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Eclectic flavor scheme from ten-dimensional string theory – I. Basic results

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ABSTRACT

In a consistent top-down approach based on orbifold compactifications, modular and traditional flavor symmetries combine nontrivially to the so-called eclectic flavor symmetry. We extend this scheme from two extra dimensions, discussed previously, to the six extra dimensions of string theory. By doing so, new insights on the nature of $\mathcal{CP}$ and its spontaneous breaking emerge. Moreover, we identify a new interpretation of R-symmetries as unbroken remnants from modular symmetries that are associated with geometrically stabilized complex structure moduli. Hence, all symmetries (i.e. modular, traditional flavor, $\mathcal{CP}$ and R) share a common origin in string theory: on a technical level, they are given by outer automorphisms of the Narain space group. The eclectic top-down approach leads to a very restrictive scheme with high predictive power. It remains a challenge to connect this with existing bottom-up constructions of modular flavor symmetry.

1. Introduction

We present a study of the eclectic flavor approach [1,2] from the point of view of ten-dimensional (10d) string theory with six compact dimensions. Eclectic flavor groups appear naturally in string constructions from the nontrivial product of modular and traditional flavor groups. This work extends previous analyses [3,4] that were based on a simplified picture with two extra dimensions (where the four additional compact dimensions of 10d string theory were treated as independent spectators), which apparently does not include all aspects of the full 10d top-down discussion. New information is especially relevant in view of $\mathcal{CP}$ and, moreover, R-symmetries that appear as remnants of the Lorentz group in six extra dimensions. With these R-symmetries, we discover that the R-charges are intrinsically related to the modular weights of the corresponding matter fields. The full picture leads to an enlargement of the eclectic flavor group as well as its subgroups that represent enhanced local flavor symmetries at specific points (or other sub-loci) in moduli space. These enhanced symmetries appear due to the alignment of vacuum expectation values (vevs) of so-called moduli fields and might include $\mathcal{CP}$-like symmetries. They are spontaneously broken at generic regions in moduli space.

In this paper we present the outcome of our analysis without going into technical details. We shall illustrate our results in a specific benchmark model with a sub-sector based on the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. The generic situation and the full technical discussion will be relegated to an upcoming companion paper [5]. In section 2 we shall review and summarize the results of the eclectic approach that are known so far. These include the origin of traditional and modular flavor symmetries, their inclusion in the eclectic picture, as well as the restrictions on the low-energy effective action represented by the superpotential and the Kähler potential. Section 3.1 will be devoted to the completion of flavor symmetries within the full six-dimensional orbifold picture. These are mainly due to remnant symmetries of the modular group of the complex structure moduli and they enhance the eclectic flavor symmetry by R-symmetries. Moreover, the nature of $\mathcal{CP}$-like transformations is only fully uncovered in six (and not two) extra dimensions. In section 4 we shall discuss the enhancements of the flavor groups at specific locations in moduli space. Section 5 will give conclusions and outlook for future research.

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2. Status of flavor symmetries in string orbifolds

Traditional flavor symmetries. Discrete flavor symmetries appear naturally in string theories [6] with the low energy particle content of the Standard Model of particle physics [7]. Furthermore, they are instrumental as an explanation of mass hierarchies and mixings of quarks and leptons in bottom-up extensions of the Standard Model [8]. In particular, flavor symmetries based on non-Abelian finite groups are promising, especially if they allow for three-dimensional irreducible representations corresponding to the three generations of quarks and/or leptons. A crucial step in flavor model building is the (spontaneous) breaking of the non-Abelian flavor symmetry. Traditionally, this breaking is achieved by the vevs of scalars, called flavon fields, that are uncharged under the Standard Model but transform in a higher-dimensional representation of the flavor symmetry.

Modular symmetries. Recently, a new bottom-up approach based on modular groups and their realizations as finite modular groups has become popular, inspired by the seminal work of Feruglio [9]. The modular group $\mathrm{SL}(2,\mathbb{Z})$ can be defined by two abstract generators S and T, subject to the relations

$$\mathrm{S}^4 = (\mathrm{S}\,\mathrm{T})^3 = \mathbb{1}\,, \qquad \mathrm{S}^2\,\mathrm{T} = \mathrm{T}\,\mathrm{S}^2\,. \tag{1}$$

A choice of its generators S and T is given by

$$\mathrm{S} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{T} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

In these terms, all elements $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ are expressed as 2×2 matrices with integer entries and unit determinant. The modular group $\mathrm{SL}(2,\mathbb{Z})$ is accompanied by a special scalar field, called the modulus M, that transforms as

$$M \xrightarrow{\mathrm{S}} -\frac{1}{M} \quad \text{and} \quad M \xrightarrow{\mathrm{T}} M+1\,. \tag{3}$$

Moreover, the modular group $\mathrm{SL}(2,\mathbb{Z})$ acts on matter fields with unitary representation matrices that generate a finite group, the so-called finite modular group. For example, the finite modular groups Γ'_N for $N \in \{2,3,4,5\}$ can be defined by the abstract definition (1) combined with the additional relation $\mathrm{T}^N = \mathbb{1}$, see ref. [3] for $N=3$ and ref. [10] for general N. In the scheme of modular symmetries, the crucial step of flavon alignment from the traditional approach finds a natural explanation through couplings $\hat{Y}(M)$ that depend solely on the modulus M but still transform in higher-dimensional representations of the finite modular group: couplings become so-called modular forms of M. Thus, the alignment of the coupling $\hat{Y}(M)$ in flavor space depends only on the vev $\langle M\rangle$ of the single modulus M. Interestingly, the modular approach can be naturally extended to include $\mathcal{CP}$-like transformations such that the vev $\langle M\rangle$ can also be a measure of spontaneous $\mathcal{CP}$ breaking, see refs. [3,11,12].

Unified picture of traditional and modular flavor symmetry in string theory. The approach based on a traditional flavor symmetry and the one utilizing a (finite) modular symmetry seem orthogonal from a bottom-up perspective. One can distinguish between two cases: a symmetry transformation can either leave the modulus invariant, giving rise to a traditional flavor transformation, or transform the modulus nontrivially, yielding a modular transformation.

However, in a unified picture based on string theory, they turn out to be intimately related. First of all, it is well-known that string theory compactified on a two-torus $\mathbb{T}^2$ is equipped with two modular groups $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$ associated with two moduli: the so-called Kähler modulus T and the complex structure modulus U, respectively. Furthermore, in toroidal orbifold compactifications, both, traditional and modular symmetries, originate from outer automorphisms of the so-called Narain space group [3], which encodes a string compactification on an orbifold geometry, see ref. [13] for a technical discussion of Narain space groups. This avenue of outer automorphisms of the Narain space group was initially discussed in a general study of two-dimensional orbifolds $\mathbb{T}^2/\mathbb{Z}_N$ [3,4], where the four orthogonal compact dimensions are considered as spectators. In what follows we shall illustrate the results in a specific example with $N=3$.

Symmetries of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. In this case, the complex structure modulus is stabilized at $\langle U\rangle = \omega := \exp(2\pi\mathrm{i}/3)$, such that the two-torus $\mathbb{T}^2$ exhibits a $\mathbb{Z}_3$ rotational symmetry. For the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold, the modular group $\mathrm{SL}(2,\mathbb{Z})_U$ associated with the complex structure modulus U is broken. However, there remain several unbroken symmetries, as summarized in Table 1: there are two generators of translational outer automorphisms of the $\mathbb{T}^2/\mathbb{Z}_3$ Narain space group, denoted by A and B. Since translations leave all moduli invariant, they belong to the traditional flavor symmetry. It turns out that A and B generate $\Delta(27)$. Then, depending on the geography of strings in extra dimensions, matter fields build various representations of $\Delta(27)$: strings that live in the bulk of the extra dimensions are singlets, while so-called twisted strings that are localized in extra dimensions at the fixed points of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold are triplets or anti-triplets. In addition, there exists an unbroken rotational outer automorphism of the $\mathbb{T}^2/\mathbb{Z}_3$ Narain space group, denoted by C, which leaves the Kähler modulus invariant and thus belongs to the traditional flavor symmetry. C enlarges $\Delta(27)$ to $\Delta(54)$, which is known to be the flavor symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold [6]. However, it is important to note that C is realized as a 180° rotation in the extra dimensions of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold [3], i.e. C is an unbroken remnant of the higher-dimensional Lorentz symmetry. Consequently, $\Delta(54)$ is an example of a non-Abelian discrete R-symmetry of $\mathcal{N}=1$ supersymmetry [14] arising from string theory, where the superpotential transforms as a nontrivial singlet $\mathbf{1}'$ of $\Delta(54)$ [4].

Similar to the toroidal case, the outer automorphisms of the $\mathbb{T}^2/\mathbb{Z}_3$ Narain space group include transformations that act nontrivially on the Kähler modulus T. One can identify two generators, denoted also by S and T, which satisfy the defining relations (1) of the modular group $\mathrm{SL}(2,\mathbb{Z})_T$. When acting on localized matter fields, this modular group is realized as the finite modular group $T' \cong \Gamma'_3 \cong \mathrm{SL}(2,3)$. Therefore, matter fields transform under modular transformations in representations of T' [15,16]. For example, the $\Delta(54)$ triplet $\Phi_{-2/3}$ of

Table 1
Eclectic flavor group $\Omega(1) \cong \Delta(54) \cup T' \cong [648, 533]$ for a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. Note that the order-3 translational outer automorphisms A and B generate $\Delta(27)$, which contains the $\mathbb{Z}_3 \times \mathbb{Z}_3$ point and space group selection rules [24–26]. The rotational outer automorphism C is special as it belongs to both, the traditional flavor symmetry $\Delta(54)$ and the finite modular symmetry T'.

<table>
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<tr><td rowspan="5">eclectic</td><td rowspan="2">modular</td><td>rotation $\mathrm{S} \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_4$</td><td colspan="2" rowspan="2">T'</td><td rowspan="5">$\Omega(1)$</td></tr>
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<tr><td rowspan="3">traditional flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td></tr>
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</table>

twisted strings localized at the three fixed points of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold transform as $\mathbf{2}' \oplus \mathbf{1}$ of T' and the subscript $-2/3$ of $\Phi_{-2/3}$ gives the $\mathrm{SL}(2,\mathbb{Z})_T$ modular weight (see ref. [2] for details).

The eclectic group and local flavor unification. The traditional flavor group $\Delta(54)$ does not commute with the finite modular group T'. This fact gives rise to the so-called eclectic flavor group [1,2]: the nontrivial combination of traditional flavor and finite modular groups, where the finite modular group corresponds to outer automorphisms of the traditional flavor group. For the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold one obtains the eclectic flavor group as the multiplicative closure of $\Delta(54)$ and T', i.e.

$$\Omega(1) \;\cong\; [648, 533] \;\cong\; \Delta(54) \;\cup\; T' \,, \tag{4}$$

where [648, 533] denotes the GAP ID [17] of $\Omega(1)$. Note that the eclectic flavor group has a similar structure as the one discussed in ref. [18]. Some phenomenological properties of $\Omega(1)$ have been studied [19,20], considering this group as a traditional flavor symmetry of bottom-up models. In contrast, in our eclectic approach, $\Omega(1)$ is nonlinearly realized on the Kähler modulus T, except for the subgroup $\Delta(54)$, as one can see from the action eq. (3) on the modulus $M = T$. However, there are certain points (or regions, in general) in moduli space, where larger subgroups of $\Omega(1)$ are realized linearly: for certain vevs of T, some modular transformations leave $\langle T\rangle$ invariant and remain unbroken. They give rise to the stabilizer subgroup $H_{\langle T\rangle} \subset \mathrm{SL}(2,\mathbb{Z})_T$. Then, embedding the stabilizer subgroup into T' yields an enhancement of the traditional flavor symmetry $\Delta(54)$ to a so-called unified flavor symmetry at some specific values of $\langle T\rangle$ in moduli space. In detail, one can identify two independent points $\langle T\rangle$ with a nontrivial stabilizer subgroup $H_{\langle T\rangle}$:

$$\text{at } \langle T\rangle \;=\; \mathrm{i} \quad : \quad H_{\mathrm{i}} \;=\; \{\mathrm{S}^k \text{ for } k = 0,1,2,3\} \qquad\qquad \cong\; \mathbb{Z}_4 \,, \tag{5a}$$

$$\text{at } \langle T\rangle \;=\; \omega \quad : \quad H_{\omega} \;=\; \{(\mathrm{S\,T})^k \text{ for } k = 0,1,2\} \times \{\mathbb{1}, \mathrm{S}^2\} \;\cong\; \mathbb{Z}_3 \times \mathbb{Z}_2 \,, \tag{5b}$$

see also refs. [21,22]. In both cases, a $\mathbb{Z}_2$ subgroup is generated by S^2. On the level of outer automorphisms of the Narain space group, S^2 coincides with the transformation C discussed earlier and thus belongs to the traditional flavor symmetry $\Delta(54)$. Therefore, the enhancement of $\Delta(54)$ by the stabilizer subgroup increases the order of the group only by a factor of two (three) in the first (second) case, resulting in the following enhancements [1]:

$$\text{at } \langle T\rangle \;=\; \mathrm{i} \quad : \quad \Delta(54) \;\to\; \Sigma(36 \times 3) \;\cong\; [108, 15] \;\subset\; \Omega(1) \,, \tag{6a}$$

$$\text{at } \langle T\rangle \;=\; \omega \quad : \quad \Delta(54) \;\to\; \tilde{Y}(0) \qquad\quad \cong\; [162, 10] \;\subset\; \Omega(1) \,, \tag{6b}$$

using ref. [23] for naming conventions of finite groups. This peculiar feature of eclectic flavor models has been named local flavor unification, as different (enhanced) flavor symmetries are realized at different points in moduli space. Further enhancements are possible if $\mathcal{CP}$ is taken into account as well [3,4].

Eclectic restrictions on superpotential and Kähler potential. The eclectic flavor group severely constrains the allowed couplings in the theory for both the superpotential and the Kähler potential [2]. Let us illustrate this in a specific example. If one demands a modular symmetry $\mathrm{SL}(2,\mathbb{Z})_T$ with finite modular group T', the superpotential has to transform as

$$\mathcal{W} \xrightarrow{\gamma_T} (c\,T + d)^{-1}\,\mathcal{W} \,, \qquad \text{under} \quad \gamma_T \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})_T \,. \tag{7}$$

In other words, even though the superpotential $\mathcal{W}$ does transform nontrivially under $\gamma_T \in \mathrm{SL}(2,\mathbb{Z})_T$, it is invariant under T', where matter fields and Yukawa couplings $\hat{Y}(T)$ transform with automorphy factors $(c\,T + d)^n$ with modular weights n under $\mathrm{SL}(2,\mathbb{Z})_T$ and, simultaneously, in representations of T'. Now, we consider the trilinear superpotential $\mathcal{W}$ of three copies of twisted strings $\Phi^i_{-2/3}$, where $n = -2/3$ and $i = 1,2,3$, localized at the three fixed points of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold [2]. If we impose only invariance under the finite modular group T' the superpotential $\mathcal{W}$ can be parameterized by four coefficients. These correspond to four independent couplings that are undetermined from a bottom-up perspective based solely on the modular flavor symmetry. If we now consider the full eclectic group $\Omega(1)$, which includes the extension of T' by the traditional flavor symmetry $\Delta(54)$, these four coefficients from the T' theory are forced to be identical and there remains only one independent parameter. Similarly, the bilinear Kähler potential (which is essentially unconstrained in bottom-up models based solely on (finite) modular symmetries [27]) is very sensitive to the eclectic extension: starting with only T', the Kähler potential of localized strings contains non-diagonal terms, which are removed by the traditional flavor symmetry $\Delta(54)$ to all orders in the Kähler modulus T. Therefore, eclectic flavor symmetries are highly predictive and their phenomenological implications are worth to be studied in more detail.

3. New insights from six compact dimensions

The eclectic approach to flavor symmetries has been discovered by the study of two-dimensional $\mathbb{T}^2/\mathbb{Z}_N$ orbifolds and discussed in detail for $N=3$. As we show next, the generalization to six extra dimensions might seem straightforward, but it still yields new insights concerning R-symmetries and $\mathcal{CP}$.

3.1. Flavor R-symmetries from modular symmetries

It is known that compactifications of heterotic strings on factorizable orbifolds $\mathbb{T}^6/P$ lead to various discrete R-symmetries [28–32]. They can be understood as discrete remnants of the originally ten-dimensional Lorentz symmetry that are described by so-called *sublattice rotations* in the compactified extra dimensions. A sublattice rotation R is defined as a discrete rotational symmetry of the orbifold that is not part of the point group, i.e. $\mathrm{R}\notin P$. As such, R has to map the torus $\mathbb{T}^6$ underlying the orbifold $\mathbb{T}^6/P$ to itself. For factorizable orbifolds, the six-torus is a direct product $\mathbb{T}^6=\mathbb{T}^2\times\mathbb{T}^2\times\mathbb{T}^2$, where an Abelian point group P acts diagonally in the three two-tori. Then, a separate rotation only in one $\mathbb{T}^2$ sub-torus of $\mathbb{T}^6$ is a symmetry of the orbifold geometry. For instance, the factorizable $\mathbb{T}^6/\mathbb{Z}_6$-II orbifold geometry allows three sublattice rotations: a separate $\mathbb{Z}_6$, $\mathbb{Z}_3$ and $\mathbb{Z}_2$ rotation in the first, second and third two-torus, respectively. It turns out that under these discrete rotations localized strings carry fractional charges. Taking these fractional charges into account, the three sublattice rotations yield a $\mathbb{Z}_{36}^R\times\mathbb{Z}_9^R\times\mathbb{Z}_4^R$ R-symmetry for the $\mathbb{T}^6/\mathbb{Z}_6$-II orbifold geometry [30] (where we give the order of the R-symmetries for the superfields; in this normalization, fermions can have half-integer R-charges). In this section, we present an alternative interpretation of these discrete R-symmetries using the example of the $\mathbb{Z}_9^R$ R-symmetry associated with the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector (corresponding to the second torus of the $\mathbb{T}^6/\mathbb{Z}_6$-II orbifold) discussed above: we show that sublattice rotations and, hence, their corresponding discrete R-symmetries can be understood as discrete remnants of modular symmetries. In this way, R-symmetries appear on the same footing as modular and flavor symmetries, with common origin in the outer automorphisms of the Narain space group. Moreover, it is obvious that these R-symmetries are linked to the Lorentz group of the six-torus, and we would like to understand why these symmetries had not been identified in the earlier discussions [3,4] that just considered an (orbifolded) two-torus. In the following we shall clarify this situation.

A two-torus $\mathbb{T}^2$ can be parameterized by a Kähler modulus T and a complex structure modulus U. For the U modulus there are two outer automorphisms of the corresponding Narain lattice, denoted here by S_U and T_U, which satisfy the defining relations (1) of $\mathrm{SL}(2,\mathbb{Z})_U$. They act on the modulus as $U\xrightarrow{\mathrm{S}_U}-1/U$ and $U\xrightarrow{\mathrm{T}_U}U+1$, while leaving T untouched. One can show that this action translates into the transformations

$$e_1 \xrightarrow{\mathrm{S}_U} -e_2\,,\quad e_2 \xrightarrow{\mathrm{S}_U} e_1\,,\qquad \text{and}\qquad e_1 \xrightarrow{\mathrm{T}_U} e_1\,,\quad e_2 \xrightarrow{\mathrm{T}_U} e_1+e_2 \tag{8}$$

of the $\mathbb{T}^2$ basis vectors e_1 and e_2. In addition, similar to eq. (7), the superpotential transforms as

$$\mathcal{W} \xrightarrow{\gamma_U} (c\,U+d)^{-1}\,\mathcal{W}\,,\qquad \text{under}\quad \gamma_U\;=\;\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z})_U\,. \tag{9}$$

Next, we observe from eq. (8) that we can express a $\mathbb{Z}_N$ rotational symmetry R of the two-torus as a modular transformation $\mathrm{R}=\gamma_{(N)}\in\mathrm{SL}(2,\mathbb{Z})_U$ of order N, i.e. $(\gamma_{(N)})^N=\mathbb{1}$. This modular transformation $\gamma_{(N)}$ acts on the superpotential as in eq. (9). However, the automorphy factor of a $\mathbb{Z}_N$ rotation $\gamma_{(N)}$ is just a phase of order N because $(\gamma_{(N)})^N=\mathbb{1}$ implies $(c\,U+d)^{-N}=1$. Here, one already realizes that (for $N\neq 2$) the modulus U has to be stabilized to a fixed value $\langle U\rangle$ such that $(c\,\langle U\rangle+d)^{-N}=1$ is valid. Indeed, one can show that U has to be invariant under a modular transformation that corresponds to a $\mathbb{Z}_N$ rotational symmetry of the two-torus if $N\neq 2$. For example, the modular transformation $S_U\,T_U\in\mathrm{SL}(2,\mathbb{Z})_U$ generates a $\mathbb{Z}_3$ rotation if e_1 and e_2 have equal lengths and enclose an angle of 120°, i.e. if U is stabilized at $\omega=\exp\left(2\pi\mathrm{i}/3\right)$.

Let us now consider the case of a (six-dimensional) orbifold geometry $\mathbb{T}^6/P$ and focus on a sublattice rotation of a $\mathbb{T}^2/\mathbb{Z}_N$ orbifold sector, where $N\in\{2,3,4,6\}$. Due to the point group action, the complex structure modulus U of the $\mathbb{T}^2/\mathbb{Z}_N$ orbifold sector has to be stabilized geometrically at some value $\langle U\rangle$ depending on $N\neq 2$ (see ref. [33] for a related discussion). This vev breaks $\mathrm{SL}(2,\mathbb{Z})_U$ to the stabilizer group $H_{\langle U\rangle}\subset\mathrm{SL}(2,\mathbb{Z})_U$ that leaves $\langle U\rangle$ invariant, and the stabilizer group $H_{\langle U\rangle}$ contains the sublattice rotation $\mathrm{R}=\gamma_{(N)}$ that maps the $\mathbb{T}^2/\mathbb{Z}_N$ orbifold sector to itself. Moreover, due to the nontrivial transformation of the superpotential eq. (9), each unbroken modular transformation $\mathrm{R}=\gamma_{(N)}$ generates a discrete Abelian R-symmetry. Thus, the sublattice rotation $\gamma_{(N)}\in H_{\langle U\rangle}\subset\mathrm{SL}(2,\mathbb{Z})_U$ can be identified with an automorphism of the Narain space group. Is this transformation an independent symmetry of the orbifold? If the orbifold is two-dimensional, R belongs to the point group P. Hence, R is an inner automorphism in the two-dimensional case and, consequently, not an independent symmetry of the theory. This is different in the six-dimensional case, as we shall explain now. The key observation is that in a six-dimensional orbifold compactification which preserves $\mathcal{N}=1$ supersymmetry, the point group consists of rotations that act simultaneously on more than two compact dimensions and not only on one $\mathbb{T}^2/\mathbb{Z}_N$ orbifold sector. Thus, a two-dimensional sublattice rotation $\mathrm{R}=\gamma_{(N)}\in H_{\langle U\rangle}\subset\mathrm{SL}(2,\mathbb{Z})_U$ cannot belong to the point group P of a six-dimensional orbifold geometry. Hence, R is an independent outer automorphism of the Narain space group of six-dimensional orbifolds and therefore an authentic symmetry of the theory. Since it leaves all Kähler and complex structure moduli invariant, it contributes to the traditional flavor symmetry.

Let us now consider our example of a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector embedded in a six-dimensional orbifold $\mathbb{T}^6/P$. We sketch the main results here and do not enter a detailed technical discussion which will be included in ref. [5]. In this case, the associated complex structure modulus is fixed, for example, at $\langle U\rangle=\omega$. Consequently, the stabilizer group is $H_{\langle U\rangle}\cong\mathbb{Z}_3\times\mathbb{Z}_2$, where the $\mathbb{Z}_2$ factor is generated by S_U^2. It

Table 2
Eclectic flavor group $\Omega(2)$ for orbifolds $\mathbb{T}^6/P$ that contain a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. In this case, $\mathrm{SL}(2,\mathbb{Z})_U$ of the stabilized complex structure modulus $U = \exp(2\pi\mathrm{i}/3)$ is broken, resulting in a remnant $\mathbb{Z}_9^R$ R-symmetry. Including $\mathbb{Z}_9^R$ enhances the traditional flavor group $\Delta(54)$ to $\Delta'(54,2,1) \cong [162,44]$ and, thereby, the eclectic group to $\Omega(2) \cong [1944,3448]$. Note that $\Omega(1) \subset \Omega(2)$.

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<tr><td rowspan="4">traditional flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td><td rowspan="4">$\Delta'(54,2,1)$</td></tr>
<tr><td>translation B</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td>rotation $\mathrm{C} = \mathrm{S}^2 \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td colspan="2">$\mathbb{Z}_2^R$</td></tr>
<tr><td>rotation $\mathrm{R} = \gamma_{(3)} \in \mathrm{SL}(2,\mathbb{Z})_U$</td><td colspan="3">$\mathbb{Z}_9^R$</td></tr>
</table>

is known from the Narain formalism that $\mathrm{S}_U^2 = \mathrm{S}^2 = \mathrm{C}$, i.e. $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$ share a common element [4]. This $\mathbb{Z}_2$ factor is already included in both, the traditional flavor symmetry $\Delta(54)$ and the modular symmetry $\mathrm{SL}(2,\mathbb{Z})_T$. On the other hand, the $\mathbb{Z}_3$ factor of $H_{\langle U\rangle}$ is generated by the $\mathrm{SL}(2,\mathbb{Z})_U$ element $\mathrm{R} := \mathrm{S}_U\mathrm{T}_U$. It is represented by the 2×2 matrix

$$\mathrm{R} \;=\; \mathrm{S}_U\,\mathrm{T}_U \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \;\in\; H_{\langle U\rangle} \;\subset\; \mathrm{SL}(2,\mathbb{Z})_U\,, \tag{10}$$

which acts, according to eq. (8), on the torus lattice as a $\mathbb{Z}_3$ sublattice rotation: $e_1 \xrightarrow{\mathrm{R}} -e_1 - e_2$ and $e_2 \xrightarrow{\mathrm{R}} e_1$. Eq. (9) evaluated at $\langle U\rangle = \omega$ implies that the transformation R given in eq. (10) acts on the superpotential as $\mathcal{W} \xrightarrow{\mathrm{R}} (-\omega-1)^{-1}\mathcal{W} = \omega\,\mathcal{W}$. Modular transformations from the stabilizer group $H_{\langle U\rangle}$, which include in this case the $\mathbb{Z}_3$ sublattice rotation R, leave the moduli and hence the Kähler potential invariant. Thus, the Grassmann number ϑ must transform under the $\mathbb{Z}_3$ sublattice rotation as $\vartheta \xrightarrow{\mathrm{R}} \omega^{1/2}\,\vartheta$, which implies that R generates a discrete R-symmetry. In fact, this discrete R-symmetry corresponds to the $\mathbb{Z}_9^R$ symmetry previously identified in ref. [30]. To see this, recall that the modular weights n_U of matter fields are multiples of $1/3$, $n_U = m/3$ for $m \in \mathbb{Z}$, see e.g. ref. [34,35]. Consequently, their transformation under R induces a phase

$$(c\,\langle U\rangle + d)^{n_U} \;=\; \omega^{-m/3}\,, \tag{11}$$

originating from the automorphy factor of $\mathrm{SL}(2,\mathbb{Z})_U$. This means that i) the $\mathbb{Z}_3$ sublattice rotational symmetry in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector of the six-dimensional $\mathbb{T}^6/P$ orbifold is realized as a $\mathbb{Z}_9^R$ R-symmetry, and ii) the discrete R-charges of matter fields are related to their $\mathrm{SL}(2,\mathbb{Z})_U$ modular weights n_U. Finally, note that the $\mathbb{Z}_3$ subgroup of $\mathbb{Z}_9^R$ is not an R-symmetry as it maps $\mathcal{W} \xrightarrow{\mathrm{R}^3} \mathcal{W}$. Moreover, R^3 corresponds to the point group selection rule [24–26] contained already in $\Delta(54)$, see also ref. [36].

These observations lead to an extension of the eclectic flavor group, summarized in Table 2. On the one hand, the traditional flavor symmetry becomes now the multiplicative closure of $\Delta(54)$ and $\mathbb{Z}_9^R$, which is known as

$$\Delta'(54,2,1) \;\cong\; [162,44] \;\cong\; \Delta(54) \,\cup\, \mathbb{Z}_9^R\,. \tag{12}$$

On the other hand, combining this group with the T' finite modular flavor symmetry arising from $\mathrm{SL}(2,\mathbb{Z})_T$, one arrives at the full eclectic flavor symmetry of a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector without $\mathcal{CP}$, being

$$\Omega(2) \;\cong\; \Omega(1) \,\cup\, \mathbb{Z}_9^R \;\cong\; \Delta'(54,2,1) \,\cup\, T'\,. \tag{13}$$

$\Omega(2) \cong [1944,3448]$ can also be written as $\Omega(2) \cong Z(3,2) \rtimes T'$, where $Z(3,2) \cong [81,14]$ is generated by A, B and R.

3.2. $\mathcal{CP}$ for six-dimensional orbifolds $\mathbb{T}^6/P$

As shown in refs. [3,4] for two compact dimensions, the string construction of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold automatically yields a $\mathbb{Z}_2$ $\mathcal{CP}$-like transformation that maps string states to their $\mathcal{CP}$-conjugated partner states. In more detail, let us consider a string state that is localized in the first twisted sector of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold geometry. Then, the $\mathcal{CP}$-partner originates from the second twisted sector. Hence, a $\mathcal{CP}$-like transformation has to be given by an outer automorphism of the Narain space group that interchanges the twisted sectors of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold.

How does this situation change for six-dimensional orbifolds $\mathbb{T}^6/P$? For simplicity, let us consider an Abelian point group P that yields exactly three bulk Kähler moduli T_i, $i = 1,2,3$, such as $P = \mathbb{Z}_3 \times \mathbb{Z}_3$, see for example appendix C.1 in ref. [37]. These Kähler moduli parameterize the sizes (and B-fields) of the three two-dimensional planes underlying the six-torus $\mathbb{T}^6$ in which P acts diagonally. Then, each Kähler modulus T_i is associated with its own modular group $\mathrm{SL}(2,\mathbb{Z})_{T_i}$ for $i = 1,2,3$. Moreover, as in the two-dimensional discussion, a $\mathcal{CP}$-like transformation has to interchange simultaneously all twisted sectors of the six-dimensional $\mathbb{T}^6/P$ orbifold that contain $\mathcal{CP}$-conjugated partner states. Consequently, the $\mathcal{CP}$-like transformation has to act in all six extra dimensions simultaneously and not only in two. In this case, one can show that under the $\mathcal{CP}$-like transformation the three Kähler moduli transform simultaneously,

$$T_i \;\xrightarrow{\mathcal{CP}}\; -\bar{T}_i\,, \qquad \text{for} \qquad i = 1,2,3\,, \tag{14}$$

i.e. there is no $\mathcal{CP}$-like transformation that acts nontrivially only on one modulus (e.g. $T_1 \to -\bar{T}_1$) while leaving the other moduli invariant (e.g. $T_2 \to T_2$ and $T_3 \to T_3$). Only the combined action eq. (14) is possible in these orbifold constructions (as one might have expected from the holomorphicity of the superpotential). An interesting phenomenological consequence is that, if any modulus is stabilized away from a so-called self-dual point or region in moduli space, $\mathcal{CP}$ is broken spontaneously.

4. Local flavor unification with modular *R*-symmetries

As discussed in section 2, at special points $\langle T\rangle$ in moduli space of orbifolds with a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector there is a non-trivial stabilizer group $H_{\langle T\rangle} \subset \mathrm{SL}(2,\mathbb{Z})_T$ that leaves $\langle T\rangle$ (as well as the stabilized complex structure modulus $\langle U\rangle$) invariant. Thus, $H_{\langle T\rangle}$ enhances the traditional flavor group to a discrete unified flavor group, which is realized linearly as a subgroup of the eclectic flavor group. This holds also for the traditional flavor group $\Delta'(54,2,1) \cong [162,44]$ which includes $\Delta(54)$ and the remnant $\mathbb{Z}_9^R$ R-symmetry from $\mathrm{SL}(2,\mathbb{Z})_U$ in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. In this case, one finds different subgroups of the full eclectic flavor group $\Omega(2) \cong [1944, 3448]$ to be realized linearly for different specific values $\langle T\rangle$ of the Kähler modulus. Let us consider this enhancement for the two points with non-trivial stabilizer groups discussed in eq. (5) (in the case where $\mathcal{CP}$-like transformations were not taken into account).

According to eq. (5a), the stabilizer group at the point $\langle T\rangle = \mathrm{i}$ is given by $H_{\langle T\rangle=\mathrm{i}} \cong \mathbb{Z}_4$, generated by the element S of $\mathrm{SL}(2,\mathbb{Z})_T$. In this case, the traditional flavor group $\Delta'(54,2,1)$ gets enhanced to the unified flavor group $\Xi(2,2) \cong [324,111] \subset \Omega(2)$. It is generated by the $\Delta'(54,2,1)$ generators A, B and R, as well as the stabilizer generator S at the point $\langle T\rangle = \mathrm{i}$. Note that C is not an independent generator because it satisfies $\mathrm{C} = \mathrm{S}^2$. This relation also explains why the order of the unified flavor group is just twice (and not four times) the order of the traditional flavor symmetry group. The group $\Xi(2,2)$ contains the group $\Sigma(36\times 3) \cong [108,15] \subset \Omega(1)$ found in the absence of the $\mathbb{Z}_9^R$ R-symmetry (see eq. (6a)).

At the point $\langle T\rangle = \omega$ in moduli space, the unbroken stabilizer group is $H_{\langle T\rangle=\omega} \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ (see eq. (5b)), where only the $\mathbb{Z}_3$ factor, generated by ST, is independent since again the $\mathbb{Z}_2$ factor is already included in the traditional flavor group as it coincides with C. Hence, considering the generators A, B, C, R and ST, the unified flavor group at $\langle T\rangle = \omega$ is given by $H(3,2,1) \cong [486,125] \subset \Omega(2)$. As before, we find that this unified flavor group contains $\tilde{Y}(0) \cong [162,10] \subset \Omega(1)$ that appears at $\langle T\rangle = \omega$ when ignoring the existence of the $\mathbb{Z}_9^R$ modular remnant symmetry (see eq. (6b)).

In summary, we find that the traditional flavor group $\Delta'(54,2,1)$ is enhanced at different points of moduli space to the following unified flavor groups:

$$\text{at } \langle T\rangle = \mathrm{i} \quad : \quad \Delta'(54,2,1) \to \Xi(2,2) \quad \cong [324,111] \subset \Omega(2)\,, \tag{15a}$$

$$\text{at } \langle T\rangle = \omega \quad : \quad \Delta'(54,2,1) \to H(3,2,1) \cong [486,125] \subset \Omega(2)\,, \tag{15b}$$

whose orders, comparing with eqs. (6), are enlarged by a factor of three because the discrete $\mathbb{Z}_9^R$ R-symmetry is included. Some of the phenomenological properties of these unified flavor groups have been addressed in ref. [20].

As mentioned earlier, $\mathcal{CP}$-like transformations are included naturally in string-derived models. After stabilizing all moduli, a $\mathbb{Z}_2$ $\mathcal{CP}$-like transformation remains unbroken if it leaves all moduli invariant, cf. eq. (14). In particular, at the points $\langle T\rangle = \mathrm{i}$ and $\langle T\rangle = \omega$ in Kähler moduli space, there exist $\mathcal{CP}$-like transformations leaving $\langle T\rangle$ invariant. Thus, assuming that all other moduli are also stabilized at some $\mathcal{CP}$ invariant points in moduli space, the unified flavor groups of eq. (15) get further enhanced. Moreover, there are additional points (and lines) beside $\langle T\rangle = \mathrm{i}$ and $\langle T\rangle = \omega$ that are left invariant by elements of the $\mathcal{CP}$-enhanced modular group, as shown in earlier papers [3,4], enriching the structure of the possible flavor groups and the possibilities of spontaneous $\mathcal{CP}$ breakdown. We shall discuss the details of these possibilities in a companion paper [5].

5. Conclusions and outlook

We have discussed the top-down (TD) approach of eclectic flavor symmetries from the point of view of ten-dimensional string theory with six compact spatial dimensions. Compared to previous studies that were essentially based on two compact extra dimensions, we can report several new observations:

- There is a further enlargement of the eclectic flavor group, as illustrated in the difference between Table 1 and Table 2.
- We find a new interpretation of R-symmetries connected to both, the modular symmetry of the complex structure modulus and the Lorentz symmetry in six extra dimensions, as explained in section 3.1.
- New insights on the nature of $\mathcal{CP}$-symmetry and its breakdown are discussed in section 3.2.
- R-charges of quark and lepton fields are inherently associated with their $\mathrm{SL}(2,\mathbb{Z})_U$ modular weights.

This leads to even more restrictions for TD model building that include strict constraints on the Kähler potential and the superpotential as well as the possible modular weights and the representations of the finite modular flavor group of matter fields. It implies that TD model building is very restrictive, with enhanced predictive power. Therefore, it remains a challenge to bridge the gap between the TD approach and existing bottom-up constructions of modular flavor symmetries.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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