$\mathcal{H}_2$ suboptimal output synchronization of heterogeneous multi-agent systems☆

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ABSTRACT

This paper deals with the $\mathcal{H}_2$ suboptimal output synchronization problem for heterogeneous linear multi-agent systems. Given a multi-agent system with possibly distinct agents and an associated $\mathcal{H}_2$ cost functional, the aim is to design output feedback based protocols that guarantee the associated cost to be smaller than a given upper bound while the controlled network achieves output synchronization. A design method is provided to compute such protocols. For each agent, the computation of its two local control gains involves two Riccati inequalities, each of dimension equal to the state space dimension of the agent. A simulation example is provided to illustrate the performance of the proposed protocols.

1. Introduction

Over the last two decades, the problems of designing protocols that achieve consensus or synchronization in multi-agent systems have attracted much attention in the field of systems and control, see e.g. [1–3] and [4]. The essential feature of these problems is that, while each agent makes use of only local state or output information to implement its own local controller, the resulting global protocol will achieve consensus or synchronization for the global controlled multi-agent network [5,6]. One of the challenging problems in this context is the problem of designing protocols that minimize given quadratic cost criteria while achieving consensus or synchronization, see e.g. [7–10] and [11]. Due to the structural constraints imposed on the protocols, such optimal control problems are non-convex and very difficult to solve. It is also unclear whether in general closed form solutions exist.

In the past, many efforts have been devoted to designing distributed protocols for *homogeneous* multi-agent systems that guarantee suboptimal or optimal performance and achieve *state* synchronization or consensus. In [9], this was done for distributed linear quadratic control of multi-agent systems with *single integrator* agent dynamics, see also [12]. In [11] and [7], multi-agent systems with general agent dynamics and a global linear quadratic cost functional were considered. In [10] and [13], an inverse optimal approach was adopted to address the distributed

linear quadratic control problem, see also [14]. For $\mathcal{H}_2$ cost functionals of a particular form, [15] and [16] proposed distributed suboptimal protocols that stabilize the controlled multi-agent network. In [17], a distributed $\mathcal{H}_2$ suboptimal control problem was addressed using static state feedback. The results in [17] were then generalized in [8] to the case of dynamic output feedback.

More recently, *output* synchronization problems for *heterogeneous* multi-agent systems have also attracted much attention. In [18], it was shown that solvability of certain regulator equations is a necessary condition for output synchronization of heterogeneous multi-agent systems, and suitable protocols were proposed, see also [19]. By embedding an internal model in the local controller of each agent, in [20] dynamic output feedback based protocols were proposed for a class of heterogeneous uncertain multi-agent systems. In [21], it was shown that the outputs of the agents can be synchronized by a networked protocol if and only if these agents have certain dynamics in common. Later on, in [22] a linear quadratic control method was adopted for computing output synchronizing protocols. In [23], an $\mathcal{L}_2$-gain output synchronization problem was addressed by casting this problem into a number of $\mathcal{L}_2$-gain stabilization problems for certain linear systems, where the state space dimensions of these systems are equal to that of the agents. In [24], a model-free approach based on reinforcement learning algorithms was proposed to obtain output synchronizing protocols, see also [25]. For related work, we also mention [26–29] and [30], to name a few.

Up to now, little attention has been paid in the literature to problems of designing output synchronizing protocols for heterogeneous multi-agent systems that guarantee a certain performance. In particular, to the authors' best knowledge, none of the

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existing publications considers the problem of output synchronization while obtaining a guaranteed $\mathcal{H}_2$ performance. Therefore, in the present paper we will deal with the problem of $\mathcal{H}_2$ optimal output synchronization for heterogeneous linear multi-agent systems, i.e. the problem of minimizing a given $\mathcal{H}_2$ cost functional over all protocols that achieve output synchronization.

Instead of addressing this *optimal* control problem, we will address a version of this problem that requires *suboptimality*. More specifically, we will extend previous results in [8] for homogeneous multi-agent systems to the case of heterogeneous multi-agent systems. The main contributions of the present paper are the following.

(1) We consider a generalization of the type of heterogeneous multi-agent systems that was considered before in [18]. For this type of multi-agent systems we establish a design procedure for protocols that achieve output synchronization.
(2) We show that the $\mathcal{H}_2$ suboptimal output synchronization problem for heterogeneous multi-agent systems can be cast as a simultaneous $\mathcal{H}_2$ suboptimal control problem for a number of low-dimensional systems.
(3) We provide a method for computing $\mathcal{H}_2$ suboptimal dynamic output feedback protocols for heterogeneous multi-agent systems.

The outline of this paper is as follows. In Section 2, we provide some notation and graph theory used throughout this paper. In Section 3, we formulate the $\mathcal{H}_2$ suboptimal output synchronization problem. In order to solve this problem, in Section 4 we review some basic material on $\mathcal{H}_2$ suboptimal control by dynamic output feedback for linear systems, and some relevant results on output synchronization of heterogeneous multi-agent systems. In Section 5, we solve the problem introduced in Section 3 and provide a design method for obtaining $\mathcal{H}_2$ suboptimal protocols. To illustrate the performance of our proposed protocols, a simulation example is provided in Section 6. Finally, Section 7 concludes this paper.

2. Notation and graph theory

2.1. Notation

We denote by $\mathbb{R}$ the field of real numbers and by $\mathbb{C}$ the field of complex numbers. The space of n dimensional real vectors is denoted by $\mathbb{R}^n$. We denote by $\mathbf{1}_n \in \mathbb{R}^n$ the vector with all its entries equal to 1. For a symmetric matrix P, we denote $P > 0$ if P is positive definite and $P < 0$ if P is negative definite. The identity matrix of dimension $n \times n$ is denoted by I_n. The trace of a square matrix A is denoted by $\mathrm{tr}(A)$. A matrix is called Hurwitz if all its eigenvalues have negative real parts. We denote by $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ the $n \times n$ diagonal matrix with $d_1, d_2, \ldots, d_n$ on the diagonal. For given matrices $M_1, M_2, \ldots, M_n$, we denote by $\mathrm{blockdiag}(M_1, M_2, \ldots, M_n)$ the block diagonal matrix with diagonal blocks M_i. The Kronecker product of two matrices A and B is denoted by $A \otimes B$.

2.2. Graph theory

A directed weighted graph is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the finite nonempty node set, $\mathcal{E} = \{e_1, e_2, \ldots, e_M\}$ with $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}]$ is the adjacency matrix with nonnegative elements a_{ij}, called the edge weights. The entry a_{ji} is nonzero if and only if $(i, j) \in \mathcal{E}$. A graph is called simple if $a_{ii} = 0$ for all i. It is called *a weighted undirected* graph if $(j, i) \in \mathcal{E}$ whenever $(i, j) \in \mathcal{E}$, and $a_{ji} = a_{ij}$ for all (i, j). Given a graph $\mathcal{G}$, a path from node 1 to node p is a sequence of edges $(k, k+1), k = 1, 2, \ldots, p-1$. A graph is said to contain a *spanning*

tree if it contains a node such that there exists a path from this node to every other node. Throughout this paper it will be a standing assumption that the communication between the agents of the network is represented by a weighted directed graph that contains a spanning tree. Unless stated otherwise, in this paper the term 'graph' will refer to a weighted directed graph.

Given a graph $\mathcal{G}$, the degree matrix of $\mathcal{G}$ is defined by $\mathcal{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ with $d_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix is defined as $L := \mathcal{D} - \mathcal{A}$. The eigenvalues of the Laplacian matrix L have nonnegative real part. In particular, L has a zero eigenvalue with eigenvector $\mathbf{1}_N$. A graph contains a spanning tree if and only if its Laplacian matrix has rank $N - 1$. In this case, the zero eigenvalue has multiplicity one and all other eigenvalues have positive real part.

For any graph with M edges and N nodes, we define the incidence matrix $R \in \mathbb{R}^{N \times M}$ as the matrix $R = (r_1, r_2, \ldots, r_M)$ with columns $r_k \in \mathbb{R}^N$. Each column r_k corresponds to exactly one edge $e_k = (i, j)$, and the ith and jth entry of r_k are equal to 1 and -1, respectively. The remaining entries of r_k are equal to 0. We also define the matrix

$$W = \mathrm{diag}(w_1, w_2, \ldots, w_M) \tag{1}$$

as the $M \times M$ diagonal matrix, where w_k is the weight on the edge e_k ($k = 1, 2, \ldots, M$). The positive semi-definite matrix $RWR^\top$ can be considered as the Laplacian of an associated undirected graph, and will be denoted in this paper by L_{new}.

3. Problem formulation

In this paper, we consider a heterogeneous linear multi-agent system consisting of N possibly distinct agents. The dynamics of the ith agent is represented by the linear time-invariant system

$$\begin{aligned}
\dot{x}_i &= A_i x_i + B_i u_i + E_i d_i, \\
y_i &= C_{1i} x_i + D_{1i} d_i, \qquad i = 1, 2, \ldots, N, \\
z_i &= C_{2i} x_i + D_{2i} u_i,
\end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$ is the coupling input, $d_i \in \mathbb{R}^{q_i}$ is an unknown external disturbance input, $y_i \in \mathbb{R}^{r_i}$ is the measured output and $z_i \in \mathbb{R}^p$ is the output to be synchronized. The matrices A_i, B_i, C_{1i}, D_{1i}, C_{2i}, D_{2i} and E_i are of suitable dimensions. Throughout this paper we assume that the pairs (A_i, B_i) are stabilizable and the pairs (C_{1i}, A_i) are detectable. Since in (2) the agents may have non-identical dynamics, in particular the state space dimensions of the agents may differ. Therefore, one cannot expect to achieve *state* synchronization for the network. Instead, in the context of heterogeneous networks it is natural to consider *output* synchronization, see e.g. [18,19] and [21].

Remark 1. Note that the system (2) representing the ith agent is more general than the one in [18]. Indeed, our agents contain two types of outputs, namely measured outputs y_i and outputs z_i to be synchronized, while the agents in [18] only contain outputs to be synchronized.

It was shown in [18] that solvability of certain *regulator equations* is necessary for output synchronization of heterogeneous linear multi-agent systems, see also [19,23,30] and [31]. Following up on this, throughout this paper we make the standard assumption that there exists a positive integer r such that the regulator equations

$$\begin{aligned}
A_i \Pi_i + B_i \Gamma_i &= \Pi_i S, \\
C_{2i} \Pi_i + D_{2i} \Gamma_i &= R, \quad i = 1, 2, \ldots, N
\end{aligned} \tag{3}$$

have solutions $\Pi_i \in \mathbb{R}^{n_i \times r}$, $\Gamma_i \in \mathbb{R}^{m_i \times r}$, $R \in \mathbb{R}^{p \times r}$ and $S \in \mathbb{R}^{r \times r}$, where the eigenvalues of S lie on the imaginary axis and the pair (R, S) is observable.

Following [18], we assume that the agents (2) should be interconnected by a protocol of the form

$$\dot{w}_i = A_i w_i + B_i u_i + G_i(y_i - C_{1i} w_i),$$

$$\dot{v}_i = S v_i + \sum_{i=1}^{N} a_{ij}(v_j - v_i),$$

$$u_i = F_i(w_i - \Pi_i v_i) + \Gamma_i v_i, \quad i = 1, 2, \ldots, N, \tag{4}$$

where $v_i \in \mathbb{R}^r$ and $w_i \in \mathbb{R}^{n_i}$ are the states of the ith local controller, the matrices S, Π_i and Γ_i are solutions of (3), and the matrices $F_i \in \mathbb{R}^{m_i \times n_i}$ and $G_i \in \mathbb{R}^{n_i \times r_i}$ are control gains to be designed. The coefficients a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ of the communication graph. We briefly explain the structure of this protocol. The first equation in (4) has the structure of an asymptotic observer for the state of the ith agent. The second equation represents an auxiliary system associated with the ith agent. Each auxiliary system receives the relative state values with respect to its neighboring auxiliary systems. In this way, the network of auxiliary systems will reach state synchronization. The third equation in (4) is a static gain, it feeds back the value $w_i - \Pi_i v_i$ and the state v_i of the associated auxiliary system to the ith agent. The idea of the protocol (4) is that, as time goes to infinity, the state x_i of the ith agent and its estimate w_i converge to $\Pi_i v_i$ due the first equation in (3). Subsequently, as a consequence of the second equation in (3), the outputs z_i of the agents will reach synchronization.

Denote by $\mathbf{x} = (x_1^\top, x_2^\top, \ldots, x_N^\top)^\top$ the aggregate state vector and likewise define $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{d}$. Denote by A the block diagonal matrix

$$A = \text{blockdiag}(A_1, A_2, \ldots, A_N) \tag{5}$$

and likewise define B, C_1, C_2, D_1, D_2 and E. The multi-agent system (2) can then be written in compact form as

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + E\mathbf{d},$$

$$\mathbf{y} = C_1 \mathbf{x} + D_1 \mathbf{d}, \tag{6}$$

$$\mathbf{z} = C_2 \mathbf{x} + D_2 \mathbf{u}.$$

Similarly, denote

$$F = \text{blockdiag}(F_1, F_2, \ldots, F_N)$$

and likewise define G, Γ and Π. The protocol (4) can be written in compact form as

$$\dot{\mathbf{w}} = A\mathbf{w} + B\mathbf{u} + G(\mathbf{y} - C_1 \mathbf{w}),$$

$$\dot{\mathbf{v}} = (I_N \otimes S - L \otimes I_r)\mathbf{v}, \tag{7}$$

$$\mathbf{u} = F\mathbf{w} + (\Gamma - F\Pi)\mathbf{v}.$$

Next, denote

$$\mathbf{x}_o = (\mathbf{x}^\top, \mathbf{w}^\top, \mathbf{v}^\top)^\top.$$

By interconnecting the system (6) and the protocol (7), the controlled network is then represented in compact form by

$$\dot{\mathbf{x}}_o = A_o \mathbf{x}_o + E_o \mathbf{d},$$

$$\mathbf{z} = C_o \mathbf{x}_o, \tag{8}$$

where

$$A_o = \begin{pmatrix} A & BF & B\Gamma - BF\Pi \\ GC_1 & A + BF - GC_1 & B\Gamma - BF\Pi \\ 0 & 0 & I_N \otimes S - L \otimes I_r \end{pmatrix},$$

$$C_o = \begin{pmatrix} C_2 & D_2 F & D_2 \Gamma - D_2 F\Pi \end{pmatrix}, \quad E_o = \begin{pmatrix} E \\ GD_1 \\ 0 \end{pmatrix}.$$

Foremost, we want the protocol (4) to achieve output synchronization for the overall network:

Definition 1. The protocol (4) is said to achieve $\mathbf{z}$-output synchronization for the network (8) if, for all $i, j = 1, 2, \ldots, N$, we

have $z_i(t) - z_j(t) \to 0$, $v_i(t) - v_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.

In the context of output synchronization, we are interested in the differences of the output values of the agents in the controlled network. Since the differences of the output values of communicating agents are captured by the incidence matrix R of the communication graph [32], we define a performance output variable as

$$\zeta = (W^{\frac{1}{2}} R^\top \otimes I_p)\mathbf{z},$$

where W is the weight matrix defined in (1). The output ζ reflects the weighted disagreement between the outputs of the agents in accordance with the weights of the edges connecting these agents. Subsequently, we have the following equations for the controlled network

$$\dot{\mathbf{x}}_o = A_o \mathbf{x}_o + E_o \mathbf{d},$$

$$\mathbf{z} = C_o \mathbf{x}_o, \tag{9}$$

$$\zeta = C_p \mathbf{x}_o,$$

where

$$C_p = (W^{\frac{1}{2}} R^\top \otimes I_p)C_o.$$

The impulse response matrix of the disturbance $\mathbf{d}$ to the performance output ζ is given by

$$T_d(t) = C_p e^{A_o t} E_o. \tag{10}$$

The performance of the network is now quantified by the $\mathcal{H}_2$-norm of this impulse response. Thus we define the associated $\mathcal{H}_2$ cost functional as

$$J := \int_0^\infty \text{tr}\left[T_d^\top(t) T_d(t) \right] dt. \tag{11}$$

Note that the cost functional (11) is a function of the gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$.

The $\mathcal{H}_2$ optimal output synchronization problem is now defined as the problem of minimizing the cost functional (11) over all protocols (4) that achieve output synchronization. Since the protocol (4) has a particular structure imposed by the communication topology, the $\mathcal{H}_2$ optimal output synchronization problem is a non-convex optimization problem, and it is unclear whether a closed form solution exists in general. Therefore, in this paper we will address a version of this problem that only requires *suboptimality*. The aim of this paper is then to design a protocol of the form (4) that guarantees the associated cost (11) to be smaller than an a priori given upper bound while achieving $\mathbf{z}$-output synchronization for the network. More concretely, the problem we will address is the following:

Problem 1. Let $\gamma > 0$ be a given tolerance. Design gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$ such that the resulting protocol (4) achieves $\mathbf{z}$-output synchronization and its associated cost (11) satisfies $J < \gamma$.

To solve Problem 1, in the next section we will first review some preliminary results on $\mathcal{H}_2$ suboptimal control for linear systems and on output synchronization of heterogeneous linear multi-agent systems. It will become clear later on that these preliminary results are necessary ingredients to address Problem 1.

4. Preliminary results

4.1. $\mathcal{H}_2$ Suboptimal control for linear systems by dynamic output feedback

In this subsection, we will review the $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback for linear systems, see

e.g. [33–36] and [8]. In particular, we will review the results from [8] on separation principle based $\mathcal{H}_2$ suboptimal control for continuous-time linear systems.

Consider the system

$$
\begin{aligned}
\dot{x} &= \bar{A}x + \bar{B}u + \bar{E}d, \\
y &= \bar{C}_1 x + \bar{D}_1 d, \\
z &= \bar{C}_2 x + \bar{D}_2 u,
\end{aligned}
\tag{12}
$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^q$ is an unknown external disturbance input, $y \in \mathbb{R}^r$ is the measured output, and $z \in \mathbb{R}^p$ is the output to be controlled. The matrices $\bar{A}$, $\bar{B}$, $\bar{C}_1$, $\bar{C}_2$, $\bar{D}_1$, $\bar{D}_2$ and $\bar{E}$ are of suitable dimensions. We assume that the pair $(\bar{A}, \bar{B})$ is stabilizable and the pair $(\bar{C}_1, \bar{A})$ is detectable. We consider dynamic output feedback controllers of the form

$$
\begin{aligned}
\dot{w} &= \bar{A}w + \bar{B}u + G\left(y - \bar{C}_1 w\right), \\
u &= Fw,
\end{aligned}
\tag{13}
$$

where $w \in \mathbb{R}^n$ is the state of the controller, $F \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times r}$ are gain matrices to be designed. By interconnecting the controller (13) and the system (12), we obtain the controlled system

$$
\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} \bar{A} & \bar{B}F \\ G\bar{C}_1 & \bar{A} + \bar{B}F - G\bar{C}_1 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} \bar{E} \\ G\bar{D}_1 \end{pmatrix} d, \\
z &= \begin{pmatrix} \bar{C}_2 & \bar{D}_2 F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}.
\end{aligned}
\tag{14}
$$

Denote $A_e = \begin{pmatrix} \bar{A} & \bar{B}F \\ G\bar{C}_1 & \bar{A} + \bar{B}F - G\bar{C}_1 \end{pmatrix}$, $E_e = \begin{pmatrix} \bar{E} \\ G\bar{D}_1 \end{pmatrix}$, $C_e = \begin{pmatrix} \bar{C}_2 & \bar{D}_2 F \end{pmatrix}$. The impulse response matrix of the disturbance d to the output z is given by $T_{F,G}(t) = C_e e^{A_e t} E_e$. We define the $\mathcal{H}_2$ cost functional as

$$
J(F, G) := \int_0^\infty \operatorname{tr}\left[T_{F,G}^\top(t)T_{F,G}(t)\right] dt.
\tag{15}
$$

The $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback is the problem of finding a controller of the form (13) such that the associated cost (15) is smaller than an a priori given upper bound and the controlled system (14) is internally stable. The following lemma provides a design method for computing such a controller, see also [8, Theorem 4].

Lemma 1. *Let $\gamma > 0$ be a given tolerance. Assume that $\bar{D}_1 \bar{E}^\top = 0$, $\bar{D}_2^\top \bar{C}_2 = 0$ and $\bar{D}_1 \bar{D}_1^\top = I_r$, $\bar{D}_2^\top \bar{D}_2 = I_m$. Let $P > 0$ and $Q > 0$ satisfy the Riccati inequalities*

$$
\begin{aligned}
\bar{A}^\top P + P\bar{A} - P\bar{B}\bar{B}^\top P + \bar{C}_2^\top \bar{C}_2 &< 0, \\
\bar{A}Q + Q\bar{A}^\top - Q\bar{C}_1^\top \bar{C}_1 Q + \bar{E}\bar{E}^\top &< 0.
\end{aligned}
$$

If, in addition, such P and Q satisfy

$$
\operatorname{tr}\left(\bar{C}_1 QPQ\bar{C}_1^\top\right) + \operatorname{tr}\left(\bar{C}_2 Q\bar{C}_2^\top\right) < \gamma,
$$

then the controller (13) with $F = -\bar{B}^\top P$ and $G = Q\bar{C}_1^\top$ internally stabilizes the system (12) and is suboptimal, i.e. $J(F, G) < \gamma$.

For a proof of Lemma 1, we refer to [8, Theorem 4].

4.2. Output synchronization of heterogeneous linear multi-agent systems

In this subsection, we will study output synchronization of heterogeneous linear multi-agent systems, see also [18–20] and [21].

Consider a heterogeneous linear multi-agent system consisting of N possibly distinct agents. The dynamics of the ith agent is represented by the linear time-invariant system

$$
\begin{aligned}
\dot{x}_i &= A_i x_i + B_i u_i, \\
y_i &= C_{1i} x_i, \qquad\qquad i = 1, 2, \ldots, N. \\
z_i &= C_{2i} x_i + D_{2i} u_i,
\end{aligned}
\tag{16}
$$

As we have mentioned before, the heterogeneous system (2) is in fact a generalization of the heterogeneous system that was considered in [18]. The agents (16) will be interconnected by a protocol of the form (4), where the matrices S, Γ_i and Π_i are assumed to satisfy the regulator equations (3). The multi-agent system (16) can be written in compact form as

$$
\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}, \\
\mathbf{y} &= C_1 \mathbf{x}, \\
\mathbf{z} &= C_2 \mathbf{x} + D_2 \mathbf{u},
\end{aligned}
\tag{17}
$$

and the protocol (4) can be written as (7). By interconnecting the system (17) and the protocol (7), the controlled network is then given by

$$
\begin{aligned}
\dot{\mathbf{x}}_o &= A_o \mathbf{x}_o, \\
\mathbf{z} &= C_o \mathbf{x}_o.
\end{aligned}
\tag{18}
$$

The following lemma yields conditions under which the controlled network (18) achieves $\mathbf{z}$-output synchronization.

Lemma 2. *Consider the multi-agent system (16) and the protocol (4). Let gain matrices F_i and G_i be such that the matrices $A_i + B_i F_i$ and $A_i - G_i C_{1i}$ are Hurwitz. Then the associated protocol (4) achieves $\mathbf{z}$-output synchronization for the network.*

A proof of Lemma 2 can be given along the lines of the proof of [18, Theorem 5].

We are now ready to deal with the $\mathcal{H}_2$ suboptimal output synchronization problem formulated in Problem 1.

5. Design of $\mathcal{H}_2$ suboptimal output synchronization protocols using dynamic output feedback

In this section, we will resolve Problem 1. More specifically, we will establish a design method for computing gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$ such that the associated protocol (4) achieves $\mathbf{z}$-output synchronization and guarantees $J < \gamma$.

In the sequel, we will first show that this problem can be simplified by transforming it into $\mathcal{H}_2$ suboptimal control problems for N auxiliary systems. The suboptimal gains F_i and G_i for these N separate problems will turn out to also yield a suboptimal protocol for the heterogeneous network.

To this end, we introduce the following N auxiliary systems

$$
\begin{aligned}
\dot{\xi}_i &= A_i \xi_i + B_i v_i + E_i \delta_i, \\
\vartheta_i &= C_{1i} \xi_i + D_{1i} \delta_i, \\
\eta_i &= C_{2i} \xi_i + D_{2i} v_i, \quad i = 1, 2, \ldots, N,
\end{aligned}
\tag{19}
$$

where $\xi_i \in \mathbb{R}^{n_i}$ is the state, $v_i \in \mathbb{R}^{m_i}$ is the coupling input, $\delta_i \in \mathbb{R}^{q_i}$ is an unknown external disturbance input, $\vartheta_i \in \mathbb{R}^{r_i}$ is the measured output and $\eta_i \in \mathbb{R}^p$ is the output to be controlled. For given gain matrices F_i and G_i, consider the dynamic output feedback controllers

$$
\begin{aligned}
\dot{\omega}_i &= A_i \omega_i + B_i v_i + G_i(\vartheta_i - C_{1i} \omega_i), \\
v_i &= F_i \omega_i, \quad i = 1, 2, \ldots, N,
\end{aligned}
\tag{20}
$$

where $\omega_i \in \mathbb{R}^n$ is the state of the ith controller.

By interconnecting the systems (19) and the controllers (20), we obtain the N controlled auxiliary systems

$$\begin{pmatrix} \dot{\xi}_i \\ \dot{\omega}_i \end{pmatrix} = \begin{pmatrix} A_i & B_iF_i \\ G_iC_{1i} & A_i + B_iF_i - G_iC_{1i} \end{pmatrix} \begin{pmatrix} \xi_i \\ \omega_i \end{pmatrix} + \begin{pmatrix} E_i \\ G_iD_{1i} \end{pmatrix} \delta_i,$$

$$\eta_i = \begin{pmatrix} C_{2i} & D_{2i}F_i \end{pmatrix} \begin{pmatrix} \xi_i \\ \omega_i \end{pmatrix}, \quad i = 1, 2, \ldots, N. \tag{21}$$

For $i = 1, 2, \ldots, N$, denote

$$\bar{A}_i = \begin{pmatrix} A_i & B_iF_i \\ G_iC_{1i} & A_i + B_iF_i - G_iC_{1i} \end{pmatrix},$$

$$\bar{C}_i = \begin{pmatrix} C_{2i} & D_{2i}F_i \end{pmatrix}, \quad \bar{E}_i = \begin{pmatrix} E_i \\ G_iD_{1i} \end{pmatrix}.$$

The impulse response matrix of the disturbance δ_i to the output η_i is equal to

$$T_{\delta i}(t) = \bar{C}_i e^{\bar{A}_i t} \bar{E}_i,$$

and an associated $\mathcal{H}_2$ cost functional is defined as

$$J_i = \int_0^\infty \text{tr}[T_{\delta i}^\top(t) T_{\delta i}(t)] dt. \tag{22}$$

The following lemma holds.

Lemma 3. *Let $\gamma > 0$ be a given tolerance. Assume, for $i = 1, 2, \ldots, N$, the systems (21) are internally stable and the costs (22) satisfy*

$$\sum_{i=1}^N J_i < \frac{\gamma}{\lambda_N}, \tag{23}$$

where λ_N is the largest eigenvalue of the matrix L_{new} given by $L_{new} = RWR^\top$. Then the protocol (4) achieves z-output synchronization for the network (9) and the associated cost (11) satisfies $J < \gamma$.

Proof. First, note that the systems (21) are internally stable if and only if the matrices $A_i + B_iF_i$ and $A_i - G_iC_{1i}$ are Hurwitz, see e.g. [37, Section 3.12]. Hence, by Lemma 2, if the systems (21) are internally stable, then the network controlled using the protocol (4) reaches z-output synchronization.

Next, we will show that if (23) holds, then $J < \gamma$. Note that (23) is equivalent to

$$\lambda_N \sum_{i=1}^N \int_0^\infty \text{tr}[T_{\delta i}^\top(t) T_{\delta i}(t)] dt < \gamma. \tag{24}$$

In turn, the inequality (24) holds if and only if

$$\lambda_N \int_0^\infty \text{tr}[\bar{T}_d^\top(t) \bar{T}_d(t)] dt < \gamma \tag{25}$$

holds, where

$$\bar{T}_d = \bar{C}_o e^{\bar{A}_o t} \bar{E}_o$$

with

$$\bar{A}_o = \begin{pmatrix} A & BF \\ GC_1 & A + BF - GC_1 \end{pmatrix}, \quad \bar{E}_o = \begin{pmatrix} E \\ GD_1 \end{pmatrix},$$

$$\bar{C}_o = \begin{pmatrix} C_2 & D_2F \end{pmatrix}.$$

Recall that the matrix A is the block diagonal matrix defined in (5), similarly for the matrices $B, C_1, C_2, D_1, D_2, E, F$ and G. Using the fact that $\lambda_N I_{pN} - L_{new} \otimes I_p \geq 0$, it can be shown that (25) implies

$$\int_0^\infty \text{tr}[\bar{T}_d^\top(t)(L_{new} \otimes I_p)\bar{T}_d(t)] dt < \gamma. \tag{26}$$

On the other hand,

$$\int_0^\infty \text{tr}[\bar{T}_d^\top(t)(L_{new} \otimes I_p)\bar{T}_d(t)] dt = \int_0^\infty \text{tr}\left[T_d^\top(t) T_d(t)\right] dt \tag{27}$$

with $T_d(t)$ given by (10). Note that the right hand side of (27) is exactly the cost J given by (11) associated with the network (9). It follows that $J < \gamma$. This completes the proof. $\square$

By the previous, if the gain matrices F_i and G_i are such that $A_i + B_iF_i$ and $A_i - G_iC_{1i}$ are Hurwitz and (23) holds, then the protocol (4) using these F_i and G_i yields z-output synchronization and $J < \gamma$. In the next theorem, we will provide a method for computing gain matrices F_i and G_i such that the above holds.

Theorem 4. *Let $\gamma > 0$ be a given tolerance. For $i = 1, 2, \ldots, N$, assume that $D_{1i}E_i^\top = 0$, $D_{2i}^\top C_{2i} = 0$, $D_{1i}D_{1i}^\top = I_{r_i}$ and $D_{2i}^\top D_{2i} = I_{m_i}$. Let $P_i > 0$ satisfy*

$$A_i^\top P_i + P_i A_i - P_i B_i B_i^\top P_i + C_{2i}^\top C_{2i} < 0. \tag{28}$$

Let $Q_i > 0$ satisfy

$$A_i Q_i + Q_i A_i^\top - Q_i C_{1i}^\top C_{1i} Q_i + E_i E_i^\top < 0. \tag{29}$$

If, in addition, such P_i and Q_i satisfy

$$\text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top) + \text{tr}(C_{2i}Q_iC_{2i}^\top) < \frac{\gamma}{N\lambda_N}, \tag{30}$$

then the protocol (4) with $F_i := -B_i^\top P_i$ and $G_i := Q_iC_{1i}^\top$ achieves z-output synchronization for the network (9) and guarantees $J < \gamma$.

Proof. Note that (28) is equivalent to

$$(A_i - B_iB_i^\top P_i)^\top P_i + P_i(A_i - B_iB_i^\top P_i) + P_iB_iB_i^\top P_i + C_{2i}^\top C_{2i} < 0 \tag{31}$$

and (29) is equivalent to

$$(A_i - Q_iC_{1i}^\top C_{1i})Q_i + Q_i(A_i - Q_iC_{1i}^\top C_{1i})^\top + Q_iC_{1i}^\top C_{1i}Q_i + E_iE_i^\top < 0. \tag{32}$$

Taking $F_i := -B_i^\top P_i$ and $G_i := Q_iC_{1i}^\top$, it then follows that $A_i + B_iF_i$ and $A_i - G_iC_{1i}$ are Hurwitz.

Next, by (30), it follows from Lemma 1 that

$$J_i < \frac{\gamma}{N\lambda_N}, \quad i = 1, 2, \ldots, N.$$

Thus we have (23), and the conclusion then follows from Lemma 3. $\square$

We note that the conditions $D_{1i}E_i^\top = 0, D_{2i}^\top C_{2i} = 0, D_{1i}D_{1i}^\top = I_{r_i}$ and $D_{2i}^\top D_{2i} = I_{m_i}$ are made here to simplify notation, and can be relaxed to the regularity conditions $D_{1i}D_{1i}^\top > 0$ and $D_{2i}^\top D_{2i} > 0$ alone.

Remark 2. Although Theorem 4 is an extension to the case of heterogeneous systems of our previous results in [8] on homogeneous multi-agent systems, the structure of the proposed protocol (4) is different from the one proposed in [8]. In particular, in Theorem 4, one needs to compute $2N$ control gains $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$, while in [8] one needs to compute only two control gains.

Remark 3. In Theorem 4, in order to select γ, the following steps could be taken. For $i = 1, 2 \ldots, N$:

(i) Compute positive definite solutions P_i and Q_i of the Riccati inequalities (28) and (29). Such solutions exist.
(ii) Denote $S_i = \text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top) + \text{tr}(C_{2i}Q_iC_{2i}^\top)$.
(iii) Choose γ such that $N\lambda_N S_i < \gamma$.

Note that the smaller S_i or λ_N is, the smaller such feasible γ is allowed to be. Unfortunately, the problem of minimizing S_i over

all $P_i > 0$ and $Q_i > 0$ that satisfy (28) and (29) is a non-convex optimization problem. However, since smaller Q_i leads to smaller $\text{tr}(C_{2i}Q_iC_{2i}^\top)$ and smaller P_i and Q_i lead to smaller $\text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top)$, and consequently smaller feasible γ, we could try to find P_i and Q_i as small as possible. In fact, one can find $P_i = P_i(\epsilon_i) > 0$ to (28) by solving the Riccati equation

$$A_i^\top P_i + P_iA_i^\top - P_iB_iB_i^\top P_i + C_{2i}^\top C_{2i} + \epsilon_i I_{n_i} = 0$$

with $\epsilon_i > 0$ arbitrary. Similarly, one can find $Q_i = Q_i(\sigma_i) > 0$ to (29) by solving the dual Riccati equation

$$A_iQ_i + Q_iA_i^\top - Q_iC_{1i}^\top C_{1i}Q_i + E_iE_i^\top + \sigma_i I_{n_i} = 0$$

with $\sigma_i > 0$ arbitrary. By using a standard argument, it can be shown that $P_i(\epsilon_i)$ and $Q_i(\sigma_i)$ decrease as ϵ_i and σ_i decrease, respectively. So ϵ_i and σ_i should be taken close to 0 to get smaller P_i and Q_i.

Remark 4. As a final remark, we note that the assumption that all eigenvalues of S lie on the imaginary can be relaxed. The assumption that none of the eigenvalues of S has a have negative real part is made to exclude the trivial case that the outputs of the agents converge to zero as time goes to infinity, see e.g. [18]. The assumption that none of the eigenvalues of S has positive real part is due to the second equation in protocol (4). If S has an eigenvalue with positive real part, then synchronization is not guaranteed.

The assumption that none of the eigenvalues of S has positive real part can be removed by instead considering the protocol:

$$\dot{w}_i = A_iw_i + B_iu_i + G_i(y_i - C_{1i}w_i),$$
$$\dot{v}_i = Sv_i + K\sum_{i=1}^{N} a_{ij}(v_j - v_i), \qquad (33)$$
$$u_i = F_i(w_i - \Pi_iv_i) + \Gamma_iv_i, \quad i = 1, 2, \dots, N,$$

where K is a control gain to be designed, see e.g. [23]. Methods exist to compute a suitable gain matrix K, see e.g. [38].

6. Simulation example

In this section, we will give a simulation example based on the example in [18] to illustrate the design method of Theorem 4.

Consider a network of $N = 6$ heterogeneous agents. The dynamics of the agents are given by

$$\dot{x}_i = A_ix_i + B_iu_i + E_id_i,$$
$$y_i = C_{1i}x_i + D_{1i}d_i, \qquad i = 1, 2, \dots, 6,$$
$$z_i = C_{2i}x_i + D_{2i}u_i,$$

where $A_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c_i \\ 0 & -f_i & -a_i \end{pmatrix}$, $B_i = \begin{pmatrix} 0 \\ 0 \\ b_i \end{pmatrix}$, $E_i = \begin{pmatrix} 0 & 0.2 \\ 0 & 0 \\ 0 & 0.2 \end{pmatrix}$,

$C_{1i} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $D_{1i} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $C_{2i} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $D_{2i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The parameters a_i, b_i, c_i and f_i are chosen to be

$a_i = 2$, $c_i = 1$, $i = 1, 2, \dots, 6$,
$b_1 = b_4 = 1$, $b_2 = b_5 = 2$, $b_3 = b_6 = 3$,
$f_1 = f_4 = 1$, $f_2 = f_5 = 2$, $f_3 = f_6 = 3$.

The pairs (A_i, B_i) are stabilizable and the pairs (C_{1i}, A_i) are detectable. We also have that $D_{1i}E_i^\top = 0$, $D_{2i}^\top C_{2i} = 0$, $D_{1i}D_{1i}^\top = 1$ and $D_{2i}^\top D_{2i} = 1$. The communication graph between the six agents is assumed to be a weighted undirected cycle graph with all edge weights equal to 1. Its Laplacian matrix is denoted by L. It turns out that $L_{new} = 2L$, and the largest eigenvalue of L_{new} is $\lambda_6 = 8$.

We choose the matrices S and R in the regulator equations (3) to be

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues of S are on the imaginary axis and the pair (R, S) is observable. We solve the equations (3) and compute

$$\Pi_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad i = 1, 2, \dots, 6.$$

The objective is to design a protocol of the form (4) such that the associated cost (11) satisfies $J < \gamma$ while achieving **z**-output synchronization. Let the desired upper bound be $\gamma = 36$.

Following the design method in Theorem 4, for $i = 1, 2, \dots, 6$, we compute a positive definite solution P_i to (28) by solving the Riccati equation

$$A_i^\top P_i + P_iA_i^\top - P_iB_iB_i^\top P_i + C_{2i}^\top C_{2i} + \epsilon I_{n_i} = 0$$

with $\epsilon = 0.001$. We also compute a positive definite solution Q_i to (28) by solving the dual Riccati equation

$$A_iQ_i + Q_iA_i^\top - Q_iC_{1i}^\top C_{1i}Q_i + E_iE_i^\top + \sigma I_{n_i} = 0$$

with $\sigma = 0.001$. Accordingly, we compute the associated gain matrices F_i and G_i to be

$$F_1 = F_4 = \begin{pmatrix} -1.0005 & -1.7329 & -0.7326 \end{pmatrix},$$
$$F_2 = F_5 = \begin{pmatrix} -1.0005 & -1.2345 & -0.4951 \end{pmatrix},$$
$$F_3 = F_6 = \begin{pmatrix} -1.0005 & -1.0327 & -0.3982 \end{pmatrix},$$

and

$$G_1 = G_4 = \begin{pmatrix} 0.3290 & 0.0341 & 0.0028 \end{pmatrix}^\top,$$
$$G_2 = G_5 = \begin{pmatrix} 0.2804 & 0.0193 & 0.0007 \end{pmatrix}^\top,$$
$$G_3 = G_6 = \begin{pmatrix} 0.2578 & 0.0132 & 0.0002 \end{pmatrix}^\top.$$

As an example, we take the initial states of the agents to be $x_{10} = \begin{pmatrix} 1.0 & 1.4 & 1.6 \end{pmatrix}^\top$, $x_{20} = \begin{pmatrix} 1.2 & -1.7 & 0.5 \end{pmatrix}^\top$, $x_{30} = \begin{pmatrix} 1.3 & -1.2 & 1.3 \end{pmatrix}^\top$, $x_{40} = \begin{pmatrix} 0.6 & 1.6 & -1.3 \end{pmatrix}^\top$, $x_{50} = \begin{pmatrix} 1.8 & 1.5 & 1.6 \end{pmatrix}^\top$, $x_{60} = \begin{pmatrix} -1.1 & 1.7 & 0.9 \end{pmatrix}^\top$. We take the initial states w_i to be zero, and the initial states v_i to be $v_{10} = \begin{pmatrix} 0.9 & 1.1 \end{pmatrix}^\top$, $v_{20} = \begin{pmatrix} 0.8 & 1.4 \end{pmatrix}^\top$, $v_{30} = \begin{pmatrix} -1.0 & 0.9 \end{pmatrix}^\top$, $v_{40} = \begin{pmatrix} 1.8 & 1.1 \end{pmatrix}^\top$, $v_{50} = \begin{pmatrix} -1.6 & 1.4 \end{pmatrix}^\top$, $v_{60} = \begin{pmatrix} 1.1 & -1.2 \end{pmatrix}^\top$. In Figs. 1 and 2, we have plotted the trajectories of the output vectors z_i, $i = 1, 2, \dots, 6$ of the controlled network. The proposed protocol indeed achieves **z**-output synchronization for the network. Moreover, for $i = 1, 2, \dots, 6$, we compute

$$S_i = \text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top) + \text{tr}(C_{2i}Q_iC_{2i}^\top),$$

and obtain that

$$S_1 = S_4 = 0.6621, \ S_2 = S_5 = 0.4379, \ S_3 = S_6 = 0.3637.$$

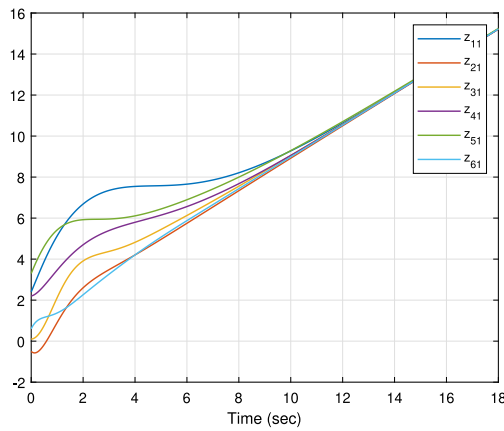Note that, for all $i = 1, 2, \dots, 6$, we have

$$S_i < \frac{\gamma}{N\lambda_N} = 0.75,$$

it then follows from Theorem 4 that the designed protocol is suboptimal, i.e. the associated cost is indeed smaller than the desired tolerance $\gamma = 36$.

7. Conclusions and future work

In this paper, we have studied the $\mathcal{H}_2$ suboptimal output synchronization problem for heterogeneous linear multi-agent

Fig. 1. Plots of trajectories of the first component of the output vectors $z_1, z_2, \ldots, z_6$.
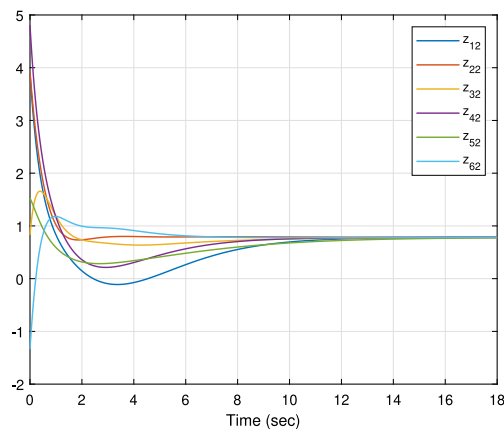
Fig. 2. Plots of trajectories of the second component of the output vectors $z_1, z_2, \ldots, z_6$.

systems. Given a heterogeneous multi-agent system and an associated $\mathcal{H}_2$ cost functional, we have provided a design method for computing dynamic output feedback based protocols that guarantee the associated cost to be smaller than a given upper bound while the controlled network achieves output synchronization. For each agent, its two local control gains are given in terms of solutions of two Riccati inequalities, each of dimension equal to that of the agent dynamics.

As a possibility for future research, we mention the generalization of the results in this paper on fixed directed graphs to that of switching graphs, using, for example, methods from [39] or [40]. It would also be interesting to extend the results in this paper to $\mathcal{H}_\infty$ suboptimal output synchronization.

CRediT authorship contribution statement

Junjie Jiao: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing - original draft, Writing - review & editing. **Harry L. Trentelman:** Formal analysis, Methodology, Writing - original draft, Writing - review & editing, Supervision. **M. Kanat Camlibel:** Formal analysis, Methodology, Writing - original draft, Writing - review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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