue and corresponding eigenfunction, it turns out that the eigenfunction corresponding to the spectral gap will in general yield the most precise bounds. Again, we remark that the idea of using eigenfunctions to estimate the mixing time is already present in [46], where Diaconis and Shahshahani obtain the complete spectrum of the random transposition shuffle, and apply a second moment method to prove cutoff. However, note that for many models, often no exact descriptions of the eigenvalues and eigenfunctions are available. In this case, it is useful to work with a generalized version of Wilson's lemma, which allows for an approximation of eigenvalues and eigenfunctions. This idea was introduced by Nam and Nestoridi in [105] for discrete-time Markov chains. It transfers to our setup as follows.

Lemma 4.10 (Generalized Wilson's lemma)**.** *Let* $(X_t)_{t\geq 0}$ *be an irreducible continuous-time Markov chain with finite state space* S *and generator* $\mathcal{A}$. *Let* $F\colon S \to \mathbb{R}$ *be a function with*

$$|(-\mathcal{A}F)(y) - \lambda F(y)| \leq c \quad \textit{for all } y \in S, \tag{4.29}$$

with constants $\lambda \geq c > 0$. *Moreover, we assume that the quadratic variation* $(\langle M\rangle_t)_{t\geq 0}$ *of the associated martingale defined in* (4.25) *satisfies for some* $R > 0$ *and all* $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\langle M\rangle_t\right] \leq R\,. \tag{4.30}$$

Then for all $\varepsilon \in (0,1)$, *the* ε*-mixing time* $t_{\text{mix}}(\varepsilon)$ *of* $(X_t)_{t\geq 0}$ *satisfies*

$$t_{\text{mix}}(1-\varepsilon) \geq \frac{1}{\lambda}\log\left(\|F\|_\infty\right) - \frac{1}{2\lambda}\log\left(\frac{16(3c\|F\|_\infty + \max(R,c))}{\lambda\varepsilon}\right). \tag{4.31}$$

We call F an **approximate eigenfunction** if (4.29) holds. We will see an application of the Lemma 4.10 in Section 6.4, where we discuss lower bounds for the symmetric simple exclusion process with open boundaries. We now prove Lemma 4.10 combining ideas from [105] with martingale techniques used for Lemma 2.2 in [83].

Proof of Lemma 4.10. For $X_0 = \eta$, let $f(t) := \mathbb{E}[F(X_t)] = \mathbb{E}_\eta[F(X_t)]$ for all $t \geq 0$. Now

$$f'(t) = \mathbb{E}\left[(\mathcal{A}F)(X_t)\right] \in [-\lambda f(t) - c, -\lambda f(t) + c] \quad \text{for all } t \geq 0$$

using the martingale property of $(M_t)_{t\geq 0}$ and (4.29). Apply Gronwall's lemma to get for all $t \geq 0$

$$f(t) \leq f(0)\mathrm{e}^{-\lambda t} + \int_0^t c\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s \leq f(0)\mathrm{e}^{-\lambda t} + \frac{c}{\lambda};$$

see Lemma 2.7 in [135]. Similarly, apply Gronwall's lemma to $-f$ to conclude that

$$\left|f(t) - \mathrm{e}^{-\lambda t}f(0)\right| \leq \frac{c}{\lambda} \tag{4.32}$$

for all $t \geq 0$. Next, we define $g(t) := \mathbb{E}[(F(X_t))^2]$. Observe that $(F(X_t))_{t\geq 0}$ is a semimartingale. Thus, we apply Itô's formula to see that

$$\begin{aligned} F^2(X_t) - F^2(X_0) = 2\int_0^t F(X_s)\mathrm{d}\big[F(X_s) - \int_0^s (\mathcal{A}F)(X_r)\mathrm{d}r\big] \\ + 2\int_0^t F(X_s)\mathrm{d}\big[\int_0^s (\mathcal{A}F)(X_r)\mathrm{d}r\big] + \frac{1}{2}\int_0^t 2\ \mathrm{d}\langle M\rangle_s \end{aligned}$$

holds; see also Theorem 5.33 in [96]. Next, we obtain

$$g(t) - g(0) = 2\int_0^t \mathbb{E}\left[F(X_s)(\mathcal{A}F)(X_s)\right]\mathrm{d}s + \mathbb{E}\left[\langle M\rangle_t\right]$$

for all $t \geq 0$ by taking expectations and changing the order of integration. Taking derivatives yields

$$g'(t) = 2\mathbb{E}\left[F(X_t)(\mathcal{A}F)(X_t)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\langle M\rangle_t]$$

for all $t \geq 0$. Using (4.29), we obtain that

$$2\mathbb{E}\left[F(X_t)(\mathcal{A}F)(X_t)\right] \leq -2\lambda g(t) + 2c||F||_\infty$$

holds. By applying Gronwall's lemma and using (4.30), a calculation shows that

$$g(t) \leq g(0)\mathrm{e}^{-2\lambda t} + \frac{c||F||_\infty}{\lambda} + \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}[\langle M\rangle_s]\right)\mathrm{e}^{-2\lambda(t-s)}\mathrm{d}s \leq g(0)\mathrm{e}^{-2\lambda t} + \frac{c||F||_\infty + R}{\lambda}$$

for all $t \geq 0$. Together with (4.32) and the fact that $g(0) = f(0)^2$, we deduce

$$\mathrm{Var}(F(X_t)) = \mathrm{Var}_\eta(F(X_t)) = g(t) - f(t)^2 \leq \frac{3c||F||_\infty + R}{\lambda} \tag{4.33}$$

for any initial state $\eta \in S$, and all $t \geq 0$. Recall the total-variation distance from (4.1) and let $d_\eta(t)$ denote the total-variation distance between the law of X_t started from η and its stationary distribution. Note that for all $t \geq 0$ and any initial state η,

$$d_\eta(t) \geq \mathbb{P}\left(F(X_t) \geq \frac{1}{2}\mathbb{E}[F(X_t)]\right) - \mathbb{P}\left(F(X_\infty) \geq \frac{1}{2}\mathbb{E}[F(X_t)]\right).$$

Here, X_∞ is a random variable whose law is the stationary distribution of $(X_t)_{t\geq 0}$. Using Chebyshev's inequality, we see that

$$\begin{aligned} d_\eta(t) &\geq 1 - \frac{4\mathrm{Var}(F(X_t))}{\mathbb{E}[F(X_t)]^2} - \frac{4\mathbb{E}[F(X_\infty)^2]}{\mathbb{E}[F(X_t)]^2} \\ &\geq 1 - 4\frac{\mathrm{Var}(F(X_t)) + \mathrm{Var}(F(X_\infty)) + \mathbb{E}[F(X_\infty)]^2}{\mathbb{E}[F(X_t)]^2} \end{aligned} \tag{4.34}$$

The goal is to show that for t equal to the right-hand side of (4.31), the right-hand side of (4.34) is $\geq 1 - \varepsilon$, which implies (4.31). Let η be such that $|F(\eta)| = ||F||_\infty$ holds. We bound the denominator in (4.34) for t equal to the right-hand side of (4.31) by

$$\mathbb{E}[F(X_t)] \geq \mathrm{e}^{-\lambda t}F(X_0) - \frac{c}{\lambda} = \mathrm{e}^{-\lambda t}||F||_\infty - \frac{c}{\lambda} \geq \frac{1}{2}\mathrm{e}^{-\lambda t}||F||_\infty\,,$$

where the last inequality is due to our choice of t. To bound the nominator of the last term in (4.34), take $t \to \infty$ in (4.32) and (4.33) to see that $|\mathbb{E}[F(X_\infty)]| \leq c/\lambda$ and $|\mathrm{Var}[F(X_\infty)]| \leq (3c||F||_\infty + R)\lambda^{-1}$. A calculation shows that indeed the right-hand side of (4.34) is $\geq 1 - \varepsilon$. □

4.4 The censoring inequality

Next, we discuss the censoring inequality which we use to establish upper bounds on the mixing time. The censoring inequality is a very recent technique, first established by Peres and Winkler [110] for spin systems, and then later applied to the symmetric simple exclusion process on the segment by Lacoin [84]. Informally speaking, this inequality says that leaving out transitions of the exclusion process along certain edges only increases the distance from equilibrium. More precisely, we will assume the existence of a partial ordering on the state space, and compare the laws of processes with and without censoring in terms of stochastic domination. This will allow us to compare the maximal distance from equilibrium in both dynamics. Let ν, ν' be two probability measures defined on a common probability space $\mathcal{S}$, which is equipped with a partial order $\succeq$. We say that ν **stochastically dominates** ν' with respect to $\succeq$, and write $\nu \succeq \nu'$, if there exists a coupling P with $X \sim \nu$ and $Y \sim \nu'$ such that $P(X \succeq Y) = 1$. Equivalently, $\nu \succeq \nu'$ holds whenever

$$\int f \, \mathrm{d}\nu \geq \int f \, \mathrm{d}\tilde{\nu} \tag{4.35}$$

for all increasing functions $f\colon \mathcal{S} \to \mathbb{R}$; see Theorem B.9 in [94].

Slightly generalizing the definition from [110], we say that a **censoring scheme** $\mathcal{C}$ for an exclusion process $(\eta_t)_{t\geq 0}$ on a finite graph $G = (V, E)$ is a random càdlàg function

$$\mathcal{C}\colon \mathbb{R}_0^+ \to \mathcal{P}(E) \tag{4.36}$$

which does not depend on the process $(\eta_t)_{t\geq 0}$. In the censored dynamics $(\eta_t^{\mathcal{C}})_{t\geq 0}$, a transition along an edge e at time t is performed if and only if $e \notin \mathcal{C}(t)$; see Figure 6. In order to state the censoring inequality, recall the partial order $\succeq_{\mathrm{h}}$ from (3.29) as well as the height function representation from (3.33). We assume that the state space of the exclusion process has a unique maximal element $\wedge$ and a unique minimal element $\vee$ with respect to $\succeq_h$. Further, assume that the dynamics obey a monotone grand coupling with respect to $\succeq_h$. By Proposition 3.12, we note that these assumptions are satisfied for the simple exclusion process on the segment. The following lemma is an immediate consequence of the proofs of Theorem 1.1 and Lemma 2.1 in [110], and can be found for the symmetric simple exclusion process as Proposition 6.2 in [84]; see also Lemma 2.12 in [66] for a general version.

Lemma 4.11. *Let $\mathcal{C}$ be a censoring scheme for the exclusion process. For an initial state η and a time $t \geq 0$, let $P_\eta(\eta_t \in \cdot)$ and $P_\eta(\eta_t^{\mathcal{C}} \in \cdot)$ denote the law of $(\eta_t)_{t\geq 0}$ and its censored dynamics $(\eta_t^{\mathcal{C}})_{t\geq 0}$, respectively. Under the above assumptions, for all $t \geq 0$*

$$P_\wedge(\eta_t^{\mathcal{C}} \in \cdot\,) \succeq_{\mathrm{h}} P_\wedge(\eta_t \in \cdot\,) \quad \textit{and} \quad P_\vee(\eta_t^{\mathcal{C}} \in \cdot\,) \preceq_{\mathrm{h}} P_\vee(\eta_t \in \cdot\,)\,. \tag{4.37}$$

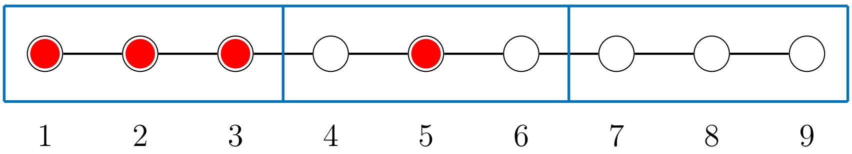

Figure 6: Visualization of a censoring scheme for the simple exclusion process, where the particles can only move within the blue boxes. In particular, only the particle in the middle box can move and performs a random walk on $\{4, 5, 6\}$.

Moreover, the density function $\eta \mapsto \frac{1}{\mu(\eta)} P_\wedge(\eta_t = \eta)$ *is increasing with respect to the partial order* $\succeq_{\mathrm{h}}$ *and*

$$\|P_\wedge(\eta_t^{\mathcal{C}} \in \cdot\,) - \mu\|_{\mathrm{TV}} \geq \|P_\wedge(\eta_t \in \cdot\,) - \mu\|_{\mathrm{TV}} \tag{4.38}$$

and

$$\|P_\vee(\eta_t^{\mathcal{C}} \in \cdot\,) - \mu\|_{\mathrm{TV}} \geq \|P_\vee(\eta_t \in \cdot\,) - \mu\|_{\mathrm{TV}} \tag{4.39}$$

holds for all $t \geq 0$.

We will see applications of Lemma 4.11 in Section 5.3.1 as well as in Sections 6.6 and 6.8. Note that the statement (4.37) in Lemma 4.11 extends to the asymmetric simple exclusion process $\mathbb{Z}$ when starting from a ground state ϑ_n for some $n \in \mathbb{Z}$ by using finite systems as an approximation. This yields the following corollary.

Corollary 4.12. *Using the partial order* $\succeq_{\mathrm{h}}$ *from* (3.25) *for the simple exclusion process on the integers restricted to* A_n *for some* $n \in \mathbb{Z}$*, we see that*

$$P_{\vartheta_n}(\eta_t \in \cdot\,) \succeq_h P_{\vartheta_n}(\eta_t^{\mathcal{C}} \in \cdot\,) \tag{4.40}$$

holds for all $t \geq 0$ *and* $n \in \mathbb{Z}$.

4.5 The disagreement process

So far, we focused on the case where for a graph $G = (V, E)$ the state space Ω is given by a subset of $\{0, 1\}^V$, i.e., we choose $S = \{0, 1\}$ as the set of spins. In this section, we allow for more general sets S. We start with the case where $S = \{0, 1, 2\}$, which leads to the **exclusion process with second class particles**. Intuitively, each site is still either occupied by a particle or left empty. However, we now allow for different kinds of particles, namely first class particles, which will correspond to our original particles, and second class particles.

We give a brief introduction to the notion of second class particles for the exclusion process on a locally finite graph $G = (V, E)$; see [94, Part III, Section 1] for a more comprehensive discussion. For a configuration $\xi \in \{0, 1, 2\}^V$, we say that a vertex $x \in V$

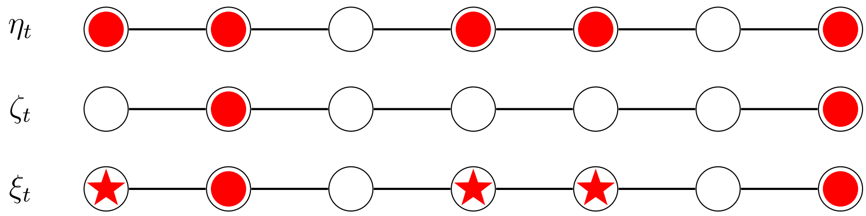

Figure 7: Two configurations $\eta_t \succeq_{\mathrm{h}} \zeta_t$ with disagreement process ξ_t at time $t \geq 0$.

is occupied by a **first class particle** if $\eta(x) = 1$ holds, by a **second class particle** if $\eta(x) = 2$ holds and **vacant** in the remaining case $\eta(x) = 0$. We assign priorities to the vertices. Sites with first class particles get the highest priority, then sites with second class particles and then empty sites. For given transition rates $(p(x,y))_{x,y \in V}$, the **(simple) exclusion process with second class particles**, sometimes also called **two-species exclusion process**, is the Markov process $(\xi_t)_{t \geq 0}$ on the state space $\{0,1,2\}^V$ with the following construction.

Recall the graphical representation of the simple exclusion process from Section 3.3. Whenever a clock associated to $p(x,y)$ rings, we swap the spins of x and y, provided that x has a higher priority than y. In addition, we may allow for rules of flipping spins at certain sites in general exclusion processes; see for example Section 6.3.1.

Second class particles for the simple exclusion process were intensively studied over the last decades as they are closely related to current fluctuations and shocks; see [11, 51, 53, 54]. In the context of mixing times, second class particles were first used by Benjamini et al. in [19] for the asymmetric simple exclusion process on the segment. We will follow their ideas when using second class particle arguments for the simple exclusion process in marginal nestling environments in Section 5.3.2, and for the asymmetric simple exclusion process with one blocked entry in Section 6.6.

In the following, our main application for second class particles is to describe the difference between two exclusion processes. More precisely, for two exclusion processes $(\eta_t)_{t \geq 0}$ and $(\zeta_t)_{t \geq 0}$ on a locally finite graph $G = (V,E)$, we define, with a slight abuse of notation, the **disagreement process** $(\xi_t)_{t \geq 0}$ between $(\eta_t)_{t \geq 0}$ and $(\zeta_t)_{t \geq 0}$ by

$$\xi_t(x) := \mathbb{1}_{\{\eta_t(x) = \zeta_t(x) = 1\}} + 2\mathbb{1}_{\{\eta_t(x) \neq \zeta_t(x)\}} \tag{4.41}$$

for all $x \in V$ and all $t \geq 0$. In words, we keep the current value if the processes $(\eta_t)_{t \geq 0}$ and $(\zeta_t)_{t \geq 0}$ agree and place a second class particle otherwise; see Figure 7. Note that this definition does not rely on the transition rules or the coupling of the two exclusion processes. However, when both exclusion processes have the same transition rules, obey the canonical coupling, and are initially ordered according to $\succeq_{\mathrm{c}}$, then the disagreement process yields an exclusion process with second class particles.

When the underlying graph $G = (V, E)$ is finite, we can use the disagreement process $(\xi_t)_{t\geq 0}$ on $\{0, 1, 2\}^{|V|}$ to provide upper bounds on the mixing time of the exclusion process on G. The next lemma is immediate from Proposition 4.1.

Lemma 4.13. *For a pair of initial configurations η, ζ, let $(\xi_t^{\eta,\zeta})_{t\geq 0}$ be the disagreement process between two exclusion processes with respect to the canonical coupling started from η and ζ, respectively. We denote by $\tau(\eta, \zeta)$ the first time at which $(\xi_t^{\eta,\zeta})_{t\geq 0}$ contains no second class particles. Then for any $\varepsilon \in (0, 1)$, if*

$$\max_{\eta,\zeta} \mathbb{P}\left(\tau(\eta, \zeta) > t\right) \leq \varepsilon \tag{4.42}$$

holds, the ε-mixing time of the exclusion process on G satisfies $t_{\text{mix}}(\varepsilon) \leq t$.

Note that in the same way, we can allow any totally ordered finite set of spins S. We use a similar construction by assigning priorities and applying the canonical coupling, but now for more than two different hierarchies of particles. The resulting process is called **multi-species exclusion process**. Multi-species exclusion processes are of huge recent interest; see [27, 61]. We will see a modified multi-species exclusion processes in Section 6.8 where the priorities only obey a partial ordering.

Suppose that the simple exclusion process on a graph G has symmetric transition rates; see (4.23). If all sites are occupied and have different spins, we refer to the resulting multi-species exclusion process under the above construction as the **interchange process** on G. Note that since all particles can be distinguished in the interchange process, the spins along any undirected edge will be swapped at rate 1. Hence, the generator of the interchange process has the form in (3.17), which was stated for the special case where the underlying graph is $\mathbb{Z}$.

4.6 Notation for asymptotic estimates of mixing times

Before we come to our main results on mixing times, we give some remarks on the used notation. Whenever we consider a family of Markov chains indexed by $N \in \mathbb{N}$, we will use the following notation for asymptotic estimates. For functions $f, g \colon \mathbb{N} \to \mathbb{R}$

$$f \lesssim g \quad \Leftrightarrow \quad \exists c > 0 \text{ s.t. } \limsup_{N\to\infty} \left|\frac{f(N)}{g(N)}\right| \leq c \tag{4.43}$$

$$f \gtrsim g \quad \Leftrightarrow \quad \exists c > 0 \text{ s.t. } \liminf_{N\to\infty} \left|\frac{f(N)}{g(N)}\right| \geq c\,. \tag{4.44}$$

We write $f \asymp g$ if and only if $f \lesssim g$ and $f \gtrsim g$ holds. Moreover, let $f \ll g$ or $f = o(g)$ if and only if (4.43) is satisfied for all $c > 0$. We say that an asymptotic estimate holds **with high probability** if for some fixed $c > 0$, the respective inequality in (4.43) or (4.44) holds with probability tending to one.

5 The simple exclusion process in random environment

5.1 Introduction

In Section 4.2, we discussed mixing times for the simple exclusion process with homogeneous or symmetric transition rates. In the following, we investigate mixing times for the simple exclusion process when the underlying transition rates are inhomogeneous and asymmetric. This section is in large parts based on Section 5 of [119]. We will impose the following assumptions on the choice of the transition rates: we will allow only nearest-neighbor jumps, such that each particle tries to perform a jump at a total rate of 1. This is called the **constant speed model** of the simple exclusion process. In the beginning, we choose for each site the rate to perform a jump to the right i.i.d. according to some distribution on $(0,1]$. We assume that the distribution is such that a single particle on $\mathbb{Z}$ with i.i.d. chosen jump rates according to this distribution has almost sure a positive linear speed. We refer to this as the **ballistic regime**; see (5.1). In the following, we focus on mixing times for the simple exclusion process in the ballistic regime when the underlying distribution is supported on $A \subseteq [\frac{1}{2}, 1]$ with $\frac{1}{2} \in A$. This ensures that the particles have a macroscopic drift to the right-hand side and microscopically no bias to the left-hand side. We call this regime the **marginal nestling case**.

5.1.1 Definition of the model

We now give a formal introduction to the simple exclusion process in random environments. First, we define the simple exclusion process on the segment in a fixed environment. The **simple exclusion process in environment $\boldsymbol{\omega}$** for $\omega = (\omega(x))_{x\in[N]}$ on a segment of size N with k particles is a continuous-time Markov chain $(\eta_t)_{t\geq 0}$ on the state space $\Omega_{N,k}$ from (3.27). It is generated by the closure of

$$\begin{aligned}\mathcal{L}f(\eta) = &\sum_{x=1}^{N-1} \omega(x)\ \eta(x)(1-\eta(x+1)) \left[f(\eta^{x,x+1}) - f(\eta)\right] \\ &+ \sum_{x=2}^{N} (1-\omega(x))\ \eta(x)(1-\eta(x-1)) \left[f(\eta^{x,x-1}) - f(\eta)\right] ,\end{aligned}$$

where we assume $\omega(x) \in (0,1]$ for all $x \in [N]$. For the simple exclusion process on $\Omega_{N,k}$ in a random environment, we choose the transition probabilities $(\omega(x))_{x\in[N]}$ to be i.i.d. according to some probability distribution on $(0,1]$ and denote the law of the environment by $\mathbb{P}$. Since $\omega(x) \in (0,1]$ for all $x \in [N]$, the simple exclusion process has a unique essential class, and hence a unique stationary distribution μ; see (3.28). We

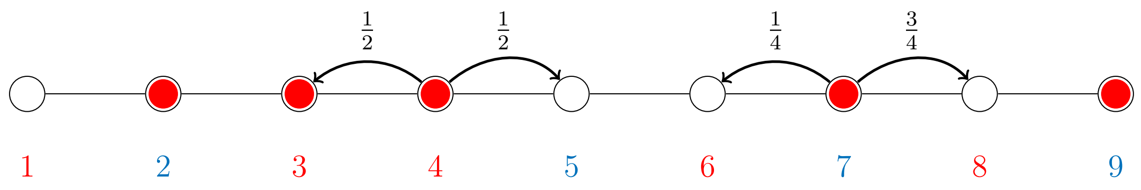

Figure 8: Example of the simple exclusion process in a marginal nestling environment ω where $\mathbb{P}(\omega(1) = \frac{1}{2}) = \mathbb{P}(\omega(1) = \frac{3}{4}) = \frac{1}{2}$. For positions x with $\omega(x) = \frac{1}{2}$, we mark the label in red, and in blue otherwise.

denote the quenched law of the exclusion process in a fixed environment ω with starting distribution ν by $P_{\omega,\nu}$. If ν is the Dirac measure on some configuration $\psi \in \Omega_{N,k}$, we will write $P_{\omega,\psi}$. Define the **quenched ε-mixing time** of the exclusion process $(\eta_t)_{t\geq 0}$

$$t_{\text{mix}}^{\omega,N}(\varepsilon) := \inf\left\{ t \geq 0 \,:\, \max_{\psi\in\Omega_{N,k}} \|P_{\omega,\psi}(\eta_t \in \cdot\,) - \mu\|_{\text{TV}} < \varepsilon \right\}$$

for $\varepsilon \in (0,1)$. Again, we refer to $t_{\text{mix}}^{\omega,N} = t_{\text{mix}}^{\omega,N}(\frac{1}{4})$ simply as the mixing time. Our goal is to study the order of $t_{\text{mix}}^{\omega,N}$ when N tends to infinity.

In the following, let $(\eta_t)_{t\geq 0}$ be the simple exclusion process in a random environment ω with state space $\Omega_{N,k}$ and mixing time $t_{\text{mix}}^{\omega,N}$. We study the **ballistic regime** with a drift to the right, i.e.,

$$\mathbb{E}\left[\frac{1-\omega(1)}{\omega(1)}\right] < 1 \tag{5.1}$$

holds. Solomon [132] showed that under assumption (5.1), a single particle has almost surely a positive linear speed to the right-hand side; see [141] for a survey on random walks in random environment. We distinguish three different cases. When

$$\mathbb{P}\left(\omega(1) \geq \frac{1}{2} + \delta\right) = 1 \tag{5.2}$$

holds for some $\delta > 0$, we say that we are in the **non-nestling case**. When

$$\mathbb{P}\left(\omega(1) \geq \frac{1}{2}\right) = 1 \tag{5.3}$$

holds, but (5.2) is not satisfied, we say that we are in the **marginal nestling case**; see Figure 8 for an example. If neither (5.2) nor (5.3) holds, but the environment law satisfies (5.1), we are in the **plain nestling case**. Note that for plain nestling environments, we must have

$$\mathbb{P}\left(\omega(1) < \frac{1}{2}\right) > 0\,, \tag{5.4}$$

i.e., with positive probability a site has a bias against the macroscopic drift direction. The terms non-nestling, marginal nestling and plain nestling are taken from [142].

5.1.2 Related literature

For a discussion of mixing times for the simple exclusion process in homogeneous environments, we refer to Section 4.2. However, let us emphasis at this point that the results by Oliveira, stated as Theorem 4.8, do in general not apply for the simple exclusion process in random environments. When assumption (3.15) holds, like in the setup of Theorem 4.8, we say that we consider the **variable speed model** of the simple exclusion process. This assumption is in general not satisfied in the constant speed model of the simple exclusion process, where the particles attempt jumps at a total rate of 1.

For the special case of a single particle, i.e., when $k = 1$ holds for all $N \in \mathbb{N}$, mixing times were studied by Gantert and Kochler [63]. Under some mild assumptions on the environment law, they show that in the ballistic regime, we see with high probability with respect to the law of the environment that the mixing time is linear in the size of the segment. Moreover, the cutoff phenomenon occurs. Furthermore, they study mixing times in the transient regime, where

$$\mathbb{P}\left(\log\left(\frac{1-\omega(1)}{\omega(1)}\right)\right) < 0 \tag{5.5}$$

holds. Solomon showed that assumption (5.5) guarantees that a single particle in an i.i.d. environment on $\mathbb{Z}$, which satisfies (5.5), will have a drift to the right-hand side and almost surely return to the origin only finitely often [132]. When an environment is transient but not ballistic, a single particle escapes to infinity at a sublinear speed. This case is called the **subballistic regime**. Again, under some mild conditions on the environment law, including the assumption that

$$\mathbb{E}\left[\left(\frac{1-\omega(1)}{\omega(1)}\right)^{\lambda}\right] = 1 \tag{5.6}$$

holds for some $\lambda \in (0, 1]$, Gantert and Kochler prove that in the subballistic regime, we see with high probability a mixing time of order $N^{\frac{1}{\lambda}}$, and no cutoff.

For the simple exclusion process in a ballistic random environment with a positive fraction of particles, i.e., when the number of particles $k = k(N)$ satisfies

$$0 < \liminf_{N\to\infty} \frac{k(N)}{N} \leq \limsup_{N\to\infty} \frac{k(N)}{N} < 1\,, \tag{5.7}$$

lower bounds on the mixing time were obtained in the author's unpublished Master thesis [118]; see also Sections 3 and 4 in [119] for a published version of these results. The lower bounds on the mixing times can be summarized as follows.

Theorem 5.1. *Let $(\eta_t)_{t\geq 0}$ denote the simple exclusion process in random environment ω with state space $\Omega_{N,k}$ and mixing time $t_{\text{mix}}^{\omega,N}$. Suppose that assumption* (5.7) *holds. We distinguish three cases:*

- *Suppose that we are in the non-nestling case. Then $\mathbb{P}$-almost surely the mixing time satisfies $t_{\text{mix}}^{\omega,N} \gtrsim N$.*
- *Suppose that we are in the marginal nestling case. Then with high probability, the mixing time satisfies $t_{\text{mix}}^{\omega,N} \gg N$.*
- *Suppose that we are in the plain nestling case. Then there exists some $\delta > 0$ such that with high probability, the mixing time satisfies $t_{\text{mix}}^{\omega,N} \gtrsim N^{1+\delta}$.*

For the simple exclusion process in the marginal nestling regime with a positive fraction of particles, the mixing time has with high probability with respect to the environment law a superlinear growth. Moreover, in the plain nestling regime, we see that the mixing time of a single particle and the mixing time for a positive fraction of particles differ with high probability by a polynomial factor in the size of the segment, using the results of [63]. For the non-nestling case, it is immediate that $t_{\text{mix}}^{\omega,N} \asymp N$ for $\mathbb{P}$-almost all environments, using an extension of the canonical coupling; see Section 5.2 and Theorem 4.7.

Very recently, the lower bounds were improved and extended for the simple exclusion process in the subballistic regime by Lacoin and Yang [86]. Further, they show that in the ballistic and subballistic regime, with high probability with respect to the law of the environment, the mixing time is polynomial in the size of the segment.

5.1.3 Main result

We present now our main result on the mixing time of the simple exclusion process in a marginal nestling random environment.

Theorem 5.2. *Let $(\eta_t)_{t\geq 0}$ denote the simple exclusion process in environment ω with state space $\Omega_{N,k}$ and mixing time t_{mix}^{N} for some $k = k(N) = [N-1]$. Then for all marginal nestling environments, which satisfy* (5.3) *but not* (5.2), *$t_{\text{mix}}^{N} \lesssim N \log^3(N)$ holds with high probability with respect to the environment law $\mathbb{P}$.*

Together with the results of Gantert and Kochler in [63] on the mixing time of a single particle in a ballistic random environment, this shows that with high probability, the mixing time of the simple exclusion process in a marginal nestling environment is within a polylogarithmic factor of the mixing time of a single particle; see Figure 9 for a simulation of the corresponding height function over time.

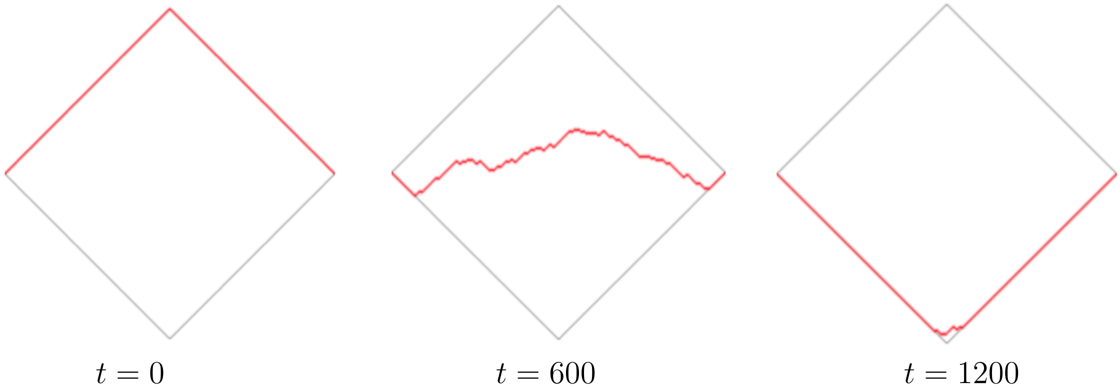

Figure 9: Simulation of the path of the height function for a simple exclusion process in the marginal nestling case. We have a segment of size $N = 200$, $k = 100$ particles, and the environment law satisfies $\mathbb{P}(\omega(1) = \frac{1}{2}) = \mathbb{P}(\omega(1) = \frac{3}{4}) = \frac{1}{2}$.

5.1.4 Outline of the proof

We will now give an overview of the strategy for the proof of Theorem 5.2. We start by recalling the canonical coupling for the simple exclusion process from Section 3.3, and present an extension for inhomogeneous environments in Section 5.2. This modified canonical coupling will preserve the partial order $\succeq_{\mathrm{h}}$ using height functions for the simple exclusion process and a partial order for the environment.

In Section 5.3, we study the speed of the particles on the segment when starting from the configuration $\theta_{N,k}$ with all particles at the left-hand side. In general, the speed will no longer be linear in N. However, when we extend the line segment to a larger size, say N^2, we can show that with high probability, the particles have traveled a distance of $N\log(N)$ until a time of order $N\log^3(N)$. We formalize this observation in Proposition 5.4 using the censoring technique; see Section 4.4. More precisely, we partition the segment into boxes according to a censoring scheme such that with high probability, each box contains at most one particle at a time. The isolated particles perform independent random walks within their boxes. In this way, we control the particle movements with respect to their local equilibria simultaneously.

The remaining part of the proof follows the ideas of Benjamini et al. in [19]. Using Theorem 4.7, we extend the simple exclusion process to the integers and study the hitting time of the ground state. We will then use the exclusion process with second class particles; see Section 4.5. We obtain an upper bound on the hitting time, which is of order $N\log^3(N)$ plus the hitting time of the ground state in a system with a different starting configuration; see Proposition 5.9. We iterate this argument until the remaining hitting time is with high probability of order at most N.

5.2 The canonical coupling for inhomogeneous environments

Recall the canonical coupling from Section 3.3. We present now a modified canonical coupling for the simple exclusion process in general environments. It is defined with respect to a common space of all possible initial configurations and environments. It will be monotone with respect to the partial order $\succeq_{\mathrm{h}}$ on $\Omega_{N,k}$ from (3.29) as well as with respect to the partial order $\succeq_{\mathrm{e}}$ on the set of all possible environments given by

$$\omega \preceq_{\mathrm{e}} \bar{\omega} \;\Leftrightarrow\; \omega \succeq_{\mathrm{c}} \bar{\omega} \;\Leftrightarrow\; 1-\omega(x) \leq 1-\bar{\omega}(x) \;\text{ for all } x \in [N] \tag{5.8}$$

for environments ω and $\bar{\omega}$, and the component-wise order $\succeq_{\mathrm{c}}$ from (3.13). We will now construct the modified canonical coupling $(\eta_t, \zeta_t)_{t\geq 0}$ for two simple exclusion processes $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ in environments ω and $\bar{\omega}$, respectively.

We place rate 2 Poisson clocks on all sites $x \in [N]$. Whenever the clock at a site x rings at time t, we flip a fair coin and sample a Uniform-$[0,1]$ random variable U independently. We proceed as follows:

Suppose that the coin shows "head" and it holds that $x \neq N$, $U \leq \omega(x)$ and $\eta_t(x) = 1-\eta_t(x+1) = 1$. Then we move the particle from site x to site $x+1$ in η_t. Moreover, if $U \leq \bar{\omega}(x)$ as well as $\zeta_t(x) = 1-\zeta_t(x+1) = 1$ are satisfied, then move the particle from site x to site $x+1$ in the configuration ζ_t. Similarly, suppose that the coin shows "tail", and $x \neq 1$, $U > \omega(x)$ and $\eta_t(x) = 1-\eta_t(x-1) = 1$. Then we move the particle from site x to site $x-1$ in η_t. Again, if $U > \bar{\omega}(x)$ as well as $\zeta_t(x) = 1-\zeta_t(x-1) = 1$ are satisfied, then move the particle from site x to site $x-1$ in configuration ζ_t. If none of these rules apply, we leave the configurations unchanged.

The following proposition is immediate from the construction of this modified canonical coupling for the simple exclusion process. With a slight abuse of notation, we call this coupling again the canonical coupling and denote its law by $\mathbf{P}$.

Proposition 5.3. *Let $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ be exclusion processes in environments ω and $\bar{\omega}$, respectively, according to the canonical coupling. If $\eta_0 \preceq_{\mathrm{h}} \zeta_0$ and $\omega \preceq_{\mathrm{e}} \bar{\omega}$, then*

$$\mathbf{P}\left(\eta_t \preceq_{\mathrm{h}} \zeta_t \;\forall t \geq 0\right) = 1\,.$$

We will see another possible modification of the canonical coupling for the TASEP on trees; see Section 10.2. Note that using the canonical coupling, together with Corollary 4.2 and Theorem 4.7, we immediately see that for any deterministic environment ω with some $\delta > 0$ such that $\omega(x) \geq \frac{1}{2} + \delta$ for all $x \in [N]$, the mixing time is linear in the size of the segment; see also [118] and Theorem 1.1(i) in [119].

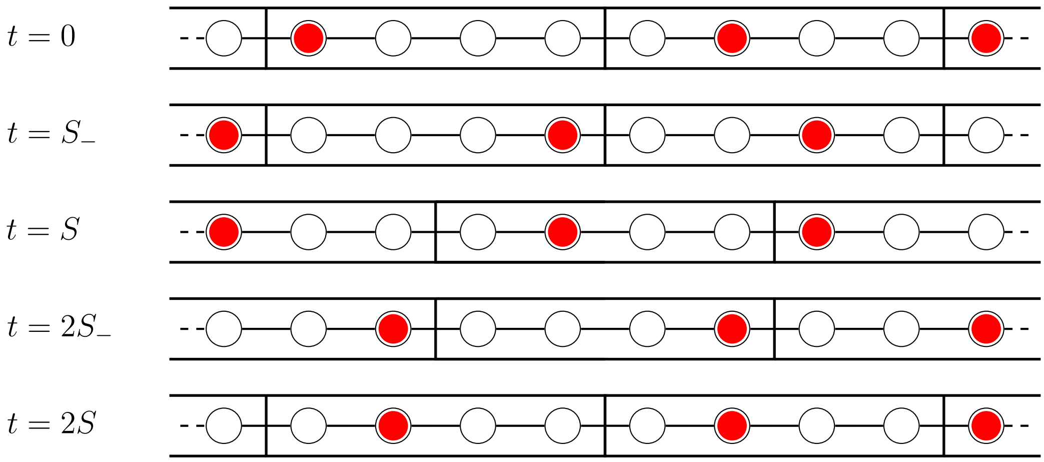

Figure 10: Illustration of the censoring scheme used in the proof of Proposition 5.4 with $U = 2$. During each period $[iS, (i+1)S)$ for $i \in \mathbb{N}_0 = \{0, 1, \dots\}$, the particles shown in red are only allowed to move within their assigned boxes.

5.3 Upper bound for marginal nestling environments

We will now show Theorem 5.2, following the strategy outlined in Section 5.1.4.

5.3.1 Speed of the particles in the marginal nestling case

Recall the censoring inequality from Section 4.4. By Proposition 5.3, we see that the modified canonical coupling for the simple exclusion process in random environment is a monotone grand coupling with respect to the partial order $\succeq_{\text{h}}$. Hence, we can in the following apply the censoring inequality from Lemma 4.11 to get a lower bound on the speed of the particles. In order to define the speed on a suitable scale, we will, for the moment, enlarge the underlying segment. More precisely, we will assume that the simple exclusion process $(\eta_t)_{t \geq 0}$ in a marginal nestling environment is defined with respect to the line segment of size N^2 and $k \in [N-1]$ particles. In Section 5.3.2, we will see how the results for the exclusion process on this enlarged segment transfer to bounds on the mixing time for the simple exclusion process on a segment of size N. In the following, we denote by $L(\eta)$ the position of the leftmost particle and by $R(\eta)$ the position of the rightmost empty site in a configuration η, respectively.

Proposition 5.4. *Consider the simple exclusion process $(\eta_t)_{t \geq 0}$ with initial configuration $\theta_{N^2,k}$ from* (3.31) *for $k \in [N-1]$. Then with $\mathbb{P}$-probability at least $1 - N^{-2}$*

$$P_{\omega,\theta_{N^2,k}}\left(L(\eta_{T_N}) \geq N \log(N) + N\right) \geq 1 - \frac{2}{N^2} \tag{5.9}$$

holds for $T_N = cN \log^3(N)$, where $c > 0$ is a sufficiently large constant.

In order to show Proposition 5.4, we will now provide a censoring scheme $\mathcal{C}$ for the simple exclusion process; see Figure 10. Intuitively, we alternate between two partitions of the line segment into boxes of logarithmic size. Moreover, in every second iteration, we release a particle at the left-hand side as long as there are particles available. The time between the switches of the two partitions and the size of the boxes will be chosen such that with high probability, up to time T_N, all particles move to the right half of the box within each iteration. Formally, we define $\mathcal{C}$ as follows:

The censoring scheme $\mathcal{C}$ remains constant within the intervals $[iS,(i+1)S)$ for all $i \in \mathbb{N}_0$ and some $S = S(N)$, which we choose later. For i even, $\mathcal{C}$ contains all edges $e = \{x, x+1\}$ such that $x = 2jU$ for some $j \in \mathbb{N}$ and $x \leq N^2 - 2U$. Again, the value of $U = U(N)$ will be determined later on. For i odd, $\mathcal{C}$ consists of all edges $e = \{x, x+1\}$ such that $x = (2j+1)U$ for some $j \in \mathbb{N}$ as well as $x \leq N^2 - 2U$. In both cases, whenever $i < 2k$, we let $e = \{x, x+1\}$ be the unique edge in $\mathcal{C}$ with the smallest x such that $k - \lfloor \frac{i}{2} \rfloor \leq x$ holds. We remove e from $\mathcal{C}$ and add the edge $\{k - \lfloor \frac{i}{2} \rfloor - 1, k - \lfloor \frac{i}{2} \rfloor\}$. This ensures that the i^{th} particle from the right will only move from time $2(i-1)S$ onward.

We refer to Figure 10 for an illustration. Our goal is to control the particles within the boxes in the censoring scheme $\mathcal{C}$. Whenever a particle is allowed to move, it is isolated in a box of size $2U$ during an iteration. Only the first and the last box might be larger due to boundary effects, but are at most of size $4U$. Consider the i^{th} particle and condition on its position at time jS for the largest j such that $t \geq jS$ holds. Let $B = B(i,t)$ denote the interval according to the censoring scheme in which the i^{th} particle is placed at time $t \geq 0$. Further, let $C = C(i,t)$ denote the set of the rightmost U vertices in B. Let $\mathcal{B}$ be the set of all $B(i,t)$ for some $t \geq 0$ and $i \in [k]$. The next lemma gives an estimate on the invariant measure and the mixing time of the random walk within a box $B \in \mathcal{B}$.

Lemma 5.5. *Let $\mu_{\text{RW}}^{\omega,B}$ denote the invariant measure of the random walk on $B \in \mathcal{B}$ in environment $\omega|_B$. There exists a constant $u > 0$ such that for $U = u\log(N)$, with $\mathbb{P}$-probability at least $1 - N^{-2}$*

$$\mu_{\text{RW}}^{\omega,B}(C) \geq 1 - N^{-5} \tag{5.10}$$

holds for all $B \in \mathcal{B}$ and $N \in \mathbb{N}$. For this choice of U, let $t_{\text{RW}}^{\omega,B}(\varepsilon)$ denote the ε-mixing time of the random walk on $B \in \mathcal{B}$ in environment $\omega|_B$. There exists a constant $s > 0$ such that for $S = s\log^3(N)$ and almost every environment ω

$$t_{\text{RW}}^{\omega,B}(N^{-5}) \leq S \tag{5.11}$$

holds for all $B \in \mathcal{B}$ and $N \in \mathbb{N}$. Hence, with $\mathbb{P}$-probability at least $1 - N^{-2}$, a random walk started at some point in B is contained in the respective set C after time S with probability at least $1 - 2N^{-5}$ for all $B \in \mathcal{B}$.

Proof. Observe that $\mathcal{B}$ contains at most N^3 elements by construction of the censoring scheme. For the random walk on B, the stationary distribution $\mu_{\text{RW}}^{\omega,B}$ is given by

$$\mu_{\text{RW}}^{\omega,B}(y) = \frac{1}{Z_B} \prod_{i=1}^{y} \frac{\omega(i)}{1-\omega(i+1)}$$

for all $y \in B$ and some normalization constant Z_B. Using condition (5.1), we know that $\mathbb{E}[\mu_{\text{RW}}^{\omega,B}(y)]$ is exponentially increasing in y, where $\mathbb{E}[\,\cdot\,]$ denotes the expectation with respect to $\mathbb{P}$. Hence, let $u > 0$ be such that with $\mathbb{P}$-probability at least $1 - N^{-5}$

$$\mu_{\text{RW}}^{\omega,B}(C) \geq 1 - N^{-5}$$

holds for every $B \in \mathcal{B}$ fixed and $N \in \mathbb{N}$. Taking a union bound over all elements in $\mathcal{B}$ gives (5.10). In order to show (5.11), recall that $|B| \leq 4U$ holds for all $B \in \mathcal{B}$. We claim that the mixing time of the random walk in B satisfies

$$t_{\text{RW}}^{\omega,B}\left(\frac{1}{4}\right) \leq 64U^2$$

for all $B \in \mathcal{B}$ and almost every environment ω. Using Corollary 4.2 for $k = 1$, it suffices to give a bound on the hitting time of the rightmost site in B when starting the random walk from the leftmost site in B. Using Proposition 5.3, this hitting time is dominated by the respective hitting time of a symmetric simple random walk on B, which has mean $|B|^2$. Hence, we obtain (5.11) from Corollary 4.2 and Markov's inequality. □

Proof of Proposition 5.4. We start by making the following observation. Suppose that for all $i \in [k]$ and $j \in \mathbb{N}_0$ with $2(i-1) \leq j \leq T_N/S$, the i^{th} particle, counted from the right-hand side, is contained in the set $C(i, jS)$ at time $((j+1)S)_-$, i.e., up to time T_N all the particles reach the right half of their respective boxes within time S whenever they are able to move. By construction of the censoring scheme $\mathcal{C}$, we then have that up to time T_N, each box contains at most one particle at a time. Moreover, each particle has moved a distance of at least $U(T_N/S - 2k)$ to the right-hand side.

Let $U = U(N)$ and $S = S(N)$ from Lemma 5.5 be the size of the boxes and the time between the switches of the partitions in the censoring scheme $\mathcal{C}$, respectively. We set $T_N := S(U^{-1}(N\log(N) + N) + 2k)$ for all $N \in \mathbb{N}$. Note that we have at most N particles and each particle is contained in at most N^2 different boxes up to time T_N for all N sufficiently large. Using Lemma 5.5 and the above observation, we obtain that with $\mathbb{P}$-probability at least $1 - N^{-2}$

$$P_{\omega,\theta_{N^2,k}}^{\mathcal{C}}\left(L(\eta_{T_N}^{\mathcal{C}}) \geq N\log(N) + N\right) \geq 1 - \frac{2}{N^2}$$

holds. Since the event in (5.9) is decreasing for $\succeq_{\text{h}}$, we conclude using Lemma 4.11. □

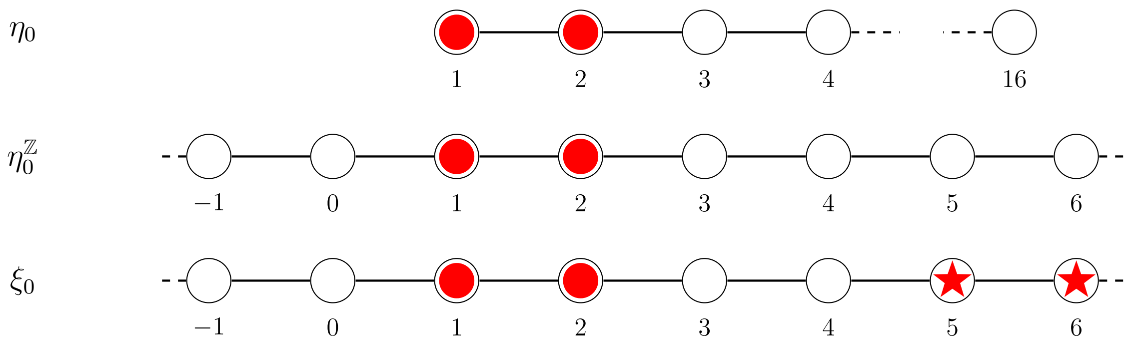

Figure 11: Visualization of the initial configurations $\theta_{N^2,k}$, $\theta_{\mathbb{Z},k}$, and ξ_0 for the different processes $(\eta_t)_{t\geq 0}$, $(\eta^{\mathbb{Z}}_{t\geq 0})$, and $(\xi_{t\geq 0})$, respectively, for $N=4$ and $k=2$, which are involved in the proof of Theorem 5.2.

5.3.2 Comparison to the exclusion process on the integers

Next, we want to compare the simple exclusion process $(\eta_t)_{t\geq 0}$ on $\{0,1\}^{N^2}$ to a simple exclusion process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ on the integers; see Figure 11. Formally, the simple exclusion process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ in environment $\omega \in (0,1]^{\mathbb{Z}}$ is a Feller process on $\{0,1\}^{\mathbb{Z}}$ generated by the closure of

$$\begin{aligned}\tilde{\mathcal{L}}f(\eta) = &\sum_{x\in\mathbb{Z}} \omega(x)\ \eta(x)(1-\eta(x+1))\left[f(\eta^{x,x+1})-f(\eta)\right] \\ &+\sum_{x\in\mathbb{Z}} (1-\omega(x))\ \ \eta(x)(1-\eta(x-1))\left[f(\eta^{x,x-1})-f(\eta)\right] . \qquad (5.12)\end{aligned}$$

Note that Theorem 3.1 ensures that (5.12) gives a Feller process. With a slight abuse of notation, we will use the same notation for the quenched law of $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ as for the simple exclusion process on the segment. Note that the partial order $\succeq_{\mathrm{h}}$ as well as the canonical coupling from Section 5.2 naturally extend to $\mathbb{Z}$ when the number of particles is finite, i.e., we compare the positions of the i^{th} particles in both configurations. However, we loose the existence of a unique maximal or minimal element.

In the following, we assume that the environment $\omega \in (0,1]^{\mathbb{Z}}$ for $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ is marginal nestling, i.e., $(\omega(x))_{x\in\mathbb{Z}}$ are i.i.d. and their law satisfies condition (5.3), but not (5.2). Let $\theta_{\mathbb{Z},k}$ denote the configuration in $\{0,1\}^{\mathbb{Z}}$, where we place exactly k particles on $[k]$.

Lemma 5.6. *Let $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ be the simple exclusion process on the integers in environment ω started from $\theta_{\mathbb{Z},k}$. Then for all $k\in[N-1]$ with $\mathbb{P}$-probability at least $1-N^{-2}$*

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(L(\eta^{\mathbb{Z}}_{T_N}) \geq N\log(N)\right) \geq 1-\frac{4}{N^2}$$

holds for all N large enough, where T_N is taken from Proposition 5.4.

Proof. For the simple exclusion process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ on the integers in environment ω, we consider its projection to the environment $\tilde{\omega} := \omega|_{[-N+1,N^2-N]}$. Observe that $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ is uniquely determined by its values on $\tilde{\omega}$ whenever no particle reaches the sites $-N+1$ or N^2-N. We claim that for almost all environments ω the statements

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(\exists t \in [0,T_N]\colon \max\left\{i \geq 0 : \eta_t^{\mathbb{Z}}(i) = 1\right\} \geq N^2 - N\right) \leq \frac{1}{N^2} \tag{5.13}$$

and

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(\exists t \in [0,T_N]\colon L(\eta_t^{\mathbb{Z}}) \leq -N+1\right) \leq \frac{1}{N^2} \tag{5.14}$$

hold for all N large enough. The first statement is immediate when we consider the motion of the rightmost particle in $(\eta_t^{\mathbb{Z}})_{t\geq 0}$. For the second statement, notice that the position of the leftmost particle in $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ stochastically dominates the position of the leftmost particle in a symmetric simple exclusion process on $\mathbb{Z}$ with the same initial condition. Recall from Section 3.4.1 that the symmetric simple exclusion process can be seen as an interchange process in which the particles swap positions along each edge irrespectively of the configuration at the endpoints. In this case, the particles perform symmetric simple random walks on $\mathbb{Z}$, and we use Chernoff bounds to conclude.

Whenever the events in (5.13) and (5.14) occurs, the simple exclusion process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ with initial configuration $\theta_{\mathbb{Z},k}$ has on the set $[-N+1, N^2-N]$ up to time T_N the same law as a simple exclusion process $(\eta_t)_{t\geq 0}$ on the segment of size N^2 in environment $\tilde{\omega}$, started from the configuration where all k particles are placed on the positions $\{N+1,\ldots,N+k\}$ of the segment $[N^2]$. Hence, using the stochastic domination from Proposition 5.3 together with Proposition 5.4 gives the desired result. □

Lemma 5.6 shows that the particles in $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ started from $\theta_{\mathbb{Z},k}$ have passed a distance of at least $N\log(N)$ to the right-hand side until time T_N. We will now ensure that also for times larger than T_N, the particles escape fast enough. In particular, we study the motion of the particles at different time scales, which will be increasing exponentially with respect to N.

Lemma 5.7. *Let $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ be the simple exclusion process in environment ω started from $\theta_{\mathbb{Z},k}$. For all $k \in [N-1]$, with $\mathbb{P}$-probability at least $1-2N^{-2}$,*

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(\forall t \geq T_N\colon L(\eta_t^{\mathbb{Z}}) > t^{\frac{2}{3}} + N\right) \geq 1 - \frac{10}{N^2}$$

holds for all N large enough, where T_N is taken from Proposition 5.4.

Proof. For a given $N \in \mathbb{N}$, we define the sequences $(N_i)_{i\in\mathbb{N}_0}$ and $(t_i)_{i\in\mathbb{N}_0}$ to be

$$N_i := \exp\left(\left(\frac{4}{3}\right)^i \log(N)\right) \quad \text{and} \quad t_i := \sum_{j=0}^{i} T_{N_j}\,. \tag{5.15}$$

By Lemma 5.6, with $\mathbb{P}$-probability at least $1 - N_0^{-2}$

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(L(\eta^{\mathbb{Z}}_{t_0}) \geq N_0 \log(N_0)\right) \geq 1 - \frac{4}{N_0^2} \tag{5.16}$$

holds. Suppose that the event in (5.16) occurs. Then without loss of generality, we can assume that the particles are placed on the sites in $[N_0 \log(N_0), N_0 \log(N_0) + k]$ at time t_0 by Proposition 5.3. Starting from this configuration, we can apply Lemma 5.6 again to obtain that with $\mathbb{P}$-probability at least $1 - N_0^{-2} - N_1^{-2}$

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(L(\eta^{\mathbb{Z}}_{t_i}) \geq N_i \log(N_i) \text{ for } i \in \{0,1\}\right) \geq 1 - 4\left(\frac{1}{N_0^2} + \frac{1}{N_1^2}\right)$$

holds. Iterating this argument along the sequence $(N_i)_{i\in\mathbb{N}_0}$, we see that $\mathbb{P}$-probability at least $1 - 2N^{-2}$

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(L(\eta^{\mathbb{Z}}_{t_i}) \geq N_i \log(N_i) \text{ for } i \in \mathbb{N}_0\right) \geq 1 - \frac{8}{N^2}$$

is satisfied. Observe that

$$N_i \log(N_i) > (t_i)^{\frac{2}{3}} + N$$

holds for all $i \in \mathbb{N}_0$ and N large enough. Hence, it remains to consider the case of $t \in (t_i, t_{i+1})$ for some $i \in \mathbb{N}_0$. Using the same arguments as for the proof of (5.14) in Lemma 5.6, we obtain that for $\mathbb{P}$-almost every environment ω

$$P_{\omega,\theta_{\mathbb{Z},k}}\left(L(\eta^{\mathbb{Z}}_t) \geq N_i \ \forall t \in [t_i, t_{i+1}] \mid L(\eta^{\mathbb{Z}}_{t_i}) \geq N_i \log(N_i)\right) \geq 1 - \frac{1}{N_{i+1}^2}$$

holds for all $i \in \mathbb{N}$ and N sufficiently large. Since

$$(t_{i+1})^{\frac{2}{3}} + N < N_i$$

holds for all $i \geq 1$ and N sufficiently large, we obtain the desired result. □

We will now work towards using the above estimates for mixing times. Recall from (3.23) that $\vartheta_0 \in \{0,1\}^{\mathbb{Z}}$ denotes the ground state on A_0, and that $\Theta_{N,k}$ from (4.20) is the projection of $\theta_{N,k}$ to A_0, for $N \in \mathbb{N}$ and $k \in [N-1]$. Further, recall (4.8) and let

$$t^{\omega,N}_{\text{hit}}(\varepsilon) := \inf\left\{t \geq 0 \colon P_{\omega,\Theta_{N,k}}\left(\tau^{\Theta_{N,k}}_{\text{hit}}(\vartheta_0) > t\right) \leq \varepsilon\right\} \tag{5.17}$$

be the **ε-hitting time** of $(\eta^{\mathbb{Z}}_t)_{t\geq 0}$ for all $\varepsilon \in (0,1)$. We now relate the ε-hitting time of $(\eta^{\mathbb{Z}}_t)_{t\geq 0}$ to the ε-mixing time $t^{\omega|_{[N]}}_{\text{mix}}(\varepsilon)$ of a simple exclusion process on the line segment of size N in environment $\omega|_{[N]}$; see also Section 4.1.1. The following statement is the analogue of Theorem 4.7 for inhomogeneous environments. Again, it follows directly by extending the canonical coupling to $\mathbb{Z}$, so we omit the proof.

Lemma 5.8. *For almost all environments $\omega \in (0,1]^{\mathbb{Z}}$, and for all $\varepsilon > 0$*

$$t_{\text{mix}}^{\omega|_{[N]},N}(\varepsilon) \le t_{\text{hit}}^{\omega,N}(\varepsilon) \,.$$

For the remainder of the proof of Theorem 5.2, we will follow the ideas of Benjamini et al. [19]. We will show that whenever the particles in the simple exclusion process on $\mathbb{Z}$ have with high probability passed a distance of at least N to the right-hand side, an associated exclusion process on the line segment has "almost" reached the ground state. This will be our main idea for the proof of Proposition 5.9, which states a recursion for the ε-hitting time. For an environment $\omega \in (0,1]^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we denote by ω_n the environment shifted to the right-hand side by n, i.e., for all $x \in \mathbb{Z}$

$$\omega_n(x) := \omega(x-n) \,. \tag{5.18}$$

Proposition 5.9. *For a given $N \in \mathbb{N}$ and $k = k(N) \in [N-1]$, we set $N' = N^{3/4}$ and $k' = k(N') = \frac{1}{2}N^{3/4}$. Consider the simple exclusion process on $\mathbb{Z}$ in a marginal nestling environment ω. Set $n = N - k - N' + k'$ and recall T_N from Proposition 5.4. Then with $\mathbb{P}$-probability at least $1 - 2N^{-2}$,*

$$t_{\text{hit}}^{\omega,N}(\varepsilon) \le T_N + t_{\text{hit}}^{\omega_n,N'}\left(\varepsilon - 12N^{-2}\right)$$

holds for all $\varepsilon > 0$ and N large enough.

In words, Proposition 5.9 states that the ε-hitting time of the ground state can with high probability be bounded from above by T_N plus the $(\varepsilon - 12N^{-2})$-hitting time with respect to N' and k' of the ground state for the simple exclusion process in the shifted environment ω_n.

In order to show Proposition 5.9, recall the notion of second class particles from Section 4.5. Let $(\xi_t)_{t\ge 0}$ be a simple exclusion process on $\mathbb{Z}$ with second class particles. We will now define two projections of $(\xi_t)_{t\ge 0}$ onto $\{0,1\}^{\mathbb{Z}}$. Let $(\xi_t^{2\to 1})_{t\ge 0}$ be the process given by

$$\xi_t^{2\to 1}(x) := \begin{cases} 1 & \text{if } \xi_t(x) \neq 0 \\ 0 & \text{if } \xi_t(x) = 0 \end{cases} \tag{5.19}$$

for all $x \in \mathbb{Z}$ and $t \ge 0$. Similarly, $(\xi_t^{2\to 0})_{t\ge 0}$ denotes the process where

$$\xi_t^{2\to 0}(x) := \begin{cases} 1 & \text{if } \xi_t(x) = 1 \\ 0 & \text{if } \xi_t(x) \neq 1 \end{cases} \tag{5.20}$$

for all $x \in \mathbb{Z}$ and $t \ge 0$. We refer to $(\xi_t^{2\to 1})_{t\ge 0}$ and $(\xi_t^{2\to 0})_{t\ge 0}$ as **particle blindness** and **second class-empty site blindness**, respectively. A visualization of the two projections is given in Figure 12.

In addition, we define a third projection $(\xi_t^{\mathrm{X}})_{t\geq 0}$ onto $\{0,1\}^{\mathbb{Z}}$ by removing all first class particles as well as the sites corresponding to the particles, and then applying projection $(\xi_t^{2\to 1})_{t\geq 0}$. Since the resulting process is only well-defined up to translations, we initially place a tagged particle in the origin. Note that a formal introduction to the tagged particle process on general graphs will be given in Section 7. Here, in the special case of the simple exclusion process on $\mathbb{Z}$ in random environment, we use the following construction. Assume that a given configuration $\xi \in \{0,1,2\}^{\mathbb{Z}}$ satisfies

$$|\{i \in \mathbb{Z} \colon \xi(i) = 2\}| = \infty \, .$$

Let $u \colon \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$ be an enumeration of the sites in ξ such that

$$u(0) := \begin{cases} \inf\{i \leq 0 \colon \xi(i) = 2\} & \text{if } -\infty < \inf\{i \leq 0 \colon \xi(i) = 2\} < +\infty \\ \inf\{i > 0 \colon \xi(i) = 2\} & \text{otherwise.} \end{cases}$$

We define the positions $u(j)$ and $u(-j)$ for all $j \in \mathbb{N}$ recursively by

$$u(j) := \inf\{i > u(j-1) \colon \xi(i) \neq 1\}$$

and

$$u(-j) := \inf\{i > u(-j+1) \colon \xi(i) \neq 1\} \, .$$

We can now define ξ^{X} as

$$\xi^{\mathrm{X}}(i+1) := \begin{cases} 1 & \text{if } \xi(u(i)) = 2 \\ 0 & \text{if } \xi(u(i)) = 0 \end{cases} \tag{5.21}$$

for all $i \in \mathbb{Z}$. In order to obtain a stochastic process $(\xi_t^{\mathrm{X}})_{t\geq 0}$, we denote by $u_t(i)$ the position of the particle at time t which is in position $u(i)$ in ξ_0, and then apply (5.21) accordingly. Again, we refer to Figure 12 for a visualization of the projection. The proof of Proposition 5.9 will now be an interplay of the three projections $(\xi_t^{2\to 1})_{t\geq 0}$, $(\xi_t^{2\to 0})_{t\geq 0}$ and $(\xi_t^{\mathrm{X}})_{t\geq 0}$ of a simple exclusion process with second class particles $(\xi_t)_{t\geq 0}$.

Proof of Proposition 5.9. Let $(\xi_t)_{t\geq 0}$ be the simple exclusion process with second class particles in environment ω with

$$\xi_0(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \in [k] \\ 0 & \text{if } x \in [k+1, N] \\ 2 & \text{if } x > N \end{cases}$$

as initial configuration. Observe that the process $(\xi_t^{2\to 1})_{t\geq 0}$ has the same law as a simple exclusion process in environment ω_{N-k} started from the configuration $\Theta_{N,k}$.

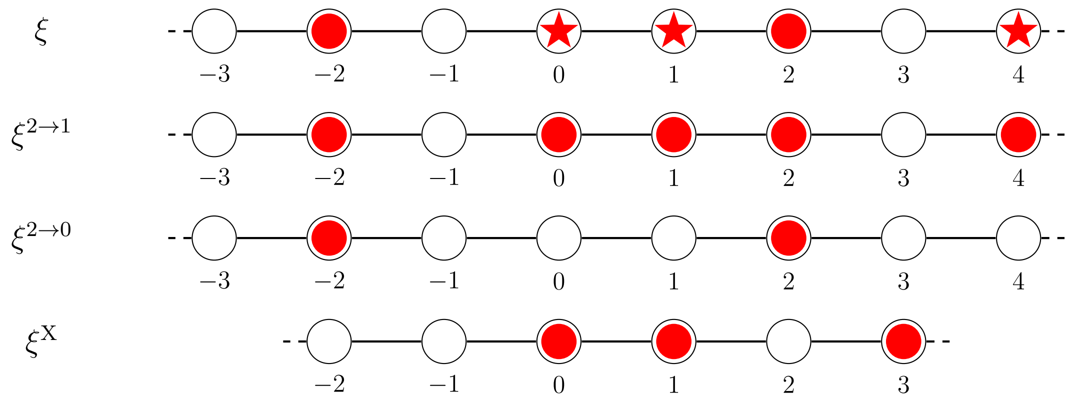

Figure 12: Visualization of the particle blindness $\xi^{2\to1}$, the second-class empty site blindness $\xi^{2\to0}$, and the projection ξ^{X} (with $u(0)=0$) for a configuration $\xi \in \{0,1,2\}^{\mathbb{Z}}$ of the simple exclusion process with second class particles.

Our goal is to bound the hitting time of the ground state for the process $(\xi_t^{2\to1})_{t\geq0}$. We make the following observation: Suppose that at time $t \geq 0$, the two events

$$K_1 := \left\{\inf\{x\in\mathbb{Z}\colon \xi_t(x)=1\} \geq t^{\frac{2}{3}}+N\right\}$$
$$K_2 := \left\{\xi_t^{X}(x) = \mathbb{1}_{\{x\geq0\}}\ \forall x\in\mathbb{Z}\right\}$$

occur. Then $\xi_t^{2\to1} = \vartheta_{N-k}$ holds. To see this, note that if K_1 occurs, then there exists a second class particle which is on the left-hand side of the leftmost first-class particle in ξ_t. If K_2 occurs, then all empty sites are placed on the left-hand side of the leftmost second class particle in ξ_t. We claim that with $\mathbb{P}$-probability at least $1-2N^{-2}$,

$$P_{\omega,\xi_0}\left(K_1 \text{ holds for all } t\geq T_N\right) \geq 1-10N^{-2} \tag{5.22}$$

holds. Note that the process $(\xi_t^{2\to0})_{t\geq0}$ has the same law as a simple exclusion process in environment ω started from configuration $\theta_{\mathbb{Z},k}$, and so (5.22) follows by Lemma 5.7.

We now want give an upper bound on the first time $t\geq T_N$ such that the event K_2 occurs. We claim that for almost every marginal nestling environment ω

$$P_{\omega,\xi_0}\left(\sup\{i\geq0\colon \xi_t^{X}=0\} < t^{\frac{2}{3}}\ \forall t\geq T_N\right) \geq 1-\frac{1}{N^2} \tag{5.23}$$

holds. To see this, note that by Proposition 5.3 and the censoring inequality, the position of the rightmost empty site in the process $(\xi_t^{X})_{t\geq0}$ is stochastically dominated by the position of the rightmost empty site for a symmetric simple exclusion process with starting configuration $\psi\in\{0,1\}^{\mathbb{Z}}$ given by $\psi(x)=\mathbb{1}_{x\leq0}$ for all $x\in\mathbb{Z}$. Recall from Section 3.4.1 that the symmetric simple exclusion process can be seen as an interchange process on $\mathbb{Z}$ and hence, we obtain (5.23) by applying Chernoff bounds. Using the same argument for the position of the leftmost second class particle,

$$P_{\omega,\xi_0}\left(\inf\{i \leq 0 \colon \xi_t^{\mathrm{X}} = 1\} > -t^{\frac{2}{3}} \ \forall t \geq T_N\right) \geq 1 - \frac{1}{N^2} \tag{5.24}$$

holds for almost every environment ω. Note that when the events in (5.22) and (5.23) hold, no first-class particle will be next to an empty site for any time $t \geq T_N$. Since transitions between first and second class particles do not change a configuration in $(\xi_t^{\mathrm{X}})_{t\geq T_N}$, the process $(\xi_t^{\mathrm{X}})_{t\geq T_N}$ then has the law of a simple exclusion process in environment ω_n. Hence, the hitting time of the ground state in $(\xi_t^{\mathrm{X}})_{t\geq T_N}$ started from $\xi_{T_N}^{\mathrm{X}}$ gives an upper bound on the hitting time of the ground state for $(\xi_t^{2\to 1})_{t\geq 0}$.

We now show that it suffices to consider the hitting time of the ground state for $(\xi_t^{\mathrm{X}})_{t\geq T_N}$ started from $\Theta_{N',k'}$ at time T_N. Provided that the events in (5.23) and (5.24) occur, $\xi_{T_N}^{\mathrm{X}} \preceq_h \Theta_{N',k'}$ holds for all N sufficiently large, where we recall $\succeq_h$ from (3.25). Note that the canonical coupling extended for the exclusion process on $\mathbb{Z}$ preserves the partial order $\succeq_{\mathrm{h}}$ on A_0. Combining these observations, we obtain that the hitting time of the ground state for $(\xi_t^{2\to 1})_{t\geq 0}$ is stochastically dominated by the hitting time of the ground state in $(\xi_t^{\mathrm{X}})_{t\geq T_N}$ started from $\Theta_{N',k'}$ at time T_N whenever the events in (5.22), (5.23) and (5.24) occur. This gives the desired result. □

The next lemma gives a bound on the ε-hitting time of the ground state when the parameters in the initial configuration of the simple exclusion process on $\mathbb{Z}$ are increasing slowly.

Lemma 5.10. *For all $\varepsilon > 0$, we find a sequence $(M_N)_{N\in\mathbb{N}}$ with $\lim_{N\to\infty} M_N = \infty$ such that the ε-hitting time of the ground state for a simple exclusion process on the integers with initial configuration $\Theta_{M_N,M_N/2}$ satisfies*

$$\mathbb{P}\left(t_{\mathrm{hit}}^{\omega,M_N}(\varepsilon) < N\right) \geq 1 - \frac{1}{M(N)}$$

for all N sufficiently large.

Proof. Note that by Theorem 1.1(b) of [77] and Theorem B.52 of [94], the simple exclusion process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ restricted to A_0 is an ergodic Markov chain for $\mathbb{P}$-almost every environment ω. Hence, for all $m \in \mathbb{N}$ and all $\varepsilon > 0$, the ε-hitting time of the ground state for a simple exclusion process started from $\Theta_{m,m/2}$ satisfies

$$\lim_{N\to\infty} \mathbb{P}\left(t_{\mathrm{hit}}^{\omega,m}(\varepsilon) < N\right) = 1$$

by the Poincaré recurrence theorem. For every $N \in \mathbb{N}$, we set

$$M_N := \max\left\{m \in \mathbb{N} \colon \mathbb{P}\left(t_{\mathrm{hit}}^{\omega,m}(\varepsilon) < N\right) \geq 1 - m^{-1}\right\}$$

in order to obtain a sequence $(M_N)_{N\in\mathbb{N}}$ as stated in Lemma 5.10. □

Proof of Theorem 5.2. By Lemma 5.8, it suffices to show that

$$\lim_{N\to\infty} \mathbb{P}\left(t_{\text{hit}}^{\omega,N}\left(\frac{1}{4}\right) < CN\log^3(N)\right) = 1$$

holds for some constant $C > 0$. Recall the definition of N_i from (5.15), and extend it for $i \in \mathbb{Z}$. For $N \in \mathbb{N}$ large enough and M_N of Lemma 5.10 with respect to $\varepsilon = \frac{1}{8}$, define

$$I_N := \min\{i \in \mathbb{N} \colon N_{-i} < M_N\} .$$

We iterate Proposition 5.9 now I_N many times to obtain that with $\mathbb{P}$-probability at least $1 - 4M_N^{-3/4}$

$$t_{\text{hit}}^{\omega,N}\left(\frac{1}{4}\right) \leq \sum_{i=0}^{I_N} T_{N_{-i}} + t_{\text{hit}}^{\omega_l,M_N}\left(\frac{1}{4} - \sum_{i=0}^{I_N}\frac{1}{N_{-i}}\right) \leq 2T_N + t_{\text{hit}}^{\omega_l,M_N}\left(\frac{1}{8}\right)$$

holds for all N sufficiently large and some $l \in \mathbb{Z}$ depending only on N and k. Since the shifted environment ω_l has the same law as ω, we conclude that with $\mathbb{P}$-probability at least $1 - 5M_N^{-3/4}$

$$t_{\text{hit}}^{\omega,N}\left(\frac{1}{4}\right) \leq 2T_N + N \leq CN\log^3(N)$$

holds for some $C > 0$ and N large enough. This finishes the proof of Theorem 5.2. □

5.4 Open problems

In Theorem 5.2, we prove an upper bound on the mixing time for the simple exclusion process in the marginal nestling case. This raises the following question.

Question 5.11. *What is the correct order of the mixing time for the simple exclusion process in a marginal nestling random environment?*

For single particle in the ballistic regime, Gantert and Kochler show that with high probability cutoff occurs; see Theorem 1.5 in [63].

Question 5.12. *Does the simple exclusion process in the ballistic regime exhibit cutoff for any number of particles?*

The last open problem of this section concerns the exclusion process in the constant speed model on general graphs. In the varying speed model on $G = (V, E)$, the mixing time of the simple exclusion process differs from the mixing time of the random walk on G by at most a factor of order $\log(|V|)$; see [107]. Theorem 5.1 shows that this relation does in general not hold in the constant speed model.

Question 5.13. *Does a similar relation as in Theorem 4.8 hold for the constant speed model of the simple exclusion process?*

6 The simple exclusion process with open boundaries

6.1 Introduction

In Sections 4.2 and 5, we studied mixing times for the simple exclusion process when the number of particles is preserved. In this case, under mild assumptions on the transition rates, the simple exclusion process has a unique reversible distribution. We now study mixing times for the simple exclusion process with open boundaries. The presented material is based in large parts on [66], which is joint work with Nina Gantert and Evita Nestoridi. In the simple exclusion process with open boundaries, we allow particles to jump in and out of the system at the ends of the segment according to given rates. In particular, note that the number of particles is no longer preserved over time. Moreover, in general, the simple exclusion process with open boundaries no longer has a reversible measure, and the mean displacement of the particles does not stay fixed in equilibrium. This makes the simple exclusion process with open boundaries one of the most fundamental examples of a non-equilibrium particle system from a statistical mechanics' perspective.

The simple exclusion process with open boundaries is intensively studied in various fields, including probability theory, statistical mechanics, and combinatorics [21, 89, 94, 140]. Informally speaking, the simple exclusion process with open boundaries is an approximation of an infinite system, which captures surprising phenomena such as phase transitions in the current and the formation of shocks; see [45, 51, 54, 55, 137]. In the course of this chapter, we will see examples for these phenomena, e.g., in Section 6.3.2 we discuss current results for the asymmetric simple exclusion process with open boundaries, and in Section 6.6.5, we prove the shock wave phenomenon using an asymmetric simple exclusion process on the half-line.

Note that, in general, a major difficulty is to write down explicitly the stationary distribution of the simple exclusion process with open boundaries. Over the last decades, various different descriptions of the stationary measure were achieved. We refer to Section 6.1.2 for an overview of the available literature. When the stationary measure is hard to describe, an alternative can be to simulate it using Markov chains. We determine in this section how many steps of running the specific Markov chain given by the dynamics of the simple exclusion process with open boundaries are required to have mixed. In particular, we show in several cases that a number of steps proportional to the length of the path is necessary and sufficient for the process to be close to its equilibrium.

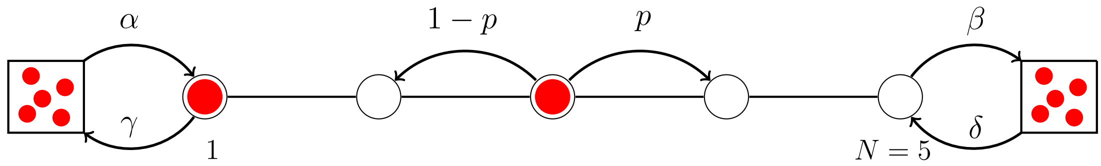

Figure 13: Simple exclusion process with open boundaries for parameters $p, \alpha, \beta, \gamma, \delta$.

6.1.1 Definition of the model

We will now define the simple exclusion process with open boundaries. Recall the simple exclusion process on the segment with drift $p > 0$ from Section 3.4.3, and let

$$\begin{aligned}\mathcal{L}_{\text{ex}} f(\eta) = &\sum_{x=1}^{N-1} p\ \eta(x)(1-\eta(x+1))\left[f(\eta^{x,x+1}) - f(\eta)\right] \\ &+ \sum_{x=2}^{N} (1-p)\ \eta(x)(1-\eta(x-1))\left[f(\eta^{x,x-1}) - f(\eta)\right]\end{aligned}$$

be the generator of the simple exclusion process on the closed segment. For the **simple exclusion process with open boundaries** $(\eta_t)_{t\geq 0}$, we in addition allow creation of particles from reservoirs at the endpoints of the segment. Moreover, particles can be annihilated at the endpoints. More precisely, for parameters $\alpha, \beta, \gamma, \delta \geq 0$, $(\eta_t)_{t\geq 0}$ is the exclusion process with state space $\Omega_N := \{0,1\}^N$ for some $N \in \mathbb{N}$ and generator

$$\begin{aligned}\mathcal{L} f(\eta) = \mathcal{L}_{\text{ex}} f(\eta) &+ \alpha(1-\eta(1))\ \left[f(\eta^1) - f(\eta)\right] + \gamma\eta(1)\ \ \left[f(\eta^1) - f(\eta)\right] \\ &+ \delta(1-\eta(N))\left[f(\eta^N) - f(\eta)\right] + \beta\eta(N)\left[f(\eta^N) - f(\eta)\right] \qquad (6.1)\end{aligned}$$

for all cylinder functions f and configurations $\eta \in \Omega_N$; see Figure 13. Note that in contrast to the simple exclusion process on the closed segment, the number of particles will in general no longer be preserved over time. In the remainder, we assume that the above parameters $\alpha, \beta, \gamma, \delta, p \geq 0$ are chosen such that $(\eta_t)_{t\geq 0}$ has a unique stationary distribution μ; see also (3.28). Again, let us stress that μ is not necessarily reversible for $(\eta_t)_{t\geq 0}$. In the following, our goal is to investigate the speed of convergence towards μ in terms of the ε-mixing time $t^N_{\text{mix}}(\varepsilon)$ of $(\eta_t)_{t\geq 0}$ when N goes to infinity.

6.1.2 Related literature

Depending on the parameters, the simple exclusion process with open boundaries can be found under many different names. In the physics literature, it is called partially asymmetric simple exclusion process when $p \in (\frac{1}{2}, 1)$ and totally asymmetric simple exclusion process for $p = 1$. When $p = \frac{1}{2}$ and the boundary parameters are chosen in such a way that the simple exclusion process with open boundaries has no reversible measure, the process is typically called boundary driven simple exclusion process.

When the simple exclusion process with open boundaries is not reversible, it can be seen as one of the simplest examples of a non-equilibrium system, i.e., the expected position of the particles in equilibrium does not stay fixed over time. This observation is quantified by studying currents for the simple exclusion process with open boundaries; see Section 6.3.2. For the symmetric simple exclusion process, currents are investigated in [87]. For the simple exclusion process with open boundaries and general parameters, the leading order term of the current was determined in [137] using Askey–Wilson polynomials, extending the results of [22]. More recently, current fluctuations for the asymmetric simple exclusion process with open boundaries are studied in the physics literature [69, 88]. Note that the moments of the current for the simple exclusion process are closely linked to the motion of second class particles; see [10, 11, 53, 112] for a discussion when the underlying graph is $\mathbb{Z}$. Recently, currents are studied for the simple exclusion process with open boundaries containing second class particles; see [40, 136]. Note that second class particles can also be used to identify shocks; see for example [51, 54, 55], and we will see an application of second class particle arguments when proving a shock wave behavior for the asymmetric simple exclusion process with one blocked entry in Section 6.6.5.

Despite the fact that the invariant measure of the simple exclusion process with open boundaries has in general not a simple closed form, several representations were achieved in the last decades. An essential tool is the matrix product ansatz, which allows us to study various observables like the current, the density profile and correlations between the occupation variables on the segment [117, 137, 138]. Intuitively, we represent in the matrix product ansatz each configuration by a weight which consists of a product of matrices and vectors, satisfying the so-called the DEHP algebra [45]. The vectors and matrices are, in general, infinite-dimensional, and the fact that such a representation exists is non-trivial. We refer to [94, Part III, Section 3] for an introduction. Historically, a similar idea to express the stationary distribution in a recursive way was already discovered by Liggett in [92]. We will revisit parts of the argument in the course of Section 10. The formal framework of the matrix product ansatz was first introduced in [45] for the simple exclusion process with open boundaries in the totally asymmetric case. The question of representing the weights in the matrix product ansatz also gained recent attention in combinatorics, and descriptions such as weighted Catalan paths and staircase tableaux were found; see [23, 33, 101].

We conclude this paragraph on related results by noting that a discussion on the mixing times of the simple exclusion process on the closed segment can be found in Section 4.2. Again, let us point out that the discussed results have in common that the underlying simple exclusion process is reversible, while the simple exclusion process with open boundaries is in general not reversible. This is a crucial difference since reversibility is required for many techniques which give precise mixing time bounds.

6.2 Main results

In the following, we investigate mixing times for the simple exclusion process with open boundaries. Our goal is to provide sharp bounds on the leading order of the mixing time, and investigate whether pre-cutoff or cutoff occurs; see Section 4.1. We claim that without loss of generality, we can assume $p \in [\frac{1}{2}, 1]$. This is due to the symmetry in the definition of $(\eta_t)_{t\geq 0}$ with respect to the boundary parameters. Moreover, we assume that $\max(\alpha, \beta, \gamma, \delta) > 0$ holds. When all boundary parameters are zero, we refer to Section 4.2 for precise bounds on the mixing times of the simple exclusion process on the closed segment.

6.2.1 Symmetric simple exclusion process with open boundaries

We start with the case when all transitions in the bulk are symmetric, i.e., $p = \frac{1}{2}$ holds, but we allow now for general boundary parameters.

Theorem 6.1. *For $p = \frac{1}{2}$, the ε-mixing time of the symmetric simple exclusion process with open boundaries satisfies*

$$\frac{1}{\pi^2} \leq \liminf_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N^2 \log(N)} \leq \limsup_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N^2 \log(N)} \leq C \tag{6.2}$$

for all $\varepsilon \in (0,1)$ and some constant $C = C(\alpha, \beta, \gamma, \delta)$ In particular, the symmetric simple exclusion process with open boundaries has pre-cutoff.

When all boundary parameters are zero and we have a positive fraction of particles, recall (5.7), we note that by Theorem 4.5, the lower bound in (6.2) gives the asymptotic behavior of the ε-mixing time for the simple exclusion process in first-order. However, the next theorem shows that when particles enter and exit only at a single side of the segment, we see a different constant in the leading order.

Theorem 6.2. *For $p = \frac{1}{2}$, suppose that $\max(\alpha, \gamma) = 0$ and $\min(\beta, \delta) > 0$ holds. Then for all $\varepsilon \in (0,1)$, the ε-mixing time of the symmetric simple exclusion process with open boundaries satisfies*

$$\lim_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N^2 \log(N)} = \frac{4}{\pi^2}\,. \tag{6.3}$$

By symmetry, (6.3) holds for $p = \frac{1}{2}$, $\min(\alpha, \gamma) > 0$ and $\max(\beta, \delta) = 0$ as well. In particular, the symmetric simple exclusion process with one open boundary exhibits cutoff.

Together with Theorem 6.1, it is a natural conjecture that cutoff occurs for all choices of $\alpha, \beta, \gamma, \delta \geq 0$ when $p = \frac{1}{2}$ holds; see also Section 6.10 for a discussion.

6.2.2 Asymmetric simple exclusion process with one blocked entry

Next, consider the asymmetric simple exclusion process with $p > \frac{1}{2}$. When $\alpha > 0$, let

$$a = a(\alpha, \gamma, p) := \frac{1}{2\alpha}\left(2p - 1 - \alpha + \gamma + \sqrt{(2p-1-\alpha+\gamma)^2 + 4\alpha\gamma}\right) \tag{6.4}$$

and similarly, for $\beta > 0$, we set

$$b = b(\beta, \delta, p) := \frac{1}{2\beta}\left(2p - 1 - \beta + \delta + \sqrt{(2p-1-\beta+\delta)^2 + 4\beta\delta}\right) . \tag{6.5}$$

We study the case of one blocked entry, i.e., $\min(\alpha, \beta) = 0$ and $\max(\alpha, \beta) > 0$.

Theorem 6.3. *Let $p > \frac{1}{2}$ and $\gamma, \delta \geq 0$. If $\alpha = 0$ and $\beta > 0$, then for all $\varepsilon \in (0, 1)$, the ε-mixing time of the simple exclusion process with open boundaries satisfies*

$$\lim_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} = \frac{(\max(b,1)+1)^2}{(2p-1)\max(b,1)} . \tag{6.6}$$

For $\alpha > 0$ and $\beta = 0$

$$\lim_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} = \frac{(\max(a,1)+1)^2}{(2p-1)\max(a,1)} . \tag{6.7}$$

In particular, we see in both cases that cutoff occurs.

A key ingredient for the proof of Theorem 6.3 is to understand the creation of shocks. The shocks will travel at a linear speed. Heuristically, the mixing time corresponds to the time at which the shock hits the boundary, which justifies a sharp mixing behavior.

6.2.3 The reverse bias phase for the simple exclusion process

In contrast to the simple exclusion process where all boundary parameters are zero, there exists a regime of the asymmetric simple exclusion process with open boundaries with an exponentially large ε-mixing time. This case is known in the literature as the **reverse bias phase**; see [21]. This terminology is justified as the particles are forced by the boundary conditions to move against their natural drift direction.

Theorem 6.4. *Suppose that $\max(\alpha, \beta) = 0$ and $p \in \left(\frac{1}{2}, 1\right)$. Then for all $\varepsilon \in (0, \frac{1}{2})$*

$$\lim_{N\to\infty} \frac{\log\left(t^N_{\text{mix}}(\varepsilon)\right)}{N} = \log\left(\frac{p}{1-p}\right) \tag{6.8}$$

holds whenever $\min(\gamma, \delta) = 0$ and $\max(\gamma, \delta) > 0$. If $\min(\gamma, \delta) > 0$ holds, then

$$\lim_{N\to\infty} \frac{\log\left(t^N_{\text{mix}}(\varepsilon)\right)}{N} = \frac{1}{2}\log\left(\frac{p}{1-p}\right) . \tag{6.9}$$

6.2.4 The high and low density phase for the simple exclusion process

Now suppose that $\min(\alpha, \beta) > 0$ and $p > \frac{1}{2}$, so the quantities a and b from (6.4) and (6.5) are both well-defined. We distinguish three different regimes according to the density within the stationary distribution; see Section 6.3.3 with Figure 16 for more details. The regime $a > \max(b, 1)$ is called the **low density phase** of the exclusion process, while we refer to the regime $b > \max(a, 1)$ as the **high density phase**. The remaining case where $\max(a, b) \leq 1$ holds is called the **maximal current phase**. A visualization of these three phases in terms of the parameters a and b is given in Section 6.3.2 with Figure 15. Intuitively, the invariant distribution is an interpolation between two Bernoulli-product measures with densities $\frac{1}{1+a}$ and $\frac{b}{1+b}$, respectively, and we will see a justification of this claim in Lemma 6.12. The terminology low density phase, respectively high density phase, will be justified in Lemma 6.13, since the average density within the invariant measure stays below, respectively above $\frac{1}{2}$.

Theorem 6.5. *For parameters $\alpha, \beta > 0$ and $\gamma, \delta \geq 0$, as well as $p > \frac{1}{2}$, suppose that we are in the high density phase. Then there exists some $C_h = C_h(a, b, p) > 0$ such that the ε-mixing time of the simple exclusion process with open boundaries satisfies*

$$\frac{1}{2p-1} \leq \liminf_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} \leq \limsup_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} \leq C_h \tag{6.10}$$

for all $\varepsilon \in (0, 1)$. Similarly, when we are in the low density phase with parameters $\alpha, \beta > 0$ and $\gamma, \delta \geq 0$, as well as $p > \frac{1}{2}$, the ε-mixing time of the simple exclusion process with open boundaries satisfies

$$\frac{1}{2p-1} \leq \liminf_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} \leq \limsup_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} \leq C_\ell \tag{6.11}$$

for some $C_\ell = C_\ell(a, b, p) > 0$, and all $\varepsilon \in (0, 1)$. In particular, pre-cutoff occurs.

6.2.5 The triple point of the simple exclusion process

An interesting special case is the **triple point** where $p > \frac{1}{2}$ and $a = b = 1$ holds. Intuitively, the low-density phase, the high density phase and maximal current phase coexist at the triple point, and it can be shown that the process gives raise to the KPZ equation under a suitable scaling [39, 108]. We have the following bound on the mixing time; see also [120] for a recent improvement to $N^{3/2}$ when $p = 1$ and $\gamma = \delta = 0$.

Theorem 6.6. *Suppose that $p > \frac{1}{2}$ and $a = b = 1$ holds. Then there exists some constant $C = C(\alpha, \beta, \gamma, \delta, p)$ such that for all $\varepsilon \in (0, 1)$, the ε-mixing time of the simple exclusion process with open boundaries satisfies*

$$t^N_{\text{mix}}(\varepsilon) \leq CN^3 \,. \tag{6.12}$$

6.3 Preliminaries on the simple exclusion process with open boundaries

In this section, we collect basic properties and techniques which are specific for the simple exclusion process with open boundaries. This includes a modified canonical coupling, currents and invariant measures as well as a hitting time bound. Motivations and applications of these techniques come from probability theory, statistical mechanics and combinatorics. We will give a brief background to the different techniques and point out where we require generalizations. For a discussion of general techniques to estimate the mixing time of exclusion processes, we refer to Section 4.

6.3.1 The canonical coupling for the SEP with open boundaries

In Section 3.3, we introduced the canonical coupling for the simple exclusion process on the segment. We will now present a modified canonical coupling for the simple exclusion process with open boundaries:

We place rate 1 Poisson clocks on all edges $e \in E$. Whenever the clock of an edge $e = \{x, x+1\}$ rings, we sample a Uniform-$[0,1]$-random variable U independently of all previous samples and distinguish two cases.

- If $U \leq p$ and $\eta(x) = 1 - \eta(x+1) = 1$ holds, we move the particle at site x to site $x+1$ in configuration η.
- If $U > p$ and $\eta(x) = 1 - \eta(x+1) = 0$ holds, we move the particle at site $x+1$ to site x in configuration η.

In addition, we place a rate α Poisson clock (a rate γ Poisson clock) on the vertex 1. Whenever a clock rings, we place a particle (an empty site) at site 1, independently of the current value of $\eta(1)$. Similarly, we put a rate β Poisson clock (a rate δ Poisson clock) on the vertex N. Whenever this clock rings, we place an empty site (a particle) at site N independently of the current value of $\eta(N)$.

Observe that we obtain a grand coupling by using the same clocks for the exclusion processes. Moreover, we can extend this construction to simple exclusion processes with open boundaries, which differ in the parameters $p, \alpha, \beta, \gamma, \delta$; see also Section 5.2 for a similar extension for the simple exclusion process in random environment. More precisely, suppose that two simple exclusion processes $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ with open boundaries on a segment of size N agree in all parameters, except for $\alpha > 0$ in $(\eta_t)_{t\geq 0}$ and some $\alpha' > \alpha$ in $(\zeta_t)_{t\geq 0}$. We then use the same rate α Poisson clocks to determine in both processes when a particle enters at the left-hand side boundary. In addition, insert particles at the leftmost site in $(\zeta_t)_{t\geq 0}$ according to an independent rate $(\alpha' - \alpha)$ Poisson clock. A similar construction applies for the remaining parameters.

The component-wise partial order revisited

We saw in Proposition 3.8 that the canonical coupling of the simple exclusion process preserves the component-wise partial order $\succeq_{\mathrm{c}}$ from (3.13). A similar statement holds for the modified canonical coupling, even if we allow for some differences in the boundary parameters $\alpha, \beta, \gamma, \delta$. These observations are formalized in the following lemma. Since it follows immediately from the above construction, we omit a formal proof. With a slight abuse of notation, we denote the law of this modified canonical coupling for the simple exclusion process with open boundaries again by $\mathbf{P}$.

Lemma 6.7. *Consider two exclusion processes $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ on the segment of size N with parameters $(p, \alpha, \beta, \gamma, \delta)$ and $(p, \alpha', \beta', \gamma', \delta')$, respectively. Suppose that*

$$\alpha \geq \alpha' \quad \beta \leq \beta' \quad \gamma \leq \gamma' \quad \text{and} \quad \delta \geq \delta' \tag{6.13}$$

holds, then the canonical coupling $\mathbf{P}$ satisfies

$$\mathbf{P}\left(\eta_t \succeq_{\mathrm{c}} \zeta_t \text{ for all } t \geq 0 \mid \eta_0 \succeq_{\mathrm{c}} \zeta_0\right) = 1\,. \tag{6.14}$$

Let $\mathbf{1}$ and $\mathbf{0}$ be the configurations in Ω_N containing only particles and empty sites, respectively, and observe that these two configurations form the unique maximal and minimal elements with respect to the partial order $\succeq_{\mathrm{c}}$ on Ω_N. Hence, the following lemma is a consequence of Lemma 6.7 and Corollary 4.2.

Lemma 6.8. *For a simple exclusion process with open boundaries and ε-mixing time $t^N_{\text{mix}}(\varepsilon)$, let τ denote the first time, at which the processes started from $\mathbf{1}$ and $\mathbf{0}$, respectively, agree within the canonical coupling $\mathbf{P}$. If for some $s \geq 0$*

$$\mathbf{P}(\tau \geq s) \leq \varepsilon \tag{6.15}$$

holds, then the ε-mixing time satisfies $t^N_{\text{mix}}(\varepsilon) \leq s$.

The partial order via height functions revisited

When $\max(\alpha, \gamma) = 0$ or $\max(\beta, \delta) = 0$ holds, we will see that the partial order $\succeq_{\mathrm{h}}$ from (3.29) extends to the simple exclusion process with open boundaries on Ω_N. More precisely, when $\max(\alpha, \gamma) = 0$ holds, we use the definition (3.29) for all configurations $\eta, \zeta \in \Omega_N$ to define the partial order $\succeq_{\mathrm{h}}$ on Ω_N. For $\max(\beta, \delta) = 0$, we apply the definition (3.29) to the simple exclusion process with open boundaries and parameters $(1-p, 0, \gamma, 0, \alpha)$. Note that the partial order $\succeq_{\mathrm{h}}$ again arises from the height function representation; see (3.33). In the following, it will be convenient to treat the height function of a given configuration $\eta \in \{0,1\}^N$ as a function $h_\eta \colon \{0, 1, \dots, 2N\} \to \mathbb{R}$ on

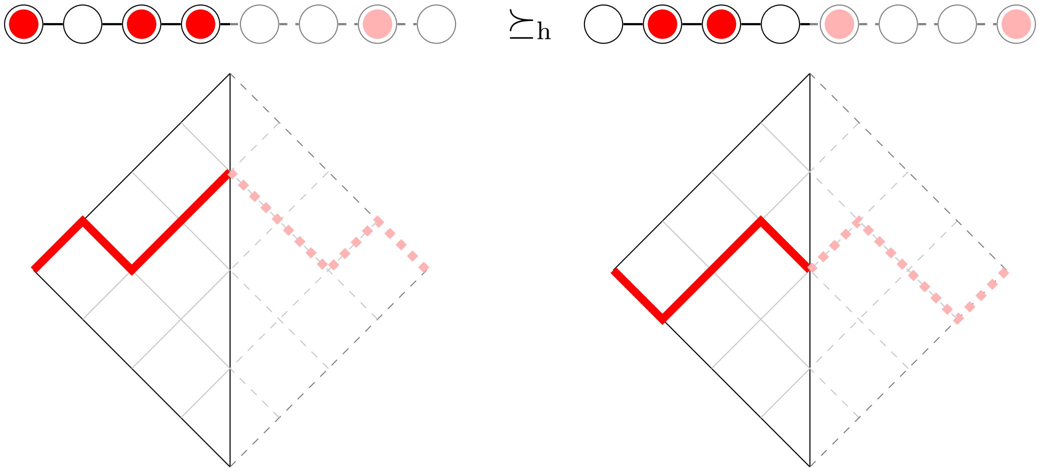

Figure 14: Paths of the height function for two ordered instances of the simple exclusion process for $N = 4$ and with particles entering and exiting only at the right-hand side of the segment.

the segment of size $2N$ given by

$$h_\eta(x) := \sum_{i=1}^{x} 2\left[\eta(i)\mathbb{1}_{\{i\leq N\}} + (1-\eta(2N+1-i))\mathbb{1}_{\{i>N\}}\right] - x \tag{6.16}$$

for all $x \in \{0,1,\dots,2N\}$. Note that $h_\eta(0) = h_\eta(2N) = 0$ by construction. Intuitively, we can think of the height function as having a mirror image on $\{N+1,\dots,2N\}$; see Figure 14 for a visualization using the path representation of the height function. For all $N \in \mathbb{N}$, we see that a pair of configurations satisfies $\eta \succeq_{\mathrm{h}} \zeta$ if and only if $h_\eta(x) \geq h_\zeta(x)$ holds for all $x \in [N]$. Again, the canonical coupling will be monotone with respect to the partial order $\succeq_{\mathrm{h}}$. In addition, similar to Lemma 6.7, we allow for differences in the parameters $p, \alpha, \beta, \gamma, \delta$. This is formalized in the next lemma. Since the statement is again immediate from the above construction of the modified canonical coupling, so we omit the proof.

Lemma 6.9. *Consider two exclusion processes $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ on the segment of size N with parameters $(p,\alpha,\beta,\gamma,\delta)$ and $(p',\alpha',\beta',\gamma',\delta')$, respectively. Suppose that*

$$p \leq p' \quad \alpha = 0 \quad \beta \leq \beta' \quad \gamma = 0 \quad \delta \geq \delta' \tag{6.17}$$

or

$$p \leq p' \quad \alpha \geq \alpha' \quad \beta = 0 \quad \gamma \leq \gamma' \quad \delta = 0 \tag{6.18}$$

holds. Then the canonical coupling $\mathbf{P}$ *of the two processes satisfies*

$$\mathbf{P}\left(\eta_t \succeq_{\mathrm{h}} \zeta_t \text{ for all } t \geq 0 \mid \eta_0 \succeq_{\mathrm{h}} \zeta_0\right) = 1\,. \tag{6.19}$$

6.3.2 The current of the SEP with open boundaries

In this section, we study of the current of the simple exclusion process with open boundaries. Currents are one of the main objects for the exclusion process in statistical mechanics with deep connections to second class particles; see [11, 53, 137]. Intuitively, the current formalizes the way of counting the number of particles which pass through the segment over time; see also Section 7.2 for a more general definition. We will use current arguments to prove the upper bounds in Theorem 6.3 and Theorem 6.5.

For $p \in \left(\frac{1}{2}, 1\right]$, assume that $\min(\alpha, \beta) > 0$ holds. On the segment of size N, let J_t^{N+} be the number of particles which have entered at the left-hand side of the segment by time t and let J_t^{N-} be the number of particles which have exited at the left-hand side of the segment by time t. Let $(J_t^N)_{t \geq 0}$ with

$$J_t^N := J_t^{N+} - J_t^{N-} \quad \text{for all } t \geq 0 \tag{6.20}$$

be the **current** of the simple exclusion process with open boundaries. Similarly, one could define the current with respect to the net number of particles crossing the right-hand side of the segment, leading to the same long-term behavior. The following lemma states an asymptotic bound on the current. We obtain it directly from the results in Section 6 of [137] and Theorem 3.5.

Lemma 6.10. *Recall the definition of a and b from* (6.4) *and* (6.5)*, and set*

$$J = J(a, b, p) := \begin{cases} (2p-1)\frac{a}{(1+a)^2} & \text{if } a > \max(b, 1) \\ (2p-1)\frac{b}{(1+b)^2} & \text{if } b > \max(a, 1) \\ (2p-1)\frac{1}{4} & \text{if } \max(a, b) \leq 1\,. \end{cases} \tag{6.21}$$

Then the current $(J_t^N)_{t \geq 0}$ of the simple exclusion process with open boundaries satisfies

$$\lim_{t \to \infty} \frac{J_t^N}{t} = J_N \tag{6.22}$$

almost surely for some deterministic sequence $(J_N)_{N \in \mathbb{N}}$ with $\lim_{N \to \infty} J_N = J$. In particular, the current is maximized when $\max(a, b) \leq 1$; see also Figure 15.

We refer to J_N as the **flux** of the simple exclusion process with open boundaries.

6.3.3 Invariant measures of the SEP with open boundaries

In this section, we focus on the stationary distribution μ of the simple exclusion process with open boundaries. The following result, which is adopted from [23], shows that under certain conditions on the boundary parameters, the invariant distribution has a product structure. In general, μ can not be stated in a simple closed form; see [25, 44].

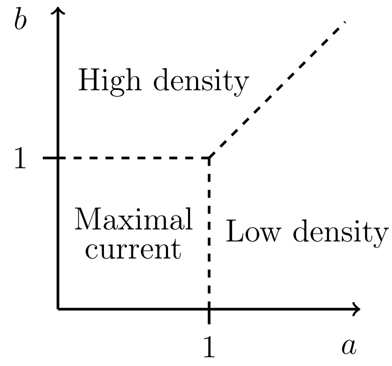

Figure 15: Visualization of the three main different regimes of current of the simple exclusion process with open boundaries.

Lemma 6.11 (c.f. [23], Proposition 2)**.** *Suppose that* $\min(\alpha,\beta)>0$ *and* $a=\frac{1}{b}$ *holds for* a *and* b *given in* (6.4) *and* (6.5)*. Then for every configuration* $\eta\in\Omega_N$*,*

$$\mu(\eta)=\frac{1}{(\alpha+\beta+\gamma+\delta)^N}(\alpha+\delta)^{|\eta|}(\beta+\gamma)^{N-|\eta|}=\left(\frac{1}{1+a}\right)^{|\eta|}\left(\frac{a}{1+a}\right)^{N-|\eta|} \tag{6.23}$$

where $|\eta|:=\sum_{i=1}^N\eta(i)$ *denotes the number of particles in configuration* η*.*

Next, we compare the stationary measure μ to Bernoulli-ρ-product measures ν_ρ for some $\rho\in[0,1]$ on Ω_N. Recall the notion of stochastic domination from (4.35).

Lemma 6.12. *Suppose that* $\min(\alpha,\beta)>0$ *holds. Then the stationary distribution* μ *of the simple exclusion process with open boundaries satisfies*

$$\nu_{c_{\max}}\succeq_{\mathrm{c}}\mu\succeq_{\mathrm{c}}\nu_{c_{\min}} \tag{6.24}$$

where

$$c_{\min}:=\min\left(\frac{1}{1+a},\frac{b}{1+b}\right)\quad\textit{and}\quad c_{\max}:=\max\left(\frac{1}{1+a},\frac{b}{1+b}\right). \tag{6.25}$$

Proof. We consider only $\mu\succeq_{\mathrm{c}}\nu_{c_{\min}}$ for $c_{\min}=\frac{b}{1+b}$ as the remaining cases are similar. In this case, we have $a\le\frac{1}{b}$, and we recall $a=a(\alpha,\gamma,p)$ from (6.4). Observe that a is decreasing in α and note that we can choose some $\alpha'\in(0,\alpha]$ such that $a':=a(\alpha',\gamma,p)$ satisfies $a'=\frac{1}{b}$. We conclude using Lemma 6.7 and Lemma 6.11. □

Note that Lemma 6.12 is motivated by treating the simple exclusion process with open boundaries as having reservoirs at both ends with densities $\frac{1}{1+a}$ and $\frac{b}{1+b}$, respectively, and μ interpolating between both sides. The next result characterizes how the interpolation within the stationary distribution μ is realized. Using Lemma 6.10 and Lemma 6.12, it follows from the same arguments as Theorem 3.29 in [94, Part III].

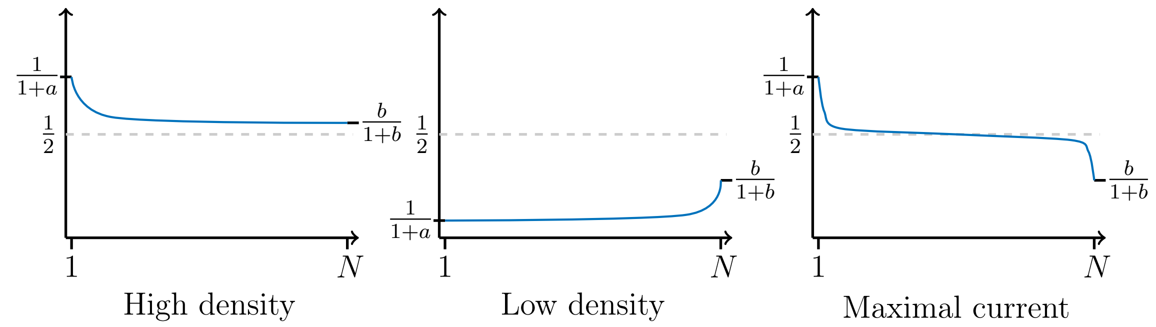

Figure 16: Visualization of the three main different regimes of densities of the simple exclusion process with open boundaries on a segment of size N. In the high density phase, we see that the average density stays above $\frac{1}{2}$, in the low density phase, the average density stays below $\frac{1}{2}$, and in the maximum current phase, the average density is close to $\frac{1}{2}$; see also Lemma 6.13.

Lemma 6.13. *Suppose that* $\min(\alpha, \beta) > 0$ *holds. Let* $(x_N)_{N\in\mathbb{N}}$ *be a sequence with* $\min(x_N, \frac{N}{2} - x_N) \to \infty$ *for* $N \to \infty$. *Further, let* μ_N *denote the measure on* $\{0,1\}^{\mathbb{N}}$ *given on the sites* $1, \ldots, N - 2x_N$ *by the restriction of* μ *to* $[x_N, N - x_N]$, *and by the Dirac measure on empty sites everywhere else. Then*

$$\lim_{N\to\infty} \mu_N = \begin{cases} \nu_{\frac{1}{1+a}} & \text{if } a > \max(b,1) \\ \nu_{\frac{b}{1+b}} & \text{if } b > \max(a,1) \\ \nu_{\frac{1}{2}} & \text{if } \max(a,b) \leq 1, \end{cases} \tag{6.26}$$

where the limit is with respect to weak convergence, and the product measures $\nu_{\cdot}$ *are defined on* $\{0,1\}^{\mathbb{N}}$*; see also* (3.9).

When particles are allowed to enter and exit only from one side of the segment, the measure μ is reversible and can be given explicitly. Suppose that particles are only allowed to enter and exit at the right-hand side, i.e., $\max(\alpha, \gamma) = 0$ holds. A similar formula will hold when $\max(\beta, \delta) = 0$. For $p \in (0, 1]$ and $\min(\beta, \delta) > 0$, consider μ with

$$\mu(\eta) = \frac{1}{Z_N} \left(\frac{\delta}{\beta}\right)^{|\eta|} \cdot \prod_{i=1}^{|\eta|} \left(\frac{1-p}{p}\right)^{z_i} \quad \text{for all } \eta \in \Omega_N, \tag{6.27}$$

where z_i denotes the distance of the i^{th} particle from site N and Z_N is a normalization constant. Then μ is reversible for the process $(\eta_t)_{t\geq 0}$. When $\min(\beta, \delta) = 0$ holds, μ is the Dirac measure on $\mathbf{1}$ if $\beta = 0$, and on $\mathbf{0}$ if $\delta = 0$. Note that for the simple exclusion process on the segment where all boundary parameters are equal to zero, a similar formula for the reversible measure is given in (3.28).

6.3.4 Hitting times for the SEP in the blocking measure

The last preliminary result, which we will present in this section, is a bound on the position of the leftmost particle and the rightmost empty site in the simple exclusion process on $\mathbb{Z}$. Intuitively, this allows us to determine for which time horizons we can couple the simple exclusion process on the segment with the simple exclusion process on $\mathbb{Z}$. Recall from Section 5.3.1 that $L(\eta)$ and $R(\eta)$ denote the positions of the leftmost particle and the rightmost empty site in a configuration η, respectively. In Sections 6.6 and 6.7, we will use the following lemma which bounds $L(\cdot)$ and $R(\cdot)$ when starting in the blocking measure $\nu_{(0)}$ on the set A_0, defined in Section 3.4.2.

Lemma 6.14. *For $p \in (\frac{1}{2}, 1)$, let $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ denote the simple exclusion process in A_0 with initial distribution $\nu_{(0)}$. There exists a constant $C = C(p) > 0$ such that for any $\varepsilon \in (0, \frac{1}{2})$ and all $x \geq 0$ sufficiently large,*

$$\mathbb{P}_{\nu_{(0)}}\left(\max\left(R(\eta_t^{\mathbb{Z}}), -L(\eta_t^{\mathbb{Z}})\right) \leq x \text{ for all } t \in \left[0, \frac{\varepsilon C}{x}\left(\frac{p}{1-p}\right)^{x}\right]\right) \geq 1 - 2\varepsilon\,. \tag{6.28}$$

Recall (4.8). In the following, we abbreviate $\tau_0 := \tau_{\text{hit}}^{\cdot}(\vartheta_0)$, and write τ_A for the hitting time of a set of configurations A.

Lemma 6.15. *For $x \in \mathbb{N}$, recall $\Theta_{2x+1,x} \in A_0$ from (4.20), where the particles are placed on $\{-x, \dots, -1\} \cup \{x+1, \dots\}$. Let $\Theta_{2x+1,x}$ be the initial state for the simple exclusion process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ on A_0. Then there exists some $c > 0$ such that $\mathbb{E}_{\Theta_{2x+1,x}}[\tau_0] \leq cx$ holds for all $x \geq 0$.*

Proof. For all $x \geq 0$, we define B_x to be the set of configurations

$$B_x := \{\eta \in A_0 \colon \max(R(\eta), -L(\eta)) > x\} \tag{6.29}$$

and denote for all $s \geq 0$ by $\tau^s_{B_x^{\mathrm{c}}} := \inf\{t \geq s \colon \eta_t \notin B_x\}$ the first time after time s when we hit the set B_x^{c}. We claim that there exists some $\tilde{c} > 0$ such that for all $x, s \geq 0$

$$E_{\Theta_{2x+1,x}}[\tau^s_{B_x^{\mathrm{c}}}] - s \leq \tilde{c}\,. \tag{6.30}$$

To see this, let $(\eta_t^x)_{t\geq 0}$ and $(\eta_t^{-x-1})_{t\geq 0}$ be two exclusion processes on A_x and A_{-x-1}, started from the blocking measure, respectively. Using Corollary 4.12, we note that

$$\mathbf{P}\left(R(\eta_t^{\mathbb{Z}}) \leq R(\eta_t^x) \text{ and } L(\eta_t^{\mathbb{Z}}) \geq L(\eta_t^{-x-1}) \text{ for all } t \geq 0\right) = 1 \tag{6.31}$$

holds with respect to the canonical coupling $\mathbf{P}$; see also Figure 17. Moreover, note that $(\eta_t^x, \eta_t^{-x-1})_{t\geq 0}$ is a stationary and ergodic Feller process for which the state $(\vartheta_x, \vartheta_{-x-1})$ has a strictly positive probability in equilibrium, and that $\tau^s_{B_x^{\mathrm{c}}} \leq T$ whenever $(\eta_t^x, \eta_t^{-x-1})_{t\geq 0}$ is in the state $(\vartheta_x, \vartheta_{-x-1})$ at time $T \geq s$. We conclude (6.30) using Kac's lemma for the embedded discrete chain of $(\eta_t^x, \eta_t^{-x-1})_{t\geq 0}$; see Theorem 21.12 in [91], and a time-change. Next, by Theorem 1.9 in [19],

$$\mathbb{P}_{\Theta_{2x+1,x}}(\tau_0 \leq c_1 x) \geq c_2 \tag{6.32}$$

holds for all $x \geq 0$ with constants $c_1, c_2 > 0$. We claim that together with (6.30), this yields

$$\mathbb{E}_{\Theta_{2x+1,x}}[\tau_0] \leq c_2 c_1 x + (1-c_2)\left(c_1 x + \tilde{c} + \mathbb{E}_{\Theta_{2x+1,x}}[\tau_0]\right) . \tag{6.33}$$

To see this, note that with probability at least c_2, we hit ϑ_0 by time $c_1 x$. Suppose that ϑ_0 was not hit by time $c_1 x$, then we can wait until hitting B_x^{c} and use (6.30). Since $\eta \preceq_{\mathrm{h}} \Theta_{2x+1,x}$ holds for all $\eta \in B_x^{\mathrm{c}}$, the hitting time of ϑ_0 starting from the configuration at time $\tau_{B_x^{\mathrm{c}}}^{c_1 x}$ is stochastically dominated by the hitting time of ϑ_0 when starting from $\Theta_{2x+1,x}$. Now take expectations to get (6.33). Since $\mathbb{E}_{\Theta_{2x+1,x}}[\tau_0] < \infty$, we conclude by solving (6.33) for $\mathbb{E}_{\Theta_{2x+1,x}}[\tau_0]$. □

Next, we study the **return time** $\tau_{B_x}^{+} := \inf\left\{t \geq \tau_{B_x^{\mathrm{c}}} : \eta_t \in B_x\right\}$ to the set B_x.

Lemma 6.16. *There exists some $C > 0$ such that for all $x \geq 0$*

$$E_{\nu_{(0)}}[\tau_{B_x}^{+}] \geq \nu_{(0)}(\vartheta_0)\mathbb{E}_{\vartheta_0}[\tau_{B_x}] \geq \frac{C}{x}\left(\frac{p}{1-p}\right)^x . \tag{6.34}$$

Proof. Observe that an exclusion process in B_x^{c} can change its state if and only if a clock on the sites $[-x,x]$ rings. Hence, using Kac's lemma for the embedded discrete chain, we see that

$$\mathbb{E}_{\nu_{(0)}(\,\cdot\,|B_x)}[\tau_{B_x}^{+}] \geq \frac{1}{(2x+1)\nu_{(0)}(B_x)} \geq \frac{c_1}{x}\left(\frac{p}{1-p}\right)^x \tag{6.35}$$

holds for all $x \geq 0$ and some constant $c_1 > 0$. Since $\vartheta_0 \preceq_{\mathrm{h}} \eta$ for all $\eta \in B_x^{\mathrm{c}}$

$$\begin{aligned}\mathbb{E}_{\nu_{(0)}(\,\cdot\,|B_x)}[\tau_{B_x}^{+}] &= \sum_{\zeta\in B_x^{\mathrm{c}}}\left(\mathbb{E}_{\nu_{(0)}(\,\cdot\,|B_x)}\left[\tau_\zeta \mid \eta_{\tau_{B_x^{\mathrm{c}}}} = \zeta\right] + \mathbb{E}_\zeta[\tau_{B_x}]\right)\mathbb{P}_{\nu_{(0)}(\,\cdot\,|B_x)}(\eta_{\tau_{B_x^{\mathrm{c}}}} = \zeta) \\ &\leq \mathbb{E}_{\nu_{(0)}(\,\cdot\,|B_x)}[\tau_0] + \mathbb{E}_{\vartheta_0}[\tau_{B_x}] .\end{aligned} \tag{6.36}$$

Recall that $\eta \preceq_{\mathrm{h}} \Theta_{2x+1,x}$ for all $\eta \in B_x^{\mathrm{c}}$. Note that there exists some $c_2 > 0$ such that

$$\begin{aligned}\mathbb{E}_{\nu_{(0)}(\,\cdot\,|B_x)}[\tau_0] &= \sum_{y\geq x}\sum_{\eta\in B_y\setminus B_{y+1}} \mathbb{E}_\eta[\tau_0]\nu_{(0)}(\eta|B_x) \\ &\leq \sum_{y\geq x}\mathbb{E}_{\Theta_{2y+3,y+1}}[\tau_0]\nu_{(0)}(B_y|B_x) \leq c_2 x\end{aligned} \tag{6.37}$$

holds for all $x \geq 0$, using Lemma 6.15 and the fact that $\nu_{(0)}(B_y|B_x) \leq c_3((1-p)/p)^{y-x}$ for some $c_3 > 0$ in the last inequality. By (6.35), (6.36) and (6.37), we conclude. □

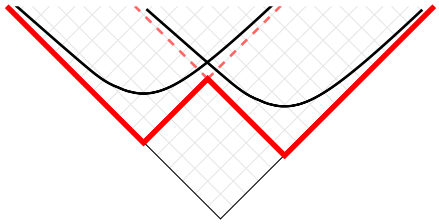

Figure 17: The initial state $\Theta_{2x+1,x}$ of $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ is shown in red. The position of the leftmost particle in $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ is stochastically dominated by the position of the leftmost particle in $(\eta_t^{-x-1})_{t\geq 0}$ shown in black. A similar statement holds for the position of the rightmost empty site in $(\eta_t^{\mathbb{Z}})_{t\geq 0}$.

Proof of Lemma 6.14. We will prove Lemma 6.14 by contradiction. Take $C>0$ from Lemma 6.16 and assume that (6.28) is not true. Then, using the general fact that for arbitrary events A and B, the inequality $\mathbb{P}(A\cap B)\geq \mathbb{P}(A)-\mathbb{P}(B^{\mathrm{c}})$ holds,

$$q := \mathbb{P}_{\nu_{(0)}}\left(\eta_t \in B_x \text{ for some } t \in \left[0, \frac{\varepsilon C}{x}\left(\frac{p}{1-p}\right)^x\right] \text{ and } \eta_0 \in B_x^{\mathrm{c}}\right) > 2\varepsilon - \nu_{(0)}(B_x)\,.$$

A similar argument as for (6.33) yields

$$\mathbb{E}_{\nu_{(0)}}[\tau^+_{B_x}] \leq q\frac{\varepsilon C}{x}\left(\frac{p}{1-p}\right)^x + (1-q)\left(\frac{\varepsilon C}{x}\left(\frac{p}{1-p}\right)^x + \mathbb{E}_{\nu_{(0)}}[\tau^+_{B_x}]\right)\,. \tag{6.38}$$

Solving (6.38) for $\mathbb{E}_{\nu_{(0)}}[\tau^+_{B_x}]$ and using the definition of $\nu_{(0)}$ for q, we see that for all x large enough $\mathbb{E}_{\nu_{(0)}}[\tau^+_{B_x}] < \varepsilon C x^{-1}(p/(1-p))^x$ holds. This contradicts Lemma 6.16. □

6.3.5 Outline of the proof of the main results

In order to show the main results on the mixing time for the simple exclusion process with open boundaries, we build on the results from Sections 3 and 4 as well as on the above preliminaries for the simple exclusion process with open boundaries.

In Sections 6.4 and 6.5, we study mixing times of the symmetric simple exclusion process with open boundaries. Lower bounds are achieved using a continuous-time version of a generalization of Wilson's lemma from [105]; see Lemma 4.10. A general upper bound follows from a comparison to independent simple random walks using the interchange process. This bound is then refined in the special case of one open boundary, following closely the ideas of Lacoin in [84].

The analysis of mixing times for the asymmetric simple exclusion process is carried out in Sections 6.6 to 6.9. In Section 6.6, we use second class particles, current estimates and a comparison to the exclusion process on $\mathbb{Z}$ to investigate mixing times for the ASEP with one blocked entry. The reverse bias phase is considered in Section 6.7

requiring second class particle estimates, hitting time bounds for the simple exclusion process on the integers. In Section 6.8 we study the simple exclusion process in the high density phase and in the low density phase using multi-species exclusion processes, stochastic orderings and the censoring inequality. The triple point for the simple exclusion process is treated in Section 6.9 using a symmetrization argument. We conclude this section with a discussion of related open problems and conjectures.

6.4 Lower bounds for the SSEP with open boundaries

In this section, we prove the lower bounds in Theorems 6.1 and 6.2 using a generalized version of Wilson's lemma with approximate eigenfunctions; see Lemma 4.10. For the construction of the approximate eigenfunctions, we have the following intuition. Observe that for all choices of boundary parameters and initial configurations η, $(f_\eta(x,t))_{x\in[N],t\geq 0}$ given by

$$f_\eta(x,t) := \mathbb{E}_\eta[\eta_t(x)] \qquad \text{for all } x \in [N] \text{ and } t \geq 0$$

solves a discrete heat equation, where we see either discrete Neumann or Dirichlet boundary conditions for closed or open endpoints, respectively. In the following, we consider a simple exclusion process with open boundaries at both endpoints and compare it to a simple exclusion process on the circle of length $2N$ with N particles. On the circle, the eigenfunctions are sine and cosine waves, where the length of the circle is a multiple of the period length; see Lemma 2.2 in [85] and Lemma 1 in [139]. Our approximate eigenfunctions will be stretched and shifted versions of these eigenfunctions. With a slight abuse of notation, extend each $\eta \in \Omega_N$ to $\Omega_{2N,N}$ given in (3.27) by

$$\eta(x) := 1 - \eta(2N+1-x) \ \text{ for all } x \in \{N+1,\dots,2N\}.$$

Lemma 6.17. *Recall that* $p = \frac{1}{2}$ *and assume that* $\max(\alpha,\gamma) > 0$ *and* $\max(\beta,\delta) > 0$ *holds. Set*

$$C := \frac{1}{2(\alpha+\gamma)} - \frac{1}{2} \quad \text{and} \quad D := \frac{1}{2} - \frac{1}{2(\beta+\delta)} \tag{6.39}$$

and define $M := N + C + D$. *Let* $\phi\colon \mathbb{Z}/(2N)\mathbb{Z} \to \mathbb{R}$ *be given by*

$$\phi(x) := \sin\left(\left(x + C - \frac{1}{2}\right)\frac{\pi}{M}\right) \tag{6.40}$$

for all $x \in [N]$, *and set* $\phi(x) = -\phi(2N+1-x)$ *for all* $x \in \{N+1,\dots,2N\}$. *Moreover, we let* $\lambda_N := 1 - \cos(\frac{\pi}{M})$ *and define*

$$\Phi_N(\eta) := \sum_{x=1}^{2N} \eta(x)\phi(x) + \frac{\phi(1)}{\lambda_N}(1-2\alpha) + \frac{\phi(N)}{\lambda_N}(1-2\delta) \tag{6.41}$$

for all $\eta \in \Omega_N$. Then Φ_N satisfies the conditions of Lemma 4.10 for $\lambda = \lambda_N$, for some c of order N^{-3}, some R of order N^{-1} and $\|\Phi_N\|_\infty$ of order N. In particular, under the above assumptions the lower bound stated in Theorem 6.1 holds.

Proof. Using trigonometric identities, $(\Delta\phi)(x) = -\lambda_N \phi(x)$ holds for $x \in \{2, \dots, N-1\}$. Here, Δ is the discrete Laplace operator on the circle of length $2N$, i.e., for all functions $f \colon \mathbb{Z}/(2N)\mathbb{Z} \to \mathbb{R}$, we set

$$(\Delta f)(x) := \frac{1}{2}(f(x-1) + f(x+1)) - f(x) \quad \text{for all } x \in \mathbb{Z}/(2N)\mathbb{Z}. \tag{6.42}$$

By our choice of C and D, note that for all N large enough and some $c_1, c_2 > 0$

$$\begin{aligned} |(\Delta\phi)(1) + (1-\alpha-\gamma)\phi(1) + \lambda_N \phi(1)| &\leq \frac{c_1}{M^3} \\ |(\Delta\phi)(N) + (1-\beta-\delta)\phi(N) + \lambda_N \phi(N)| &\leq \frac{c_2}{M^3} \end{aligned}$$

holds using the Taylor expansion of the sine and trigonometric identities. Since

$$\begin{aligned} \sum_{x=1}^{N} (\mathcal{L}\eta)(x)\phi(x) = \sum_{x=1}^{N} (\Delta\eta)(x)\phi(x) &+ \phi(1)\Big(\eta(1)(1-\alpha-\gamma) + \alpha - \frac{1}{2}\Big) \\ &+ \phi(N)\Big(\eta(N)(1-\beta-\delta) + \gamma - \frac{1}{2}\Big) \end{aligned}$$

and $(\mathcal{L}\eta)(x) = -(\mathcal{L}\eta)(2N+1-x)$ for all $x \in [N]$, we see that

$$|(-\mathcal{L})\Phi_N(\eta) - \lambda_N \Phi_N(\eta)| \leq \frac{2(c_1+c_2)}{M^3}$$

holds. This gives condition (4.29) in Lemma 4.10. To verify condition (4.30), we follow the ideas of the proof of Lemma 2.2 in [85] for the simple exclusion process on the circle.

Observe that the process $(\Phi(\eta_t))_{t \geq 0}$ can change its value only whenever an edge or a boundary vertex is updated. This happens at rate N' where $N' := N - 1 + \alpha + \beta + \gamma + \delta$. For two configurations η and η' which differ by at most one transition,

$$|\Phi(\eta) - \Phi(\eta')| \leq 2 \max_{x \in [2N]} |\phi(x) - \phi(x+1)| \leq \frac{c_3}{M}$$

holds for some constant $c_3 = c_3(C, D) > 0$. Combining these observations, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\left[\langle M \rangle_t\right] \leq N' \left(\frac{c_3}{M}\right)^2 \quad \text{for all } t \geq 0.$$

This gives the desired bound on R of order N^{-1}. Since $\max(|\Phi_N(\mathbf{1})|, |\Phi_N(\mathbf{0})|)$ is of order N, Lemma 4.10 yields the lower bound stated in Theorem 6.1. □

Next, we consider the case of the simple exclusion process with open boundaries when particles are allowed to enter and exit the segment only at one side. Without loss of generality, assume that $\max(\alpha,\gamma)=0$ and $\max(\beta,\delta)>0$ holds. We will again construct approximate eigenfunctions for the simple exclusion process using the height function representation.

Lemma 6.18. *Recall that $p=\frac{1}{2}$ and assume that $\max(\alpha,\gamma)=0$ and $\max(\beta,\delta)>0$ holds. For D defined in (6.39), let $\tilde{\phi}\colon \mathbb{Z}/(2N)\mathbb{Z}\to\mathbb{R}$ be*

$$\tilde{\phi}(x) := \sin\left(\frac{x\pi}{2(N-D)}\right) \tag{6.43}$$

for all $x\in[N-1]$, and $\tilde{\phi}(x)=\tilde{\phi}(2N+1-x)$ for $x\in\{N+1,\dots,2N\}$. Further, set $\tilde{\lambda}_N := 1-\cos(\frac{\pi}{2(N-D)})$ and define

$$\tilde{\phi}(N) := \frac{1}{\beta+\delta-\tilde{\lambda}_N}\tilde{\phi}(N-1)\,. \tag{6.44}$$

Recall the height function for the simple exclusion process defined in (6.16) and set

$$\tilde{\Phi}_N(\eta) := \sum_{x=1}^{2N} h_\eta(x)\tilde{\phi}(x) + \frac{\tilde{\phi}(N)}{\tilde{\lambda}_N}(\delta-\beta) \tag{6.45}$$

for all $\eta\in\Omega_N$. Then $\tilde{\Phi}_N$ satisfies the conditions of Lemma 4.10 for $\lambda=\tilde{\lambda}_N$, some c of order N^{-4}, some R of order N and $\|\tilde{\Phi}_N\|_\infty$ of order N^2. In particular, the lower bound in Theorem 6.2 holds for $\max(\alpha,\gamma)=0$ and $\max(\beta,\delta)>0$.

Proof. A similar computation as in Lemma 6.17 shows that for N large enough

$$\left|(-\mathcal{L})\tilde{\Phi}_N(\eta)-\tilde{\lambda}_N\tilde{\Phi}_N(\eta)\right| \leq \frac{c_1}{(N-D)^4}$$

holds for some $c_1>0$ using the Taylor expansion of the sine and trigonometric identities. This gives condition (4.29) of Lemma 4.10 for some c of order N^{-4}. For condition (4.30), we again follow the ideas of the proof of Lemma 2.2 in [85]. Note that the process $(\tilde{\Phi}(\eta_t))_{t\geq 0}$ may change its value only when an edge or boundary vertex is updated. This happens at a rate N' Poisson clock for $N'=N-1+\beta+\delta$. For two configurations η and η' which differ by at most one transition, observe that $\tilde{\Phi}(\eta)$ and $\tilde{\Phi}(\eta')$ differ by at most 2. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\langle M\rangle_t\right] \leq 4N'.$$

holds for all $t\geq 0$.This gives the desired bound on R of order N. Since we have that $\max(|\tilde{\Phi}(\mathbf{1})|,|\tilde{\Phi}(\mathbf{0})|)$ is of order N^2, Lemma 4.10 yields the desired lower bound. □

Combining Lemma 6.17 and Lemma 6.18, this finishes the proof of the lower bounds in Theorem 6.1 and Theorem 6.2.

6.5 Upper bounds for the SSEP with open boundaries

In this section, we prove the upper bounds in Theorems 6.1 and 6.2. We start with a general upper bound for the simple exclusion process with open boundaries for $p = \frac{1}{2}$ and arbitrary boundary rates with $\max(\alpha, \beta, \gamma, \delta) > 0$. This bound is refined in Section 6.5.2 when particles enter and exit only at one side of the segment.

6.5.1 A general upper bound on the mixing time

We now prove the upper bound in Theorem 6.1. Without loss of generality, assume that $\max(\alpha, \beta, \gamma, \delta) = \alpha$ holds as we can flip the segment and use the particle-empty site symmetry, otherwise. Let $(\xi_t)_{t \geq 0}$ be the disagreement process when starting from $\mathbf{1}$ and $\mathbf{0}$, and let τ be the first time at which all second class particles have left. By Lemma 4.13, taking $t = \log \frac{N}{\varepsilon}$, it suffices to show that for some $c > 0$, and all $t \geq 0$,

$$\mathbf{P}\left(\tau > ctN^2\right) \leq N\mathrm{e}^{-t} . \tag{6.46}$$

Since $p = \frac{1}{2}$ and particles of the same type are indistinguishable, we can also describe the dynamics along the edges such that the values of the endpoints are swapped at rate 1, independently; see also (3.17). From this perspective, the second class particles perform continuous-time simple random walks with absorption on at least one of the boundaries. Using a comparison to the Gambler's ruin problem on $[N]$ with reflection at the right-hand side, we see that with probability at least $\frac{1}{2}$, a given second class particle gets either absorbed or reaches site 1 by time $2N^2$. Note that this bound does not depend on the starting point of the particle. Moreover, for a second class particle at site 1 at time t, with probability at least $\frac{1-\mathrm{e}^{-\alpha}}{e}$ the particle gets absorbed at the boundary until time $t+1$. Thus,

$$\mathbf{P}\left(\tau_* > 2N^2 + 1\right) \leq \frac{1 - \mathrm{e}^{-\alpha}}{2\mathrm{e}} \tag{6.47}$$

holds, where τ_* denotes the absorption time of a fixed second class particle in the above dynamics. Using (6.47), we see that

$$\mathbf{P}\left(\tau_* > t\frac{2e}{1 - \mathrm{e}^{-\alpha}}(2N^2 + 1)\right) \leq \left(1 - \frac{1 - \mathrm{e}^{-\alpha}}{2\mathrm{e}}\right)^{t\frac{2\mathrm{e}}{1-\mathrm{e}^{-\alpha}}} \leq \mathrm{e}^{-t} . \tag{6.48}$$

for all $t \in \mathbb{N}$. The inequality (6.46), and hence the upper bound in Theorem 6.1 follows from a union bound on the events in (6.48), and choosing c accordingly.

Remark 6.19. *Note that by a standard argument, the bound in* (6.46) *implies that any eigenvalue λ of the generator of the symmetric simple exclusion process with open boundaries must satisfy $|\lambda|^{-1} \leq cN^2$; see Corollary 12.7 in [91] for a similar statement for reversible, discrete-time Markov chains.*

6.5.2 Cutoff for the SSEP with one open boundary

In this section, we prove the upper bound in Theorem 6.2 using the ideas and results of [84]. Since large parts of the proof will follow verbatim from the arguments in Section 8 of [84] for the simple exclusion process, we will focus on presenting the required adjustments in the proof rather than giving full details. We start by collecting some technical results on the simple exclusion process with open boundaries. Together with Section 6.3, this will cover the corresponding preliminaries on the simple exclusion process in Section 6 of [84]. We then highlight how these results are used to adapt the arguments of [84] for the simple exclusion process with one open boundary.

Correlation properties of the SSEP with one open boundary

Our first preliminary result is the FKG-inequality as well as a corollary of Holley's inequality for the simple exclusion process with $p = \frac{1}{2}$ and one open boundary. For $\eta, \zeta \in \Omega_N$, we let $\min(\eta, \zeta)$ and $\max(\eta, \zeta)$ be the configurations in Ω_N which satisfy

$$h_{\min(\eta,\zeta)}(x) := \min\left(h_\eta(x), h_\zeta(x)\right) \quad \text{and} \quad h_{\max(\eta,\zeta)}(x) := \max\left(h_\eta(x), h_\zeta(x)\right) \quad (6.49)$$

for all $x \in [N]$, respectively. Note that $\min(\eta, \zeta)$ and $\max(\eta, \zeta)$ are indeed elements of Ω_N. Further, Ω_N equipped with these operations is a distributive lattice. By (6.27)

$$\mu(\min(\eta, \zeta)) = \min(\mu(\eta), \mu(\zeta)) \quad \text{and} \quad \mu(\max(\eta, \zeta)) = \max(\mu(\eta), \mu(\zeta))$$

holds when $\delta \geq \beta$ and similarly for $\delta < \beta$. With these insights, the next result follows from the same arguments as Proposition 6.1 in [84].

Lemma 6.20 (c.f. [84], Proposition 6.1)**.** *For any two functions f and g on Ω_N which are increasing with respect to the partial order $\succeq_{\mathrm{h}}$ on Ω_N,*

$$\int fg d\mu \geq \int f d\mu \int g d\mu \,. \quad (6.50)$$

Moreover, we have for any two increasing subsets $A \subseteq B$ of Ω_N with

$$\{\min(\eta, \zeta) | \eta \in A, \zeta \in B\} \subseteq B \quad (6.51)$$

that $\frac{1}{\mu(A)} \int\limits_A f d\mu \geq \frac{1}{\mu(B)} \int\limits_B f d\mu$ holds for any increasing function f.

Mean of the height function of the SSEP with one open boundary

Next, we give an estimate on the mean of the height function of the simple exclusion process with $p = \frac{1}{2}$ and one open boundary. For $\eta \in \Omega_N$, we define, recalling (6.16)

$$h^*_\eta(x) := h_\eta(x) - \min(x, 2N + 1 - x)\frac{\delta - \beta}{\delta + \beta} \quad (6.52)$$

for all $x \in [2N]$. Intuitively, h^*_η is the height function of η after subtracting the mean of the height function according to stationary measure.

Lemma 6.21 (c.f. [84], Lemma 6.4). *For all N large enough*

$$\max_{x\in\{0,\dots,2N\}} \left|\mathbb{E}_\eta\left[h^*_{\eta_t}(x)\right]\right| \leq 3N\mathrm{e}^{-\lambda t} \tag{6.53}$$

holds for all $t \geq 0$ and initial states $\eta \in \Omega_N$, where $\lambda = 1 - \cos(\frac{\pi}{2N+(\beta+\delta)^{-1}})$.

Proof. Observe that the function $f_\eta \colon \{0,\dots,2N\} \times \mathbb{R}^+_0 \to \mathbb{R}$ with $f_\eta(x,t) := \mathbb{E}_\eta[h^*_{\eta_t}(x)]$ for some initial state $\eta \in \Omega_N$ is a solution to the system of equations

$$\begin{cases} \partial_t f_\eta = (\mathbb{1}_{\{x\neq N\}} + \mathbb{1}_{\{x=N\}}(\beta+\delta))\Delta f_\eta \\ f_\eta(0,t) = f_\eta(2N,t) = 0 \end{cases} \tag{6.54}$$

for all $t \geq 0$ and $x \in \{0,\dots,2N\}$, with initial condition $f_\eta(x,0) = h^*_\eta(x)$. Here, Δ denotes the discrete Laplace operator which is defined in (6.42). Using Taylor expansion and a continuity argument, we see that there exists some $c_N \in [\frac{1}{2(\beta+\delta)} - 1, \frac{1}{2(\beta+\delta)}]$ such that for all N large enough, the function $g \colon \{0,1,\dots,2N\} \to \mathbb{R}$ with

$$g(x,t) := \left(\mathbb{1}_{\{x\leq N\}} \sin\left(\frac{x\pi}{2(N+c_N)}\right) + \mathbb{1}_{\{x>N\}} \sin\left(\frac{(2N-x)\pi}{2(N+c_N)}\right)\right)\mathrm{e}^{-\lambda_N t}$$

for all $x \in \{0,1,\dots,2N\}$, $t \geq 0$ and $\lambda_N := 1 - \cos\left(\frac{\pi}{2(N+c_N)}\right)$, is a solution to (6.54), with initial condition $g(x,0)$. Note that $\sin(z\pi/2) \geq \min(z, 2-z)$ holds for all $z \in [0,2]$, and $c_N \geq -1$. Hence, we obtain for sufficiently large N that

$$|h^*_\eta(x)| \leq 2\min(x, 2N-x) \leq 3Ng(x,0)$$

for all $x \in \{0,1,\dots,2N\}$ and $\eta \in \Omega_N$. Since this relation is preserved in (6.54) over time, we conclude. □

Scaling limits for the SSEP with one open boundary

We now study the law of the height function in equilibrium.

Lemma 6.22 (c.f. [84], Lemma 8.5). *Let η be a configuration sampled according to the stationary distribution μ of the simple exclusion process with $p = \frac{1}{2}$ and one open boundary. Then*

$$\left(\frac{\beta+\delta}{\sqrt{N\beta\delta}} h^*_\eta(xN)\right)_{x\in[0,1]} \tag{6.55}$$

converges for $N \to \infty$ in law to a standard Brownian motion on the interval $[0,1]$.

Proof. Using the explicit form of the invariant distribution μ in (6.27) for $p = \frac{1}{2}$ and the Binomial theorem, we see that the total number of particles $|\eta|$ in a configuration

η according to μ is Binomial-$(N, \frac{\delta}{\beta+\delta})$-distributed. Conditioning on the number of particles in the segment, observe that the number of particles in η until position y is Binomial-$(y, \frac{\delta}{\beta+\delta})$-distributed. The convergence for all finite marginals follows from the De Moivre-Laplace theorem. Together with a tightness argument, we obtain the convergence in law to a standard Brownian motion on $[0,1]$. □

Proof of the upper bound in Theorem 6.2

The upper bound in Theorem 6.2 is shown in two steps. First, we give an upper bound on the time it takes to reach equilibrium when starting from the two extremal configurations $\mathbf{1}$ and $\mathbf{0}$. In the next step, we consider a suitable coupling such that the exclusion processes started from $\mathbf{1}$ and $\mathbf{0}$ agree with high probability. This will be formalized in Lemma 6.23 and Lemma 6.24.

Lemma 6.23 (c.f. [84], Propositions 8.2). *Let $(\eta_t^{\mathbf{1}})_{t\geq 0}$ and $(\eta_t^{\mathbf{0}})_{t\geq 0}$ denote the simple exclusion processes with one open boundary and $p = \frac{1}{2}$ started from the configurations $\mathbf{1}$ and $\mathbf{0}$, respectively. For a given $\varepsilon > 0$, we set*

$$t_0 := \frac{4}{\pi^2} N^2 \log N \left(1 + \frac{\varepsilon}{2}\right) . \tag{6.56}$$

Then

$$\lim_{N\to\infty} \|P(\eta_{t_0}^{\mathbf{1}} \in \cdot) - \mu\|_{\mathrm{TV}} = 0 \quad \text{and} \quad \lim_{N\to\infty} \|P(\eta_{t_0}^{\mathbf{0}} \in \cdot) - \mu\|_{\mathrm{TV}} = 0 \tag{6.57}$$

holds for all $\varepsilon > 0$.

Sketch of the proof. The proof of Lemma 6.23 is divided into two main steps. First, we consider the simple exclusion process $(\eta_t)_{t\geq 0}$ with open boundaries for initial states $\mathbf{1}$ and $\mathbf{0}$ up to time t_2, where

$$t_2 := \frac{4}{\pi^2} N^2 \log N \left(1 + \frac{\varepsilon}{4}\right) . \tag{6.58}$$

We study the functions $(h^*_{\eta_t})_{t\geq 0}$, defined in (6.52), and evaluate them at $x_i := \lfloor 2iN/K \rfloor$ for $K := \varepsilon^{-1}$ and $i \in \{0, \dots, K\}$. Following [84], we call the dynamics restricted to $(x_i)_{i\in[K]}$ the **skeleton**. Our goal is to argue that when the mean of $(h^*_{\eta_t})_{t\geq 0}$ at time t_2 has at most the order of the typical fluctuations within the stationary distribution μ, the law of the skeleton at time t_2 is in total-variation distance close to equilibrium. This follows by applying the same arguments as in the proof of Lemma 8.4 in [84], replacing Proposition 6.1 and Lemma 8.5 in [84] by Lemma 6.20 and Lemma 6.22, respectively.

To conclude the first step, use Lemma 6.21 to see that for any initial state $\eta \in \Omega_N$, $\mathbb{E}_\eta[h^*_{\eta_t}(x)]$ is at most of order $\sqrt{N}$ at time $t = t_2$ for all $x \in [2N]$. In a second step, we apply the censoring inequality from Lemma 4.11 for the censoring scheme

$$\mathcal{C}(t) = \{\{x_i, x_i + 1\} : i \in [K]\} ,$$

where $t \in [t_2, t_0]$, in order to show that the dynamics mixes locally. In words, this censoring scheme ensures that the number of particles in the interval $[x_{i-1}, x_i]$ for all $i \in [K]$ remains almost surely constant between t_2 and t_0. Thus, we see K independent simple exclusion processes on a closed segment during this period. Using the above bounds at time t_2, the remainder is analogous to Proposition 8.2 in [84]. □

Note that Lemma 6.23 does not immediately imply Theorem 6.2 since there could be an initial state other than **1** or **0**, which maximizes the distance from equilibrium. However, using Lemma 6.23, we obtain the following result which allows us to conclude the upper bound in Theorem 6.2 using Lemma 6.8.

Lemma 6.24 (c.f. [84], Propositions 8.1)**.** *For a given $\varepsilon > 0$, we set*

$$t_1 := \frac{4}{\pi^2} N^2 \log N(1 + \varepsilon) . \tag{6.59}$$

Then there exists a coupling $\tilde{P}$ which respects $\succeq_{\mathrm{h}}$ such that for all $\varepsilon > 0$

$$\lim_{N \to \infty} \tilde{P}(\eta^{\mathbf{1}}_{t_1} \neq \eta^{\mathbf{0}}_{t_1}) = 0 . \tag{6.60}$$

Sketch of the proof. In order to show Lemma 6.24 using Lemma 6.23, we consider a coupling which is monotone with respect to $\succeq_{\mathrm{h}}$ and maximizes the fluctuations of $(h^*_{\eta_t})_{t \geq 0}$. We use the construction of the alternative coupling defined in [84, Section 8.4]. However, for all transitions where particles enter and exit the segment, we apply the update rule of the canonical coupling for the simple exclusion process with open boundaries, i.e., we use the same rate β and rate δ Poisson clocks in both simple exclusion processes to determine when a boundary vertex is updated. The proof of Lemma 6.24 follows from similar arguments as the proof of Proposition 8.1 given in [84, Section 8.4], replacing Lemma 8.5 in [84] by Lemma 6.22. □

6.6 Mixing times for ASEP with one blocked entry

In this section, we prove Theorem 6.3 for the asymmetric simple exclusion process with one blocked entry. We start by defining the simple exclusion process on the half-line as an auxiliary process, and investigate its current. By a coupling to the original dynamics, this allows us to deduce the lower bound on the mixing time. For the upper bound, we again use the simple exclusion process on the half-line to estimate the hitting time with respect to the extremal states **0** and **1**, and conclude by a shock wave argument. We will only consider $\alpha = 0$ and $\beta > 0$, and prove (6.6), since (6.7) follows from similar arguments using the symmetry between particles and empty sites.

6.6.1 The simple exclusion process on the half-line

In the following, we study the simple exclusion process $(\sigma_t)_{t\geq 0}$ on the half-line $\mathbb{N}$ with drift $p \in [\frac{1}{2}, 1]$, where particles enter at rate $\tilde{\alpha}$ and exit at rate $\tilde{\gamma}$ at site 1. Formally, $(\sigma_t)_{t\geq 0}$ is the Feller process on $\{0,1\}^{\mathbb{N}}$ generated by

$$(\mathcal{L}^{\mathbb{N}} f)(\eta) = \sum_{x=1}^{\infty} \big(p\eta(x)(1-\eta(x+1)) + (1-p)\eta(x+1)(1-\eta(x))\big)\left[f(\eta^{x,x+1}) - f(\eta)\right] \\ + \big(\tilde{\alpha}(1-\eta(1)) + \tilde{\gamma}\eta(1)\big)\left[f(\eta^1) - f(\eta)\right] \tag{6.61}$$

for all cylinder functions f. Recall the notion of the current from (6.20), and let $(J_t^{\mathbb{N}})_{t\geq 0}$ be the current of $(\sigma_t)_{t\geq 0}$, i.e., the net number of particles entering at the left-hand side boundary. Recall the component-wise order from (3.13) and the stochastic domination from (4.35). The following bound on the current of the simple exclusion process on the half-line extends the results from [92] for general boundary parameters.

Lemma 6.25. *Let $\tilde{\alpha} > 0$, $\tilde{\gamma} \geq 0$ and $p > \frac{1}{2}$. Recall $a = a(\tilde{\alpha}, \tilde{\gamma}, p)$ from (6.4). Then the simple exclusion process $(\sigma_t)_{t\geq 0}$ started from the empty initial configuration $\mathbf{0}$ satisfies*

$$\mathbb{P}_{\mathbf{0}}\left(\sigma_t \in \cdot\right) \preceq_{\mathrm{c}} \nu_{\frac{1}{1+a}}\,, \tag{6.62}$$

where $\nu_{\frac{1}{1+a}}$ denotes the Bernoulli-$\frac{1}{1+a}$-product measure on $\mathbb{N}$. Furthermore,

$$\lim_{t\to\infty} \frac{J_t^{\mathbb{N}}}{t} = (2p-1)\frac{\max(a,1)}{(\max(a,1)+1)^2} \tag{6.63}$$

holds almost surely.

Proof. Note that the measure $\nu_{\frac{1}{1+a}}$ is invariant for the simple exclusion process on the half-line. This can be seen by a direct calculation using the generator in (6.61), or alternatively, by Lemma 6.11 when taking $b = 1/a$ and letting the size of the segment go to infinity. Hence, (6.62) follows using the canonical coupling and monotonicity for the simple exclusion process on the half-line when starting from $\nu_{\frac{1}{1+a}}$.

For $\leq$ in (6.63), we compare $(\sigma_t)_{t\geq 0}$ to a simple exclusion process $(\eta_t)_{t\geq 0}$ on the segment of size N with drift p and boundary parameters $\alpha = \tilde{\alpha}$, $\beta = p$, $\gamma = \tilde{\gamma}$, $\delta = 0$, which is started from the empty configuration. We adjust now the canonical coupling $\mathbf{P}$ such that we use the same Poisson clocks in both processes on the sites $[N-1]$, and try to remove a particle in $(\eta_t)_{t\geq 0}$ at site N whenever the clock for performing a jump from site N to $N+1$ in $(\sigma_t)_{t\geq 0}$ rings. In particular, the coupling ensures that when $\eta_t(x) = 1$ holds for some $x \in [N]$, then $\sigma_t(x) = 1$, provided that both processes agreed initially on $[N]$. Therefore, the current $(J_t^N)_{t\geq 0}$ of $(\eta_t)_{t\geq 0}$ satisfies $J_t^N \geq J_t^{\mathbb{N}}$ for all $t \geq 0$ and $N \in \mathbb{N}$, $\mathbf{P}$-almost surely, and we conclude $\leq$ in (6.63) by Lemma 6.10 and taking $N \to \infty$.

For $\geq$ in (6.63), we assume without loss of generality that $a \geq 1$. This is due to the fact that for $a = a(\tilde{\alpha}, \tilde{\gamma}, p) < 1$, we can decrease $\tilde{\alpha}$ continuously until we reach $a = 1$, and apply a half-line version of Lemma 6.7 to see that this will only decrease the current. Using Lemma 6.7 again, and the canonical coupling $\mathbf{P}$, we see that the current in (6.63) is bounded from below by the current in $(\sigma_t)_{t\geq 0}$ when starting initially from $\nu_{\frac{1}{1+a}}$. In order to conclude (6.63), it suffices to show that $\nu_{\frac{1}{1+a}}$ is extremal invariant for $(\sigma_t)_{t\geq 0}$; see Theorem 3.5. We follow the arguments of Theorem 1.17 in [94, Part III] and relate the extremal invariant measures of the asymmetric simple exclusion process on $\mathbb{Z}$ to those of the symmetric simple exclusion process on $\mathbb{Z}$.

It suffices now to show that $\nu_{\frac{1}{1+a}}$ is extremal invariant for the simple exclusion process on the half-line with $p = \frac{1}{2}$, where particles enter at rate $\frac{1}{2}(\tilde{\alpha} + a^{-1}\tilde{\gamma})$ and exit at rate $\frac{1}{2}(\tilde{\gamma} + a\tilde{\alpha})$, respectively. As observed in Section 2 of [77], a sufficient condition for some distribution ν to be extremal invariant is that for any finite set $A \subseteq \mathbb{N}$

$$\lim_{t\to\infty} \mathbb{P}_{\nu_1^A}\left(\sigma_t \in B\right) = \lim_{t\to\infty} \mathbb{P}_{\nu_0^A}\left(\sigma_t \in B\right) \qquad \text{for all finite } B \subseteq \mathbb{N}, \tag{6.64}$$

where $\nu_1^A(\,\cdot\,) := \nu(\,\cdot \mid \eta(x) = 1\ \forall x \in A)$ and $\nu_0^A(\,\cdot\,) := \nu(\,\cdot \mid \eta(x) = 0\ \forall x \in A)$. The process $(\sigma_t)_{t\geq 0}$ can be realized as a disagreement process within the canonical coupling, where we start with with initial laws ν_1^A and ν_0^A. Since $\nu_{\frac{1}{1+a}}$ is a product measure, we have initially second class particles on A, and a Bernoulli-$\frac{1}{1+a}$-product measure everywhere else. Since $p = \frac{1}{2}$, we can view the dynamics as an interchange process, where all second class particles perform symmetric simple random walk on $\mathbb{N}$, with absorption at site 1 at rate at least $(\tilde{\alpha} + \tilde{\gamma})/2$. This allows us to conclude (6.64). □

6.6.2 Lower bound for the ASEP with one blocked entry

We will now show $\geq$ in (6.7). First, we assume $\alpha = \gamma = 0$ as well as $\beta > 0$. Using (6.27), the stationary distribution $\mu = \mu_N$ of $(\eta_t)_{t\geq 0}$ satisfies

$$\lim_{N\to\infty} \mu_N\left(\exists x \in \{1, \dots, N - \sqrt{N}\} \colon \eta(x) = 1\right) = 0\,. \tag{6.65}$$

Suppose we start from the configuration $\mathbf{1}$ with all sites being initially occupied. Using (4.1), we see that in order to prove an asymptotic lower bound t_N on $t^N_{\text{mix}}(\varepsilon)$ for all $\varepsilon \in (0,1)$, it suffices to show that with probability tending to 1, no more than $N - \sqrt{N}$ particles have exited the segment by time t_N. Using the symmetry between particles and empty sites, the number of particles which have exited the segment by time t_N is dominated by the current of the simple exclusion process on the half-line with drift p and boundary parameters $\tilde{\alpha} = \beta$, $\tilde{\gamma} = \delta$, evaluated at time t_N. Hence, we can conclude the lower bounds on the mixing time in Theorem 6.3 using Lemma 6.25. Note that for $\gamma > 0$, the statement (6.65) holds as well, due to Lemma 6.7. More precisely, consider the initial state $\eta^N \in \Omega_N$ given by

$$\eta^N(x) = \mathbb{1}_{x \geq \sqrt{N}} \tag{6.66}$$

for all $x \in [N]$. Comparing the process started from η^N to the blocking measure on $\mathbb{Z}$, we see that almost surely no particle reaches site 1 by time N^2 for all N sufficiently large, due to Lemma 6.14. Hence, we can again use the previous bound via the current for the simple exclusion process on the half-line to conclude.

6.6.3 An a priori upper bound on the hitting time

In order to show $\leq$ in (6.7), we will bound the hitting time $\tau_{\mathbf{0}} := \tau_{\text{hit}}^{\mathbf{1}}(\mathbf{0})$, i.e., the first time the process reaches $\mathbf{0}$, when all sites are initially occupied. We start with an a priori bound when the starting configuration contains a small number of particles and the particles are concentrated on the right-hand side.

Lemma 6.26. *Let $\alpha = \gamma = 0$ and $\beta > 0$. For $k \in [N-1]$, assume that $\eta \in \Omega_N$ satisfies $\eta(i) = 0$ for all $i \in [N-k]$. There exists $c = c(\beta, \delta, p) > 0$ such that for all k and N*

$$\mathbb{E}_\eta[\tau_{\mathbf{0}}] \leq \exp(ck^3)\,. \tag{6.67}$$

Proof. Suppose that $\delta > 0$ and $p < 1$. Recall from Section 6.3.4 the return time $\tau_{\mathbf{0}}^+$

$$\tau_{\mathbf{0}}^+ = \inf\{t \geq \tau_{\Omega_N \setminus \{\mathbf{0}\}} \colon \eta_t = \mathbf{0}\} \tag{6.68}$$

for the simple exclusion process with open boundaries, where for a set $A \subseteq \Omega_N$, we let τ_A denote the hitting time of A. Note that for all $\eta \in \Omega_N \setminus \{\mathbf{0}\}$

$$\mathbb{E}_\eta[\tau_{\mathbf{0}}] \leq \mathbb{E}_{\mathbf{0}}[\tau_{\mathbf{0}}^+] \left(\mathbb{P}_{\mathbf{0}}(\tau_\eta < \tau_{\mathbf{0}}^+)\right)^{-1}\,.$$

Further, by Kac's lemma, $\mathbb{E}_{\mathbf{0}}[\tau_{\mathbf{0}}^+] = (\mu(\mathbf{0}))^{-1}$ holds. Thus, $\mathbb{E}_{\mathbf{0}}[\tau_{\mathbf{0}}^+]$ is bounded uniformly in N due to (6.27), noting that Z_N in (6.27) is bounded uniformly in N. Starting from $\mathbf{0}$, there exists a sequence of at most k^2 updates to reach η involving only the rightmost $k+1$ edges and the right-hand side boundary. Moreover, this sequence can be chosen in such a way that all other updates do not affect the evolution of the process. Thus, forcing the rate 1 Poisson clocks along these edges to ring according to a given order, we see that

$$\mu(\mathbf{0})\mathbb{P}_{\mathbf{0}}(\tau_\eta < \tau_{\mathbf{0}}^+) \geq \mu(\mathbf{0}) \left(\frac{\min(1-p,\delta)}{k+\beta+\delta}\right)^{k^2} \geq \exp(-ck^3)$$

holds for some $c > 0$. For $\delta = 0$ or $p = 1$, use Lemma 6.9 to bound $\mathbb{E}_\eta[\tau_{\mathbf{0}}]$ by the expected hitting time for a simple exclusion process with the same parameters, except for some different choices of $\delta > 0$ and $p < 1$. $\square$

Figure 18: Visualization of the shock wave phenomenon for the asymmetric simple exclusion process with one blocked entry for different times where $N = 100$ with $p = 0.75$, and $\alpha = \gamma = 0$ as well as $\beta = \delta = 1$.

6.6.4 Upper bound for the ASEP with one blocked entry

We now prove $\leq$ in (6.7) for the asymmetric simple exclusion process $(\eta_t)_{t\geq 0}$ with one blocked entry. We will bound the hitting time $\tau_{\mathbf{0}}$ of the configuration $\mathbf{0}$, starting from configuration $\mathbf{1}$ where all sites are occupied, and conclude by Lemma 6.8 since $\tau_{\mathbf{0}}$ stochastically dominates the coupling time between the states $\mathbf{0}$ and $\mathbf{1}$. By Lemma 6.7, we can assume without loss of generality that $\gamma = 0$. For all $k \in [N]$, let θ_k be the configuration in Ω_N where the rightmost k sites are occupied and all other sites are empty. Recall that $L(\eta)$ denotes the position of the leftmost particle in η. We set

$$c_{b,p} := (\max(b,1)+1)^2/((2p-1)\max(b,1))\,. \tag{6.69}$$

Our key tool is the following lemma, showing that the particles travel like a "shock wave" at linear speed until a time $\tilde{\tau}$, which is defined later on; see Figure 18 for an illustration. In particular, $\tilde{\tau}$ will be a stopping time which describes that enough particles exited.

Lemma 6.27 (Shock wave phenomenon)**.** *Assume $\alpha = \gamma = 0$ and $\beta > 0$. Further, let $p > \frac{1}{2}$ and $\delta \geq 0$. Let $\varepsilon, \tilde{\varepsilon} > 0$. Then there exist $N_0, k_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and $k \geq k_0$ with $k = k(N) \in [N]$, we find a stopping time $\tilde{\tau}$ such that $(\eta_t)_{t\geq 0}$ on Ω_N started from θ_k satisfies*

$$\mathbb{P}\left(\left|L\left(\eta_{\tilde{\tau}}\right) - N\right| \leq \log^3 k \text{ and } \tilde{\tau} \leq (1+\varepsilon)c_{b,p}k \mid \eta_0 = \theta_k\right) \geq 1 - \tilde{\varepsilon}\,. \tag{6.70}$$

Moreover, for all $t \geq 0$,

$$\mathbb{P}\left(\tau_{\mathbf{0}} \geq (1+\varepsilon)c_{b,p}k + t \mid \eta_0 = \theta_k\right) \leq \mathbb{P}\left(\tau_{\mathbf{0}} \geq t \mid \eta_0 = \theta_{\lfloor \log^3 k \rfloor}\right) + \tilde{\varepsilon}\,. \tag{6.71}$$

The proof of Lemma 6.27 is deferred to the upcoming Section 6.6.5. We conclude this paragraph by showing Theorem 6.3 under the assumption that Lemma 6.27 holds.

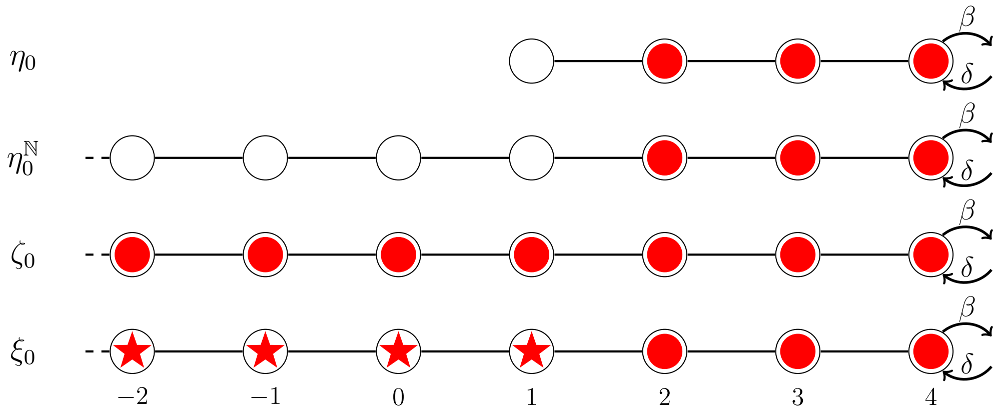

Figure 19: Visualization of the initial configurations of the different processes used in the proof of the upper bound in Theorem 6.3 for $N = 4$ and $k = 3$.

Proof of Theorem 6.3. Fix $\varepsilon, \tilde{\varepsilon} > 0$. For N sufficiently large, we apply Lemma 6.27 twice, for $k = N$ with $t = 2\varepsilon c_{b,p} N$ as well as for $k = \log^3 N \leq (1+\varepsilon)^{-1}\varepsilon N$ with $t = \varepsilon c_{b,p} N$, respectively, to see that

$$\begin{aligned}\mathbb{P}\left(\tau_{\mathbf{0}} \geq (1+3\varepsilon)c_{b,p}N \mid \eta_0 = \mathbf{1}\right) &\leq \mathbb{P}\left(\tau_{\mathbf{0}} \geq 2\varepsilon c_{b,p}N \mid \eta_0 = \theta_{\lfloor \log^3 N \rfloor}\right) + \tilde{\varepsilon} \\ &\leq \mathbb{P}\left(\tau_{\mathbf{0}} \geq \varepsilon c_{b,p}N \mid \eta_0 = \theta_{\lfloor \log^3 (\lfloor \log^3 N \rfloor) \rfloor}\right) + 2\tilde{\varepsilon}\,. \quad (6.72)\end{aligned}$$

By Lemma 6.26 and Markov's inequality, the right-hand side of (6.72) is bounded by $3\tilde{\varepsilon}$ for all N large enough. Since ε and $\tilde{\varepsilon}$ are arbitrary, apply Lemma 6.8 to conclude. □

6.6.5 Proof of the shock wave phenomenon

To show Lemma 6.27, we introduce three auxiliary exclusion processes, which are intertwined by the canonical coupling $\mathbf{P}$. A visualization can be found in Figure 19. We define $(\eta_t^{\mathbb{N}})_{t\geq 0}$ by extending the simple exclusion process $(\eta_t)_{t\geq 0}$ on the segment of size N to the half-line $(-\infty, N]$. More precisely, let $(\eta_t^{\mathbb{N}})_{t\geq 0}$ be the simple exclusion process on the half-line $(-\infty, N] \cap \mathbb{Z}$ with drift p, and particles exiting and entering at site N at rates β and δ, respectively. On all positive integers, we use the same clocks for the processes $(\eta_t)_{t\geq 0}$ and $(\eta_t^{\mathbb{N}})_{t\geq 0}$. When both processes agree initially on the sites in $[N]$, this construction will ensure that the position $L(\cdot)$ of the leftmost particle satisfies $L(\eta_t^{\mathbb{N}}) \leq L(\eta_t)$ almost surely for all $t \geq 0$. In the following, we will assume that $(\eta_t)_{t\geq 0}$ and $(\eta_t^{\mathbb{N}})_{t\geq 0}$ are started with exactly the rightmost k sites being occupied.

Next, we let $(\zeta_t)_{t\geq 0}$ be the exclusion process on $(-\infty, N] \cap \mathbb{Z}$ with the same transition rules as $(\eta_t^{\mathbb{N}})_{t\geq 0}$, but started from the all full configuration. Under the canonical coupling $\mathbf{P}$, let $(\xi_t)_{t\geq 0}$ denote the disagreement process between $(\eta_t^{\mathbb{N}})_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$. Note that $(\xi_t)_{t\geq 0}$ is again an exclusion process on $(-\infty, N] \cap \mathbb{Z}$ where the rightmost k sites are initially occupied by first class particles, and all other sites by second class

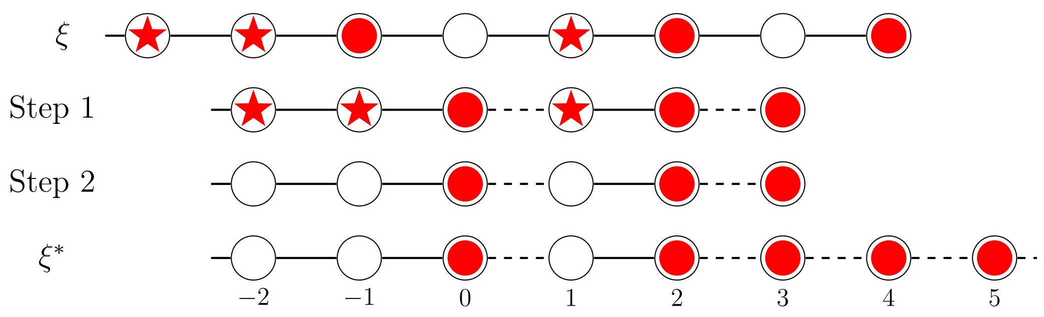

Figure 20: Construction of $\xi^* \in A_0$ from ξ. All censored edges are drawn dashed.

particles. If ξ_t has a positive number of first class particles, let $L_1(\xi_t)$ be the position of its leftmost first class particle. For all $x \in [k]$, let $\tilde{\tau}(x)$ be the first time at which $k - x$ particles have exited in $(\zeta_t)_{t\geq 0}$. The next lemma shows that $L_1(\xi_t)$ is close to the boundary at time $t = \tilde{\tau}(\lceil \log^2 k \rceil)$. Indeed $\tilde{\tau}(\lceil \log^2 k \rceil)$ will be the stopping time $\tilde{\tau}$ whose existence we claim in Lemma 6.27.

Lemma 6.28. *Let $\tilde{\varepsilon} > 0$. Then there exists some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $N \in \mathbb{N}$, under the above canonical coupling* $\mathbf{P}$

$$\mathbf{P}\left(|L_1(\xi_{\tilde{\tau}(\lceil \log^2 k \rceil)}) - N| \leq \log^3 k \right) \geq 1 - \tilde{\varepsilon} \,. \tag{6.73}$$

Proof. Note that by construction, $(\xi_t)_{t\geq 0}$ must contain at least $\log^2 k$ first class particles until time $\tilde{\tau}(\lceil \log^2 k \rceil)$, and hence $L_1(\xi_{\tilde{\tau}(\lceil \log^2 k \rceil)})$ is well-defined. In order to show (6.73), we use similar ideas as Benjamini et al. in [19] for the closed segment; see also $(\xi_t^{\mathrm{X}})_{t\geq 0}$ in (5.21). Recall (3.22) and define a process $(\xi^*_t)_{t\geq 0}$ on A_0 from $(\xi_t)_{t\geq 0}$ as follows: For every $t \geq 0$, consider the sequence which we obtain by first deleting all sites which are empty in ξ_t, merging certain edges if necessary, and then replacing all second class particles by empty sites. We let ξ^*_t be the unique configuration in A_0 which contains this sequence, and has only empty sites to its left and only first class particles to its right; see Figure 20. Note that $\xi^*_0 = \vartheta_0$ holds by construction.

Our key observation is that up to the first time τ^* at which a second class particle exits at the right-hand side boundary in $(\xi_t)_{t\geq 0}$, the process $(\xi^*_t)_{t\geq 0}$ has the law of a simple exclusion process on A_0 with censoring. More precisely, an edge e is censored in ξ^*_t at time t if and only if either one of its endpoints is $> N$ or the edge is merged in the first step of the construction from two edges which are adjacent to an empty site; see Figure 20. Note that this censoring scheme does not depend on the process $(\xi^*_t)_{t\geq 0}$, since in order to determine the positions of the empty sites in $(\xi_t)_{t\geq 0}$, we do not need to distinguish between first and second class particles. Recall $c_{b,p}$ from (6.69). For all $k \in \mathbb{N}$ and $\varepsilon > 0$, we define

$$B^1_k := \left\{ \tilde{\tau}(\lceil \log^2 k \rceil) \leq (1+\varepsilon) c_{b,p} k \right\} \quad \text{and} \quad B^2_k := \left\{ \tilde{\tau}(\lceil \log^2 k \rceil) \leq \tau^* \right\} \tag{6.74}$$

to be the events that there exists some time before $(1+\varepsilon)c_{b,p}k$ at which at least $k-\lceil \log^2 k\rceil$ particles in $(\zeta_t)_{t\geq 0}$ exited, and that no second class particle exited in $(\xi_t)_{t\geq 0}$ before that time, respectively. Observe that the total number of particles which left in $(\zeta_t)_{t\geq 0}$ by time t has the same law as the current of a simple exclusion process on the half-line at time t with boundary parameters $\tilde{\alpha}=\beta$ and $\tilde{\gamma}=\delta$; see (6.61). Hence, Lemma 6.25 implies that $\mathbf{P}(B_k^1)=1-\tilde{\varepsilon}/4$ for all k sufficiently large. Moreover, when

$$\left\{\tau^* \leq \tilde{\tau}(\lceil \log^2 k\rceil) \leq (1+\varepsilon)c_{b,p}k\right\}$$

holds, there must be an empty site in $(\xi_t^*)_{t\geq 0}$ at position $\log^2 k$ until time τ^*. Thus, we can conclude that $\mathbf{P}(B_k^2 \mid B_k^1) \geq 1-\tilde{\varepsilon}/4$ by combining Lemma 6.14, Corollary 4.12, and the above key observation on the law of $(\xi_t^*)_{t\geq 0}$ for k large enough. In particular, this yields $\mathbf{P}(B_k^2)\geq 1-\tilde{\varepsilon}/2$. Recall that $R(\cdot)$ denotes the position of the rightmost empty site. Again, by Lemma 6.14, we see that the events

$$B_k^3 := \left\{|L(\xi_t^*)-R(\xi_t^*)| \leq \log^2 k \text{ for } t=\tilde{\tau}_{\lceil \log^2 k\rceil}\right\}$$

satisfy $\mathbf{P}(B_k^3 \mid B_k^1 \cap B_k^2) \geq 1-\tilde{\varepsilon}/4$ for all k sufficiently large. Note that whenever the events B_k^1, B_k^2 and B_k^3 occur, a sufficient condition for the statement in Lemma 6.28 to hold is that the event

$$B_k^4 := \left\{|x\in[N-\log^3 k, N]\colon \xi_t(x)\neq 0| \geq 2\log^2 k \text{ for all } t\in[0,(1+\varepsilon)c_{b,p}k]\right\}$$

occurs. Using the particle-empty site symmetry, we see by (6.62) in Lemma 6.25 that the law of ζ_t dominates a Bernoulli-$\frac{b}{b+1}$-product measure for all $t\in[0,(1+\varepsilon)c_{b,p}k]$. Hence, we can conclude that $\mathbf{P}(B_k^4 \mid B_k^1\cap B_k^2\cap B_k^3)\geq 1-\tilde{\varepsilon}/4$ for all k large enough. □

Proof of Lemma 6.27. Note that (6.70) with $\tilde{\tau}=\tilde{\tau}(\lceil \log^2 k\rceil)$ follows immediately from Lemma 6.25 and Lemma 6.28, recalling that $L(\eta_t^{\mathbb{N}}) \leq L(\eta_t)$ holds almost surely for all $t\geq 0$. For the second statement (6.71), we apply the strong Markov property for $(\eta_t)_{t\geq 0}$ with respect to the stopping time $\tilde{\tau}(\lceil \log^2 k\rceil)$. Note that by adding additional particles to the process $(\eta_t)_{t\geq 0}$ at some time $t\leq \tau_{\mathbf{0}}$, we will only increase the hitting time $\tau_{\mathbf{0}}$ of the state $\mathbf{0}$. Hence, whenever the event in (6.70) holds with respect to $\tilde{\tau}(\lceil \log^2 k\rceil)$, we see that the hitting time of $\mathbf{0}$ starting from $\eta_{\tilde{\tau}(\lceil \log^2 k\rceil)}$ is stochastically dominated by the hitting time when starting from $\theta_{\lceil \log^3 k\rceil}$. This yields (6.71). □

6.7 Mixing times for the reverse bias phase

In this section, we prove upper and lower bounds on the mixing time of the simple exclusion process in the reverse bias phase. Recall that $\frac{1}{2}<p<1$ and $\alpha=\beta=0$ holds, i.e., the particles have a drift to the right-hand side, but can neither exit at the right-hand side nor enter at the left-hand side boundary. Intuitively, the particles have to move against their natural drift direction. We will see that this results in an exponentially large mixing time. For the lower bound, we consider two exclusion processes with different initial states and show that with high probability, they have a disjoint support even at exponentially large times. For the upper bound, we compare the disagreement process for initial states **0** and **1** to a birth-and-death chain.

6.7.1 Lower bounds for the reverse bias phase

We start with the lower bound when $\min(\gamma,\delta)>0$ holds. Recall the total-variation distance from (4.1) and note that by the triangle inequality

$$\max_{\zeta\in\{\theta,\theta'\}}\|P_\zeta(\eta_t\in\cdot)-\mu\|_{\mathrm{TV}}\geq\frac{1}{2}\|P_\theta(\eta_t\in\cdot)-P_{\theta'}(\eta_t\in\cdot)\|_{\mathrm{TV}} \tag{6.75}$$

for any initial states $\theta,\theta'\in\Omega_N$ of the simple exclusion process $(\eta_t)_{t\geq0}$ with open boundaries. We define

$$\theta(x):=\mathbb{1}_{\{x\geq\lfloor\frac{N}{2}\rfloor\}}\quad\text{and}\quad\theta'(x):=\mathbb{1}_{\{x\geq\lfloor\frac{N}{2}\rfloor+1\}}\quad\text{for all } x\in[N]. \tag{6.76}$$

Since the total-variation distance of two distributions with disjoint support is 1, the right-hand side of (6.75) is bounded from below by 1 minus the probability that at least one particle enters or exits in at least one of the exclusion processes started from θ and θ'. We estimate this probability by comparing the simple exclusion processes started from θ and θ', respectively, to the simple exclusion processes on $\mathbb{Z}$ via the embedding

$$\tilde{\eta}(x):=\begin{cases}\eta(x) & \text{if } x\in[N]\\ 0 & \text{if } x\leq 0\\ 1 & \text{if } x>N\end{cases} \tag{6.77}$$

for all $x\in\mathbb{Z}$ and $\eta\in\Omega_N$. In particular, note that $\tilde{\theta}$ and $\tilde{\theta}'$ are the ground states in A_n and A_{n+1} for $n=\lfloor N/2\rfloor$, respectively. Using the canonical coupling and the censoring inequality, the simple exclusion processes started from $\tilde{\theta}$ and $\tilde{\theta}'$ are stochastically dominated by the respective exclusion processes started from the blocking measures on A_n and A_{n+1}; see Corollary 4.12. Thus, we obtain the lower bound in (6.9) from Lemma 6.14 for the simple exclusion process on $\mathbb{Z}$ with $x=N-1$ and $\varepsilon=N^{-1}$.

In the case where particles can exit only from one side of the segment, a similar argument holds. Using the particle-empty site symmetry, it suffices to consider $\gamma>0$ and $\alpha,\beta,\delta=0$. The stationary distribution μ is then the Dirac measure on **0**. Consider the initial state ζ with $\zeta(x)=\mathbb{1}_{\{x=N\}}$ for all $x\in[N]$ and note that $\tilde{\zeta}$ is the ground state on A_{N-1}. Similar to the previous case, we obtain the lower bound in (6.8) by applying Lemma 6.14 for the simple exclusion process on $\mathbb{Z}$ with $x=N-2$ and $\varepsilon=N^{-1}$. This concludes the proof of the lower bounds in Theorem 6.4.

6.7.2 Upper bounds for the reverse bias phase

We now show the upper bounds in Theorem 6.4. By Lemma 4.13, it suffices to consider the disagreement process between **1** and **0**, and study the time it takes until all second class particles have left the segment. In the following, we enumerate the second class particles from left to right, and let $(X_t^{(i)})_{t\geq 0}$ for $i \in [N]$ denote the trajectory of the i^{th} second class particle in the disagreement process. Moreover, denote its exit time of the segment by τ_i^{ex}. In order to bound these exit times, we compare $(X_t^{(i)})_{t\geq 0}$ to a certain continuous-time birth-and-death chain $(B_t)_{t\geq 0}$ with state space $[n]$ for some $n \in \mathbb{N}$ which will be determined later on. Similarly to (6.68), we let for all $j \in [n]$

$$\tau_j^+ := \inf\left\{t \geq \tau_{[n]\setminus\{j\}} : B_t = j\right\} \tag{6.78}$$

be the return time of $(B_t)_{t\geq 0}$ to the state j.

Lemma 6.29. *Consider a birth-and-death chain $(B_t)_{t\geq 0}$ on $[n]$ for some $n \in \mathbb{N}$ with transition rates $1-p$ to the right and p to the left. Then the return time τ_n^+ to the site n satisfies*

$$\mathbb{E}_k\left[\tau_n^+\right] \leq \frac{1}{Z}\left(\frac{p}{1-p}\right)^n \tag{6.79}$$

for any initial state $k \in [n]$, with a constant $Z > 0$.

Proof. Observe that the stationary distribution μ' of the birth-and-death chain satisfies $\mu'(x) = \frac{1}{Z'}\left(\frac{1-p}{p}\right)^x$ for all $x \in [n]$, with a normalization constant $Z' > 0$. Moreover,

$$\mathbb{E}_k\left[\tau_n^+\right] \leq \mathbb{P}_n\left(\tau_k^+ < \tau_n^+\right)^{-1}\mathbb{E}_n\left[\tau_n^+\right] = \mathbb{P}_n\left(\tau_k^+ < \tau_n^+\right)^{-1} Z'\left(\frac{p}{1-p}\right)^n$$

holds for all $k \in [n-1]$. Note that $\mathbb{P}_n\left(\tau_k^+ < \tau_n^+\right)$ is bounded from below by some $c > 0$ uniformly in k and n. We obtain (6.79) for $Z = c^{-1}Z'$. □

We start with the case where particles can enter only at one side of the segment. Without loss of generality, assume that $\delta > 0$ and $\gamma = 0$ holds. The stationary distribution μ is then the Dirac measure on the configuration **1**. Observe that each second class particle moves to the right at least at rate $1-p$, and to the left at most at rate p independently of the remaining particle configuration. Thus, $(X_t^{(i)})_{t\geq 0}$ stochastically dominates the process $(B_t)_{t\geq 0}$ from Lemma 6.29 until time τ_i^{ex} for $n = N$ and started from $B_0 = X_0^{(i)}$, i.e., we find a coupling such that $X_t^{(i)} \geq B_t$ holds almost surely for all $t < \tau_i^{\text{ex}}$. Moreover, when a second class particle reaches site N at time t, with probability at least $\frac{1-\mathrm{e}^{-\delta}}{\mathrm{e}}$, it has exited the segment by time $t+1$. Thus, with respect to the canonical coupling, there exists some $c > 0$ such that for all $i \in [N]$

$$\mathbf{E}\left[\tau_i^{\text{ex}}\right] \leq c\left(\frac{p}{1-p}\right)^N .$$

Moreover, by Markov's inequality, we see that for all $i \in [N]$

$$\mathbf{P}\left(\tau_i^{\mathrm{ex}} > cN^2\left(\frac{p}{1-p}\right)^N\right) \leq \frac{1}{N^2}\,.$$

We conclude the upper bound in (6.8) of Theorem 6.4 using a union bound for the event that some second class particle has not left the segment by time $cN^2(p/(1-p))^N$.

Now suppose that $\min(\gamma,\delta) > 0$ holds. Note that each second class particle has a distance of at most $\lfloor N/2 \rfloor$ to either the site 1 or the site N. Consider the family of processes $(Y_t^{(i)})_{t\geq 0}$ given by

$$Y_t^{(i)} := \max(X_t^{(i)} - \lceil N/2 \rceil, \lceil N/2 \rceil + 1 - X_t^{(i)}) \tag{6.80}$$

for all $t \geq 0$ and $i \in [N]$. Note that $(Y_t^{(i)})_{t\geq 0}$ increases by 1 at most at rate p and decreases by 1 at least at rate $1-p$. Hence, for all $i \in [N]$, the process $(Y_t^{(i)})_{t\geq 0}$ is stochastically dominated by the birth-and-death process in Lemma 6.29 for $n = \lfloor N/2 \rfloor$ and $B_0 = Y_0^{(i)}$. A similar argument as for the one-sided case yields Theorem 6.4.

6.8 Mixing times in the high and low density phase

In this section, we prove Theorem 6.5 for the asymmetric simple exclusion process in the high density phase and in the low density phase; see Figure 21 for a simulation of the height function over time. We will focus on showing an upper bound of order N. The lower bound of order N follows from a comparison to a single particle in the process using the fact that the invariant measure has a positive density; see also Section 6.6.2. Moreover, we only consider the high density phase. For the low density phase, similar arguments apply using the particle-empty site symmetry.

6.8.1 Construction of two disagreement processes

Assume that we are in the high density phase of the simple exclusion process with parameters $(p,\alpha,\beta,\gamma,\delta)$, i.e., $a = a(p,\alpha,\gamma)$ and $b = b(p,\beta,\delta)$ defined in (6.4) and (6.5) satisfy $b > \max(a,1)$. We have the following strategy to show the upper bound (6.11) in Theorem 6.5. For $j \in [4]$, we study simple exclusion processes $(\eta_t^j)_{t\geq 0}$ with open boundaries within the canonical coupling $\mathbf{P}$. The processes $(\eta_t^1)_{t\geq 0}$, $(\eta_t^2)_{t\geq 0}$ and $(\eta_t^3)_{t\geq 0}$ are defined with respect to the parameters $(p,\alpha,\beta,\gamma,\delta)$. They are started at states $\mathbf{1}$, $\mathbf{0}$ and from the stationary distribution μ, respectively.

In order to define $(\eta_t^4)_{t\geq 0}$, note that b is decreasing and continuous in β. Thus, we can choose some $\beta' > \beta$ such that $b > b' > \max(a,1)$ holds for $b' := b(p,\beta',\delta)$. We let $(\eta_t^4)_{t\geq 0}$ be the simple exclusion process with open boundaries for parameters

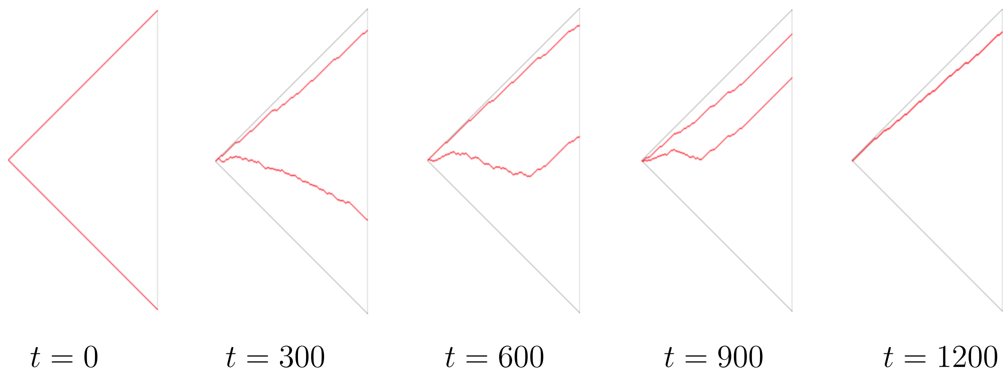

Figure 21: Simulation of the path representation of the height function for the simple exclusion process with open boundaries in the high density phase at different times $t \geq 0$. We set $N = 200$ and $p = 0.85$ as well as $\alpha = 3$, $\beta = 0.1$ and $\gamma = \delta = 1$. We start initially from the configurations $\mathbf{1}$ and $\mathbf{0}$, respectively.

$(p, \alpha, \beta', \gamma, \delta)$ started from its equilibrium. Using Lemma 6.7, note that we can choose the initial configurations in $(\eta^3_t)_{t\geq 0}$ and $(\eta^4_t)_{t\geq 0}$ such that

$$\mathbf{P}\left(\eta^3_t \succeq_{\mathrm{c}} \eta^4_t \text{ for all } t \geq 0\right) = 1\,. \tag{6.81}$$

We define $(\xi_t)_{t\geq 0}$ to be the disagreement process between $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$. Further, we let $(\zeta_t)_{t\geq 0}$ be the disagreement process between $(\eta^3_t)_{t\geq 0}$ and $(\eta^4_t)_{t\geq 0}$. Using the canonical coupling, note that $(\xi_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ can be seen as Markov chains on $\{0,1,2\}^N$, and $(\zeta_t)_{t\geq 0}$ is started from equilibrium. Further, in $(\xi_t)_{t\geq 0}$, no second class particles can enter the segment. In $(\zeta_t)_{t\geq 0}$, second class particles can enter only at site N provided that N is occupied by a first class particle. In Lemma 6.31, we will see that if enough second class particles have exited at the left-hand side in $(\zeta_t)_{t\geq 0}$, then with probability tending to 1, $(\xi_t)_{t\geq 0}$ has no second class particles.

For $i \in \{0,1,2\}$, let $(J^{(i)}_t)_{t\geq 0}$ denote the current of objects of type i, i.e., for a given time $t \geq 0$, $J^{(i)}_t$ denotes the number of objects of type i which have entered by time t minus the number of objects of type i which have exited by time t at the left-hand side boundary in $(\zeta_t)_{t\geq 0}$; see also (6.20). The next lemma shows that the current of second class particles in $(\zeta_t)_{t\geq 0}$ is linear when starting from its stationary measure μ'.

Lemma 6.30. *Let $(\zeta_t)_{t\geq 0}$ have initial distribution μ'. There exists some constant $c = c(b, b', p) > 0$ such that for all $t = t(N) \geq cN$,*

$$\lim_{N\to\infty} \mathbf{P}\left(-J^{(2)}_{t(N)} > 4N\right) = 1\,. \tag{6.82}$$

Proof. Let $(\zeta^{2\to 1}_t)_{t\geq 0}$ and $(\zeta^{2\to 0}_t)_{t\geq 0}$ denote the processes which we obtain from $(\zeta_t)_{t\geq 0}$ by projecting all second class particles to first class particles and empty sites, respectively. By construction, $(\zeta^{2\to 1}_t)_{t\geq 0}$ and $(\zeta^{2\to 0}_t)_{t\geq 0}$ are stationary simple exclusion processes

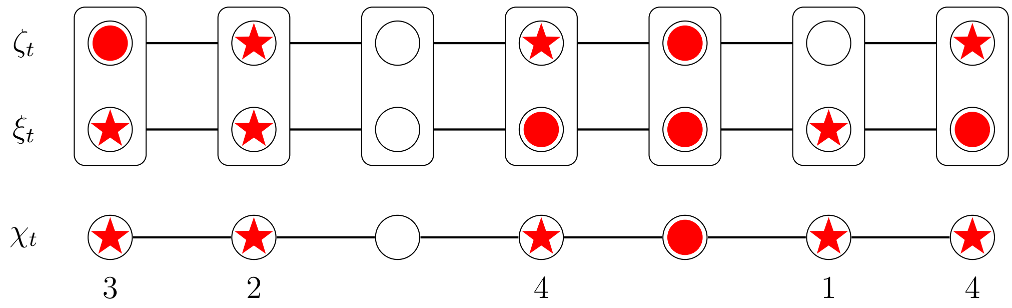

Figure 22: Coupling $(\chi_t)_{t\geq 0}$ between the processes $(\zeta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ for $N = 7$.

with parameters $(p, \alpha, \beta, \gamma, \delta)$ and $(p, \alpha, \beta', \gamma, \delta)$, respectively; see (6.81). Observe that $(J_t^{(1)})_{t\geq 0}$ is given by the current of particles in $(\zeta_t^{2\to 0})_{t\geq 0}$ while $(J_t^{(0)})_{t\geq 0}$ is given by the current of empty sites in $(\zeta_t^{2\to 1})_{t\geq 0}$. Thus, we see that for some $c > 0$,

$$\lim_{N\to\infty} \mathbf{P}\left(J_{cN}^{(0)} + J_{cN}^{(1)} < -4N\right) = 1$$

holds due to Lemma 6.10 and the ergodic theorem. Since

$$J_t^{(0)} + J_t^{(1)} + J_t^{(2)} = 0 \quad \text{for all } t \geq 0$$

and $(J_t^{(2)})_{t\geq 0}$ is decreasing in t, we conclude. □

6.8.2 Comparison via a multi-species exclusion process

Next, we relate the current of second class particles in $(\zeta_t)_{t\geq 0}$ to the motion of the second class particles in $(\xi_t)_{t\geq 0}$. The following lemma shows that when at least $4N$ second class particles have exited at the left-hand side boundary in $(\zeta_t)_{t\geq 0}$, all second class particles must have left in $(\xi_t)_{t\geq 0}$, with probability tending to 1.

Lemma 6.31. *For all N large enough and $T = T(N) \leq N^2$,*

$$\mathbf{P}\left(\xi_T(x) \neq 2 \text{ for all } x \in [N] \;\middle|\; -J_{T(N)}^{(2)} > 4N\right) \geq 1 - \frac{1}{N}, \tag{6.83}$$

where $(J_t^{(2)})_{t\geq 0}$ is defined with respect to $(\zeta_t)_{t\geq 0}$.

Proof of Theorem 6.5 assuming Lemma 6.31. The upper bound in Theorem 6.5 follows from Lemma 6.30 and Lemma 6.31 together with Lemma 4.13. □

In order to show Lemma 6.31, we require a bit of setup. Define the process $(\chi_t)_{t\geq 0} = (\zeta_t, \xi_t)_{t\geq 0}$ and note that under the canonical coupling, $(\chi_t)_{t\geq 0}$ is a Markov process with state space S^N where $S := \{0, 1, 2\}^2$. In the following, we will use an alternative interpretation of the process $(\chi_t)_{t\geq 0}$ on the state space $\{0, 1, 2\}^N$. By

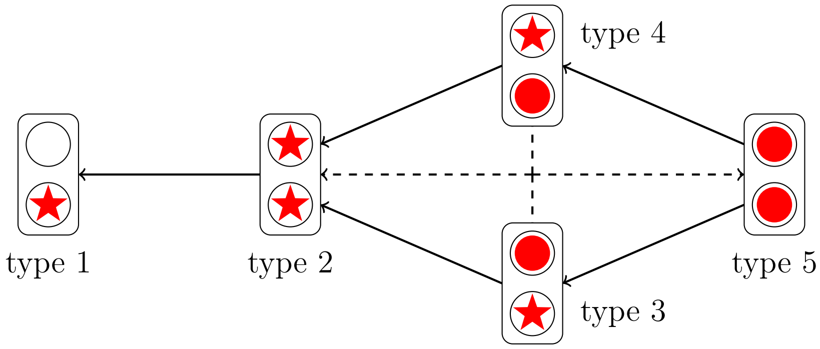

Figure 23: Visualization of the different types of second class particles. The tip of an arrow between two types indicates which type has the lower priority. The dashed arrows signalize that updating an edge with two second class particles of types 3 and 4 creates two second class particles of types 2 and 5, respectively. Note that in contrast to the multi-type exclusion process from Section 4.5, the different types of particles satisfy only a partial ordering.

construction, every site in $(\chi_t)_{t\geq 0}$ which is not occupied by two first class particles or by two empty sites, must be of the form $(0,2),(2,2),(1,2)$ or $(2,1)$. For example, note that the configuration $(2,0)$ is not attained since whenever a second class particle is created in $(\zeta_t)_{t\geq 0}$ at the boundary, there has to be a first class particle in $(\xi_t)_{t\geq 0}$. We refer to these configurations as second class particles of types 1 to 4, respectively; see Figures 22 and 23.

By definition, χ_0 contains only second class particles of types $1,2$ and 3, while all second class particles which enter at site N must have type 4. Among each other, the second class particles of types i and j respect the canonical coupling, i.e., a particle of type j has a higher priority than a particle of type i if $i<j$. For example, a second class particle of type 1 associated to $(0,2)$ has in both components a lower priority than a second class particle of type 4 which is associated to $(2,1)$. However, there is one exception: When two second class particles of types 3 and 4 are updated, they create the configurations $(2,2)$ and $(1,1)$. In this update mechanism, we call $(1,1)$ a second class particle of type 5; see Figure 23. To the other configuration values $(1,1)$ and $(0,0)$ in $(\chi_t)_{t\geq 0}$, we refer as first class particles and empty sites, respectively. Note that when ignoring the labels of the second class particles, the process $(\chi_t)_{t\geq 0}$ has the same transition rates as $(\zeta_t)_{t\geq 0}$. In particular, entering and exiting of first class particles and empty sites in $(\chi_t)_{t\geq 0}$ is not affected by the types of the second class particles.

We will now investigate the behavior of the different types of second class particles in $(\chi_t)_{t\geq 0}$ among each other using an auxiliary process $(\chi_t^\star)_{t\geq 0}$. This process has a similar construction as $(\xi_t^*)_{t\geq 0}$ in Section 6.6.5. Intuitively, for each $t\geq 0$, we obtain $\chi_t^\star$ by deleting all sites which are either empty or occupied by a first class particle in χ_t, merging certain edges if necessary, and replacing all second class particles of types

$1, 2$ or 3 with empty sites, as well as all second class particles of types 4 or 5 with first class particles. We then extend χ_t to a configuration on $\{0,1\}^{\mathbb{Z}}$ by adding particles on the right-hand side, and empty sites as well as a finite number of particles on the left-hand side of the segment.

We will see from the formal construction below that $\chi^\star_0 = \vartheta_0$, recall (3.23), and that $(\chi^\star_t)_{t\geq 0}$ has the law of a simple exclusion process on $\mathbb{Z}$ with censoring, in which the rightmost empty site $R(\chi^\star_t)$ is replaced by a first class particle whenever the corresponding second class particle in $(\chi_t)_{t\geq 0}$ exits at site N at time t. An edge e is censored for $\chi^\star_t$ at time t if and only if it was merged in χ_t in the deletion step, or if one of its endpoints is occupied by a particle which is not present in χ_t, and thus was only added in the construction when extending the configuration to $\mathbb{Z}$. Note that this censoring scheme does not depend on the different types of the second class particles in $(\chi_t)_{t\geq 0}$.

For a formal construction of $(\chi^\star_t)_{t\geq 0}$, we use the following procedure to obtain $\chi^\star = \chi^\star(v) \in \{0,1\}^{\mathbb{Z}}$ from $\chi \in \{0,1,2\}^N$ for every $v = \{0,1\}^k$ and $k \in \mathbb{N} \cup \{0\}$.

Step 1 Delete all vertices in χ which are empty or contain a first class particle.

Step 2 Concatenate the vector v at the left-hand side of the diminished segment.

Step 3 Turn all second class particles to empty sites if they are of type $1, 2$ or 3 and turn them into first class particles if they are of type 4 or 5.

Step 4 Extend to a configuration $\chi^\star \in \{0,1\}^{\mathbb{Z}}$ by adding empty sites at the left-hand side and first class particles at the right-hand side of the segment.

An illustration is given in Figure 24. Note that $\chi^\star$ in this procedure is only defined up to translations on $\mathbb{Z}$. We use this additional degree of freedom when we define the process $(\chi^\star_t)_{t\geq 0}$ from $(\chi_t)_{t\geq 0}$. For all $t \geq 0$, let $v = v(t)$ denote the vector of all second class particles which have left the segment at the left-hand side boundary by time t. More precisely, we place a 1 at position i in v if the i^{th} second class particle exiting is of type 4 or 5, and we put a 0, otherwise. For all $t \geq 0$, we obtain $\chi^\star_t$ up to translations by applying the above procedure for χ_t and $v(t)$. In order to determine the specific translation of $\chi^\star_t$ in $(\chi^\star_t)_{t\geq 0}$, we proceed as follows. We choose $\chi^\star_0 \in A_0$ where A_0 is defined in (3.22). In particular, note that $\chi^\star_0 = \vartheta_0$ holds. For $t > 0$, suppose that $\chi^\star_t \in A_n$ holds for some $n \in \mathbb{Z}$. If at time t a second class particle of type $1, 2$ or 3 exits at the right-hand side boundary in χ_t, we choose the updated configuration such that $\chi^\star_{t+} \in A_{n-1}$ holds. In all other cases, we choose $\chi^\star_{t+} \in A_n$. The next lemma states that the position of the leftmost particle $(L(\chi^\star_t))_{t\geq 0}$ is close to the position of the rightmost empty site $(R(\chi^\star_t))_{t\geq 0}$.

Lemma 6.32. *There exists a constant $c > 0$ such that for all N sufficiently large and $T \leq N^2$*

$$\mathbf{P}\left(|R(\chi^\star_T) - L(\chi^\star_T)| > c \log N + N\right) \leq \frac{1}{N} \,. \tag{6.84}$$

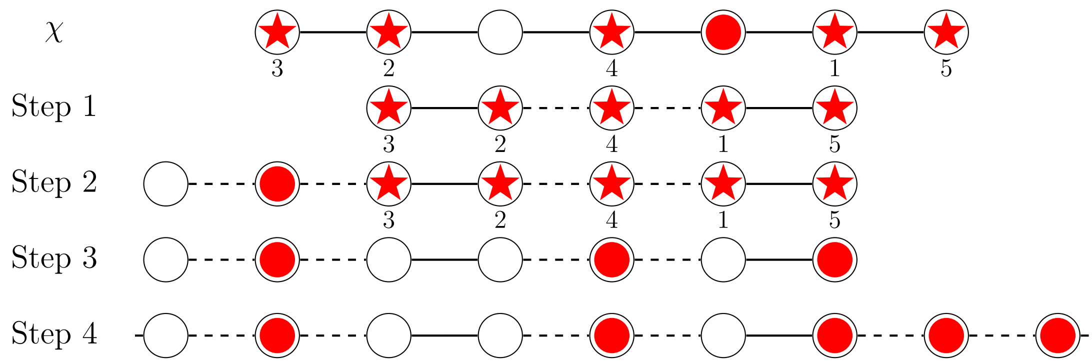

Figure 24: Construction of $\chi^\star$ from χ for $v = (0,1)$. Censored edges are drawn dashed.

Proof. Let $(\eta_t^0)_{t\geq 0}$ and $(\eta_t^{-N})_{t\geq 0}$ be two simple exclusion processes on A_0 and A_{-N} with initial states ϑ_0 and ϑ_{-N}, respectively. We let $(\eta_t^0)_{t\geq 0}$ and $(\eta_t^{-N})_{t\geq 0}$ be canonically coupled to $(\chi_t^\star)_{t\geq 0}$, and apply the same censoring scheme. Since $(\eta_t^0)_{t\geq 0}$ and $(\chi_t^\star)_{t\geq 0}$ differ only by the fact that in $(\chi_t^\star)_{t\geq 0}$ occasionally the right-most empty site is replaced by a particle, we see that $R(\chi_t^\star) \leq R(\eta_t^0)$ holds almost surely for all $t \geq 0$. Further, we claim that $L(\chi_t^\star) \geq L(\eta_t^{-N})$ holds almost surely for all $t \geq 0$. This can be seen by conditioning on the at most N times at which the rightmost empty site in $(\chi_t^\star)_{t\geq 0}$ gets replaced, and then using an induction argument. Since the above way of prohibiting updates in $(\chi_t^\star)_{t\geq 0}$ is indeed a censoring scheme in the sense of Section 4.4, we use the censoring inequality from Corollary 4.12 to see that the laws of $(\eta_t^0)_{t\geq 0}$ and $(\eta_t^{-N})_{t\geq 0}$ are stochastically dominated by the blocking measures on A_0 and A_{-N}, respectively, with respect to the partial order $\succeq_{\text{h}}$. In order to see that the statement in (6.84) holds, we apply Lemma 6.14. □

Proof of Lemma 6.31. Note that when the current of second class particles is at most $-4N$ at time T, we know that at least $4N$ second class particles are absorbed at the left-hand side boundary in $(\chi_t)_{t\geq 0}$ at time T. Note that in this case, at least $2N$ of them must be of type 4 since all second class particles created at site N are of type 4, and there are at most N second class particles of types $1,2,3$ initially in the segment. By Lemma 6.32, we see that with probability at least $1 - N^{-1}$, each second class particle of type 1, 2 or 3 in χ_T has at most $c \log N + N$ second class particles of type 4 or 5 to its left, counting also particles which have exited at site 1. Hence for all N large enough, all second class particles in $(\chi_t)_{t\geq 0}$ of type 1, 2 or 3, and thus also all second class particles in $(\xi_t)_{t\geq 0}$, have left the segment by time T with probability at least $1 - N^{-1}$. □

Remark 6.33. *For the simple exclusion process in the maximal current phase, we conjecture that a similar analysis of the disagreement process between* $\mathbf{1}$ *and* $\mathbf{0}$ *yields the order of the* ε*-mixing time, see also [120] for recent progress. In particular, the fluctuations with exponent* $\frac{2}{3}$ *for a second class particle on* $\mathbb{Z}$ *in a Bernoulli-*$\frac{1}{2}$*-product measure suggest that the time until all second class particles leave the segment is* *of order $N^{3/2}$. Furthermore, note that the exponent $\frac{2}{3}$ is the KPZ relaxation scale which was shown by Baik and Liu for periodic models in [7] as well as by Corwin and Dimitrov for the ASEP on $\mathbb{Z}$ in [35], and more broadly is present in all KPZ class models. Moreover, Corwin and Shen, as well as Parekh showed that under a weakly asymmetry scaling, the height function (suitably normalized) of the simple exclusion process with open boundaries in the triple point converges to a solution of the KPZ equation; see [39, 108]. Using Proposition 4.4, this indicates that cutoff does not occur.*

6.9 Mixing times in the triple point

In this section, we prove Theorem 6.6 for the simple exclusion process $(\eta_t)_{t\geq 0}$ with open boundaries and parameters $(p,\alpha,\beta,\gamma,\delta)$ in the triple point. We use a symmetrization argument, similar to the one presented in [60] for the case of the totally asymmetric simple exclusion process on the circle. The main technique used is a Nash inequality as introduced in [47]. We compare the total-variation distance between the law of $(\eta_t)_{t\geq 0}$ and its stationary distribution μ to the spectral gap of a reversible process $(\zeta_t)_{t\geq 0}$; see also Section 4.1.2. Define the **adjoint** $\mathcal{L}^\star$ of the generator $\mathcal{L}$ of the simple exclusion process $(\eta_t)_{t\geq 0}$ with open boundaries as the linear operator with

$$\sum_{\eta\in\Omega_N} f(\eta)(\mathcal{L}g)(\eta)\mu(\eta) = \sum_{\eta\in\Omega_N} (\mathcal{L}^\star f)(\eta)g(\eta)\mu(\eta)$$

for all functions $f,g\colon \Omega_N \to \mathbb{R}$. In particular, note that for reversible processes $\mathcal{L}=\mathcal{L}^\star$ holds; see (3.11). By Lemma 6.11, the stationary distribution μ of $(\eta_t)_{t\geq 0}$ is the Uniform distribution on Ω_N. Hence, observe that the simple exclusion process with open boundaries and parameters $(1-p,\gamma,\delta,\alpha,\beta)$ has generator $\mathcal{L}^*$. We consider now the additive symmetrization of the simple exclusion process $(\eta_t)_{t\geq 0}$ with open boundaries with generator $\mathcal{L}$ and the simple exclusion process generated by its adjoint $\mathcal{L}^*$. More precisely, we let $(\zeta_t)_{t\geq 0}$ be the Feller process on Ω_N generated by $\frac{1}{2}(\mathcal{L}^\star+\mathcal{L})$. Observe that $(\zeta_t)_{t\geq 0}$ is reversible with respect to μ. Moreover, $(\zeta_t)_{t\geq 0}$ has the law of a simple exclusion process with open boundaries for parameters

$$p' = \frac{1}{2}, \quad \alpha'=\gamma'=\frac{\alpha+\gamma}{2} \quad \text{and} \quad \beta'=\delta'=\frac{\beta+\delta}{2}\,.$$

The next lemma relates the total-variation distance of $(\eta_t)_{t\geq 0}$ to the spectral gap of the process $(\zeta_t)_{t\geq 0}$. It is an immediate consequence of Theorem 2.14 in [60].

Lemma 6.34. *Let λ denote the spectral gap of $(\zeta_t)_{t\geq 0}$. Then*

$$\|\mathbb{P}_\xi(\eta_t\in\cdot)-\mu\|_{\mathrm{TV}} \leq 2^{N/2+1}\exp(-\lambda t) \tag{6.85}$$

holds for all initial states $\xi\in\Omega_N$ and $t\geq 0$.

Proof of Theorem 6.6. By Remark 6.19, we see that $\lambda^{-1} \leq CN^2$ holds for some constant $C = C(\alpha,\beta,\gamma,\delta)$, and we conclude by applying Lemma 6.34. □

6.10 Open problems

We conclude this section with a discussion of open problems. We saw in Theorem 6.1 and Theorem 6.2 that the symmetric simple exclusion process exhibits pre-cutoff.

Conjecture 6.35. *Let* $p = \frac{1}{2}$ *and* $\alpha,\beta,\gamma,\delta \geq 0$ *with* $\max(\alpha,\gamma) > 0$ *and* $\max(\beta,\delta) > 0$. *Then the lower bound in* (6.2) *is sharp, and cutoff occurs.*

In the high density and low density phase, we have the following conjecture.

Conjecture 6.36. *Under the assumptions of Theorem 6.5, the mixing time in the high-density phase satisfies for all* $\varepsilon \in (0,1)$

$$\lim_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} = \frac{(b+1)(\hat{a}^2(2b-1)+\hat{a}(b-3)+b)}{(b-\hat{a})(2p-1)} \tag{6.86}$$

where $\hat{a} := \max(a,1)$. *A similar statement holds for the low density phase.*

Let us give some heuristics on this conjecture for the high density phase. Suppose we start from the empty initial configuration, and wait until we see the equilibrium density of $\frac{b}{b+1}$ within the segment; see Lemma 6.10. Similar to the hydrodynamic limits in [81], we expect at time $(b+1)(b-\hat{a})^{-1}(2p-1)^{-1}n$ to see a density which is $\frac{1}{\hat{a}+1}$ at 1, $\frac{1}{b+1}$ at n and linearly interpolated in between. After this time, the right boundary creates a shock wave traveling to site 1. This supports the conjecture of cutoff; see also Section 6.6. The total travel time of this shock can be computed by comparing the current at both endpoints. Note that in the maximum current phase, no such shock is created, and the particles can travel at the maximal possible speed of $\frac{1}{4}(2p-1)$. The mixing time is expected to be governed by second class particle fluctuations; see Remark 6.33 and Section 4 in [120].

Conjecture 6.37. *When* $\max(a,b) \leq 1$ *holds, including the triple point, the* ε*-mixing time of the simple exclusion process with open boundaries is of order* $N^{3/2}$ *for all* $\varepsilon \in (0,1)$. *Moreover, the cutoff phenomenon does not occur.*

Very recently, partial progress on this conjecture was made by the author when $p=1$ and $\gamma=\delta=0$ holds [120]. We have an upper bound of order $N^{3/2}\log(N)$ and a lower bound of order $N^{3/2}$ on the mixing time. In the triple point $\alpha=\beta=\frac{1}{2}$, the lower bound is sharp, i.e., the mixing times is shown to be of order $N^{3/2}$.

For $a=b>1$ and $p>\frac{1}{2}$, called the **coexistence line**, we see that the right-hand side of (6.86) in Conjecture 6.36 blows up.

Question 6.38. *What is the order of the* ε*-mixing time of the simple exclusion process with open boundaries at the coexistence line?*

Part III

Limit theorems for exclusion processes on trees

7 Prelimiaries on limit theorems for exclusion processes

In this part, we investigate limit theorems for different observables of the exclusion process on trees. More precisely, we are interested in the motion of a tagged particle, i.e., the trajectory of a particle in the exclusion process traced over time. Moreover, we study the current, which expresses the number of particles passing through a given site over time. As a motivation and to put the results from Sections 8 to 11 in a general context, we start with an overview on similar results for exclusion processes on $\mathbb{Z}^d$.

7.1 Limit theorems for the tagged particle in exclusion processes on $\mathbb{Z}^d$

Let $G = (V, E, o)$ be a rooted graph with vertex set V, edge set E and root o. We assume that G is locally finite and equipped with transition rates $p(x, y)$ for $x, y \in V$ and $c \equiv 0$ such that the corresponding simple exclusion process is a Feller process. Suppose that we are given an initial distribution ν for the exclusion process on G. We denote by ν^* the **Palm measure**, which we get from ν by conditioning on a particle at the root, i.e.,

$$\nu^*(\cdot) := \nu\left(\,\cdot \mid \eta(o) = 1\right) . \tag{7.1}$$

The particle starting at o is called the **tagged particle**. We follow the evolution of the tagged particle over time and denote by $(X_t)_{t\geq 0}$ its position. We now collect some well-known results about tagged particles.

Let the underlying graph be $\mathbb{Z}^d$ with the root at the origin. We assume that the transition rates $p(x, y)$ for $x, y \in \mathbb{Z}^d$ are **translation invariant**, i.e.,

$$p(x, y) = p(0, x - y) \tag{7.2}$$

holds for all $x, y \in \mathbb{Z}^d$. Moreover, we assume that the rates are of **finite range**, i.e., $p(0, z)$ is non-zero for at most finitely many $z \in \mathbb{Z}^d$. In particular, Theorem 3.1 guarantees that the corresponding exclusion process is in this case a Feller process.

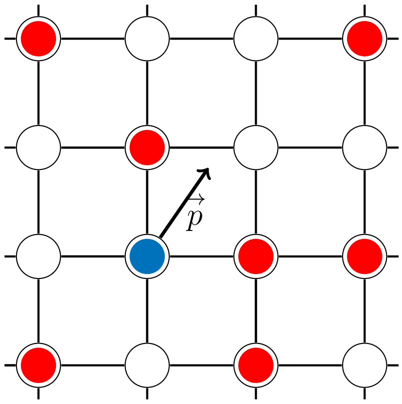

Figure 25: Visualization of a tagged particle, drawn in blue, in an asymmetric simple exclusion process on $\mathbb{Z}^d$ with drift vector $\vec{p}$.

Further, we assume that the transitions rates are **irreducible**, i.e., for every site $x \in \mathbb{Z}^d$, we can find a sequence $x_1, x_2, \dots, x_n$ of vertices such that

$$p(0, x_1)p(x_1, x_2)p(x_2, x_3) \cdots p(x_{n-1}, x_n)p(x_n, x) > 0 \tag{7.3}$$

holds. Note that since the rates satisfy a flow rule, recall (3.10), Theorem 3.6 yields that the Bernoulli-ρ-product measures ν_ρ are invariant for all $\rho \in [0,1]$.

Consider now the case where $d = 1$ with 0 as the root and ν_ρ^* as initial distribution for some $\rho \in [0,1]$. In the special case where the transition rates are nearest-neighbor and symmetric, the fluctuations of the tagged particle were determined by Arratia [4].

Theorem 7.1. *Suppose that the transition rates on $\mathbb{Z}$ satisfy $p(0,1) = p(0,-1) > 0$, and $p(0,x) = 0$, otherwise. Then for any continuous, bounded function f*

$$\lim_{t \to \infty} \mathbb{E}\left[f\left(\frac{X_t}{t^{\frac{1}{4}}} \right) \right] = \mathbb{E}[f(Y)], \tag{7.4}$$

where Y is a Gaussian with mean 0 and variance $2\sqrt{p(0,1)/\pi}(1-\rho)\rho^{-1}$.

In particular, the tagged particle has a subdiffusive scaling. The following result by Kipnis and Varadhan shows that this behavior is exceptional [79].

Theorem 7.2. *Suppose that the transition rates are symmetric, translation invariant, irreducible, and of finite range, but exclude the case of one-dimensional nearest-neighbor transition rates. Starting from ν_ρ^* for some $\rho \in [0,1]$, we see that for any continuous, bounded function $f\colon \mathbb{R}^d \to \mathbb{R}^d$*

$$\lim_{t \to \infty} \mathbb{E}\left[f\left(\frac{X_t}{t^{\frac{1}{2}}} \right) \right] = \mathbb{E}[f(Y)], \tag{7.5}$$

where Y is a multivariate Gaussian with mean 0 and non-degenerated covariance.

We remark that Theorem 7.2 states only a special case of Theorem 1.8 in [79]. In general, the results of Kipnis and Varadhan yield a central limit theorem for certain functionals of reversible Markov processes, which satisfy the so-called $\mathcal{H}_{-1}$-condition; see Proposition 8.11 for a formal description.

In the case where the transition rates are translation invariant, but not symmetric, we have a law of large numbers due to Kipnis for $d = 1$ and nearest-neighbor transition rates, and Saada for general $d \geq 1$ and finite range transition rates; see [78, 114] and Figure 25. We note that a central limit theorem in the special case of $p = 1$ and nearest-neighbor transition rates is attributed the Kesten; see also [133].

Theorem 7.3. *Suppose that the transition rates are translation invariant, irreducible, and of finite range. Then the tagged particle $(X_t)_{t\geq 0}$ satisfies*

$$\lim_{t\to\infty} \frac{X_t}{t} = (1-\rho) \sum_{x\in\mathbb{Z}^d} xp(0,x) \tag{7.6}$$

almost surely, provided that we start from ν_ρ^* for some $\rho \in [0,1]$.

In the setup of Theorem 7.3, a central limit theorem for the tagged particle was shown by Kipnis in [78] for $d = 1$ and by Sethuraman et al. in [130] for $d \geq 3$. The case $d = 2$ remains open, although partial progress was achieved by Sethuraman proving a diffusive scaling [129]. We note that similar results on the motion of a tagged particle can be achieved under weaker assumptions on the transition rates; see [80] for an overview and more comprehensive treatment of this question.

7.2 Limit theorems for the current in exclusion processes

Note that the results in Section 7.1 require that we start from an invariant measure of the simple exclusion process, conditioned on having the tagged particle in the root. If the process is not in equilibrium, or if it is too complicated to analyze the motion of a tagged particle directly, a different observable, which may be accessible, is the number of particles passing through a given site over time. This is formalized using the notion of current, which we introduced for the simple exclusion process with open boundaries in Section 6.3.2. For general graphs $G = (V, E)$, we use the following definition.

For each edge $e \in E$, we fix an orientation $\overrightarrow{e}$, and denote by $\overrightarrow{E}$ the set of all directed edges. We let $J_x^+(t)$ for a site $x \in V$ and $t \geq 0$ be the number of particles which jump to x until time t via an edge $\overrightarrow{e}$ with $\overrightarrow{e} = (\cdot, x)$. The same particle may be counted multiple times in $J_x^-(t)$, each time when it traverses an edge $\overrightarrow{e} = (\cdot, x)$. Similarly, let $J_x^+(t)$ be the number of particles which jump away from x until time t via $\overrightarrow{e} = (x, \cdot)$.

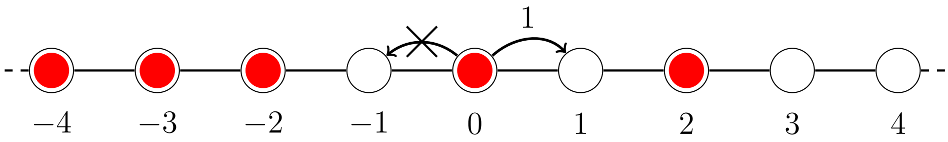

Figure 26: Visualization of the totally asymmetric simple exclusion process on the integers, where particles can only jump to the right-hand side.

The **current** through x until time t is given by

$$J_x(t) := J_x^+(t) - J_x^-(t) \tag{7.7}$$

for all $x \in V$ and $t \geq 0$. Note that we reobtain the current defined in (6.20) for the simple exclusion process with open boundaries on a segment of size N by choosing the orientation $\overrightarrow{e} = (x, x+1)$ of edges for all $x \in [N-1]$, and treating the entering and exiting of particles as incoming and outgoing directed edges, respectively. Observe that for uniformly bounded nearest neighbor rates, and when starting from an extremal invariant distribution ν, Theorem 3.5 ensures that almost surely for fixed $x \in V$

$$\lim_{t\to\infty} \frac{J_x(t)}{t} = \sum_{y \in V} (2\mathbb{1}_{(y,x)\in \overrightarrow{E}} - 1) p(y,x) \nu(\eta(y) = 1, \eta(x) = 0) \,. \tag{7.8}$$

Consider now the **totally asymmetric simple exclusion process (TASEP)** on the integers. It is the asymmetric simple exclusion process on the integers from Section 3.4.2 with $p = 1$; see Figure 26 and Section 10 for a generalization to trees. Fix the orientation $\overrightarrow{e} = (z, z+1)$ for all $z \in \mathbb{Z}$. The current through z until time t is the number of times a particle jumps from site $z-1$ to z until time t. Suppose that we start from a Bernoulli-ρ-product measure. The following seminal result is due to Ferrari and Fontes [53] for the diffusive case, where current fluctuations are of order $\sqrt{t}$, and due to Ferrari and Spohn [59] for the sub-diffusive case, where we see a scaling of order $t^{\frac{1}{3}}$.

Theorem 7.4. *Let $x = x(t) = ct$ for some constant $c \neq 1 - 2\rho$. Then for every bounded, continuous function f,*

$$\mathbb{E}\left[f\left(\frac{J_x(t) - \rho(1-\rho)t}{t^{\frac{1}{2}}} \right) \right] = \mathbb{E}\left[f\left(Y\right) \right] \tag{7.9}$$

holds, where Y is Normal-distributed with mean 0 and non-degenerated variance. When $c = 1 - 2\rho$, there exists a constant $\tilde{c} > 0$ such that

$$\mathbb{E}\left[f\left(\frac{J_x(t) - \rho(1-\rho)t}{\tilde{c}t^{\frac{1}{3}}} \right) \right] = \mathbb{E}\left[f\left(Z\right) \right] \tag{7.10}$$

holds, where Z is Baik–Rains-distributed; see [8] for a definition of the law of Z.

Note that $1-2\rho$ is the speed of single second class particle within the Bernoulli-ρ-product measure, which has fluctuations of order $t^{2/3}$ [10, 51, 112]. The scaling of order $t^{\frac{1}{3}}$ in the case $c=1-2\rho$ for the current fluctuations is characteristic for the TASEP as a model belonging to the KPZ universality class; see [34] for an introductory survey. We conclude this paragraph by noting that investigating the current of the TASEP for different initial conditions or for the partially asymmetric case $p \in (\frac{1}{2}, 1)$ is a topic of huge recent interest; see [1, 11, 17, 36, 57] for a selection of recent progress.

8 The simple exclusion process on regular trees

8.1 Introduction

In the following, our goal is to investigate the motion of a tagged particle on graphs different from $\mathbb{Z}^d$. We focus on the tagged particle $(X_t)_{t\geq 0}$ on rooted d-regular trees. The presented material is based in large parts on [29], which is joint work with Dayue Chen, Peng Chen, and Nina Gantert. Intuitively, for d large enough, the exclusion process on a d-regular tree can be seen as an approximation of the exclusion process on $\mathbb{Z}^d$. However, a crucial difference compared to $\mathbb{Z}^d$ is that the mean displacement of a tagged particle is non-zero when measured with respect to the shortest-path distance from the root. This will be resolved using ideas from Lyons et al. for the speed of a simple random walk on Galton–Watson trees [97]. Intuitively, we treat the particles in the exclusion process as a dynamic random environment for the tagged particle to apply the arguments from [97]. Let us stress that it is crucial for the results on tagged particles on $\mathbb{Z}^d$ that we have a translation invariant system. We will see in Section 8.2 that the translation invariance of regular trees allows for a simple description of the process "seen from the tagged particle". We discuss in Section 9 how this can be relaxed for the simple exclusion process on Galton–Watson trees.

8.1.1 Definition of the model

For $d \in \mathbb{N}$ with $d \geq 2$, let $T^d = (V, E, o)$ denote the **d-regular tree** with root o. We assume that the transition rates $p(x,y) \geq 0$ are irreducible, of finite range, and depend for all $x, y \in V$ only on the shortest path distance $|x-y|$ between x and y in T^d. With a slight abuse of notation, we will write $p(|x-y|) := p(x,y)$ in the following. Recall the operator $\mathcal{L}$ in (3.3) with $c \equiv 0$. We let the **exclusion process on T^d** be the exclusion process $(\eta_t)_{t\geq 0}$ with state space $\{0,1\}^V$ and generator $\mathcal{L}$ given in (3.3) for the above choice of transition rates $p(\cdot,\cdot)$ and $c \equiv 0$; see Figure 27. By Theorem 3.1, the exclusion process on T^d is under the above assumptions indeed a Feller process.

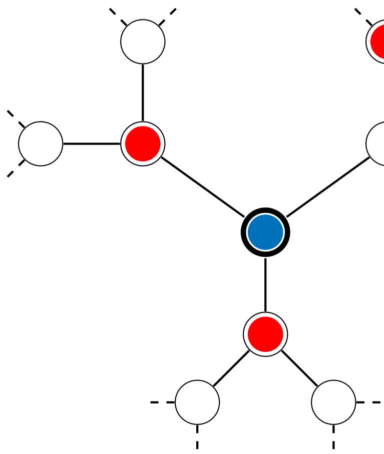

Figure 27: Visualization of the tagged particle, drawn in blue, in a configuration of the simple exclusion process on a d-regular tree.

We claim that by Theorem 3.7, the Bernoulli-ρ-product measures ν_ρ from (3.9) on $\{0,1\}^V$ with $\rho \in [0,1]$ are reversible for the exclusion process on $T^d = (V, E)$. To see this, note that we can extend the exclusion process on T^d to a simple exclusion process by adding only finitely many edges to each site. Moreover, under the above assumption that the transition rates $p(x, y)$ are symmetric, we can set $\pi(x) = 1$ in (3.12) for all $x \in V$ to obtain the claim. Note that due to Theorem 3.9, the collection $\{\nu_\rho \colon \rho \in [0,1]\}$ of measures is also extremal invariant for $(\eta_t)_{t\geq 0}$. Recall the Palm measure ν_ρ^* from (7.1) with respect to the Bernoulli-ρ-product measures ν_ρ for $\rho \in (0,1]$. Observe that ν_ρ^* is a Bernoulli-ρ-product measure on all sites other than o. In the following, our goal is to study the evolution of the tagged particle $(X_t)_{t\geq 0}$ with $X_0 = o$ and initial distribution $\nu_\rho^* = \nu_\rho(\,\cdot \mid \eta(o) = 1)$.

8.1.2 Related literature

Proving limit laws for the position of a tagged particle in exclusion processes is a classical problem, which is intensively studied when the underlying graph is $\mathbb{Z}^d$; see Section 7.1 for an overview. For the tagged particle process on other translation invariant graphs, less results are known. We note that the results of Kipnis and Varadhan on a central limit theorem for the tagged particle can be extended to exclusion processes on more graphs; see [79, 130] as well as Section 8.5 for a discussion. In the case of the d-dimensional ladder graph, a central limit theorem for the position of the tagged particle is due to Zhang [143]. However, the question of a law of large numbers for the speed of a tagged particle on trees is not resolved by these observations. Hence, we require different approaches such as analyzing the process "seen from the tagged particle"; see [114]. We conclude this paragraph by noting that also other interacting particle systems, as for example the contact process, are studied on regular trees; see for example [103, 109, 144].

8.1.3 Main results

We consider now the exclusion process on a d-regular tree for $d \geq 2$, and establish a law of large numbers with respect to the shortest path distance from the root. Moreover, for $d \geq 3$, we show that a central limit theorem holds. We start with a law of large numbers for the position of the tagged particle $(X_t)_{t\geq 0}$. In the following, we write $|z| := |z - o|$ for all $z \in V$.

Theorem 8.1. *For $d \geq 2$, let $(\eta_t)_{t\geq 0}$ on T^d have initial distribution ν^*_ρ for some $\rho \in (0,1]$. The position of the tagged particle $(X_t)_{t\geq 0}$ satisfies a law of large numbers, i.e.,*

$$\lim_{t\to\infty} \frac{|X_t|}{t} = (1-\rho)(d-2)\sum_{i\in\mathbb{N}_0} ip(i) =: v$$

almost surely. In particular, we see a speed of $(1-\rho)(d-2)d^{-1}$ in the case of the simple exclusion process on T^d, where we have $p(1) = d^{-1}$.

Recall that a similar relation holds for exclusion processes with drift on $\mathbb{Z}^d$; see Theorem 7.3. If ρ tends to 0, we obtain the speed of the random walk on T^d with transition rates $p(\cdot,\cdot)$. If $\rho = 1$, then $v = 0$ holds, and in between the speed is linear in $1-\rho$. For $d \geq 3$, we show that the tagged particle has a diffusive behavior. Note that when $d = 2$, the d-regular tree equals $\mathbb{Z}$ and the tagged particle has a subdiffusive behavior; see Theorem 7.1.

Theorem 8.2. *For $d \geq 3$ and $\rho \in (0,1)$, the tagged particle $(X_t)_{t\geq 0}$ on T^d satisfies*

$$\frac{|X_t| - tv}{\sqrt{t}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,\sigma^2)$$

for some $\sigma = \sigma(d,\rho,p(\cdot)) \in (0,\infty)$, and v from Theorem 8.1.

8.1.4 Outline of the proof

The proof of the main results is structured as follows: In Section 8.2, we introduce the environment process, which can be interpreted as the exclusion process "seen from the tagged particle". As a first step in the proof of Theorem 8.1, we show in Section 8.3 that the Palm measures ν^*_ρ are ergodic for the environment process, following the approach in [114] for the tagged particle in exclusion processes on $\mathbb{Z}^d$ with drift. This is a result of independent interest. In Section 8.4, we study the tagged particle process in more detail. We show that the tagged particle is transient using a martingale decomposition which can be found in [94, Part III, Section 4]. We then deduce Theorem 8.1 following the ideas of Lyons et al. in [97]. In Section 8.5, we prove Theorem 8.2 using the results of Kipnis and Varadhan as well as Sethuraman et al.; see [79, 130].

8.2 Construction of the environment process

We will now introduce the environment process, which is a classical tool in order to establish a law of large numbers; see [114, 133] for exclusion processes and [141] for random walks in random environment. For its definition, we require the following notation. The d-regular tree $T^d = (V, E, o)$ has a natural interpretation in terms of Cayley graphs. For $I = \{1, \ldots, d\}$, let

$$\mathcal{G} := \langle a_i, i \in I | a_i^2 = e \text{ for all } i \in I \rangle$$

denote the free group over all $i \in I$ for the two-element groups $\{e, a_i\}$ with the relation $a_i^2 = e$ and neutral element e. The tree T^d can be now be identified with the Cayley graph of $\mathcal{G}$ with respect to the generator $S = \{a_1, \ldots, a_d\}$. Note that the vertex set V is isomorphic to $\mathcal{G}$ with $e \cong o$ and two corresponding elements $b, c \in \mathcal{G}$ are neighbored if and only if $ba = c$ holds for some $a \in S$. The group structure of T^d allows us to extend this relation and define

$$b + c := bc \quad \text{as well as} \quad b - c := bc^{-1} \tag{8.1}$$

for $b, c \in \mathcal{G}$. In the same way, we write $x + y = z$ and $x - y = z$ for $x, y, z \in V$ if the corresponding elements in $\mathcal{G}$ satisfy (8.1). Let the maps τ_x on configurations $\eta \in \{0,1\}^V$ be given as

$$\tau_x \eta(y) := \eta(x + y)$$

for all $x, y \in V$. Equipped with this notation, we define the **environment process** $(\zeta_t)_{t \geq 0}$ as

$$\zeta_t(x) := \tau_{X_t} \eta_t(x) \tag{8.2}$$

for all $t \geq 0$ and $x \in V$. Note that $(\zeta_t)_{t \geq 0}$ is again a Feller process on the state space $\{\zeta \in \{0,1\}^V : \zeta(o) = 1\}$ generated by the closure of

$$\begin{aligned} Lf(\zeta) = &\sum_{x,y \neq o} p(|x-y|)\zeta(x)(1-\zeta(y)) \left[f(\zeta^{x,y}) - f(\zeta) \right] \\ &+ \sum_{x \in V} p(|x|)(1-\zeta(x)) \left[f(\tau_x \zeta) - f(\zeta) \right] . \end{aligned} \tag{8.3}$$

Note that each transition in $(\eta_t)_{t \geq 0}$ involving the root is a transition in $(\zeta_t)_{t \geq 0}$ followed by a translation. In the following, our goal is to investigate the set of invariant measures of $(\zeta_t)_{t \geq 0}$. The next proposition follows from the same arguments as Proposition 4.3 in [94, Part III] for the exclusion process on $\mathbb{Z}^d$, so we omit the proof.

Proposition 8.3. *The measure ν_ρ^* is invariant for $(\zeta_t)_{t \geq 0}$ for all $\rho \in [0,1]$.*

In order to calculate the speed of the tagged particle, we will now show that $(\zeta_t)_{t \geq 0}$ started from ν_ρ^* is a stationary and ergodic process.

8.3 Ergodicity for the environment process

In order to derive ergodicity with respect to ν_ρ^* and $\rho \in (0,1)$, we closely follow the arguments of Saada in [114]. For simplicity of notation, we only consider the case of a simple exclusion process on T^d as for general $p(\cdot)$ the same arguments apply.

Proposition 8.4. *For $d \geq 3$, the stationary environment process $(\zeta_t)_{t\geq 0}$ with initial law ν_ρ^* is ergodic for all $\rho \in (0,1)$.*

To show Proposition 8.4, we proceed with a proof by contraction. Suppose that $(\zeta_t)_{t\geq 0}$ is not ergodic. By Theorem 3.5, there exists $A \subseteq \big\{\zeta \in \{0,1\}^V : \zeta(o) = 1\big\}$ with

$$0 < \nu_\rho^*(A) < 1 \tag{8.4}$$

such that A is invariant, i.e.,

$$\mathbb{P}\big(\zeta_t \in A \bigm| \zeta_0 = \tilde{\zeta}\big) = 1$$

holds for almost all $\tilde{\zeta} \in A$. Hence, A is a non-trivial invariant set for $(\zeta_t)_{t\geq 0}$. Define $B := \big\{\zeta \in \{0,1\}^V : \zeta(o) = 1\big\} \setminus A$ and note that B is a non-trivial, invariant set for $(\zeta_t)_{t\geq 0}$ as well. Recall from Theorem 3.9 that ν_ρ is extremal invariant for the simple exclusion process $(\eta_t)_{t\geq 0}$, and hence $(\eta_t)_{t\geq 0}$ started from ν_ρ is ergodic; see Theorem 3.5. We want to use this observation to establish a contradiction. Let the sets $\tilde{A}$ and $\tilde{B}$ be

$$\tilde{A} := \bigcup_{x\in V,\ \zeta\in A} \tau_x\zeta \quad \text{and} \quad \tilde{B} := \bigcup_{x\in V,\ \zeta\in B} \tau_x\zeta\,.$$

Then, $\tilde{A}$ and $\tilde{B}$ are invariant for $(\eta_t)_{t\geq 0}$. Since $A \subseteq \tilde{A}$ and $B \subseteq \tilde{B}$, by (8.4)

$$\nu_\rho(\tilde{A}) = \nu_\rho(\tilde{B}) = 1 \tag{8.5}$$

holds. In particular, the sets $\tilde{A}$ and $\tilde{B}$ are not disjoint. We now deduce that A and B are not disjoint, contradicting the definition of B, by using the next lemma. In the following, let $\sim$ denote the relation of two sites being adjacent.

Lemma 8.5. *For almost every η distributed according to ν_ρ, there exist integers n, m, l and sites*

$$w, x, y, z; \quad x_1, x_2, \dots, x_n; \quad y_1, y_2, \dots, y_m; \quad z_1, z_2, \dots, z_l$$

with the following properties:

(i) $\tau_x\eta \in A$, $\tau_w\eta \in B$

(ii) $\eta(y) = \eta(z) = \eta(x_1) = \ldots = \eta(x_n) = 0$

(iii) *x, y, z are located in pairwise different branches with respect to w in T^d*

(iv) w is connected to x via the path $x_1 \sim x_2 \sim \cdots \sim x_n$*, connected to y via the path* $y_1 \sim y_2 \sim \cdots \sim y_m$ *and connected to z via the path* $z_1 \sim z_2 \sim \cdots \sim z_l$*.*

Proof. By (8.5), there almost surely exist sites $x, w \in V$ such that $\tau_x\eta \in A$ and $\tau_w\eta \in B$ holds. Let $x_1, x_2, \ldots, x_n$ denote the shortest path connecting x and w, which may be empty for $x \sim w$. Without loss of generality, we assume that $\eta(x_1) = \ldots = \eta(x_n) = 0$ holds. More precisely, note that A and B form a partition of $\{\zeta \in \{0,1\}^V : \zeta(o) = 1\}$ and so $\tau_w\eta \in A \cup B$ holds for all occupied sites $w \in V$. Among the occupied sites along the path from x to w, there exist two sites $\tilde{x}, \tilde{y}$ with $\tau_{\tilde{x}}\eta \in A$, $\tau_{\tilde{y}}\eta \in B$ with only vacant sites in between of them. Take $\tilde{x}, \tilde{y}$ as new choices for x and y.

In order to show that properties (iii) and (iv) hold, let $C(x,w)$ and $D(x,w)$ denote the vertices of two arbitrary branches of w different from the one containing x. Since $C(x,w)$ and $D(x,w)$ contain infinitely many sites, for ν^*_ρ-almost every η there are infinitely many y in $C(x,w)$ and z in $D(x,w)$ such that $\eta(y) = \eta(z) = 0$ holds. Choose two of these sites as y and z arbitrarily and define $y_1, y_2, \ldots, y_m$ and $z_1, z_2, \ldots, z_l$ to be the shortest paths connecting them to w, respectively. □

Proof of Proposition 8.4. Take an η satisfying the properties in Lemma 8.5 for sites

$$N := \{w, x, y, z, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_l\} .$$

Fix an arbitrary time $t_0 > 0$. Let $\tilde{\eta}$ denote the configuration that agrees with η on N while on the complement of N, $\tilde{\eta}$ has the distribution of a simple exclusion process $(\eta_t)_{t\geq 0}$ at time t_0 which is started from η and where all moves involving the sites N are suppressed. In the following, we consider two ways of transforming η into $\eta^{x,y}$. Since the transformations use only transitions in N, they also provide two ways of transforming $\tilde{\eta}$ into $\tilde{\eta}^{x,y}$ for any fixed $t_0 > 0$.

(a) First, move the particle from w to z along $z_1, z_2, \ldots, z_l$, i.e., for $\{i_j, 1 \leq j \leq J\}$ being successive values of i such that $\eta(z_{i_j}) = 1$, move the particle from z_{i_J} to z, then from z_{i_J} to $z_{i_{J-1}}$ and so on. Next, move the particle from x to y along $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ in the same way. Finally, move the particle from z back to w along $z_1, z_2, \ldots, z_l$.

(b) Move the particle from w to y along $y_1, y_2, \ldots, y_m$, then the particle from x to w along $x_1, x_2, \ldots, x_n$.

A visualization of the transformations in (a) and (b) is given in Figure 28. Note that in (a), the particle originally at w moves back to w. Since $\tau_w\eta \in B$ and B is invariant for the process $(\zeta_t)_{t\geq 0}$, we conclude that $\tau_w\tilde{\eta}^{x,y} \in B$ holds almost surely. In transformation (b), the particle originally at x moved to w. Since $\tau_x\eta \in A$ and A is invariant for $(\zeta_t)_{t\geq 0}$, we conclude that $\tau_w\tilde{\eta}^{x,y} \in A$ holds almost surely. Using the graphical representation, observe that η_{t_0} agrees with $\tilde{\eta}^{x,y}$ with positive probability. Hence, we obtain a contradiction to A and B being disjoint. □

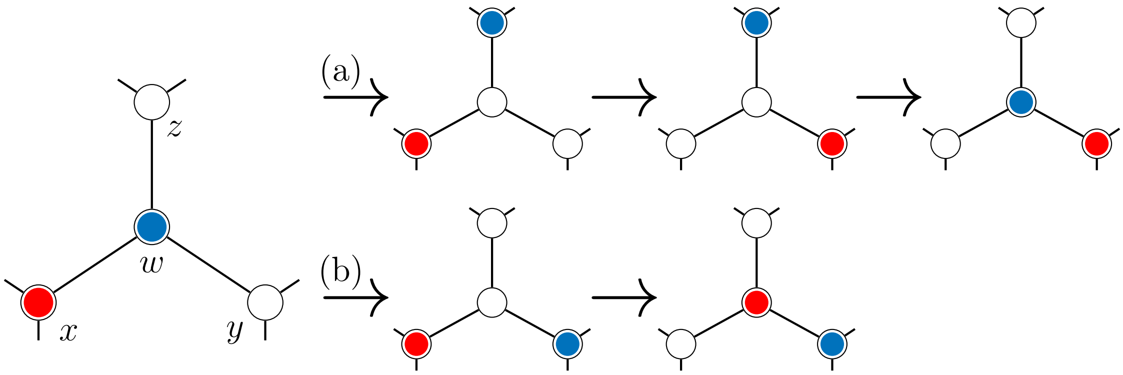

Figure 28: Transformations for η to $\eta^{x,y}$ in T^3 where x, y, z are neighbors of w.

8.4 Speed of the tagged particle

In this section, we prove that the tagged particle $(X_t)_{t\geq 0}$ on T^d for $d \geq 3$ satisfies a law of large numbers. As a first step, we show that $(X_t)_{t\geq 0}$ is **transient** for $d \geq 3$, i.e., $(X_t)_{t\geq 0}$ visits the root of T^d almost surely only finitely many times. To do so, we use the following framework introduced by Lyons et al. in [97] to study random walks on Galton–Watson trees. An infinite path $(x_0, x_1, \dots)$ of sites in T^d will be denoted by $\vec{x}$. We say that a path $\vec{x}$ is a **ray** ξ if it never backtracks, i.e., $x_i \neq x_j$ for all $i \neq j$. The set of rays starting at the root is called the **boundary** ∂T^d of the tree T^d. We say that a path $\vec{x}$ **converges** to a ray ξ if $\vec{x}$ visits every site at most finitely many times and ξ is the unique ray which is intersected infinitely often. For a site x and a ray ξ, let $[x, \xi]$ denote the unique ray starting in x and converging to ξ. Moreover, for two distinct sites $x, y \in V$, let $x \wedge_\xi y$ denote the site where $[x, \xi]$ and $[y, \xi]$ meet for the first time.

For two vertices $x, y \in V$, recall that $|x - y|$ denotes the shortest path distance between x and y. We define their **horodistance** with respect to some given ray ξ as the signed distance

$$\langle y - x\rangle_\xi := |y - x \wedge_\xi y| - |x - x \wedge_\xi y| \,, \tag{8.6}$$

see Figure 31 in Section 9 for a visualization of the horodistance on general trees. We set $\langle x\rangle_\xi := \langle x - o\rangle_\xi$ with respect to the root o of T^d. Throughout the rest of this section, let $\xi \in \partial T^d$ be an arbitrary, but fixed boundary point of T^d, which will in the following be omitted as a subscript in the notation of the horodistance. Note that without loss of generality, we can define the addition on T^d such that the horodistance defines a group homomorphism between $(T^d, +)$ and $(\mathbb{Z}, +)$, i.e., for all sites $x, y \in V$

$$\langle x + y\rangle = \langle x\rangle + \langle y\rangle \,. \tag{8.7}$$

Our goal is to show a law of large numbers for the stochastic process $(\langle X_t\rangle)_{t\geq 0}$ from which we will deduce Theorem 8.1. We define

$$\psi(\zeta) := \sum_{z\in V} p(|z|)(1-\zeta(z))\langle z\rangle \tag{8.8}$$

to be the **local drift at the root** for a configuration $\zeta \in \{0,1\}^V$ with $\zeta(o) = 1$. Recall the definition of the environment process $(\zeta_t)_{t\geq 0}$ in (8.2). We want to express $(\langle X_t\rangle)_{t\geq 0}$ in terms of $(\zeta_t)_{t\geq 0}$. Observe that $(X_t, \zeta_t)_{t\geq 0}$ is a Feller process whose generator is given by the closure of

$$\begin{aligned}\tilde{\mathcal{L}}f(x,\zeta) &= \sum_{y,z\neq o} p(|z-y|)\zeta(y)(1-\zeta(z))\left[f(x,\zeta^{y,z}) - f(x,\zeta)\right] \\ &\quad + \sum_{y\in V} p(|x-y|)(1-\zeta(y))\left[f(y,\tau_{y-x}\zeta) - f(x,\zeta)\right],\end{aligned}$$

and let $(\mathcal{F}_t)_{t\geq 0}$ denote the respective σ-algebra. Note that the process $(X_t)_{t\geq 0}$ on its own is in general not Markovian. We now decompose the process $(\langle X_t\rangle)_{t\geq 0}$ into a martingale and a function depending only on the environment process. This follows the ideas of Proposition 4.1 in [94, Part III].

Lemma 8.6. *For all $t \geq 0$, it holds that*

$$\langle X_t\rangle = \int_0^t \psi(\zeta_s)\mathrm{d}s + M_t\,, \tag{8.9}$$

where $(M_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Proof. Define the function $f(x,\zeta) := \langle x\rangle$. Observe that for this choice of f,

$$\tilde{\mathcal{L}}f(x,\zeta) = \sum_{y\in V} p(|x-y|)(1-\zeta(y))\left[f(y,\tau_{y-x}\zeta) - f(x,\zeta)\right] = \psi(\zeta) \tag{8.10}$$

holds using (8.7). It remains to show that the process $(M_t)_{t\geq 0}$ defined via the relation in (8.9) is indeed a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. In particular, for all $s < t$, we need to verify that

$$\mathbb{E}\left[M_t - M_s|\mathcal{F}_s\right] = 0$$

holds. Using the Markov property of $(X_t, \zeta_t)_{t\geq 0}$, we obtain that

$$\begin{aligned}\mathbb{E}\left[M_t - M_s|\mathcal{F}_s\right] &= \mathbb{E}\left[\langle X_t\rangle - \langle X_s\rangle - \int_s^t \psi(\zeta_r)\mathrm{d}r|\mathcal{F}_s\right] \\ &= \mathbb{E}_{(X_s,\zeta_s)}\left[\langle X_{t-s}\rangle - \langle X_0\rangle - \int_0^{t-s} \psi(\zeta_r)\mathrm{d}r\right],\end{aligned}$$

where $\mathbb{E}_{(X_s,\zeta_s)}[\,\cdot\,]$ denotes the expectation when starting the process from (X_s,ζ_s). In particular, it suffices to show that for fixed $x\in V$ and $\zeta\in\{0,1\}^V$,

$$\mathbb{E}_{(x,\zeta)}\left[\langle X_t\rangle-\langle x\rangle\right]-\int_0^t \mathbb{E}_{(x,\zeta)}\left[\psi(\zeta_r)\right]\mathrm{d}r=0$$

holds for all $t\geq 0$. Using (8.10), this follows immediately by Dynkin's formula; see (4.25) or Chapter 3 in [96]. □

Applying the results of Section 8.2, we obtain the following lemma as an immediate consequence, and as an analogue of Corollaries 4.5 and 4.16 in [94, Part III].

Lemma 8.7. *Suppose that $(\eta_t)_{t\geq 0}$ has initial distribution ν_ρ^* for some $\rho\in[0,1]$. Then the martingale $(M_t)_{t\geq 0}$ in Lemma 8.6 has stationary and ergodic increments.*

Proof. Observe that $\langle X_t\rangle$ can be expressed as a function F_t of $\{\zeta_s, 0\leq s\leq t\}$ for all $t\geq 0$ since all transitions of $(\langle X_t\rangle)_{t\geq 0}$ correspond precisely to the shifts in the environment process. In particular,

$$\langle X_t\rangle-\langle X_0\rangle=F_t(\zeta_s,0\leq s\leq t)$$

holds. Using that $(\zeta_t)_{t\geq 0}$ is stationary, Lemma 8.6 yields

$$M_t-M_s=F_{t-s}(\zeta_r,s\leq r\leq t)+\int_s^t\psi(\zeta_s)\mathrm{d}s$$

for all $s<t$. Recall from Propositions 8.3 and 8.4 that $(\zeta_t)_{t\geq 0}$ started from ν_ρ^* is stationary and ergodic, and hence, the claimed statement follows. □

We now show a law of large numbers for the process $(\langle X_t\rangle)_{t\geq 0}$. A similar statement for the tagged particle on $\mathbb{Z}^d$ can be found as Theorem 4.17 in [94, Part III].

Proposition 8.8. *For $d\geq 2$, let $(\eta_t)_{t\geq 0}$ on T^d have initial distribution ν_ρ^* for some $\rho\in(0,1)$. Then the associated tagged particle $(X_t)_{t\geq 0}$ satisfies*

$$\mathbb{E}\left[\langle X_t\rangle\right]=(1-\rho)(d-2)\sum_{i\in\mathbb{N}_0} ip(i)\cdot t=v\cdot t \tag{8.11}$$

for all $t\geq 0$. Moreover, almost surely

$$\lim_{t\to\infty}\frac{\langle X_t\rangle}{t}=v \tag{8.12}$$

holds. In particular, the tagged particle $(X_t)_{t\geq 0}$ on T^d is transient for $d\geq 3$.

Proof. For $d=2$, (8.11) with $v=0$ follows by symmetry and this easily implies (8.12), so we assume that $d\geq 3$. Taking expectations on both sides of (8.9) in Lemma 8.6

yields that

$$\mathbb{E}\left[\langle X_t\rangle\right] = \int_0^t \mathbb{E}\left[\psi(\zeta_s)\right] \mathrm{d}s + \mathbb{E}[M_t]\,.$$

Observe that $(\psi(\zeta_t))_{t\geq 0}$ is a stationary sequence. Hence,

$$\mathbb{E}\left[\psi(\zeta_t)\right] = \mathbb{E}\left[\psi(\zeta_0)\right] = (1-\rho)(d-2)\sum_{i\in\mathbb{N}_0} ip(i)$$

holds for all $t \geq 0$. Since $(M_t)_{t\geq 0}$ is a martingale, the statement in (8.11) follows. In order to show (8.12), recall Proposition 8.4 and Lemma 8.7, and apply the ergodic theorem to both terms on the right-hand side of (8.9), respectively, to conclude. □

As an immediate consequence of the transience of the tagged particle $(X_t)_{t\geq 0}$ for $d \geq 3$, and T^d being spherically symmetric, we obtain the following corollary.

Corollary 8.9. *For $d \geq 3$ and $\rho \in (0,1)$, let $\vec{x}$ denote the trajectory of the tagged particle $(X_t)_{t\geq 0}$ on T^d. Then $\vec{x}$ converges almost surely to a unique boundary point $x_{+\infty} \in \partial T^d$. Moreover, for any deterministic choice of $\xi \in \partial T^d$, $x_{+\infty} \neq \xi$ holds almost surely.*

Proof of Theorem 8.1. By Corollary 8.9, we almost surely have for all $t \geq 0$ sufficiently large that

$$|X_t| = \langle X_t\rangle + 2|w|\,, \tag{8.13}$$

where w is the last common vertex of $x_{+\infty}$ and ξ. Since ξ was arbitrary, but fixed at the beginning, w is well defined and $|w|$ is almost surely finite. Since $|w|$ does not depend on t, we obtain Theorem 8.1 from Proposition 8.8. □

8.5 Diffusivity of the tagged particle

In order to prove Theorem 8.2, we show a central limit theorem for the process $(\langle X_t\rangle)_{t\geq 0}$. Recall from (8.9) that $(\langle X_t\rangle)_{t\geq 0}$ can be decomposed into a martingale $(M_t)_{t\geq 0}$ and a process $\int_0^t \psi(\zeta_s)\mathrm{d}s$. For $p(\cdot)$ and v taken from Section 8.1.3, we define

$$\bar{\psi}(\zeta) := \psi(\zeta) - v = \sum_{x\in V} p(|x|)\langle x\rangle(\rho - \zeta(x))\,.$$

Our goal is to establish a similar decomposition for the process $\int_0^t \bar{\psi}(\zeta_s)\mathrm{d}s$. Let $L^2(\nu_\rho^*)$ denote the Hilbert space of square integrable functions with respect to ν_ρ^* and scalar product

$$\langle f,g\rangle_{\nu_\rho^*} := \int fg\mathrm{d}\nu_\rho^*\,.$$

Observe that the environment process $(\zeta_t)_{t\geq 0}$ with generator L from (8.3) is reversible with respect to ν_ρ^* for all $\rho \in [0,1]$. For a function $f \in L^2(\nu_\rho^*)$ in the domain of L, we define its **$\|\cdot\|_1$-norm** to be

$$\|f\|_1 := \sqrt{\langle f, (-L)f \rangle_{\nu_\rho^*}}\,.$$

Let $\mathcal{H}_1$ denote the respective Hilbert space generated by all local functions f of finite $\|\cdot\|_1$-norm. We define its dual space $\mathcal{H}_{-1}$ to be the Hilbert space generated by all local functions which have a finite norm with respect to

$$\|f\|_{-1} := \inf \left\{ C \geq 0 \colon \left| \int f g \mathrm{d}\nu_\rho^* \right| \leq C \|g\|_1 \text{ for all local functions } g \right\}.$$

The following result was shown by Sethuraman et al. for the exclusion process on $\mathbb{Z}^d$ with $d \geq 3$, and carries over to T^d for $d \geq 3$; see Lemma 2.1 in [130].

Proposition 8.10. *For $d \geq 3$, we have $\bar{\psi} \in \mathcal{H}_{-1}$.*

Note that the proof in [130] only uses the transience of the simple random walk on the underlying graph T^d as well as the fact that $\bar{\psi}$ is a bounded, local function of zero mean. The next proposition is a special case of the seminal result by Kipnis and Varadhan on additive functionals of reversible Markov processes; see also Theorem 1.8 in [79].

Proposition 8.11. *Assume that $\bar{\psi} \in L^2(\nu_\rho^*) \cap \mathcal{H}_{-1}$ has mean zero. Then $\int_0^t \bar{\psi}(\zeta_s)\mathrm{d}s$ can be decomposed into a square integrable martingale $(N_t)_{t\geq 0}$ with stationary increments and a stochastic process $(R_t)_{t \geq 0}$, i.e.,*

$$\int_0^t \bar{\psi}(\zeta_s)\mathrm{d}s = N_t + R_t\,,$$

where $(R_t)_{t\geq 0}$ satisfies $\lim_{t\to\infty} t^{-1} \cdot \mathbb{E}\left[R_t^2\right] = 0$.

Proof of Theorem 8.2. A simple computation shows that the martingale $(M_t)_{t\geq 0}$ satisfies a central limit theorem with non-degenerate limit variance; see Proposition 4.19 in [94, Part III]. Combining Propositions 8.10 and 8.11, we can now apply a martingale central limit theorem to the process $(M_t + N_t)_{t\geq 0}$. To see that the limit variance of this process is non-degenerate, observe that by Lemma 3.8 in [130], there exists a constant $C > 0$ such that

$$\left| \int \bar{\psi} g \mathrm{d}\nu_\rho^* \right| \leq C \sqrt{\mathcal{D}_{\mathrm{ex}}(g)}$$

holds for all local functions g, where

$$\mathcal{D}_{\mathrm{ex}}(g) := \frac{1}{4} \int \sum_{x,y \neq o} p(x,y) \left[g(\eta_{x,y}) - g(\eta) \right]^2 \mathrm{d}\nu_\rho^*\,.$$

We then apply the same arguments which were used in the proof of Theorem 4.55 in [94, Part III]. Together with (8.13), this yields Theorem 8.2. □

9 The simple exclusion process on Galton–Watson trees

9.1 Introduction

So far, we studied the simple exclusion process when the underlying graph has a regular structure, i.e., it is the integer lattice or a regular tree. In the following, we allow that the underlying graph is chosen randomly, namely as a supercritical, augmented Galton–Watson tree without extinction. We will consider two different models of simple exclusion processes, the variable speed model where a particle attempts to cross all adjacent edges with rate 1, respectively, and the constant speed model where particles wait according to rate 1 Poisson clocks before attempting a jump to some uniformly chosen neighbor. Precise descriptions for both cases are given in Section 9.1.1. After choosing the tree, we keep the tree fixed and start from a stationary distribution, where we condition on initially having a tagged particle in the root. In both models, we study the evolution of the tagged particle over time. The presented material is based in large parts on [67], which is joint work with Nina Gantert.

Our motivation is two-fold. On the one hand, the speed of a tagged particle in exclusion processes on $\mathbb{Z}^d$ was extensively studied; see Section 7.1. In particular, recall the simple exclusion process on the d-dimensional lattice in Theorem 7.3, where the transition probabilities are given by a random walk with drift, and where we start from a Bernoulli-ρ-product measure. In this setup, the speed of the tagged particle is $1 - \rho$ times the speed of a single particle. Intuitively, this is plausible as the density of empty sites is $1 - \rho$, and thus only a proportion of $1 - \rho$ of the steps is carried out. We saw in Theorem 8.1 that the same formula continues to hold for exclusion processes on regular trees. In this case, a crucial tool are the explicitly known invariant measures. We consider Galton–Watson trees as an example for a random environment, where invariant measures for the exclusion process are known. In the variable speed model, the Bernoulli-ρ-product measures are invariant distributions and the speed of the tagged particle is again $1 - \rho$ times the speed of a single particle. For the constant speed model, we have invariant measures which are still product measures, but not with identical marginals, and we see a different formula for the speed.

On the other hand, random walks on Galton–Watson trees are intensively investigated. We refer to the seminal work [97] and Section 9.1.2 for a discussion of related results. We consider tagged particles in exclusion processes as a natural generalizations of random walks, treating the tagged particle as a random walk in a dynamical random environment. One difficulty is that the position of the tagged particle is not a Markov chain, and thus the proof of its transience is not straightforward; see Section 9.4.

9.1.1 Definition of the model

Our goal is to study the simple exclusion process on randomly chosen trees. In particular, we choose the underlying graph to be an **augmented Galton–Watson tree** T with vertex set $V(T)$, edge set $E(T)$, and root $o \in V(T)$. More precisely, we have the following construction.

Let $(p_k)_{k\in\mathbb{N}_0}$ be a sequence of non-negative numbers with $\sum_{k=0}^{\infty} p_k = 1$, which defines the **offspring distribution** of the tree, i.e., a random variable according to the offspring distribution takes the value k with probability p_k for all $k \in \mathbb{N}_0$. We construct T in such a way that each site has precisely $k+1$ neighbors with probability p_k for all $k \in \mathbb{N}_0$ independently of all other sites. To do so, define a starting vertex o, the root, and recursively, starting from o, let every site have a number of descendants drawn independently according to the offspring distribution. The resulting tree is called **Galton–Watson tree**. Since in this construction, the root has on average one neighbor less than all other sites, we add one additional descendant to o and apply the same recursion in order to obtain an **augmented Galton–Watson tree**. We will in the following assume that the underlying Galton–Watson branching process is supercritical and without extinction, i.e.,

$$p_0 = 0 \qquad \text{and} \qquad \mathfrak{m} := \sum_{k\geq 1} k p_k \in (1,\infty) \tag{9.1}$$

holds. In particular, the corresponding augmented Galton–Watson tree is almost surely locally finite since every vertex has only a finite number of descendants.

Recall the construction of a simple exclusion process from (3.3) for $c \equiv 0$ and nearest-neighbor transition rates $p(x,y)$. Let $T = (V,E)$ be a supercritical, augmented Galton–Watson tree without extinction and consider the following two ways of defining a simple exclusion process on T. For $p(x,y) = \mathbb{1}_{\{x,y\}\in E}$, we refer to the resulting process $(\eta_t^{\mathrm{v}})_{t\geq 0}$ as the **variable speed model**, for $p(x,y) = \deg(x)^{-1}\mathbb{1}_{\{x,y\}\in E}$, we call $(\eta_t^{\mathrm{c}})_{t\geq 0}$ the **constant speed model** of the simple exclusion process; see Section 5.1. The terms "variable speed model" and "constant speed model" go back to [12] who consider random walks among random conductances. In words, each particle in the variable speed model at a site x has an exponential waiting time with parameter $\deg(x)$ independently of all other particles. When the time is up, it jumps to one of its neighbors uniformly at random under an exclusion rule. In the constant speed model, the particles have an exponential waiting time with parameter 1. They then choose one neighbor uniformly at random and jump to the selected site if it is vacant. Note that the two models of the simple exclusion process agree for regular graphs up to a deterministic time change. The next result states that the simple exclusion process on Galton–Watson trees in both models is almost surely a Feller process.

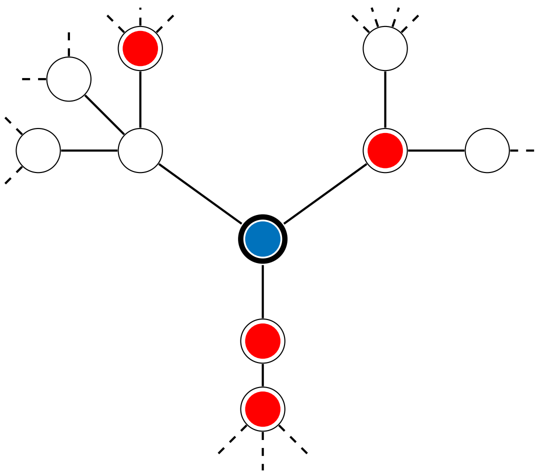

Figure 29: Visualization of the tagged particle, drawn in blue, in a configuration of the simple exclusion process on a Galton–Watson tree.

Proposition 9.1. *For almost every Galton–Watson tree T, the simple exclusion processes according to the variable speed model and the constant speed model on T are well-defined Feller processes whose generators are given by* (3.3)*.*

Proof. Observe that bond percolation with parameter p on a Galton–Watson tree with offspring mean $\mathfrak{m}$ gives a Galton–Watson tree of mean $\mathfrak{m}p$. Hence, $p_G = \frac{1}{\mathfrak{m}} \in (0,1)$ holds for almost every supercritical Galton–Watson tree without extinction; see also Proposition 5.9 in [99]. We conclude Proposition 9.1 from Theorem 3.2. □

For a given realization T of an augmented Galton–Watson tree, we describe a parametrized set of invariant measures with respect to both models of the simple exclusion process. For $\rho \in [0,1]$, let $\nu_{\rho,T}$ denote the Bernoulli-ρ-product measure on $\{0,1\}^{V(T)}$, i.e.,

$$\nu_{\rho,T}\left(\eta(x)=1\right)=\rho \tag{9.2}$$

for all $x \in V(T)$. For $\alpha \in [0,\infty)$, let $\tilde{\nu}_{\alpha,T}$ denote the product measure on $\{0,1\}^{V(T)}$ with marginals

$$\tilde{\nu}_{\alpha,T}\left(\eta(x)=1\right)=\frac{\alpha\deg(x)}{1+\alpha\deg(x)} \tag{9.3}$$

for all $x \in V(T)$. The fact that the measures $\nu_{\rho,T}$ and $\tilde{\nu}_{\alpha,T}$ are reversible, and hence invariant, for the simple exclusion processes $(\eta_t^{\mathrm{v}})_{t\geq 0}$ in the variable speed model and $(\eta_t^{\mathrm{c}})_{t\geq 0}$ in the constant speed model, respectively, whenever $\rho \in [0,1]$ or $\alpha \in [0,\infty)$ holds, follows from Theorem 3.7. When we condition to initially have a particle in the root, recall that we denote the resulting Palm measures by $\nu_{\rho,T}^*$ and $\tilde{\nu}_{\alpha,T}^*$ on $\{0,1\}^{V(T)}$; see (7.1). For $\rho=0$ and $\alpha=0$, we use the convention that the simple exclusion process started from $\nu_{0,T}^*$, respectively $\tilde{\nu}_{0,T}^*$, is the simple random walk on T in the respective model starting in the root o. In the following, our goal is to investigate the motion of the tagged particle in both models, where we denote by $(X_t^{\mathrm{v}})_{t\geq 0}$ the position of the tagged particle in T in the variable speed model and by $(X_t^{\mathrm{c}})_{t\geq 0}$ its position in T in the constant speed model of the simple exclusion process; see also Figure 29.

9.1.2 Related literature

In the last decades, many results for random walks on Galton–Watson trees were achieved. The study of random walks on Galton–Watson trees goes back to Grimmett and Kesten who proved that the simple random walk on supercritical Galton–Watson trees conditioned on non-extinction is almost surely transient [70]. Lyons et al. showed that this random walk has almost surely a positive linear speed and calculated the speed explicitly [97]. The case of a simple random walk on Galton–Watson trees with bias was studied by Lyons et al. in [98]. More recent progress on the speed of random walks on Galton–Watson trees includes [3, 5, 64], among others. An introduction to random walks on Galton–Watson trees can be found in Chapter 17 of [99].

For the simple exclusion process in a one-dimensional random environment, Chayes and Liggett investigated the set of invariant distributions [28]. The tagged particle in one-dimensional random environments was studied by Jara and Landim [75]. To our best knowledge, this is the first time the tagged particle in the simple exclusion process is studied on random graphs. Note that a crucial difference for the study of the tagged particle in an exclusion process on augmented Galton–Watson trees compared to $\mathbb{Z}^d$ or T^d from Sections 7.1 and 8 is that the underlying graphs are no longer transitive. However, we will be able to recover the transitivity of the graphs “in distribution” since all sites have the same degree distribution by construction.

9.1.3 Main result

Our main result is to establish a law of large numbers for the tagged particle in the simple exclusion process when starting from a Palm measure on an augmented Galton–Watson tree. For a rooted tree T and $x \in V(T)$, recall that we write $|x|$ for the shortest path distance from the root.

Theorem 9.2. *Let Z be distributed according to the offspring distribution. Then for almost every augmented Galton–Watson tree T, the following holds:*

(i) Variable speed: Let $(\eta_t^{\mathrm{v}})_{t\geq 0}$ on T have initial distribution $\nu^*_{\rho,T}$ for some $\rho \in [0,1)$. Then

$$\lim_{t\to\infty} \frac{|X_t^{\mathrm{v}}|}{t} = (1-\rho)\mathbb{E}\left[\frac{Z-1}{Z+1}\right]\left(\mathbb{E}\left[\frac{1}{Z+1}\right]\right)^{-1} \tag{9.4}$$

almost surely.

(ii) *Constant speed: Let $(\eta_t^{\mathrm{c}})_{t\geq 0}$ on T have initial distribution $\tilde{\nu}^*_{\alpha,T}$ for some $\alpha \in [0,\infty)$. Then*

$$\lim_{t\to\infty} \frac{|X_t^{\mathrm{c}}|}{t} = \mathbb{E}\left[\frac{Z-1}{Z+1}\frac{1}{\alpha(Z+1)+1}\right] \tag{9.5}$$

almost surely.

In particular, the tagged particle has almost surely a strictly positive speed.

If the tree is regular, i.e., $Z \equiv \mathfrak{m}$, then $1-\rho$ corresponds to $\frac{1}{\alpha(\mathfrak{m}+1)+1}$, comparing with (9.2) and (9.3). In particular, the two formulas (9.4) and (9.5) agree up to the deterministic time change $\left(\mathbb{E}\left[\frac{1}{Z+1}\right]\right)^{-1}$. We make the following remarks.

Remark 9.3. *(i) In the constant speed model for $\alpha \to 0$, we recover the result of Lyons et al. on the speed of a random walk on supercritical Galton–Watson trees without extinction [97].*

(ii) For the variable speed model, we see a linear scaling in the density $1-\rho$ of empty sites. Similar results hold for an exclusion process with drift on $\mathbb{Z}^d$ and without drift on the regular tree; see Theorems 7.3 and 8.1.

(iii) In the constant speed model

$$\mathbb{E}\left[\frac{Z-1}{Z+1}\frac{1}{\alpha(Z+1)+1}\right] \leq \mathbb{E}\left[\frac{Z-1}{Z+1}\right]\mathbb{E}\left[\frac{1}{\alpha(Z+1)+1}\right]$$

holds, with strict inequality unless $Z \equiv \mathfrak{m}$. Hence, in general the scaling of the speed is lower than linear in the averaged density of empty sites.

9.1.4 Outline of the proof

Our formulas for the speed of the tagged particle rely on explicitly given invariant measures for the environment seen from the tagged particle. We show that the environment process started from these invariant measures is ergodic. We use two sets of techniques, one coming from random walks on Galton–Watson trees and the other one from exclusion processes. In contrast to the case of regular trees from Section 8, our proof requires to intertwine the techniques from both sets.

In particular, we have the following strategy in order to prove Theorem 9.2. In Section 9.2, we define a common probability space for locally finite, rooted trees and the respective exclusion processes on them. This will allow us to study the environment process in Section 9.3, which can be interpreted as the exclusion process "seen from the tagged particle". We provide stationary measures for the environment process in both models of the simple exclusion process. The arguments in this section are based on the ideas of Lyons et al. for studying random walks on Galton–Watson trees.

Since the motion of the tagged particle is itself not a Markov process, a crucial step is to show that the tagged particle is transient. This is accomplished in Section 9.4 by combining the results of Section 9.3 on Galton–Watson trees and martingale techniques from [94, Part III, Section 4] for the motion of a tagged particle in exclusion processes. A similar approach was used in Proposition 8.8 to show the transience of the tagged particle on regular trees.

In Section 9.5, we show ergodicity for the environment process. We do this by intertwining different techniques coming from random walks on Galton–Watson trees and interacting particle systems, i.e., we combine coupling arguments of Saada in [114] for the exclusion process on $\mathbb{Z}^d$ with drift, with regeneration time arguments of Lyons and Peres in Chapter 17 of [99] for the random walk on Galton–Watson trees. Using the ergodicity of the environment process, we deduce a law of large numbers for the position of the tagged particle in Section 9.6, and then conclude this section with an outlook on related open problems.

9.2 Spaces and measures for trees

In this section, we introduce spaces and measures for rooted trees which allow us to study the simple exclusion process and locally finite, rooted trees on a common probability space. We write $(T, o) \in \mathcal{T}$ for a tree T with root o, where $\mathcal{T}$ denotes the space of all rooted, locally finite trees. Recall from Section 3.5 that we denote by $B_r(T, o)$ the ball of radius r around the root of T with respect to the graph distance. We say that two rooted trees $(T, o), (T', o') \in \mathcal{T}$ are **isomorphic** on a ball of radius r, and write $B_r(T, o) \cong B_r(T', o')$, if there exists a bijection $\phi\colon B_r(T, o) \to B_r(T', o')$ such that $\phi(o) = o'$ and $\{x, y\} \in E(T)$ for $x, y \in B_r(T, o)$ holds if and only if $\{\phi(x), \phi(y)\} \in E(T')$. In words, two trees are isomorphic on a ball of radius r around the root when the sites of distance at most r from the root can be mapped one-to-one such that the adjacency structure of the tree is preserved.

The space $\mathcal{T}$ will be equipped with the **local topology**, that is the topology introduced by the distance function $\tilde{d}_{\text{loc}}$ on $\mathcal{T}$ given by

$$\tilde{d}_{\text{loc}}((T, o), (T', o')) := \frac{1}{1 + \tilde{R}}$$

for all trees $(T, o), (T', o') \in \mathcal{T}$, where

$$\tilde{R} = \sup\{r \in \mathbb{N}_0 \colon B_r(T, o) \cong B_r(T', o')\} .$$

In particular, the open sets on $\mathcal{T}$ will be generated by the sets of all trees which are isomorphic on a ball of radius r for some $r \geq 0$. Note that $\tilde{d}_{\text{loc}}$ is not a metric, but only a pseudo-metric, since trees which are isomorphic on $B_r(T,o)$ for every $r \geq 0$ have $\tilde{d}_{\mathrm{loc}}$-distance 0. In order to turn $\mathcal{T}$ together with $\tilde{d}_{\mathrm{loc}}$ into a metric space, we consider isomorphism classes of trees. We say that two trees $(T,o),(T',o') \in \mathcal{T}$ are **isomorphic** if $\tilde{d}_{\mathrm{loc}}((T,o),(T',o')) = 0$, and write $[\mathcal{T}]$ for the set of isomorphism classes. It is a well-known result that $([\mathcal{T}], \tilde{d}_{\mathrm{loc}})$ forms a Polish space; see [97].

With a slight abuse of notation, we denote in the following the space of all 0/1-colored, locally finite, rooted trees by

$$\Omega := \left\{(T,o,\eta) : \eta \in \{0,1\}^{V(T)}, (T,o) \in \mathcal{T}\right\} , \tag{9.6}$$

as our (extended) state space for the simple exclusion process. We let $B_r(T,o,\eta)$ denote the ball of radius r around the root o of T where each site receives a color 0 or 1 according to η. Similar to the case of unlabeled trees, we say that (T,o,η) and (T',o',η') are **isomorphic** on a ball of radius r, and write $B_r(T,o,\eta) \cong B_r(T',o',\eta')$, if $B_r(T,o) \cong B_r(T',o')$ for some bijection ϕ as well as $\eta(v) = \eta'(\phi(v))$ holds for all $v \in B_r(T,o)$. The space Ω is equipped with the topology induced by

$$d_{\mathrm{loc}}((T,o,\eta),(T',o',\eta')) := \frac{1}{1+R}$$

with

$$R = \sup\{r \in \mathbb{N}_0 \colon B_r(T,o,\eta) \cong B_r(T',o',\eta')\}$$

for all $(T,o,\eta),(T',o',\eta') \in \Omega$. Again, we will in the following consider isomorphism classes of 0/1-colored trees. As before, one can show that this yields a Polish space $([\Omega], d_{\mathrm{loc}})$; see Lemma 2.3 in [115] for a proof.

For a fixed tree $(T,o) \in \mathcal{T}$, we define

$$\Omega_T := \left\{(T,o,\eta) \in \Omega \colon \eta \in \{0,1\}^{V(T)}\right\} \subseteq \Omega$$

to be the space of 0/1-configurations on (T,o). Moreover, let

$$\tilde{\Omega}_T := \left\{(T,x,\eta) \in \Omega \colon \eta \in \{0,1\}^{V(T)}, x \in V(T)\right\} \subseteq \Omega \tag{9.7}$$

be the space of 0/1-configurations on (T,o) and on all isomorphism classes of trees obtained from (T,o) by shifting the root. In addition, we denote by

$$\Omega^* := \{(T,o,\eta) \in \Omega \colon \eta(o) = 1\} \subseteq \Omega$$

the set of configurations in Ω with occupied root, and define Ω_T^* and $\tilde{\Omega}_T^*$ similarly. Note that forming the above subspaces is consistent under taking isomorphism classes of trees, e.g., $[\Omega_T \cap \Omega] = [\Omega_T] \cap [\Omega]$. From now on, we will only work with isomorphism classes of trees and drop the brackets in the notation. Let us stress that we define all probability measures on the subspaces of $(\mathcal{T}, \tilde{d}_{\text{loc}})$ and (Ω, d_{loc}) with respect to the Borel-σ-algebra.

Let **GW** denote the **Galton–Watson measure** on $\mathcal{T}$ which is induced by the Galton–Watson branching process; see Chapter 4 of [99]. More precisely, we define **GW** for families of rooted trees

$$\mathcal{T}_T(r) := \{(T', o') \in \mathcal{T} \colon B_r(T', o') = B_r(T, o)\}$$

with $r \in \mathbb{N}$ and $(T, o) \in \mathcal{T}$ fixed. The measure **GW** assigns now to $\mathcal{T}_T(r)$ the probability that the genealogical tree of a branching process according to the offspring distribution agrees with $B_r(T, o)$ up to generation r. A standard extension argument yields the probability measure **GW** on $\mathcal{T}$. In the same way, we define **AGW** to be the **augmented Galton–Watson measure** on $\mathcal{T}$ by taking a branching process where the first particle has one additional child. One may also define **AGW** directly on $\mathcal{T}$ by choosing two independent trees according to **GW** and joining their roots by an edge; see Chapter 17 of [99].

The simple exclusion process in the variable speed model is now a process on Ω with the initial distribution

$$\mathbb{P}^{\text{v}}_{\rho} := \mathbf{AGW} \times \nu^*_{\rho,T} \tag{9.8}$$

being a semi-direct product of **AGW** on $\mathcal{T}$ and $\nu^*_{\rho,T}$. Similarly, the simple exclusion process in the constant speed model is a process on Ω with the initial distribution

$$\mathbb{P}^{\text{c}}_{\alpha} := \mathbf{AGW} \times \tilde{\nu}^*_{\alpha,T} \tag{9.9}$$

being a semi-direct product of **AGW** on $\mathcal{T}$ and $\tilde{\nu}^*_{\alpha,T}$. In particular, this construction as a semi-direct product defines $\mathbb{P}^{\text{v}}_{\rho}$ for 0/1-colored balls of radius r. However, in contrast to $\mathbb{P}^{\text{v}}_{\rho}$, we can not determine $\mathbb{P}^{\text{c}}_{\alpha}$ for 0/1-colored balls of radius r in a direct way. This is due to the fact that in order to determine the color of a vertex at distance r according to $\tilde{\nu}^*_{\alpha,T}$, one has to know the number of its adjacent sites at distance $r+1$ from the root. To remedy this problem, we condition according to the number of children in the $(r+1)^{\text{th}}$ generation for each site at level r. For the resulting balls of radius $r+1$ with colors only up to level r, we can now make sense of the measure $\mathbb{P}^{\text{c}}_{\alpha}$. We conclude this section by noting that for a fixed augmented Galton–Watson tree $(T, o) \in \mathcal{T}$, the simple exclusion process on (T, o) is **AGW**-almost surely a Feller process with values in the space Ω_T for both models. Hence, instead of working with the measures $\tilde{\nu}^*_{\alpha,T}$ and $\nu^*_{\rho,T}$ on a fixed Galton–Watson tree $(T, o) \in \mathcal{T}$, we will from now on study the measures $\mathbb{P}^{\text{c}}_{\alpha}$ and $\mathbb{P}^{\text{v}}_{\rho}$ on the space Ω, and restrict the space to Ω_T whenever we condition on a certain underlying tree $(T, o) \in \mathcal{T}$.

9.3 Stationarity for the environment process

In Section 8.2, we introduced the environment process for exclusion processes on regular trees as the process "seen from the tagged particle". We will now study a corresponding environment process for the simple exclusion process on augmented Galton–Watson trees. Intuitively, it is given as the following Markov process with values in Ω^*. The state of the environment process at time t is the 0/1-colored tree given by the configuration of the exclusion process on the original tree whose root is shifted to the position of the tagged particle at time t. Its state can change in two ways: either the coloring outside the root changes according to the exclusion process, or the root of the tree is shifted. The latter happens if and only if the tagged particle moves; see below for precise definitions.

Recall that the environment process is a common approach to prove a law of large numbers for random walks in random environment or tagged particles in exclusion processes; see [141] and Theorems 7.3 and 8.1. However, a crucial difference to these examples is that the underlying tree for the environment process on augmented Galton–Watson trees changes with time, which requires a more detailed analysis. Recall that $\sim$ denotes the relation of two sites being adjacent. For the simple exclusion process on Ω with transition rates $p(\cdot,\cdot)$, we define the corresponding environment process to be the Feller process with state space Ω^* generated by the closure of

$$\begin{aligned} Lf(T,o,\zeta) = &\sum_{x,y\neq o} p(x,y)\zeta(x)(1-\zeta(y))\left[f(T,o,\zeta^{x,y}) - f(T,o,\zeta)\right] \\ &+ \sum_{z\sim o} p(o,z)(1-\zeta(z))\left[f(T,z,\zeta^{o,z}) - f(T,o,\zeta)\right] \end{aligned} \tag{9.10}$$

for all cylinder functions f. We write L^{v} and L^{c} for the generators of the environment process of the simple exclusion process in the variable speed model and in the constant speed model, respectively. Note that the generator can be split into two parts, namely into transitions which do only exchange particles and do not change the underlying tree, as well as into transitions which involve the root of the tree. More precisely, we define the generators

$$\begin{aligned} (L^{\mathrm{c}}_{\mathrm{ex}}f)(T,o,\zeta) &:= \sum_{x,y\neq o} \frac{1}{\deg(x)}\zeta(x)(1-\zeta(y))\left[f(T,o,\zeta^{x,y}) - f(T,o,\zeta)\right] \\ (L^{\mathrm{c}}_{\mathrm{sh}}f)(T,o,\zeta) &:= \sum_{z\sim o} \frac{1}{\deg(o)}(1-\zeta(z))\left[f(T,z,\zeta^{o,z}) - f(T,o,\zeta)\right] \end{aligned}$$

for the environment process in the constant speed model for $(T,o,\zeta) \in \Omega^*$ and all cylinder functions f. The generators $L^{\mathrm{v}}_{\mathrm{ex}}$ and $L^{\mathrm{v}}_{\mathrm{sh}}$ for the environment process in the variable speed model are defined analogously.

We want to investigate the invariant measures of the environment process. We provide two classes of reversible measures for the environment process, $\mathbb{Q}^{\mathrm{v}}_{\rho}$ for $\rho \in (0,1)$ and $\mathbb{Q}^{\mathrm{c}}_{\alpha}$ for $\alpha \in (0,\infty)$, such that $\mathbb{Q}^{\mathrm{v}}_{\rho}$ and $\mathbb{P}^{\mathrm{v}}_{\rho}$, respectively $\mathbb{Q}^{\mathrm{c}}_{\alpha}$ and $\mathbb{P}^{\mathrm{c}}_{\alpha}$, are equivalent, i.e., mutually absolutely continuous, for all $\rho \in (0,1)$ and $\alpha \in (0,\infty)$. Let us stress once again that we work on isomorphism classes of trees in order to properly define stationary measures for the environment process.

For the environment process in the variable speed model, we will use the ideas of Aldous and Lyons [3]. Consider the **unimodular Galton–Watson measure UGW** which we obtain from **AGW** by weighting a tree according to the reciprocal of the degree of its root, i.e.,

$$\frac{\mathrm{d}\mathbf{UGW}}{\mathrm{d}\mathbf{AGW}}(T,o) = \left(\mathbb{E}\left[\frac{1}{Z+1}\right]\right)^{-1} \cdot \frac{1}{\deg(o)} \tag{9.11}$$

for $(T,o) \in \mathcal{T}$, where Z is distributed according to the offspring distribution. We define $\mathbb{Q}^{\mathrm{v}}_{\rho}$ on Ω^* to be the probability measure given as the semi-direct product

$$\mathbb{Q}^{\mathrm{v}}_{\rho} := \mathbf{UGW} \times \nu^*_{\rho,T} \tag{9.12}$$

for all $\rho \in (0,1)$. As pointed out by the authors of [3], the measure **AGW** on $\mathcal{T}$ gives the environment process a natural bias proportional to the degree of the root. This bias is compensated by the Radon–Nikodym derivative in (9.11). For the environment process in the constant speed model with parameter $\alpha \in (0,\infty)$, we let $\mathbb{Q}^{\mathrm{c}}_{\alpha}$ denote the probability measure on Ω^* which is absolutely continuous with respect to $\mathbb{P}^{\mathrm{c}}_{\alpha}$ and satisfies

$$\frac{\mathrm{d}\mathbb{Q}^{\mathrm{c}}_{\alpha}}{\mathrm{d}\mathbb{P}^{\mathrm{c}}_{\alpha}}(T,o,\zeta) = \left(\mathbb{E}\left[\frac{1}{\alpha(Z+1)+1}\right]\right)^{-1} \cdot \frac{1}{\alpha\deg(o)+1} \tag{9.13}$$

for all $(T,o,\zeta) \in \Omega^*$, where Z is distributed according to the offspring distribution. We want to provide some intuition for the Radon–Nikodym derivative in (9.13). Observe that the semi-direct product $\mathbf{AGW} \times \tilde{\nu}_{\alpha,T}$ satisfies

$$(\mathbf{AGW} \times \tilde{\nu}_{\alpha,T})\,(\deg(o) = k | \zeta(o) = 1) = \frac{\alpha k}{\alpha k + 1} \cdot \frac{p_{k-1}}{\sum_{k\geq 1} \frac{\alpha k}{\alpha k+1} p_{k-1}}$$

for all $k \geq 1$. Since the root is always occupied in the environment process, we expect to see a similar weighting of the degree of o within $\mathbb{Q}^{\mathrm{c}}_{\alpha}$. Recall $\mathbf{AGW}\,(\deg(o) = k) = p_{k-1}$ for all $k \in \mathbb{N}$. Since **AGW** provides for the environment process a natural bias proportional to the degree of the root o, it remains to include the factor of $\frac{1}{\alpha k+1}$ for $\mathbb{Q}^{\mathrm{c}}_{\alpha}$. We now show that $\mathbb{Q}^{\mathrm{v}}_{\rho}$ and $\mathbb{Q}^{\mathrm{c}}_{\alpha}$ are indeed reversible measures for the environment process for all $\rho \in (0,1)$ and $\alpha \in (0,\infty)$, respectively. For an introduction to reversibility of Feller processes, we refer to Section 3.2.2.

Proposition 9.4. *Fix parameters $\rho \in (0,1)$ and $\alpha \in (0,\infty)$ for the measures $\mathbb{Q}^{\mathrm{v}}_\rho$ and $\mathbb{Q}^{\mathrm{c}}_\alpha$, respectively. Then the following statements hold.*

(i) The measure $\mathbb{Q}^{\mathrm{v}}_\rho$ is reversible for the environment process generated by L^{v}.

(ii) The measure $\mathbb{Q}^{\mathrm{c}}_\alpha$ is reversible for the environment process generated by L^{c}.

In particular, the measures $\mathbb{Q}^{\mathrm{v}}_\rho$ and $\mathbb{Q}^{\mathrm{c}}_\alpha$ are invariant for the environment process in the variable speed model and the constant speed model, respectively.

Proof. It suffices to show reversibility with respect to the different parts of the generators L^{v} and L^{c}. By construction, the processes on Ω^* associated to the generators $L^{\mathrm{v}}_{\mathrm{ex}}$ and $L^{\mathrm{c}}_{\mathrm{ex}}$, respectively, leave the underlying tree unchanged and ignore all moves involving the root. Recall that for a fixed $(T,o) \in \mathcal{T}$, the measures $\nu_{\rho,T}$ and $\tilde{\nu}_{\alpha,T}$ are invariant for the simple exclusion process on (T,o) for all $\rho \in (0,1)$ in the variable speed model as well as for all $\alpha \in (0,\infty)$ in the constant speed model, respectively; see Theorem 3.4. By Theorem 3.7, we obtain the reversibility of the measures $\nu_{\rho,T}$ and $\tilde{\nu}_{\alpha,T}$, i.e., for all $(T,o) \in \mathcal{T}$, and for all cylinder functions f and g

$$\int f(\eta)(\mathcal{L}^{\mathrm{v}} g)(\eta)\mathrm{d}\nu_{\rho,T} = \int (\mathcal{L}^{\mathrm{v}} f)(\eta) g(\eta)\mathrm{d}\nu_{\rho,T}$$

as well as

$$\int f(\eta)(\mathcal{L}^{\mathrm{c}} g)(\eta)\mathrm{d}\tilde{\nu}_{\alpha,T} = \int (\mathcal{L}^{\mathrm{c}} f)(\eta) g(\eta)\mathrm{d}\tilde{\nu}_{\alpha,T}$$

holds for all $\rho \in (0,1)$ and $\alpha \in (0,\infty)$, where $\mathcal{L}^{\mathrm{v}}$ and $\mathcal{L}^{\mathrm{c}}$ denote the generators of the variable speed model and the constant speed model of the simple exclusion process on (T,o), respectively. Since the Palm measures $\nu^*_{\rho,T}$ and $\tilde{\nu}^*_{\alpha,T}$ have the same law as $\nu_{\rho,T}$ and $\tilde{\nu}_{\alpha,T}$ except at the root, this shows reversibility of the measures $\mathbb{Q}^{\mathrm{v}}_\rho$ and $\mathbb{Q}^{\mathrm{c}}_\alpha$ for the processes generated by $L^{\mathrm{v}}_{\mathrm{ex}}$ and $L^{\mathrm{c}}_{\mathrm{ex}}$, respectively.

We now show reversibility with respect to the operators $L^{\mathrm{v}}_{\mathrm{sh}}$ and $L^{\mathrm{c}}_{\mathrm{sh}}$, following the ideas of Lyons et al. in [97] for the random walk on Galton–Watson trees. For 0/1-colored trees $(T,o,\zeta),(T',o',\zeta') \in \Omega$, let $(T \bullet\!- T', o, \zeta \bullet\!- \zeta')$ denote the tree, where we join the roots of T and T' by an edge and let the resulting tree have its root at o. For Borel sets $C, D \subseteq \Omega$, we define

$$C \bullet\!- D := \{(T \bullet\!- T', o, \zeta \bullet\!- \zeta') \in \Omega \,:\, (T,o,\zeta) \in C, (T',o',\zeta') \in D\}\,.$$

For disjoint trees $(T_1,o_1,\zeta_1),\dots,(T_k,o_k,\zeta_k) \in \Omega$, let $\left(\bigvee_{i=1}^k T_i, o', \bigvee_{i=1}^k \zeta_i\right) \in \Omega^*$ denote the tree where we connect the roots to a new vertex o' forming the new root, and to which we assign color 1. Similarly, we define

$$\bigvee_{i=1}^k F_i := \left\{ \left(\bigvee_{i=1}^k T_i, o', \bigvee_{i=1}^k \zeta_i \right) : (T_i,o_i,\zeta_i) \in F_i \right\}$$

for Borel sets $F_1, \dots, F_k \subseteq \Omega$. Note that $\bigvee_{i=1}^k F_i$ is again a Borel set of 0/1-colored trees. Moreover, for a set of trees $F \subseteq \Omega$, we define

$$\bar{F} := \{(T, o, \zeta^o) \in \Omega \colon (T, o, \zeta) \in F\} ,$$

where $\zeta^o \in \{0,1\}^V$ denotes the configuration in which we flip the color in $\zeta \in \{0,1\}^V$ at the root o. Observe that the processes generated by L_{sh}^{v} and L_{sh}^{c} on Ω^* yield the transition rates

$$q_{\text{sh}}^{\text{v}}((T, o, \zeta), B) := |\{z \in V(T) \colon z \sim o, (T, z, \zeta^{o,z}) \in B\}|$$

for the variable speed model and

$$q_{\text{sh}}^{\text{c}}((T, o, \zeta), B) := \frac{1}{\deg(o)} |\{z \in V(T) \colon z \sim o, (T, z, \zeta^{o,z}) \in B\}|$$

for the constant speed model for all $(T, o, \zeta) \in \Omega^*$, respectively. We define for Borel sets $A, B \subseteq \Omega^*$

$$\begin{aligned} q_{\text{sh}}^{\text{v}}(A, B) &:= \int_A q_{\text{sh}}^{\text{v}}((T, o, \zeta), B) \mathrm{d}\mathbb{Q}_\rho^{\text{v}}(T, o, \zeta) \\ q_{\text{sh}}^{\text{c}}(A, B) &:= \int_A q_{\text{sh}}^{\text{c}}((T, o, \zeta), B) \mathrm{d}\mathbb{Q}_\alpha^{\text{c}}(T, o, \zeta) . \end{aligned}$$

Note that in order to prove reversibility it suffices to show that for almost all Borel sets $A, B \subseteq \Omega^*$

$$\begin{aligned} q_{\text{sh}}^{\text{v}}(A, B) &= q_{\text{sh}}^{\text{v}}(B, A) \\ q_{\text{sh}}^{\text{c}}(A, B) &= q_{\text{sh}}^{\text{c}}(B, A) . \end{aligned}$$

Without loss of generality, we assume that A and B have the form $A = C \bullet\!\!- \bar{D}$ and $B = D \bullet\!\!- \bar{C}$ for

$$C = \bigvee_{i=1}^{k} C_i \quad \text{and} \quad D = \bigvee_{j=1}^{l} D_j \tag{9.14}$$

with integers k, l such that $C, C_1, \dots, C_k, D, D_1, \dots, D_l \subseteq \Omega$ are disjoint Borel sets. More precisely, for two independent samples $(T, o), (T', o') \in \mathcal{T}$ of trees according to $\mathbf{GW}$, we have almost surely that $\tilde{d}_{\text{loc}}((T, o), (T', o')) > 0$ holds. Thus, for $\mathbb{Q}_\rho^{\text{v}}$ and $\mathbb{Q}_\alpha^{\text{c}}$, the underlying Borel-$\sigma$-algebra on $(\Omega^*, d_{\text{loc}})$ is generated up to nullsets when taking only disjoint Borel-sets C and D of 0/1-colored trees into account. A similar argument applies for all remaining pairs of sets $C, C_1, \dots, C_k, D, D_1, \dots, D_l$. A visualization of the sets $C \bullet\!\!- \bar{D}$ and $D \bullet\!\!- \bar{C}$ is given in Figure 30. In the variable speed model, we claim

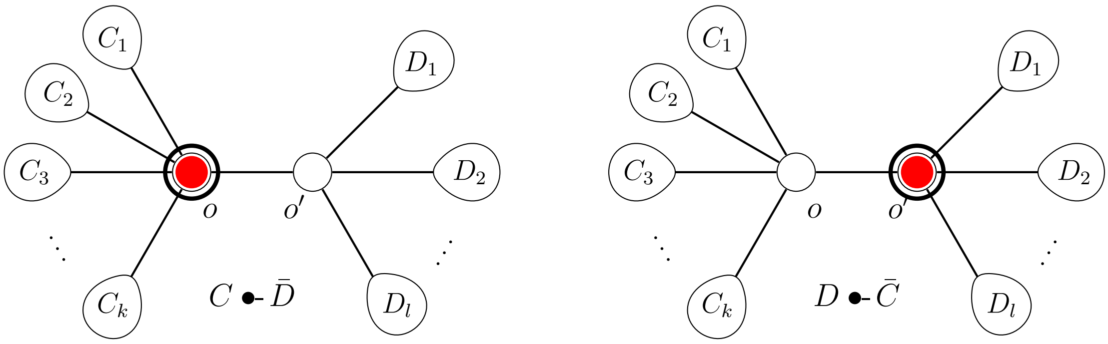

Figure 30: Visualization of the Borel sets $C \bullet\!\text{-}\, \bar{D} \subseteq \Omega^*$ and $D \bullet\!\text{-}\, \bar{C} \subseteq \Omega^*$, where the tagged particle is marked in red.

that we obtain

$$\begin{aligned}\mathbb{Q}_\rho^{\mathrm{v}}(A) =&(k+1)! p_k (l!) p_l \prod_{i=1}^{k} (\mathbf{GW} \times \nu_{\rho,T})(C_i) \prod_{j=1}^{l} (\mathbf{GW} \times \nu_{\rho,T})(D_j) \\ &\cdot (1-\rho) \cdot \frac{1}{k+1} \cdot \left(\mathbb{E}\left[\frac{1}{Z+1} \right] \right)^{-1} . \end{aligned} \tag{9.15}$$

In particular, (9.15) implies that $\mathbb{Q}_\rho^{\mathrm{v}}(A) = \mathbb{Q}_\rho^{\mathrm{v}}(B)$. In order to show (9.15), consider a tree $(T \bullet\!\text{-}\, T', o, \zeta \bullet\!\text{-}\, \zeta') \in A$ with $(T, o, \zeta) \in C$ and $(T', o', \zeta') \in \bar{D}$ where C and D are given in (9.14). By construction, the tree (T, o, ζ) must have degree k at the root o before the tree (T', o', ζ') is attached. There are now $(k+1)!$ ways of attaching the subtrees belonging to $C_1, \ldots, C_k, D$ to the root o. Furthermore, the subtree (T', o', ζ') must have degree l at o' before being connected to o, and o' has to be empty. There are $l!$ possibilities to attach to o' the trees belonging to $D_1, \ldots, D_l$. By construction, under $\mathbb{Q}_\rho^{\mathrm{v}}$ the subtrees belonging to $C_1, \ldots, C_k$ and $D_1, \ldots, D_l$ are i.i.d. with law $(\mathbf{GW} \times \nu_{\rho,T})$. The factor $(1-\rho)$ is the probability that the site o' is vacant. Together with (9.11), this gives the above formula for $\mathbb{Q}_\rho^{\mathrm{v}}(A)$. Since

$$q_{\mathrm{sh}}^{\mathrm{v}}((T, o, \zeta), B) = q_{\mathrm{sh}}^{\mathrm{v}}((T', o', \zeta'), A)$$

for all $(T, o, \zeta) \in A, (T', o', \zeta') \in B$, we obtain claim (i) of Proposition 9.4.

For a given tree $(T, o) \in \mathcal{T}$ and $\alpha \in (0, \infty)$, let $\bar{\nu}_{\alpha,T}$ be the product measure on $\{0,1\}^{V(T)}$ with marginals

$$\bar{\nu}_{\alpha,T}(\eta \colon \eta(x) = 1) = \begin{cases} \tilde{\nu}_{\alpha,T}(\eta(x) = 1) & \text{if } x \neq o \\ \frac{\alpha(\deg(o)+1)}{\alpha(\deg(o)+1)+1} & \text{if } x = o \,. \end{cases}$$

In words, we obtain $\bar{\nu}_{\alpha,T}$ by taking $\tilde{\nu}_{\alpha,T}$ except that we add 1 to the degree of the root. Using a similar decomposition as for $\mathbb{Q}_\rho^{\mathrm{v}}$, and taking (9.13) into account, we can write

$\mathbb{Q}^{\mathrm{c}}_\alpha(A)$ in the constant speed model as

$$\mathbb{Q}^{\mathrm{c}}_\alpha(A) =(k+1)!p_k(l!)p_l \prod_{i=1}^{k}(\mathbf{GW}\times\bar{\nu}_{\alpha,T})(C_i)\prod_{j=1}^{l}(\mathbf{GW}\times\bar{\nu}_{\alpha,T})(D_j) \cdot \frac{1}{\alpha(k+1)+1}\cdot\frac{1}{\alpha(l+1)+1}\cdot\left(\mathbb{E}\left[\frac{1}{\alpha(Z+1)+1}\right]\right)^{-1}. \tag{9.16}$$

Here, $\frac{1}{\alpha(l+1)+1}$ is the probability that o' is vacant, and the factor $\frac{1}{\alpha(k+1)+1}$ comes from (9.13). In particular, (9.16) implies that

$$\frac{\mathbb{Q}^{\mathrm{c}}_\alpha(A)}{k+1}=\frac{\mathbb{Q}^{\mathrm{c}}_\alpha(B)}{l+1}$$

holds. Since

$$q^{\mathrm{c}}_{\mathrm{sh}}((T,o,\zeta),B)=\frac{1}{k+1} \quad\text{and}\quad q^{\mathrm{c}}_{\mathrm{sh}}((T',o',\zeta'),A)=\frac{1}{l+1}$$

for all $(T,o,\zeta)\in A$ and $(T',o',\zeta')\in B$, we obtain claim (ii) of Proposition 9.4. □

9.4 Transience of the tagged particle

Recall that the position of the tagged particle in the simple exclusion process is denoted by $(X^{\mathrm{v}}_t)_{t\geq 0}$ in the variable speed model and by $(X^{\mathrm{c}}_t)_{t\geq 0}$ in the constant speed model. Let $P_{\mathbb{P}^{\mathrm{v}}_\rho}$ be the law of the simple exclusion process started from $\mathbb{P}^{\mathrm{v}}_\rho$ in the variable speed model for some $\rho\in(0,1)$. Similarly, let $P_{\mathbb{P}^{\mathrm{c}}_\alpha}$ denote the law of the simple exclusion process started from $\mathbb{P}^{\mathrm{c}}_\alpha$ in the constant speed model for some $\alpha\in(0,\infty)$. For both models, we say that the tagged particle is **transient** if $(X^{\mathrm{v}}_t)_{t\geq 0}$, respectively $(X^{\mathrm{c}}_t)_{t\geq 0}$, hits the root $P_{\mathbb{P}^{\mathrm{v}}_\rho}$-almost surely, respectively $P_{\mathbb{P}^{\mathrm{c}}_\alpha}$-almost surely, only finitely many times.

Proposition 9.5. *The tagged particle is transient for the simple exclusion process in the variable speed model with initial distribution $\mathbb{P}^{\mathrm{v}}_\rho$ and in the constant speed model with initial distribution $\mathbb{P}^{\mathrm{c}}_\alpha$ for all $\rho\in(0,1)$ and $\alpha\in(0,\infty)$.*

In order to show Proposition 9.5, we recall the framework by Lyons et al. in [97] for the simple random walks on Galton–Watson trees, which we discussed for the exclusion process on regular trees in Section 8.4, and which we will now use again in order to study the simple exclusion process on augmented Galton–Watson trees. Fix a tree $(T,o)\in\mathcal{T}$. Recall that we write $\vec{x}$ for a path $(x_0,x_1,\dots)$ in (T,o) and say that a path is a ray ξ if it never backtracks. We denote by $\partial(T,o)$ the boundary of a tree (T,o) and note that $\partial(T,o)$ consists **AGW**-almost surely of infinitely many elements, as we see **AGW**-almost surely infinitely many sites of degree at least 3 by our assumptions on the offspring distribution. Recall that a path $\vec{x}$ converges to $\xi\in\partial(T,o)$ if ξ is the only

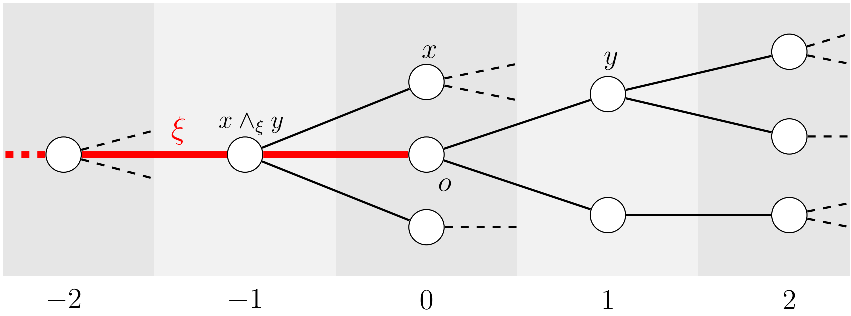

Figure 31: Visualization of the horodistance on a tree (T, o) with ray $\xi \in \partial(T, o)$. In this example, $\langle x\rangle_{(T,o)} = 0$ and $\langle y - x\rangle_{(T,o)} = 1$ holds.

ray which is intersected infinitely often and $\vec{x}$ visits every site at most finitely many times. We let $[x, \xi]$ denote the unique ray starting at a site $x \in V(T)$ and converging to $\xi \in \partial(T, o)$. Note that this ray can be constructed for general rooted trees by taking the shortest path connecting x and o, following this path starting from x until the first time a vertex of ξ is hit and then following the ray ξ in the direction pointing away from o. For sites $x, y \in V(T)$ and a ray ξ, let $x \wedge_\xi y$ be the first site at which $[x, \xi]$ and $[y, \xi]$ meet. In particular, as $[x, \xi]$ and $[y, \xi]$ are both rays converging to ξ, they must agree for all but finitely many vertices with ξ.

For two sites $x, y \in V(T)$, recall from (8.6) that we denote by $\langle y - x\rangle_\xi$ their horodistance with respect to some given ray ξ of (T, o); see Figure 31 for a visualization, and we write $\langle x\rangle_\xi := \langle x - o\rangle_\xi$. In the following, we will fix for every infinite tree $(T, o) \in \mathcal{T}$ a ray $\xi = \xi(T, o) \in \partial(T, o)$ and write

$$\langle y - x\rangle_{(T,o)} := \langle y - x\rangle_{\xi(T,o)}$$

for all $x, y \in V(T)$. Similar to (8.7), we require that the choice of the ray is consistent under performing shifts of the root, i.e., for a given tree $(T, o) \in \mathcal{T}$

$$\langle y - x\rangle_{(T,z)} = \langle y\rangle_{(T,x)} \tag{9.17}$$

holds for all $x, y, z \in V(T)$. In contrast to the simple exclusion process on regular trees, we can not exploit a group structure of the underlying trees to achieve (9.17), so we use the following construction for consistent rays instead. We first choose a tree $(T, o) \in \mathcal{T}$. Let $\xi(T, o) \in \partial(T, o)$ now be an arbitrary, but fixed ray. For all (T, z) with $z \in V(T)$, we then set $\xi(T, z) := [z, \xi(T, o)]$. Now choose another tree not in this collection and iterate. Observe that for this choice of rays, the relation in (9.17) indeed holds as the vertex $x \wedge_{\xi(T,z)} y$ remains the same for any choice of z; see also Figure 31. Our construction ensures that we have the same ray on all trees which one can get from a given tree by shifting its root. For $(T, x, \zeta) \in \Omega^*$, we define the **local drift** at x in the variable speed model to be

$$\psi^{\mathrm{v}}(T,x,\zeta) := \sum_{z\sim x}(1-\zeta(z))\langle z-x\rangle_{(T,x)}\,, \tag{9.18}$$

see also (8.8). Similarly, the local drift at x in the constant speed model is denoted by

$$\psi^{\mathrm{c}}(T,x,\zeta) := \sum_{z\sim x}\frac{1}{\deg(x)}(1-\zeta(z))\langle z-x\rangle_{(T,x)}\,. \tag{9.19}$$

Using these notions, we rewrite the position of the tagged particle as a martingale and a function depending only on the environment process in a ball of radius 1 around its root. This follows the ideas of Proposition 4.1 in [94, Part III] and Lemma 8.6.

Lemma 9.6. *Fix $\rho\in(0,1)$ and $\alpha\in(0,\infty)$. Then the following two statements hold:*

(i) Let $(T_t,o_t,\zeta_t)_{t\geq 0}$ be the environment process in the variable speed model with initial distribution $\mathbb{Q}^{\mathrm{v}}_\rho$ and natural filtration $(\mathcal{F}^{\mathrm{v}}_t)_{t\geq 0}$. Then

$$\langle o_t\rangle_{(T_0,o_0)} = \int_0^t \psi^{\mathrm{v}}(T_s,o_s,\zeta_s)\mathrm{d}s + M^{\mathrm{v}}_t \tag{9.20}$$

holds for all $t\geq 0$, where $(M^{\mathrm{v}}_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}^{\mathrm{v}}_t)_{t\geq 0}$.

(ii) Let $(T_t,o_t,\zeta_t)_{t\geq 0}$ be the environment process in the constant speed model with initial distribution $\mathbb{Q}^{\mathrm{c}}_\alpha$ and natural filtration $(\mathcal{F}^{\mathrm{c}}_t)_{t\geq 0}$. Then

$$\langle o_t\rangle_{(T_0,o_0)} = \int_0^t \psi^{\mathrm{c}}(T_s,o_s,\zeta_s)\mathrm{d}s + M^{\mathrm{c}}_t \tag{9.21}$$

holds for all $t\geq 0$, where $(M^{\mathrm{c}}_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}^{\mathrm{c}}_t)_{t\geq 0}$.

Proof. We only show part (i) of Lemma 9.6 as for part (ii) the same arguments apply. For a given tree $(T,o)\in\mathcal{T}$, we define

$$\mathbb{Q}^{\mathrm{v}}_{\rho,T} := \mathbb{Q}^{\mathrm{v}}_\rho\left((T',o',.)\in\cdot\;|(T',o')=(T,o)\right)\,.$$

Note that we can write for all $\mathbb{Q}^{\mathrm{v}}_\rho$-measurable sets A

$$\begin{aligned}\mathbb{Q}^{\mathrm{v}}_\rho(A) &= \int_{\Omega^*}\mathbb{1}_{\{(T,o,\zeta)\in A\}}\mathrm{d}\mathbb{Q}^{\mathrm{v}}_\rho(T,o,\zeta)\\ &= \int_{\mathcal{T}}\int_{\{0,1\}^{V(T)}}\mathbb{1}_{\{(T,o,\zeta)\in A\}}\mathrm{d}\nu^*_{\rho,T}(\zeta)\mathrm{d}\mathbf{UGW}(T,o)\\ &= \int_{\mathcal{T}}\int_{\tilde{\Omega}^*_T}\mathbb{1}_{\{(T',o',\zeta)\in A\}}\mathrm{d}\mathbb{Q}^{\mathrm{v}}_{\rho,T}(T',o',\zeta)\mathrm{d}\mathbf{UGW}(T,o)\,.\end{aligned} \tag{9.22}$$

Thus, it suffices to show that

$$E_{\mathbb{Q}^{\mathrm{v}}_{\rho,T}}\left[\langle o_t\rangle_{(T,o)} - \langle o_s\rangle_{(T,o)} - \int_s^t \psi^{\mathrm{v}}(T_r, o_r, \zeta_r)\mathrm{d}r \middle| \mathcal{F}^{\mathrm{v}}_s\right] = 0$$

holds for all $t > s \geq 0$ and **UGW**-almost every tree $(T, o) \in \mathcal{T}$, where $E_{\mathbb{Q}^{\mathrm{v}}_{\rho,T}}$ denotes the expectation with respect to the environment process started from $\mathbb{Q}^{\mathrm{v}}_{\rho,T}$. Recall that the choice of the ray for a tree in $\mathcal{T}$ is consistent under performing shifts of the root. Moreover, note that an environment process started from $\mathbb{Q}^{\mathrm{v}}_{\rho,T}$ remains in $\tilde{\Omega}^*_T$ almost surely. Using the Markov property of the environment process as well as the fact that $\mathbb{Q}^{\mathrm{v}}_{\rho}$ is stationary for the environment process by Proposition 9.4, for all $s \geq 0$,

$$\begin{aligned} &E_{\mathbb{Q}^{\mathrm{v}}_{\rho,T}}\left[\langle o_t\rangle_{(T,o)} - \langle o_s\rangle_{(T,o)} - \int_s^t \psi^{\mathrm{v}}(T_r, o_r, \zeta_r)\mathrm{d}r \middle| \mathcal{F}^{\mathrm{v}}_s\right] = \\ &E_{\mathbb{Q}^{\mathrm{v}}_{\rho,T_s}}\left[\langle o_{t-s}\rangle_{(T,o)} - \langle o_0\rangle_{(T,o)} - \int_0^{t-s} \psi^{\mathrm{v}}(T_r, o_r, \zeta_r)\mathrm{d}r\right] . \end{aligned}$$

Hence, it suffices to show that for **UGW**-almost every $(T, o) \in \mathcal{T}$

$$E_{\mathbb{Q}^{\mathrm{v}}_{\rho,T}}\left[\langle o_t\rangle_{(T,o)} - \langle o_0\rangle_{(T,o)}\right] - \int_0^t E_{\mathbb{Q}^{\mathrm{v}}_{\rho,T}}\left[\psi^{\mathrm{v}}(T_s, o_s, \zeta_s)\right]\mathrm{d}s = 0 \qquad (9.23)$$

is satisfied for all $t \geq 0$. For a tree $(T, o) \in \mathcal{T}$, let g be the function on $\tilde{\Omega}^*_T$ given by

$$g(T, x, \zeta) := \langle x\rangle_{(T,o)}$$

for all $(T, x, \zeta) \in \tilde{\Omega}^*_T$. Plugging g into the generator in (9.10), we note that

$$L^{\mathrm{v}} g(T, x, \zeta) = \sum_{y\sim x}(1 - \zeta(y))\left[g(T, y, \zeta^{x,y}) - g(T, x, \zeta)\right] = \psi^{\mathrm{v}}(T, x, \zeta)$$

holds for all $(T, x, \zeta) \in \tilde{\Omega}^*_T$. For the second equality, we use (9.17) together with the relation

$$\langle y - x\rangle_{(T,x)} = \langle y - x\rangle_{(T,o)} = \langle y\rangle_{(T,o)} - \langle x\rangle_{(T,o)}$$

for all $x, y \in V(T)$, which follows from the construction of the horodistance; see also Figure 31. We obtain (9.23) by applying Dynkin's formula. □

Proof of Proposition 9.5. We will only prove transience for the tagged particle in the variable speed model of the simple exclusion process. For the tagged particle in the constant speed model of the simple exclusion process, similar arguments apply. We will show that the tagged particle has $P_{\mathbb{P}^{\mathrm{v}}_{\rho}}$-almost surely a strictly positive speed with respect to the horodistance, which implies transience. Observe that the martingale $(M^{\mathrm{v}}_t)_{t\geq 0}$ defined via the relation (9.20) has stationary increments by Proposition 9.4, and thus satisfies a law of large numbers. By Lemma 9.6

$$E_{\mathbb{Q}^{\mathrm{v}}_\rho}\left[\lim_{t\to\infty}\frac{\langle o_t\rangle_{(T_0,o_0)}}{t}\right] = E_{\mathbb{Q}^{\mathrm{v}}_\rho}\left[\psi^{\mathrm{v}}(T_0,o_0,\zeta_0)\right]$$

holds using that $\mathbb{Q}^{\mathrm{v}}_\rho$ is stationary for the environment process. Moreover,

$$\sum_{x\sim o_0} E_{\mathbb{Q}^{\mathrm{v}}_\rho}\left[\langle x\rangle_{(T_0,o_0)}\middle|\deg(o_0)=k\right] = k-2$$

is satisfied for all $k\geq 2$. Thus, combining these two observations yields

$$\begin{aligned} E_{\mathbb{Q}^{\mathrm{v}}_\rho}\left[\lim_{t\to\infty}\frac{\langle o_t\rangle_{(T_0,o_0)}}{t}\right] &= (1-\rho)\int_{\mathcal{T}}(\deg(o)-2)\mathrm{d}\mathbf{UGW}(T,o) \\ &= (1-\rho)\mathbb{E}\left[\frac{Z-1}{Z+1}\right]\left(\mathbb{E}\left[\frac{1}{Z+1}\right]\right)^{-1}, \end{aligned} \tag{9.24}$$

where Z has the offspring distribution. Since $(\langle o_t\rangle_{(T_0,o_0)})_{t\geq 0}$ is the horodistance of the tagged particle from the root within the environment process, with positive $\mathbb{Q}^{\mathrm{v}}_\rho$-probability, the tagged particle has a strictly positive speed. Now in order to show that the tagged particle is transient with respect to the initial distribution $\mathbb{P}^{\mathrm{v}}_\rho$, suppose that there exists an initial set of 0/1-colored trees $B\subseteq\Omega^*$ with $\mathbb{Q}^{\mathrm{v}}_\rho(B)>0$ for which the tagged particle has speed zero. Note that B can be chosen such that it forms an invariant set for the environment process. Let $E_{\mathbb{Q}^{\mathrm{v}}_\rho(\,\cdot\,|B)}$ denote the expectation of the environment process started from $\mathbb{Q}^{\mathrm{v}}_\rho(\,\cdot\,|B)$. Using the same arguments as in the proof of Lemma 9.6, the environment process must satisfy

$$\begin{aligned} 0 &= \int_0^t E_{\mathbb{Q}^{\mathrm{v}}_\rho(\,\cdot\,|B)}\left[\psi^{\mathrm{v}}(T_s,o_s,\zeta_s)\right]\mathrm{d}s \\ &= \int_0^t \sum_{k\geq 2}\sum_{x\sim o_s} E_{\mathbb{Q}^{\mathrm{v}}_\rho(\,\cdot\,|B)}\left[\langle x\rangle_{(T_s,o_s)}\middle|\deg(o_s)=k\right]\mathbb{Q}^{\mathrm{v}}_\rho(\deg(o_s)=k,\zeta_s(x)=0|B)\mathrm{d}s \end{aligned}$$

for all $t\geq 0$. Again, from the construction of the horodistance,

$$\sum_{x\sim o_s} E_{\mathbb{Q}^{\mathrm{v}}_\rho(\,\cdot\,|B)}\left[\langle x\rangle_{(T_s,o_s)}\middle|\deg(o_s)=k\right] = k-2$$

holds for all $k\geq 2$. Moreover, note that the conditional probability

$$\mathbb{Q}^{\mathrm{v}}_\rho(\deg(o_s)=k,\zeta_s(x)=0|B) \tag{9.25}$$

does not depend on the particular choice of $x\sim o_s$. Hence,

$$0 = \int_0^t \sum_{k\geq 2}(k-2)\mathbb{Q}^{\mathrm{v}}_\rho(\deg(o_s)=k,\zeta_s(x)=0|B)\mathrm{d}s$$

must hold for all $t \geq 0$ and $x \sim o_s$. However, this gives a contradiction as the term in (9.25) is non-negative for all $k \geq 2$ and strictly positive for at least one $k \geq 3$. Otherwise, the underlying augmented Galton–Watson tree would almost surely be restricted to a copy of $\mathbb{Z}$. Thus, the tagged particle has a strictly positive speed $\mathbb{Q}^{\mathrm{v}}_{\rho}$-almost surely. We conclude since $\mathbb{P}^{\mathrm{v}}_{\rho}$ and $\mathbb{Q}^{\mathrm{v}}_{\rho}$ are equivalent for all $\rho \in (0,1)$. □

Corollary 9.7. *Choose an initial configuration* $(T, o, \zeta) \in \Omega^*$ *according to* $\mathbb{Q}^{\mathrm{v}}_{\rho}$, *respectively according to* $\mathbb{Q}^{\mathrm{c}}_{\alpha}$. *Consider the tagged particles within two independently sampled environment processes* $(T_t, o_t, \zeta_t)_{t\geq 0}$ *and* $(T'_t, o'_t, \zeta'_t)_{t\geq 0}$, *which are both started from* (T, o, ζ). *Then the trajectories of the respective tagged particles converge almost surely to two distinct rays* $\xi, \xi' \in \partial(T, o)$.

Proof. By Proposition 9.5, the tagged particles in both environment processes are almost surely transient, and hence their trajectories converge almost surely to unique rays $\xi, \xi' \in \partial(T, o)$, respectively. It remains to show that $\xi \neq \xi'$ holds almost surely. In the proof of Proposition 9.5, we only require the ray of a tree to be consistent under performing shifts of the root and to be fixed at the beginning. Since $(T_t, o_t, \zeta_t)_{t\geq 0}$ and $(T'_t, o'_t, \zeta'_t)_{t\geq 0}$ are evaluated independently, we can first sample $(T_t, o_t, \zeta_t)_{t\geq 0}$ in which the tagged particle almost surely converges to some ray $\xi \in \partial(T, o)$. For the process $(T'_t, o'_t, \zeta'_t)_{t\geq 0}$, we then define the horodistance with respect to this ray ξ. Since almost surely

$$\lim_{t\to\infty} \langle o_t \rangle_{(T_0, o_0)} = \infty$$

holds, and the tagged particle in $(T'_t, o'_t, \zeta'_t)_{t\geq 0}$ converges almost surely to some ray $\xi' \in \partial(T, o)$, the two rays ξ and ξ' can almost surely not be the same. □

Remark 9.8. *In* (9.24), *we saw that the averaged speed of the tagged particle in the environment process in the variable speed model is given by*

$$E_{\mathbb{Q}^{\mathrm{v}}_{\rho}}\left[\lim_{t\to\infty} \frac{\langle o_t \rangle_{(T_0,o_0)}}{t}\right] = (1-\rho)\mathbb{E}\left[\frac{Z-1}{Z+1}\right]\left(\mathbb{E}\left[\frac{1}{Z+1}\right]\right)^{-1} \tag{9.26}$$

for Z *having the law of the offspring distribution, and* $\rho \in (0,1)$. *Similarly, one derives that the averaged speed of the tagged particle in the environment process in the constant speed model is given by*

$$E_{\mathbb{Q}^{\mathrm{c}}_{\alpha}}\left[\lim_{t\to\infty} \frac{\langle o_t \rangle_{(T_0,o_0)}}{t}\right] = \mathbb{E}\left[\frac{Z-1}{Z+1}\frac{1}{\alpha(Z+1)+1}\right] \tag{9.27}$$

for $\alpha \in (0,\infty)$. *We will show in Section 9.6 that* (9.26) *and* (9.27) *give the speed of the tagged particle* $P_{\mathbb{P}^{\mathrm{v}}_{\rho}}$*-almost surely, respectively* $P_{\mathbb{P}^{\mathrm{c}}_{\alpha}}$*-almost surely, using an ergodicity argument for the environment process.*

9.5 Ergodicity for the environment process

In this section, we show that the environment process started from $\mathbb{Q}^{\mathrm{v}}_\rho$ in the variable speed model and from $\mathbb{Q}^{\mathrm{c}}_\alpha$ in the constant speed model, respectively, is ergodic for all $\rho \in (0,1)$ and $\alpha \in (0,\infty)$. The proof will have two main ingredients. First, we show that every invariant set A can be represented by a set of trees, which we obtain by dropping the 0/1-coloring in every configuration of A. This step follows the ideas of Saada for the exclusion process on $\mathbb{Z}^d$; see [114], and Lemma 8.5 for regular trees. We then deduce ergodicity using regeneration points, following the ideas of Lyons and Peres in Chapter 17 of [99] for the simple random walk on Galton–Watson trees.

Proposition 9.9. *Fix parameters $\rho \in (0,1)$ and $\alpha \in (0,\infty)$ for the measures $\mathbb{Q}^{\mathrm{v}}_\rho$ and $\mathbb{Q}^{\mathrm{c}}_\alpha$, respectively. The following two statements hold.*

(i) The measure $\mathbb{Q}^{\mathrm{v}}_\rho$ is ergodic for the environment process generated by L^{v}.

(ii) The measure $\mathbb{Q}^{\mathrm{c}}_\alpha$ is ergodic for the environment process generated by L^{c}.

We will only show part (i) of Proposition 9.9, i.e., we will prove that $\mathbb{Q}^{\mathrm{v}}_\rho(A) \in \{0,1\}$ holds for any set A which is invariant under the environment process in the variable speed model. For part (ii) of Proposition 9.9, similar arguments apply. The following lemma says that in order to determine if $(T,o,\zeta) \in A$ holds, it suffices $\mathbb{Q}^{\mathrm{v}}_\rho$-almost surely to know the underlying tree $(T,o) \in \mathcal{T}$.

Lemma 9.10. *Let $A \subseteq \Omega^*$ be an invariant set for the environment process started from $\mathbb{Q}^{\mathrm{v}}_\rho$. Then for* **UGW***-almost every tree $(T,o) \in \mathcal{T}$*

$$\int_{\tilde{\Omega}^*_T} \mathbb{1}_{\{(T',o',\zeta)\in A\}} \mathrm{d}\mathbb{Q}^{\mathrm{v}}_{\rho,T}\,(T',o',\zeta) \in \{0,1\} \tag{9.28}$$

holds. Moreover, we can find a Borel set of rooted trees $U \subseteq \mathcal{T}$ which is invariant under the environment process such that

$$\int_{\tilde{\Omega}^*_T} \mathbb{1}_{\{(T',o',\zeta)\in A\}} \mathrm{d}\mathbb{Q}^{\mathrm{v}}_{\rho,T}\,(T',o',\zeta) = \mathbb{1}_{\{(T,o)\in U\}} \tag{9.29}$$

is satisfied.

In order to show Lemma 9.10, we follow the arguments of Saada in [114]. A similar approach can be found in [29] for the simple exclusion process on regular trees. A key tool in [114] for showing ergodicity of the environment process of the simple exclusion process on $\mathbb{Z}^d$ with drift is to use that the Bernoulli-ρ-product measures are extremal invariant for the simple exclusion process on $\mathbb{Z}^d$ with drift for all $\rho \in [0,1]$.

Similarly, our arguments are based on the fact that we have ergodicity for the simple exclusion process started from $\nu_{\rho,T}$ for **AGW**-almost every initial tree $(T,o) \in \mathcal{T}$.

More precisely, by Theorem 3.9 the measures $\nu_{\rho,T}$ are extremal invariant for the simple exclusion on $\{0,1\}^{V(T)}$ for **AGW**-almost every tree $(T,o) \in \mathcal{T}$, for all $\rho \in (0,1)$. By Theorem 2.1 of [77], a similar statement holds for the measures $\tilde{\nu}_{\alpha,T}$ in the constant speed model. Since, for **AGW**-almost all trees, the simple exclusion process on a given tree is a Markov process, this implies that the measures $\nu_{\rho,T}$ are ergodic for the simple exclusion process on a **AGW**-almost every tree (T,o); see Theorem 3.5.

We will now show that the environment process on $\tilde{\Omega}^*_T$ with initial law $\mathbb{Q}^{\mathrm{v}}_{\rho,T}$ is ergodic for **AGW**-almost every tree $(T,o) \in \mathcal{T}$, using a proof by contradiction. Suppose that for some $\rho \in (0,1)$, the set A satisfies

$$0 < \mathbb{Q}^{\mathrm{v}}_{\rho,T}(A) < 1\,.$$

Since the set A is invariant for the environment process with starting distribution $\mathbb{Q}^{\mathrm{v}}_{\rho}$, it has to be invariant for the environment process on $\tilde{\Omega}^*_T$ with initial law $\mathbb{Q}^{\mathrm{v}}_{\rho,T}$ for **AGW**-almost every tree $(T,o) \in \mathcal{T}$. Define $B := \tilde{\Omega}^*_T \setminus (A \cap \tilde{\Omega}^*_T)$ and note that B is a non-trivial, invariant set for the environment process started from $\mathbb{Q}^{\mathrm{v}}_{\rho,T}$. Moreover, we let the sets $\tilde{A}, \tilde{B} \subseteq \Omega_T$ be given as

$$\tilde{A} := \bigcup_{(\tilde{T},v,\zeta)\in A\cap\tilde{\Omega}^*_T\colon (\tilde{T},v)=(T,v)} \{(T,o,\zeta)\}$$

and

$$\tilde{B} := \bigcup_{(\tilde{T},v,\zeta)\in B\colon (\tilde{T},v)=(T,v)} \{(T,o,\zeta)\}\,.$$

In words, $\tilde{A}$ is the set of all 0/1-colorings of (T,o) which we obtain by taking all 0/1-colored trees in $A \cap \tilde{\Omega}^*_T$ and considering their colorings of (T,o). Observe that the sets $\tilde{A}$ and $\tilde{B}$ are invariant for the simple exclusion process with initial distribution

$$\mathbb{P}^{\mathrm{v}}_{\rho,T} := \delta_{(T,o)} \times \nu_{\rho,T} \tag{9.30}$$

where $\delta_{(T,o)}$ denotes the Dirac measure on $\mathcal{T}$ concentrated on (T,o). Moreover, since $\mathbb{Q}^{\mathrm{v}}_{\rho,T}$ is absolutely continuous with respect to $\mathbb{P}^{\mathrm{v}}_{\rho,T}$ for all $\rho \in (0,1)$,

$$\mathbb{P}^{\mathrm{v}}_{\rho,T}(\tilde{A}) > 0 \quad\text{and}\quad \mathbb{P}^{\mathrm{v}}_{\rho,T}(\tilde{B}) > 0$$

is satisfied for **AGW**-almost every tree $(T,o) \in \mathcal{T}$. From [77], using the ergodicity of the simple exclusion process on Ω_T, we conclude that

$$\mathbb{P}^{\mathrm{v}}_{\rho,T}(\tilde{A}) = \mathbb{P}^{\mathrm{v}}_{\rho,T}(\tilde{B}) = 1\,. \tag{9.31}$$

In particular, the sets $\tilde{A}$ and $\tilde{B}$ are not disjoint. From this, we want to deduce that A and B are not disjoint as well. For a tree $(T,o) \in \mathcal{T}$ and $x, y \in V(T)$, let $[x,y]$ be the sites in the shortest path connecting x and y in (T,o). The following lemma is the analogue of Lemma 4 in [114] and Lemma 8.5 for regular trees.

Lemma 9.11. *For $\mathbb{P}^{\mathrm{v}}_{\rho,T}$-almost every $(T,o,\eta) \in \Omega_T$, there exist sites $v,w,x,y,z \in V(T)$ with the following properties:*

(i) $(T,y,\eta) \in A$, $(T,z,\eta) \in B$

(ii) $\eta(v) = \eta(w) = 0$ *and* $\eta(a) = 0$ *for all* $a \in [y,z] \setminus \{y,z\}$

(iii) x *and* z *are located in different branches with respect to* y *in* T.

(iv) v,w,y *are located in pairwise different branches with respect to* x *in* T.

(v) The path $[v,x]$ *contains at least* $|x-y|+1$ *vacant sites.*

Proof. Using (9.31), there almost surely exist sites $y,z \in V(T)$ such that $(T,y,\eta) \in A$ and $(T,z,\eta) \in B$ holds. Without loss of generality, the shortest path connecting y and z can be assumed to consist only of vacant sites. To see this, observe that for any $(T,a,\eta) \in \tilde{\Omega}^*_T$ with $\eta(a) = 1$ and $a \in V(T)$, either $(T,a,\eta) \in A$ or $(T,a,\eta) \in B$ holds. Hence, along the shortest path connecting y and z, there must be a pair of occupied sites y' and z' with $(T,y',\eta) \in A$ and $(T,z',\eta) \in B$, and only vacant sites on the shortest path between them. Thus, we can take these sites y' and z' as our new choices of y and z which satisfy (i). By our assumptions on the augmented Galton–Watson tree, there almost surely exists a site x in a branch of y different from the one containing z with degree at least 3. Let $C(x,y)$ and $D(x,y)$ denote the vertices of two distinct branches of x which do not intersect the path $[x,y]$. Using a Borel–Cantelli argument, we see that $C(x,y)$ and $D(x,y)$ both contain $\mathbb{P}^{\mathrm{v}}_{\rho}$-almost surely a ray starting at x with infinitely many vacant sites. Let w be the first vacant site along that ray in $C(x,y)$. Let v be the first vacant site along that ray in $D(x,y)$ such that there are $|x-y|+1$ empty sites along the path $[x,v]$. □

Proof of Lemma 9.10. Take a configuration $(T,o,\eta) \in \Omega_T$ according to $\mathbb{P}^{\mathrm{v}}_{\rho,T}$ which satisfies the properties (i) to (v) of Lemma 9.11 with sites v,w,x,y,z and set

$$N := \{v,w,x,y,z,[v,x],[w,x],[x,y],[y,z]\} \ . \tag{9.32}$$

We fix a time $t_0 > 0$ and define a 0/1-coloring $\tilde{\eta} \in \{0,1\}^{V(T)}$ as follows. We let $\tilde{\eta}$ agree with η on N. On $V(T) \setminus N$, let $\tilde{\eta}$ have the law of a simple exclusion process at time t_0 started from η, where all moves involving a site in N are suppressed. We will now provide two ways of transforming η into $\eta^{w,z}$ which only involve the sites in N; see Figure 32. This also provides two ways of changing $\tilde{\eta}$ into $\tilde{\eta}^{w,z}$ for any fixed $t_0 > 0$. At the beginning, we assume for both transformations that all particles in $[x,y] \setminus \{y\}$ are moved into the empty sites within $[v,x] \setminus \{v,x\}$ in an arbitrary way using only nearest neighbor moves within N. In a next step, the two transformations differ in performing the following transitions.

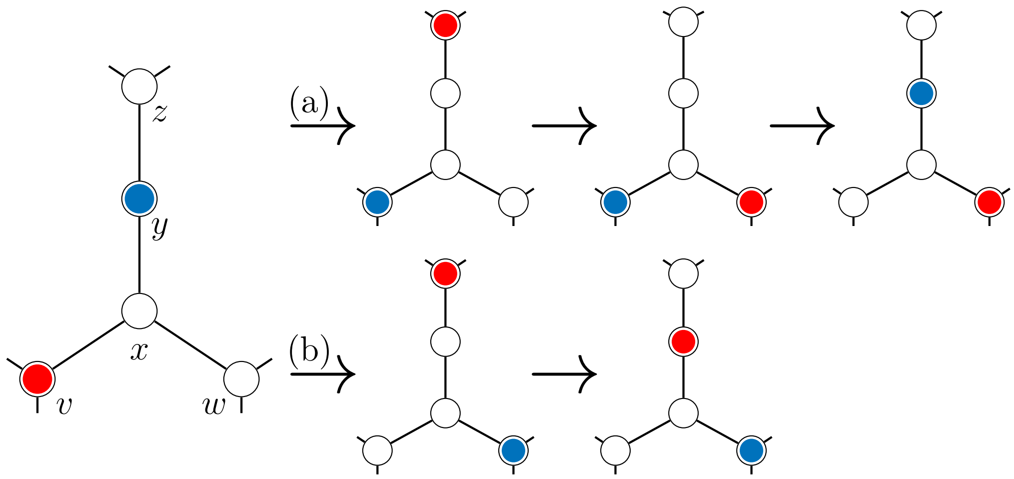

Figure 32: Transformations for η to $\eta^{w,z}$ in a sample of an augmented Galton–Watson tree for the special case when v, w and y are neighbors of x and $[x,w]$ as well as $[x,v]$ are empty.

(a) Push the particles along the path $[y,v]$ towards v, i.e., for $\{v_i, 1 \leq i \leq k\}$ being successive vertices in $[v,y]$ with $\eta(v_i) = 1$, move the particle from v_1 to v, then the particle from v_2 to v_1 and so on. Next, we push the particles in $[z,w]$ towards w in the same way. Afterwards, push the particles along $[v,y]$ towards y.

(b) Push the particles along the path $[y,w]$ towards w in the same way as described in (a). Afterwards, move the particle at z to y along $[z,y]$.

At the end, in both transformations all particles which were moved to the empty sites in $[v,x] \setminus \{x\}$ at the beginning are moved back to their original positions.

Following the transformation according to (a), we see that $(T, y, \tilde{\eta}^{w,z}) \in A$ holds $\mathbb{P}^{\mathrm{v}}_{\rho,T}$-almost surely, using that A is invariant for the environment process and (i) of Lemma 9.11. For the transformation according to (b), note that $(T, y, \tilde{\eta}^{w,z}) \in B$ holds following the trajectory of the particle originally at z and using that B is invariant for the environment process. Observe that at time t_0, the simple exclusion process started from (T, o, η) agrees with $(T, o, \tilde{\eta}^{w,z})$ with positive probability using the graphical representation. Hence, we obtain the desired contradiction of A and B not being disjoint.

For the second statement in Lemma 9.10, we let S denote the set of trees which we obtain by deleting all 0/1-colorings in the elements of A, i.e.,

$$U := \{(T,o) \colon (T,o,\zeta) \in A\} \subseteq \mathcal{T}\,. \tag{9.33}$$

From the construction of the σ-algebras on Ω and $\mathcal{T}$, we see that U forms a Borel set of trees. Using the first statement of Lemma 9.10, we obtain that U is invariant for the environment process started from $\mathbb{Q}^{\mathrm{v}}_{\rho}$. □

Next, we show that $\mathbf{UGW}(U) \in \{0,1\}$ holds for the set U defined in (9.33). This yields Proposition 9.9 since

$$\mathbb{Q}_\rho^{\mathrm{v}}(A) = \int_{\mathcal{T}} \mathbb{1}_{\{(T,o)\in U\}} \mathrm{d}\mathbf{UGW}(T,o)$$

by Lemma 9.10. We follow the same arguments as in the proof of Theorem 17.13 in [99], which are used to establish ergodicity for the environment process of the simple random walk on supercritical Galton–Watson trees.

Lemma 9.12. *Let $(T_t, o_t, \zeta_t)_{t\geq 0}$ denote the environment process with state space Ω^* and initial distribution $\mathbb{Q}_\rho^{\mathrm{v}}$. The corresponding dynamical system is mixing in the tree-component, i.e.,*

$$\mathbf{UGW}((T_0,o_o) \in C, (T_t,o_t) \in D) \overset{t\to\infty}{\longrightarrow} \mathbf{UGW}((T_0,o_o) \in C)\mathbf{UGW}((T_t,o_t) \in D)$$

holds for all Borel-sets $C, D \subseteq \mathcal{T}$. In particular, $\mathbf{UGW}(\tilde{U}) \in \{0,1\}$ holds for any set of trees $\tilde{U}$ which is invariant for the environment process.

To prove Lemma 9.12, we will need some preliminaries. Recall that the σ-algebras on $\mathcal{T}$ and Ω are generated by sets of trees which agree within a ball of finite radius around the root. Including all finite unions and intersections of these balls, we see that the balls generating the σ-algebra on $\mathcal{T}$ form a semi-algebra. Hence, using a well-known result from ergodic theory, it suffices to show mixing in the tree component for the sets C and D which take into account only a finite range of the tree around its root; see Exercise 2.7.3(1) in [50].

For a given tree $(T,o) \in \mathcal{T}$, let $\overset{\leftrightarrow}{x} = (\dots, x_{-1}, x_0, x_1, \dots)$ be a bi-infinite path in (T,o) with $x_0 = o$. We denote by $\overset{\leftrightarrow}{T}$ the space of all such bi-infinite paths for which both ends converge to distinct rays in $\partial(T,o)$. Define the **path space** of trees to be

$$\mathrm{PathsInTrees} := \left\{ (\overset{\leftrightarrow}{x}, T) \colon \overset{\leftrightarrow}{x} \in \overset{\leftrightarrow}{T}, (T,o) \in \mathcal{T} \right\} . \tag{9.34}$$

Let S be the map which shifts $\overset{\leftrightarrow}{x}$ to the right and changes the root of T to x_1, i.e.,

$$(S\overset{\leftrightarrow}{x})_n = x_{n+1}$$

holds for all $n \in \mathbb{Z}$ and

$$S(\overset{\leftrightarrow}{x}, T) := (S\overset{\leftrightarrow}{x}, T) .$$

As in Corollary 9.7, choose an initial configuration $(T,o,\zeta) \in \Omega^*$ according to $\mathbb{Q}_\rho^{\mathrm{v}}$. We consider two independently sampled environment processes $(T_t, o_t, \zeta_t)_{t\geq 0}$ and $(T'_t, o'_t, \zeta'_t)_{t\geq 0}$, which are both started from (T,o,ζ). We let $\vec{x} = (x_0, x_1, \dots)$ and $\vec{y} = (y_0, y_1, \dots)$ denote the trajectories of the tagged particles in the environment processes $(T_t, o_t, \zeta_t)_{t\geq 0}$ and $(T'_t, o'_t, \zeta'_t)_{t\geq 0}$, respectively. We join the trajectories $\vec{x}$ and $\vec{y}$ to obtain a bi-infinite path $\overset{\leftrightarrow}{x}$ by

$$\overset{\leftrightarrow}{x} := (\dots, y_2, y_1, x_0, x_1, x_2, \dots)\,,$$

and denote the corresponding law of $(\overset{\leftrightarrow}{x}, T)$ in PathsInTrees by $\mathrm{EX} \times \mathbb{Q}^{\mathrm{v}}_\rho$. Note that $\overset{\leftrightarrow}{x} \in \overset{\leftrightarrow}{T}$ holds almost surely, since by Corollary 9.7, the two trajectories of the tagged particles converge almost surely to two distinct rays. The path space is equipped with the σ-algebra $\mathcal{F}$ induced by the environment processes. Since by Proposition 9.4, the environment process is a reversible Feller process with respect to $\mathbb{Q}^{\mathrm{v}}_\rho$, we observe that

$$\big(\mathrm{PathsInTrees}, \mathcal{F}, \mathrm{EX} \times \mathbb{Q}^{\mathrm{v}}_\rho, S\big)$$

forms a measure-preserving system, i.e.,

$$\mathrm{EX} \times \mathbb{Q}^{\mathrm{v}}_\rho(F) = \mathrm{EX} \times \mathbb{Q}^{\mathrm{v}}_\rho(S^{-1}F) \tag{9.35}$$

holds for all $F \in \mathcal{F}$. Define the event of having a **regeneration point** at x_0 to be

$$\mathrm{Regen} := \Big\{(\overset{\leftrightarrow}{x}, T) \in \mathrm{PathsInTrees} \text{ s.t. } \forall n \leq 0\colon x_n \neq x_1 \text{ and } \forall n \geq 1\colon x_n \neq x_0\Big\}\,.$$

In words, x_0 is a regeneration point if the edge $\{x_0, x_1\}$ is traversed precisely once.

We will see that regeneration points are random points in time, which allow us to determine when the tagged particle visits a new part of the tree which is "independent of its past". The following lemma is the analogue of Proposition 17.12 in [99] for the simple random walk on Galton–Watson trees.

Lemma 9.13. *For* $\mathrm{EX} \times \mathbb{Q}^{\mathrm{v}}_\rho$ *almost every configuration* $(\overset{\leftrightarrow}{x}, T)$*, we find infinitely many* $n \in \mathbb{Z}$ *such that* $S^n(\overset{\leftrightarrow}{x}, T) \in \mathrm{Regen}$ *holds.*

In order to show Lemma 9.13, observe that the event of having a regeneration point at x_0 can be written as the intersection of the event of having a **fresh point** at x_0

$$\mathrm{Fresh} := \Big\{(\overset{\leftrightarrow}{x}, T) \in \mathrm{PathsInTrees} \text{ s.t. } \forall n \leq 0\colon x_n \neq x_1\Big\}$$

and the event of having an **exit point** at x_0

$$\mathrm{Exit} := \Big\{(\overset{\leftrightarrow}{x}, T) \in \mathrm{PathsInTrees} \text{ s.t. } \forall n \geq 1\colon x_n \neq x_0\Big\}\,.$$

Using reversibility of the environment process with respect to $\mathbb{Q}^{\mathrm{v}}_\rho$ together with (9.35), we see that

$$\mathrm{EX} \times \mathbb{Q}^{\mathrm{v}}_\rho(\mathrm{Fresh}) = \mathrm{EX} \times \mathbb{Q}^{\mathrm{v}}_\rho(\mathrm{Exit}) \tag{9.36}$$

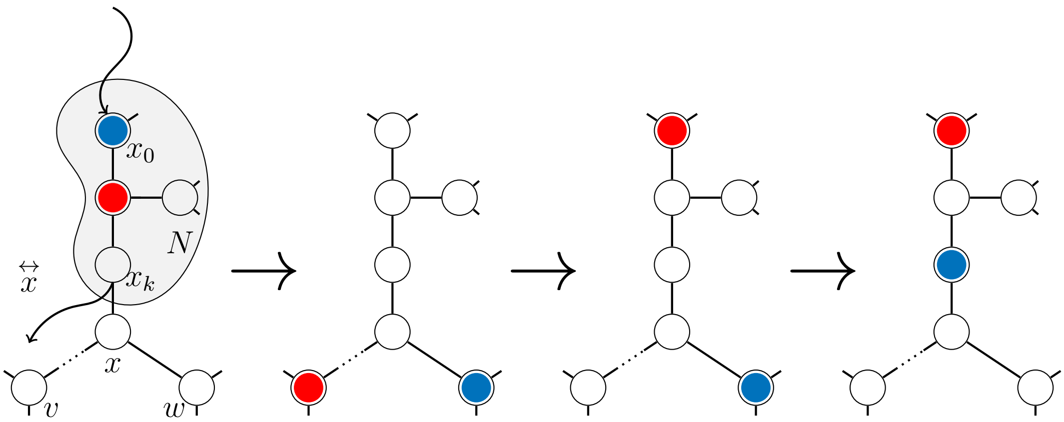

Figure 33: Construction of a regeneration point at x_0 in a sample of an augmented Galton–Watson tree when w and x_k are neighbors of x, and $[x,v]$ as well as $[x,w]$ are empty. The tagged particle is drawn in blue.

holds. Using the transience of the tagged particle together with Corollary 9.7, there are $\mathrm{EX} \times \mathbb{Q}_\rho^{\mathrm{v}}$-almost surely infinitely many fresh points, i.e.,

$$\mathrm{EX} \times \mathbb{Q}_\rho^{\mathrm{v}}\Big(\exists n \geq m \text{ s.t. } S^n(\overleftrightarrow{x}, T) \in \mathrm{Fresh}\Big) = 1$$

holds for any $m \geq 0$. Since the probability of the event of having a fresh point at x_0 is invariant under shifts according to S, we conclude that the probabilities in (9.36) must be strictly positive. Moreover, this shows that $\mathrm{EX} \times \mathbb{Q}_\rho^{\mathrm{v}}$-almost surely, we see infinitely many exit points. For $m, n \in \mathbb{Z}$ with $m \leq n$, we define the event

$$H_{m,n} := \Big\{S^m(\overleftrightarrow{x}, T) \in \mathrm{Fresh}, S^n(\overleftrightarrow{x}, T) \in \mathrm{Exit}, [x_m, x_n] \cap \{x_i, i \in \mathbb{Z} \setminus [m,n]\} = \emptyset\Big\} .$$

In words, $H_{m,n}$ is the event that x_m is a fresh point, x_n is an exit point and the shortest path connecting x_m and x_n does not intersect the remaining trajectory.

Lemma 9.14. *There exists some $k \geq 0$ such that $\mathrm{EX} \times \mathbb{Q}_\rho^{\mathrm{v}}(H_{0,k}) > 0$ holds.*

Proof. Observe that for $\mathrm{EX} \times \mathbb{Q}_\rho^{\mathrm{v}}$-almost every $(\overleftrightarrow{x}, T) \in \mathrm{PathsInTrees}$, the tagged particles in the environment processes converge to different rays $\xi_1, \xi_2 \in \partial(T, x_0)$. Let $a \in V(T)$ be the last common vertex of ξ_1 and ξ_2. Using transience, we observe that a is hit almost surely only finitely often. We choose m such that $S^m(\overleftrightarrow{x}, T) \in \mathrm{Fresh}$ with $a \notin \{\dots, x_{m-1}, x_m\}$, and n such that $S^n(\overleftrightarrow{x}, T) \in \mathrm{Exit}$ with $a \notin \{x_n, x_{n+1}, \dots\}$. For these choices of m and n, we see that $(\overleftrightarrow{x}, T) \in H_{m,n}$. Note that we find such m and n for $\mathrm{EX} \times \mathbb{Q}_\rho^{\mathrm{v}}$-almost every element of PathsInTrees. Hence, $\mathrm{EX} \times \mathbb{Q}_\rho^{\mathrm{v}}(H_{m,n}) > 0$ must hold for some deterministic choice of m and n. Set $k = n - m$ and use the fact that we have a measure-preserving system to conclude. □

For a given configuration $(\overleftrightarrow{x}, T) \in H_{0,k}$ with $k \geq 0$ from Lemma 9.14, let $(T_t, o_t, \zeta_t)_{t \geq 0}$ be the underlying environment process in positive time direction. We recursively define a sequence of almost surely finite stopping times by $\tau_0 := 0$ and

$$\tau_i := \inf\{t \geq \tau_{i-1} \colon o_t \neq o_{\tau_{i-1}}\}$$

for all $i \geq 1$. We show that $(\overleftrightarrow{x}, T)$ is contained in the set of regeneration points at x_0 with positive probability using a similar argument as in the proof of Lemma 9.10. Let $(T'_t, o'_t, \zeta'_t)_{t\geq 0}$ be the environment process with the same initial configuration as $(T_t, o_t, \zeta_t)_{t\geq 0}$ but where all moves involving the tagged particle are suppressed. Using the graphical representation, note that ζ'_{τ_k} and ζ_{τ_k} differ almost surely in at most finitely many values. Let N denote the sites in the minimal spanning tree consisting of $\{x_0, x_1, \dots, x_k\}$ together with all sites in which ζ'_{τ_k} and ζ_{τ_k} differ. The proof of the following lemma uses similar arguments as the proof of Lemma 9.11.

Lemma 9.15. *For almost every configuration $(\overleftrightarrow{x}, T)$ and 0/1-colorings ζ'_{τ_k} and ζ_{τ_k} differing only in sites within N, there exist $v, w, x \in V(T)$ with the following properties:*

(i) $v, w, x \notin N$, $x_0 \notin [x, x_k]$

(ii) $\zeta_{\tau_k}(w) = 0$.

(iii) x_k, v *and* w *are located in pairwise different branches with respect to* x *in* T.

(iv) The path $[x, v]$ *contains at least* $|x - x_k| + k + 1$ *vacant sites.*

Proof. Let $C(x_k)$ be a branch of x_k which does not contain x_0. Using a Borel–Cantelli argument, there almost surely exists a site $x \in C(x_k)$ with $\deg(x) \geq 3$ which is not contained in the set N. Let $C(x)$ and $D(x)$ be two different branches of x which are disjoint of $[x, x_k]$. Note that $C(x)$ and $D(x)$ are disjoint from the set N and contain almost surely an infinite number of vacant sites. Let w be the first site in $C(x)$ which is empty. Similarly, let v be the first site in $D(x)$ such that condition (iv) holds. □

Proof of Lemma 9.13. If Lemma 9.14 holds for $k = 0$, we conclude by Poincaré's recurrence theorem. For $k \geq 1$, we will use Lemma 9.15 to provide a way of transforming ζ'_{τ_k} into ζ_{τ_k} by finitely many transitions; see Figure 33 for a visualization. In this transformation, the tagged particle will not come back to x_0 once it has left its starting point. Let $N_k := N \cup [x_k, v] \cup [x_k, w]$ for v, w from Lemma 9.15. We start by moving all particles on the sites $[x_0, x] \setminus \{x_0\}$ into empty positions in $[x, v] \setminus \{x\}$ using only nearest neighbor transitions in N_k which do not involve x_0. In a next step, we push the tagged particle from x_0 towards w along the path $[x_0, w]$, i.e., for $\{w_i, 1 \leq i \leq n\}$ being successive vertices in $[w, x_0]$ with $\eta(w_i) = 1$, move the particle from w_1 to w, then the particle from w_2 to w_1 and so on. In particular, note that after this push, the tagged particle is contained in $[x, w] \setminus \{x\}$. Next, we perform nearest neighbor moves involving only the sites $N \cup [x_k, v]$ such that $z \in N \setminus [x_k, x]$ is occupied if and only if $\zeta_{\tau_k}(z) = 1$ holds and all sites in $[x_k, x]$ are empty. We now push the particles along the path $[w, x_k]$ towards x_k in the same way as described before. Note at this point that the tagged particle is located in x_k and the constructed configuration may differ from ζ_{τ_k} only at sites $[x_k, v] \setminus \{x_k\}$. Since the number of particles in N_k is preserved, we can now perform nearest neighbor moves using only the particles in $[x_k, v] \setminus \{x_k\}$ to obtain the configuration ζ_{τ_k}. Note that for almost every pair of configurations ζ'_{τ_k} and ζ_{τ_k}, this provides a way of transforming ζ'_{τ_k} into ζ_{τ_k} by modifying the exclusion process only between times 0 and τ_k on an almost surely finite set of vertices N_k. Thus, under the measure $\text{EX} \times \mathbb{Q}^{\text{v}}_{\rho}(\ \cdot \mid H_{0,k})$, with a positive probability all transitions among the sites N_k follow precisely the above described transformation from ζ_0 to ζ_{τ_k} between times 0 and τ_k. Since in this case, we have by construction a regeneration point at x_0,

$$\text{EX} \times \mathbb{Q}^{\text{v}}_{\rho}(\text{Regen}|H_{0,k}) > 0$$

holds. Using Lemma 9.14, we conclude by Poincaré's recurrence theorem. □

Proof of Lemma 9.12. We follow similar arguments as in the proof of Proposition 17.12 in [99]. For a tree $(T, o) \in \mathcal{T}$ and $x \in V(T)$, let T^x denote the subtree of (T, o) rooted at x, containing the sites which become disconnected from o when x is removed. For $(\overset{\leftrightarrow}{x}, T) \in \text{Regen}$, let the **first return time** n_{Regen} be

$$n_{\text{Regen}}(\overset{\leftrightarrow}{x}, T) := \inf \left\{ n > 0 \colon S^n(\overset{\leftrightarrow}{x}, T) \in \text{Regen} \right\}$$

and note that n_{Regen} is almost surely finite; see (6.68) for a similar definition of return times for Markov chains. For $n = n_{\text{Regen}}$, we define the associated **slab**

$$\text{Slab}(\overset{\leftrightarrow}{x}, T) := \left((x_0, \dots, x_{n-1}), T \setminus (T^{x_{-1}} \cup T^{x_n}) \right)$$

and set $S_{\text{Regen}} := S^{n_{\text{Regen}}}$. This yields an i.i.d. sequence $\left(\text{Slab}(S^k_{\text{Regen}}(\overset{\leftrightarrow}{x}, T))\right)_{k \in \mathbb{Z}}$ generating $(\overset{\leftrightarrow}{x}, T)$. Recall that we have to prove mixing in the tree-component only for Borel-sets $C, D \subseteq T$ which take into account a finite range of the tree around the root. Since the tagged particle is transient, we see that for all t sufficiently large, the events $\{(T_0, o_0) \in C\}$ and $\{(T_t, o_t) \in D\}$ depend on disjoint sets of slabs. This gives Lemma 9.12 and hence also Proposition 9.9. □

9.6 Speed of the tagged particle

Combining the results of the previous sections, we have all ingredients to prove Theorem 9.2. As pointed out in Remark 9.8, we will use the arguments from Section 9.4 to show transience of the tagged particle in order to determine the almost-sure speed of the tagged particle with respect to $P_{\mathbb{P}^{\text{v}}_{\rho}}$ and $P_{\mathbb{P}^{\text{c}}_{\alpha}}$. Recall from Lemma 9.6 that we can rewrite the horodistance of the tagged particle in terms of the environment process in a ball of radius 1 around its root and a martingale. Using the results of Section 9.3 and Section 9.5, we obtain the following lemma; see also Lemma 8.7 for regular trees.

Lemma 9.16. *For any $\rho \in (0,1)$ in the variable speed model and for any $\alpha \in (0,\infty)$ in the constant speed model of the simple exclusion process, the martingales $(M_t^{\mathrm{v}})_{t\geq 0}$ and $(M_t^{\mathrm{c}})_{t\geq 0}$ in Lemma 9.6 have stationary and ergodic increments.*

Proof. We only prove the case of the variable speed model. For each $t \geq 0$, the random variable $\langle o_t \rangle_{(T_0,o_0)}$ can be expressed as a function F_t of $\{(T_s, o_s, \zeta_s), 0 \leq s \leq t\}$ by following the shifts of the root, i.e.,

$$\langle o_t \rangle_{(T_0,o_0)} - \langle o_0 \rangle_{(T_0,o_0)} = F_t((T_s, o_s, \zeta_s), 0 \leq s \leq t)\,.$$

Since the environment process is a stationary process when starting from $\mathbb{Q}_\rho^{\mathrm{v}}$,

$$M_t^{\mathrm{v}} - M_s^{\mathrm{v}} = F_{t-s}((T_r, o_r, \zeta_r), s \leq r \leq t) + \int_s^t \psi^{\mathrm{v}}(T_r, o_r, \zeta_r)\mathrm{d}r$$

holds for all $s < t$. From Propositions 9.4 and 9.9, we know that the environment process with respect to $\mathbb{Q}_\rho^{\mathrm{v}}$ is stationary and ergodic, and so the claimed statement follows. □

Proof of Theorem 9.2. Using Proposition 9.9 and Lemma 9.16, we can apply the ergodic theorem for both terms on the right-hand side of (9.20), respectively, to see that

$$\lim_{t\to\infty} \frac{\langle o_t \rangle_{(T_0,o_0)}}{t} = (1-\rho)\mathbb{E}\left[\frac{Z-1}{Z+1}\right]\left(\mathbb{E}\left[\frac{1}{Z+1}\right]\right)^{-1}$$

holds almost surely for $\mathbb{Q}_\rho^{\mathrm{v}}$-almost every initial configuration in the variable speed model and $\rho \in (0,1)$. Similarly,

$$\lim_{t\to\infty} \frac{\langle o_t \rangle_{(T_0,o_0)}}{t} = \mathbb{E}\left[\frac{Z-1}{Z+1}\frac{1}{\alpha(Z+1)+1}\right]$$

holds almost surely for $\mathbb{Q}_\alpha^{\mathrm{c}}$-almost every initial configuration in the constant speed model and $\alpha > 0$. Recall that the measures $\mathbb{Q}_\rho^{\mathrm{v}}$ and $\mathbb{P}_\rho^{\mathrm{v}}$, respectively $\mathbb{Q}_\alpha^{\mathrm{c}}$ and $\mathbb{P}_\alpha^{\mathrm{c}}$, are equivalent for all $\rho \in (0,1)$ and $\alpha \in (0,\infty)$. Since $(o_t)_{t\geq 0}$ describes the position of the tagged particle within the environment process, we conclude that

$$\lim_{t\to\infty} \frac{\langle X_t^{\mathrm{v}} \rangle_{(T_0,o_0)}}{t} = (1-\rho)\mathbb{E}\left[\frac{Z-1}{Z+1}\right]\left(\mathbb{E}\left[\frac{1}{Z+1}\right]\right)^{-1}$$

holds $P_{\mathbb{P}_\rho^{\mathrm{v}}}$-almost surely for $\mathbb{P}_\rho^{\mathrm{v}}$-almost every initial configuration in the variable speed model and all $\rho \in (0,1)$. Similarly,

$$\lim_{t\to\infty} \frac{\langle X_t^{\mathrm{c}} \rangle_{(T_0,o_0)}}{t} = \mathbb{E}\left[\frac{Z-1}{Z+1}\frac{1}{\alpha(Z+1)+1}\right]$$

holds $P_{\mathbb{P}^{\mathrm{c}}_\alpha}$-almost surely for $\mathbb{P}^{\mathrm{c}}_\alpha$-almost every initial configuration in the constant speed model and all $\alpha \in (0,\infty)$. Note that in both models of the simple exclusion process, the tagged particle converges almost surely to a ray $\xi' \in \partial(T_0, o_0)$ different from $\xi = \xi(T_0,o_0)$. Let a denote the last common vertex of ξ and ξ' in the variable speed model and observe that

$$|X^{\mathrm{v}}_t| = \langle X^{\mathrm{v}}_t \rangle_{(T_0,o_0)} + 2|a|$$

holds for all $t \geq 0$ sufficiently large. A similar statement is true for the tagged particle in the constant speed model. We conclude since $|a|$ does not depend on t. □

9.7 Open problems

In this section, we studied the speed of a tagged particle when the particles perform simple random walks under an exclusion rule on supercritical, augmented Galton–Watson trees without extinction. It is a natural question if a similar result holds for Galton–Watson trees which may die out.

Conjecture 9.17. *On supercritical Galton–Watson trees conditioned on survival, the tagged particles in the constant speed model and in the variable speed model of the simple exclusion process have almost surely a positive linear speed.*

Another extension of this model is to consider random walks with different transition probabilities. In particular, it is interesting to understand the case where the particles perform biased simple random walks; see Sections 10 and 11 for an approach towards this problem by studying currents for the TASEP on trees.

Question 9.18. *What is the speed of a tagged particle when the particles perform biased random walks on augmented Galton–Watson trees under the exclusion rule?*

A classical problem for exclusion processes is the question if the tagged particle satisfies a central limit theorem. In the case where the augmented Galton–Watson tree is a d-regular tree with $d \geq 3$, a central limit theorem holds; see Theorem 8.2.

Conjecture 9.19. *For any $\rho \in (0,1)$, there exists a constant $\sigma_{\mathrm{v}} \in (0,\infty)$, depending only on ρ and the offspring distribution, such that on almost every supercritical augmented Galton–Watson tree without leaves, the tagged particle $(X^{\mathrm{v}}_t)_{t\geq 0}$ in the variable speed model satisfies*

$$\frac{|X^{\mathrm{v}}_t| - E_{\mathbb{P}^{\mathrm{v}}_\rho}[|X^{\mathrm{v}}_t|]}{\sqrt{t}} \xrightarrow{(d)} \mathcal{N}(0,\sigma_{\mathrm{v}}^2)\,.$$

Similarly, for every $\alpha \in (0,\infty)$, there exists a constant $\sigma_{\mathrm{c}} \in (0,\infty)$, depending only on α and the offspring distribution, such that on almost every supercritical augmented Galton–Watson tree without leaves, the tagged particle $(X^{\mathrm{c}}_t)_{t\geq 0}$ in the constant speed model satisfies

$$\frac{|X^{\mathrm{c}}_t| - E_{\mathbb{P}^{\mathrm{c}}_\alpha}[|X^{\mathrm{c}}_t|]}{\sqrt{t}} \xrightarrow{(d)} \mathcal{N}(0,\sigma_{\mathrm{c}}^2)\,.$$

10 The TASEP on trees in equilibrium

10.1 Introduction

In Sections 8 and 9, we studied exclusion processes on trees, where the particles choose one of its neighbors uniformly at random when attempting a jump. In this case, a natural bias of the particles away from the root is entirely created by the underlying tree structure. We will now consider the case, where the particles themselves have a bias away from the root. We will focus on a totally asymmetric setup, i.e., the particles can only jump in the direction pointing away from the root. The presented material is based in large parts on [62], which is joint work with Nina Gantert and Nicos Georgiou.

In one dimension, i.e., when the underlying tree is $\mathbb{N}$ or $\mathbb{Z}$, the TASEP is among the most investigated interacting particle systems. It is a classical model to describe particle movements or traffic jams, which is studied from various different perspectives; see [21, 94, 140]. Recall Section 7.2, where we gave an introduction to the TASEP on the integers. We now study the TASEP on trees as a natural generalization, and investigate the set of invariant measures as well as the convergence to equilibrium. In particular, we are interested in the long-term behavior of the current across the root. Note that the TASEP on trees is a natural way to describe transport on irregular structures, like blood, air or water circulations systems, which can also be found in the physics literature; see [14, 104, 131].

10.1.1 Definition of the model

Recall from Section 9.2 that we denote by $\mathcal{T}$ the space of all locally finite rooted trees. In the following, we fix a tree $(T,o) \in \mathcal{T}$ with $T=(V,E)$, and let all edges in E be directed and pointing away from the root. The **totally asymmetric simple exclusion process (TASEP)** $(\eta_t)_{t\geq 0}$ on T with a reservoir of intensity $\lambda > 0$ at the root and transition rates $(r_{x,y})_{(x,y)\in E}$ has the following construction; see Figure 34. A particle at site x tries to move to site y at rate $r_{x,y}$, provided that $(x,y) \in E$. In addition, we have a particle source of rate $\lambda > 0$ at the root. Formally, the TASEP on trees is the exclusion process $(\eta_t)_{t\geq 0}$ with state space $\{0,1\}^V$ and generator

$$\mathcal{L}f(\eta) = \lambda(1-\eta(o))[f(\eta^o) - f(\eta)] + \sum_{(x,y)\in E} r_{x,y}(1-\eta(y))\eta(x)[f(\eta^{x,y}) - f(\eta)] \quad (10.1)$$

for all cylinder functions f. In the following, we assume that the underlying tree and the transition rates are such that we obtain by Theorem 3.2 a Feller process in (10.1). In particular, this includes Galton–Watson trees when the rates are uniformly bounded.

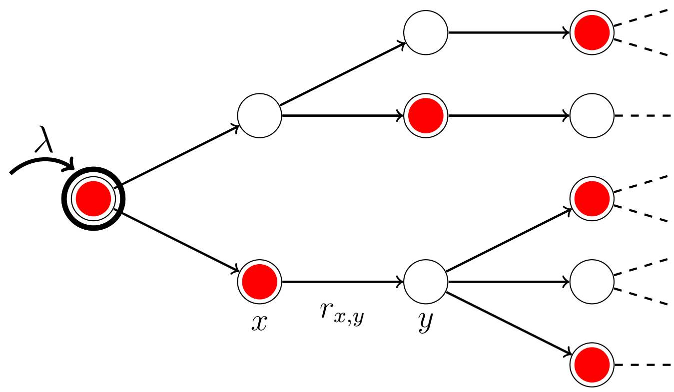

Figure 34: Visualization of the TASEP on trees, where particles are generated at rate λ at the root, and can only move in the direction away from the root. In particular, the edge (x, y) has a rate $r_{x,y}$ Poisson clock, and a particle at x performs a jump to y under the exclusion rule when this clock rings.

In order to state our main results, we require the following notation for the TASEP on trees and its current. For any pair of sites $x, y \in V$, we say that x is below y (and write $x \leq y$) if there exists a directed path in T connecting x to y. Moreover, we set

$$\mathcal{Z}_\ell := \{x \in V : \ |x| = \ell\} \tag{10.2}$$

and refer to $\mathcal{Z}_\ell$ as the ℓ^{th} **generation** or **level** of the tree, for all $\ell \in \mathbb{N}_0$. Recall the current from (7.7). When the starting configuration η_0 of $(\eta_t)_{t\geq 0}$ has only finitely many particles, let the current $(J_x(t))_{t\geq 0}$ across $x \in V$ be given by

$$J_x(t) := \sum_{y:\, x\leq y} \eta_t(y) - \sum_{y:\, x\leq y} \eta_0(y) = \sum_{y:\, x\leq y} (\eta_t(y) - \eta_0(y)) \tag{10.3}$$

for all $t \geq 0$. Similarly, we define the **aggregated current** $(J_\ell(t))_{t\geq 0}$ at generation ℓ by

$$J_\ell(t) := \sum_{x\in\mathcal{Z}_\ell} J_x(t) \tag{10.4}$$

for all $\ell \in \mathbb{N}_0$ and $t \geq 0$. Intuitively, the current, respectively the aggregated current, denotes the number of particles that have reached site x, respectively level ℓ, by time t. Our goal is to prove a law of large numbers for the aggregated current through a fixed generation. In particular, we are interested when we have an aggregated current which is linear in time. We will see that an answer to this question depends on the local structure of the rates. In particular, we have to compare the incoming rates to a site $x \in V$ and, for all $x \in V$, the sum of the outgoing rates

$$r_x := \sum_{(x,y)\in E} r_{x,y} \, . \tag{10.5}$$

10.1.2 Main results

We will now present our main results on the long-term behavior of the TASEP on trees. Let the TASEP start from ν_0, where we recall that ν_ρ denotes the Bernoulli-ρ-product measure on $\{0,1\}^V$ for all $\rho \in [0,1]$. For all $x \in V(T)$, we define the **net flow** $q(x)$ through x to be

$$q(x) := \begin{cases} r_x - r_{\bar{x},x} & x \neq o \\ r_o & x = o\,, \end{cases} \tag{10.6}$$

where $\bar{x}$ denotes the unique parent of x. We say that the rates satisfy a **superflow rule** if $q(x) \geq 0$ holds for all $x \in V(T) \setminus \{o\}$. Recall the definition of a flow rule from (3.10). With a slight abuse of notation, we say that a **flow rule** holds for the TASEP on trees if $q(x) = 0$ for all $x \in V(T) \setminus \{o\}$, and we let $q(o)$ be the **strength** of the flow. Furthermore, we say that the rates satisfy a **subflow rule** if

$$\lim_{m\to\infty} \sum_{x\in\mathcal{Z}_m} r_x = 0\,. \tag{10.7}$$

Theorem 10.1. *Let $(S_t)_{t\geq 0}$ be the semi-group of the TASEP $(\eta_t)_{t\geq 0}$ on a fixed tree $(T,o) \in \mathcal{T}$. We assume that T is infinite, without leaves, and has uniformly bounded rates $(r_{x,y})$. Suppose that particles are generated at the root at rate $\lambda > 0$. Then there exists a stationary measure π_λ of $(\eta_t)_{t\geq 0}$ with*

$$\lim_{t\to\infty} \nu_0 S_t = \pi_\lambda \tag{10.8}$$

with respect to weak convergence of probability measures on $\{0,1\}^V$. If a superflow rule holds, then $\pi_\lambda \neq \nu_1$ and the current $(J_o(t))_{t\geq 0}$ through the root satisfies

$$\varliminf_{t\to\infty} \frac{J_o(t)}{t} \geq \lambda \pi_\lambda(\eta(o) = 0) > 0\,. \tag{10.9}$$

Moreover, if in addition $\lambda < r_o$ as well as

$$\lim_{n\to\infty} |\mathcal{Z}_{n-m}| \min_{x\in\mathcal{Z}_n} r_{x,y} = \infty \tag{10.10}$$

holds for all $m \geq 0$ fixed, we see a fan behavior, i.e.,

$$\lim_{n\to\infty} \frac{1}{|\mathcal{Z}_n|} \sum_{x\in\mathcal{Z}_n} \pi_\lambda(\eta(x) = 1) = 0\,. \tag{10.11}$$

When the rates satisfy a subflow rule, we see blockage, i.e., $\pi_\lambda = \nu_1$ and almost surely

$$\lim_{t\to\infty} \frac{J_o(t)}{t} = 0\,. \tag{10.12}$$

10.1.3 Parallels between the TASEP on the integers and on trees

For the TASEP on trees and on the integers, an essential first step is to understand their equilibrium measures. For the TASEP on $\mathbb{Z}$, the extremal invariant measures are Bernoulli-ρ-product measures and the Dirac measures on blocking configurations; see Theorem 3.10. In Lemma 10.6, we will see that for the TASEP on trees with a reservoir at the root, a flow rule on the rates implies that there exists a non-trivial invariant Bernoulli-ρ-product measure. This should be compared to Theorem 3.6, where we show that the Bernoulli-ρ-product measures are invariant for all $\rho \in [0, 1]$ for simple exclusion processes with a flow rule, as well as to the proof of Lemma 6.25, where we discuss extremal invariant measures for the asymmetric simple exclusion process on the half-line with a particle source at the endpoint.

A great strength of various particle systems are their explicit hydrodynamic limits, as the macroscopic and the microscopic behavior are connected; see [52] for a survey on the TASEP on the integers, and references therein. Hydrodynamic limits for the one-dimensional TASEP, in the sense of a rigorous connection to the Burgers equation, were established by Rost in [113], who studied the rarefaction fan. The result were then extended in various ways in [121, 122, 123, 124]. Similar ideas for the TASEP on regular trees with a flow rule and spherically symmetric rates can be found in the physics literature [104]. However, to our best knowledge, hydrodynamic limits are not available for the TASEP on general trees, and we refer to Conjecture 10.13 for a possible approach to hydrodynamic limits in the case of flow rules.

Depending on the initial particle configuration, the macroscopic evolution of the particle density in the TASEP on the integers may show a shock or a rarefaction fan, as one can see from the limiting partial differential equation. Moreover, in a simple two-phase example, starting from macroscopically constant initial conditions, one can see the simultaneous development of blockage and fans, depending on the common value of the density [68]. In Theorem 10.1, we describe blockage and fans in the limiting particle distribution for the TASEP on trees even without having access to hydrodynamic limits. In particular, we show that blockage occurs in the case of a subflow rule, starting with all sites being empty.

A common tool is to approximate the TASEP on the integers using a finite exclusion process with open boundaries. We saw in Section 6 how the speed of convergence to the equilibrium for the TASEP with open boundaries can be analyzed in terms of mixing times. We will see an approximation of the TASEP on trees using a finite TASEP with open boundaries in Section 10.4. However, determining the mixing time of the TASEP on finite trees remains an open problem; see Question 11.28.

10.1.4 Outline of the proof

In order to show Theorem 10.1, we have the following strategy. We start by introducing a modified canonical coupling for the TASEP on trees. In Section 10.3, we prove that the law of the TASEP on trees converges for any choice of the rates when we start from the empty initial configuration; see Proposition 10.4. We show in Section 10.4 that for superflow rules, we have a positive linear current; see Proposition 10.8. Furthermore, the particle density vanishes; see Proposition 10.10. For subflow rules, we show in Section 10.5 that blockage occurs; see Proposition 10.12. We conclude with an outlook on open problems.

10.2 The canonical coupling for the TASEP on trees

Recall the canonical coupling for the simple exclusion process from Section 3.3. As in Section 5.2 for the simple exclusion process in random environment, we will now define a modified canonical coupling for the TASEP on trees.

Let $(\eta_t^1)_{t\geq 0}$ and $(\eta_t^2)_{t\geq 0}$ denote two totally asymmetric simple exclusion processes on $T=(V,E)$ with transition rates $(r_{x,y})$, where particles are generated at the root at rates λ_1 and λ_2, respectively. Assume $\lambda_1 \leq \lambda_2$. The modified canonical coupling is the joint evolution $(\eta_t^1, \eta_t^2)_{t\geq 0}$ of the two TASEPs according to the following description.

For every edge $e=(x,y)\in E$, consider independent rate $r_{x,y}$ Poisson clocks. Whenever a clock rings at time t for an edge (x,y), we try in both processes to move a particle from x to y, provided that $\eta_t^1(x) = 1-\eta_t^1(y) = 1$ or $\eta_t^2(x) = 1-\eta_t^2(y)=1$ holds. We place a rate λ_1 Poisson clock at the root. Whenever the clock rings, we try to place a particle at the root in both processes. Furthermore, if $\lambda_1 < \lambda_2$, we place an additional independent rate $(\lambda_2-\lambda_1)$ Poisson clock at the root. Whenever this clock rings, we try to place a particle at the root in $(\eta_t^2)_{t\geq 0}$.

Recall from (3.13) that $\succeq_{\mathrm{c}}$ denotes the component-wise partial order on $\{0,1\}^V$, and that, with a slight abuse of notation, we denote by $\mathbf{P}$ the law of the modified canonical coupling.

Lemma 10.2. *Let $(\eta_t^1)_{t\geq 0}$ and $(\eta_t^2)_{t\geq 0}$ be two TASEPs on trees within the above canonical coupling. Suppose that $\lambda_1 \leq \lambda_2$ holds, then*

$$\mathbf{P}\left(\eta_t^1 \preceq_{\mathrm{c}} \eta_t^2 \text{ for all } t\geq 0 \mid \eta_0^1 \preceq_{\mathrm{c}} \eta_0^2\right) = 1. \tag{10.13}$$

Remark 10.3. *In a similar way, we can define the modified canonical coupling for the TASEP on trees when we allow reservoirs of intensities λ_1^v and λ_2^v at all sites $v\in V$, respectively. The canonical coupling preserves the partial order $\succeq_{\mathrm{c}}$ provided that $\lambda_1^v \leq \lambda_2^v$ holds for all sites $v \in V$.*

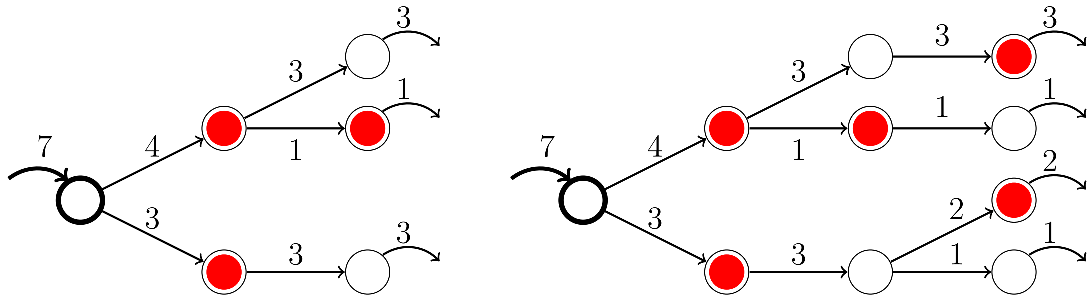

Figure 35: Approximation of a tree T satisfying a flow rule of strength 7 by trees T_2, depicted on the left-hand side, and T_3, depicted on the right-hand side.

10.3 Construction of the invariant measure

Recall that we assume $(T, o) \in \mathcal{T}$ to be a locally finite, rooted tree on which the TASEP is a Feller process with respect to a given family of rates $(r_{x,y})$. Moreover, we start the TASEP on T from ν_0, where all sites are initially empty.

Proposition 10.4. *Let $(S_t)_{t \geq 0}$ be the semi-group of the TASEP $(\eta_t)_{t \geq 0}$ where particles are generated at the root at rate λ for some $\lambda > 0$. There exists a probability measure π_λ on $\{0,1\}^V$ such that*

$$\lim_{t \to \infty} \nu_0 S_t = \pi_\lambda . \tag{10.14}$$

In particular, π_λ is a stationary measure for $(\eta_t)_{t \geq 0}$.

In order to show Proposition 10.4, we adopt a sequence of results from Liggett [92]. Let T_n denote the tree restricted to level n, where particles exit from the tree at $x \in \mathcal{Z}_n$ at rate r_x; see Figure 35. For every n, let π_λ^n denote the invariant distribution of the dynamics $(\eta_t^n)_{t \geq 0}$ on T_n with semi-group $(S_t^n)_{t \geq 0}$. We extend each measure π_λ^n to a probability measure on $\{0,1\}^{V(T)}$ by taking the Dirac measure on 0 for all sites $x \in V(T) \setminus V(T_n)$. Recall from (4.35) the stochastic domination for probability measures with respect to $\succeq_{\mathrm{c}}$.

Lemma 10.5 (c.f. Proposition 3.7 in [92])**.** *For any initial distribution $\tilde{\pi}$, the laws of the TASEPs $(\eta_t^n)_{t \geq 0}$ and $(\eta_t^{n+1})_{t \geq 0}$ on T_n and T_{n+1}, respectively, satisfy*

$$\tilde{\pi} S_t^n = \mathbb{P}(\eta_t^n \in \cdot) \preceq_{\mathrm{c}} \mathbb{P}(\eta_t^{n+1} \in \cdot) = \tilde{\pi} S_t^{n+1} \tag{10.15}$$

for all $t \geq 0$. In particular, $\pi_\lambda^n \preceq_{\mathrm{c}} \pi_\lambda^{n+1}$ holds for all $n \in \mathbb{N}$.

Proof. We follow the arguments in the proof of Theorem 2.13 in [92]. We note that for all $n \in \mathbb{N}$, the generators $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$ of the TASEPs on T_n and T_{n+1} satisfy

$$\mathcal{L}_{n+1} f(\eta) - \mathcal{L}_n f(\eta) = \sum_{x \in \mathcal{Z}_n, y \in \mathcal{Z}_{n+1}} [f(\eta^x) - f(\eta)] \, r_{x,y} \, (-\eta(x)\eta(y)) \geq 0$$

for any increasing function f which does only depend on $V(T_n)$, for all $\eta \in \{0,1\}^{V(T)}$. Using the extension arguments from Theorem 2.3 and Theorem 2.11 in [92], we obtain that

$$\int f \mathrm{d}\left[\tilde{\pi} S_t^n\right] \leq \int f \mathrm{d}\left[\tilde{\pi} S_t^{n+1}\right] \tag{10.16}$$

for any increasing function f which only depends on $V(T_n)$, for all $t \geq 0$. It suffices now to show that (10.16) holds for all increasing functions f which only depend on $V(T_{n+1})$. This follows verbatim the proof of Theorem 2.13 in [92] by decomposing f according to its values on $V(T_{n+1}) \setminus V(T_n)$. □

Lemma 10.5 implies that the probability distribution π_λ given by

$$\pi_\lambda := \lim_{n\to\infty} \pi_\lambda^n \tag{10.17}$$

exists; see also Theorem 3.10 (a) in [92]. More precisely, Lemma 10.5 guarantees for every increasing cylinder function f that

$$\lim_{n\to\infty} \int f \mathrm{d}\pi_\lambda^n = \int f \mathrm{d}\pi_\lambda \,.$$

Since the set of increasing functions is a determining class, (10.17) follows. Furthermore, since $S_t^n f$ converges uniformly to $S_t f$ for any cylinder function f, π_λ is invariant for $(\eta_t)_{t\geq 0}$; see Proposition 2.2 and Theorem 4.1 in [92]. We now have all tools to show Proposition 10.4.

Proof of Proposition 10.4. Since we know that π_λ is invariant, we apply the modified canonical coupling from Lemma 10.2 to see that for all $t \geq 0$,

$$\nu_0 S_t \preceq_{\mathrm{c}} \pi_\lambda \,.$$

Moreover, by Lemma 10.5, for all $t \geq 0$ and all $n \in \mathbb{N}$

$$\nu_0 S_t^n \preceq_{\mathrm{c}} \nu_0 S_t \,.$$

To prove Proposition 10.4, it suffices to show that

$$\lim_{t\to\infty} \int f \mathrm{d}\left[\nu_0 S_t\right] = \int f \mathrm{d}\pi_\lambda$$

holds for any increasing cylinder function f. Combining the above observations

$$\int f \mathrm{d}\pi_\lambda^n = \liminf_{t\to\infty} \int f \mathrm{d}\left[\nu_0 S_t^n\right] \leq \liminf_{t\to\infty} \int f \mathrm{d}\left[\nu_0 S_t\right] \leq \limsup_{t\to\infty} \int f \mathrm{d}\left[\nu_0 S_t\right] \leq \int f \mathrm{d}\pi_\lambda$$

holds for every $n \in \mathbb{N}$ and for any increasing cylinder function f. We conclude the proof recalling (10.17); see also the proof of Lemma 4.3 in [92]. □

Next, we show that if the rates satisfy a flow rule then there exists an invariant Bernoulli-ρ-product measure for some $\rho \in (0,1)$ for the TASEP on the tree; see also Theorem 3.6.

Lemma 10.6. *Let T be a locally finite, rooted tree with rates satisfying a flow rule for a flow of strength q. Assume that particles are generated at the root at rate $\lambda = \rho q$ for some $\rho \in (0,1)$. Then ν_ρ is an invariant measure for the TASEP $(\eta_t)_{t\geq 0}$ on T.*

Proof. By Theorem 3.4, we have to show that for all cylinder functions f,

$$\int \mathcal{L} f \mathrm{d}\nu_\rho = 0\,.$$

Due to the linearity of $\mathcal{L}$, it suffices to consider f of the form

$$f(\eta) = \prod_{x\in A} \eta(x) \tag{10.18}$$

with $\eta \in \{0,1\}^{V(T)}$ and for A some finite subset of $V(T)$. A calculation shows that if $o \notin A$,

$$\int \mathcal{L} f \mathrm{d}\nu_\rho = (1-\rho)\rho^{|A|} \sum_{x\in A, y\notin A} \left[r_{y,x} - r_{x,y} \right]\,; \tag{10.19}$$

see also the proof of Theorem 2.1(a) in [95, Chapter VIII]. Since a flow rule holds, the sum in (10.19) is zero. Similarly, we obtain in the case $o \in A$

$$\int \mathcal{L} f \mathrm{d}\nu_\rho = (1-\rho)\rho^{|A|} \left(\sum_{x\in A, y\notin A} \left[r_{y,x} - r_{x,y} \right] + \frac{\lambda}{\rho} \right)\,.$$

We conclude using the flow rule, noting $r_o = q = \frac{\lambda}{\rho}$ and recalling the definition of r_o from (10.5). □

Remark 10.7. *Note that the measure ν_1 is always invariant for the TASEP on trees. Theorem 1 of [24] shows that the TASEP on T with a half-line attached to the root, where all edges point to the root, has an invariant product measure with densities in $(0,1)$ if and only if a flow rule holds. If a flow rule holds, a similar argument as Theorem 1.17 in [94, Part III] shows that ν_ρ is extremal invariant for all $\rho \in [0,1]$.*

10.4 Proof of positive current

Next, we consider the case where the rates do not necessarily satisfy a flow rule. In the following, we will without loss of generality assume that $\lambda < q(o)$ holds. When $\lambda \geq q(o)$, the canonical coupling in Lemma 10.5 yields that the current stochastically dominates the current of any TASEP with rate λ' for some $\lambda' < q(o)$. We now characterize the behavior of the TASEP in the superflow case.

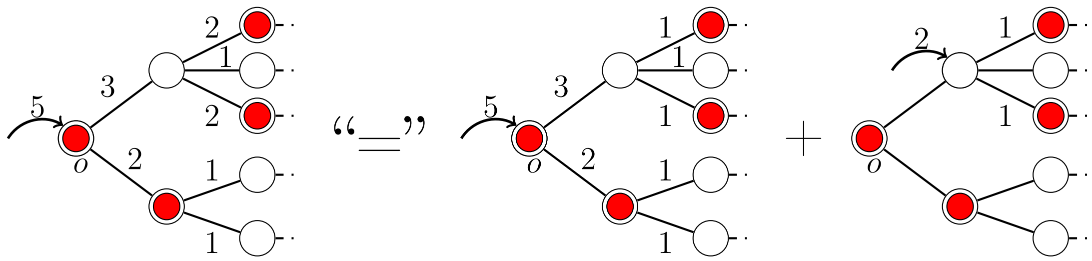

Figure 36: Visualization of the superflow decomposition used in Lemma 10.9. The superflow given at the left-hand side is decomposed into two flows of strengths 5 and 2, respectively, shown at the right-hand side.

Proposition 10.8. *Assume that a superflow rule holds. Let $(J_o(t))_{t\geq 0}$ be the current at the root for the TASEP on a tree T with a reservoir of rate $\lambda = \rho q(o)$ for some $\rho \in (0,1)$, and initial distribution ν_0. Then the current $(J_o(t))_{t\geq 0}$ through the root satisfies*

$$\lim_{t\to\infty} \frac{J_o(t)}{t} = \lambda \pi_\lambda(\eta(o) = 0) \geq q(o)\rho(1-\rho) \tag{10.20}$$

almost surely, where π_λ is given by (10.14)*.*

In order to prove Proposition 10.8, we will use the following lemma, which states that the law of the TASEP on trees is always dominated by a certain Bernoulli-ρ-product measure on the tree.

Lemma 10.9. *Assume that the rates satisfy a superflow rule and consider the TASEP $(\eta_t)_{t\geq 0}$ with a reservoir of rate $\lambda = \rho q(o)$ for some $\rho \in (0,1)$. If $\mathbb{P}(\eta_0 \in \cdot) \preceq_{\mathrm{c}} \nu_\rho$ holds, then*

$$\mathbb{P}(\eta_t \in \cdot) \preceq_{\mathrm{c}} \nu_\rho \tag{10.21}$$

for all $t \geq 0$. In particular, the measure π_λ from (10.14) *satisfies $\pi_\lambda \preceq_{\mathrm{c}} \nu_\rho$.*

Proof. In order to show (10.21), we decompose the rates satisfying a superflow rule into flows starting at different sites. More precisely, we claim that there exists a family of transition rates $((r^z_{x,y})_{(x,y)\in E(T)})_{z\in V(T)}$ with the following two properties. For every $i \in V(T)$ fixed, the rates $(r^z_{x,y})_{(x,y)\in E(T)}$ satisfy a flow rule for a flow of strength $q(z)$ for the tree rooted in z. Moreover, for all $(x,y) \in E(T)$,

$$\sum_{z\in V(T)} r^z_{x,y} = r_{x,y}\,;$$

see also Figure 36. We construct such a family of transition rates as follows. We start with the root o and choose a set of rates $(r^o_{x,y})_{(x,y)\in E(T)}$ according to an arbitrary rule such that the rates satisfy a flow rule for a flow of strength $q(o)$ starting at o, and $r^o_{x,y} \leq r_{x,y}$ for all $(x,y) \in E(T)$. Next, we consider the neighbors of o in the tree. For every $z \in V(T)$ with $|z| = 1$, we choose a set of rates $(r^z_{x,y})_{(x,y)\in E(T)}$ according to an arbitrary rule such that the rates satisfy a flow rule for a flow of strength $q(z)$ starting at z. Moreover, we require that

$$r^z_{x,y} \leq r_{x,y} - r^o_{x,y}$$

holds for all $(x,y) \in E(T)$. The existence of the flow is guaranteed by the superflow rule. More precisely, we use the following observation. Whenever the rates satisfy a superflow rule, we can treat the rates as maximal capacities and find a flow $(r^o_{x,y})$ of strength $q(o)$ which does not exceed these capacities. Note that the reduced rates $(r_{x,y} - r^o_{x,y})$ must again satisfy a superflow rule, but now on the connected components of the graph with vertex set $V(T) \setminus \{o\}$. This is due to the fact that the net flow vanishes on all sites $V(T) \setminus \{o\}$. We then iterate this procedure to obtain the claim.

Let $(\tilde{\eta}_t)_{t\geq 0}$ be the exclusion process with rates $(r_{x,y})_{(x,y)\in E(T)}$, where in addition, we create particles at every site $x \in V(T)$ at rate $q(x)\rho$. Due to the above decomposition of the rates and Lemma 10.6, we claim that the measure ν_ρ is invariant for $(\tilde{\eta}_t)_{t\geq 0}$. To see this, we define a family of generators $(\mathcal{L}_z)_{z\in V(T)}$ on the state space $\{0,1\}^{V(T_z)}$. Here, the trees T_z are the subtrees of T rooted at z, consisting of all sites which can be reached from site z using a directed path. For all cylinder functions f, we set

$$\mathcal{L}_z f(\eta) = \rho q(z)(1-\eta(z))[f(\eta^z) - f(\eta)] + \sum_{(x,y)\in E(T_z)} r^z_{x,y}(1-\eta(y))\eta(x)[f(\eta^{x,y}) - f(\eta)]$$

and thus by Lemma 10.6

$$\int \mathcal{L}_z f(\eta) \mathrm{d}\nu_\rho = 0 \tag{10.22}$$

holds. Note that the generator $\tilde{\mathcal{L}}$ of the process $(\tilde{\eta}_t)_{t\geq 0}$ satisfies

$$\tilde{\mathcal{L}} f(\eta) = \sum_{z\in V(T)} \mathcal{L}_z f(\eta) \tag{10.23}$$

for all cylinder functions f on $\{0,1\}^{V(T)}$, and that at most finitely many terms in the sum in (10.23) are non-zero since f is a cylinder function. Hence, we obtain that ν_ρ is an invariant measure of $(\tilde{\eta}_t)_{t\geq 0}$ by combining (10.22) and (10.23). Using Remark 10.3, we see that the modified canonical coupling $\mathbf{P}$ for the TASEP on trees satisfies

$$\mathbf{P}\left(\eta_t \preceq_{\mathrm{c}} \tilde{\eta}_t \text{ for all } t \geq 0 \mid \eta_0 \preceq_{\mathrm{c}} \tilde{\eta}_0\right) = 1\,.$$

Thus, we let $(\tilde{\eta}_t)_{t\geq 0}$ be started from ν_ρ and conclude using Strassen's theorem [134]. □

Proof of Proposition 10.8. Combining Proposition 10.4, Remark 10.7, and Lemma 10.9, we obtain (10.20) by applying the ergodic theorem for Markov processes. □

Proposition 10.10. *Consider the TASEP* $(\eta_t)_{t\geq 0}$ *on the tree* $T = (V, E)$ *for some* $\lambda = \rho q(o) > 0$ *with* $\rho \in (0, 1)$*. Moreover, assume that a superflow rule holds and that* (10.10) *is satisfied. Then the measure* π_λ *from Proposition 10.4 satisfies*

$$\lim_{n\to\infty} \frac{1}{|\mathcal{Z}_n|} \sum_{x\in\mathcal{Z}_n} \pi_\lambda(\eta(x) = 1) = 0\,. \tag{10.24}$$

Proof. Note that (10.10) is equivalent to assuming

$$\lim_{n\to\infty} |\mathcal{Z}_n| \min_{\substack{(x,y)\in E\\ |x|\in[n,n+m]}} r_{x,y} = \infty \tag{10.25}$$

for any $m \geq 0$ fixed. Moreover, note that

$$J_o(t) - J_n(t) \leq \sum_{i\in[n]} |\mathcal{Z}_i| \tag{10.26}$$

for any $n \in \mathbb{N}$ and $t \geq 0$. Using Proposition 10.4, we see that

$$\lambda \geq \lim_{t\to\infty} \frac{J_o(t)}{t} = \lim_{t\to\infty} \frac{J_n(t)}{t} = \sum_{x\in\mathcal{Z}_n\colon (x,y)\in E} \pi_\lambda(\eta(x) = 1, \eta(y) = 0) r_{x,y}$$

holds for all $n \in \mathbb{N}_0$. In particular, for all $n, m \in \mathbb{N}_0$

$$\sum_{|x|\in[n,n+m]} \sum_{(x,y)\in E} \pi_\lambda(\eta(x) = 1, \eta(y) = 0) \leq m\lambda\Big(\min_{\substack{(x,y)\in E\\ |x|\in[n,n+m]}} r_{x,y}\Big)^{-1}. \tag{10.27}$$

Let $\delta > 0$ be arbitrary and fix some $m \in \mathbb{N}$ such that $\rho^m \leq \frac{\delta}{2}$. Moreover, for all $x \in \mathcal{Z}_n$, fix a sequence of sites $(x = x_1, x_2, \dots, x_m)$ with $(x_i, x_{i+1}) \in E$ for all $i \in [m-1]$. Note that the sites $(x_i)_{i\in[m]}$ are disjoint for different $x \in \mathcal{Z}_n$ and that by Lemma 10.9

$$\pi_\lambda(\eta(x_i) = 1 \text{ for all } i \in [m]) \leq \delta/2 \tag{10.28}$$

for all $x \in \mathcal{Z}_n$. For $x \in \mathcal{Z}_n$, we decompose according to the value on $(x_i)_{i\in[m]}$ to get

$$\sum_{x\in\mathcal{Z}_n} \pi_\lambda(\eta(x) = 1) \leq \sum_{x\in\mathcal{Z}_n} \pi_\lambda(\eta(x_i) = 1\ \forall i \in [m]) + \sum_{\substack{(x,y)\in E\\ |x|\in[n,n+m]}} \pi_\lambda(\eta(x) = 1, \eta(y) = 0)\,.$$

Hence, combining (10.25), (10.27) and (10.28), we see that for all n sufficiently large,

$$\sum_{x\in\mathcal{Z}_n} \pi_\lambda(\eta(x) = 1) \leq \frac{\delta}{2} |\mathcal{Z}_n| + m\lambda\Big(\min_{\substack{(x,y)\in E\\ |x|\in[n,n+m]}} r_{x,y}\Big)^{-1} \leq \delta\, |\mathcal{Z}_n|\,.$$

Since $\delta > 0$ was arbitrary, we conclude. □

We use a similar argument to determine when we have a positive averaged density.

Corollary 10.11. *Suppose that a superflow rule holds. Consider the TASEP* $(\eta_t)_{t\geq 0}$ *on the tree* $T=(V,E)$ *for some* $\lambda = \rho q(o) > 0$ *with* $\rho \in (0,1)$*. Moreover, assume that* T *has maximum degree* Δ*, and that*

$$\limsup_{n\to\infty} |\mathcal{Z}_n| \min_{(x,y)\in E, x\in\mathcal{Z}_n} r_{x,y} \leq c \tag{10.29}$$

holds for some constant $c>0$*. Then*

$$\liminf_{n\to\infty} \frac{1}{|\mathcal{Z}_n|} \sum_{x\in\mathcal{Z}_n} \pi_\lambda(\eta(x)=1) > 0\,. \tag{10.30}$$

Proof. Observe that for every $x\in\mathcal{Z}_n$ and $n\in\mathbb{N}$, we can choose a neighbor $y\in\mathcal{Z}_{n+1}$ of x such that

$$\frac{1}{\Delta}\limsup_{t\to\infty}\frac{J_x(t)}{t} \leq \limsup_{t\to\infty}\frac{J_y(t)}{t} = \pi_\lambda(\eta(x)=1,\eta(y)=0)r_{x,y}$$

holds. Together with (10.29)

$$\sum_{x\in\mathcal{Z}_n}\pi_\lambda(\eta(x)=1) \geq \sum_{x\in\mathcal{Z}_n}\frac{1}{\Delta r_{x,y}}\limsup_{t\to\infty}\frac{J_x(t)}{t} \geq \frac{1}{c\Delta}|\mathcal{Z}_n|\limsup_{t\to\infty}\frac{J_n(t)}{t}\,.$$

Since the rates satisfy a superflow rule, we conclude by applying Proposition 10.8. □

10.5 Proof of blockage

Next, we consider the case where the rates in the tree decay too fast, i.e., when a subflow rule holds; see (10.7) and Figure 37. We show that the current is sublinear.

Proposition 10.12. *Suppose that the rates satisfy a subflow rule. Then the current* $(J_o(t))_{t\geq 0}$ *of the TASEP* $(\eta_t)_{t\geq 0}$ *on a tree* $T=(V,E)$ *with a reservoir of rate* $\lambda>0$ *satisfies*

$$\lim_{t\to\infty}\frac{J_o(t)}{t} = 0 \tag{10.31}$$

almost surely. Moreover, the limit measure π_λ *of Lemma 10.4 is the Dirac measure* ν_1*. In particular,* $(\eta_t)_{t\geq 0}$ *has a unique invariant measure.*

Proof. By (10.26), it suffices for (10.31) to prove that for every $\varepsilon>0$, there exists some $m=m(\varepsilon)$ such that the aggregated current $(J_m(t))_{t\geq 0}$ at generation m satisfies

$$\limsup_{t\to\infty}\frac{J_m(t)}{t} \leq \varepsilon\,.$$

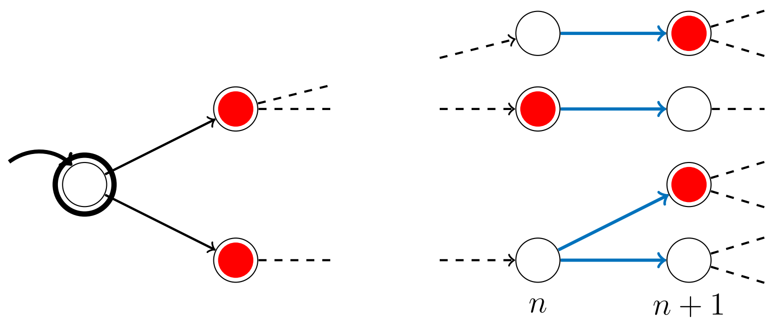

Figure 37: Visualization of the proof of blockage for the TASEP on trees when a subflow rule holds. For any given ε, we can choose n large enough such that the total rates along the edges shown in blue sum up to at most ε.

Recall r_x from (10.5) for all $x \in V$, and let $(X_t^x)_{t\geq 0}$ be a rate r_x Poisson clock, indicating how often the clock of an outgoing edge from x rang until time t. In order to bound $(J_m(t))_{t\geq 0}$, recall that we start with all sites being empty, and observe that the current can only increase by one if a clock at an edge connecting level $m-1$ to level m rings. Thus, we see that

$$0 \leq \limsup_{t\to\infty} \frac{J_m(t)}{t} \leq \limsup_{t\to\infty} \frac{1}{t} \sum_{x\in\mathcal{Z}_{m-1}} X_t^x = \sum_{x\in\mathcal{Z}_{m-1}} r_x$$

holds almost surely. Using the subflow rule, we can choose $m = m(\varepsilon)$ sufficiently large to conclude (10.31). To prove that π_λ is the Dirac measure on all sites being occupied, use Proposition 10.8 to see that (10.31) holds if and only if $\pi_\lambda(\eta(o) = 1) \in \{0, 1\}$. Since the rate λ at which particles are generated is strictly positive and π_λ is an invariant measure, we conclude that $\pi_\lambda(\eta(o) = 1) = 1$. Using the ergodic theorem, we see that almost surely for all neighbors z of o,

$$\pi_\lambda(\eta(o) = 1, \eta(z) = 0) r_{o,z} = \lim_{t\to\infty} \frac{J_z(t)}{t} \leq \lim_{t\to\infty} \frac{J_o(t)}{t} = 0\,.$$

Hence, we obtain that $\pi_\lambda(\eta(z) = 1) = 1$ holds for all $z \in V$ with $|z| = 1$ as well. We iterate this argument to conclude. □

10.6 Open problems

In Theorem 10.1, we study the stationary measure π_λ for the TASEP on trees when starting from the empty configuration. We expect the following property of π_λ, which is similar to the TASEP on the half-line; see Lemma 4.3 in [92].

Conjecture 10.13. *Consider TASEP with a reservoir of rate $\lambda = \rho q$ for some constant $\rho \in (0, \infty)$ such that a flow rule holds for some flow of strength $q > 0$. Then for $\rho \leq \frac{1}{2}$,* *the invariant measure* π_λ *from* (10.14) *satisfies* $\pi_\lambda = \nu_\rho$. *For* $\rho > \frac{1}{2}$, *it holds that*

$$\lim_{|x|\to\infty} \pi_\lambda(\eta(x) = 1) = \frac{1}{2}\,. \tag{10.32}$$

Note that in Theorem 10.1, a key assumption is that either a superflow or a subflow rule holds. When both rules do not apply, the question whether a single local microscopic change in the rates affects the current is already a difficult task for the one-dimensional TASEP, known as the "slow bond problem". Originally, the "slow bond problem" was introduced in [73, 74] on a finite segment with open boundaries. On $\mathbb{Z}$, progress was made in [125], and the problem was solved in [16].

Question 10.14. *What can we say about the current through the root for the TASEP on trees when the rates neither satisfy a superflow rule nor a subflow rule?*

11 The TASEP on trees out of equilibrium

11.1 Introduction

In this section, we study the TASEP on trees out of equilibrium. The presented material is based in large parts on [62], which is joint work with Nina Gantert and Nicos Georgiou. Our goal is to investigate the aggregated current through a given generation of the tree. In contrast to Section 10, we consider the current through the ℓ^{th} generation for some $\ell = \ell(t)$ which dependents on the time $t \geq 0$. We will assume that $\ell(t) \to \infty$ for $t \to \infty$. Throughout this section, we start the TASEP on trees from the configuration where all sites are empty, and focus on the motion of the first n particles entering the tree. In particular, we are interested in establishing a time interval in which we see a transition from no particles to approximately n particles at a given generation. Conversely, we aim at proving a dual theorem where we give a window of generations in which most of the first n particles can be located at a given point in time.

Our investigations are motivated by similar results for the one-dimensional TASEP; see Sections 7.2 and 10.1.3. We consider the TASEP on directed rooted trees as a natural generalization, where we can mimic this total asymmetry. An important observation and difference to the one-dimensional setup is that once two particles are located on distinct branches of the tree, they do not affect the transitions of each other. We make use of this observation by locating where the particle trajectories disentangle and the particles start to move independently. Quantifying the location of disentanglement is a key step in our analysis. Since the arguments require several assumptions on the underlying structure of the trees, we will restrict our attention to the TASEP on supercritical Galton–Watson trees without extinction.

11.1.1 Definition of the model

Recall the definition of Galton–Watson trees from Section 9. We refer to Chapter 4 of [99] for a more comprehensive introduction. In the following, we assume that all Galton–Watson trees are supercritical and without extinction, i.e., the conditions in (9.1) hold, and recall that the Galton–Watson branching process induces a probability measure $\mathbf{GW}$ on the space of all locally finite rooted trees $\mathcal{T}$; see Section 9.2.

Fix a tree $(T, o) \in \mathcal{T}$ drawn according to $\mathbf{GW}$. On this tree $T = (V, E)$, we study the TASEP $(\eta_t)_{t \geq 0}$ with a reservoir of intensity $\lambda > 0$ and transition rates $(r_{x,y})_{(x,y) \in E}$. Formally, $(\eta_t)_{t \geq 0}$ is the exclusion process on $\{0,1\}^V$ with respect to the generator given in (10.1). Note that the same arguments as in Proposition 9.1 for the simple exclusion process on Galton–Watson trees ensure that for uniformly bounded transition rates, the TASEP on T is $\mathbf{GW}$-almost surely a Feller process. For a tree $(T, o) \in \mathcal{T}$, let P_T denote the law of the TASEP on T. Furthermore, we let in the following

$$\mathbb{P} = \mathbf{GW} \times P_T$$

be the semi-direct product, where we first choose a tree $(T, o) \in \mathcal{T}$ according to $\mathbf{GW}$ and then perform the TASEP on T.

11.1.2 Related literature

In Section 10.1.3, we discussed parallels between the TASEP on trees and the TASEP on the integers with a focus on their equilibrium measures and hydrodynamic limits. For the one-dimensional TASEP started out of equilibrium, a key tool to obtain sharp results is an alternative representation as a two-dimensional exponential corner growth model; see also Section 3.4.3. For example, the TASEP on $\mathbb{Z}$ in the rarefaction fan, where the negative integers are initially occupied and the positive integers are initially empty, as well as the TASEP on $\mathbb{N}$ with a rate 1 particle source at site 1 and initially only empty sites, both have representations as exponential corner growth models with i.i.d. weights on the first quadrant, respectively on the half-quadrant, of $\mathbb{Z}^2$.

In the seminal work [76], Johansson showed that the TASEP on $\mathbb{Z}$ in the rarefaction fan has Tracy–Widom weak limits associated with the Kardar–Parisi–Zhang universality class; see [17, 58] for more general initial conditions. Similar results were recently achieved for the TASEP on $\mathbb{N}$; see [6, 20]. More generally, when viewing one-dimensional interacting particle systems like the TASEP on the integers as queues in series, one can often utilize Burke's theorem to find a family of corresponding invariant last passage models [9]. Burke-type theorems usually imply that the model in question is an integrable example of the KPZ universality class; see [34] for an overview and [13, 31, 38, 106, 127] for other lattice examples having Burke's property. This means that we are often endowed with precise estimates for various statistics of the process, for example on the current fluctuations [76, 112].

Note that the variety and precision of exact results for the TASEP on the integers and related models is so far not available for the TASEP on trees. Hence, in the following, our goal is to make a first step into this direction by providing quantitative results for observables of the TASEP on trees. More precisely, when starting the TASEP on a supercritical Galton–Watson tree with all sites being initially empty, we give estimates in terms of the aggregated current on the locations of the first n particles which enter the tree.

11.1.3 Main results

In the following, let the rates be bounded uniformly from above for **GW**-almost every Galton–Watson tree, and let the tree be initially empty. We start with an upper bound on the first generation at which the first n particles are located in different branches of the tree, and hence behave like independent random walks. Throughout this section, we will impose the following two conditions on the transition rates. Our first assumption on $(r_{x,y})$ is a non-degeneracy condition, which ensures that the particle system can in principle explore the whole tree.

Assumption 11.1 (Uniform Ellipticity (UE))**.** *The transition rates on T are* ***uniformly elliptic****, i.e., there exists an $\varepsilon \in (0,1]$ such that*

$$\inf\left\{\frac{r_{x,y}}{r_{x,z}}\colon (x,y),(x,z)\in E\right\} \geq \varepsilon\,. \tag{11.1}$$

Note that (UE) guarantees that the first n particles will eventually move on different subtrees of T and behave as independent random walks after a certain generation; see Proposition 11.8. To state our next assumption, we define

$$r_\ell^{\min} = \min\{r_{x,y}\colon x\in\mathcal{Z}_\ell, y\in\mathcal{Z}_{\ell+1}\} \quad\text{and}\quad r_\ell^{\max} = \max\{r_{x,y}\colon x\in\mathcal{Z}_\ell, y\in\mathcal{Z}_{\ell+1}\} \tag{11.2}$$

to be the minimal and maximal transition rates in generation ℓ for all $\ell\in\mathbb{N}_0$. The following assumption guarantees that the rates are not decaying too fast, which may cause certain branches of the tree to become blocked for the particles.

Assumption 11.2 (Exponential decay (ED))**.** *The transition rates* ***decay at most exponentially*** *fast, i.e., there exist constants $c_{\text{low}},\kappa>0$ such that for all $\ell\geq 0$*

$$r_\ell^{\min} \geq \kappa\exp(-c_{\text{low}}\ell)\,.$$

Equipped with these two assumptions, we will now introduce some notation to state our main results. In the following, we let

$$d_{\min} := \min\{i \colon p_i > 0\} \quad \text{and} \quad \tilde{\mathfrak{m}} := \left(\sum_{k=2}^{\infty} p_k \right)^{-1} \sum_{k=2}^{\infty} k p_k \tag{11.3}$$

be the minimal number of offspring and the mean number of offspring when conditioning on having at least two offspring, respectively. Let

$$c_o := \begin{cases} (5 + \log_2 \tilde{\mathfrak{m}})(\log(1+p_1) - \log(2p_1))^{-1} & \text{if } d_{\min} = 1\,, \\ 1/\log d_{\min} & \text{if } d_{\min} > 1\,, \end{cases} \tag{11.4}$$

and define the integer function

$$\mathcal{D}_n := \inf\left\{ m \in \mathbb{N} \colon r_\ell^{\max} \le n^{-(2+c_{\text{low}} c_o)} (\log n)^{-3} \text{ for all } \ell \ge m \right\} \tag{11.5}$$

for all $n \in \mathbb{N}$, where we use the convention $\inf\{\emptyset\} = \infty$. In words, $(\mathcal{D}_n)_{n\in\mathbb{N}}$ denotes a sequence of generations along which all rates decay at least polynomially fast. The order of the underlying polynomial depends on the structure of the tree. In particular, for exponentially fast decaying rates, $\mathcal{D}_n$ will be of order $\log(n)$. We are now ready to quantify the generation where decoupling of the first n particles is guaranteed.

Theorem 11.3. *Consider the TASEP on a Galton–Watson tree and assume that the transition rates satisfy assumptions* (UE) *and* (ED) *with ε from* (11.1)*. Let $\delta > 0$ be arbitrary, but fixed, and define $\mathcal{M}_n$ for all $n \in \mathbb{N}$ as follows.*

1. *When $\limsup_{n\to\infty} \frac{\mathcal{D}_n}{\log n} < \infty$ holds, set*

$$\mathcal{M}_n := \begin{cases} (c_o + 1)\mathcal{D}_n + c_o(2+\delta)\log_{1+\varepsilon} n & \text{if } d_{\min} = 1\,, \\ \frac{d_{\min}}{d_{\min}-1}\mathcal{D}_n + (2+\delta)\log_{1+\varepsilon} n & \text{if } d_{\min} > 1\,. \end{cases} \tag{11.6}$$

2. *When $\liminf_{n\to\infty} \frac{\mathcal{D}_n}{\log n} = +\infty$ holds, set*

$$\mathcal{M}_n := \left(c_o \mathbb{1}_{\{d_{\min}=1\}} + \frac{1}{d_{\min}-1} \mathbb{1}_{\{d_{\min}>1\}} + (1+\delta) \right) \min\{\mathcal{D}_n, n\}\,. \tag{11.7}$$

Then $\mathbb{P}$-almost surely, the trajectories of the first n particles decouple after generation $\mathcal{M}_n$ for n large enough, i.e., the first n particles visit distinct sites at level $\mathcal{M}_n$.

At this point, let us give two remarks on this disentanglement theorem, including a comparison to independent random walks on the tree.

Remark 11.4. *Using the pigeonhole principle, we see that for* **GW***-almost every tree and any family of rates, the first generation of decoupling of* n *particles will be at least of order* $\log n$*. When the rates decay exponentially fast, the disentanglement theorem ensures that order* $\log n$ *generations are sufficient to decouple* n *particles. In particular, in this case the bounds in Theorem 11.3 are sharp up to constant factors.*

Remark 11.5. *A similar result on the disentanglement of the particles holds when we replace the reservoir by dynamics which generate almost surely a linear amount of particles. This may for example be a TASEP on a half-line attached to the root and started from a Bernoulli-*ρ*-product measure for some* $\rho \in (0,1)$*.*

Using the disentanglement estimates from Theorem 11.3, we now study the current for the TASEP on Galton–Watson trees. Recall the notation from Section 10.1.1 for the TASEP on trees. For $m \geq \ell \geq 0$, we define

$$R^{\min}_{\ell,m} := \sum_{i=\ell}^{m} \big(\min_{x\in\mathcal{Z}_i} r_x\big)^{-1}, \qquad R^{\max}_{\ell,m} := \sum_{i=\ell}^{m} \big(\max_{x\in\mathcal{Z}_i} r_x\big)^{-1} \tag{11.8}$$

and set $R^{\min}_{\ell} := R^{\min}_{\ell,\ell}$ as well as $R^{\max}_{\ell} := R^{\max}_{\ell,\ell}$. Intuitively, $R^{\min}_{\ell,m}$ and $R^{\max}_{\ell,m}$ are the expected waiting times to pass from generation ℓ to m when choosing the slowest, respectively the fastest, rate in every generation.

In the following, we will state our main results on the current only for the special case of exponentially decaying rates, i.e., we assume that there exists some $c_{\text{up}} > 0$ such that

$$R^{\max}_{\ell} \geq \exp(c_{\text{up}}\ell) \tag{11.9}$$

holds for all $\ell \in \mathbb{N}$. We provide more general statements in Section 11.4 from which the next two theorems directly follow. Fix now an integer sequence $(\ell_n)_{n\in\mathbb{N}}$ with $\ell_n \geq \mathcal{M}_n$ for all $n \in \mathbb{N}$, where $\mathcal{M}_n$ is taken from Theorem 11.3. For every $n \in \mathbb{N}$, we define a time window $[t_{\text{low}}, t_{\text{up}}]$ in which we study the current through the ℓ_n^{th} level of the tree, and where we see a number of particles proportional to n passing through $\mathcal{Z}_{\ell_n}$.

Theorem 11.6. *Suppose that* (UE) *and* (ED) *holds, and that the rates satisfy* (11.9)*. Then for all* $\delta \in (0,1)$*, there exists a constants* $c, \tilde{c} > 0$ *such that for all* $t_{\text{low}} = t_{\text{low}}(n)$ *and* $t_{\text{up}} = t_{\text{up}}(n)$ *with*

$$t_{\text{up}} \leq c(nR^{\min}_{\mathcal{M}_n} + R^{\min}_{\ell_n}) \leq n^{\tilde{c}c_{\text{low}}} + n\exp(c_{\text{low}}\ell_n)\,, \quad t_{\text{low}} \geq \exp\left(\frac{1}{2}c_{\text{up}}\ell_n\right), \tag{11.10}$$

for $n \in \mathbb{N}$*, we see that* $\mathbb{P}$*-almost surely*

$$\lim_{n\to\infty} J_{\ell_n}(t_{\text{low}}) = 0\,, \qquad \liminf_{n\to\infty} \frac{1}{n} J_{\ell_n}(t_{\text{up}}) \geq 1-\delta\,. \tag{11.11}$$

Next, we let t be a fixed time horizon and define an interval $[L_{\text{low}}, L_{\text{up}}]$ of generations. Recall the notation of the disentanglement generation $\mathcal{M}_n$ from Theorem 11.3 for the first n particles and define

$$n_t := \sup \left\{ n \in \mathbb{N}_0 : (n + \mathcal{M}_n)(\min_{|x| \leq \mathcal{M}_n} r_x)^{-1} \leq t \right\} . \tag{11.12}$$

Note that for exponentially decaying rates, the quantity n_t will be a polynomial in t. For large times t, the next theorem gives a window of generations where we expect to see a positive fraction of the first n_t particles which entered the tree.

Theorem 11.7. *Suppose that* (UE) *and* (ED) *holds, and the rates satisfy* (11.9). *Then there exists a constant $c > 0$ such that for all $L_{\text{low}} = L_{\text{low}}(t)$ and $L_{\text{up}} = L_{\text{up}}(t)$ with*

$$L_{\text{low}} \geq c \log t \, , \qquad L_{\text{up}} \leq \frac{2}{c_{\text{up}}} \log t \, , \tag{11.13}$$

for $t \geq 0$, we see that $\mathbb{P}$-almost surely

$$\limsup_{t \to \infty} J_{L_{\text{up}}}(t) = 0 \, , \qquad \liminf_{t \to \infty} \frac{1}{n_t} J_{L_{\text{low}}}(t) > 0 \, . \tag{11.14}$$

Note that the precision of the results strongly depends on the transition rates and the structure of the tree. The above theorems can be sharpened when we have more information about the rates and the tree. This trade-off is illustrated in Section 11.5 for the special case of regular trees T^d some integer $d \geq 3$. When the rates decay polynomially, we determine a regime such that the lower and upper bounds in the time window t_{low} and t_{up} agree in first-order. Similarly, when the rates decay exponentially fast, we refine the arguments used in Section 11.4 to show Theorem 11.7, and give conditions such that the lower and upper bounds in the location window L_{low} and L_{up} agree in the leading order.

11.1.4 Outline of the proof

In order to show the different theorems presented in Section 11.1.3, we will proceed as follows. We start in Section 11.2 with the proof of the disentanglement theorem. The proof combines combinatorial arguments, geometric properties of Galton–Watson trees and large deviation estimates on the particle movements. In Section 11.3, we introduce two comparisons of the TASEP on trees to related models, which will be helpful in the proof of the remaining theorems. This includes a coupling to independent random walks and a comparison to a slowed down TASEP on trees, which can be studied using inhomogeneous last passage percolation. We apply these tools in Section 11.4 to prove a generalized version of Theorems 11.6 and 11.7; see Theorems 11.21 and 11.22. We show in Section 11.5 that the current bounds can be sharpened in certain cases of the TASEP on regular trees, and conclude with an outlook on open problems.

11.2 The disentanglement theorem

The proof of Theorem 11.3 will be divided into four parts. First, we give an a priori argument on the level where the particles disentangle, requiring assumption (UE). We then study geometric properties of Galton–Watson trees. Afterwards, we estimate the time of n particles to enter the tree. This will require only assumption (ED). In a last step, the ideas are combined in order to prove Theorem 11.3.

11.2.1 An a priori bound on the disentanglement

In this section, we give an a priori bound on the disentanglement of the trajectories within the exclusion process. This bound relies on a purely combinatorial argument, where we count the number of times a particle performing TASEP has a chance to disentangle from a particle ahead. Recall that we start from the configuration where all sites are empty. For a given infinite, locally finite rooted tree T and $x, y \in V(T)$, recall that we denote by $[x, y]$ the set of vertices in the shortest path in T connecting x and y. We set

$$F(o, x) := |\{z \in [o, x] \setminus \{x\} \colon \deg(z) \geq 3\}| \tag{11.15}$$

to be the number of vertices in $[o, x] \setminus \{x\}$ with degree at least 3. For any fixed tree $(T, o) \in \mathcal{T}$, let d_T be the smallest possible number of offspring a site can have. Note that when T is a Galton–Watson tree, $d_T = d_{\min}$ holds **GW**-almost surely for $d_{\min}$ from (11.3). For all $i, m \in \mathbb{N}$, let $z_i^m \in \mathcal{Z}_m$ denote the unique site in generation m which is visited by the i^{th} particle which enters the tree.

Proposition 11.8. *For a given tree $(T, o) \in \mathcal{T}$ consider the TASEP on T where n particles are generated at the root according to an arbitrary rule. Assume that* (UE) *holds for some $\varepsilon > 0$. Then*

$$P_T\left(z_i^m \neq z_j^m \text{ for all } i, j \in \{1, \ldots, n\} \text{ with } i \neq j\right) \geq 1 - n^2 \left(\frac{1}{\varepsilon + 1}\right)^{F_n(m)}, \tag{11.16}$$

where for all $m, n \in \mathbb{N}$, we set

$$F_n(m) := \begin{cases} \min\{F(o, x) \colon x \in \mathcal{Z}_m\} - n & \text{if } d_T = 1\,, \\ m - \lceil n(d_T - 1)^{-1} \rceil & \text{if } d_T \geq 2\,. \end{cases} \tag{11.17}$$

With this proposition, we control the probability that two particles have the same exit point at generation $\mathcal{Z}_m$ in a summable way, provided that $F_n(m) \geq c \log(n)$ for some $c = c(\varepsilon) > 0$. Note that this bound can in general be quite rough. For example on Galton–Watson trees, if instead of TASEP we have independent random walks, we expect to see disentanglement of n particles already after order $\log n$ generations.

Proof of Proposition 11.8. Consider the j^{th} particle for some $j \in [n] := \{1, \dots, n\}$ which enters the tree. We show that the probability of particle j to exit from $x \in \mathcal{Z}_m$ satisfies

$$P_T\left(z_j^m = x\right) \leq \left(\frac{1}{1+\varepsilon}\right)^{F_n(m)} \tag{11.18}$$

for all $j \in [n]$. Note that if particle j exits through x, it must follow the unique path $[o, x]$; see also Figure 38. Our goal is to find a generation m large enough that guarantees that on any ray the particle will have enough opportunities to escape this ray.

For $d_T \geq 3$, we argue that any particle will encounter at least $F_n(m)$ many locations on $[o, x]$ which have at least 2 holes in front when the particle arrives. To see this, suppose that particle j encounters at least $n(d_T - 1)^{-1}$ generations among the first n generations with no two empty sites in front of it when arriving at that generation. In other words, this means that particle j sees at least $d_T - 1$ particles directly in front of its current position when reaching such a generation. Since particle j may follow the trajectory of at most one of these particles, this implies that particle j encounters at least $(d_T - 1) \cdot \frac{n}{d_T - 1} = n$ different particles in total until reaching level n. This is a contradiction as $j \leq n$ and the tree was originally empty.

For $d_T \in \{1, 2\}$ we apply a similar argument. We need to find m large enough so that every possible trajectory has $\min_{x \in \mathcal{Z}_m} F(o, x) \geq n$ locations where, when a particle arrives there are at least two children, and there is no particle ahead. By definition, every possible trajectory has at least $F(o, x) \geq F_n(m) + n$ sites with at least two children. Observe that in order to follow the trajectory $[o, x]$ for some $x \in \mathcal{Z}_m$, the first accepted transition at every stage must be along $[o, x]$. But there can be at most n sites at which the first attempt was not to follow $[o, x]$ and this attempt was suppressed. This is because in order to block an attempt of leaving $[o, x]$, the blocking particle cannot be on $[o, x]$ and thus block only a single attempt of particle j to jump. Hence, there must be at least $F_n(m)$ sites of degree at least 3 accepting the first attempted transition.

Now we prove (11.18). Suppose that particle j is at one of the $F_n(m)$ many locations, say $y \in \mathcal{Z}_\ell$, on $[o, x]$ where two children z_1, z_2 of y are vacant. At most one of them belongs to $[o, x]$, say z_1. Using (UE), the probability of selecting z_1 is bounded from above by $(1 + \varepsilon)^{-1}$. To stay on $[o, x]$, we must pick the unique site in $[o, x]$ at least $F_n(m)$ many times, independently of the past trajectory. This shows (11.18). Since particle i is not influenced by the motion of particle j for all $j > i$, we conclude

$$P_T\left(\exists i, j \in [n], i \neq j, : z_i^m = z_j^m\right) \leq \sum_{1 \leq i < j \leq n} P_T\left(z_i^m = z_j^m\right) \leq n^2 \left(\frac{1}{1+\varepsilon}\right)^{F_n(m)},$$

applying (11.18) for the last inequality. □

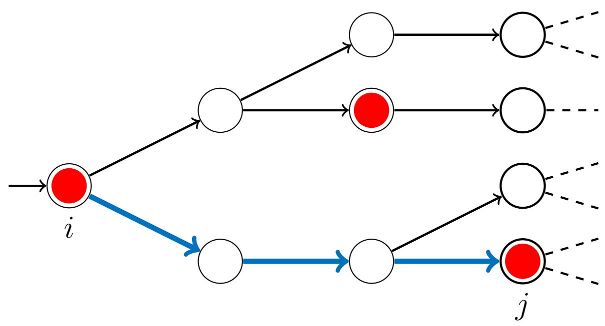

Figure 38: Visualization of the key idea for the proof of the a priori bound on the disentanglement. When (UE) holds, the probability that particle i follows the blue trajectory of particle j is at most $\left(\frac{1}{1+\varepsilon}\right)^2$.

11.2.2 Geometric properties of the Galton–Watson tree

Next, we give an estimate on the number $F(o,x)$, defined in (11.15), which will be essential in the proof of Theorem 11.3 when there is a positive probability to have exactly one offspring.

We define the **core** of a Galton–Watson tree to be the Galton–Watson tree, which we obtain by conditioning in the offspring distribution with respect to $(p_k)_{k\in\mathbb{N}}$ on producing at least 2 sites. Intuitively, we obtain the core from a given tree by collapsing all linear segments to single edges. On the other hand, given a core $\tilde{T}$ according to the conditioned offspring distribution, we can reobtain a Galton–Watson tree with the original offspring distribution according to $(p_k)_{k\in\mathbb{N}}$, by extending every edge $\tilde{e}$ to a line segment of size $G_{\tilde{e}}$ where $(G_{\tilde{e}})_{\tilde{e}\in E(\tilde{T})}$ are i.i.d. Geometric-$(1-p_1)$-distributed random variables supported on $\mathbb{N}_0$. Moreover, we have to attach a line segment $[o,\tilde{o}]$ of Geometric-$(1-p_1)$-size to the root $\tilde{o}$ of $\tilde{T}$ and declare o to be the new root of the tree. An illustration of this procedure is given in Figure 39.

We now give an estimate on how much the tree is stretched when extending the core with the conditioned offspring distribution to a Galton–Watson tree with an offspring distribution with respect to $(p_k)_{k\in\mathbb{N}}$.

Lemma 11.9. *Let $(H_n)_{n\in\mathbb{N}}$ be an increasing sequence that goes to infinity and assume that $p_1\in(0,1)$. Recall $\tilde{\mathfrak{m}}$ from (11.3). Set $M_n := \lceil \alpha H_n \rceil$ for all $n\in\mathbb{N}$, where*

$$\alpha := \frac{5+\log_2 \tilde{\mathfrak{m}}}{\log_2(1+p_1)-\log_2(2p_1)} \,. \tag{11.19}$$

Then we have that

$$\mathbf{GW}\Big(\inf_{x\in\mathcal{Z}_{M_n}} \sum_{v\in[o,x)} \mathbb{1}\{\deg(v)\geq 3\} \geq \lceil H_n\rceil\Big) \geq 1-2^{-2H_n+1}\,. \tag{11.20}$$

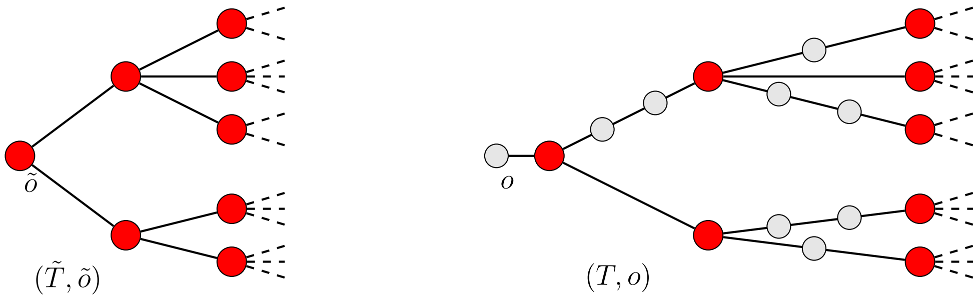

Figure 39: A core $(\tilde{T}, \tilde{o})$ and one of its corresponding Galton–Watson trees (T, o). We obtain the Galton–Watson tree from the core (the core from the Galton–Watson tree) by adding (removing) the smaller vertices depicted in gray.

Proof. Note that it suffices to bound the probability that all sites at generation H_n of $\tilde{T}$ are mapped to a generation less or equal than M_n in the corresponding Galton–Watson tree. Using Markov's inequality, we see that

$$\mathbf{GW}(|x \in V(\tilde{T})\colon\ |x| = H_n| \geq \tilde{\mathfrak{m}}^{H_n} 2^{2H_n}) \leq 2^{-2H_n} . \tag{11.21}$$

Note that each site x at level H_n in $\tilde{T}$ is mapped to a generation given as the sum of H_n-many independent Geometric-$(1 - p_1)$-distributed random variables $(G_i)_{i \in [H_n]}$. Using Chebyshev's inequality, we see that

$$P\Big(\sum_{i=1}^{H_n} G_i \geq M_n\Big) \leq \mathrm{e}^{-tM_n} \left(\frac{1 - p_1}{1 - p_1 \mathrm{e}^t}\right)^{H_n} = \left(\frac{1 + p_1}{2p_1}\right)^{-M_n} 2^{H_n} \tag{11.22}$$

when we set $t = \log(\frac{1+p_1}{2p_1})$. Fix some site $\tilde{x} \in \mathcal{Z}_{M_n}$. Now condition on the number of sites at level H_n in $\tilde{T}$ and apply (11.21) together with a union bound to see that

$$\begin{aligned}
&\mathbf{GW}\Big(\exists x \in \mathcal{Z}_{M_n}\colon \sum_{v \in [o,x)} \mathbb{1}\{\deg(v) \geq 3\} \leq \lceil H_n \rceil\Big) \\
&\qquad \leq \tilde{\mathfrak{m}}^{H_n} 2^{2H_n} \mathbf{GW}\Big(\sum_{v \in [o,\tilde{x})} \mathbb{1}\{\deg(v) \geq 3\} \leq \lceil H_n \rceil\Big) + 2^{-2H_n} \\
&\qquad \leq \tilde{\mathfrak{m}}^{H_n} 2^{2H_n} P\Big(\sum_{i=1}^{H_n} G_i \geq M_n\Big) + 2^{-2H_n} \leq 2^{-2H_n+1}
\end{aligned}$$

using (11.22) and the definition of M_n for the last two steps. □

11.2.3 Entering times of the particles in the tree

We now define an inverse for the current. For any $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, we set

$$\tau_m^n := \inf\{t \geq 0 : J_m(t) \geq n\} . \tag{11.23}$$

In words, τ_m^n gives the time that the aggregated current across generation m becomes n, or equivalently, precisely n particles reached $\mathcal{Z}_m$. Hence, the following two events are equal:

$$\{\tau_m^n \leq t\} = \{J_m(t) \geq n\} .$$

The main goal of this section is to give a bound on the first time τ_0^n at which n particles have entered the tree. Note that this random time τ_0^n depends on the underlying tree as well as on the evolution of the exclusion process.

Proposition 11.10. *Fix a number of particles n. Consider a supercritical Galton–Watson tree without extinction and assume that* (ED) *holds for some constant c_{low}. Recall c_o from* (11.4). *There exists a constant $c > 0$ such that*

$$P_T\big(\tau_0^n < cn^{c_{\text{low}}c_o+1}\log n\big) \geq 1 - \frac{2}{n^2} \tag{11.24}$$

holds with **GW***-probability at least $1 - 2n^{-2}$ for all n sufficiently large.*

In order to show Proposition 11.10, we require a bit of setup. Let $\mathcal{Z}_m^{(x)}$ be the m^{th} generation of the subtree T_x rooted at x. For a tree $(T, o) \in \mathcal{T}$ and a site x, we say that the exclusion process on T has **depth of traffic** $D_x(t) \in \mathbb{N}_0$ with

$$D_x(t) = \inf\{m \geq 0 : \eta_t(z) = 0, \text{ for some } z \in \mathcal{Z}_m^{(x)}\} , \tag{11.25}$$

at site x at time t. In words, $D_x(t)$ is the distance to the first generation ahead of x which contains an empty site. Note that for any fixed x, the process $D_x(t)$ is a non-negative integer process. It takes the value 0 when $\eta_t(x) = 0$ and it becomes positive when $\eta_t(x) = 1$. Since particles move only in a directed way, the process is non-increasing until it hits 0. The following lemma gives a bound on the depth of traffic at the root in Galton–Watson trees.

Lemma 11.11. *Let $H_n = \log_2(n)$ and recall M_n and $\tilde{\mathfrak{m}}$ from Lemma 11.9. Then*

$$\mathbf{GW}\Big(P_T\big(D_o(t) \leq M_n + 1 \textit{ for all } t \leq \tau_0^n\big) = 1\Big) \geq 1 - \frac{2}{n^2} . \tag{11.26}$$

In words, this means that with **GW**-probability at least $1 - 2n^{-2}$, the depth at the root is smaller than M_n whenever no more than n particles have entered the tree.

Proof of Lemma 11.11. Observe that the root can only have depth ℓ when all vertices until level ℓ are occupied and that there are at most n particles until time τ_0^n. Note that Lemma 11.9 guarantees, with our choice of H_n, that with probability at least $1 - 2n^{-2}$, the tree up to generation M_n contains more than n sites. Hence, there is at least one empty site until generation M_n by the definition of τ_0^n. □

Next, we give a bound on the renewal times of the process $(D_o(t))_{t\geq 0}$. For $t \geq 0$ and $x \in V$, we define the first **availability time** $\psi_x(t)$ after time t to be

$$\psi_x(t) = \inf\{s > t : D_x(s) = 0\} - t \geq 0\,.$$

This is the time it takes until x is empty, observing the process from time t onward.

Lemma 11.12. *Fix a tree $(T, o) \in \mathcal{T}$ with root o, and assume that* (ED) *holds for some c_{low}, $\kappa > 0$. Moreover, let $t = t(\ell) \geq 0$ satisfy $0 \leq D_o(t) \leq \ell$. Then for all $c > 0$*

$$P_T\Big(\psi_o(t) > (1+c)(\ell+1)\kappa^{-1}\mathrm{e}^{c_{\text{low}}(\ell+1)}\Big) \leq \exp\Big(-(c - \log(1+c))\ell\Big)\,. \tag{11.27}$$

Proof of Lemma 11.12. Since $D_o(t) \leq \ell$, there exists a site y with $|y| \leq \ell + 1$ and $\eta_t(y) = 0$, such that the ray connecting y to x is fully occupied by particles. Thus, $\psi_o(t)$ is bounded by the time a hole at level $\ell + 1$ needs to travels to o. By (ED),

$$\psi_o(t) \leq \kappa^{-1}\exp(c_{\text{low}}(\ell+1))\sum_{i=1}^{\ell+1}\omega_i$$

holds for independent Exponential-1-distributed random variables $(\omega_i)_{i\in[l+1]}$. Now

$$P\left(\sum_{i=1}^{\ell+1}\omega_i > (1+c)(\ell+1)\right) \leq \exp\left(-(c - \log(1+c))\ell\right)$$

by using Cramér's theorem yields an upper bound on the left-hand side in (11.27). □

Proof of Proposition 11.10. Recall that a particle can enter the tree if and only if the root is empty, and that particles are created at the root at rate λ. Thus

$$\tau_0^i - \tau_0^{i-1} \leq \psi_o(\tau_0^{i-1}) + \lambda^{-1}\omega_i \tag{11.28}$$

holds for all $i \in [n]$ for some sequence $(\omega_i)_{i\in[n]}$ of i.i.d. Exponential-1-distributed random variables. Recall (11.4) where for $d_{\min} > 1$, we take c_o such that $M_n = c_o \log n$, and set $c_o = 1/\log(d_{\min})$ otherwise. Rewriting τ_0^n as a telescopic sum yields

$$\begin{aligned} P_T(\tau_0^n > cn^{c_{\text{low}}c_o+1}\log n) &\leq P_T(\exists i \in [n] \colon \tau_0^i - \tau_0^{i-1} > cn^{c_{\text{low}}c_o}\log n) \\ &\leq n\max_{i\in[n]} P_T\left(\psi_o(\tau_0^{i-1}) > (c - 3\lambda^{-1})n^{c_{\text{low}}c_o}\log n\right) + nP_T(\omega_1 > 3\log n)\,. \end{aligned}$$

Together with Lemma 11.11 and Lemma 11.12 for $\ell = c_o \log n$, we obtain that

$$n\max_{i\in[n]} P_T\left(\psi_o(\tau_0^{i-1}) > (c - 3\lambda^{-1})n^{c_{\text{low}}c_o}\log n\right) + nP_T(\omega_1 > 3\log n) \leq \frac{1}{n^2} + \frac{1}{n^2}$$

holds for some $c > 0$ with **GW**-probability at least $1 - 2n^2$ for all n sufficiently large. □

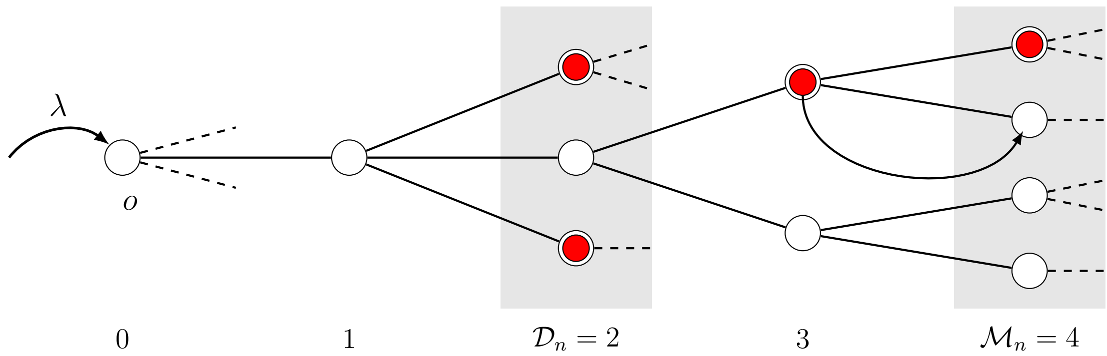

Figure 40: Visualization of the TASEP on trees and the different generations $\mathcal{D}_n$ and $\mathcal{M}_n$ involved in the proof for $n = 4$. The particles are drawn in red. Note that it depends on the next successful jump of the particle at generation 3 if the first 4 particles are disentangled at generation $\mathcal{M}_n = 4$, i.e., they will disentangle if the particle jumps at the location indicated by the arrow.

11.2.4 Proof of the disentanglement theorem

For the proof of Theorem 11.3 we have the following strategy. We wait until all n particles have entered the tree. We then consider a level in the tree which was reached by no particle yet. For every vertex at that level as a starting point, we use the a priori bound on the disentanglement from Proposition 11.8; see also Figure 40.

Starting from the empty initial configuration, we study the maximal generation which is reached until time τ_0^n. The next lemma gives an estimate on the degrees of the vertices along the possible trajectories of the particles.

Lemma 11.13. *Let $(L_n)_{n\in\mathbb{N}}$ be an integer sequence such that $L_n \geq \tilde{c}\log n$ holds for some $\tilde{c} > 0$ and $n \in \mathbb{N}$. Then we can find a sequence $(\delta_n)_{n\in\mathbb{N}}$ with δ_n tending to 0 with n such that the following statement holds with* **GW***-probability at least $1 - n^{-2}$ for all n large enough: for every site $x \in \mathcal{Z}_{\lceil L_n(1+\delta_n)\rceil}$, there exists a site $y \leq x$, i.e., y is on a directed path from the root to x, with $|y| \geq L_n$ and $\deg(y) \leq \log\log n$.*

Proof. It suffices to consider the case where the offspring distribution has infinite support. Using Markov's inequality, we see that with **GW**-probability at least $1-(2n)^{-2}$, the Galton–Watson tree contains at most $(2n)^2\mathfrak{m}^{L_n}$ sites at generation L_n. We denote by $(T_i)_{i\in[|\mathcal{Z}_{L_n}|]}$ the trees with roots o_i attached to these sites. We claim that with **GW**-probability at least $1 - (2n)^{-4}\mathfrak{m}^{-L_n}$, every ray $[o_i, x]$ for x at level $\lceil \delta_n L_n\rceil$ of T_i contains at least one vertex which has at most $\log\log n$ neighbors. To see this, we use a comparison to a different offspring distribution. Recall that the mean of the offspring distribution is $\mathfrak{m} < \infty$, and that p_i is the probability of having precisely i offspring.

We define another offspring distribution for weights $(\bar{p}_i)_{i\in\{0,1,\dots\}}$, where

$$\bar{p}_i := \begin{cases} p_i & \text{for } i > \log\log n \\ 1 - \sum_{i=1}^{\lfloor \log\log n\rfloor} p_i & \text{for } i = 0 \\ 0, & \text{else.} \end{cases}$$

Let $\bar{\mathfrak{m}}_n$ denote the mean of the distribution given by $(\bar{p}_i)_{i\in\{0,1,\dots\}}$, and note that $\bar{\mathfrak{m}}_n \to 0$ holds when $n \to \infty$. Observe that the probability that all rays up to generation $\lceil \delta_n L_n \rceil$ contain at least one vertex of degree at most $\log\log n$ is equal to the probability that the tree with offspring distribution drawn according to $(\bar{p}_i)_{i\in\{0,1,\dots\}}$ dies out until generation $\lceil \delta_n L_n \rceil$. Using a standard estimate for Galton–Watson trees, this probability is at least $1 - \bar{\mathfrak{m}}_n^{\lceil \delta_n L_n \rceil}$. Set

$$\delta_n = -\frac{2L_n + 4\log_{\mathfrak{m}}(2n)}{L_n \log_{\mathfrak{m}} \bar{\mathfrak{m}}_n} \tag{11.29}$$

and note that $\delta_n \to 0$ holds when $n \to \infty$. From this, and $L_n \geq \tilde{c}\log n$ for some $\tilde{c} > 0$, for all n large enough

$$\bar{\mathfrak{m}}_n^{\lceil \delta_n L_n \rceil} \leq (2n)^{-4}\mathfrak{m}^{-L_n}$$

follows. We conclude with a union bound over all trees T_i at level L_n. $\square$

Next, for all $t \geq 0$, we let $\mathcal{S}(t)$ denote the generation

$$\mathcal{S}(t) = \max\{\ell \geq 0 : J_\ell(t) = 1\}$$

when starting from the configuration where all sites are empty.

Lemma 11.14. *Recall* $(\mathcal{D}_n)$ *from* (11.5) *and* (δ_n) *from* (11.29). *Then* $\mathbb{P}$*-almost surely*

$$\mathcal{S}(\tau_0^n) \leq (1+\delta_n)\mathcal{D}_n \tag{11.30}$$

for all n *sufficiently large.*

Proof. By Lemma 11.13, with **GW**-probability at least $1 - n^{-2}$, there exists some generation $\ell \geq \mathcal{D}_n$ such that for every $i \in [n]$, the i^{th} particle has at most $\log\log n$ neighbors. Let ζ_i be the holding time at this generation for particle i and note that ζ_i satisfies with respect to the stochastic domination $\succeq_{\text{c}}$ from (4.35)

$$\zeta_i \succeq_{\text{c}} \omega_i \sim \operatorname{Exp}(r_{\mathcal{D}_n}^{\max} \log\log n)\,.$$

Set $t = cn^{c_{\text{low}}c_o+1}\log n$ for $c > 0$ sufficiently large such that for all n large enough

$$\mathbf{GW}\Big(P_T\big(\mathcal{S}(t) \geq \mathcal{S}(\tau_0^n)\big) \geq 1 - \frac{2}{n^2}\Big) \geq \mathbf{GW}\Big(P_T\big(\tau_0^n \leq t\big) \geq 1 - \frac{2}{n^2}\Big) \geq 1 - 2n^{-2}$$

using that $\mathcal{S}(\cdot)$ is monotone increasing for the first inequality, and Proposition 11.10 for the second step. For the same choice of t and using the definitions of $\mathcal{D}_n$ and $\mathcal{S}(t)$

$$P_T(\mathcal{S}(t) > \mathcal{D}_n(1+\delta_n)) \leq P_T\Big(\min_{1\leq i\leq n} \zeta_i < t\Big) \leq P_T\Big(\min_{1\leq i\leq n} \omega_i < t\Big) \leq \frac{c_1 \log\log n}{n \log^2 n}$$

holds for some constant $c_1 > 0$ and all n sufficiently large, with **GW**-probability at least $1 - n^{-2}$. An integral test shows that all error terms in the above estimates are summable with respect to n, and we obtain (11.30) by the Borel–Cantelli lemma. □

Proof of Theorem 11.3. Note that when the event in Lemma 11.14 occurs, $\mathbb{P}$-almost surely no ray contains more than $\mathcal{D}_n(1+\delta_n)$ particles out of the first n particles for all n sufficiently large. We will use this observation to apply the a priori bound from Proposition 11.8 for all trees (T^i) rooted at generation $\mathcal{D}_n(1+\delta_n)$ which eventually contain at least one of the first n particles. In the following, we assume that $\mathcal{D}_n < n$. For $\mathcal{D}_n \geq n$, we directly apply Proposition 11.8 for the original tree T with n particles.

We start with the case where $d_{\min} \geq 2$ holds. Let $\delta \in (0,1)$ be fixed and set

$$\tilde{\mathcal{M}}_n = \frac{1}{d_{\min}-1}(\mathcal{D}_n(1+\delta_n)) + (2+\delta)\log_{1+\varepsilon}(n\mathcal{D}_n)\,. \tag{11.31}$$

Moreover, we fix a tree T^i rooted at generation $\mathcal{D}_n(1+\delta_n)$ which eventually contains a particle. We claim that by Proposition 11.8, all of the at most $\mathcal{D}_n(1+\delta_n)$ particles entering T^i are disentangled after $\tilde{\mathcal{M}}_n$ generations in T^i with P_T-probability at least $1 - cn^{-2-\delta}$ for some constant $c > 0$. To see this, recall (11.17) and observe that

$$F_{\mathcal{D}_n(1+\delta_n)}(\tilde{\mathcal{M}}_n) \leq (2+\delta)\log_{1+\varepsilon}(n\mathcal{D}_n)\,.$$

We then apply (11.16) to obtain the claim. Note that this holds for **GW**-almost every tree $(T,o) \in \mathcal{T}$. Moreover, the events that the particles disentangle on the trees (T^i) are mutually independent, and we conclude using a union bound for the trees (T^i).

Now suppose that $d_{\min} = 1$ holds. Recall c_o from (11.4) and that $\delta \in (0,1)$ is fixed. Note that $\delta_n \leq \delta$ holds for all n sufficiently large and set

$$\tilde{\mathcal{M}}_n = c_o(\mathcal{D}_n(1+\delta)) + (2+\delta)c_o\log_{1+\varepsilon}(n\mathcal{D}_n)\,. \tag{11.32}$$

Observe that $(2+\delta)\log_{1+\varepsilon} n \geq \log_2 n$ for all n using the definition of ε in (UE). Let $H_n = \mathcal{D}_n(1+\delta) + (2+\delta)\log_{1+\varepsilon}(n\mathcal{D}_n)$. Similar to the case $d_{\min} \geq 2$, we now combine Proposition 11.8 and Lemma 11.9 to see that $\mathbb{P}$-almost surely, all of the at most $\mathcal{D}_n(1+\delta)$ particles entering T^i are disentangled after $\tilde{\mathcal{M}}_n$ generations in T^i for all $i \in [n]$ and n large enough. Compare (11.31) and (11.32) with $\mathcal{M}_n$ in (11.6) and (11.7) to conclude. □

11.3 Two comparisons using couplings

In this section, we discuss two ways of comparing the TASEP on trees to related processes via couplings. We start with a comparison to independent random walks on the tree. This coupling is used to prove a lower bound on the time window in Theorem 11.6 and an upper bound on the window of generations in Theorem 11.7. Our second model is a slowed down TASEP which is studied using an inhomogeneous last passage percolation model. It is used to give an upper bound on the time window in Theorem 11.6 and a lower bound on the window of generations in Theorem 11.7. In both cases, recall that we fix the respective tree $(T,o) \in \mathcal{T}$ with $T=(V,E)$, and a family of rates $(r_{x,y})_{x,y\in E}$ such that the TASEP on T is a Feller process.

11.3.1 A comparison with independent random walks

We start by comparing the TASEP $(\eta_t)_{t\geq 0}$ on T to independently moving biased random walks on T. Assume that the TASEP is started from some state η, which is — in contrast to our previous assumptions — not necessarily the configuration with only empty sites. We enumerate the particles according to an arbitrary rule and denote by z_t^i the position of the i^{th} particle at time $t \geq 0$. We define the waiting time $\sigma_\ell^{(i)}$ in level ℓ for all $i \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ to be the time particle i spends on generation ℓ once it sees at least one empty site. Recall $R_\ell^{\max}$ from (11.8) and the stochastic domination $\succeq_{\mathrm{c}}$ from (4.35). Then

$$\sigma_\ell^{(i)} \succeq_{\mathrm{c}} R_\ell^{\max}\omega_\ell^{(i)} \tag{11.33}$$

holds for all $i \in [n]$ and $\ell \geq 0$, where $\omega_\ell^{(i)}$ are independent Exponential-1-distributed random variables. We now define the independent random walks $(\tilde{\eta}_t)_{t\geq 0}$ started from η.

Each particle at level ℓ waits according to independent rate $(R_\ell^{\max})^{-1}$ Poisson clocks, and jumps to a neighbor in generation $\ell+1$ chosen uniformly at random when the clock rings. When a particle is created in $(\eta_t)_{t\geq 0}$, create a particle in $(\tilde{\eta}_t)_{t\geq 0}$ as well.

Note that in these dynamics, a site can be occupied by multiple particles at a time. Let $\tilde{z}_t^i$ denote the position of the i^{th} particle in $(\tilde{\eta}_t)_{t\geq 0}$ at time $t \geq 0$ and denote by $(\tilde{J}_\ell(t))_{t\geq 0}$ the aggregated current of $(\tilde{\eta}_t)_{t\geq 0}$ at generation $\ell \in \mathbb{N}_0$. The following lemma is immediate from (11.33) and the construction of the random walks $(\tilde{\eta}_t)_{t\geq 0}$.

Lemma 11.15. *There exists a coupling $\tilde{\mathbb{P}}$ between the TASEP $(\eta_t)_{t\geq 0}$ on T and the corresponding independent random walks $(\tilde{\eta}_t)_{t\geq 0}$ such that*

$$\tilde{\mathbb{P}}\left(\left|z_t^i\right| \leq \left|\tilde{z}_t^i\right| \text{ for all } i \in \mathbb{N}\right) = 1 \tag{11.34}$$

holds for any common initial configuration. In particular, $J_\ell(t) \leq \tilde{J}_\ell(t)$ holds for all $\ell \in \mathbb{N}_0$ and $t \geq 0$.

Using the comparison to independent random walks, we can give bounds on the current using estimates on weighted sums of Exponential random variables. We will frequently use the following estimates.

Lemma 11.16. *For $\ell \in \mathbb{N}$ and $c_0, c_1, c_2, \ldots, c_\ell, t \geq 0$, set $S := \sum_{i=0}^{\ell} c_i^{-1}$ as well as $c := \min_{i \in \{0,1,\ldots,\ell\}} c_i$. Let $(\omega_i)_{i \in \{0,1,\ldots,\ell\}}$ be independent Exponential-1-distributed random variables. Then for any $\delta \in (0,1)$,*

$$1 - \frac{\mathrm{e}^{-\delta c t}}{(1-\delta)^{cS}} \leq P\Big(\sum_{i=0}^{\ell} \frac{1}{c_i}\omega_i \leq t\Big) \leq \min\left(\frac{\mathrm{e}^{\delta c t}}{(1+\delta)^{cS}}, \mathrm{e}^{\ell(1+\log\frac{t}{\ell}) + \sum_{i=0}^{\ell}\log c_i}\right) . \quad (11.35)$$

Proof. By Chebyshev's inequality, we see that

$$P\Big(\sum_{i=0}^{\ell} \frac{1}{c_i}\omega_i \leq t\Big) \leq \mathrm{e}^{\ell} \prod_{i=0}^{\ell} E\left[\exp\left(-\frac{\ell}{tc_i}\omega_i\right)\right] = \mathrm{e}^{\ell} \exp\left(-\sum_{i=0}^{\ell} \log\left(1 + \frac{\ell}{tc_i}\right)\right)$$

holds. Since the logarithm is increasing, we can rearrange the sums to get the second upper bound. For the first upper bound, again apply Chebyshev's inequality for

$$P\Big(\sum_{i=0}^{\ell} \frac{1}{c_i}\omega_i \leq t\Big) \leq \mathrm{e}^{\delta c t} \exp\left(-\sum_{i=0}^{\ell} \log\left(1 + \frac{\delta c}{c_i}\right)\right) . \quad (11.36)$$

Using concavity of the logarithm, we obtain for all $i \in \{0,1,\ldots,\ell\}$ and all $x > -1$ that

$$\log\left(1 + \frac{xc}{c_i}\right) \geq \log(1+x)\frac{c}{c_i} . \quad (11.37)$$

For $x = \delta$ in (11.37), together with (11.36), this yields the first upper bound. For the lower bound, we use again Chebyshev's inequality and (11.37) with $x = -\delta$ to get that

$$P\Big(\sum_{i=0}^{\ell} \frac{1}{c_i}\omega_i \geq t\Big) \leq \mathrm{e}^{-\delta c t} \exp\left(-\sum_{i=0}^{\ell} \log\left(1 - \frac{c\delta}{c_i}\right)\right) \leq \frac{\mathrm{e}^{-\delta c t}}{(1-\delta)^{cS}} .$$

This finishes the proof of the lemma. □

Remark 11.17. *Note that the bounds in Lemma 11.16 are in general not sharp, and can be refined, for example when for some $k > 1$ the weights satisfy $c_i = k^i$ for all $i \geq 1$; see Theorem 2.3 in [49].*

11.3.2 A comparison with an inhomogeneous LPP model

In this section, we compare the TASEP on T to a slowed down exclusion process, which we study using last passage percolation (LPP) in an inhomogeneous environment. To describe this model, we will now give a brief introduction to last passage percolation, and refer the reader to [126, 128] for a more comprehensive discussion.

Consider the lattice $\mathbb{N}\times\mathbb{N}$, and let $(\omega_{i,j})_{i,j\in\mathbb{N}}$ be independent Exponential-1-distributed random variables. Let $\pi_{m,n}$ be an up-right **lattice path** from $(1,1)$ to (m,n), i.e.,

$$\pi_{m,n} = \{u_1 = (1,1), u_2, \ldots, u_{m+n} = (m,n) : u_{i+1} - u_i \in \{(1,0),(0,1)\} \text{ for all } i\}.$$

The set of all up-right lattice paths from $(1,1)$ to (m,n) is denoted by $\Pi_{m,n}$. The **last passage time** in an environment ω is defined as

$$G^{\omega}_{m,n} = \max_{\pi_{m,n}\in\Pi_{m,n}} \sum_{u\in\pi_{m,n}} \omega_u , \tag{11.38}$$

for all $m,n\in\mathbb{N}$. Equivalently, the last passage times are defined recursively as

$$G^{\omega}_{m,n} = \max\{G^{\omega}_{m-1,n}, G^{\omega}_{m,n-1}\} + \omega_{m,n} , \tag{11.39}$$

with boundary conditions for all $k,\ell\in\mathbb{N}$ given by

$$G^{\omega}_{1,\ell} = \sum_{j=1}^{\ell} \omega_{1,j} , \qquad G^{\omega}_{k,1} = \sum_{i=1}^{k} \omega_{i,1} . \tag{11.40}$$

In the following, we will restrict the space of lattice paths, i.e., we consider the set of paths $A_m := \{u = (u^1,u^2) : u^2 \geq u^1 - m\} \cap \mathbb{N}\times\mathbb{N}$. For any (i,j) in $\mathbb{N}\times\mathbb{N}$, we define

$$G^{\omega}_{i,j}(A_m) = \max_{\pi\in\Pi_{i,j}(A_m)} \sum_{u\in\pi} \omega_u ,$$

where $\Pi_{i,j}(A_m)$ contains all up-right paths from $(1,1)$ to (i,j) that do not exit A_m, i.e.,

$$\Pi_{i,j}(A_m) = \Big\{\pi = \{(1,1) = u_1, \ldots u_{i+j} = (i,j)\} : u_{i+1} - u_i \in \{(1,0),(0,1)\}, u_i \in A_m\Big\}.$$

Based on the environment ω, we define an environment $\tilde{\omega} = \{\omega_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{N}}$ by

$$\tilde{\omega}_{i,j} := \begin{cases} \frac{1}{r^{\min}_{i-j-1}}\omega_{i,j} & \text{if } j < i , \\ \lambda^{-1}\omega_{i,j} & \text{if } j = i , \\ 0, & \text{else;} \end{cases} \tag{11.41}$$

see Figure 41 for a visualization. The next lemma shows that the last passage times in $\tilde{\omega}$ can be used to study the entering time of the n^{th} particle in the TASEP on trees.

Lemma 11.18. *Let $m,n\in\mathbb{N}$ be such that $m \leq \mathcal{M}_n$ holds, where $\mathcal{M}_n$ is defined in Theorem 11.3. Then there exists a coupling between $G^{\tilde{\omega}}_{n,n+m}$ and the time τ^n_m of the TASEP on trees, defined in* (11.23)*, such that $\mathbb{P}$-almost surely, for all n large enough*

$$G^{\tilde{\omega}}_{n+m,n}(A_{\mathcal{M}_n}) \geq \tau^n_m . \tag{11.42}$$

0	0	0	0	λ	$r_0^{\min}$	$r_1^{\min}$
0	0	0	λ	$r_0^{\min}$	$r_1^{\min}$	$r_2^{\min}$
0	0	λ	$r_0^{\min}$	$r_1^{\min}$	$r_2^{\min}$	$r_3^{\min}$
0	λ	$r_0^{\min}$	$r_1^{\min}$	$r_2^{\min}$	$r_3^{\min}$	$r_4^{\min}$
λ	$r_0^{\min}$	$r_1^{\min}$	$r_2^{\min}$	$r_3^{\min}$	$r_4^{\min}$	$r_5^{\min}$

Figure 41: Visualization of the environment which is used to describe the slowed down TASEP as a last passage percolation model. The numbers in the cells are the parameters of the respective Exponential-distributed random variables. The square at the bottom left of the grid corresponds to the cell $(1,1)$.

In order to show Lemma 11.18, we require a bit of setup. Consider the event

$$D_n := \{\text{the first } n \text{ particles disentangle by generation } \mathcal{M}_n\} \tag{11.43}$$

which holds for all n large enough by Theorem 11.3. In particular, note that if D_n holds, whenever one of the first n particles reaches generation $\mathcal{M}_n$, it no longer blocks any of the first n particles. Moreover, observe that when it is possible to jump for particle i from generation ℓ, the time $\sigma_\ell^{(i)}$ until this jump is performed is stochastically dominated by an Exponential-distributed random variable with the smallest possible rate out from generation ℓ. In other words, the inequality

$$\sigma_\ell^{(i)} \preceq_{\mathrm{c}} \frac{1}{r_\ell^{\min}} \omega_{\ell+i+1,i}$$

holds for all $i, \ell \in \mathbb{N}$.

We construct now a slowed down TASEP $(\tilde{\eta}_t)_{t\geq 0}$ in which particles use the times $(r_\ell^{\min})^{-1}\omega_{\ell+i+1,i}$ to jump from generation ℓ to $\ell+1$, but only after particle $i-1$ left generation $\ell+1$. Moreover, we assume without loss of generality that all particles follow the trajectories of the original dynamics $(\eta_t)_{t\geq 0}$. As before, let z_t^i and $\tilde{z}_t^i$ denote the position of the i^{th} particle in $(\eta_t)_{t\geq 0}$ and $(\tilde{\eta}_t)_{t\geq 0}$, respectively. The following lemma is immediate from the construction of the two processes.

Lemma 11.19. *There exists a coupling $\tilde{\mathbb{P}}$ between the TASEP $(\eta_t)_{t\geq 0}$ on T and the corresponding slowed down dynamics $(\tilde{\eta}_t)_{t\geq 0}$ such that*

$$\tilde{\mathbb{P}}\left(\left|\tilde{z}_t^i\right| \leq \left|z_t^i\right| \textit{ for all } i \in [n]\right) = 1 \tag{11.44}$$

holds for any common initial configuration of the two processes.

Proof of Lemma 11.18. It suffices to show that the time in which the n^{th} particle reaches generation m in the slowed down dynamics has the same law as $G^{\tilde{\omega}}_{n+m,n}(A_{\mathcal{M}_n})$. Let $\widetilde{G}_{m,n}$ be the time the n^{th} particle jumped $n-m$ times in the slowed down process and note that for all m,n

$$\widetilde{G}_{m,n} = \max(\widetilde{G}_{m-1,n}, \widetilde{G}_{m,n-1}) + \tilde{\omega}_{m,n}\,.$$

Moreover,

$$\widetilde{G}_{0,m} = \sum_{\ell=1}^{m} \tilde{\omega}_{0,\ell}, \qquad \widetilde{G}_{\ell,1} = \sum_{k=1}^{\ell} \tilde{\omega}_{k,1}\,.$$

The right-hand side of the last three stochastic equalities are the recursive equations and initial conditions for the one-dimensional TASEP, in which particle i waits on site ℓ for $(r^{\min}_{\ell})^{-1}\omega_{\ell,\ell+1}$ amount of time, after $\ell+1$ becomes vacant. Note that any maximal path from $(0,1)$ up to $(n, n+\mathcal{M}_n)$ will never touch the sites for which the environment is 0, so the passage times in environment (11.39) and (11.40) coincide with those in environment (11.41), as long as we restrict the set of paths to not cross the line $\ell - i = \mathcal{M}_n$. For any time $t \geq 0$, on the event D_n, this yields

$$P_T(J_m(t) \leq n, D_n) \leq P_T(\widetilde{J}_m(t) \leq n, D_n) \leq P_T(G^{\tilde{\omega}}_{n+m,n}(A_{\mathcal{M}_n}) \geq t)\,.$$

We set $t = \tau^n_m$ and conclude as D_n holds $\mathbb{P}$-almost surely for all n large enough. □

We use this comparison to an inhomogeneous LPP model to give a rough estimate on the time τ^n_m for general transition rates. Note that this bound can be refined when we have more detailed knowledge about the structure of the rates.

Lemma 11.20. *Recall $\mathcal{M}_n$ from Theorem 11.3 and fix $\alpha > 0$. Then*

$$P_T\Big(G^{\tilde{\omega}}_{n+\mathcal{M}_n,n}(A_{\mathcal{M}_n}) \leq \frac{4(1+\alpha)}{\min\limits_{|x|\leq \mathcal{M}_n} r_x}(n+\mathcal{M}_n)\Big) \geq 1 - \mathrm{e}^{-cn} \tag{11.45}$$

holds for some constant $c = c(\alpha) > 0$ with $\lim_{\alpha\to\infty} c(\alpha) = \infty$.

Proof. Let $G^{(1)}_{m,n}$ be the passage time up to (m,n) in an i.i.d. environment with Exponential-1-distributed weights. Observe that we have the stochastic domination

$$G^{\tilde{\omega}}_{n+\mathcal{M}_n,n}(A_{\mathcal{M}_n}) \preceq_{\mathrm{c}} G^{\tilde{\omega}}_{n+\mathcal{M}_n,n+\mathcal{M}_n}(A_{\mathcal{M}_n}) \preceq_{\mathrm{c}} \big(\min_{|x|\leq \mathcal{M}_n} r_x\big)^{-1} G^{(1)}_{n+\mathcal{M}_n,n+\mathcal{M}_n}\,. \tag{11.46}$$

For all $\alpha > 0$, we obtain from Theorem 4.1 in [122] that

$$P_T\Big(G^{(1)}_{M,M} \leq 4(1+\alpha)M\Big) \geq 1 - \mathrm{e}^{-cM} \tag{11.47}$$

holds for some $c = c(\alpha) > 0$ with $\lim_{\alpha\to\infty} c(\alpha) = \infty$ and all $M \in \mathbb{N}$, where the constant $c(\alpha)$ is an explicitly known rate function. Combine (11.46) and (11.47) to conclude. □

11.4 Proof of the current theorems

We have now all tools to prove Theorem 11.6 and Theorem 11.7. In fact, we will prove more general theorems which allow for any transition rates $(r_{x,y})$ satisfying the assumptions (UE) and (ED). We start with a generalization of Theorem 11.6 on the current in a time window $[t_{\text{low}}, t_{\text{up}}]$. Recall (11.8), and set

$$\rho_\ell := \min_{i \leq \ell} \max_{x \in \mathcal{Z}_i} r_x \,. \tag{11.48}$$

For the lower bound, let

$$t_{\text{low}} := \max\left(t_1^{\text{low}}, t_2^{\text{low}}\right)$$

with

$$t_1^{\text{low}} := R_{0,\ell_n}^{\max}\left(1 - 2\left(R_{0,\ell_n}^{\max}\rho_{\ell_n}\right)^{-\frac{1}{3}} \log R_{0,\ell_n}^{\max}\right) \tag{11.49}$$

$$t_2^{\text{low}} := \frac{\ell_n}{2} \exp\left(\frac{1}{\ell_n + 1}\sum_{i=0}^{\ell_n} \log R_i^{\max}\right). \tag{11.50}$$

Note that both terms in the maximum can give the main contribution in the definition of t_{low}, depending on the rates. For the upper bound, we define

$$\theta := \liminf_{n\to\infty} (\min_{\mathcal{M}_n < |x| \leq \ell_n} r_x) R_{\mathcal{M}_n,\ell_n}^{\min} \in [0,\infty] \tag{11.51}$$

and fix some $\delta \in (0,1)$. We let $t_{\text{up}} = t_{\text{up}}(\delta)$ be

$$t_{\text{up}} := \frac{5(n + \mathcal{M}_n)}{\min\limits_{|x| \leq \mathcal{M}_n} r_x} + \left[\mathbb{1}_{\{\theta<\infty\}}\left(1 + \delta - \frac{2\log\delta}{\theta\delta}\right) + \mathbb{1}_{\{\theta=\infty\}}(1+\theta_n)\right] R_{\mathcal{M}_n,\ell_n}^{\min} \tag{11.52}$$

with some sequence $(\theta_n)_{n\in\mathbb{N}}$ tending to 0 satisfying

$$\liminf_{n\to\infty} \frac{1}{\theta_n} (\min_{\mathcal{M}_n < |x| \leq \ell_n} r_x) R_{\mathcal{M}_n,\ell_n}^{\min} = \infty \tag{11.53}$$

when $\theta = \infty$. Consider the first n particles which enter the tree, starting with the configuration which contains only empty sites. The following theorem states that we see at least an aggregated current in $[t_{\text{low}}, t_{\text{up}}]$ of order n.

Theorem 11.21. *Suppose that* (UE) *and* (ED) *hold and let* $(\ell_n)_{n\in\mathbb{N}}$ *be a sequence of generations with* $\ell_n \geq \mathcal{M}_n$ *for all* $n \in \mathbb{N}$. *Fix* $\delta \in (0,1)$ *and let* t_{low} *and* $t_{\text{up}} = t_{\text{up}}(\delta)$ *be given in* (11.49), (11.50) *and* (11.52). *Then* $\mathbb{P}$*-almost surely*

$$\lim_{n\to\infty} J_{\ell_n}(t_{\text{low}}) = 0\,, \qquad \liminf_{n\to\infty} \frac{1}{n} J_{\ell_n}(t_{\text{up}}) \geq 1 - \delta\,. \tag{11.54}$$

In particular, for rates which satisfy (11.9), *the estimates in Theorem 11.6 hold.*

Proof. We start with the lower bound involving t_{up}. Recall D_n from (11.43) as the event that the first n particles are disentangled at generation $\mathcal{M}_n$, and $\tau_n^{\mathcal{M}_n}$ from (11.23) as the first time such that the first n particles have reached generation $\mathcal{M}_n$. Set

$$t_1 = 5(n + \mathcal{M}_n)\big(\min_{|x| \le \mathcal{M}_n} r_x\big)^{-1}$$

and define $t_2 := t_{\text{up}} - t_1$. Combining Theorem 11.3, Lemma 11.18 and Lemma 11.20, we see that

$$D_n \cap \{\tau_n^{\mathcal{M}_n} \le t_1\} \tag{11.55}$$

holds $\mathbb{P}$-almost surely for all n sufficiently large. In words, this means that all particles have reached generation $\mathcal{M}_n$ by time t_1 and perform independent random walks after level $\mathcal{M}_n$. We claim that it suffices to show that

$$p := P_T\left(\sum_{i=\mathcal{M}_n}^{\ell_n} \frac{\omega_i}{r_i^{\min}} > t_2\right) < \delta \tag{11.56}$$

holds, where (ω_i) are independent Exponential-1-distributed random variables. To see this, let B_i be the indicator random variable of the event that the i^{th} particle did not reach level ℓ_n by time t_{up}. From (11.56), we obtain that $(B_i)_{i\in[n]}$ are stochastically dominated by independent Bernoulli-p-random variables when conditioning on the event in (11.55). Hence, we obtain that

$$\begin{aligned} P_T\left(J_{\ell_n}(t_{\text{up}}) \ge (1-\delta)n \;\middle|\; D_n, \tau_n^{\mathcal{M}_n} \le t_1\right) &\le P_T\left(\sum_{i=1}^{n} B_i \ge \delta n \;\middle|\; D_n, \tau_n^{\mathcal{M}_n} \le t_1\right) \\ &\le \mathrm{e}^{-\delta n}(1 + \mathrm{e}^{pn}) \end{aligned}$$

holds using Chebyshev's inequality for the second step. Together with a Borel–Cantelli argument and (11.55), this proves the claim. In order to verify (11.56), we distinguish two cases depending on the value of θ defined in (11.51). Suppose that $\theta < \infty$ holds. Then by Lemma 11.16 and a calculation, we obtain that

$$\begin{aligned} P_T\left(\sum_{i=\mathcal{M}_n}^{\ell_n} \frac{\omega_i}{r_i^{\min}} > t_2\right) &< \exp\Big(\big(\min_{\mathcal{M}_n < |x| \le \ell_n} r_x\big) R^{\min}_{\mathcal{M}_n,\ell_n}\big(-\delta - \delta^2 + \frac{2\log\delta}{\theta} - \log(1-\delta)\big)\Big) \\ &\le \exp\big(\theta\big(-\delta\big(1 - \theta^{-1}\log\delta\big) + \delta^2\big)\big) \le \delta \end{aligned}$$

holds for all n large enough and $\delta \in (0,1)$, using the Taylor expansion of the logarithm for the second step. Similarly, when $\theta = \infty$, we apply Lemma 11.16 to see that

$$P_T\left(\sum_{i=\mathcal{M}_n}^{\ell_n} \frac{\omega_i}{r_i^{\min}} > t_2\right) \le \exp\Big(\big(\min_{\mathcal{M}_n < |x| \le \ell_n} r_x\big) R^{\min}_{\mathcal{M}_n,\ell_n}\big(-\delta\big(\theta_n \log\delta\big) + \delta^2\big)\Big) \tag{11.57}$$

holds for all n large enough and some sequence $(\theta_n)_{n\in\mathbb{N}}$ according to (11.53). In this case, we obtain that for any fixed $\delta \in (0,1)$, the right-hand side in (11.57) converges to 0 when $n \to \infty$. Thus, we obtain that (11.56) holds for both cases depending on θ, which gives the lower bound.

Next, for the upper bound, we use a comparison to the independent random walks $(\tilde{\eta}_t)_{t\geq 0}$ defined in Section 11.3.1. By Lemma 11.15,

$$P_T\left(J_{\ell_n}(t_{\text{low}}) \leq \delta\right) \geq P_T\left(\tilde{J}_{\ell_n}(t_{\text{low}}) \leq \delta\right)$$

holds for all $\delta > 0$, where $(\tilde{J}_t)_{t\geq 0}$ denotes the current with respect to $(\tilde{\eta}_t)_{t\geq 0}$. Fix some $\delta > 0$ and let $(\omega_i)_{i\in\mathbb{N}_0}$ be independent Exponential-1-distributed random variables. We claim that the probability for a particle in $(\tilde{\eta}_t)_{t\geq 0}$ to reach level ℓ_n is bounded by

$$P_T\left(\sum_{i=0}^{\ell_n} \frac{\omega_i}{r_i^{\max}} \leq t_{\text{low}}\right) \leq \frac{1}{2\lambda t_{\text{low}}} \tag{11.58}$$

for all n sufficiently large, where we recall that particles enter the tree at rate $\lambda > 0$. To see this, we distinguish two cases. Recall the construction of t_{low} in (11.49) and (11.50), and assume that $t_{\text{low}} = t_1^{\text{low}}$. By the first upper bound in Lemma 11.16,

$$t_{\text{low}} P_T\left(\sum_{i=0}^{\ell_n} \frac{\omega_i}{r_i^{\max}} \leq t_{\text{low}}\right) \leq t_1^{\text{low}} \exp\left(\delta \rho_{\ell_n} t_1^{\text{low}} - \rho_{\ell_n} R_{0,\ell_n}^{\max} \log(1+\delta)\right)$$

holds for all $\delta \in (0,1)$. For $\delta = (\rho_{\ell_n} R_{0,\ell_n}^{\max})^{-1/2}$ and using the Taylor expansion of the logarithm, we see that the right-hand side in (11.58) converges to 0 when $n \to \infty$. Similarly, for $t_{\text{low}} = t_2^{\text{low}}$ the second upper bound in Lemma 11.16 yields

$$t_{\text{low}} P_T\left(\sum_{i=0}^{\ell_n} \frac{\omega_i}{r_i^{\max}} \leq t_{\text{low}}\right) \leq t_2^{\text{low}} \exp\left(\ell_n(1 + \log t_2^{\text{low}} - \log \ell_n) - \sum_{i=0}^{\ell_n} \log R_i^{\max}\right),$$

where the right-hand side converges to 0 for $n \to \infty$ using the definition of t_2^{low} and comparing the leading order terms. Since particles enter in both dynamics at the root at rate λ, note that for all n large enough, at most $\frac{5}{4}\lambda t_{\text{low}}$ particles have entered by time t_{low}. By Chebyshev's inequality together with (11.58), $\mathbb{P}$-almost surely no particle has reached generation ℓ_n by time t_{low} for all n sufficiently large. □

Now let t be a fixed time horizon and define an interval $[L_{\text{low}}, L_{\text{up}}]$ of generations. Recall $\mathcal{M}_n$ from Theorem 11.3 and define the generations

$$L_{\text{low}} := \mathcal{M}_{n_t} \quad \text{and} \quad L_{\text{up}} := \min(L_1^{\text{up}}, L_2^{\text{up}}) \tag{11.59}$$

for n_t from (11.12) and, recalling (11.2),

$$L_1^{\mathrm{up}} := \inf\left\{\ell : \log \ell - \frac{1}{\ell+1}\sum_{i=1}^{\ell} \log r_i^{\max} \geq \log t + 2\right\}, \; L_2^{\mathrm{up}} := \inf\left\{\ell : R_{0,\ell}^{\max} \geq t + t^{\frac{2}{3}}\right\}.$$

Since $r_i^{\max}$ is bounded from above uniformly in i, L_1^{up} and L_2^{up} are both finite. The next theorem is the dual result of Theorem 11.21. Recall n_t from (11.12). We are interested in a window of generations $[L_{\mathrm{low}}, L_{\mathrm{up}}]$ where we can locate the first n_t particles.

Theorem 11.22. *Suppose that* (UE) *and* (ED) *hold. Then the aggregated current through generations* L_{low} *and* L_{up} *satisfies* $\mathbb{P}$*-almost surely*

$$\limsup_{t\to\infty} J_{L_{\mathrm{up}}}(t) = 0\,, \qquad \liminf_{t\to\infty} \frac{1}{n_t} J_{L_{\mathrm{low}}}(5t) \geq 1\,. \tag{11.60}$$

Note that Theorem 11.22 implies Theorem 11.7 for rates which satisfy (11.9), keeping in mind that in the setup of Theorem 11.7, there exists some $c > 0$ such that $n_{5t} \leq c n_t$ holds for all $t \geq 0$.

Proof. Let us start with the bound involving L_{up}. Let $(\omega_i)_{i\in\mathbb{N}_0}$ be independent Exponential-1-distributed random variables. Note that $\mathbb{P}$-almost surely, no more than $2\lambda t$ particles have entered the tree by time t for all $t > 0$ large enough. Using a similar argument as after (11.58) in the proof of Theorem 11.21, it suffices to show that

$$\lim_{t\to\infty} 2\lambda t P_T\left(\sum_{i=0}^{L_{\mathrm{up}}} \frac{\omega_i}{r_i^{\max}} \leq t\right) = 0\,. \tag{11.61}$$

By Lemma 11.16 and using the definition of L_1^{up}

$$tP_T\left(\sum_{i=0}^{L_1^{\mathrm{up}}} \frac{\omega_i}{r_i^{\max}} \leq t\right) \leq \exp\left(L_1^{\mathrm{up}}(1 + \log t - \log L_1^{\mathrm{up}}) + \log t + \sum_{i=0}^{L_1^{\mathrm{up}}} \log r_i^{\max}\right),$$

where the right-hand side converges to 0 for $t \to \infty$. Moreover, by Lemma 11.16

$$tP_T\left(\sum_{i=0}^{L_2^{\mathrm{up}}} \frac{\omega_i}{r_i^{\max}} \leq t\right) \leq \frac{t\exp(\delta \rho_{L_2^{\mathrm{up}}} t)}{\exp\left(\rho_{L_2^{\mathrm{up}}} R_{0,L_2^{\mathrm{up}}}^{\max} \log(1+\delta)\right)} \tag{11.62}$$

holds for any $\delta \in (0,1)$ which may also depend on t. Note that $\sup_{\ell\in\mathbb{N}} \rho_\ell < \infty$ holds by our assumptions that the transition rates are uniformly bounded from above. Set $\delta = 2(t^{2/3}\rho_{L_2^{\mathrm{up}}})^{-1}\log t$ for all t large enough. Using the definition of L_2^{up} and the Taylor expansion of the logarithm, we conclude that the right-hand side in (11.62) converges to 0 for $t \to \infty$. Since $L_{\mathrm{up}} = \min(L_1^{\mathrm{up}}, L_2^{\mathrm{up}})$, we obtain (11.61).

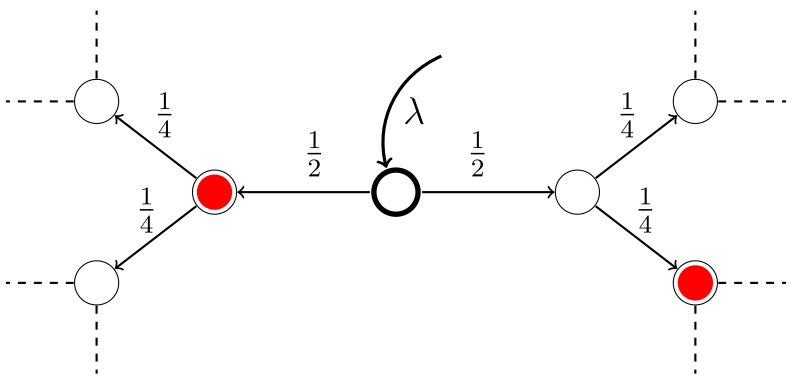

Figure 42: The 3-regular tree satisfying a flow rule with equal splitting as an example for a tree with exponential decaying rates.

For the remaining bound in Theorem 11.22, recall the slowed down exclusion process from Section 11.3.2. By Lemma 11.18 and Lemma 11.20, note that for some $c > 0$

$$P_T(J_{L_{\text{low}}}(5t) < n_t) \leq P_T\Big(G^{\tilde{\omega}}_{n_t+L_{\text{low}},n_t} \geq 5t\Big) \leq \mathrm{e}^{-cn_t} \tag{11.63}$$

holds $\mathbb{P}$-almost surely when t is sufficiently large. Consider a sequence of times $(t_i)_{i\in\mathbb{N}}$ such that $t_i \to \infty$ as $i \to \infty$ and

$$\lim_{i\to\infty} \frac{J_{L_{\text{low}}(5t_i)}(5t_i)}{n_{t_i}} = \liminf_{t\to\infty} \frac{J_{L_{\text{low}}(5t)}(5t)}{n_t} \tag{11.64}$$

By possibly removing some of the t_i's, we can assume without loss of generality that $n_{t_i} < n_{t_{i+1}}$. This way, $n_{t_i} \geq i$ for all $i \in \mathbb{N}$. Therefore by (11.63) and the Borel–Cantelli lemma, we obtain that

$$J_{L_{\text{low}}}(5t_i) \geq n_{t_i}$$

holds almost surely for all i large enough. Theorem 11.22 follows from (11.64). □

Remark 11.23. *Note that the bound in Theorem 11.22 involving L_{low} continues to hold when we replace n_t by some n with $n_t \geq n > c' \log t$.*

11.5 Current theorems for the TASEP on regular trees

In this section, we let the underlying tree be a d-regular tree, i.e., we assume that the offspring distribution is the Dirac measure on $d-1$ for some $d \geq 3$; see Figure 42. Our goal is to show how the results of Theorems 11.6 and 11.7, and more generally of Theorems 11.21 and 11.22 can be refined when we have detailed knowledge about the structure of the tree and the rates. This is illustrated in Section 11.5.1 for polynomially decaying rates, and in Section 11.5.2 for exponentially decaying rates.

11.5.1 The regular tree with polynomially decaying rates

Consider the d-regular tree and homogeneous polynomial rates, i.e. we assume that we can find some $p>0$ such that the rates satisfy

$$\frac{1}{j^p} = r_j^{\min} = r_j^{\max} \tag{11.65}$$

for all $j \in \mathbb{N}$. For this choice of the rates, we want to show how the bounds in Theorem 11.21 on a time window can be improved. In the following, we will write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n(b_n)^{-1} = 1$. Note that $\mathcal{D}_n$ and $\mathcal{M}_n$ from (11.5) and (11.7) satisfy

$$\mathcal{D}_n \sim \left(n^{2+c_o c_{\text{low}}} \log^3 n\right)^{\frac{1}{p}} \qquad \text{and} \qquad \mathcal{M}_n \sim \frac{d-1+\delta}{d-2} \min(\mathcal{D}_n, n)$$

for all $p>0$ and $\delta>0$. Recall that we are free in the choice of the sequence of generations $(\ell_n)_{n\in\mathbb{N}}$ with $\ell_n \geq \mathcal{M}_n$ for all $n \in \mathbb{N}$ along which we observe the current created by the first n particles entering the tree. We assume that $(\ell_n)_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to\infty} \frac{\mathcal{M}_n}{\ell_n^p} = a\,, \qquad \lim_{n\to\infty} \frac{n\mathcal{M}_n^p}{\ell_n^{p+1}} = b \tag{11.66}$$

for some $a \in [0,1)$ and $b \in [0,\infty)$. We apply now Theorem 11.21 in this setup.

Proposition 11.24. *Consider the TASEP on the d-regular tree with polynomial weights from* (11.65) *for some $p>0$, and a and b as in* (11.66) *for some sequence of generations $(\ell_n)_{n\in\mathbb{N}}$. Let $t_{\text{up}}, t_{\text{low}}$ be taken from* (11.49), (11.50) *and* (11.52). *For $a \in [0,1)$ and $b=0$,*

$$\lim_{n\to\infty} \frac{t_{\text{up}}}{t_{\text{low}}} = \lim_{n\to\infty} t_{\text{up}} \frac{(d-1)(1+p)}{(1-a)\ell_n^{p+1}} = 1\,. \tag{11.67}$$

For $a \in [0,1)$ and $b \in (0,\infty)$,

$$c \leq \liminf_{n\to\infty} \frac{t_{\text{up}}}{t_{\text{low}}} \leq \limsup_{n\to\infty} \frac{t_{\text{up}}}{t_{\text{low}}} \leq c' \tag{11.68}$$

holds for some constants $c, c' > 0$.

Proof. For $b \in (0,\infty)$, we observe that for the above choice of transitions rates

$$\Big(\min_{\mathcal{M}_n < |x| \leq \ell_n} r_x\Big) R^{\min}_{\mathcal{M}_n,\ell_n} = r_{\ell_n} \sum_{k=\mathcal{M}_n}^{\ell_n} \frac{1}{r_k} = \frac{1}{\ell_n^p} \sum_{k=\mathcal{M}_n}^{\ell_n} k^p \sim \frac{1}{\ell_n^p} \int_{\mathcal{M}_n}^{\ell_n} x^p \mathrm{d}x \sim \frac{1-a}{1+p}\ell_n$$

holds, and hence $\theta = \infty$ in (11.51). Thus, we see that

$$t_{\text{up}} \sim 5(n(\mathcal{M}_n)^p + (\mathcal{M}_n)^{p+1}) + \frac{1-a}{(d-1)(1+p)}\ell_n^{p+1}\,. \tag{11.69}$$

A similar computation for $b=0$ shows that $t_{\text{up}} \sim (1-a)((d-1)(1+p))^{-1}\ell_n^{p+1}$ holds. For the lower bound t_{low}, we use that $t_{\text{low}} \geq t_1^{\text{low}}$ with t_1^{low} in (11.49) to see that

$$t_{\text{low}} \sim R_{0,\ell_n}^{\min} \sim \frac{1-a}{(d-1)(1+p)} \ell_n^{p+1} \tag{11.70}$$

holds. Therefore, combining (11.69) and (11.70), we obtain a sharp time window where we see a current of order n when $b=0$. We obtain the correct leading order for the time window to observe a current linear in n in the case of $0<b<\infty$. □

11.5.2 The regular tree with exponentially decaying rates

We now study the d-regular tree with exponentially decaying rates, i.e. the rates satisfy

$$\kappa \mathrm{e}^{-c_{\text{up}}\ell} = r_\ell^{\min} = r_\ell^{\max}$$

for all $\ell \in \mathbb{N}$ and some constants $\kappa, c_{\text{up}} > 0$. In this setup, our goal is to improve the bounds on the window of generations in Theorem 11.22. Let $(N_t)_{t\geq 0}$ be some integer sequence and assume that for some $c_{\exp} \in [0,1)$.

$$\lim_{t\to\infty} \frac{\log N_t}{\log t} = c_{\exp} \,. \tag{11.71}$$

Proposition 11.25. *Consider the TASEP on the d-regular tree with exponentially decaying rates, and fix some $\delta \in (0,1)$. We set*

$$\tilde{L}_{\text{up}} := \left\lceil \frac{1}{c_{\text{up}}} \log t \left(1 + \log^{-\frac{1}{3}} t\right) \right\rceil \quad \text{and} \quad \tilde{L}_{\text{low}} := \frac{1-\delta}{c_{\text{up}}} \log t \,. \tag{11.72}$$

Then there exists some $C = C(\delta, c_{\text{up}}) > 0$ such that if $c_{\exp} \leq C$, then

$$\lim_{t\to\infty} J_{\tilde{L}_{\text{up}}}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{N_t} J_{\tilde{L}_{\text{low}}}(t) = \infty \,. \tag{11.73}$$

In particular, for $c_{\exp} = 0$, we can choose $\tilde{L}_{\text{up}}$ and $\tilde{L}_{\text{low}}$ such that $\tilde{L}_{\text{up}} \sim \tilde{L}_{\text{low}}$ holds.

Proof. We start with the lower bound $\tilde{L}_{\text{low}}$. Observe that by Theorem 11.3, there exists some $C = C(\delta, c_{\text{up}}) \in (0,1)$ such that the first $\lceil t^C \rceil$ particles are $\mathbb{P}$-almost surely disentangled at generation L_{low} for all t sufficiently large. Since $J_m(t)$ is decreasing in the generation m, and $\mathcal{M}_n$ is increasing in the number of particles n, we apply Theorem 11.22 and Remark 11.23 to conclude the second statement in (11.73). For the first statement, we follow the proof of Theorem 11.22. It suffices to show

$$\lim_{t\to\infty} 2\lambda t P_T \left(\sum_{i=0}^{\tilde{L}_{\text{up}}} \frac{\omega_i}{r_i^{\max}} < t \right) = 0 \,, \tag{11.74}$$

where (ω_i) are independent Exponential-1-distributed random variables. Using Chebyshev's inequality, we obtain that

$$P_T\left(\sum_{i=0}^{\tilde{L}_{\text{up}}} \frac{\omega_i}{r_i^{\max}} < t\right) \leq \exp\left(\tilde{L}_{\text{up}} - \sum_{i=0}^{\tilde{L}_{\text{up}}} \log\left(1 + \frac{\tilde{L}_{\text{up}}}{t}\kappa \exp(c_{\text{up}} i)\right)\right) .$$

Since $\tilde{L}_{\text{up}} t^{-1} \exp(c_{\text{up}} i) \geq 0$ holds for all $i \in \mathbb{N}$, we see that

$$tP_T\left(\sum_{i=0}^{\tilde{L}_{\text{up}}} \frac{\omega_i}{r_i^{\max}} < t\right) \leq \exp\left(\log t + \tilde{L}_{\text{up}} - \sum_{i=\lfloor \tilde{L}_{\text{up}} - \sqrt{\tilde{L}_{\text{up}}} \rfloor}^{\tilde{L}_{\text{up}}} \left(c_{\text{up}} i + \log(\kappa \tilde{L}_{\text{up}}) - \log t\right)\right) .$$

Plugging in the definition of $\tilde{L}_{\text{up}}$ from (11.72), a computation shows that the right hand side converges to 0 when $t \to \infty$. This yields (11.74). □

11.6 Open problems

We saw that under certain assumption on the transition rates, the first n particles in the TASEP will eventually disentangle and continue to move as independent random walks. Intuitively, one expects for small times that the particles in the TASEP block each other, and hence force each other not to follow their predecessors. This raises the following question.

Question 11.26. *Consider the TASEP $(\eta_t)_{t\geq 0}$ on T started from the configuration where all sites are empty. Let $(\tilde{\eta}_t)_{t\geq 0}$ be the dynamics on T where we start n independent random walks at the root. Let $p_{n,\ell}$ and $\tilde{p}_{n,\ell}$ denote the P_T-probability that the first n particles are disentangled at level ℓ in $(\eta_t)_{t\geq 0}$ and $(\tilde{\eta}_t)_{t\geq 0}$, respectively. Does $\tilde{p}_{n,\ell} \leq p_{n,\ell}$ hold for all $\ell, n \in \mathbb{N}$?*

It is not hard to see that this is true for $n = 2$. However, already the case $n = 3$ remains unclear. The next question concerns the current in the TASEP on trees through generations which increase in their distance from the root with time.

Question 11.27. *Does the aggregated current of the TASEP on trees from Theorem 11.6 for a given sequence of generations increasing with time satisfy a law of large numbers?*

We saw that in Part II of this thesis that a natural tool to investigate the transition from a non-equilibrium starting distribution to equilibrium are mixing times for finite exclusion processes. One possibility to get a family of finite approximations of the TASEP on trees is by truncating the tree at generation n for $n \in \mathbb{N}$; see Section 10.3.

Question 11.28. *What is the mixing time of the TASEP on trees, when the tree is truncated at generation n?*

List of Figures

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