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# Pointwise Estimates for Finite Element Approximations of the Stokes Problem and Applications in Optimal Control

Niklas Behringer

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Vorsitzende(r): Prof. Dr. Christian Kühn

Prüfer(innen) der Dissertation:

1. Prof. Dr. Boris Vexler
2. Prof. Dmitriy Leykekhman, Ph.D.
3. Prof. Dr. Karl Kunisch

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**Abstract**

We derive pointwise best-approximation results for the stationary Stokes problem. For the instationary problem, we show approximation error estimates pointwise in time and in $L^2$ in space. To that end, we employ weighted norm and discrete resolvent estimates. We discuss discrete maximal regularity results for the instationary Stokes problem. Furthermore, we present applications of the stationary best-approximation results to optimal control problems with pointwise tracking or sparse control.

**Zusammenfassung**

Wir zeigen punktweise Bestapproximationsraten für das stationäre Stokes Problem. Für das instationäre Problem zeigen wir Approximationsraten punktweise in der Zeit und in $L^2$ im Ort. Dabei setzen wir gewichtete Norm- und diskrete Resolventenabschätzungen ein. Wir diskutieren diskrete maximale Regularität für das instationäre Stokes Problem. Des Weiteren präsentieren wir Anwendungen dieser Ergebnisse im stationären Fall für punktuelle Minimierungs- oder sparse Kontrollprobleme.

# Contents

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# Chapter 1.

# Introduction

How to approach and numerically analyze stationary Stokes optimal control problems with certain sparsity properties? That is one premise of this thesis and to that end, we derive pointwise stability estimates for finite element approximations of the Stokes problem and show applications for sparse and pointwise tracking optimal control problems. Furthermore, we discuss the extension of this to the instationary case.

We consider the following Stokes optimal control problem.

$$\text{Minimize } J(\boldsymbol{u}, \boldsymbol{q}) \text{ subject to} \qquad \begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{q} && \text{in } \Omega, && (1.1.1a)\\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, && (1.1.1b)\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. && (1.1.1c) \end{aligned}$$

For our applications, $J(\boldsymbol{u}, \boldsymbol{q})$ denotes a convex cost functional, consisting of a tracking-type term $\|\boldsymbol{u} - \boldsymbol{u}_d\|$ and a regularization term $\|\boldsymbol{q}\|$. The choice of norms and powers of these terms is left open for the moment. Here $(\boldsymbol{u}, p)$ denotes the state, $\boldsymbol{u}_d$ the desired state or data, and $\boldsymbol{q}$ the control. We assume that the domain $\Omega$ is a two- or three-dimensional polyhedron and denote the dimension by $d$. The problem given by the equations (1.1.1a)–(1.1.1c) is called Stokes problem. In the context of the Stokes problem, $\boldsymbol{u}$ is also called velocity and $p$ pressure. The objective of Problem (1.1.1) is to determine a control $\boldsymbol{q}$ such that $J(\boldsymbol{u}, \boldsymbol{q})$ is minimal for all admissible $\boldsymbol{u}$ and $\boldsymbol{q}$. We will define what that means more precisely later.

Now, why would we be interested in solving optimal control problems governed by the Stokes equation? And how is that related to the solution of sparse problems? Naturally, one can find applications for the examples mentioned above, for instance, simulation of lava flow based on measurement data. But solving the Stokes optimal control problem, the Stokes equation being a linearized version of the more general Navier-Stokes problem, can be seen as an intermediate step to solving Navier-Stokes optimal control problems. After all, to solve the non-linear Navier-Stokes optimal control problem, one has to solve linearized subproblems, e.g., as discussed in [39].

Sparsity for Stokes optimal control problems comes into play when we consider cost functionals $J(\boldsymbol{u}, \boldsymbol{q})$ involving point evaluations, for example, pointwise measurements of $\boldsymbol{u}_d$, or regularization in measure spaces.

The motivation for the numerical analysis of such problems is also driven in part by closely related optimal control problems governed by the Poisson equation, for which we state an example next.

$$\text{Minimize } J(u,\check{q}) \text{ subject to} \qquad -\Delta u = \check{q} \quad \text{in } \Omega, \tag{1.1.2a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.1.2b}$$

Note that, compared to the Stokes case, $u$ and $\check{q}$ are scalar-valued. The problem given by the equations (1.1.2a) and (1.1.2b) is called Poisson problem.

Apart from applications like optimal control of temperature, modifications of Problem (1.1.2) are often used as academic examples for new approaches in mathematical optimal control because of its simple structure and well-known underlying PDE. There are already many results known for Problem (1.1.2) and in this thesis we will see that in some cases similar results can be derived for the Stokes problem (1.1.1).

To see what is potentially possible, we give some examples from the literature regarding problems like Problem (1.1.2), focusing in particular on numerical analysis. There is on the one hand the scenario of imposing additional constraints on the control, e.g., in the form of upper and lower bounds $a \le \check{q} \le b$ for real numbers $a < b$ (cf. [32, 97, 107]), which leads to some kind of a projection formula. And on the other hand, one might consider constraints on the state $u$, e.g., $a \le u \le b$ for real numbers $a < 0 < b$. This is in some sense more challenging than control constraints because the question of the existence of a solution $(\bar{q}, \bar{u})$ to the problem (1.1.2) is less straightforwardly answered. In particular, the case of pointwise state constraints leads to Lagrange multipliers in measure spaces (cf. [25, 26, 42, 96]).

Other variants of Problem (1.1.2) that have been under investigation in the past are using modified cost functionals to control the influence of the data term $u_d$ or the regularity of the solution $\check{q}$. Relevant examples for this work concern regularizing $\check{q}$ in the $L^1$ norm or over the even larger space of regular Borel measures $\mathcal{M}(\Omega)$ in [27, 36, 103, 118, 125]. For the latter one, the cost functional reads

$$J(u,\check{q}) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \alpha\|\check{q}\|_{\mathcal{M}(\Omega)}.$$

Here $\alpha > 0$ denotes the control cost parameter. The second variation of the cost functional, which we would like to mention, is a modification of the norm of $u - u_d$. Depending on the application, one might only consider minimizing the difference pointwise at multiple locations. Here we would like to mention the work done in [10, 18, 22, 33, 69]. The cost functional takes the form

$$J(u,\check{q}) = \frac{1}{2}\sum_{i\in\mathcal{I}}(u(\boldsymbol{x}_i) - \xi_i)^2 + \frac{\alpha}{2}\|\check{q}\|^2_{L^2(\Omega)},$$

where $\{\boldsymbol{x}_i\}_{i\in\mathcal{I}} \subset \Omega$ and $\{\xi_i\}_{i\in\mathcal{I}}$ represent the data at the points $\boldsymbol{x}_i$ for the finite index set $\mathcal{I} \subset \mathbb{N}$.

Note that for the numerical analysis of the state constrained problem, optimization in measure spaces and pointwise tracking, the use of pointwise approximation error estimates is an important ingredient when showing approximation error estimates in the control and state variables. For example, as soon as we work in the measure space, where we consider a measure $\mu$ as the right-hand side of (1.1.2a) and (1.1.2b), we often argue by duality, e.g., using a test function $\varphi \in C_0(\Omega)$, where $C_0(\Omega)$ denotes the space of continuous functions which are zero at the boundary. We can then estimate $\langle \varphi, \mu \rangle \leq \|\varphi\|_{C_0(\Omega)} \|\mu\|_{\mathcal{M}(\Omega)}$, where $\mathcal{M}(\Omega)$ is the space of regular Borel measures on $\Omega$. The norm $\|\varphi\|_{C_0(\Omega)}$ is then equivalent to $\|\varphi\|_{L^\infty(\Omega)}$ for continuous $\varphi$. The pointwise approximation error estimates become relevant if $\varphi$ is chosen to be the difference between a state solution to (1.1.2a) and (1.1.2b) and its finite element solution.

Motivated by those examples, we now turn to the optimal control problem governed by the Stokes equations. Having a different PDE in Problem (1.1.1) is a relatively big step away from the comparably simple world of (1.1.2). The Stokes problem is not fundamentally different from the Poisson problem but certain properties and tools, e.g., harmonic properties, are no longer available.

The questions posed here are: Can we achieve similar results as for the Poisson case? Are there limitations in transferring techniques for the Poisson problem to the Stokes problem? Where do we need new approaches? We want to understand, where the structural differences between Stokes and Poisson problem actually matter. After all, the two partial differential equations are quite similar. The Stokes problem can be seen as an extension of the Poisson problem by an additional constraint on the velocity, the incompressibility condition (1.1.1b), requiring the introduction of a Lagrange multiplier, which we get to know as the pressure. Obviously, there is also the matter of the Poisson equation involving scalar function spaces and the Stokes equations involving vector function spaces. These two points represent some major differences between those two problems and later we will see that in our investigations these two points are the main trouble makers. But let us go back to the question if we can "recycle" ideas from the Poisson case. When analyzing the velocity term, we typically are able to use techniques developed for the Poisson problem without much modification. For dealing with the pressure term, we require a different approach. When possible, this amounts to transforming the pressure term back to the more amiable velocity case, but we will also encounter situations, where adequate estimates of the pressure, even in the $L^2$ norm, are not possible.

Naturally, before we can dive into the analysis of the optimal control problem (1.1.1) we first need to get a grip on the underlying partial differential equation (1.1.1a)–(1.1.1c). In recent years, knowledge has been expanded for the Stokes problem in the numerically interesting setting of convex polyhedral domains. The results, which form the basis of this research, are the publications of Maz'ya and Rossmann [94, 108] in 2010. There the authors give essential Hölder-type regularity estimates for the solution of the Stokes problem on polyhedral domains. Their results led to new research on maximum norm estimates of the gradient of the numerically computed velocity and on the maximum norm of the numerically computed pressure in [65, 72], improving, in particular, previously known results on three-dimensional domains in [64]. Prompted by their results, our research seeks to extend these estimates to maximum norm estimates for the velocity itself (*not* the gradient), since as much as the results by Maz'ya and Rossmann are the basis for the maximum norm estimates, the maximum norm estimates for the numerically computed velocity are the basis for the analysis of optimal control problems with sparse components. We show estimates of the following form in Chapter 3:

$$\|\boldsymbol{u}_h\|_{L^\infty(\Omega)} \le C|\ln h|\Big(\|\boldsymbol{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)}\Big), \tag{1.1.3}$$

as well as respective local versions. Here $(\boldsymbol{u}_h, p_h)$ denotes the solution of the discrete Stokes problem based on finite elements. The result in (1.1.3) allows one to immediately derive pointwise best-approximation error estimates for the finite element discretization of the Stokes problem. It thus solves a problem which also has been discussed for the Poisson equation in [100, 101, 104, 111].

This estimate and its localized version are crucial for tackling versions of Problem (1.1.1) concerned with sparse controls, pointwise tracking or state constraints. We consider a problem sparse if the value of interest has very small or even measure zero support. Typically, the appearing parameters offer little in terms of regularity, e.g., them being only measure-valued. Estimate (1.1.3) is then very helpful in the sense that it allows us to handle those sparse parameters using duality arguments. In particular, since the arising quantities have such small support, the localized version of (1.1.3) is going to be especially helpful, allowing us to sideline interferences, e.g., from non-smooth boundaries.

We first consider the case of the pointwise tracking cost functional in Chapter 4

$$J(\boldsymbol{u}, \boldsymbol{q}) = \frac{1}{2}\sum_{i\in\mathcal{I}}(\boldsymbol{u}(\boldsymbol{x}_i) - \boldsymbol{\xi}_i)^2 + \frac{\alpha}{2}\|\boldsymbol{q}\|^2_{L^2(\Omega)} \tag{1.1.4}$$

for a control $\boldsymbol{q} \in L^2(\Omega)^3$ with box constraints, i.e., every component of $\boldsymbol{q}$ is bounded from below and above. Problem (1.1.4) allows one to model a typical measurement scenario, where we have a system governed by the Stokes equations and want to determine the forcing term on the right-hand side by taking measurements $\{\boldsymbol{\xi}_i\}_{i\in\mathcal{I}}$ of the velocity at the points $\{\boldsymbol{x}_i\}_{i\in\mathcal{I}}$. The sparse behavior of the problem arises here from the point evaluations in the cost functional $J$, which lead to the occurrence of Dirac measures on the right-hand side of the so-called adjoint problem which is used to determine optimality conditions for (1.1.4). We conduct numerical analysis for the three-dimensional case of Problem (1.1.4) and show the following convergence rate for the approximation error:

$$\|\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)} \le C|\ln h|^{1/3}h^{5/6},$$

where $\bar{\boldsymbol{q}}$ is the solution to the continuous optimal control problem and $\bar{\boldsymbol{q}}_h$ is the solution of the discrete optimal control problem, limited to the space of piecewise constant functions. For the so-called variational discretization, we show a convergence rate of $|\ln h|h$. We support our theoretical results with respective numerical experiments.

The second optimal control problem that we consider in Chapter 5 for an application of the pointwise estimate in (1.1.3) is a problem with a measure norm in the regularization term. We assume $\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega)$, the vector space of regular Borel measures, and we consider both the two- and three-dimensional cases. This is realized using the following cost functional:

$$J(\boldsymbol{u}, \boldsymbol{q}) = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_d\|^2_{L^2(\Omega)} + \alpha\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)}.$$

The setting of this problem promotes sparse structures in the control because of the norm $\|\cdot\|_{\mathcal{M}(\Omega)}$ in the cost functional. The computational aspects of this problem are accomplished via a standard finite element discretization in $\boldsymbol{u}_h$ and representation of $\boldsymbol{q}_h$ by Dirac measures on each node of the underlying mesh and respective coefficients. We then give error approximation estimates for $J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}})$ and $J(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h)$:

$$|J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}}) - J(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h)| \leq C|\ln h|^{2+r} h^{4-d},$$

with $r = -1/3$ for $d = 3$ and $r = 1$ in the case $d = 2$. Here $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}_h$ are the optimal state and the respective discrete optimal state. The approximation error for $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}_h$ can be bounded as

$$\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{L^2(\Omega)} \leq C|\ln h|^{1+r/2} h^{2-d/2},$$

with $r$ and $d$ as above. Again we obtain supportive numerical results except for the three-dimensional case, where we actually observe

$$\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{L^2(\Omega)} \leq Ch$$

for $\boldsymbol{u}_d \in L^\infty(\Omega)^3$.

Finally, we discuss the instationary Stokes problem in Chapter 6 which is given as

$$\begin{aligned} \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } I \times \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } I \times \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } I \times \partial\Omega, \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 && \text{in } \Omega, \end{aligned}$$

where $\boldsymbol{u}_0$ denotes the initial state of the problem at time zero. Here we have to deal with the time derivative $\partial_t \boldsymbol{u}$ on the interval $I = [0, \mathscr{T}]$, with $\mathscr{T} > 0$. Similar to before, we can introduce discrete versions of the velocity $\boldsymbol{u}_{\tau h}$ and pressure $p_{\tau h}$ and again we would like to consider sparse optimal control problems. Compared to the stationary case, it is more challenging in this scenario to derive estimates of the kind in (1.1.3).

Following the ideas from the parabolic problem, governed in the spatial part by the Poisson equation, one first seeks to prove the discrete version of maximal regularity for $\boldsymbol{u}_{\tau h}$, e.g., an estimate of the form:

$$\left( \sum_{m=1}^{M} \|\partial_t \boldsymbol{u}_{\tau h}\|_{L^s(I_m; L^p(\Omega))} \right)^{1/s} + \|\Delta_h \boldsymbol{u}_{\tau h}\|_{L^s(I; L^p(\Omega))} \leq C\|\boldsymbol{f}\|_{L^s(I; L^p(\Omega))}, \tag{1.1.6}$$

where $\{I_m\}_{1 \leq m \leq M}$ is a subdivision of $I$ and $\Delta_h$ is the discrete Laplace operator. Estimate (1.1.6) is a short version of the estimate we discuss in Chapter 6. The full estimate also involves terms including the Stokes operator and jump terms, for whose precise definition we refer to Chapter 6. Estimate (1.1.6) is important since it again allows us to give approximation error estimates in the Bochner space $L^s(I; L^p(\Omega))$.

To arrive at an estimate as (1.1.6), one considers the resolvent problem

$$z\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1.7a}$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1.7b}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{1.1.7c}$$

where $z$ is a complex number, $\boldsymbol{f}$ a suitable vector-valued function, and the function spaces for $\boldsymbol{u}$ and $p$ are respectively adapted to the complex case. While recently successful approaches have been made to tackle estimates for (1.1.7a)–(1.1.7c) even in $L^\infty(\Omega)$ in [2], a bound for the discrete resolvent problem is still an open problem.

Compared to resolvent estimates for the Poisson problem, the obstacles in the way of deriving respective estimates for the Stokes problem seem to be related to the Dirichlet boundary conditions and the divergence constraint (1.1.7b). In particular it turns out to be challenging to provide a bound for the pressure term in $L^2(\Omega)$ (in the two-dimensional case). One would hope for an estimate of the kind

$$\|p\|_{L^2(\Omega)} \leq \|\boldsymbol{f}\|_{H^{-1}(\Omega)},$$

but it has been shown in [123] that such an estimate does not necessarily hold even for domains with very smooth boundaries.

We discuss the problems that arise when trying to derive estimates for the discrete resolvent problem and give discrete maximal regularity results in $L^\infty(I; L^2(\Omega))$ and semi-discrete results in time in $L^s(I; L^p(\Omega))$. This then gives rise to approximation error estimates of the following form in the fully discrete case:

$$\|\boldsymbol{u} - \boldsymbol{u}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \leq C \ln \frac{\mathscr{T}}{\tau} \Big(\tau \|\boldsymbol{u}\|_{W^{1,\infty}(I;L^2(\Omega))} + h^2 \Big(\|\boldsymbol{u}\|_{L^\infty(I;H^2(\Omega))} + \|p\|_{L^\infty(I;H^1(\Omega))}\Big)\Big).$$

We compare this result with results from the available literature.

Beyond the analysis above, we discuss technical difficulties involving the best-approximation result from Chapter 3 in Appendix B and an extension of the results from Chapter 3 to the *mini* element in Appendix A.

The overview above reflects the presentation of the material in the following chapters. In each chapter we treat a specific problem including a more detailed survey of the literature on the topic and, if relevant, also discuss numerical experiments.

We note that some of the material has already been published. In particular, this concerns:

- Chapter 3 based on [18].
- Chapter 4 based on [16].

Next, we introduce basic notation and recall relevant results from the literature.

# Chapter 2.

# Notation, regularity results, and finite element discretization for the Stokes problem

In this section we collect some notation and results which are used throughout the thesis to avoid redundancies. Notation and results which are only required for some particular application are introduced in the respective chapter.

In the following, the constant $C$ is a non-negative real number, independent of the parameters it appears with, e.g., the mesh size $h$, the dimension $d$ etc., unless explicitly stated otherwise. We denote vectors or tensors with bold symbols (e.g., $\boldsymbol{u}$).

We try to follow the commonly used conventions and definitions from the literature on numerical analysis and optimal control. In particular, we use the usual notation for the Lebesgue, Sobolev, and Hölder spaces.

## 2.1. Spaces and domain

Let $\Omega$ be a bounded open Lipschitz domain in $\mathbb{R}^d$ (unless stated otherwise), with $d \in \{2, 3\}$ the dimension. We denote the boundary of $\Omega$ by $\partial\Omega$. For the most part $\Omega$ will be also a convex polygonal or polyhedral domain. Here $C^{n,\zeta}$ is the space of $n$ times $\zeta$ Hölder differentiable functions. We define $n$ as a non-negative integer and $\zeta \in (0, 1]$.

The symbols $L^s(\Omega)$, $W^{n,s}(\Omega)$, and $H^n(\Omega)$ denote the usual Lebesgue and Sobolev spaces with $1 \leq s \leq \infty$ and $s'$ its Hölder conjugate. Then, $W_0^{1,s}(\Omega)$ and $H_0^1(\Omega)$ are the respective Sobolev spaces consisting of functions which are zero on the boundary. A precise definition of these spaces can be found in [5, Section 3.27]. $L_0^s(\Omega)$ denotes the functions in $L^s(\Omega)$ with mean value zero.

With $W^{-1,s'}(\Omega)$ (or $H^{-1}(\Omega)$) we denote the dual space of $W_0^{1,s}(\Omega)$ (or $H_0^1(\Omega)$).

These spaces can be extended in a straightforward manner to vector functions, with the same notation but with the following modification for the norm in the non-Hilbert case: if $\boldsymbol{u} = (u_1, u_2, u_3)$, we then set

$$\|\boldsymbol{u}\|_{L^r(\Omega)} = \left[\int_\Omega |\boldsymbol{u}(\boldsymbol{x})|^r d\boldsymbol{x}\right]^{1/r},$$

where $|\cdot|$ denotes the Euclidean vector norm for vectors or the Frobenius norm for tensors. We write vector-valued function spaces with a superscript, e.g., $L^2(\Omega)^d$.

We denote by $(\cdot\,,\cdot)$ the $L^2(\Omega)$ inner product and specify subdomains by subscripts in the case they are not equal to the whole domain. In case of Banach space duality products, we write $\langle\cdot\,,\cdot\rangle$, where we specify the respective spaces in the subscript if they are not immediately clear from the context.

## 2.2. Weak form and regularity results

Next, we recall some regularity results for solutions to the stationary Stokes problem. We state the results for an auxiliary problem, using the notation $(\boldsymbol{w}, \phi)$.

Existence and uniqueness of solutions to the weak formulation of the Stokes problem on bounded domains are shown, for example, in [58, Theorem IV.1.1] for $\boldsymbol{f} \in H^{-1}(\Omega)^d$. For $\boldsymbol{f}$ with even less regularity we provide an existence result below. The regularity results for polyhedral domains stated next can be found, e.g., in [94, Chapter 11], [38] and [79]. For $\boldsymbol{f} \in W^{-1,s}(\Omega)^d$, let $(\boldsymbol{w}, \varphi) \in W_0^{1,s}(\Omega)^d \times L_0^s(\Omega)$ solve

$$a((\boldsymbol{w}, \varphi), (\boldsymbol{v}, l)) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall (\boldsymbol{v}, l) \in W_0^{1,s'}(\Omega)^d \times L^{s'}(\Omega)/\mathbb{R}, \tag{2.2.1}$$

where

$$a((\boldsymbol{w}, \varphi), (\boldsymbol{v}, l)) = \langle \nabla \boldsymbol{w}, \nabla \boldsymbol{v}\rangle - \langle \varphi, \nabla \cdot \boldsymbol{v}\rangle + \langle \nabla \cdot \boldsymbol{w}, l\rangle. \tag{2.2.2}$$

Note that we choose $\varphi$ to have zero mean. Then, there holds for $\boldsymbol{f} \in L^2(\Omega)^d$

$$\|\boldsymbol{w}\|_{H^2(\Omega)} + \|\varphi\|_{H^1(\Omega)} \leq C\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{2.2.3}$$

In the two-dimensional case we have by [68, Theorem 7.3.3.1, Lemma 7.3.2.4] that $(\boldsymbol{w}, \varphi) \in W^{2,2+\varepsilon}(\Omega)^2 \times W^{1,2+\varepsilon}(\Omega)$ for $\varepsilon > 0$ and $\boldsymbol{f} \in L^{2+\varepsilon}(\Omega)^2$ which allows for an embedding of $(\boldsymbol{w}, \varphi) \in C^{1,\zeta}(\Omega)^2 \times C^{0,\zeta}(\Omega)$ because of Morrey's inequality (cf. [5, Section 10.13]).

*Remark* 2.1 The results in [68, Theorem 7.3.3.1, Lemma 7.3.2.4] show that for a domain $\Omega$ with interior angles $\omega_i$ such that $0 < \omega_i < \pi - \varepsilon$ (as it is the case for convex domains) there holds $(\boldsymbol{w}, \varphi) \in W^{2,s}(\Omega)^3 \times W^{1,s}(\Omega)$ for $s > 2$. This follows from the fact that the roots $\lambda_{j,i}$ of

$$\sinh^2(\lambda\omega_i) = \lambda^2 \sin^2(\omega_i),$$

an identity which characterizes the corner singularities on $\Omega$, have imaginary part $|\mathfrak{Im}(\lambda_{j,i})| \geq 1 + C(\varepsilon)$ (cf. proof of [68, Lemma 7.3.2.4]) and thus none of $\mathfrak{Im}(\lambda_{j,i})$ lie on $-2/s'$ (cf. [68, Theorem 7.3.3.1]) for $s$ close to two.

A similar argument does not hold in three dimensions without restrictions on the angles of $\Omega$ beyond convexity but similar results are shown in [94, Chapter 11]. The authors show an estimate in weighted Sobolev spaces in [94, Theorem 11.1.4] and in weighted Hölder spaces in [94, Theorem 11.1.7] for the Stokes problem. Then, they show in [94, Theorem 11.3.1, 11.3.2] (this time for the nonlinear Navier-Stokes Problem) how to use the weighted results to derive estimates in the unweighted spaces.

The argument for the Navier-Stokes problem extends to the Stokes problem we discuss here, such that for $2 < s < \infty$, $\boldsymbol{f} \in W^{-1,s}(\Omega)^3$ we have that $(\boldsymbol{w}, \varphi)$ are elements of $W_0^{1,s}(\Omega)^3 \times L_0^s(\Omega)$ and it holds

$$\|\boldsymbol{w}\|_{W^{1,s}(\Omega)} + \|\varphi\|_{L^s(\Omega)} \leq C\|\boldsymbol{f}\|_{W^{-1,s}(\Omega)}. \tag{2.2.4}$$

For $\boldsymbol{f} \in L^{3+\varepsilon}(\Omega)^3$ and $\zeta \in (0,1)$, depending on the largest interior angle of the domain, we have

$$\|\boldsymbol{w}\|_{C^{1,\zeta}(\Omega)} + \|\varphi\|_{C^{0,\zeta}(\Omega)} \leq C\|\boldsymbol{f}\|_{L^{3+\varepsilon}(\Omega)}. \tag{2.2.5}$$

A discussion of this result, in particular how to apply it based on the requirements stated in [93] (also cf. [94, Theorem 11.1.7] for the respective weighted space estimate), can be found below [65, Theorem 3].

Furthermore, we need to discuss the case of singular-valued right-hand sides $\boldsymbol{f}$, i.e., $\boldsymbol{f} \in W^{-1,s}(\Omega)^d$ for $1 < s < 2$, which includes the situation where $\boldsymbol{f}$ is measure-valued. This case is actually not covered by the existence results above.

One possible approach to establish existence is to argue via a dual problem, using the concept of very weak solutions $(\boldsymbol{w}, \varphi) \in L^2(\Omega)^d \times H^{-1}(\Omega)$, which has been done in [24, 25] for a general elliptic problem and for the Stokes problem on smooth domains in [7, 8, 9]. This approach would lead to existence of a solution $(\boldsymbol{w}, \varphi) \in L^2(\Omega)^d \times H^{-1}(\Omega)$.

Fortunately we can resort here to results on Lipschitz domains for which respective problems have been discussed in [23, Theorem 2.9] for $d = 3$ and more recently, covering $d \geq 2$, in [98, Corollary 1.7] for parameters $\alpha = -1$ and $q = 2$, which are parameters for the weighted spaces used in [98]. Referring to [98, (1.52)] one has for $\boldsymbol{f} \in W^{-1,s}(\Omega)^d$ with $2d/(d+1) - \varepsilon < s < 2d/(d-1) + \varepsilon$ that there is a well-defined solution operator such that

$$\|\boldsymbol{w}\|_{W^{1,s}(\Omega)} + \|\varphi\|_{L^s(\Omega)} \leq C\|\boldsymbol{f}\|_{W^{-1,s}(\Omega)}. \tag{2.2.6}$$

Finally, we need to discuss interior regularity results, since we will encounter problems for which certain parameters are supported only away from the boundary, allowing us to work with stronger regularity estimates. In the following, we denote by $\Omega_1 \Subset \Omega_2$ that a domain $\Omega_1$ is contained in $\Omega_2$ and $\operatorname{dist}(\Omega_1, \partial\Omega_2) > \varrho > 0$.

**Proposition 2.2** (Interior regularity for Stokes) *Let* $(\boldsymbol{w}, \varphi) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ *solve*

$$\begin{aligned} -\Delta \boldsymbol{w} + \nabla\varphi &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{w} &= 0 && \text{in } \Omega, \\ \boldsymbol{w} &= \boldsymbol{0} && \text{on } \partial\Omega, \end{aligned}$$

*with $\boldsymbol{f} \in L^\infty(\Omega)^d$. Then we get for $\Omega_1 \Subset \Omega_2 \Subset \Omega$ the following semi-norm estimate*

$$|\boldsymbol{w}|_{W^{2,s}(\Omega_1)} + |\varphi|_{W^{1,s}(\Omega_1)} \leq Cs\|\boldsymbol{f}\|_{L^\infty(\Omega)} \quad \text{for all } 1 < s < \infty,$$

*with $C$ independent of $s$.*

*Proof.* The proposition, for the most part, is already available in [58, Theorem IV.4.1] with

$$|\boldsymbol{w}|_{W^{2,s}(\Omega_1)} + |\varphi|_{W^{1,s}(\Omega_1)} \leq C\Big(\|\boldsymbol{f}\|_{L^s(\Omega_2)} + \|\boldsymbol{w}\|_{W^{1,s}(\Omega_2\setminus\Omega_1)} + \|\varphi\|_{L^s(\Omega_2\setminus\Omega_1)}\Big). \tag{2.2.7}$$

Now, because of (2.2.5) we can bound $\|\boldsymbol{w}\|_{W^{1,s}(\Omega_1\setminus\Omega_2)} + \|\varphi\|_{L^s(\Omega_1\setminus\Omega_2)}$ for $s \to \infty$ by $\|\boldsymbol{f}\|_{L^\infty(\Omega)}$ and obviously $\|\boldsymbol{f}\|_{L^s(\Omega_1)}$ is as well bounded by $\|\boldsymbol{f}\|_{L^\infty(\Omega)}$, such that we get

$$|\boldsymbol{w}|_{W^{2,s}(\Omega_1)} + |\varphi|_{W^{1,s}(\Omega_1)} \leq C\|\boldsymbol{f}\|_{L^\infty(\Omega)}.$$

It remains to trace the dependency on $s$ of the constant $C$. Starting from [58, Theorem IV.4.1] we can trace the constant over [58, Theorem IV.2.1] to [58, Theorem II.11.4] and [58, Remark II.11.2] to the form stated in the theorem. □

## 2.3. Finite element discretization of the stationary Stokes problem

In the following, we follow the definitions and statements made in [66]. Let $\mathcal{T}_h$ be a regular, quasi-uniform family of triangulations, partitioning $\bar{\Omega}$, consisting of closed triangles $T$ in two dimensions or tetrahedra $T$ in three dimensions. We denote by $h = \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$ the global mesh-size. Let $\boldsymbol{V}_h \subset H_0^1(\Omega)^d$ and $M_h \subset L_0^2(\Omega)$ be a pair of finite element spaces satisfying a uniform discrete inf-sup condition,

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \frac{(q_h, \nabla\cdot\boldsymbol{v}_h)}{\|\nabla\boldsymbol{v}_h\|_{L^2(\Omega)}} \geq \tilde{\beta}\|q_h\|_{L^2(\Omega)} \quad \forall q_h \in M_h, \tag{2.3.1}$$

with a constant $\tilde{\beta} > 0$ independent of $h$. Note that this is equivalent to the existence of a Fortin projection $P_h\colon H_0^1(\Omega)^d \to \boldsymbol{V}_h$ (cf. [66, Lemma 1.1]) that preserves the divergence with respect to the discrete finite element space. For details on this operator we refer to the assumptions in the next chapter.

The respective discrete solution associated with the velocity-pressure pair $(\boldsymbol{w}, \varphi) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ is defined as the pair $(\boldsymbol{w}_h, \varphi_h) \in \boldsymbol{V}_h \times M_h$ that solves the weak formulation (2.2.1) for respective discrete solution and test spaces $\boldsymbol{V}_h \times M_h$:

$$a((\boldsymbol{w}_h, \varphi_h), (\boldsymbol{v}_h, q_h)) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h. \tag{2.3.2}$$

Throughout this thesis we will work for the most part with Taylor-Hood finite elements (cf. [119]) since the method satisfies certain requirements on the finite element space which we will introduce in the next chapter. In Appendix A we will discuss how the results for the Taylor-Hood finite elements can be extended to the mini element.

### 2.3.1. Taylor-Hood finite elements

We give now some details on the Taylor-Hood finite elements based on the definitions in [66, Section 4.2]. Here one chooses

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in C(\bar{\Omega})^d : \boldsymbol{v}|_T \in \mathcal{P}_k(T)^d \quad \forall T \in \mathcal{T}_h, \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\},$$

where $\mathcal{P}_k(T)^d$ is the space of polynomials of degree $k > 1$ on $T$. And for the pressure space one chooses

$$M_h = \{q \in C(\bar{\Omega}) : q|_T \in \mathcal{P}_{k-1}(T) \quad \forall T \in \mathcal{T}_h\} \cap L^2_0(\Omega).$$

The inf-sup condition can then be proven by partitioning $\Omega$ into macro elements consisting of multiple cells $T$. To that end, one assumes that $\mathcal{T}_h$ has an appropriate set of interior nodes (cf. [66, (4.17)], [67]). Since typically a mesh is constructed via repeated mesh refinements, this is not a strong restriction.

Finally, note that the Taylor-Hood finite elements are very similar to the Lagrange elements used for elliptic problems, which allows us to extend some properties of the Lagrange finite element solution to Taylor-Hood finite elements in Chapter 3.

# Chapter 3.

# Global and local pointwise approximation error estimates

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The following text is the result of collaborative work with Dmitriy Leykekhman and Boris Vexler.

## 3.1. Introduction

In the introduction and the major part of this chapter we focus on the three-dimensional setting. However, our results are also valid in two dimensions and we comment on that at the end of the chapter. We assume $\Omega \subset \mathbb{R}^3$ is a convex polyhedral domain, on which we consider the following Stokes problem:

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{3.1.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{3.1.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{3.1.1c}$$

with $\boldsymbol{f} = (f_1, f_2, f_3)$ such that $\boldsymbol{u} \in (H_0^1(\Omega) \cap L^\infty(\Omega))^3$ for the pointwise error estimates or respectively $\boldsymbol{u} \in (H_0^1(\Omega) \cap W^{1,\infty}(\Omega))^3$ and $p \in L^\infty(\Omega)$ for the gradient error estimates. The solution $p$ is unique up to a constant, we choose $p \in L_0^2(\Omega)$, i.e., $p$ has zero mean.

In this chapter, we give a new $L^\infty$ stability result of the form

$$\|\boldsymbol{u}_h\|_{L^\infty(\Omega)} \le C|\ln h|\Big(\|\boldsymbol{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)}\Big). \tag{3.1.2}$$

In a second step we prove respective local versions of (3.1.2) and of the corresponding $W^{1,\infty}$ results from [65, 72]. These estimates take the form

$$\|\nabla \boldsymbol{u}_h\|_{L^\infty(D_1)} + \|p_h\|_{L^\infty(D_1)} \\ \leq C\left(\|\nabla \boldsymbol{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\right) + C_\varrho\left(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\right)$$

and

$$\|\boldsymbol{u}_h\|_{L^\infty(D_1)} \leq C|\ln h|\left(\|\boldsymbol{u}\|_{L^\infty(D_2)} + h\|p\|_{L^\infty(D_2)}\right) \\ + C_\varrho|\ln h|\left(\|\boldsymbol{u}\|_{L^2(\Omega)} + h\|\boldsymbol{u}\|_{H^1(\Omega)} + h\|p\|_{L^2(\Omega)}\right),$$

where $\tilde{\boldsymbol{x}} \in \Omega$, $D_1 = B_r(\tilde{\boldsymbol{x}}) \cap \Omega$, $D_2 = B_{\tilde{r}}(\tilde{\boldsymbol{x}}) \cap \Omega$, $\tilde{r} > r > 0$, and $C_\varrho$ depends on $\varrho = |r - \tilde{r}| > \bar{\kappa} h$, with $\bar{\kappa}$ being a fixed positive number.

Global pointwise error estimates for the Stokes system similar to (3.1.2) have been thoroughly discussed in recent years. The three-dimensional $W^{1,\infty}$ case was first discussed in [34, 64] under smoothness assumptions on the domain or limiting angles in non-smooth domains. Later on, using new results on convex polyhedral domains, e.g., from [92, 94, 108], the limitations on the domain were weakened in [65, 72]. The $L^\infty$ bounds were first discussed for $\Omega \subset \mathbb{R}^2$ in [47] and for dimensions greater than one and smooth domains in [34] but with the $W^{1,\infty}$ norm appearing on the right-hand side and using weighted norms, which is not sufficient for the applications we have in mind.

Interior (or local) maximum norm estimates are well-known for elliptic equations, see, e.g., [84, 111], and are particularly useful when dealing with scenarios, where the solution has low regularity close to the boundary or on local subsets of $\Omega$, e.g., for optimal control problems with pointwise state constraints, sparse optimal control, and pointwise best-approximation results for the time dependent problem, see [40, 86, 103]. For the Stokes system, the only pointwise interior error estimates are available on regular translation invariant meshes in two dimensions in [99]. To our best knowledge, the interior results presented here are novel and have not been discussed before.

We want to point out that there are some differences between our local results and the classical results of Schatz and Wahlbin [111, 112] for elliptic problems. There, the pollution terms are still in the discrete (or error) form, but in a weaker norm and still local. In our results, the pollution terms are in continuous (or approximation form), global, but in a weaker norm and valid all the way to the boundary. Although the pollution terms in the estimates of Schatz and Wahlbin appear to be sharper, they are much more technical to obtain and we see no apparent benefits for potential applications. Such pollution terms still need to be estimated, usually by a global duality argument.

Let us quickly comment on one property specific to the Stokes problem. Regularity results typically appear as combined estimates for the velocity-pressure pair, where the pressure has weaker norm, e.g., $\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)}$ and $\|p\|_{L^\infty(\Omega)}$. This pair can then be estimated as in [65, 72]. Thus, we only supply estimates for $\|\boldsymbol{u}_h\|_{L^\infty(\Omega)}$ in the max-norm estimate since bounds for $\|p_h\|_{W^{-1,\infty}(\Omega)}$ would add another layer of complexity and to our knowledge have no apparent advantages.

In three dimensions our proof of the local estimates is essentially based on $L^1$ and weighted estimates of regularized Green's functions. For $W^{1,\infty}$ it is enough to slightly adapt the results from [72] for the regularized Green's function of velocity and pressure.

In the case of $L^\infty$, we prove the respective estimates using the local energy estimates given in [72] and estimates for Green's matrix of the Stokes system, see, e.g., [94]. Furthermore, another important element of the proof for $L^\infty$ is a pointwise estimate of the Ritz projection (cf. [83, 85]). The stability results proven there significantly simplify the analysis. Thus, we avoid a technical step of integrating by parts over each element and dealing with jump terms as it was done in [85].

In two dimensions our approach for the local estimates follows the lines of the three-dimensional case. Here the estimates for the regularized Green's functions and the Ritz projection are all known from the literature, see [47, 64, 109]. The results from [47, 64] are derived using an alternative technique, the global weighted approach as introduced in [100, 104]. For the global weighted approach we need similar but slightly different assumptions on the finite element space than for the local energy estimate technique in the three-dimensional setting. Thus, to keep the notation simple, we deal with the two-dimensional case in a separate section at the end of this work.

Several important applications from Navier-Stokes free surface flows to the numerical analysis of finite element schemes for non-Newtonian flows have already been noted in [64]. As mentioned, interior estimates play a role specifically for optimal control problems with state constraints, e.g., in [40]. Stokes optimal control problems are also closely related to subproblems in optimal control of Navier-Stokes systems, where for Newton iterations one has to solve linearized optimal control subproblems in each step, see, e.g., [39].

An outline of this chapter is as follows. In Section 3.2, we introduce some basic results and state assumptions on the approximation operators as well as the main results of our analysis. Section 3.3 gives key arguments for the proof of the main theorems for the velocity and reduces them to the estimates of regularized Green's functions, which are derived in Section 3.4. Based on these results, we deal with bounds for the pressure in Section 3.5. Finally, in the last section we show the local estimates in two dimensions.

## 3.2. Assumptions and main results in three dimensions

In the analysis, we make use of the weight $\sigma = \sigma_{\boldsymbol{x}_0,h}(\boldsymbol{x}) = \sqrt{|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (\kappa h)^2}$, for which $\boldsymbol{x}_0$ and $\kappa$ will be defined later on.

### 3.2.1. Basic estimates

Next, we prove or respectively recall some basic results, which are essential for our analysis.

### Local $H^2$ stability estimates

In the following analysis we require the following localized $H^2$ stability estimates. By $B_r(\boldsymbol{x})$ we denote the open ball with radius $r$ and center $\boldsymbol{x}$.

**Lemma 3.1** *Let $A_1 = B_r(\tilde{\boldsymbol{x}}) \cap \Omega$, $A_2 = B_{\tilde{r}}(\tilde{\boldsymbol{x}}) \cap \Omega$ for $\tilde{\boldsymbol{x}} \in \Omega$, and $\tilde{r} > r > 0$. We denote the difference of the radii by $\varrho = |\tilde{r} - r|$. Furthermore, let $(\boldsymbol{u}, p) \in H^2(\Omega)^3 \cap H_0^1(\Omega)^3 \times H^1(\Omega) \cap L_0^2(\Omega)$ be the solution to* (3.1.1a)–(3.1.1c) *for $\boldsymbol{f} \in L^2(\Omega)^3$. Then, it holds*

$$\|\boldsymbol{u}\|_{H^2(A_1)} + \|p\|_{H^1(A_1)} \leq C\Big(\|\boldsymbol{f}\|_{L^2(A_2)} + \frac{1}{\varrho}\|\nabla \boldsymbol{u}\|_{L^2(A_2)} + \frac{1}{\varrho^2}\|\boldsymbol{u}\|_{L^2(A_2)} + \frac{1}{\varrho}\|p\|_{L^2(A_2)}\Big).$$

*Proof.* Let $\omega \in C^\infty(\Omega)$ be a smooth cut-off function with $\omega = 1$ on $A_1$ and $\omega = 0$ on $\Omega \backslash A_2$ such that

$$|\nabla^k \omega| \sim \frac{1}{\varrho^k} \quad \text{for } k = 0, 1, 2. \tag{3.2.1}$$

We consider $\tilde{\boldsymbol{u}} = \omega \boldsymbol{u}$ and $\tilde{p} = \omega p$. Then, we get the following weak formulation for $\boldsymbol{\varphi} \in H_0^1(\Omega)^3$

$$\begin{aligned}
(\nabla \tilde{\boldsymbol{u}}, \nabla \boldsymbol{\varphi}) &= (\nabla \omega \otimes \boldsymbol{u} + \omega \nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) \\
&= -(\nabla \cdot (\nabla \omega \otimes \boldsymbol{u}), \boldsymbol{\varphi}) + (\nabla \boldsymbol{u}, \nabla(\omega \boldsymbol{\varphi})) - (\nabla \boldsymbol{u}, \nabla \omega \otimes \boldsymbol{\varphi}) \\
&= -(\nabla \cdot (\nabla \omega \otimes \boldsymbol{u}), \boldsymbol{\varphi}) + (\omega \boldsymbol{f}, \boldsymbol{\varphi}) + (p, \nabla \cdot (\omega \boldsymbol{\varphi})) - (\nabla \boldsymbol{u}, \nabla \omega \otimes \boldsymbol{\varphi}) \\
&= -(\nabla \cdot (\nabla \omega \otimes \boldsymbol{u}), \boldsymbol{\varphi}) + (\omega \boldsymbol{f}, \boldsymbol{\varphi}) + (\omega p, \nabla \cdot \boldsymbol{\varphi}) + (\nabla \omega p, \boldsymbol{\varphi}) - (\nabla \boldsymbol{u} \nabla \omega, \boldsymbol{\varphi}),
\end{aligned}$$

where we used (3.1.1a) and in addition we get $\nabla \cdot \tilde{\boldsymbol{u}} = \nabla \omega \cdot \boldsymbol{u}$. Thus, $\tilde{\boldsymbol{u}}$ and $\tilde{p}$ solve the following boundary value problem in the weak sense

$$\begin{aligned}
-\Delta \tilde{\boldsymbol{u}} + \nabla \tilde{p} &= \omega \boldsymbol{f} - \nabla \cdot (\nabla \omega \otimes \boldsymbol{u}) + \nabla \omega p - \nabla \boldsymbol{u} \nabla \omega && \text{in } \Omega, \\
\nabla \cdot \tilde{\boldsymbol{u}} &= \nabla \omega \cdot \boldsymbol{u} && \text{in } \Omega, \\
\tilde{\boldsymbol{u}} &= \boldsymbol{0} && \text{on } \partial\Omega.
\end{aligned}$$

Thus, according to [38, Theorem 9.20] and the fact that $\nabla \cdot \tilde{\boldsymbol{u}}$ is zero on $\partial\Omega$, the $H^2(\Omega)$ regularity result (2.2.3) holds in this situation as well, and we obtain

$$\begin{aligned}
\|\tilde{\boldsymbol{u}}\|_{H^2(\Omega)} &+ \|\tilde{p}\|_{H^1(\Omega)} \\
&\leq C\Big(\|\omega \boldsymbol{f}\|_{L^2(\Omega)} + \|\nabla \omega \nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla^2 \omega \boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla \omega p\|_{L^2(\Omega)}\Big) \\
&\leq C\Big(\|\boldsymbol{f}\|_{L^2(A_2)} + \frac{1}{\varrho}\|\nabla \boldsymbol{u}\|_{L^2(A_2)} + \frac{1}{\varrho^2}\|\boldsymbol{u}\|_{L^2(A_2)} + \frac{1}{\varrho}\|p\|_{L^2(A_2)}\Big),
\end{aligned}$$

where we used (3.2.1). Hence,

$$\begin{aligned}
\|\boldsymbol{u}\|_{H^2(A_1)} + \|p\|_{H^1(A_1)} &= \|\tilde{\boldsymbol{u}}\|_{H^2(A_1)} + \|\tilde{p}\|_{H^1(A_1)} \leq \|\tilde{\boldsymbol{u}}\|_{H^2(\Omega)} + \|\tilde{p}\|_{H^1(\Omega)} \\
&\leq C\left(\|\boldsymbol{f}\|_{L^2(A_2)} + \frac{1}{\varrho}\|\nabla \boldsymbol{u}\|_{L^2(A_2)} + \frac{1}{\varrho^2}\|\boldsymbol{u}\|_{L^2(A_2)} + \frac{1}{\varrho}\|p\|_{L^2(A_2)}\right).
\end{aligned}$$

□

Using a covering argument, we may show the following corollary.

**Corollary 3.2** *Let* $\Omega_1 \subset \Omega_2 \subset \Omega$ *with* $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho$*, then it holds for* $(\boldsymbol{u}, p)$ *the solution to* (3.1.1a)–(3.1.1c) *that*

$$\|\boldsymbol{u}\|_{H^2(\Omega_1)} + \|p\|_{H^1(\Omega_1)} \leq C\Big(\|\boldsymbol{f}\|_{L^2(\Omega_2)} + \frac{1}{\varrho}\|\nabla\boldsymbol{u}\|_{L^2(\Omega_2)} + \frac{1}{\varrho^2}\|\boldsymbol{u}\|_{L^2(\Omega_2)} + \frac{1}{\varrho}\|p\|_{L^2(\Omega_2)}\Big).$$

*Proof.* We can construct a covering $\{K_i\}_{i=1}^M$ of $\Omega_1$, with $K_i = B_{\varrho/2}(\tilde{\boldsymbol{x}}_i) \cap \Omega$ such that

(1) $\Omega_1 \subset \bigcup_{i=1}^M K_i$.

(2) $\tilde{\boldsymbol{x}}_i \in \bar{\Omega}_1$ for $1 \leq i \leq M$.

(3) Let $L_i = B_\varrho(\tilde{\boldsymbol{x}}_i) \cap \Omega$. There exists a fixed number $N$ such that each point $\boldsymbol{x} \in \bigcup_{i=1}^M L_i$ is contained in at most $N$ sets from $\{L_j\}_{j=1}^M$.

Now, due to $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho$ and (2), we have that $\bigcup_{i=1}^M L_i \subset \Omega_2$. We can apply Lemma 3.1 to the pairs $K_i \subset L_i$:

$$\begin{aligned}
\|\boldsymbol{u}\|_{H^2(\Omega_1)} + \|p\|_{H^1(\Omega_1)} &\leq \sum_{i=1}^M \|\boldsymbol{u}\|_{H^2(K_i)} + \|p\|_{H^1(K_i)} \\
&\leq \sum_{i=1}^M C\Big(\|\boldsymbol{f}\|_{L^2(L_i)} + \frac{1}{\varrho}\|\nabla\boldsymbol{u}\|_{L^2(L_i)} + \frac{1}{\varrho^2}\|\boldsymbol{u}\|_{L^2(L_i)} + \frac{1}{\varrho}\|p\|_{L^2(L_i)}\Big) \\
&\leq NC\Big(\|\boldsymbol{f}\|_{L^2(\Omega_2)} + \frac{1}{\varrho}\|\nabla\boldsymbol{u}\|_{L^2(\Omega_2)} + \frac{1}{\varrho^2}\|\boldsymbol{u}\|_{L^2(\Omega_2)} + \frac{1}{\varrho}\|p\|_{L^2(\Omega_2)}\Big),
\end{aligned}$$

where we used (3) in the third line. □

### Green's matrix estimate

We also need estimates for the respective Green's matrix for the Stokes problem. For this, we refer to [94, Section 11.5]. Let $\phi \in C^\infty(\bar{\Omega})$ be vanishing in a neighborhood of the edges and $\int_\Omega \phi(\boldsymbol{x})d\boldsymbol{x} = 1$. The matrix $G(\boldsymbol{x}, \boldsymbol{y}) = (G_{i,j}(\boldsymbol{x}, \boldsymbol{y}))_{i,j=1,2,3,4}$ is the Green's matrix for problem (3.1.1a)–(3.1.1c) if the vector functions $\boldsymbol{G}_j = (G_{1,j}, G_{2,j}, G_{3,j})^T$ and $G_{4,j}$ are solutions of the problem

$$\begin{aligned}
-\Delta_{\boldsymbol{x}}\boldsymbol{G}_j(\boldsymbol{x}, \boldsymbol{y}) + \nabla_{\boldsymbol{x}}G_{4,j}(\boldsymbol{x}, \boldsymbol{y}) &= \delta(\boldsymbol{x} - \boldsymbol{y})(\eta_{1,j}, \eta_{2,j}, \eta_{3,j})^t && \text{for } \boldsymbol{x}, \boldsymbol{y} \in \Omega, \\
-\nabla_{\boldsymbol{x}} \cdot \boldsymbol{G}_j(\boldsymbol{x}, \boldsymbol{y}) &= (\delta(\boldsymbol{x} - \boldsymbol{y}) - \phi(\boldsymbol{x}))\eta_{4,j} && \text{for } \boldsymbol{x}, \boldsymbol{y} \in \Omega, \\
\boldsymbol{G}_j(\boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{0} && \text{for } \boldsymbol{x} \in \partial\Omega, \boldsymbol{y} \in \Omega,
\end{aligned}$$

where $\delta$ denotes the Dirac delta and $\eta_{i,j}$ is the Kronecker symbol. In addition, $G_{4,j}$ satisfies the condition

$$\int_\Omega G_{4,j}(\boldsymbol{x}, \boldsymbol{y})\phi(\boldsymbol{x})d\boldsymbol{x} = 0 \quad \text{for } \boldsymbol{y} \in \Omega, j = 1, 2, 3, 4.$$

For the existence and uniqueness of such a matrix, we again refer to [94]. If now $\boldsymbol{f} \in H^{-1}(\Omega)^3$ and the uniquely determined solutions of the Stokes system given by $(\boldsymbol{u}, p) \in H_0^1(\Omega)^3 \times L_2(\Omega)$ satisfy the condition

$$\int_\Omega p(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x} = 0, \tag{3.2.3}$$

then the components of $(\boldsymbol{u}, p)$ admit the representations

$$\boldsymbol{u}_i(\boldsymbol{x}) = \int_\Omega \boldsymbol{f}(\boldsymbol{\xi}) \cdot \boldsymbol{G}_i(\boldsymbol{\xi}, \boldsymbol{x})d\boldsymbol{\xi}, \quad i = 1, 2, 3, \quad p(\boldsymbol{x}) = \int_\Omega \boldsymbol{f}(\boldsymbol{\xi}) \cdot \boldsymbol{G}_4(\boldsymbol{\xi}, \boldsymbol{x})d\boldsymbol{\xi}. \tag{3.2.4}$$

To apply this result to our case, we need to find a suitable $\bar{\phi}$ such that (3.2.3) holds. We show this is possible for $p \in C^{0,\zeta}(\Omega) \cap L_0^2(\Omega)$. By [94, Theorem 11.3.2] the pressure $p$ has this regularity for data in $C^{-1,\zeta}(\Omega)$ (a characterization of this space is given, for example, below Theorem 3 in [65]).

Without loss of generality, we assume $p \neq 0$. Thus, since the mean value of $p$ is zero, there exist non-empty open sets $A, B \Subset \Omega$ such that $p > 0$ on $A$ and $p < 0$ on $B$. We then can choose $\bar{\phi}$ such that $\bar{\phi} = 0$ on $\Omega \backslash (A \cup B)$ and $\bar{\phi} > 0$ on $A$, $B$ and thus $\bar{\phi}$ is vanishing close to the edges of $\Omega$. Through suitable scaling of $\phi$ on $A$ and $B$, we get

$$\int_A p(\boldsymbol{x})\bar{\phi}(\boldsymbol{x})d\boldsymbol{x} = -\int_B p(\boldsymbol{x})\bar{\phi}(\boldsymbol{x})d\boldsymbol{x}$$

and hence we can conclude that (3.2.3) holds for $\bar{\phi}(\boldsymbol{x})$. Finally, since by assumption $\bar{\phi} > 0$, we normalize $\bar{\phi}$ with respect to the $L^1$ norm to complete the construction. This shows that we can apply the results for the Green's matrix to $(\boldsymbol{u}, p)$. Furthermore, we can also use the available results from [72].

We state estimates for the Green's matrix specific to convex polyhedral domains as it can be found in [94, Theorem 11.5.5, Corollary 11.5.6].

**Proposition 3.3** *Let $\Omega$ be a convex polyhedral domain. Then, the elements of the matrix $G(\boldsymbol{x}, \boldsymbol{\xi})$ satisfy the estimate*

$$|\partial_{\boldsymbol{x}}^\theta \partial_\xi^\beta G_{i,j}(\boldsymbol{x}, \boldsymbol{\xi})| \le c|\boldsymbol{x} - \boldsymbol{\xi}|^{-1-\eta_{i,4}-\eta_{j,4}-|\theta|-|\beta|}$$

*for $|\theta| \le 1 - \eta_{i,4}$ and $|\beta| \le 1 - \eta_{j,4}$. Furthermore, the following Hölder-type estimate holds in this setting*

$$\frac{|\partial_\xi^\theta G_{i,j}(\boldsymbol{x}, \boldsymbol{\xi}) - \partial_\xi^\theta G_{i,j}(\boldsymbol{y}, \boldsymbol{\xi})|}{|\boldsymbol{x} - \boldsymbol{y}|^\zeta} \le C\Big(|\boldsymbol{x} - \boldsymbol{\xi}|^{-1-\zeta-\eta_{j,4}-|\theta|} + |\boldsymbol{y} - \boldsymbol{\xi}|^{-1-\zeta-\eta_{j,4}-|\theta|}\Big).$$

### 3.2.2. Assumptions

Next, we make assumptions on the finite element spaces. We assume, that there exist approximation operators $P_h$ and $r_h$ as in [72], i.e., $P_h$ and $r_h$ fulfill the following properties. Let $Q \subset Q_\varrho \subset \Omega$, with $\varrho \ge \bar{\kappa}h$, for some fixed $\bar{\kappa}$ sufficiently large, and $Q_\varrho = \{\boldsymbol{x} \in \Omega : \mathrm{dist}(\boldsymbol{x}, Q) \le \varrho\}$. For $P_h \in \mathcal{L}(H_0^1(\Omega)^3; \boldsymbol{V}_h)$ and $r_h \in \mathcal{L}(L^2(\Omega); \bar{M}_h)$ with $\bar{M}_h$ corresponding to $M_h$ without the zero-mean value constraint, we assume that the following assumptions hold.

**Assumption 3.4** (Stability of $P_h$ in $H^1(\Omega)^3$) *There exists a constant $C$ independent of $h$ such that*

$$\|\nabla P_h(\boldsymbol{v})\|_{L^2(\Omega)} \leq C\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}, \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^3.$$

**Assumption 3.5** (Preservation of discrete divergence for $P_h$) *It holds*

$$(\nabla \cdot (\boldsymbol{v} - P_h(\boldsymbol{v})), q_h) = 0, \quad \forall q_h \in \bar{M}_h, \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^3.$$

**Assumption 3.6** (Inverse Inequality) *There is a constant $C$ independent of $h$ such that*

$$\|\boldsymbol{v}_h\|_{W^{1,p}(Q)} \leq Ch^{-1}\|\boldsymbol{v}_h\|_{L^p(Q_\varrho)} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, 1 \leq p \leq \infty.$$

**Assumption 3.7** ($L^2$ approximation) *For any $\boldsymbol{v} \in H^2(\Omega)^3$ and any $q \in H^1(\Omega)$ there exists $C$ independent of $h$, $\boldsymbol{v}$, and $q$ such that*

$$\begin{aligned} \|P_h(\boldsymbol{v}) - \boldsymbol{v}\|_{L^2(Q)} + h\|\nabla(P_h(\boldsymbol{v}) - \boldsymbol{v})\|_{L^2(Q)} &\leq Ch^2\|\nabla^2 \boldsymbol{v}\|_{L^2(Q_\varrho)}, \\ \|r_h(q) - q\|_{L^2(Q)} &\leq Ch\|\nabla q\|_{L^2(Q_\varrho)}. \end{aligned}$$

In the following, $\boldsymbol{e}_i$ denotes the $i$-th standard basis vector in $\mathbb{R}^3$.

**Assumption 3.8** (Approximation in the Hölder spaces) *For $\boldsymbol{v} \in \left(C^{1,\zeta}(\Omega) \cap H_0^1(\Omega)\right)^3$ and $q \in C^{0,\zeta}(\Omega)$, it holds*

$$\begin{aligned} \|\nabla(P_h(\boldsymbol{v}) - \boldsymbol{v})\|_{L^\infty(Q)} &\leq Ch^\zeta\|\boldsymbol{v}\|_{C^{1,\zeta}(Q_\varrho)}, \\ \|r_h(q) - q\|_{L^\infty(Q)} &\leq Ch^\zeta\|q\|_{C^{0,\zeta}(Q_\varrho)}, \end{aligned}$$

*where*

$$\|\boldsymbol{v}\|_{C^{1+\zeta}(Q)} = \|\boldsymbol{v}\|_{C^1(Q)} + \sup_{\substack{\boldsymbol{x},\boldsymbol{y} \in Q \\ i \in \{1,2,3\}}} \frac{|\boldsymbol{e}_i \cdot \nabla(\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y}))|}{|\boldsymbol{x} - \boldsymbol{y}|^\zeta}.$$

**Assumption 3.9** (Super-Approximation I) *Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\omega \in C_0^\infty(Q_\varrho)$ a smooth cut-off function such that $\omega \equiv 1$ on $Q$ and*

$$|\nabla^s \omega| \leq C\varrho^{-s}, \quad s = 0, 1.$$

*We assume*

$$\|\nabla(\omega^2 \boldsymbol{v}_h - P_h(\omega^2 \boldsymbol{v}_h))\|_{L^2(Q)} \leq C\varrho^{-1}\|\boldsymbol{v}_h\|_{L^2(Q_\varrho)}.$$

*For $q_h \in \bar{M}_h$, we assume*

$$\|\omega^2 q_h - r_h(\omega^2 q_h)\|_{L^2(Q)} \leq Ch\varrho^{-1}\|q_h\|_{L^2(Q_\varrho)}.$$

One common example of a finite element space satisfying the above assumptions are the $\mathcal{P}_k - \mathcal{P}_{k-1}$ Taylor-Hood finite elements for $k \geq 3$. For more details on these spaces and the respective approximation operators, we refer to [12, 64, 65, 67].

*Remark* 3.10 Here we restrict ourselves to the $\mathcal{P}_k - \mathcal{P}_{k-1}$ Taylor-Hood finite element spaces since in the following arguments we use results for finite element approximations of elliptic problems. These results are available for the usual space of Lagrange finite elements and can possibly be extended to other elements used for the Stokes problem, like, e.g., the *mini* element, which also fulfills the assumptions above. The above assumptions do not cover the lowest order Taylor-Hood elements, since the existence of the divergence preserving operator $P_h$, fulfilling the requirements above, is still open. However, using the approach in [73], a similar result can be shown for the lowest order Taylor-Hood finite element spaces as well.

Next, we state a well-known energy error estimate for an approximation of the Stokes system. For details on the proof, see, e.g., [51, Proposition 4.14].

**Proposition 3.11** *Let* $(\boldsymbol{u}, p)$ *solve* (3.1.1a)–(3.1.1c) *and* $(\boldsymbol{u}_h, p_h)$ *be its finite element approximation defined by* (2.3.2). *Under the assumptions above, there exists a constant* $C$ *independent of* $h$ *such that,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq C \min_{(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h} \left( \|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1(\Omega)} + \|p - q_h\|_{L^2(\Omega)} \right).$$

### 3.2.3. Local energy estimates

An important tool in our analysis are the local energy estimates from [72, Theorem 2].

**Proposition 3.12** *Suppose* $(\boldsymbol{v}, q) \in H_0^1(\Omega)^3 \times L^2(\Omega)$ *and* $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h$ *satisfy*

$$a((\boldsymbol{v} - \boldsymbol{v}_h, q - q_h), (\boldsymbol{\chi}, w)) = 0 \quad \forall (\boldsymbol{\chi}, w) \in \boldsymbol{V}_h \times M_h$$

*for the bilinear form* $a(\cdot, \cdot)$ *given in* (2.2.2). *Then, there exists a constant* $C$ *such that for every pair of sets* $A_1 \subset A_2 \subset \Omega$ *such that* $\operatorname{dist}(\bar{A}_1, \partial A_2 \backslash \partial\Omega) \geq \varrho \geq \bar{\kappa} h$ *(for some fixed constant* $\bar{\kappa}$ *sufficiently large) the following bound holds for* $\varepsilon > 0$*:*

$$\begin{aligned} \|\nabla(\boldsymbol{v} - \boldsymbol{v}_h)\|_{L^2(A_1)} \leq{}& C\|\nabla(\boldsymbol{v} - P_h(\boldsymbol{v}))\|_{L^2(A_2)} + C\|q - r_h(q)\|_{L^2(A_2)} \\ &+ \frac{C}{\varepsilon\varrho}\|\boldsymbol{v} - P_h(\boldsymbol{v})\|_{L^2(A_2)} + \varepsilon\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h)\|_{L^2(A_2)} + \frac{C}{\varepsilon\varrho}\|\boldsymbol{v} - \boldsymbol{v}_h\|_{L^2(A_2)}. \end{aligned}$$

### 3.2.4. Main results

In the following statements, the constant $C$ is independent of $\boldsymbol{u}$, $p$, and $h$, but may depend on the parameter $\zeta$ related to the largest interior angle of $\partial\Omega$. We start with the $W^{1,\infty}$ error estimates. The global stability result

$$\|\nabla \boldsymbol{u}_h\|_{L^\infty(\Omega)} + \|p_h\|_{L^\infty(\Omega)} \leq C \left( \|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)} \right),$$

on convex polyhedral domains was established in [72] (see also [65]). Here, we establish a localized version of it. In our results $B_r(\tilde{\boldsymbol{x}})$ denotes a ball of radius $r$ centered at $\tilde{\boldsymbol{x}} \in \Omega$.

**Theorem 3.13** (Interior $W^{1,\infty}$ estimate for the velocity and $L^\infty$ estimate for the pressure) *Let the assumptions of* (2.3.1) *and Section 3.2.2 hold. Put* $D_1 = B_r(\tilde{\boldsymbol{x}}) \cap \Omega$, $D_2 = B_{\tilde{r}}(\tilde{\boldsymbol{x}}) \cap \Omega$, $\tilde{r} > r > \bar{\kappa} h$ *(with* $\bar{\kappa}$ *large enough),* $\varrho = \tilde{r} - r \geq \bar{\kappa} h$. *If* $(\boldsymbol{u}, p) \in (W^{1,\infty}(D_2)^3 \times L^\infty(D_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ *is the solution to* (3.1.1a)–(3.1.1c) *and* $(\boldsymbol{u}_h, p_h)$ *is the solution to* (2.3.2)*, then*

$$\begin{aligned}&\|\nabla \boldsymbol{u}_h\|_{L^\infty(D_1)} + \|p_h\|_{L^\infty(D_1)} \\ &\qquad\qquad \leq C\left(\|\nabla \boldsymbol{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\right) + C_\varrho\Big(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big).\end{aligned}$$

*Here, the constant* $C_\varrho$ *depends on* $\varrho$.

Next, we state similar results for the velocity in the $L^\infty$ norm.

**Theorem 3.14** (Global $L^\infty$ estimate for the velocity) *Under the assumptions of* (2.3.1) *and Section 3.2.2, for* $(\boldsymbol{u}, p) \in (L^\infty(\Omega)^3 \times L^\infty(\Omega)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ *the solution to* (3.1.1a)–(3.1.1c) *and* $(\boldsymbol{u}_h, p_h)$ *the solution to* (2.3.2)*, it holds*

$$\|\boldsymbol{u}_h\|_{L^\infty(\Omega)} \leq C|\ln h|\Big(\|\boldsymbol{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)}\Big).$$

We also get the respective local estimates.

**Theorem 3.15** (Interior $L^\infty$ error estimate for the velocity) *Under the assumptions of* (2.3.1) *and Section 3.2.2, with* $D_1 = B_r(\tilde{\boldsymbol{x}}) \cap \Omega$, $D_2 = B_{\tilde{r}}(\tilde{\boldsymbol{x}}) \cap \Omega$, $\tilde{r} > r > \bar{\kappa} h$ *(with* $\bar{\kappa}$ *large enough),* $\varrho = \tilde{r} - r \geq \bar{\kappa} h$ *and for* $(\boldsymbol{u}, p) \in (L^\infty(D_2)^3 \times L^\infty(D_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ *the solution to* (3.1.1a)–(3.1.1c) *and* $(\boldsymbol{u}_h, p_h)$ *the solution to* (2.3.2)*, it holds*

$$\begin{aligned}\|\boldsymbol{u}_h\|_{L^\infty(D_1)} &\leq C|\ln h|\left(\|\boldsymbol{u}\|_{L^\infty(D_2)} + h\|p\|_{L^\infty(D_2)}\right) \\ &\qquad + C_\varrho|\ln h|\left(h\|\boldsymbol{u}\|_{H^1(\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)} + h\|p\|_{L^2(\Omega)}\right).\end{aligned}$$

*Here, the constant* $C_\varrho$ *depends on* $\varrho$.

Based on these theorems, we can derive the following corollaries for general subdomains $\Omega_1 \subset \Omega_2 \subset \Omega$ with $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho \geq \bar{\kappa} h$.

**Corollary 3.16** (Interior $W^{1,\infty}$ estimate for the velocity and $L^\infty$ estimate for the pressure) *Under the assumptions of* (2.3.1) *and Section 3.2.2,* $\Omega_1 \subset \Omega_2 \subset \Omega$ *with* $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho \geq \bar{\kappa} h$ *and for* $(\boldsymbol{u}, p) \in (W^{1,\infty}(\Omega_2)^3 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ *the solution to* (3.1.1a)–(3.1.1c) *and* $(\boldsymbol{u}_h, p_h)$ *the solution to* (2.3.2)*, we have*

$$\begin{aligned}\|\nabla \boldsymbol{u}_h\|_{L^\infty(\Omega_1)} + \|p_h\|_{L^\infty(\Omega_1)} &\leq C\left(\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega_2)} + \|p\|_{L^\infty(\Omega_2)}\right) \\ &\qquad + C_\varrho\Big(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big).\end{aligned}$$

*Here, the constant* $C_\varrho$ *depends on* $\varrho$.

*Proof.* We can construct a covering $\{K_i\}_{i=1}^M$ of $\Omega_1$, with $K_i = B_{\varrho/2}(\tilde{\boldsymbol{x}}_i) \cap \Omega$ such that

(1) $\Omega_1 \subset \bigcup_{i=1}^M K_i$.

(2) $\tilde{\boldsymbol{x}}_i \in \bar{\Omega}_1$ for $1 \leq i \leq M$.

(3) Let $L_i = B_\varrho(\tilde{\boldsymbol{x}}_i) \cap \Omega$. There exists a fixed number $N$ such that each point $\boldsymbol{x} \in \bigcup_{i=1}^M L_i$ is contained in at most $N$ sets from $\{L_j\}_{j=1}^M$.

Now, due to $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho$ and (2), we have that $\bigcup_{i=1}^M L_i \subset \Omega_2$. We can apply Theorem 3.13 to the pairs $K_i \subset L_i$:

$$\begin{aligned}
\|\nabla \boldsymbol{u}_h\|_{L^\infty(\Omega_1)} + \|p_h\|_{L^\infty(\Omega_1)} &\leq \max_{1\leq i\leq M} \left(\|\nabla \boldsymbol{u}_h\|_{L^\infty(K_i)} + \|p_h\|_{L^\infty(K_i)}\right) \\
&\leq \max_{1\leq i\leq M} \left(C\left(\|\nabla \boldsymbol{u}\|_{L^\infty(L_i)} + \|p\|_{L^\infty(L_i)}\right) + C_\varrho\left(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\right)\right) \\
&\leq C\left(\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega_2)} + \|p\|_{L^\infty(\Omega_2)}\right) + C_\varrho\left(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\right).
\end{aligned}$$

□

Similarly, the following corollary follows with $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho$.

**Corollary 3.17** (Interior $L^\infty$ error estimate for the velocity) *Under the assumptions of* (2.3.1) *and Section 3.2.2,* $\Omega_1 \subset \Omega_2 \subset \Omega$ *with* $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho \geq \bar{\kappa} h$ *and for* $(\boldsymbol{u}, p) \in (L^\infty(\Omega_2)^3 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ *the solution to* (3.1.1a)–(3.1.1c) *and* $(\boldsymbol{u}_h, p_h)$ *the solution to* (2.3.2)*, we have*

$$\begin{aligned}
\|\boldsymbol{u}_h\|_{L^\infty(\Omega_1)} \leq C|\ln h|&\left(\|\boldsymbol{u}\|_{L^\infty(\Omega_2)} + h\|p\|_{L^\infty(\Omega_2)}\right) \\
&+ C_\varrho|\ln h|\left(h\|\boldsymbol{u}\|_{H^1(\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)} + h\|p\|_{L^2(\Omega)}\right).
\end{aligned}$$

*Here, the constant* $C_\varrho$ *depends on* $\varrho$.

*Remark* 3.18 We may also write the results above in terms of best-approximation estimates. For example for $L^\infty$ global bounds:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty(\Omega)} \leq \inf_{(\boldsymbol{v}_h, q_h)\in \boldsymbol{V}_h \times M_h} C|\ln h|\left(\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^\infty(\Omega)} + h\|p - q_h\|_{L^\infty(\Omega)}\right).$$

Naturally, this also applies to other results in this section, for which we also refer to [65, Corollary 6].

The result is easily derived if one only considers $\bar{\boldsymbol{v}}_h$ satisfying the additional condition

$$(\nabla \cdot \bar{\boldsymbol{v}}_h, q_h) = 0 \qquad \forall q_h \in M_h,$$

since then the Stokes projection, given as $(\tilde{\boldsymbol{v}}_h, \tilde{q}_h) \in \boldsymbol{V}_h \times M_h$ which satisfy

$$a((\tilde{\boldsymbol{v}}_h - \bar{\boldsymbol{v}}_h, \tilde{q}_h - \bar{q}_h), (\boldsymbol{w}_h, \varphi_h)) = 0 \qquad \forall (\boldsymbol{w}_h, \varphi_h) \in \boldsymbol{V}_h \times M_h,$$

is invariant on such a pair $(\bar{\boldsymbol{v}}_h, \bar{q}_h)$, i.e., $(\tilde{\boldsymbol{v}}_h, \tilde{q}_h) = (\bar{\boldsymbol{v}}_h, \bar{q}_h)$. Thus, one arrives at a best-approximation result by using the theorems above and the triangle inequality on $(\boldsymbol{u}-\boldsymbol{v}_h, p-q_h)$.

For general $\boldsymbol{v}_h \in \boldsymbol{V}_h$ the argument is a bit more complex and depends on the finite element discretization. We lay out the details on the argument for a discretization with Taylor-Hood finite elements in Appendix B.

*Remark* 3.19 Using the weighted discrete inf-sup condition from [46] it is possible to extend the global estimate to the compressible case (but with worse constants). However, for the applications we have in mind the incompressible Stokes system is sufficient.

## 3.3. Proof of main theorems

In this section, we reduce the proofs of Theorems 3.13–3.15 for the velocity to certain estimates for the regularized Green's functions. The estimates for the pressure are given in Section 3.5. To introduce the regularized Green's function we first need to introduce a regularized delta function. In addition, we will require a certain weight function.

### 3.3.1. Regularized delta function and the weight function

Let $R > 0$ be such that for any $\boldsymbol{x} \in \Omega$ the ball $B_R(\boldsymbol{x})$ contains $\Omega$. Furthermore, let $\boldsymbol{x}_0$ be an arbitrary point of $\bar{\Omega}$ and $\boldsymbol{x}_0 \in T_{\boldsymbol{x}_0}$ with $T_{\boldsymbol{x}_0} \in \mathcal{T}_h$. In the following sections, we estimate $|\partial_{x_j} \boldsymbol{u}_{h,i}(\boldsymbol{x}_0)|$, $|\boldsymbol{u}_{h,i}(\boldsymbol{x}_0)|$, for arbitrary $1 \le i, j \le 3$, and $|p(\boldsymbol{x}_0)|$.

Next, we introduce the parameters for the weight function $\sigma(\boldsymbol{x})$. Parameter $\kappa > 1$ is a constant that is chosen to be large enough. Furthermore, let $h$ be sufficiently small such that $\kappa h \le R$ (see also [64, Remark 1.4]). In the following, we use a regularized Green's function to express the $L^\infty$ norm in a way such that the problem is reduced to estimating the discretization error of the Green's function in the $L^1$ norm as in [65, 72]. To that end, we define a smooth delta function $\delta_h \in C_0^1(T_{\boldsymbol{x}_0})$, which satisfies for every $\boldsymbol{v}_h \in \mathcal{P}_k(T_{\boldsymbol{x}_0})$:

$$\boldsymbol{v}_{h,i}(\boldsymbol{x}_0) = (\boldsymbol{v}_h, \delta_h \boldsymbol{e}_i)_{T_{\boldsymbol{x}_0}} \quad \text{or} \quad \partial_{x_j} \boldsymbol{v}_{h,i}(\boldsymbol{x}_0) = (\partial_{x_j} \boldsymbol{v}_h, \delta_h \boldsymbol{e}_i)_{T_{\boldsymbol{x}_0}} \tag{3.3.1}$$

$$\|\delta_h\|_{W_q^k(T_{\boldsymbol{x}_0})} \le C h^{-k-3(1-1/q)}, \quad 1 \le q \le \infty, \quad k = 0, 1. \tag{3.3.2}$$

The construction of such $\delta_h$ can be found in [64, Lemma 1.1]. We recall some properties for $\sigma$ and $\delta_h$. By construction, it follows

$$\inf_{\boldsymbol{x} \in \Omega} \sigma(\boldsymbol{x}) \ge \kappa h. \tag{3.3.3}$$

Next, we provide an estimate for the $L^2$ norm of the product of $\delta_h$ and $\sigma$.

**Lemma 3.20** *There exists a constant $C$ such that for $\nu > 0$*

$$\|\sigma^\nu \nabla^k \delta_h\|_{L^2(\Omega)} \le 2^{\nu/2} C \kappa^\nu h^{\nu-k-3/2} \quad k = 0, 1.$$

*Proof.* This follows from the fact that $\delta_h$ is only non-zero on $T_{\boldsymbol{x}_0}$, $\sigma$ is bounded on $T_{\boldsymbol{x}_0}$ by $\sqrt{2}\kappa h$ for $\kappa$ large enough and (3.3.2). □

The general strategy for proving the local results is to partition the domain into the local part and its complement. Then, we use regularized Green's function estimates in the $L^1$ norm on the local part and in the weighted $L^2$ norm on the complement. For the $L^\infty$ error estimates we additionally require an estimate for the Ritz projection.

### 3.3.2. Estimates in $W^{1,\infty}(\Omega)$

The proof of local $W^{1,\infty}(\Omega)$ error estimates is similar to the global case [65, 72] and is obtained by introducing a regularized Green's function.

#### Regularized Green's function

For the $W^{1,\infty}$ error estimates, we define the regularized Green's function $(\boldsymbol{g}_1, \lambda_1) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ as the solution to

$$-\Delta \boldsymbol{g}_1 + \nabla \lambda_1 = (\partial_{\boldsymbol{x}_j} \delta_h)\boldsymbol{e}_i \quad \text{in } \Omega, \tag{3.3.4a}$$

$$\nabla \cdot \boldsymbol{g}_1 = 0 \quad \text{in } \Omega, \tag{3.3.4b}$$

$$\boldsymbol{g}_1 = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{3.3.4c}$$

We also define the finite element approximation $(\boldsymbol{g}_{1,h}, \lambda_{1,h}) \in \boldsymbol{V}_h \times M_h$ by

$$a((\boldsymbol{g}_1 - \boldsymbol{g}_{1,h}, \lambda_1 - \lambda_{1,h}), (\boldsymbol{v}_h, q_h)) = 0 \quad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h. \tag{3.3.5}$$

#### Auxiliary results for $(\boldsymbol{g}_1, \boldsymbol{\lambda}_1)$ and $(\boldsymbol{g}_{1,h}, \boldsymbol{\lambda}_{1,h})$

To show our main interior $W^{1,\infty}$ result, we need a regularized Green's function error estimate in the $L^1$ norm which is given in [72, Lemma 5.2].

**Lemma 3.21** *There exists a constant $C$ independent of $h$ and $\boldsymbol{g}_1$ such that*

$$\|\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^1(\Omega)} \le C.$$

In addition, we also need the following weighted estimate, the proof of which follows by a minor modification of the proof in [72, Lemma 5.2].

**Corollary 3.22** *There exists a constant $C$ independent of $h$ and $\boldsymbol{g}_1$ such that*

$$\|\sigma^{3/2}\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^2(\Omega)} \le C.$$

The details on the proof of this corollary are given in Section 3.4, where we introduce the respective dyadic decomposition.

*Remark* 3.23 Alternatively, similar results as in Lemma 3.21 and Corollary 3.22 may be deduced as well from the results in [65]. But in [65] the authors use slightly different assumptions compared to the assumptions made in Section 3.2, which is why we provide a proof in our setting.

#### Localization

We reduce the proof to estimates involving $\boldsymbol{g}_1$ and $\boldsymbol{g}_{1,h}$.

*Proof of Theorem 3.13 (velocity).* Using the regularized Green's function as defined in (3.3.4a)–(3.3.4c), for $\boldsymbol{x}_0$ in the interior of $T_{\boldsymbol{x}_0} \subset D_1$, we have as in [72]

$$
\begin{aligned}
-\partial_{\boldsymbol{x}_j}(\boldsymbol{u}_h)_i(\boldsymbol{x}_0) &= (\boldsymbol{u}_h, (\partial_{\boldsymbol{x}_j}\delta_h)\boldsymbol{e}_i) && \text{(by (3.3.1))} \\
&= (\boldsymbol{u}_h, -\Delta\boldsymbol{g}_1 + \nabla\lambda_1) && \text{(by (3.3.4a))} \\
&= (\nabla\boldsymbol{u}_h, \nabla\boldsymbol{g}_1) + (\boldsymbol{u}_h, \nabla\lambda_1) \\
&= (\nabla\boldsymbol{u}_h, \nabla\boldsymbol{g}_1) + (\boldsymbol{u}_h, \nabla\lambda_{1,h}) + (\nabla\boldsymbol{u}_h, \nabla(\boldsymbol{g}_{1,h} - \boldsymbol{g}_1)) && \text{(by (3.3.5))} \\
&= (\nabla\boldsymbol{u}_h, \nabla\boldsymbol{g}_{1,h}) && \text{(discrete divergence)} \\
&= (\nabla\boldsymbol{u}, \nabla\boldsymbol{g}_{1,h}) + (p - p_h, \nabla\cdot\boldsymbol{g}_{1,h}) && \text{(by (2.3.2) and (3.1.1a))} \\
&= (\nabla\boldsymbol{u}, \nabla\boldsymbol{g}_{1,h}) + (p, \nabla\cdot\boldsymbol{g}_{1,h}) && \text{(by (3.3.5) and (3.3.4b))} \\
&= (\nabla\boldsymbol{u}, \nabla(\boldsymbol{g}_{1,h} - \boldsymbol{g}_1)) + (\nabla\boldsymbol{u}, \nabla\boldsymbol{g}_1) + (p, \nabla\cdot(\boldsymbol{g}_{1,h} - \boldsymbol{g}_1)) \\
& && \text{(continuous divergence)} \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

To treat $I_2$ we use integration by parts, the Hölder inequality, and (3.3.2)

$$
I_2 = (\boldsymbol{u}, -\Delta\boldsymbol{g}_1) + (\boldsymbol{u}, \nabla\lambda_1) = (\boldsymbol{u}, (\partial_{\boldsymbol{x}_j}\delta_h)\boldsymbol{e}_i) = (-\partial_{\boldsymbol{x}_j}\boldsymbol{u}, \delta_h\boldsymbol{e}_i) \le C\|\nabla\boldsymbol{u}\|_{L^\infty(T_{\boldsymbol{x}_0})}.
$$

Since $r - \tilde{r} > \bar{\kappa}h$ this proves the result for $I_2$.

For the other two terms, we split the domain into $D_2$ and $\Omega\backslash D_2$. Using that $\sigma^{-1} < (\bar{\kappa}(\tilde{r}-r))^{-1}$ on $\Omega\backslash D_2$ and the Hölder inequality, we have

$$
\begin{aligned}
I_1 + I_3 &\le C\left(\|\nabla\boldsymbol{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\right)\|\nabla(\boldsymbol{g}_{1,h} - \boldsymbol{g}_1)\|_{L^1(\Omega)} \\
&\quad + C\Big(\|\sigma^{-3/2}\nabla\boldsymbol{u}\|_{L^2(\Omega\backslash D_2)} + \|\sigma^{-3/2}p\|_{L^2(\Omega\backslash D_2)}\Big)\|\sigma^{3/2}\nabla(\boldsymbol{g}_{1,h} - \boldsymbol{g}_1)\|_{L^2(\Omega)} \\
&\le C\Big(\|\nabla\boldsymbol{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\Big)\|\nabla(\boldsymbol{g}_{1,h} - \boldsymbol{g}_1)\|_{L^1(\Omega)} \\
&\quad + C(\tilde{r} - r)^{-3/2}\Big(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big)\|\sigma^{3/2}\nabla(\boldsymbol{g}_{1,h} - \boldsymbol{g}_1)\|_{L^2(\Omega)}. \qquad (3.3.6)
\end{aligned}
$$

The result then follows from Lemma 3.21 and Corollary 3.22. □

### 3.3.3. Estimates in $L^\infty(\Omega)$

For this case we use the stability of the Ritz projection in the $L^\infty$ norm as shown in [85].

### Regularized Green's function

This time we define the approximate Green's function $(\boldsymbol{g}_0, \lambda_0) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ as the solution to

$$-\Delta \boldsymbol{g}_0 + \nabla \lambda_0 = \delta_h \boldsymbol{e}_i \quad \text{in } \Omega, \tag{3.3.7a}$$
$$\nabla \cdot \boldsymbol{g}_0 = 0 \quad \text{in } \Omega, \tag{3.3.7b}$$
$$\boldsymbol{g}_0 = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{3.3.7c}$$

Here, $\boldsymbol{e}_i$ is as before the $i$-th standard basis vector in $\mathbb{R}^3$. We also define the finite element approximation $(\boldsymbol{g}_{0,h}, \lambda_{0,h}) \in \boldsymbol{V}_h \times M_h$ by

$$a((\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}, \lambda_0 - \lambda_{0,h}), (\boldsymbol{v}_h, q_h)) = 0 \quad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h. \tag{3.3.8}$$

Compared to (3.3.4a)–(3.3.4c), the right-hand side of (3.3.7a) is less singular, which means we can expect faster convergence in the following auxiliary estimates.

### Auxiliary results for $(\boldsymbol{g}_0, \boldsymbol{\lambda}_0)$, $(\boldsymbol{g}_{0,h}, \boldsymbol{\lambda}_{0,h})$, and the Ritz projection

Similarly to the $W^{1,\infty}$ case, we need certain error estimates for the discretization of the regularized Green's function $(\boldsymbol{g}_0, \lambda_0)$. However, in contrast to $(\boldsymbol{g}_1, \lambda_1)$, we could not locate such results in the literature. For our purpose we need to establish the following results, for which the proofs are given in Section 3.4.

**Lemma 3.24** *Let $(\boldsymbol{g}_0, \lambda_0)$ be the solution of (3.3.7a)–(3.3.7c) and $(\boldsymbol{g}_{0,h}, \lambda_{0,h})$ the respective discrete solution. Then, it holds*

$$\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} \le C|\ln h|h.$$

The weighted norm estimate follows essentially from Lemma 3.24.

**Corollary 3.25** *Let $(\boldsymbol{g}_0, \lambda_0)$ be the solution of (3.3.7a)–(3.3.7c) and $(\boldsymbol{g}_{0,h}, \lambda_{0,h})$ the respective discrete solution. Then, it holds*

$$\|\sigma^{3/2}\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega)} \le C|\ln h|h.$$

As mentioned before, the proof is based on local and global max-norm estimates for the (vectorial) Ritz projection $R_h\boldsymbol{z}$ of $\boldsymbol{z} \in H_0^1(\Omega)^3$ which is given by

$$(\nabla R_h \boldsymbol{z}, \nabla \boldsymbol{v}_h) = (\nabla \boldsymbol{z}, \nabla \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

We state the slightly modified results [83, Theorem 5.1], [84, Theorem 4.4], and [85, Theorem 12] for the convenience of the reader. Note that these results continue to hold in the vector-valued case, since we can apply them componentwise.

**Proposition 3.26** *There exists a constant $C$ independent of $h$ such that, for $\boldsymbol{z} \in H_0^1(\Omega)^3 \cap L^\infty(\Omega)^3$, it holds*

$$\|R_h \boldsymbol{z}\|_{L^\infty(\Omega)} \leq C|\ln h|^{\bar{k}} \|\boldsymbol{z}\|_{L^\infty(\Omega)},$$

*where $\bar{k} = 1$ for $k = 1$ and $\bar{k} = 0$ for $k \geq 2$.*

**Proposition 3.27** *Let $D \subset D_\varrho \subset \Omega$, where $D_\varrho = \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, D) \leq \varrho\}$. Then, for $\boldsymbol{z} \in H_0^1(\Omega)^3 \cap L^\infty(\Omega)^3$, there exists a constant $C$, independent of $h$, such that*

$$\|R_h \boldsymbol{z}\|_{L^\infty(D)} \leq C|\ln h|^{\bar{k}} \|\boldsymbol{z}\|_{L^\infty(D_\varrho)} + C_\varrho h \|\boldsymbol{z}\|_{H^1(\Omega)},$$

*where $C_\varrho \sim \varrho^{-3/2}$ and as above $\bar{k} = 1$ for $k = 1$ and $\bar{k} = 0$ for $k \geq 2$.*

*Remark* 3.28 As we mentioned in the introduction of the chapter, in the original paper of Schatz and Wahlbin on smooth domains [111], the interior error estimate is of the form

$$\|R_h \boldsymbol{z}\|_{L^\infty(D)} \leq C|\ln h|^{\bar{k}} \|\boldsymbol{z}\|_{L^\infty(D_\varrho)} + C_\varrho \|R_h \boldsymbol{z}\|_{W_p^{-l}(D_\varrho)},$$

with $D \Subset D_\varrho \Subset \Omega$. The main difference is that the pollution error term $C_\varrho \|R_h \boldsymbol{z}\|_{W_p^{-l}(D_\varrho)}$ is still in the form of the Ritz projection, but can be taken in any negative norm and is still local. However, for our applications we do not see any benefits from this form of the results, since such a pollution term needs to be estimated by a duality argument, which essentially requires global estimates.

We will also require the following result.

**Lemma 3.29** *Let $(\boldsymbol{g}_0, \lambda_0)$ be the solution of* (3.3.7a)–(3.3.7c). *Then, it holds*

$$\|\nabla^2 \boldsymbol{g}_0\|_{L^1(\Omega)} + \|\nabla \lambda_0\|_{L^1(\Omega)} + \|\sigma^{3/2} \nabla^2 \boldsymbol{g}_0\|_{L^2(\Omega)} + \|\sigma^{3/2} \nabla \lambda_0\|_{L^2(\Omega)} \leq C|\ln h|.$$

The respective proof is given in Section 3.4.

**Max-norm estimate**

With these tools at hand, we can go ahead with the proof of the theorem.

*Proof of Theorem 3.14 (velocity).* We make the ansatz for $\boldsymbol{x}_0 \in \bar{\Omega}$, in the interior of a cell $T$

$$\begin{aligned} \boldsymbol{u}_{h,i}(\boldsymbol{x}_0) = a((\boldsymbol{u}_h, p_h), (\boldsymbol{g}_{0,h}, \lambda_{0,h})) &= a((\boldsymbol{u}, p), (\boldsymbol{g}_{0,h}, \lambda_{0,h})) && \text{(by orthogonality)} \\ &= (\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) - (p, \nabla \cdot \boldsymbol{g}_{0,h}). && (3.3.9) \end{aligned}$$

Since $\boldsymbol{g}_{0,h} \in \boldsymbol{V}_h$ we have $(\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) = (\nabla R_h \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h})$ and hence by using $\nabla \cdot \boldsymbol{g}_0 = 0$

$$\boldsymbol{u}_{h,i}(\boldsymbol{x}_0) = (\nabla R_h \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) - (p, \nabla \cdot \boldsymbol{g}_{0,h}) = (\nabla R_h \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) - (p, \nabla \cdot (\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)).$$

We can use an inverse estimate on $\nabla R_h \boldsymbol{u}$. Thus,

$$\begin{aligned}(\nabla R_h \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) &= (\nabla R_h \boldsymbol{u}, \nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)) - (R_h \boldsymbol{u}, \Delta \boldsymbol{g}_0) \\ &= (\nabla R_h \boldsymbol{u}, \nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)) - (R_h \boldsymbol{u}, -\delta_h \boldsymbol{e}_i + \nabla \lambda_0) \\ &\leq h^{-1} \|R_h \boldsymbol{u}\|_{L^\infty(\Omega)} \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)} \\ &\quad + C\|R_h \boldsymbol{u}\|_{L^\infty(\Omega)} (1 + \|\nabla \lambda_0\|_{L^1(\Omega)}).\end{aligned}$$

For the second term, we get by estimating the divergence by the gradient:

$$(p, \nabla \cdot (\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)) \leq C\|p\|_{L^\infty(\Omega)} \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)}. \tag{3.3.10}$$

Now we can apply our auxiliary results for $\|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)}$ and $\|\nabla \lambda_0\|_{L^1(\Omega)}$. Thus, we have by Lemma 3.24 and Lemma 3.29 combined with Proposition 3.26

$$\begin{aligned}|\boldsymbol{u}_{h,i}(\boldsymbol{x}_0)| &\leq C\|\boldsymbol{u}\|_{L^\infty(\Omega)} \Big(h^{-1} \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)} + 1 + \|\nabla \lambda_0\|_{L^1(\Omega)}\Big) \\ &\quad + C\|p\|_{L^\infty(\Omega)} \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)} \\ &\leq C|\ln h| \Big(\|\boldsymbol{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)}\Big).\end{aligned}$$

□

### Localization

The approach for the localization in the $L^\infty$ case is similar to $W^{1,\infty}$ but different in the sense that we again use the stability of $R_h$ in the $L^\infty$ norm.

*Proof of Theorem 3.15 (velocity).* We only consider $\boldsymbol{x}_0$ in the interior of $T_{\boldsymbol{x}_0} \subset D_1$. As before, using (2.2.2), (2.3.2) and (3.3.8) gives

$$\begin{aligned}\boldsymbol{u}_{h,i}(\boldsymbol{x}_0) &= a((\boldsymbol{u}_h, p_h), (\boldsymbol{g}_{0,h}, \lambda_{0,h})) = a((\boldsymbol{u}, p), (\boldsymbol{g}_{0,h}, \lambda_{0,h})) \qquad \text{(by orthogonality)} \\ &= (\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) - (p, \nabla \cdot \boldsymbol{g}_{0,h}) =: I_1 + I_2.\end{aligned}$$

Using the properties of the Ritz projection we first consider

$$\begin{aligned}I_1 &= (\nabla R_h \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) \\ &= (\nabla R_h \boldsymbol{u}, \nabla \boldsymbol{g}_0) + (\nabla R_h \boldsymbol{u}, \nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)) \\ &= -(R_h \boldsymbol{u}, \Delta \boldsymbol{g}_0) + (\nabla R_h \boldsymbol{u}, \nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)) \\ &= (R_h \boldsymbol{u}, \delta_h \boldsymbol{e}_i - \nabla \lambda_0) + (\nabla R_h \boldsymbol{u}, \nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)).\end{aligned}$$

Next, we apply (3.3.1) and split the domain into $D_* = B_{r+\varrho/2}(\tilde{\boldsymbol{x}}) \cap \Omega \subset D_2$ and $\Omega \backslash D_*$

$$\begin{aligned}I_1 &\leq \|R_h \boldsymbol{u}\|_{L^\infty(T_{\boldsymbol{x}_0})} + \|R_h \boldsymbol{u}\|_{L^\infty(D_*)} \|\nabla \lambda_0\|_{L^1(\Omega)} + \|\nabla R_h \boldsymbol{u}\|_{L^\infty(D_*)} \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)} \\ &\quad + \|\sigma^{-3/2} R_h \boldsymbol{u}\|_{L^2(\Omega \backslash D_*)} \|\sigma^{3/2} \nabla \lambda_0\|_{L^2(\Omega)} \\ &\quad + \|\sigma^{-3/2} \nabla R_h \boldsymbol{u}\|_{L^2(\Omega \backslash D_*)} \|\sigma^{3/2} \nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(\Omega)}.\end{aligned}$$

Using the properties of $\sigma$ and applying an inverse inequality gives

$$\begin{aligned} I_1 &\le C\|R_h\boldsymbol{u}\|_{L^\infty(D_*)}\Big(1 + \|\nabla\lambda_0\|_{L^1(\Omega)} + h^{-1}\|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)}\Big) \\ &\quad + C_\varrho\|R_h\boldsymbol{u}\|_{L^2(\Omega)}\Big(\|\sigma^{3/2}\nabla\lambda_0\|_{L^2(\Omega)} + h^{-1}\|\sigma^{3/2}\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(\Omega)}\Big). \end{aligned}$$

To estimate $R_h\boldsymbol{u}$ in the $L^\infty$ and $L^2$ norm we can apply Proposition 3.27, using $D_*$ and $D_2$, and an estimate for $\|R_h\boldsymbol{u}-\boldsymbol{u}\|_{L^2(\Omega)}$ to see together with Lemmas 3.24 and 3.29 and Corollary 3.25 that

$$\begin{aligned} I_1 &\le C\|\boldsymbol{u}\|_{L^\infty(D_2)}(1 + |\ln h|) + C_\varrho|\ln h|\Big(\|\boldsymbol{u}\|_{L^2(\Omega)} + h\|\boldsymbol{u}\|_{H^1(\Omega)}\Big) \\ &\le C|\ln h|\|\boldsymbol{u}\|_{L^\infty(D_2)} + C_\varrho|\ln h|\Big(\|\boldsymbol{u}\|_{L^2(\Omega)} + h\|\boldsymbol{u}\|_{H^1(\Omega)}\Big). \end{aligned}$$

Using similar arguments we get for

$$\begin{aligned} I_2 &= -(p, \nabla\cdot(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)) \\ &\le C\|p\|_{L^\infty(D_2)}\|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(\Omega)} + C_\varrho\|p\|_{L^2(\Omega)}\|\sigma^{3/2}\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(\Omega)} \\ &\le C|\ln h|h\|p\|_{L^\infty(D_2)} + C_\varrho|\ln h|h\|p\|_{L^2(\Omega)}, \end{aligned}$$

which concludes the proof of the theorem. □

## 3.4. Estimates for the regularized Green's function

In this section we prove Corollaries 3.22 and 3.25 and Lemmas 3.24 and 3.29 which we need in order to establish the main theorems.

### 3.4.1. Dyadic decomposition

For the proof of our results, we use a dyadic decomposition of the domain $\Omega$, which we will introduce next. Without loss of generality, we assume that the diameter of $\Omega$ is less than 1. We put $\varrho_j = 2^{-j}$ and consider the decomposition $\Omega = \Omega_* \cup \bigcup_{j=0}^J \Omega_j$, where

$$\Omega_* = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x} - \boldsymbol{x}_0| \le Kh\}, \qquad \Omega_j = \{\boldsymbol{x} \in \Omega : \varrho_{j+1} \le |\boldsymbol{x} - \boldsymbol{x}_0| \le \varrho_j\},$$

$K$ is a sufficiently large constant to be chosen later and $J$ is an integer such that

$$2^{-(J+1)} \le Kh \le 2^{-J}. \tag{3.4.1}$$

We keep track of the explicit dependence on $K$. Furthermore, we consider the following enlargements of $\Omega_j$

$$\begin{aligned} \Omega_j' &= \{\boldsymbol{x} \in \Omega : \varrho_{j+2} \le |\boldsymbol{x} - \boldsymbol{x}_0| \le \varrho_{j-1}\}, \\ \Omega_j'' &= \{\boldsymbol{x} \in \Omega : \varrho_{j+3} \le |\boldsymbol{x} - \boldsymbol{x}_0| \le \varrho_{j-2}\}, \\ \Omega_j''' &= \{\boldsymbol{x} \in \Omega : \varrho_{j+4} \le |\boldsymbol{x} - \boldsymbol{x}_0| \le \varrho_{j-3}\}. \end{aligned}$$

**Lemma 3.30** *There exists a constant $C$ independent of $\varrho_j$ such that for any $\boldsymbol{x} \in \Omega_j$,*

$$|\nabla \boldsymbol{g}_0(\boldsymbol{x})| + \varrho_j^{-1}|\boldsymbol{g}_0(\boldsymbol{x})| + |\lambda_0(\boldsymbol{x})| \le C\varrho_j^{-2}.$$

*Proof.* We start by pointing out that $(\boldsymbol{g}_0, \lambda_0) \in C^{1,\zeta}(\Omega)^3 \times C^{0,\zeta}(\Omega)$ due to (2.2.5). Because of (3.2.4) and Proposition 3.3, it holds for $\boldsymbol{x} \in \Omega_j$

$$\begin{aligned} |\lambda_0(\boldsymbol{x})| &= \Big|\int_\Omega \boldsymbol{G}_4(\boldsymbol{x},\boldsymbol{y}) \cdot \delta_h(\boldsymbol{y})\boldsymbol{e}_i d\boldsymbol{y}\Big| \le \int_{T_{\boldsymbol{x}_0}} |G_{i,4}(\boldsymbol{x},\boldsymbol{y})||\delta_h(\boldsymbol{y})|d\boldsymbol{y} \\ &\le C\int_{T_{\boldsymbol{x}_0}} \frac{|\delta_h(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2} d\boldsymbol{y} \le C\varrho_j^{-2}\|\delta_h\|_{L^1(\Omega)} \le C\varrho_j^{-2}, \end{aligned}$$

where we used that $\operatorname{dist}(\boldsymbol{x}_0, \Omega_j) \ge \varrho_j/2$. Similarly, without loss of generality, considering the $k$-th component, $1 \le k \le 3$, we have for

$$\begin{aligned} |\partial_{\boldsymbol{x}} \boldsymbol{g}_{0,k}(\boldsymbol{x})| = \Big|\int_\Omega \partial_{\boldsymbol{x}} \boldsymbol{G}_k(\boldsymbol{x},\boldsymbol{y}) \cdot \delta_h(\boldsymbol{y})\boldsymbol{e}_i d\boldsymbol{y}\Big| &\le \int_{T_{\boldsymbol{x}_0}} |\partial_{\boldsymbol{x}} G_{i,k}(\boldsymbol{x},\boldsymbol{y})||\delta_h(\boldsymbol{y})|d\boldsymbol{y} \\ &\le \int_{T_{\boldsymbol{x}_0}} \frac{|\delta_h(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2} d\boldsymbol{y} \le C\varrho_j^{-2}. \end{aligned}$$

The estimate for $\boldsymbol{g}_{0,k}(\boldsymbol{x})$ follows similarly. □

As an immediate application of the above result and Corollary 3.2 we obtain the following result.

**Corollary 3.31**

$$\|\boldsymbol{g}_0\|_{H^2(\Omega_j)} + \|\nabla\lambda_0\|_{L^2(\Omega_j)} \le C\varrho_j^{-3/2}.$$

*Proof.* By Corollary 3.2, the Hölder estimates, and Lemma 3.30 (with $\Omega_j'$ instead of $\Omega_j$), we obtain

$$\begin{aligned} \|\boldsymbol{g}_0\|_{H^2(\Omega_j)} + \|\nabla\lambda_0\|_{L^2(\Omega_j)} &\le C\varrho_j^{-1}\left(\|\lambda_0\|_{L^2(\Omega_j')} + \|\nabla \boldsymbol{g}_0\|_{L^2(\Omega_j')} + \varrho_j^{-1}\|\boldsymbol{g}_0\|_{L^2(\Omega_j')}\right) \\ &\le C\varrho_j^{1/2}\left(\|\lambda_0\|_{L^\infty(\Omega_j')} + \|\nabla \boldsymbol{g}_0\|_{L^\infty(\Omega_j')} + \varrho_j^{-1}\|\boldsymbol{g}_0\|_{L^\infty(\Omega_j')}\right) \\ &\le C\varrho_j^{-3/2}. \end{aligned}$$

□

### 3.4.2. $L^1(\Omega)$ interpolation estimate for $\lambda_0$

**Theorem 3.32** *For $(\boldsymbol{g}_0, \lambda_0)$ the solution of* (3.3.7a)–(3.3.7c)*, it holds*

$$\|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega)} \le C|\ln h|h.$$

*Proof.* Using the dyadic decomposition and the Cauchy-Schwarz inequality we deduce

$$\begin{aligned}\|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega)} &\leq \|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega_*)} + \sum_{j=0}^{J} \|\lambda_0 - r_h(\lambda_0)\|_{L^1(\Omega_j)} \\ &\leq (Kh)^{3/2} \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_*)} + C \sum_{j=0}^{J} \varrho_j^{3/2} \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j)}. \qquad (3.4.2)\end{aligned}$$

We apply Assumption 3.7 and the $H^1$ regularity as in (2.2.3), which give

$$\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega)} \leq Ch \|\nabla \lambda_0\|_{L^2(\Omega)} \leq Ch \|\delta_h\|_{L^2(\Omega)} \leq Ch^{-1/2}.$$

This implies for the first term in (3.4.2)

$$(Kh)^{3/2} \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_*)} \leq CK^{3/2} h.$$

For the second term, by the approximation estimate in Assumption 3.7, and Corollary 3.31 it follows

$$\|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j)} \leq Ch \|\nabla \lambda_0\|_{L^2(\Omega'_j)} \leq Ch \varrho_j^{-3/2}.$$

Hence, we can conclude

$$\sum_{j=0}^{J} \varrho_j^{3/2} \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j)} \leq \sum_{j=0}^{J} Ch \leq ChJ.$$

From (3.4.1), we see that $J$ scales logarithmically in $h$ and thus get the claimed result. □

### 3.4.3. Local duality argument

In the following theorem, we again consider the sub-domains $\Omega_j$ from the dyadic decomposition in a duality argument. For the error

$$\|\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}\|_{L^2(\Omega'_j)} = \sup_{\substack{\|\boldsymbol{v}\|_{L^2(\Omega)} \leq 1 \\ \boldsymbol{v} \in C_0^\infty(\Omega'_j)}} (\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}, \boldsymbol{v})$$

we can make a duality argument using the dual problem

$$-\Delta \boldsymbol{w} + \nabla \varphi = \boldsymbol{v} \quad \text{in } \Omega, \quad \nabla \cdot \boldsymbol{w} = 0 \quad \text{in } \Omega, \quad \boldsymbol{w} = 0 \quad \text{on } \partial\Omega. \qquad (3.4.3)$$

**Theorem 3.33** *For* $(\boldsymbol{g}_0, \lambda_0)$ *the solution of* (3.3.7a)–(3.3.7c) *and* $\zeta \in (0,1)$ *it holds*

$$\begin{aligned}\|\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}\|_{L^2(\Omega'_j)} \leq Ch \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega'''_j)} + Ch^{\zeta} \varrho_j^{-1/2-\zeta} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} \\ + Ch^{1+\zeta} \varrho_j^{-1/2-\zeta} |\ln h|.\end{aligned}$$

*Proof.* By using (3.4.3) and that $\boldsymbol{g}_0$ and $\boldsymbol{g}_{h,0}$ are divergence-free for $r_h(\varphi)$, the bilinear form $a(\cdot,\cdot)$ from (2.2.2) and Assumption 3.5, it follows

$$\begin{aligned}(\boldsymbol{g}_0-\boldsymbol{g}_{0,h},\boldsymbol{v}) &= (\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}),\nabla\boldsymbol{w})-(\varphi,\nabla\cdot(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}))\\ &=(\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}),\nabla(\boldsymbol{w}-P_h(\boldsymbol{w})))\\ &\quad+(\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}),\nabla P_h(\boldsymbol{w}))-(\varphi-r_h(\varphi),\nabla\cdot(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}))\\ &=(\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}),\nabla(\boldsymbol{w}-P_h(\boldsymbol{w})))\\ &\quad+(\lambda_0-\lambda_{0,h},\nabla\cdot P_h(\boldsymbol{w}))-(\varphi-r_h(\varphi),\nabla\cdot(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}))\\ &=(\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}),\nabla(\boldsymbol{w}-P_h(\boldsymbol{w})))\\ &\quad+(\lambda_0-r_h(\lambda_0),\nabla\cdot(P_h(\boldsymbol{w})-\boldsymbol{w}))-(\varphi-r_h(\varphi),\nabla\cdot(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}))\\ &=:\tau_1+\tau_2+\tau_3.\end{aligned}$$

For $\tau_1$, we split the term

$$\begin{aligned}\tau_1 &= (\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}),\nabla(\boldsymbol{w}-P_h(\boldsymbol{w})))_{\Omega_j'''}+(\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}),\nabla(\boldsymbol{w}-P_h(\boldsymbol{w})))_{\Omega\setminus\Omega_j'''}\\ &=:\tau_{11}+\tau_{12}.\end{aligned}$$

We then can estimate $\tau_{11}$ using Assumption 3.7 for $P_h$

$$\begin{aligned}\tau_{11} &\le \|\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')}\|\nabla(\boldsymbol{w}-P_h(\boldsymbol{w}))\|_{L^2(\Omega)}\\ &\le Ch\|\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')}\|\boldsymbol{w}\|_{H^2(\Omega)}\le Ch\|\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')}.\end{aligned}$$

Now we use [72, (5.11)] and Assumption 3.8 to see that

$$\tau_{12}\le Ch^{\zeta}\|\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h})\|_{L^1(\Omega)}\|\boldsymbol{w}\|_{C^{1+\zeta}(\Omega\setminus\Omega_j'')}\le Ch^{\zeta}\varrho_j^{-1/2-\zeta}\|\nabla(\boldsymbol{g}_0-\boldsymbol{g}_{0,h})\|_{L^1(\Omega)}.$$

Analogously, we split $\tau_2$

$$\begin{aligned}\tau_2 &= -(\lambda_0-r_h(\lambda_0),\nabla\cdot(\boldsymbol{w}-P_h(\boldsymbol{w}))_{\Omega_j'''}-(\lambda_0-r_h(\lambda_0),\nabla\cdot(\boldsymbol{w}-P_h(\boldsymbol{w}))_{\Omega\setminus\Omega_j'''}\\ &=:\tau_{21}+\tau_{22}.\end{aligned}$$

Then again, we use approximation results and Corollary 3.31, to see

$$\tau_{21}\le Ch^2\|\nabla\lambda_0\|_{L^2(\Omega_j''')}\|\boldsymbol{w}\|_{H^2(\Omega)}\le Ch^2\|\nabla\lambda_0\|_{L^2(\Omega_j''')}\le Ch^2\varrho_j^{-3/2}.$$

For the second term, we apply again the Hölder estimate, Theorem 3.32, and [72, (5.11)]

$$\begin{aligned}\tau_{22} &\le \|\lambda_0-r_h(\lambda_0)\|_{L^1(\Omega)}\|\nabla(\boldsymbol{w}-P_h(\boldsymbol{w}))\|_{L^\infty(\Omega\setminus\Omega_j''')}\\ &\le C|\ln h|h^{1+\zeta}\|\boldsymbol{w}\|_{C^{1+\zeta}(\Omega\setminus\Omega_j'')}\le C|\ln h|h^{1+\zeta}\varrho_j^{-1/2-\zeta}.\end{aligned}$$

It remains to deal with $\tau_3$, we split again

$$\tau_3\le|(\varphi-r_h(\varphi),\nabla\cdot(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}))_{\Omega_j'''}|+|(\varphi-r_h(\varphi),\nabla\cdot(\boldsymbol{g}_0-\boldsymbol{g}_{0,h}))_{\Omega\setminus\Omega_j'''}|=:\tau_{31}+\tau_{32}.$$

Analogously to before, we estimate

$$\tau_{31} \leq \|\varphi - r_h(\varphi)\|_{L^2(\Omega_j''')} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')} \leq Ch\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')} \quad \text{and}$$
$$\tau_{32} \leq \|\varphi - r_h(\varphi)\|_{L^\infty(\Omega\setminus\Omega_j''')} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} \leq Ch^\zeta \varrho_j^{-1/2-\zeta} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)}.$$

The estimate for $\|\varphi - r_h(\varphi)\|_{L^\infty(\Omega\setminus\Omega_j''')}$ is given in [72, p. 17]. Summing up, we have

$$\begin{aligned}\|\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}\|_{L^2(\Omega_j)} \leq Ch\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')} + Ch^\zeta \varrho_j^{-1/2-\zeta} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} \\ + h^2 \varrho_j^{-3/2} + Ch^{1+\zeta} \varrho_j^{-1/2-\zeta} |\ln h|.\end{aligned}$$

Now, because $h \leq \varrho_j$ due to (3.4.1) and $\zeta \leq 1$, it holds $h^2 \varrho_j^{-3/2} \leq h^{1+\zeta} \varrho_j^{-1/2-\zeta}$. Thus, we arrive at the conclusion of the theorem. □

### 3.4.4. $L^1(\Omega)$ estimate and weighted estimate

Now we can proceed with the proof of Lemma 3.24.

*Proof of Lemma 3.24.* We again use the dyadic decomposition and the Cauchy-Schwarz inequality to see

$$\begin{aligned}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} &\leq \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega_*)} + \sum_{j=0}^{J} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega_j)} \\ &\leq (Kh)^{3/2} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega)} + C\sum_{j=0}^{J} \varrho_j^{3/2} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j)}.\end{aligned} \tag{3.4.4}$$

Applying Proposition 3.11, Assumption 3.7, $H^2$ regularity as stated in (2.2.3), and (3.3.2) leads to the following estimate for the first term

$$\begin{aligned}h^{3/2}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega)} &\leq Ch^{5/2}\Big(\|\boldsymbol{g}_0\|_{H^2(\Omega)} + \|\lambda_0\|_{H^1(\Omega)}\Big) \\ &\leq Ch^{5/2}\|\delta_h\|_{L^2(T_{\boldsymbol{x}_0})} \leq Ch.\end{aligned}$$

In the following, we consider the second term for which we want to apply the local energy estimate from Proposition 3.12:

$$\begin{aligned}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j)} \leq\; & C\big(\|\nabla(\boldsymbol{g}_0 - P_h(\boldsymbol{g}_0))\|_{L^2(\Omega_j')} + \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j')}\big) \\ & + C(\varepsilon\varrho_j)^{-1}\|\boldsymbol{g}_0 - P_h(\boldsymbol{g}_0)\|_{L^2(\Omega_j')} + \varepsilon\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j')} \\ & + C(\varepsilon\varrho_j)^{-1}\|\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}\|_{L^2(\Omega_j')}.\end{aligned} \tag{3.4.5}$$

For the first two terms we use approximation results and Corollary 3.31, to obtain

$$\begin{aligned}\|\nabla(\boldsymbol{g}_0 - P_h(\boldsymbol{g}_0))\|_{L^2(\Omega_j')} + \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j')} &\leq Ch\Big(\|\boldsymbol{g}_0\|_{H^2(\Omega_j'')} + \|\lambda_0\|_{H^1(\Omega_j'')}\Big) \\ &\leq Ch\varrho_j^{-3/2}.\end{aligned}$$

The contribution to the sum is given by

$$\sum_{j=0}^{J} \varrho_j^{3/2} (\|\nabla(\boldsymbol{g}_0 - P_h(\boldsymbol{g}_0))\|_{L^2(\Omega_j')} + \|\lambda_0 - r_h(\lambda_0)\|_{L^2(\Omega_j')}) \le ChJ \le C|\ln h|h,$$

where due to (3.4.1) we see that $J \sim |\ln h|$. Similarly, we see

$$(\varepsilon\varrho_j)^{-1}\|\boldsymbol{g}_0 - P_h(\boldsymbol{g}_0)\|_{L^2(\Omega_j')} \le C\frac{h}{\varepsilon\varrho_j} h\varrho_j^{-3/2}. \tag{3.4.6}$$

For $\zeta > 0$, it holds

$$\sum_{j=0}^{J}\left(\frac{h}{\varrho_j}\right)^{\zeta} \le h^{\zeta}\sum_{j=0}^{J} 2^{j\zeta} \le Ch^{\zeta}2^{\zeta J} \le CK^{-\zeta}. \tag{3.4.7}$$

Thus, we get by summing up (3.4.6) and applying (3.4.7) with $\zeta = 1$ that $\sum_{j=0}^{J} C\frac{h}{\varepsilon\varrho_j}h \le C(K\varepsilon)^{-1}h$. To summarize our results so far, we define $M_j = \varrho_j^{3/2}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j)}$, $M_j' = \varrho_j^{3/2}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j')}$ and substitute into (3.4.5)

$$\sum_{j=0}^{J} M_j \le Ch|\ln h| + C(K\varepsilon)^{-1}h + \varepsilon\sum_{j=0}^{J} M_j' + C\sum_{j=0}^{J}(\varepsilon\varrho_j)^{-1}\varrho_j^{3/2}\|\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}\|_{L^2(\Omega_j')}.$$

Next, we apply Theorem 3.33 to the last term and get

$$\begin{aligned}\sum_{j=0}^{J} M_j &\le Ch|\ln h| + C(K\varepsilon)^{-1}h + \varepsilon\sum_{j=0}^{J} M_j' \\ &\quad + C\varepsilon^{-1}\sum_{j=0}^{J}\left(\varrho_j^{1/2}h\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')} + \left[\frac{h}{\varrho_j}\right]^{\zeta}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} + |\ln h|h\left[\frac{h}{\varrho_j}\right]^{\zeta}\right).\end{aligned}$$

We expand the sum over the last three terms so that we get

$$\begin{aligned}\sum_{j=0}^{J} M_j &\le C\Big(h|\ln h| + (K\varepsilon)^{-1}h + \varepsilon\sum_{j=0}^{J} M_j' + \frac{h}{\varrho_J}\varepsilon^{-1}\sum_{j=0}^{J}\varrho_j^{3/2}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')}\Big) \\ &\quad + C\varepsilon^{-1}\sum_{j=0}^{J}\left[\frac{h}{\varrho_j}\right]^{\zeta}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} + C|\ln h|h\varepsilon^{-1}\sum_{j=0}^{J}\left[\frac{h}{\varrho_j}\right]^{\zeta}.\end{aligned}$$

Now we can again use (3.4.7) on the last two summands to arrive at

$$\begin{aligned}\sum_{j=0}^{J} M_j &\le Ch|\ln h| + C\varepsilon\sum_{j=0}^{J} M_j' + CK^{-\zeta}\varepsilon^{-1}\Big(\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} + h|\ln h|\Big) \\ &\quad + C(K\varepsilon)^{-1}\sum_{j=0}^{J}\varrho_j^{3/2}\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')},\end{aligned}$$

where we also used that $h/\varrho_J \leq K^{-1}$ and $K > 1$. Now for the second and last term, we easily see

$$\sum_{j=0}^{J} M_j' + \sum_{j=0}^{J} \varrho_j^{3/2} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j''')} \leq C \sum_{j=0}^{J} M_j + C(Kh)^{3/2} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h}\|_{L^2(\Omega_*)},$$

where the last term is again bounded by $CK^{3/2}h$. Combined, this means we have for constant $K > 1$ and $\varepsilon > 0$

$$\sum_{j=0}^{J} M_j \leq C|\ln h|h + C((K\varepsilon)^{-1} + \varepsilon) \sum_{j=0}^{J} M_j + CK^{3/2}\varepsilon h + CK^{1/2}\varepsilon^{-1}h \\ + CK^{-\zeta}\varepsilon^{-1}\Big(\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} + |\ln h|h\Big).$$

We make $C\varepsilon < 1/4$ and $C(K\varepsilon)^{-1} < 1/4$ by choosing $\varepsilon$ small and $K$ big enough. After kicking back the sum to the left-hand side this leads to

$$\sum_{j=0}^{J} M_j \leq C_{K,\varepsilon} h|\ln h| + CK^{-\zeta}\varepsilon^{-1} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)}.$$

We now treat $\varepsilon$ as a constant. Finally, substituting this into (3.4.4)

$$\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} \leq C_{K,\varepsilon} h|\ln h| + CK^{-\zeta} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} \tag{3.4.8}$$

and choosing $K$ large enough such that $CK^{-\zeta} < 1/2$, we get the result. □

As a corollary of the theorem, we get the respective estimate for weighted norms.

*Proof of Corollary 3.25.* This corollary directly follows using the same techniques as above and the fact $\sigma(\boldsymbol{x}) \sim \varrho_j$ on $\Omega_j$. We start by splitting the left-hand side according to the dyadic decomposition

$$\begin{aligned} \|\sigma^{3/2}\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega)} &\leq \|\sigma^{3/2}\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_*)} + \sum_{j=0}^{J} \|\sigma^{3/2}\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j)} \\ &\leq C(\kappa h)^{3/2} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_*)} + C \sum_{j=0}^{J} \varrho_j^{3/2} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega_j)}. \end{aligned}$$

Without loss of generality, we can assume $\kappa = K$. After going through the same steps as in the proof of Lemma 3.24, particularly (3.4.4), we end up with the right-hand side of (3.4.8)

$$\|\sigma^{3/2}\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega)} \leq Ch|\ln h| + CK^{-\zeta} \|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)}.$$

Now applying Lemma 3.24 to estimate $\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)}$ we arrive at the result. □

Similarly, we can conclude the following result.

*Proof of Corollary 3.22.* Again using the fact $\sigma(\boldsymbol{x}) \sim \varrho_j$ on $\Omega_j$, we start by splitting the left-hand side according to the dyadic decomposition

$$\|\sigma^{3/2}\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^2(\Omega)} \\ \leq C(\kappa h)^{3/2}\|\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^2(\Omega_*)} + C\sum_{j=0}^{J}\varrho_j^{3/2}\|\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^2(\Omega_j)}.$$

As before, we can assume $\kappa = K$. This is equal to the term introduced by the dyadic decomposition in the proof of the result in [72]. Again, following the same steps as there, we get

$$\|\sigma^{3/2}\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^2(\Omega)} \leq C + C\|\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^1(\Omega)},$$

where $C$ depends on the constants introduced in the proof of the result in [72]. Nonetheless, applying Lemma 3.21 to estimate $\|\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^1(\Omega)}$ we arrive at the result. □

### 3.4.5. Proof of Lemma 3.29

*Proof of Lemma 3.29.* We use the dyadic decomposition introduced in the beginning of Section 3.4 to get the following estimate due to $\sigma \sim \varrho_j$ on $\Omega_j$ ($\sigma \sim Kh$ on $\Omega_*$) and $|\Omega_*| \sim Kh^3$

$$\|\sigma^{3/2}\nabla\lambda_0\|_{L^2(\Omega)} + \|\nabla\lambda_0\|_{L^1(\Omega)} \leq Ch^{3/2}\|\nabla\lambda_0\|_{L^2(\Omega)} + C\sum_{j=0}^{J}\varrho_j^{3/2}\|\nabla\lambda_0\|_{L^2(\Omega_j)}.$$

The first summand is bounded by a constant $C$ because of (2.2.3) and (3.3.2). By Corollary 3.31 we see that $\|\nabla\lambda_0\|_{L^2(\Omega_j)} \leq C\varrho_j^{-3/2}$ and as a result

$$\sum_{j=0}^{J}\varrho_j^{3/2}\|\nabla\lambda_0\|_{L^2(\Omega_j)} \leq C\sum_{j=0}^{J}1 = CJ \leq C|\ln h|.$$

This proves the result for $\lambda_0$. Similarly, we may argue for $\nabla^2\boldsymbol{g}_0$, where we have as above

$$\|\sigma^{3/2}\nabla^2\boldsymbol{g}_0\|_{L^2(\Omega)} + \|\nabla^2\boldsymbol{g}_0\|_{L^1(\Omega)} \leq Ch^{3/2}\|\nabla^2\boldsymbol{g}_0\|_{L^2(\Omega)} + \sum_{j=0}^{J}\varrho_j^{3/2}\|\nabla^2\boldsymbol{g}_0\|_{L^2(\Omega_j)}.$$

Again the application of (2.2.3) and (3.3.2) as well as Corollary 3.31 shows the result. □

## 3.5. Estimates for the pressure

We now consider estimates for the remaining component of our Stokes system, the pressure. Similarly to before, let $\delta_h$ denote a smooth delta function on the tetrahedron, where the maximum of the discrete pressure is attained. We may define the following regularized Green's function to deal with the pressure

$$-\Delta\boldsymbol{G} + \nabla\Lambda = \boldsymbol{0} \quad \text{in } \Omega, \quad \nabla\cdot\boldsymbol{G} = \delta_h - \phi \quad \text{in } \Omega, \quad \boldsymbol{G} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{3.5.1}$$

By construction we have $\int_\Omega \delta_h(\boldsymbol{x}) - \phi(\boldsymbol{x}) d\boldsymbol{x} = 0$. This also allows us to apply similar arguments as in [65, 72], only with different bounds for the appearing $\boldsymbol{u}_h$ terms.

The global case has already been discussed in [65, 72], thus we now focus on localized estimates. As before, we need some auxiliary results which we state now.

**Proposition 3.34**

$$\|\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^1(\Omega)} + \|r_h(\Lambda) - \Lambda\|_{L^1(\Omega)} \leq C.$$

A proof of this is given in [72, Lemma 5.4]. The following corollary follows by the same arguments as in the proofs of Corollary 3.22 and Corollary 3.25.

**Corollary 3.35**

$$\|\sigma^{3/2}\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma^{3/2}(r_h(\Lambda) - \Lambda)\|_{L^2(\Omega)} \leq C.$$

*Proof of Theorem 3.13 (pressure).* For this we again split the domain into $D_2$ and $\Omega \backslash D_2$ and only consider $\boldsymbol{x}_0 \in T_{\boldsymbol{x}_0} \subset D_1$.

The pointwise estimate of $p_h$ can be expanded in the following way

$$p_h(\boldsymbol{x}_0) = (p_h, \delta_h) = (p_h, \delta_h - \phi) + (p_h, \phi) = (p_h, \delta_h - \phi) + (p_h - p, \phi) + (p, \phi).$$

The last two terms we may estimate using Proposition 3.11

$$(p_h - p, \phi) + (p, \phi) \leq C\|\phi\|_{L^2(\Omega)}\Big(\|p - p_h\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big) \leq C\Big(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big).$$

By assumption $\phi$ is bounded on $\Omega$. For the first term, we can see by Assumption 3.5 that

$$\begin{aligned}(p_h, \delta_h - \phi) &= (p_h, \nabla \cdot \boldsymbol{G}) = (p_h, \nabla \cdot P_h(\boldsymbol{G})) \\ &= (p, \nabla \cdot P_h(\boldsymbol{G})) + (p_h - p, \nabla \cdot P_h(\boldsymbol{G})) =: I_1 + I_2.\end{aligned}$$

For $I_1$, we derive the following estimate

$$\begin{aligned} I_1 &= (p, \nabla \cdot (P_h(\boldsymbol{G}) - \boldsymbol{G})) + (p, \delta_h - \phi) \\ &\leq \|p\|_{L^\infty(D_2)}\Big(\|\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^1(\Omega)} + \|\phi\|_{L^1(\Omega)} + \|\delta_h\|_{L^1(\Omega)}\Big) \\ &\quad + C_\varrho\|p\|_{L^2(\Omega)}\Big(\|\sigma^{3/2}\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma^{3/2}\phi\|_{L^2(\Omega)} + \|\sigma^{3/2}\delta_h\|_{L^2(\Omega)}\Big) \\ &\leq C\|p\|_{L^\infty(D_2)} + C_\varrho\|p\|_{L^2(\Omega)}. \end{aligned}$$

To arrive at this bound, we used Lemma 3.20, Proposition 3.34, and Corollary 3.35 as well as $\|\sigma^{3/2}\phi\|_{L^2(\Omega)} \leq \|\phi\|_{L^2(\Omega)}\|\sigma^{3/2}\|_{L^\infty(\Omega)} \leq C$. Using (2.3.2) and (3.5.1) we see for $I_2$

$$\begin{aligned} I_2 &= (\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \nabla P_h(\boldsymbol{G})) = (\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \nabla \boldsymbol{G}) + (\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})) \\ &= -(\Lambda, \nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)) + (\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})) \\ &= -(\Lambda - r_h(\Lambda), \nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)) + (\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})) \\ &\leq \Big(\|\nabla \boldsymbol{u}\|_{L^\infty(D_*)} + \|\nabla \boldsymbol{u}_h\|_{L^\infty(D_*)}\Big)\Big(\|\Lambda - r_h(\Lambda)\|_{L^1(\Omega)} + \|\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^1(\Omega)}\Big) \\ &\quad + C_\varrho\Big(\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega)}\Big)\Big(\|\sigma^{3/2}(\Lambda - r_h(\Lambda))\|_{L^2(\Omega)} + \|\sigma^{3/2}\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^2(\Omega)}\Big). \end{aligned}$$

Here again we use that $\sigma^{-1}$ is bounded by $\varrho^{-1}$ on $\Omega\backslash D_2$ and choose $D_*$ appropriately such that we can apply Theorem 3.13 for the velocity, e.g., $D_* = B_{r^*}(\tilde{\boldsymbol{x}}) \cap \Omega$ with $r^* = r + \varrho/2$. Finally, a bound for $\boldsymbol{u}_h$ in $H^1$ follows by Proposition 3.11 and thus we get

$$I_2 \leq C\Big(\|\nabla \boldsymbol{u}\|_{L^\infty(D_2)} + \|p\|_{L^\infty(D_2)}\Big) + C_\varrho\Big(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big).$$

□

## 3.6. Assumptions and main results in two dimensions

In this section we give a short derivation of the respective local estimates in $L^\infty$ and $W^{1,\infty}$ for the two-dimensional case. Note that the arguments for the global and local scenario made in the three-dimensional case are independent of the dimension apart from the auxiliary estimates. For two dimensions the respective estimates of the regularized Green's functions and the Ritz projection are all available from the literature albeit under slightly different assumptions on the finite element space. Because of these slightly different assumptions in [47] and to give a concise overview of the respective references we provide the results on polygons separately in this section.

*Remark* 3.36 The technique used in the three-dimensional case to prove the auxiliary results in the previous sections should carry over to two dimensions. But to make a rigorous argument one must discuss the local energy estimates in [72] and respective Green's function estimates (as in Proposition 3.3) in the two-dimensional case. The first point seems to be attainable in a straightforward manner and the second point can be shown similarly to the Poisson problem in [45, Lemma 2.1]. Although, we are not aware of any such result in the literature, obtaining such results is straightforward, but lengthy. Since the auxiliary results in two dimensions can be shown using a weighted technique and are available in [47], we instead refer to them in the following form below.

We now state the required assumptions, the necessary auxiliary results, their references, and finally the local estimates. From now on let $\Omega \subset \mathbb{R}^2$, a convex polygonal domain, and consider the two-dimensional analogs $\boldsymbol{u}$, $p$, $\boldsymbol{f}$, and their finite element discretization as well as the respective two-dimensional function and finite element spaces. The basic results and requirements for the continuous problem in Section 3.2.1 and for the discrete problem in Section 2.3 still apply, as referenced in these sections.

As stated in [64], assume that we have approximation operators $P_h \in \mathcal{L}(H_0^1(\Omega)^2; \boldsymbol{V}_h)$ and $r_h \in \mathcal{L}(L^2(\Omega); \bar{M}_h)$ which fulfill the two-dimensional versions of Assumptions 3.4–3.7, and in addition the following super-approximation properties.

**Assumption 3.37** (Super-Approximation II) *Let* $\mu \in [2,3]$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$, *and* $\boldsymbol{\psi} = \sigma^\mu \boldsymbol{v}_h$, *then*

$$\|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi} - P_h(\boldsymbol{\psi}))\|_{L^2(\Omega)} \leq C\|\sigma^{\mu/2-1}\boldsymbol{v}_h\|_{L^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h$$

*and if* $q_h \in \bar{M}_h$ *and* $\xi = \sigma^\mu q_h$, *then*

$$\|\sigma^{-\mu/2}(\xi - r_h(\xi))\|_{L^2(\Omega)} \leq Ch\|\sigma^{\mu/2-1}q_h\|_{L^2(\Omega)} \quad \forall q_h \in \bar{M}_h.$$

As in the three-dimensional case, this holds for Taylor-Hood finite element spaces, but for $\mathcal{P}_k - \mathcal{P}_{k-1}$ with $k \geq 2$, see, e.g., [64, Theorem 6.3]. Apart from this, we need to adapt the estimates for $\delta_h$ and $\sigma$. For the two-dimensional versions we get

$$\|\delta_h\|_{W^k_q(T_{\boldsymbol{x}_0})} \leq Ch^{-k-2(1-1/q)}, \quad 1 \leq q \leq \infty,\ k = 0, 1, \quad \text{and for } \nu > 0 \tag{3.6.1}$$
$$\|\sigma^\nu \nabla_k \delta_h\|_{L^2(\Omega)} \leq 2^{\nu/2} C \kappa^\nu h^{\nu-k-1} \quad k = 0, 1.$$

Let $(\boldsymbol{g}_1, \lambda_1)$ and $(\boldsymbol{g}_0, \lambda_0)$ denote the two-dimensional regularized Green's functions, defined as in Section 3.3 but in two dimensions. Then we get the following convergence estimates for their discrete counterparts. The estimates needed when deriving $W^{1,\infty}$ velocity estimates,

$$\|\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^1(\Omega)} \leq C, \qquad \|\sigma\nabla(\boldsymbol{g}_1 - \boldsymbol{g}_{1,h})\|_{L^2(\Omega)} \leq C$$

follow from [64, Theorem 8.1] using (3.3.3) and similarly for the pressure estimates where we need

$$\|\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^1(\Omega)} + \|r_h(\Lambda) - \Lambda\|_{L^1(\Omega)} \leq C,$$
$$\|\sigma\nabla(P_h(\boldsymbol{G}) - \boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma(r_h(\Lambda) - \Lambda)\|_{L^2(\Omega)} \leq C,$$

which can be found on [64, p. 328]. In the $L^\infty$ case for the velocity we get

$$\|\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^1(\Omega)} \leq C|\ln h|h, \qquad \|\sigma\nabla(\boldsymbol{g}_0 - \boldsymbol{g}_{0,h})\|_{L^2(\Omega)} \leq C|\ln h|^{1/2}h$$

from [47, Theorem 4.1, Proof of Theorem 4.2]. One step in the proofs in [47] (in [47, Lemma 4.1, eq. (2.12)]) uses a slightly different assumption compared to Assumption 3.37, but this assumption can be replaced by Assumption 3.37.

The respective version of Lemma 3.29 is given by [47, Lemma 3.1]. Finally, the estimate for the (vectorial) Ritz projection $R_h$ in two dimensions

$$\|R_h \boldsymbol{z}\|_{L^\infty(\Omega)} \leq C|\ln h|^{\bar{k}} \|\boldsymbol{z}\|_{L^\infty(\Omega)},$$

where $\bar{k} = 1$ for $k = 1$ and $\bar{k} = 0$ for $k \geq 2$, is given in [109]. Note, that the local maximum norm estimates for the Ritz projection from [84] hold as well in two dimensions. Thus, using the same techniques as in Section 3.3 we get the following theorems for $\Omega \subset \mathbb{R}^2$.

**Theorem 3.38** (Interior $W^{1,\infty}$ estimate for the velocity and $L^\infty$ estimate for the pressure) *Under the assumptions above,* $\Omega_1 \subset \Omega_2 \subset \Omega$ *with* $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho \geq \bar{\kappa}h$ *and if* $(\boldsymbol{u}, p) \in (W^{1,\infty}(\Omega_2)^2 \times L^\infty(\Omega_2)) \cap (H^1_0(\Omega)^2 \times L^2_0(\Omega))$ *is the solution to* (3.1.1a)–(3.1.1c)*, then it holds for* $(\boldsymbol{u}_h, p_h)$ *the solution to* (2.3.2)*:*

$$\|\nabla \boldsymbol{u}_h\|_{L^\infty(\Omega_1)} + \|p_h\|_{L^\infty(\Omega_1)}$$
$$\leq C\Big(\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega_2)} + \|p\|_{L^\infty(\Omega_2)}\Big) + C_\varrho\Big(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}\Big).$$

*Here, the constant* $C_\varrho$ *depends on* $\varrho$*.*

**Theorem 3.39** (Interior $L^\infty$ error estimate for the velocity) *Under the assumptions above,* $\Omega_1 \subset \Omega_2 \subset \Omega$ *with* $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho \geq \bar{\kappa} h$ *and if* $(\boldsymbol{u}, p) \in (L^\infty(\Omega_2)^2 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^2 \times L_0^2(\Omega))$ *is the solution to* (3.1.1a)–(3.1.1c)*, then it holds for* $(\boldsymbol{u}_h, p_h)$ *the solution to* (2.3.2)*:*

$$\|\boldsymbol{u}_h\|_{L^\infty(\Omega_1)} \leq C|\ln h|\Big(\|\boldsymbol{u}\|_{L^\infty(\Omega_2)} + h\|p\|_{L^\infty(\Omega_2)}\Big) + C_\varrho |\ln h|^{1/2}\Big(h\|\boldsymbol{u}\|_{H^1(\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)} + h\|p\|_{L^2(\Omega)}\Big).$$

*Here, the constant* $C_\varrho$ *depends on* $\varrho$*.*

# Chapter 4.

# Approximation error estimates for an optimal control problem with pointwise tracking

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## 4.1. Introduction

For $\Omega \subset \mathbb{R}^3$, an open non-empty convex polyhedral domain, we consider the following pointwise tracking-type optimal control problem. Let $\{\boldsymbol{x}_i\}_{i\in\mathcal{I}} \neq \emptyset$ be a finite subset of $\Omega$ and $\{\boldsymbol{\xi}_i\}_{i\in\mathcal{I}}$ a corresponding set in $\mathbb{R}^3$. We denote the space of controls as $Q = L^2(\Omega)^3$. Then, the pointwise tracking-type optimal control problem is given by

$$\text{Minimize } J(\boldsymbol{u}, \boldsymbol{q}) = \frac{1}{2}\sum_{i\in\mathcal{I}}(\boldsymbol{u}(\boldsymbol{x}_i) - \boldsymbol{\xi}_i)^2 + \frac{\alpha}{2}\|\boldsymbol{q}\|^2_{L^2(\Omega)} \quad \text{for } \boldsymbol{q} \in Q, \text{ subject to}$$

$$-\Delta\boldsymbol{u} + \nabla p = \boldsymbol{q} \qquad \text{in } \Omega, \tag{4.1.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{4.1.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega, \text{and} \tag{4.1.1c}$$

$$\boldsymbol{a} \le \boldsymbol{q}(\boldsymbol{x}) \le \boldsymbol{b} \quad \text{componentwise for a.a. } \boldsymbol{x} \in \Omega, \tag{4.1.1d}$$

for $\boldsymbol{a} < \boldsymbol{b}$ componentwise, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, and $\alpha > 0$. We choose $p$ to have zero mean. The space of *admissible* controls fulfilling (4.1.1d) is denoted by $Q_{ad}$. Similar to the corresponding elliptic optimal control problem, which is discussed in [10, 18, 22, 33], it quickly follows that the point evaluations of $\boldsymbol{u}$ in $J$ lead to singular sources on the right-hand side of the adjoint equation

$$-\Delta\boldsymbol{z} + \nabla r = \sum_{i\in\mathcal{I}}(\boldsymbol{u}(\boldsymbol{x}_i) - \boldsymbol{\xi}_i)\delta_{\boldsymbol{x}_i} \qquad \text{in } \Omega, \tag{4.1.2a}$$

$$\nabla\cdot\boldsymbol{z} = 0 \qquad \text{in } \Omega, \tag{4.1.2b}$$

$$\boldsymbol{z} = \boldsymbol{0} \qquad \text{on } \partial\Omega, \tag{4.1.2c}$$

with $r$ having zero mean. We denote the sum on the right-hand side of (4.1.2a) as $D_\Sigma$ to simplify notation.

The Dirac source $\delta_{\boldsymbol{x}_i}$ is supported at $\boldsymbol{x}_i$. In spite of the singularities on the right-hand side it is possible to derive almost optimal convergence rates for the control in the control-constrained case when the control is discretized using piecewise constant functions.

This analysis is motivated by recent interest in the Stokes pointwise tracking-type problem in [57]. Let $h > 0$ denote the discretization parameter describing the maximal mesh size. Using cell-wise constant discretization for the control space, a $O(|\ln h|^3 h)$ convergence rate is proven in [57, Theorem 3, Remark 2] for the control approximation error in two dimensions and $O(|\ln h| h^{1/2})$ in three dimensions based on new weighted stability results in [48]. Similar analysis has already been conducted for the standard Poisson problem in [10, 18, 22, 33]. Furthermore, the authors of [57] give references for potential applications and also discuss an optimal control problem featuring Dirac sources on the right-hand side of the state equation.

Related to this problem are state constrained optimal control problems for the Stokes system as introduced in [40]. State constraints also lead to measure-valued right-hand sides of the adjoint equation. The case of state constraints has been already discussed at length for the Poisson problem, see for example [26, 96].

Using the results for local pointwise estimates of the Stokes problem in Chapter 3 we improve the estimate in three dimensions to $O(|\ln h|^{1/3} h^{5/6})$. The technique we employ is similar to the approach used in [18] but significantly different in some details, in particular in how we handle the behavior of the solution of the adjoint equation close to the singularities. While for the respective Poisson problem the absolute value of the solution of the adjoint equation grows towards infinity the closer it is to a singularity, this does not happen in the case of the Stokes problem for certain parameter settings.

Here we consider Taylor-Hood finite elements of an order greater than or equal to three for the solutions of the discrete state and discrete adjoint state equations. For the control, we consider a variational discretization as in [78] as well as discretization with cell-wise constant functions as in [18, 57].

In the following we begin our analysis by recalling and introducing basic properties of the optimal control problem (4.1.1). Next, we consider the discretized problem and multiple approximation error results for the quantities involved, finally leading up the approximation error estimates for the control.

## 4.2. Preliminaries and regularity results

### 4.2.1. Regularity of solutions to state and adjoint state equations

The $H^2$ regularity estimate (2.2.3) shows that (4.1.1) is well-defined and we can introduce for $\boldsymbol{f} \in L^2(\Omega)^3$ a linear control-to-state mapping for the velocity $S\colon L^2(\Omega)^3 \to C(\bar{\Omega})^3$ and the pressure $S^p\colon L^2(\Omega) \to L^2_0(\Omega) \cap H^1(\Omega)$ such that $S\boldsymbol{f} = \boldsymbol{w}$ and $S^p\boldsymbol{f} = \varphi$ are the components of the solution to (2.2.1).

Because of the linearity of $S$, $S^p$, the convexity of the cost functional, and the fact that $\alpha > 0$, standard arguments as in [124] lead to the existence of a unique solution to the optimal control problem (4.1.1).

Because of the regularity estimate (2.2.6) it holds that the Stokes problem is well-defined for a right-hand side $\boldsymbol{f} \in W^{-1,s}(\Omega)^3$ with $3/2 - \varepsilon < s < 3 + \varepsilon$ for $\varepsilon > 0$ sufficiently small. So in particular (4.1.2a)–(4.1.2c) is well-defined since here the right-hand side $\boldsymbol{f} \in \mathcal{M}(\Omega)^3 = \boldsymbol{\mathcal{M}}(\Omega) \subset W^{-1,3/2-\varepsilon}(\Omega)^3$. The right-hand side $\boldsymbol{f}$ consists of a linear combination of regular Borel measures in the space $\boldsymbol{\mathcal{M}}(\Omega)$ which can be motivated as the dual space of continuous functions on $\Omega$, zero at the boundary. The inclusion $\boldsymbol{\mathcal{M}}(\Omega) \subset W^{-1,3/2-\varepsilon}(\Omega)^3$ follows by the well-known Sobolev embedding theorem. Thus, based on this consideration and (2.2.6) we conclude that there exists a solution $(\boldsymbol{z}, r) \in W_0^{1,s}(\Omega)^3 \times L_0^s(\Omega)$ for $s < 3/2$.

### 4.2.2. Optimality condition and derivatives

First we consider derivatives of the cost functional, which also motivates the introduction of the adjoint problem. For $\boldsymbol{q} \in Q_{ad}$ we define the reduced cost functional as $j(\boldsymbol{q}) = J(S\boldsymbol{q}, \boldsymbol{q})$.

**Lemma 4.1** *For $\boldsymbol{q}, \delta\boldsymbol{q} \in Q$, the directional Fréchet derivative of the reduced cost functional $j$ is given by*

$$j'(\boldsymbol{q})(\delta\boldsymbol{q}) = (\alpha\boldsymbol{q} + \boldsymbol{z}, \delta\boldsymbol{q}),$$

*where $\boldsymbol{z} \in W^{1,s}(\Omega)^3$ solves*

$$\begin{aligned} -\Delta\boldsymbol{z} + \nabla r &= \sum_{i\in\mathcal{I}} (S\boldsymbol{q}(\boldsymbol{x}_i) - \boldsymbol{\xi}_i)\delta_{\boldsymbol{x}_i} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{z} &= 0 && \text{in } \Omega, \\ \boldsymbol{z} &= \boldsymbol{0} && \text{on } \partial\Omega, \end{aligned}$$

*which corresponds to* (4.1.2a)–(4.1.2c). *The second directional derivative is given for $\boldsymbol{q}, \delta\boldsymbol{q}, \tau\boldsymbol{q} \in Q$ by*

$$j''(\boldsymbol{q})(\delta\boldsymbol{q}, \tau\boldsymbol{q}) = \sum_{i\in\mathcal{I}} (S\delta\boldsymbol{q})(\boldsymbol{x}_i)(S\tau\boldsymbol{q})(\boldsymbol{x}_i) + \alpha(\delta\boldsymbol{q}, \tau\boldsymbol{q}).$$

*Proof.* The explicit derivatives follow directly from the linearity of $S$ and the definition of the Fréchet derivative. □

Using the adjoint equation, it is possible to formulate necessary and sufficient first order optimality conditions following standard arguments which can be found, e.g., in [90, 124].

**Lemma 4.2** *A control $\bar{\boldsymbol{q}} \in Q_{ad}$ with associated state $\bar{\boldsymbol{u}} = S\bar{\boldsymbol{q}} \in (H_0^1(\Omega) \cap H^2(\Omega))^3$ is an optimal solution to the problem* (4.1.1) *if and only if there exists an adjoint state $\bar{\boldsymbol{z}} \in (W_0^{1,s}(\Omega))^3$ such that $\bar{\boldsymbol{u}}$ solves* (4.1.1a)–(4.1.1c) *with right-hand side $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{z}}$ solves* (4.1.2a)–(4.1.2c) *with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on the right-hand side, where $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{z}}$ satisfy the following inequality*

$$(\bar{\boldsymbol{z}} + \alpha\bar{\boldsymbol{q}}, \boldsymbol{q} - \bar{\boldsymbol{q}}) \geq 0 \quad \forall \boldsymbol{q} \in Q_{ad}. \tag{4.2.2}$$

The variational inequality is equivalent to the following projection formula

$$\bar{\boldsymbol{q}} = P_{[\boldsymbol{a},\boldsymbol{b}]}\left(-\frac{1}{\alpha}\bar{\boldsymbol{z}}\right), \tag{4.2.3}$$

where $P_{[\boldsymbol{a},\boldsymbol{b}]}$ is applied componentwise and defined as $P_{[\boldsymbol{a},\boldsymbol{b}]}(\boldsymbol{v}) = \min(\boldsymbol{b}, \max(\boldsymbol{a}, \boldsymbol{v}))$, with min, max also being applied componentwise almost everywhere.

### 4.2.3. Regularity of the optimal solution $\bar{\boldsymbol{q}}$

We derive a regularity result based on (4.2.3) for solutions to (2.2.1) with right-hand side $\boldsymbol{f} = \boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(\Omega)$.

**Lemma 4.3** *Let $\boldsymbol{w}$ be the solution to* (2.2.1) *with right-hand side $\boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(\Omega)$. Then, $P_{[-\boldsymbol{M},\boldsymbol{M}]}(\boldsymbol{w}) \in H_0^1(\Omega)^3$ for every $\boldsymbol{M} \in \mathbb{R}_+^3$.*

*Proof.* We deduce the result similarly to [31, Lemma 3.3] but take into account the additional pressure term. Let $\{\boldsymbol{\mu}_k\}_k \subset L^2(\Omega)^3$ be a sequence, such that $\boldsymbol{\mu}_k \overset{*}{\rightharpoonup} \boldsymbol{\mu}$ and $\|\boldsymbol{\mu}_k\|_{L^1(\Omega)} \leq \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)}$. Then, let $(\boldsymbol{w}_k, \varphi_k) \in (H_0^1(\Omega) \cap H^2(\Omega))^3 \times H^1(\Omega)$ be the solution to

$$-\Delta \boldsymbol{w}_k + \nabla \varphi_k = \boldsymbol{\mu}_k \qquad \text{in } \Omega, \tag{4.2.4a}$$
$$\nabla \cdot \boldsymbol{w}_k = 0 \qquad \text{in } \Omega, \tag{4.2.4b}$$
$$\boldsymbol{w}_k = \boldsymbol{0} \qquad \text{on } \partial\Omega. \tag{4.2.4c}$$

Now since $\boldsymbol{\mathcal{M}}(\Omega)$ is compactly embedded into $W^{-1,s}(\Omega)^3$ for $s < 3/2$, it follows $\boldsymbol{w}_k \to \boldsymbol{w}$ strongly in $W^{1,s}(\Omega)^3$. Now we consider the projection $\boldsymbol{w}_k^M = P_{[-\boldsymbol{M},\boldsymbol{M}]}(\boldsymbol{w}_k)$ which by [126, Corollary 2.1.8] is continuous from $W^{1,s}(\Omega)^3 \to W^{1,s}(\Omega)^3$. Thus, we also have $\boldsymbol{w}_k^M \to \boldsymbol{w}^M$ strongly in $W_0^{1,s}(\Omega)^3$, where $\boldsymbol{w}^M$ is defined as $P_{[-\boldsymbol{M},\boldsymbol{M}]}(\boldsymbol{w})$. Using (4.2.4a)–(4.2.4c) we now can conclude

$$\begin{aligned} \|\nabla \boldsymbol{w}_k^M\|_{L^2(\Omega)}^2 &= (\nabla \boldsymbol{w}_k^M, \nabla \boldsymbol{w}_k^M) \leq (\nabla \boldsymbol{w}_k, \nabla \boldsymbol{w}_k^M) \\ &= (\boldsymbol{\mu}_k, \boldsymbol{w}_k^M) + (\varphi_k, \nabla \cdot \boldsymbol{w}_k^M) \\ &\leq \|\boldsymbol{w}_k^M\|_{L^\infty(\Omega)} \|\boldsymbol{\mu}_k\|_{L^1(\Omega)} \leq |\boldsymbol{M}| \|\boldsymbol{\mu}_k\|_{\boldsymbol{\mathcal{M}}(\Omega)}. \end{aligned} \tag{4.2.5}$$

From this follows that $\{\boldsymbol{w}_k^M\}_k$ is bounded in $H_0^1(\Omega)^3$ and there exist $\tilde{\boldsymbol{w}}^M \in H_0^1(\Omega)^3$ and a subsequence of $\{\boldsymbol{w}_k^M\}_k$ such that $\boldsymbol{w}_k^M \rightharpoonup \tilde{\boldsymbol{w}}^M$ weakly in $H_0^1(\Omega)^3$. Now due to the strong convergence of $\boldsymbol{w}_k^M$ in $W^{1,s}(\Omega)^3$ we get $\boldsymbol{w}^M = \tilde{\boldsymbol{w}}^M \in H_0^1(\Omega)^3$.

Note that in (4.2.5) we made use of [80, Theorem A.1] or [126, Corollary 2.1.8] which guarantee the existence of all weak partial derivatives. In particular those that vanish on neighborhoods on which the projection is active and the function constant. Furthermore, we used that the divergence of $\boldsymbol{w}_k$ is zero. □

Since $\delta_{\boldsymbol{x}_i} \in \boldsymbol{\mathcal{M}}(\Omega)$ we can apply this result for $\bar{\boldsymbol{q}} = P_{[\boldsymbol{a},\boldsymbol{b}]}\left(-\frac{1}{\alpha}\bar{\boldsymbol{z}}\right)$ and we conclude the following corollary.

**Corollary 4.4** *Let $\bar{\boldsymbol{q}}$ be the solution to* (4.1.1)*. Then, $\bar{\boldsymbol{q}} \in H^1(\Omega)^3$.*

*Proof.* Without loss of generality, let $\boldsymbol{b} > |\boldsymbol{a}|$. Then, $P_{[-\boldsymbol{b},\boldsymbol{b}]}(-\alpha^{-1}\bar{\boldsymbol{z}}) \in H_0^1(\Omega)^3$ and due to the continuity of $P_{[\boldsymbol{a},\boldsymbol{b}]}$ in $H^1(\Omega)^3$ (cf. [126, Corollary 2.1.8]) we get the result of the corollary. □

## 4.3. Finite element approximation and error estimates

In the following we discuss finite element spaces for the state and adjoint state equations as well as a discretization of the control space $Q$ and the set of admissible controls $Q_{ad}$.

### 4.3.1. State and control

For the finite element approximation we require the assumptions in Section 3.2.2 to hold. A suitable finite element space is given for example by Taylor-Hood finite elements of orders greater than or equal to three. More details can be found in Remark 3.10. The assumptions regarding the finite element space enable us to use Corollary 3.17 and Remark 3.18 to derive the following lemma.

**Lemma 4.5** *For $\Omega_1 \Subset \Omega_2 \Subset \Omega$ with $\operatorname{dist}(\bar{\Omega}_1, \partial\Omega_2) \geq \varrho \geq \bar{\kappa} h$ and for $(\boldsymbol{w}, \varphi) \in (L^\infty(\Omega_2)^3 \times L^\infty(\Omega_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ the solution to* (2.2.1) *with $\boldsymbol{f} \in L^\infty(\Omega)^3$ and $(\boldsymbol{w}_h, \varphi_h)$ the solution to* (2.3.2)*, we have*

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega_1)} \leq \inf_{(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h} & C|\ln h| \Big( \|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)} + h\|\varphi - l_h\|_{L^\infty(\Omega_2)} \Big) \\ & + C_\varrho |\ln h| \Big( h\|\boldsymbol{w} - \boldsymbol{v}_h\|_{H^1(\Omega)} + \|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^2(\Omega)} + h\|\varphi - l_h\|_{L^2(\Omega)} \Big). \end{aligned}$$

*Here, the constant $C_\varrho$ depends on $\varrho$.*

Similar to the exact solution in Section 4.2 we can define respective control-to-state maps in the discrete case, for the velocity $S_h \colon Q \to \boldsymbol{V}_h$ and the pressure $S_h^p \colon Q \to M_h$ such that $S_h \boldsymbol{f} = \boldsymbol{w}_h$ and $S_h^p \boldsymbol{f} = \varphi_h$ as the components of the solution to (2.3.2).

The set of discrete admissible controls is given by

$$Q_{ad,h} = Q_h \cap Q_{ad}, \tag{4.3.1}$$

where $Q_h$ is the space of piecewise constant functions

$$Q_h = \{\boldsymbol{q} \in L^2(\Omega)^3 \;:\; \boldsymbol{q}|_T \in \mathcal{P}_0(T)^3 \;\forall T \in \mathcal{T}_h\}.$$

For $Q_h$ we now introduce the $L^2$ projection $\pi_h \colon L^2(\Omega)^3 \to Q_h$ of a function $\boldsymbol{q} \in L^2(\Omega)^3$ as $\pi_h \boldsymbol{q} \in Q_h$ satisfying

$$(\pi_h \boldsymbol{q}, \boldsymbol{r}_h) = (\boldsymbol{q}, \boldsymbol{r}_h) \quad \forall \boldsymbol{r}_h \in Q_h. \tag{4.3.2}$$

Using orthogonality, $\pi_h$ can also be characterized as

$$(\pi_h \boldsymbol{q})_i\Big|_T = \frac{1}{|T|}\int_T \boldsymbol{q}_i d\boldsymbol{x}_i \quad \text{for } 1 \le i \le 3$$

on each cell $T \in \mathcal{T}_h$. Now using Poincaré's inequality [82, Theorem 12.30] on each cell $T$ we get for $1 \le s < \infty$ and $\boldsymbol{q} \in W^{1,s}(\Omega)^3$

$$\|\pi_h \boldsymbol{q} - \boldsymbol{q}\|_{L^s(T)} \le Ch\|\nabla \boldsymbol{q}\|_{L^s(T)}. \tag{4.3.3}$$

Summing up, we conclude

$$\|\pi_h \boldsymbol{q} - \boldsymbol{q}\|_{L^s(\Omega)} \le Ch\|\nabla \boldsymbol{q}\|_{L^s(\Omega)}. \tag{4.3.4}$$

Note that while the convergence result holds for $1 \le s < \infty$ we still require $\boldsymbol{q} \in L^2(\Omega)^3$ to apply property (4.3.2).

### 4.3.2. Discrete optimal control problem and optimality conditions

We can then formulate the discrete version of (4.1.1) as

$$\text{Minimize } \; J(\boldsymbol{u}_h, \boldsymbol{q}_h) \quad \text{for } \boldsymbol{q}_h \in Q_{ad,h}$$

subject to

$$a((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, l_h)) = (\boldsymbol{q}_h, \boldsymbol{v}_h) \quad \forall(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h. \tag{4.3.5a}$$

We have the following adjoint problem

$$a((\boldsymbol{z}_h, r_h), (\boldsymbol{v}_h, l_h)) = \sum_{i\in\mathcal{I}}(\boldsymbol{u}_h(\boldsymbol{x}_i) - \boldsymbol{\xi}_i)\boldsymbol{v}_h(\boldsymbol{x}_i) \quad \forall(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h, \tag{4.3.6}$$

which can again be motivated by the following derivatives of the objective functional. For $\boldsymbol{q} \in Q_{ad}$ we define the discrete reduced cost functional $j_h(\boldsymbol{q}) = J(S_h\boldsymbol{q}, \boldsymbol{q})$. We get the following first and second derivatives with respect to $\boldsymbol{q}$ for $j_h$.

**Lemma 4.6** *For $\boldsymbol{q}, \delta\boldsymbol{q} \in Q$, the first directional Fréchet derivative of the reduced cost functional $j_h$ is given by*

$$j_h(\boldsymbol{q})(\delta\boldsymbol{q}) = (\alpha\boldsymbol{q} + \boldsymbol{z}_h, \delta\boldsymbol{q}),$$

*where $\boldsymbol{z}_h \in \boldsymbol{V}_h$ solves*

$$a((\boldsymbol{z}_h, r_h), (\boldsymbol{v}_h, l_h)) = \sum_{i\in\mathcal{I}}(S_h\boldsymbol{q}(\boldsymbol{x}_i) - \boldsymbol{\xi}_i)\boldsymbol{v}_h(\boldsymbol{x}_i) \quad \forall(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h$$

*which corresponds to* (4.3.6). *The second directional derivative is given for $\boldsymbol{q}, \delta\boldsymbol{q}, \tau\boldsymbol{q} \in Q$ by*

$$j_h''(\boldsymbol{q})(\delta\boldsymbol{q}, \tau\boldsymbol{q}) = \sum_{i\in\mathcal{I}}(S_h\delta\boldsymbol{q})(\boldsymbol{x}_i)(S_h\tau\boldsymbol{q})(\boldsymbol{x}_i) + \alpha(\delta\boldsymbol{q}, \tau\boldsymbol{q}). \tag{4.3.7}$$

*Proof.* The form of the derivatives follows as for the continuous case. □

Similarly to the continuous case we then have the following optimality condition.

**Lemma 4.7** *A control $\bar{\boldsymbol{q}}_h \in Q_{ad,h}$ with associated state $\bar{\boldsymbol{u}}_h = S_h\bar{\boldsymbol{q}}_h \in \boldsymbol{V}_h$ is an optimal solution to the problem* (4.3.5) *if and only if there exists an adjoint state $\bar{\boldsymbol{z}}_h \in \boldsymbol{V}_h$ such that $\bar{\boldsymbol{u}}_h$ solves* (4.3.5a) *with right-hand side $\bar{\boldsymbol{q}}_h$ and $\bar{\boldsymbol{z}}_h$ solves* (4.3.6) *with right-hand side $\bar{\boldsymbol{u}}_h$ and $\bar{\boldsymbol{q}}_h$ satisfies the following inequality*

$$(\bar{\boldsymbol{z}}_h + \alpha\bar{\boldsymbol{q}}_h, \boldsymbol{q}_h - \bar{\boldsymbol{q}}_h) \geq 0 \quad \forall \boldsymbol{q}_h \in Q_{ad,h}. \tag{4.3.8}$$

### 4.3.3. Error estimates for the solutions to state and adjoint state equations

In this section, we consider convergence rates for the discrete Stokes problem with bounded right-hand side.

**Lemma 4.8** *Let $\Omega_1 \Subset \Omega_2 \Subset \Omega$, $\boldsymbol{w} \in H_0^1(\Omega)^3$ be the velocity solution to* (2.2.1)*, and $\boldsymbol{w}_h$ the respective finite element velocity solution. Then, for $\boldsymbol{f} \in L^\infty(\Omega)^3$ there holds*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega_1)} \leq C|\ln h|^2 h^2 \|\boldsymbol{f}\|_{L^\infty(\Omega)} + C_\varrho |\ln h| h^2 \|\boldsymbol{f}\|_{L^2(\Omega)}$$

*for* $\operatorname{dist}(\Omega_1, \partial\Omega_2) \geq \varrho > 0$.

*Proof.* Due to Lemma 4.5 we have

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega_1)} \leq \inf_{(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h} & C|\ln h| \Big( \|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)} + h\|\varphi - l_h\|_{L^\infty(\Omega_2)} \Big) \\ & + C_\varrho |\ln h| \Big( h\|\boldsymbol{w} - \boldsymbol{v}_h\|_{H^1(\Omega)} + \|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^2(\Omega)} + h\|\varphi - l_h\|_{L^2(\Omega)} \Big). \end{aligned} \tag{4.3.9}$$

The interpolation error estimate [21, Corollary 4.4.24] shows that we get the expected convergence rates for finite element functions $(\boldsymbol{v}_h, l_h)$ since $(\boldsymbol{w}, \varphi)$ are sufficiently regular. In particular, to use nodal interpolation we conclude from (2.2.5) that $(\boldsymbol{w}, \varphi) \in C^{1,\zeta}(\Omega)^3 \times C^{0,\zeta}(\Omega)$. This then shows the result for the second line in (4.3.9) because of $(\boldsymbol{w}, \varphi) \in H^2(\Omega)^3 \times H^1(\Omega)$. For the first line we can argue by [21, Corollary 4.4.24] (note that the dependence of the constant there on $p$ is good natured due to [21, Lemma 4.4.1]) and Proposition 2.2 that

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)} &+ h\|r - l_h\|_{L^\infty(\Omega_2)} \\ &\leq Ch^{2-3/p} \Big( \|\nabla^2 \boldsymbol{w}\|_{L^p(\Omega_2)} + \|\nabla\varphi\|_{L^p(\Omega_2)} \Big) \\ &\leq Cph^{2-3/p} \|\boldsymbol{f}\|_{L^\infty(\Omega)}. \end{aligned}$$

Choosing $p = |\ln h|$, we get $ph^{2-3/p} \leq C|\ln h| h^2$ and thus follows the result. □

Using this, we can now prove a “dual” result for the adjoint equation.

**Lemma 4.9** *For the velocity solution $\boldsymbol{z} \in W_0^{1,s}(\Omega)^3$ of (4.1.2a)–(4.1.2c) and $\hat{\boldsymbol{z}}_h$ the solution of the respective finite element problem with right-hand side $D_\Sigma$ there holds the following error estimate*

$$\|\boldsymbol{z} - \hat{\boldsymbol{z}}_h\|_{L^1(\Omega)} \leq C|\ln h|^2 h^2 \left( \|\boldsymbol{q}\|_{L^2(\Omega)} + \sum_{i\in\mathcal{I}} |\boldsymbol{\xi}_i| \right).$$

*Proof.* Using a dual formulation with $\boldsymbol{f} = \operatorname{sgn}(\boldsymbol{z} - \hat{\boldsymbol{z}}_h) \in L^\infty(\Omega)^3$ as the right-hand side of (2.2.1) and (2.3.2) we get

$$\begin{aligned}
\|\boldsymbol{z} - \hat{\boldsymbol{z}}_h\|_{L^1(\Omega)} &= (\boldsymbol{f}, \boldsymbol{z} - \hat{\boldsymbol{z}}_h) \\
&= (\nabla \boldsymbol{w}, \nabla(\boldsymbol{z} - \hat{\boldsymbol{z}}_h)) - (\varphi, \nabla \cdot (\boldsymbol{z} - \hat{\boldsymbol{z}}_h)) \\
&= (\nabla(\boldsymbol{w} - \boldsymbol{w}_h), \nabla(\boldsymbol{z} - \hat{\boldsymbol{z}}_h)) + (\varphi, \nabla \cdot \hat{\boldsymbol{z}}_h) + (\nabla \boldsymbol{w}_h, \nabla(\boldsymbol{z} - \hat{\boldsymbol{z}}_h)) \quad (4.3.10)\\
&= (\nabla(\boldsymbol{w} - \boldsymbol{w}_h), \nabla(\boldsymbol{z} - \hat{\boldsymbol{z}}_h)) + (\varphi, \nabla \cdot \hat{\boldsymbol{z}}_h) + (\nabla \cdot \boldsymbol{w}_h, r - \hat{r}_h) \quad (4.3.11)\\
&= (\nabla(\boldsymbol{w} - \boldsymbol{w}_h), \nabla \boldsymbol{z}) + (\varphi, \nabla \cdot \hat{\boldsymbol{z}}_h) \\
&\quad + (\nabla \cdot \boldsymbol{w}_h, r) - (\nabla(\boldsymbol{w} - \boldsymbol{w}_h), \nabla \hat{\boldsymbol{z}}_h) \quad (4.3.12)\\
&= (\nabla(\boldsymbol{w} - \boldsymbol{w}_h), \nabla \boldsymbol{z}) + (\varphi, \nabla \cdot \hat{\boldsymbol{z}}_h) \\
&\quad + (\nabla \cdot \boldsymbol{w}_h, r) - (\varphi - \varphi_h, \nabla \cdot \hat{\boldsymbol{z}}_h) \quad (4.3.13)\\
&= (\nabla(\boldsymbol{w} - \boldsymbol{w}_h), \nabla \boldsymbol{z}) + (\varphi, \nabla \cdot \hat{\boldsymbol{z}}_h) + (\nabla \cdot \boldsymbol{w}_h, r) - (\varphi, \nabla \cdot \hat{\boldsymbol{z}}_h) \\
&= (\nabla(\boldsymbol{w} - \boldsymbol{w}_h), \nabla \boldsymbol{z}) - (\nabla \cdot (\boldsymbol{w} - \boldsymbol{w}_h), r) \quad (4.3.14)\\
&= (\boldsymbol{w} - \boldsymbol{w}_h, D_\Sigma) \\
&\leq \|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega_1)} \sum_{i\in\mathcal{I}} |\boldsymbol{u}(\boldsymbol{x}_i) - \boldsymbol{\xi}_i| \|\delta_{\boldsymbol{x}_i}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \\
&\leq C|\ln h|^2 h^2 \|\boldsymbol{f}\|_{L^\infty(\Omega)} \left( \|\boldsymbol{q}\|_{L^2(\Omega)} + \sum_{i\in\mathcal{I}} |\boldsymbol{\xi}_i| \right).
\end{aligned}$$

We used (2.2.1) and (2.3.2), the fact that $\boldsymbol{z}$ is divergence-free, and inserted $\boldsymbol{w}_h$ in (4.3.10). Next, we test (4.1.2a)–(4.1.2c) and the respective finite element formulation with $\boldsymbol{w}_h$ to get (4.3.11), use in (4.3.12) and (4.3.13) that $\boldsymbol{w}_h$ is discretely divergence-free, and test (2.2.1) with $\hat{\boldsymbol{z}}_h$. To proceed, we use that $\hat{\boldsymbol{z}}_h$ is discretely divergence-free and that $\boldsymbol{w}$ is divergence-free to arrive at (4.3.14), where we apply the weak formulation of (4.1.2a)–(4.1.2c). Finally, we apply Lemma 4.8 with $\Omega_1$ containing all $\boldsymbol{x}_i$ for $i \in \mathcal{I}$. □

One also quickly surmises that $\|\boldsymbol{q}\|_{L^2(\Omega)} + \sum_{i\in\mathcal{I}} |\boldsymbol{\xi}_i|$ only depends on the prescribed values $\boldsymbol{\xi}_i$ and the control constraints.

### 4.3.4. $L^2$ projection approximation error estimates for adjoint and control

We start with a convergence result for the $L^2$ projection of $\bar{\boldsymbol{z}} \in W^{1,s}(\Omega)^3$ with $s < 3/2$. The convergence rate of the projection for a sufficiently regular function is discussed in (4.3.4). The question is now one of regularity. To analyze the dependence on $s$ when we consider the error in the $L^s$ norm we choose $s = 3/2 - \varepsilon$ and let the Hölder conjugate $s'$ be given by

$1/s + 1/s' = 1$. Then, by the Sobolev bound on the supremum norm [5, Theorem 10.10], we have $\boldsymbol{v} \in L^\infty(\Omega)^3 \cap W_0^{1,s'}(\Omega)^3$ and

$$\|\boldsymbol{v}\|_{L^\infty(\Omega)} \leq C\left( \int_{B_R(\boldsymbol{x}_0)} \frac{d\boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{x}_0|^{2s}} \right)^{1/s} \|\nabla \boldsymbol{v}\|_{L^{s'}(\Omega)}.$$

Since $\Omega$ is bounded, it is contained in the ball $B_R(\boldsymbol{x}_0)$ for $R$ large enough. Transforming to spherical coordinates, we can rewrite this as

$$\begin{aligned}
\|\boldsymbol{v}\|_{L^\infty(\Omega)} &\leq C\left( \int_0^R \rho^{2-2s} d\rho \right)^{1/s} \|\nabla \boldsymbol{v}\|_{L^{s'}(\Omega)} \\
&\leq C\left( \int_0^R \rho^{-1+2\varepsilon} d\rho \right)^{1/s} \|\nabla \boldsymbol{v}\|_{L^{s'}(\Omega)} \\
&= C\left( \left[ \frac{1}{2\varepsilon}\rho^{2\varepsilon} \right]_0^R \right)^{1/s} \|\nabla \boldsymbol{v}\|_{L^{s'}(\Omega)} \\
&\leq C\varepsilon^{-1/s} \|\nabla \boldsymbol{v}\|_{L^{s'}(\Omega)}. 
\end{aligned} \tag{4.3.15}$$

Using a duality argument we get

$$\begin{aligned}
\|D_\Sigma\|_{W^{-1,s}(\Omega)} &= \sup_{v \in W_0^{1,s'}(\Omega), \|v\|_{W_0^{1,s'}(\Omega)} \leq 1} \langle D_\Sigma, \boldsymbol{v} \rangle \\
&\leq \|D_\Sigma\|_{\boldsymbol{\mathcal{M}}(\Omega)} \|\boldsymbol{v}\|_{C_0(\Omega)} \\
&= \|D_\Sigma\|_{\boldsymbol{\mathcal{M}}(\Omega)} \|\boldsymbol{v}\|_{L^\infty(\Omega)} \\
&\leq C\varepsilon^{-1/s} \|D_\Sigma\|_{\boldsymbol{\mathcal{M}}(\Omega)}.
\end{aligned} \tag{4.3.16}$$

These considerations allow us to prove the following lemma.

**Lemma 4.10** *Let $\boldsymbol{z} \in W_0^{1,s}(\Omega)^3$ be the solution to* (4.1.2a)–(4.1.2c) *and $s$ is defined as above. Then, it holds for the $L^2$ projection $\pi_h$ to the space of cellwise constant functions*

$$\|\boldsymbol{z} - \pi_h \boldsymbol{z}\|_{L^s(\Omega)} \leq Ch\varepsilon^{-1/s} \|D_\Sigma\|_{\boldsymbol{\mathcal{M}}(\Omega)}.$$

*Proof.* This follows by applying (2.2.6), (4.3.4) and (4.3.16). □

Since the optimal solution $\bar{\boldsymbol{q}}$ to Problem (4.1.1) is given by $P_{[\boldsymbol{a},\boldsymbol{b}]}(\bar{\boldsymbol{z}})$, we obtain by Corollary 4.4 that $\bar{\boldsymbol{q}} \in (L^\infty(\Omega) \cap H^1(\Omega))^3$, thus motivating the following (suboptimal) convergence result for the $L^2$ projection onto cellwise constant functions.

**Lemma 4.11** *Let $\bar{\boldsymbol{q}} \in (L^\infty(\Omega) \cap H^1(\Omega))^3$ be the solution to the optimal control problem* (4.1.1) *and $s'$ as above. Then, it holds for the $L^2$ projection $\pi_h$ to the space of cellwise constant functions*

$$\|\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}}\|_{L^{s'}(\Omega)} \leq Ch^{2/s'} \|\nabla \bar{\boldsymbol{q}}\|_{L^2(\Omega)}^{2/s'}.$$

*Proof.* The result follows from an application of (4.3.3). To see that, we consider $\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}}$ on the cell $T$

$$\begin{aligned}\|\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}}\|_{L^{s'}(T)}^{s'} &= \int_T |\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}}|^{s'} d\boldsymbol{x} \le \|(\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}})^{s'-2}\|_{L^\infty(T)} \|\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}}\|_{L^2(T)}^2 \\ &\le Ch^2 \|\nabla \bar{\boldsymbol{q}}\|_{L^2(T)}^2.\end{aligned}$$

Since $\bar{\boldsymbol{q}} \in Q_{ad}$, summing over all cells gives the conclusion of the lemma. □

*Remark* 4.12 As mentioned, we consider this estimate suboptimal, which is due to the fact that the regularity of $\bar{\boldsymbol{q}}$, as derived in Corollary 4.4, is likely not the best possible regularity result. Compared to the elliptic problem studied in [18], where it was shown that the control actually lies in $W^{1,\infty}(\Omega)$, the Stokes fundamental solution exhibits large jumps at the singularity depending on the approach direction in certain situations. In particular, one can construct examples such that in every neighborhood of the singularity we can find an open subset, where the solution is bounded and thus the projection does not become active for the whole neighborhood, leading to less regularity for the gradient of the projected solution. Based on the behavior of the fundamental solution one can straightforwardly construct optimal control problems also exhibiting this behavior. This is visualized in Figure 4.1. Depicted is

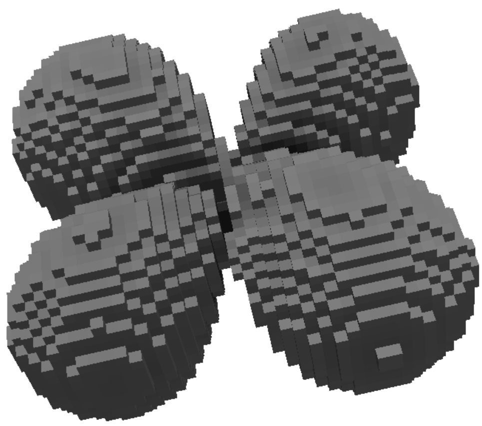

Figure 4.1.: Threshold visualization of the first component of a solution $\boldsymbol{q}_h$ to Problem (4.3.5).

a neighborhood of a point $\boldsymbol{x}_i$ for the first component of $\boldsymbol{q}_h$. Only cells where the function value is greater or respectively smaller than a threshold are visible. Note that also for this discrete solution in the neighborhood of $\boldsymbol{x}_i$ there are subsets on which the function appears to be bounded, i.e., the thresholds do not become active and the respective cells are not visible.

### 4.3.5. Error estimates for the objective functional

Next, we give an approximation result for the difference of the directional Fréchet derivatives of $j$ and $j_h$.

**Lemma 4.13** *For $\boldsymbol{q} \in Q_{ad}$ and $\delta\boldsymbol{q} \in L^\infty(\Omega)^3$, it holds*

$$|j'(\boldsymbol{q})(\delta\boldsymbol{q}) - j_h'(\boldsymbol{q})(\delta\boldsymbol{q})| \leq C|\ln h|^2 h^2.$$

*The constant $C$ depends on $\|\boldsymbol{q}\|_{L^\infty(\Omega)}$ and $\|\delta\boldsymbol{q}\|_{L^\infty(\Omega)}$.*

*Proof.* Due to Lemma 4.1 and Lemma 4.6 we get

$$|j'(\boldsymbol{q})(\delta\boldsymbol{q}) - j_h'(\boldsymbol{q})(\delta\boldsymbol{q})| = |(\boldsymbol{z} - \boldsymbol{z}_h, \delta\boldsymbol{q})|$$

with $\boldsymbol{z}$ and $\boldsymbol{z}_h$ defined as in Lemma 4.1 and Lemma 4.6. Defining $\hat{\boldsymbol{z}}_h \in \boldsymbol{V}_h$ as the solution to

$$a((\hat{\boldsymbol{z}}_h, \hat{r}_h), (\boldsymbol{v}_h, l_h)) = \sum_{i\in\mathcal{I}} (S\boldsymbol{q}(\boldsymbol{x}_i) - \boldsymbol{\xi}_i)\boldsymbol{v}_h(\boldsymbol{x}_i) \quad \forall(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h, \tag{4.3.17}$$

we obtain by the triangle inequality

$$|(\boldsymbol{z} - \boldsymbol{z}_h, \delta\boldsymbol{q})| \leq |(\boldsymbol{z} - \hat{\boldsymbol{z}}_h, \delta\boldsymbol{q})| + |(\hat{\boldsymbol{z}}_h - \boldsymbol{z}_h, \delta\boldsymbol{q})|.$$

We first consider the first term on the right-hand side. Using Hölder's inequality and Lemma 4.9 gives

$$(\boldsymbol{z} - \hat{\boldsymbol{z}}_h, \delta\boldsymbol{q}) \leq \|\boldsymbol{z} - \hat{\boldsymbol{z}}_h\|_{L^1(\Omega)} \|\delta\boldsymbol{q}\|_{L^\infty(\Omega)} \leq C|\ln h|^2 h^2 \|\delta\boldsymbol{q}\|_{L^\infty(\Omega)}. \tag{4.3.18}$$

For the second term we use the auxiliary problem (2.3.2) with right-hand side $\delta\boldsymbol{q}$. Now, since $\hat{\boldsymbol{z}}_h - \boldsymbol{z}_h \in \boldsymbol{V}_h$ and $(\nabla \cdot (\hat{\boldsymbol{z}}_h - \boldsymbol{z}_h), \varphi_h) = 0$, we can write because of (4.3.17)

$$(\hat{\boldsymbol{z}}_h - \boldsymbol{z}_h, \delta\boldsymbol{q}) = a((\hat{\boldsymbol{z}}_h - \boldsymbol{z}_h, \hat{r}_h - r_h), (\boldsymbol{w}_h, \varphi_h)) = \sum_{i\in\mathcal{I}} (S\boldsymbol{q} - S_h\boldsymbol{q})(\boldsymbol{x}_i)\boldsymbol{w}_h(\boldsymbol{x}_i).$$

Since $\boldsymbol{x}_i$ for $i \in \mathcal{I}$ does not lie on the boundary, we can choose subsets $\Omega_1 \Subset \Omega_2 \Subset \Omega$ which fulfill the requirements of Lemma 4.8. Thus, we can conclude

$$\begin{aligned}
(\hat{\boldsymbol{z}}_h - \boldsymbol{z}_h, \delta\boldsymbol{q}) &\leq C\|S\boldsymbol{q} - S_h\boldsymbol{q}\|_{L^\infty(\Omega_1)} \|\boldsymbol{w}_h\|_{L^\infty(\Omega_1)} \\
&\leq C\|S\boldsymbol{q} - S_h\boldsymbol{q}\|_{L^\infty(\Omega_1)} \Big( \|\boldsymbol{w}\|_{L^\infty(\Omega_1)} + \|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega_1)} \Big) \\
&\leq C|\ln h|^2 h^2 \|\boldsymbol{q}\|_{L^\infty(\Omega)} \Big( \|\delta\boldsymbol{q}\|_{L^2(\Omega)} + |\ln h|^2 h^2 \|\delta\boldsymbol{q}\|_{L^\infty(\Omega)} \Big).
\end{aligned}$$

Combined with (4.3.18) and the assumptions on $\boldsymbol{q}$ and $\delta\boldsymbol{q}$ this proves the lemma. □

**Lemma 4.14** *Let $\boldsymbol{p}$, $\boldsymbol{q} \in Q_{ad}$ and $\delta\boldsymbol{q} \in L^\infty(\Omega)^3$. Then, there holds*

$$|j_h'(\boldsymbol{q})(\delta\boldsymbol{q}) - j_h'(\boldsymbol{p})(\delta\boldsymbol{q})| \leq C\Big( \|\boldsymbol{q} - \boldsymbol{p}\|_{L^2(\Omega)} + |\ln h|h \Big) \Big( \|\delta\boldsymbol{q}\|_{L^2(\Omega)} + |\ln h|h \Big).$$

*The constant $C$ depends on $\|\boldsymbol{q}\|_{L^\infty(\Omega)}$, $\|\boldsymbol{p}\|_{L^\infty(\Omega)}$, and $\|\delta\boldsymbol{q}\|_{L^\infty(\Omega)}$.*

*Proof.* To show the result, we first show a maximum norm bound for $S_h$. Due to Theorem 3.14 and Remark 3.18 we obtain, e.g., for $\boldsymbol{q} \in L^\infty(\Omega)^3$

$$\begin{aligned}\|S_h\boldsymbol{q}\|_{L^\infty(\Omega)} &\leq \|S\boldsymbol{q}\|_{L^\infty(\Omega)} + \|S\boldsymbol{q} - S_h\boldsymbol{q}\|_{L^\infty(\Omega)} \\ &\leq C\|\boldsymbol{q}\|_{L^2(\Omega)} + \inf_{(\boldsymbol{v}_h,l_h)\in\boldsymbol{V}_h\times M_h} C|\ln h|\Big(\|S\boldsymbol{q} - \boldsymbol{v}_h\|_{L^\infty(\Omega)} \\ &\qquad + h\|S^p\boldsymbol{q} - l_h\|_{L^\infty(\Omega)}\Big).\end{aligned}$$

Since $(S\boldsymbol{q}, S^p\boldsymbol{q}) \in C^{1,\zeta}(\Omega)^3 \times C^{0,\zeta}(\Omega)$ due to (2.2.5) we can conclude by an interpolation error estimate that

$$\|S_h\boldsymbol{q}\|_{L^\infty(\Omega)} \leq C\|\boldsymbol{q}\|_{L^2(\Omega)} + C|\ln h|h\|\boldsymbol{q}\|_{L^\infty(\Omega)} \leq C\Big(\|\boldsymbol{q}\|_{L^2(\Omega)} + |\ln h|h\Big).$$

With this result in mind and Lemma 4.6, we then see because of the mean value theorem for $\boldsymbol{\rho} \in Q$ that

$$j_h'(\boldsymbol{q})(\delta\boldsymbol{q}) - j_h'(\boldsymbol{p})(\delta\boldsymbol{q}) = j_h''(\boldsymbol{\rho})(\boldsymbol{q}-\boldsymbol{p}, \delta\boldsymbol{q}) = \sum_{i\in\mathcal{I}} S_h(\boldsymbol{q}-\boldsymbol{p})(\boldsymbol{x}_i)S_h\delta\boldsymbol{q}(\boldsymbol{x}_i) + \alpha(\boldsymbol{q}-\boldsymbol{p}, \delta\boldsymbol{q})$$

which can be bounded as

$$\begin{aligned}|j_h'(\boldsymbol{q})(\delta\boldsymbol{q}) - j_h'(\boldsymbol{p})(\delta\boldsymbol{q})| &\leq C\|S_h(\boldsymbol{q}-\boldsymbol{p})\|_{L^\infty(\Omega)}\|S_h\delta\boldsymbol{q}\|_{L^\infty(\Omega)} + \alpha|(\boldsymbol{q}-\boldsymbol{p}, \delta\boldsymbol{q})| \\ &\leq C\Big(\|\boldsymbol{q}-\boldsymbol{p}\|_{L^2(\Omega)} + |\ln h|h\Big)\Big(\|\delta\boldsymbol{q}\|_{L^2(\Omega)} + |\ln h|h\Big).\end{aligned}$$

□

## 4.4. Error estimates for $\|\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}$

In this section, we discuss approximation error estimates for two types of control discretization. Before considering a discretization with piecewise constants as introduced in Section 4.3, we show a result for the so-called variational discretization, which was first discussed in [78].

### 4.4.1. Variational Discretization

Variational discretization means we do *not* discretize the control, i.e., $Q_{ad,h} = Q_{ad}$. It should be noted that the control nonetheless has a discrete structure due to discretization of the adjoint state and the variational inequality (4.3.8). The variational discretization allows for a more direct approach when proving the following convergence result, since we can test the discrete optimality condition with the solution to the continuous optimal control problem.

**Theorem 4.15** *Let $\bar{\boldsymbol{q}} \in Q_{ad}$ be the solution to Problem* (4.1.1) *and $\bar{\boldsymbol{q}}_h \in Q_{ad}$ the solution to the corresponding discrete Problem* (4.3.5) *with $Q_{ad,h} = Q_{ad}$. Then, it holds*

$$\|\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)} \leq C|\ln h|h.$$

*Proof.* For $\delta \boldsymbol{q} = \tau \boldsymbol{q} = \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h$ it follows from (4.3.7)

$$\alpha \|\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}^2 \leq j_h''(\boldsymbol{\rho})(\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h, \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h)$$

with $\boldsymbol{\rho} \in Q$. By the mean value theorem, (4.3.8), and (4.2.2) it follows

$$\begin{aligned}
\alpha \|\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}^2 &\leq j_h''(\boldsymbol{\rho})(\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h, \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) \\
&= j_h'(\bar{\boldsymbol{q}})(\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - j_h'(\bar{\boldsymbol{q}}_h)(\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) \\
&= (\boldsymbol{z}_h + \alpha \bar{\boldsymbol{q}}, \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - (\bar{\boldsymbol{z}}_h + \alpha \bar{\boldsymbol{q}}_h, \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) \\
&\leq (\boldsymbol{z}_h + \alpha \bar{\boldsymbol{q}}, \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - (\bar{\boldsymbol{z}} + \alpha \bar{\boldsymbol{q}}, \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) \\
&= j_h'(\bar{\boldsymbol{q}})(\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - j'(\bar{\boldsymbol{q}})(\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h).
\end{aligned}$$

Here $\boldsymbol{z}_h$ is as in Lemma 4.6 with $\boldsymbol{q} = \bar{\boldsymbol{q}}$. Applying Lemma 4.13 shows the result. □

### 4.4.2. Discretization with piecewise constant functions

**Theorem 4.16** *Let $\bar{\boldsymbol{q}} \in Q_{ad}$ be the solution to Problem* (4.1.1) *and $\bar{\boldsymbol{q}}_h \in Q_{ad,h}$ the solution to the corresponding discrete Problem* (4.3.5) *with $Q_{ad,h}$ as defined in* (4.3.1)*. Then, it holds*

$$\|\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)} \leq C |\ln h|^{1/3} h^{5/6}.$$

*Proof.* Since in this case $Q_{ad} \neq Q_{ad,h}$ we need to consider the $L^2$ projection when testing the optimality conditions. To do so, we split

$$\|\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)} \leq \|\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}}\|_{L^2(\Omega)} + \|\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}.$$

We get that the first term is bounded by $Ch \|\nabla \bar{\boldsymbol{q}}\|_{L^2(\Omega)}$ because of (4.3.4) and Corollary 4.4. For the second term we argue as in the variational case with the mean value theorem for $\boldsymbol{\rho} \in Q$

$$\begin{aligned}
\alpha \|\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}^2 &\leq j_h''(\boldsymbol{\rho})(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h, \pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) \\
&= j_h'(\pi_h \bar{\boldsymbol{q}})(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - j_h'(\bar{\boldsymbol{q}}_h)(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) \\
&\leq j_h'(\pi_h \bar{\boldsymbol{q}})(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - j'(\bar{\boldsymbol{q}})(\bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h),
\end{aligned}$$

where we used the optimality conditions (4.3.8) and (4.2.2) in the last line. We can further expand this to

$$\begin{aligned}
\alpha \|\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}^2 &\leq [j_h'(\pi_h \bar{\boldsymbol{q}})(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - j_h'(\bar{\boldsymbol{q}})(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h)] \\
&\quad + [j_h'(\bar{\boldsymbol{q}})(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h) - j'(\bar{\boldsymbol{q}})(\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h)] - j'(\bar{\boldsymbol{q}})(\bar{\boldsymbol{q}} - \pi_h \bar{\boldsymbol{q}}) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For $I_1$, since $\bar{\boldsymbol{q}}, \pi_h \bar{\boldsymbol{q}}, \bar{\boldsymbol{q}}_h \in Q_{ad}$, we can apply Lemma 4.14 and follow up with (4.3.4) and Young's inequality to see

$$\begin{aligned}
I_1 &\leq C\Big(\|\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}\|_{L^2(\Omega)} + |\ln h| h\Big)\Big(\|\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)} + |\ln h| h\Big) \\
&\leq C |\ln h|^2 h^2 + \frac{\alpha}{2} \|\pi_h \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}^2.
\end{aligned}$$

Then, $I_2$ is dealt with by Lemma 4.13 which implies

$$I_2 \leq C|\ln h|^2 h^2.$$

And finally we apply Lemma 4.10 and Lemma 4.11 to $I_3$. Recall that we chose $s = 3/2 - \varepsilon$ which implies $s' > 3$. Now by the $L^2$ orthogonality of the projection $\pi_h$, the triangle inequality, and (4.3.4) we get

$$\begin{aligned}
|I_3| &= |(\alpha\bar{\boldsymbol{q}} + \bar{\boldsymbol{z}}, \bar{\boldsymbol{q}} - \pi_h\bar{\boldsymbol{q}})| = |((\alpha\bar{\boldsymbol{q}} + \bar{\boldsymbol{z}}) - \pi_h(\alpha\bar{\boldsymbol{q}} + \bar{\boldsymbol{z}}), \bar{\boldsymbol{q}} - \pi_h\bar{\boldsymbol{q}})| \\
&\leq \|(\alpha\bar{\boldsymbol{q}} + \bar{\boldsymbol{z}}) - \pi_h(\alpha\bar{\boldsymbol{q}} + \bar{\boldsymbol{z}})\|_{L^s(\Omega)}\|\bar{\boldsymbol{q}} - \pi_h\bar{\boldsymbol{q}}\|_{L^{s'}(\Omega)} \\
&\leq \Big(\|\alpha\bar{\boldsymbol{q}} - \pi_h(\alpha\bar{\boldsymbol{q}})\|_{L^s(\Omega)} + \|\bar{\boldsymbol{z}} - \pi_h(\bar{\boldsymbol{z}})\|_{L^s(\Omega)}\Big)\|\bar{\boldsymbol{q}} - \pi_h\bar{\boldsymbol{q}}\|_{L^{s'}(\Omega)} \\
&\leq Ch^{1+2/s'}\varepsilon^{-1/s}.
\end{aligned}$$

Next, we simplify the expression for $h$ and $\varepsilon$ and choose $\varepsilon$ appropriately. For $s'$ we get

$$s' = \frac{3 - 2\varepsilon}{1 - 2\varepsilon}$$

and thus for $h^{2/s'}$, $h < 1$, and $\varepsilon$ small

$$h^{2(1-2\varepsilon)/(3-2\varepsilon)} \leq h^{2(1-2\varepsilon)/3} = h^{2/3}h^{-4\varepsilon/3}.$$

We choose $\varepsilon = 1/|\ln h| = -1/\ln(h)$. Then, it follows $h^{-4\varepsilon/3} = e^{4/3}$, implying

$$|I_3| \leq C|\ln h|^{\frac{2}{3-2/|\ln h|}} h^{5/3} \leq C|\ln h|^{2/3}h^{5/3}. \tag{4.4.1}$$

We conclude that in this convergence estimate, $I_3$ is the dominating term and therefore the statement of the lemma follows. □

*Remark* 4.17 The last inequality in (4.4.1) follows from

$$\frac{2}{3 - 2/|\ln h|} = \frac{2}{3}\left(1 + \frac{1}{3/2|\ln h| - 1}\right)$$

and

$$\lim_{h\to 0} |\ln h|^{\frac{2/3}{3/2|\ln h|-1}} = 1.$$

*Remark* 4.18 The proof shows that the estimate of $I_3$ is the limiting factor for the convergence rate estimate. To achieve an optimal convergence rate, one would require a regularity estimate $\bar{\boldsymbol{q}} = P_{[\boldsymbol{a},\boldsymbol{b}]}(-\alpha^{-1}\bar{\boldsymbol{z}}) \in W^{1,3+\varepsilon}(\Omega)^3$.

## 4.5. Numerical experiments

We conduct numerical experiments to support the result in Theorem 4.16. The optimal control problems are solved by the optimization library RoDoBo [106] and the finite element toolkit Gascoigne [60]. The empirical convergence rates are computed by comparing solutions with a solution computed on a mesh twice as fine as the finest mesh which we compare.

While in our numerical experiments we consider a slightly different setting than that introduced in Section 4.3, using local projection stabilization finite element methods on meshes of hexahedral geometry instead of Taylor-Hood finite elements on a triangulation, the results indicate better rates than in Theorem 4.16 for $h$ small enough.

Our results coincide with the output shown in [57, Fig. 2 (Ex.2)] when considering the same example problem ([57, Example 2]) which we introduce next.

**Example 4.19** Let $\Omega = (0,1)^3$, $\boldsymbol{a} = (a,a,a)^T$, $\boldsymbol{b} = (b,b,b)^T$, $\alpha = 1.99$, $\boldsymbol{\xi}_0 = (-1,-1,-1)^T$ and $\boldsymbol{x}_0 = (0.5, 0.5, 0.5)^T$ with $\mathcal{I} = \{0\}$ and $a$ and $b$ to be chosen later. Then, we consider the optimal control problem as in Problem (4.1.1) but with the state equation

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} + \boldsymbol{q} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, \end{aligned}$$

with

$$\boldsymbol{f} = \frac{1}{\pi}\Delta \text{curl}((\sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))^2 \boldsymbol{e}_1) + \nabla(x_1 x_2 x_3).$$

*Remark* 4.20 This is not precisely the same example as stated in [57, Example 2] because there the authors consider a problem with a slightly different forcing term and inhomogeneous Dirichlet boundary condition but the essential numerical behavior should be unchanged.

The resulting empirical convergence rates for different bounds $\boldsymbol{a}$ and $\boldsymbol{b}$ are shown in Figure 4.2 where $N_{\text{dof}}$ corresponds to the number of cells in the mesh and $\bar{\boldsymbol{q}}_n$ to the approximate solution computed on a finer mesh. When the constraints do not become active because $h$ is not small enough, we observe a convergence rate $h^{1/2}$ as in the case $a = -10$, $b = 2$. For the intermediate case $a = -0.4$, $b = 0.4$ we see that as soon as the constraints become active, the convergence rate increases.

Finally, for $a = -0.1$, $b = 0.1$, we immediately observe an empirical convergence rate of $O(h)$ which is faster than the result we have proven in Theorem 4.16. That is likely because of $\bar{\boldsymbol{q}}$ being in $W^{1,3+\varepsilon}$ for $\varepsilon > 0$ which is better than what we have shown with Corollary 4.4. More careful analysis of the impact of $P_{[\boldsymbol{a},\boldsymbol{b}]}$ on the Stokes fundamental solution might provide additional insights.

*Remark* 4.21 Example 4.19 is well-behaved in the sense that the singularities in the adjoint equation do not exhibit the behavior described in Remark 4.12. Additional tests ran for a modified problem also resulted in a numerical convergence rate of $O(h)$.

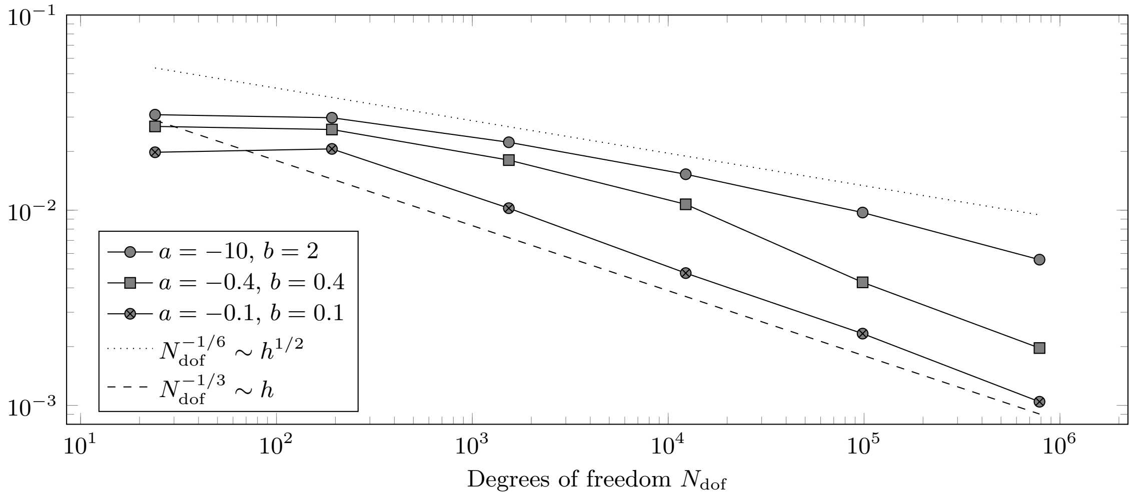


Figure 4.2.: Error $\|\bar{\boldsymbol{q}}_n - \bar{\boldsymbol{q}}_h\|_{L^2(\Omega)}$ for cellwise constant control discretization and different choices for the bounds $\boldsymbol{a}$ and $\boldsymbol{b}$. $\bar{\boldsymbol{q}}_n$ denotes the approximate solution on a finer mesh.

# Chapter 5.

# Approximation error estimates for a sparse optimal control problem

## 5.1. Introduction

For $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, an open non-empty convex polyhedral domain, we investigate the following sparse optimal control problem

$$\text{Minimize } J(\boldsymbol{u},\boldsymbol{q}) = \frac{1}{2}\|\boldsymbol{u}-\boldsymbol{u}_d\|^2_{L^2(\Omega)} + \alpha\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \tag{5.1.1a}$$

$$\text{subject to} \quad -\Delta\boldsymbol{u} + \nabla p = \boldsymbol{q} \quad \text{in } \Omega, \tag{5.1.1b}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{5.1.1c}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{5.1.1d}$$

We allow for $\boldsymbol{q}$ to be sparse, i.e., $\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega) = (\mathcal{M}(\Omega))^d$, where $\mathcal{M}(\Omega)$ denotes the space of regular Borel measures on $\Omega$ which is identified with the dual of the space of continuous functions $C_0(\Omega)$ with compact support. We will make clear later, how exactly the norm of $\boldsymbol{\mathcal{M}}(\Omega)$ is chosen. The parameters $\boldsymbol{u}_d \in L^s(\Omega)^d$ for $2 \le s \le \infty$ and $\alpha > 0$ are fixed.

Choosing $\boldsymbol{\mathcal{M}}(\Omega)$ and not, e.g., $L^1(\Omega)^d$, even though $\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)}$ coincides with $\|\boldsymbol{q}\|_{L^1(\Omega)}$ for $\boldsymbol{q} \in L^1(\Omega)^d$, is motivated by the fact that $L^1(\Omega)^d$ is missing weak-compactness properties, required for the well-posedness of the problem. Like for the $L^1(\Omega)^d$ norm, looking for $\boldsymbol{q}$ in $\boldsymbol{\mathcal{M}}(\Omega)$ promotes sparsity of the solution which is sought in many applications.

The discrete and the continuous problem have been thoroughly analyzed already in the elliptic case, e.g., in [27, 36, 103], including potential applications in [37]. Here we show that their approach to the discretization of the measure space with Dirac measure at the degrees of freedom of the mesh discretization can be extended to the Stokes problem. We show convergence rates for the approximation error in the cost functional of the kind

$$J(\bar{\boldsymbol{q}},\bar{\boldsymbol{u}}) - J(\bar{\boldsymbol{q}}_h,\bar{\boldsymbol{u}}_h) \le C|\ln h|^{2+r}h^{4-d}, \tag{5.1.2}$$

as well as a rate for the convergence in the state variable

$$\|\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}_h\|_{L^2(\Omega)} \le C|\ln h|^{1+r/2}h^{2-d/2}, \tag{5.1.3}$$

with $r = -1/3$ for $d = 3$ and $r = 1$ in the case $d = 2$.

This is comparable to the results for the Poisson problem in [103], assuming similar regularity for $\boldsymbol{u}_d$. This work is also related to [57], where the authors consider Diracs at fixed points as controls. Other results on optimal control of the Stokes problem have focused on $L^2$ controls and/or the state constrained problem, for example in [40, 107]. Efforts to analyze the respective sparse optimal control problem in the case of Navier-Stokes equations have been made recently in [30] for the two-dimensional case.

To our best knowledge, our results are novel for the Stokes problem. We are able to show the estimates (5.1.2) and (5.1.3) on convex polyhedral domains, using the local pointwise estimates from Chapter 3 and the fact that the optimal control $\bar{\boldsymbol{q}}$ is compactly supported in $\Omega$. This is an improvement compared to [103], where the authors required the boundary $\partial\Omega$ to be at least $C^{2,\zeta}$ smooth.

In the following we introduce notation and optimality conditions and then tackle the discretization error estimates. Finally, we support our theoretical results with some numerical experiments.

## 5.2. Notation and definitions

First, we briefly discuss the space $\boldsymbol{\mathcal{M}}(\Omega)$. The space $\mathcal{M}(\Omega)$ is a Banach space together with the norm

$$\|\boldsymbol{q}\|_{\mathcal{M}(\Omega)} = \sup_{\|\phi\|_{C_0(\Omega)}\leq 1} \int_\Omega \phi(\boldsymbol{x}) dq(\boldsymbol{x}).$$

Respectively for $\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega)$ and $d = 3$ we choose the norm

$$\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)} = \Big|\Big(\|\boldsymbol{q}_1\|_{\mathcal{M}(\Omega)}, \|\boldsymbol{q}_2\|_{\mathcal{M}(\Omega)}, \|\boldsymbol{q}_3\|_{\mathcal{M}(\Omega)}\Big)\Big|_{\mathbb{R}^d}.$$

Similarly, for $d = 2$. By $|\cdot|_{\mathbb{R}^d}$ we denote a respective $\mathbb{R}^d$ space norm which we here choose as the "one-norm". Thus, the norm of $\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega)$ is given as

$$\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)} = \sum_{i=1}^d \|\boldsymbol{q}_i\|_{\mathcal{M}(\Omega)},$$

where the index $i$ denotes the $i$th component of $\boldsymbol{q}$. Alternative choices are the maximum norm or the Euclidean norm, which both lead to similar structurally relevant results, e.g., in [30, Proposition 3.5]. The choice of the "one-norm" appears natural, leading to similar results for the optimality conditions and adjoint problem as in the elliptic case discussed in [27, 36, 103]. We will come back to this choice in the proof of Lemma 5.2.

## 5.3. The continuous problem

Regularity result (2.2.6) shows that (5.1.1) is well-defined and we may introduce for $\boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(\Omega)$ a linear control-to-state mapping for the velocity with $S : \boldsymbol{\mathcal{M}}(\Omega) \to W^{1,s}(\Omega)^d$ and for the pressure with $S^p : \boldsymbol{\mathcal{M}}(\Omega) \to L^s_0(\Omega)$ for $1 < s < d/(d-1)$ such that $S\boldsymbol{\mu} = \boldsymbol{w}$ and $S^p\boldsymbol{\mu} = \varphi$ as the components of the solution to (2.2.1).

Since the control-to-state mapping $S$ is injective and therefore the cost functional strictly convex, the approach in the proof of [36, Proposition 2.2] leads to the existence of a unique control solution, denoted as $\bar{\boldsymbol{q}}$, and an optimal state, denoted as $\bar{\boldsymbol{u}}$, to (5.1.1). We next consider derivatives of the tracking term and the adjoint problem, as well as optimality conditions.

In the following let $F(\boldsymbol{q}) = \frac{1}{2}\|S(\boldsymbol{q}) - \boldsymbol{u}_d\|^2_{L^2(\Omega)}$ for $\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega)$.

**Lemma 5.1** *The derivative of $F$ with respect to $\boldsymbol{q}$ in direction $\delta\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega)$ is given as*

$$\langle F'(\boldsymbol{q}), \delta\boldsymbol{q}\rangle = \langle \boldsymbol{z}, \delta\boldsymbol{q}\rangle,$$

*where $(\boldsymbol{z}, r) \in (H^2(\Omega) \cap H^1_0(\Omega))^d \times (H^1(\Omega) \cap L^2_0(\Omega))$ is the solution of the adjoint equation*

$$-\Delta\boldsymbol{z} + \nabla r = S\boldsymbol{q} - \boldsymbol{u}_d \quad \text{in } \Omega, \tag{5.3.1a}$$
$$\nabla \cdot \boldsymbol{z} = 0 \quad \text{in } \Omega, \tag{5.3.1b}$$
$$\boldsymbol{z} = 0 \quad \text{on } \partial\Omega. \tag{5.3.1c}$$

*Proof.* This follows directly from the definition of the Gâteaux differential and the linearity of $S$. ☐

We obtain the following optimality system.

**Lemma 5.2** *Let $\bar{\boldsymbol{q}}$ and $(\bar{\boldsymbol{u}}, \bar{p})$ be the optimal control and state solution for Problem (5.1.1). Then, there exists a unique adjoint state $\bar{\boldsymbol{z}} \in (H^2(\Omega) \cap H^1_0(\Omega))^d \hookrightarrow C_0(\Omega)^d$ given as a solution to (5.3.1a)–(5.3.1c) with right-hand side $S\bar{\boldsymbol{q}} - \boldsymbol{u}_d$. For $\bar{\boldsymbol{z}}$ it holds that*

$$-\frac{1}{\alpha}\bar{\boldsymbol{z}} \in \partial\|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)},$$

*where $\partial$ denotes the subdifferential (cf. [20, (2.227)]), which means in particular*

$$-\langle \boldsymbol{q} - \bar{\boldsymbol{q}}, \bar{\boldsymbol{z}}\rangle + \alpha\|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \le \alpha\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \quad \text{for all } \boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega), \tag{5.3.2}$$

$$\|\bar{\boldsymbol{z}}\|_{L^\infty(\Omega)} \le \alpha.$$

*For $1 \le i \le d$ and $\bar{\boldsymbol{q}}_i \neq 0$, the support of $\bar{\boldsymbol{q}}_i$ is contained in the set $\{\boldsymbol{x} \in \Omega \mid |\bar{\boldsymbol{z}}_i| = \alpha\}$ and for the Jordan-decomposition $\bar{\boldsymbol{q}}_i = \bar{\boldsymbol{q}}_i^+ - \bar{\boldsymbol{q}}_i^-$ we have*

$$\operatorname{supp} \bar{\boldsymbol{q}}_i^+ \subset \{\boldsymbol{x} \in \Omega \mid \bar{\boldsymbol{z}}_i(\boldsymbol{x}) = -\alpha\} \quad \text{and} \quad \operatorname{supp} \bar{\boldsymbol{q}}_i^- \subset \{\boldsymbol{x} \in \Omega \mid \bar{\boldsymbol{z}}_i(\boldsymbol{x}) = \alpha\}.$$

The proof is mostly based on standard results from convex analysis, as well as the analysis conducted for the Navier-Stokes problem in [30]. We summarize the most important steps in the following.

*Proof.* By the regularity and existence results in Chapter 2 we have the uniqueness and existence of solutions to (5.3.1a)–(5.3.1c), in particular $\bar{\boldsymbol{z}}$ is continuous and zero at the boundary. Since $\bar{\boldsymbol{u}}$ is a solution to (5.1.1a)–(5.1.1d) and $J$ defined in (5.1.1a) is a convex functional, there holds by standard results in convex analysis

$$0 \in \partial J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{q}})$$

and thus, it follows by Lemma 5.1 that

$$-\frac{1}{\alpha}\bar{\boldsymbol{z}} \in \partial\|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)}. \tag{5.3.3}$$

Here, by the definition of the subdifferential, for $\boldsymbol{\lambda} \in \partial\|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)}$ it holds that

$$\langle \boldsymbol{q} - \bar{\boldsymbol{q}}, \boldsymbol{\lambda}\rangle + \|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \leq \|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \quad \text{for all } \boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega), \tag{5.3.4}$$

showing (5.3.2) for $\bar{\boldsymbol{z}}$ and $\bar{\boldsymbol{q}}$ (note the negative sign in (5.3.3)). The statements about the support of $\bar{\boldsymbol{q}}$ can be derived from results for scalar functions. To see this, we proceed as in [30, Proposition 3.5]. Fix $1 \leq j \leq d$. In (5.3.4) we choose $\boldsymbol{q}_i = \bar{\boldsymbol{q}}_i$ for $i \neq j$. Then, due to our choice of the "one-norm" for the vector space $\boldsymbol{\mathcal{M}}(\Omega)$ in the previous section, (5.3.4) simplifies to

$$\langle \boldsymbol{q}_j - \bar{\boldsymbol{q}}_j, \lambda_j\rangle + \|\bar{\boldsymbol{q}}_j\|_{\mathcal{M}(\Omega)} \leq \|\boldsymbol{q}_j\|_{\mathcal{M}(\Omega)} \quad \text{for all } \boldsymbol{q}_j \in \mathcal{M}(\Omega). \tag{5.3.5}$$

Now we choose first $\boldsymbol{q}_j = 0$ and then $\boldsymbol{q}_j = 2\bar{\boldsymbol{q}}_j$ to conclude

$$\langle \bar{\boldsymbol{q}}_j, \lambda_j\rangle = \|\bar{\boldsymbol{q}}_j\|_{\mathcal{M}(\Omega)} \tag{5.3.6}$$

and by subtracting this from (5.3.5) it follows

$$\langle \boldsymbol{q}_j, \lambda_j\rangle \leq \|\boldsymbol{q}_j\|_{\mathcal{M}(\Omega)} \quad \text{for all } \boldsymbol{q}_j \in \mathcal{M}(\Omega). \tag{5.3.7}$$

From (5.3.7) we get $\|\lambda_j\|_{C_0(\Omega)} \leq 1$ and from (5.3.6) that $\|\lambda_j\|_{C_0(\Omega)} = 1$ since we assume $\bar{\boldsymbol{q}}_i \neq 0$. Hence, we have

$$\langle \bar{\boldsymbol{q}}_j, \lambda_j\rangle = \|\bar{\boldsymbol{q}}_j\|_{\mathcal{M}(\Omega)}\|\lambda_j\|_{C_0(\Omega)} \quad \text{for } 1 \leq j \leq d,$$

which allows us to apply the respective scalar space result [28, Lemma 3.4], delivering the result for the support of $\bar{\boldsymbol{q}}$. □

## 5.4. The discrete problem

Based on the finite element approximation given in Chapter 2, for which we assume that the assumptions in Sections 3.2 and 3.6 apply (e.g., Taylor-Hood finite elements with $k > 2$), we define a discrete linear control-to-state mapping for the velocity $S_h : \boldsymbol{\mathcal{M}}(\Omega) \to \boldsymbol{V}_h$ and the pressure $S_h^p : \boldsymbol{\mathcal{M}}(\Omega) \to M_h$ such that $S_h\boldsymbol{\mu} = \boldsymbol{w}_h$ and $S_h^p\boldsymbol{\mu} = \varphi_h$ for $\boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(\Omega)$ are the components of the solution to (2.3.2). Important to note is that in contrast to the continuous problem (5.1.1b)–(5.1.1d), the solution operator to the discrete problem is no longer injective when we take $\boldsymbol{\mathcal{M}}(\Omega)$ as the space of admissible controls. We will later see that one can remedy this by discretizing $\boldsymbol{\mathcal{M}}(\Omega)$.

Next, we introduce the discrete optimal control problem

$$\text{Minimize } J_h(\boldsymbol{u}_h, \boldsymbol{q}_h) = \frac{1}{2}\|\boldsymbol{u}_h - \boldsymbol{u}_d\|^2_{L^2(\Omega)} + \alpha\|\boldsymbol{q}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)} \quad \text{over } \boldsymbol{q}_h \in \boldsymbol{\mathcal{M}}(\Omega) \tag{5.4.1}$$

subject to

$$a((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, l_h)) = \langle \boldsymbol{q}_h, \boldsymbol{v}_h \rangle \quad \forall (\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h,$$

where $a$ is as in (2.2.2). Having chosen no discretization of $\boldsymbol{\mathcal{M}}(\Omega)$ yet, we are working so far with the so-called variational discretization [78]. Since $S_h$ is no longer injective, $J_h(\boldsymbol{u}_h, \boldsymbol{q}_h)$ is no longer strictly convex but only convex, implying the existence of an optimal solution $\tilde{\boldsymbol{q}}_h$ but not uniqueness.

As before, using the linearity of the operator $S_h$, one can derive the derivative of the tracking term and the respective adjoint problem. Similarly, to above, we let $F_h(\boldsymbol{q}) = 1/2\|S_h(\boldsymbol{q}) - \boldsymbol{u}_d\|^2_{L^2(\Omega)}$ for $\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega)$.

**Lemma 5.3** *The derivative with respect to $\boldsymbol{q}$ in direction $\delta\boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega)$ is given as*

$$\langle F_h'(\boldsymbol{q}), \delta\boldsymbol{q} \rangle = \langle \boldsymbol{z}_h, \delta\boldsymbol{q} \rangle,$$

*where $(\boldsymbol{z}_h, r_h) \in \boldsymbol{V}_h \times M_h$ is the solution of the adjoint equation*

$$a((\boldsymbol{z}_h, r_h), (\boldsymbol{v}_h, l_h)) = (S_h\boldsymbol{q} - \boldsymbol{u}_d, \boldsymbol{v}_h) \quad \forall (\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h. \tag{5.4.2}$$

As in the continuous case, we consider the following optimality conditions.

**Lemma 5.4** *Let $\tilde{\boldsymbol{q}}_h$ and $(\bar{\boldsymbol{u}}_h, \bar{p}_h)$ be a solution to the problem* (5.4.1)*. Then, there exists an adjoint state $\bar{\boldsymbol{z}}_h \in \boldsymbol{V}_h$ given as a solution to* (5.4.2) *with right-hand side $S_h\tilde{\boldsymbol{q}}_h - \boldsymbol{u}_d$. For $\bar{\boldsymbol{z}}_h$ it holds that*

$$-\frac{1}{\alpha}\bar{\boldsymbol{z}}_h \in \partial\|\tilde{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)},$$

*where $\partial$ denotes the subdifferential, which means in particular*

$$-\langle \boldsymbol{q} - \tilde{\boldsymbol{q}}_h, \bar{\boldsymbol{z}}_h \rangle + \alpha\|\tilde{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)} \leq \alpha\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \quad \textit{for all } \boldsymbol{q} \in \boldsymbol{\mathcal{M}}(\Omega), \tag{5.4.3}$$

$$\|\bar{\boldsymbol{z}}_h\|_{L^\infty(\Omega)} \leq \alpha.$$

*For $1 \leq i \leq d$ and $\tilde{\boldsymbol{q}}_{i,h} \neq 0$, the support of $\tilde{\boldsymbol{q}}_{i,h}$ is contained in the set $\{\boldsymbol{x} \in \Omega \mid |\bar{\boldsymbol{z}}_{i,h}| = \alpha\}$, and for the Jordan-decomposition $\tilde{\boldsymbol{q}}_{i,h} = \tilde{\boldsymbol{q}}^+_{i,h} - \tilde{\boldsymbol{q}}^-_{i,h}$ we have*

$$\operatorname{supp} \tilde{\boldsymbol{q}}^+_{i,h} \subset \{\boldsymbol{x} \in \Omega \mid \bar{\boldsymbol{z}}_{i,h}(\boldsymbol{x}) = -\alpha\} \quad \textit{and} \quad \operatorname{supp} \tilde{\boldsymbol{q}}^-_{i,h} \subset \{\boldsymbol{x} \in \Omega \mid \bar{\boldsymbol{z}}_{i,h}(\boldsymbol{x}) = \alpha\}.$$

*Proof.* The lemma follows as Lemma 5.2 since $J(\boldsymbol{u}_h, \boldsymbol{q}_h)$ is still a convex functional and thus the optimality condition for the subdifferential still holds. □

In order to identify a unique and (numerically) accessible solution $\bar{\boldsymbol{q}}_h \in \boldsymbol{\mathcal{M}}(\Omega)$, we now introduce a discretization of $\boldsymbol{\mathcal{M}}(\Omega)$ as proposed in [27]. Let $\{\boldsymbol{x}_j\}_{j=1}^{N(h)}$ denote points in $\Omega$ associated with the degrees of freedom of the Taylor-Hood finite element discretization (cf. [27, Remark 3.4]). For the finite element space $\boldsymbol{V}_h$ we have the basis functions $\{e_j\}_{j=1}^{N(h)}$ associated with each degree of freedom. Similarly, we now define the space

$$\boldsymbol{\mathcal{M}}_h = \left\{ \boldsymbol{q}_h \in \boldsymbol{\mathcal{M}}(\Omega) \,:\, \boldsymbol{q}_h = \sum_{j=1}^{N(h)} \boldsymbol{\lambda}_j \delta_{\boldsymbol{x}_j}, \text{ where } \{\boldsymbol{\lambda}_j\}_{j=1}^{N(h)} \subset \mathbb{R}^d \right\}.$$

We may also define the linear projection operator $\Lambda_h : \boldsymbol{\mathcal{M}}(\Omega) \to \boldsymbol{\mathcal{M}}_h$ as

$$(\Lambda_h \boldsymbol{q})_i = \sum_{j=1}^{N(h)} (\boldsymbol{q}_i, e_j)\delta_{\boldsymbol{x}_j},$$

where $(\,\cdot\,)_i$ denotes the $i$-th component of a vector. Then, analogously to [27, Theorems 3.1, 3.2] we obtain the following proposition.

**Proposition 5.5** *Problem* (5.4.1) *admits at least one solution. Among them there exists a unique* $\bar{\boldsymbol{q}}_h \in \boldsymbol{\mathcal{M}}_h$. *Moreover, any other solution* $\tilde{\boldsymbol{q}}_h \in \boldsymbol{\mathcal{M}}(\Omega)$ *satisfies* $\Lambda_h \tilde{\boldsymbol{q}}_h = \bar{\boldsymbol{q}}_h$.

In particular this means that $\bar{\boldsymbol{q}}_h$ satisfies the properties in Lemma 5.4.

## 5.5. Approximation error estimates for the state equation

**Lemma 5.6** *Let* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(\Omega)$ *be compactly supported in* $\Omega$, *i.e., on* $\Omega_1 \Subset \Omega_2 \Subset \Omega$, *where* $\operatorname{dist}(\Omega_2, \partial\Omega), \operatorname{dist}(\Omega_1, \partial\Omega_2) \geq \varrho > 0$, *with respective continuous state* $(\boldsymbol{w}, \varphi)(\boldsymbol{\mu})$ *and discrete state* $(\boldsymbol{w}_h, \varphi_h)(\boldsymbol{\mu})$ *given. Then, we have the following estimate*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^s(\Omega)} \leq C|\ln h|^{1+r} h^{2-d/s'} \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)}, \quad s \in [1,2]\,, \quad \frac{1}{s} + \frac{1}{s'} = 1,$$

*where* $r = 1$ *for* $s = 1$ *and* $r = 0$ *otherwise.*

The error bound may be improved on smooth domains and $s \in (1, d/(d-2)]$, removing the logarithm using a similar result as in [105] based on the analysis in [65, 72]. In our setting featuring convex polyhedral domains, such estimates are not easily available. Removing the logarithmic factor here does not justify the additional complexity introduced by deriving such estimates, since additional logarithmic factors will reappear later in our arguments.

Furthermore, we note that the upper bound for $s$ is due to some limitations in the following lemma, which is a modification of Corollary 3.17. Lemma 5.6 extends Corollary 3.17 to the case of unbounded pressures which appear also in the common case of the Stokes problem with $L^2$ regular right-hand side.

Before we prove Lemma 5.6 we first need to derive the following auxiliary result.

**Lemma 5.7** *For $\Omega_1 \subset \Omega_2 \subset \Omega$ with $\operatorname{dist}(\bar\Omega_1, \partial\Omega_2) \geq \varrho \geq \bar\kappa h$ and for $(\boldsymbol{w}, \varphi) \in (L^\infty(\Omega_2) \cap H_0^1(\Omega))^d \times L_0^s(\Omega)$ the solution to (2.2.1) and $(\boldsymbol{w}_h, \varphi_h)$ the solution to (2.3.2), we have for $s \in [2, \infty]$*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega_1)} \leq \inf_{(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h} C|\ln h|\Big(\|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)} + h^{1-d/s}\|\varphi - l_h\|_{L^s(\Omega_2)}\Big) + C|\ln h|^{\hat r}\Big(h\|\boldsymbol{w} - \boldsymbol{v}_h\|_{H^1(\Omega)} + \|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^2(\Omega)} + h\|\varphi - l_h\|_{L^2(\Omega)}\Big).$$

*Here, the constant $C_\varrho$ depends $\varrho$, $\hat r = 1$ in the case $d = 3$, and $\hat r = 1/2$ in the case $d = 2$.*

*Proof.* The $L^\infty$ best-approximation result for $\boldsymbol{w}$ is shown in Theorem 3.15, Corollary 3.17, and Remark 3.18. We need to modify the proof of Theorem 3.15 with respect to the estimate of the pressure $\varphi$. In our notation, this means estimating

$$I_2 = -(\varphi, \nabla \cdot (\boldsymbol{g}_{0,h} - \boldsymbol{g}_0))$$

from the proof of Theorem 3.15, where $(\boldsymbol{g}_0, \lambda_0)$ and $(\boldsymbol{g}_{0,h}, \lambda_{0,h})$ denote the regularized Green's function defined in (3.3.7a)–(3.3.7c) and its finite element discretization in (3.3.8):

$$\begin{aligned} -\Delta \boldsymbol{g}_0 + \nabla \lambda_0 &= \delta_h \boldsymbol{e}_i && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{g}_0 &= 0 && \text{in } \Omega, \\ \boldsymbol{g}_0 &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned}$$

We proceed similarly as in the proof of Theorem 3.15, adopting the notation $D_1$ and $D_2$, and splitting the domain, to get

$$I_2 = -(\varphi, \nabla \cdot (\boldsymbol{g}_{0,h} - \boldsymbol{g}_0))_{D_2} - (\varphi, \nabla \cdot (\boldsymbol{g}_{0,h} - \boldsymbol{g}_0))_{\Omega \setminus D_2}.$$

In the following $\sigma$ denotes a weight function of the form:

$$\sigma = \sigma_{\boldsymbol{x}_0,h}(\boldsymbol{x}) = \sqrt{|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (\kappa h)^2},$$

where $\kappa$ is a suitable parameter as chosen in Chapter 3. After an application of Hölder's inequality, we obtain the following bound for $I_2$

$$|I_2| \leq C\|\varphi\|_{L^s(D_2)}\|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^{s'}(\Omega)} + C_\varrho\|\varphi\|_{L^2(\Omega)}\|\sigma^{d/2}\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(\Omega)}.$$

Here, the second term can be handled as in the proof of Theorem 3.15. In the following we restrict ourselves to $s \in (2, \infty]$, since the case $s = 2$ is immediately clear from standard finite element error estimates in [66] and (3.3.2) and (3.6.1). For the first term, we examine the behavior of $\|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^{s'}(\Omega)}$ more closely. We can use Hölder's inequality and $\sigma$ to obtain

$$\begin{aligned} \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^{s'}(\Omega)}^{s'} &= \int_\Omega \sigma^{-b}\sigma^b|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)|^{s'}d\boldsymbol{x} \leq \|\sigma^{-b}\|_{L^{\frac{1}{1-s'/2}}(\Omega)}\|\sigma^b|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)|^{s'}\|_{L^{2/s'}(\Omega)} \\ &= \left(\int_\Omega \sigma^{\frac{-b}{1-s'/2}}d\boldsymbol{x}\right)^{1-s'/2}\left(\int_\Omega \sigma^{2b/s'}|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)|^2 d\boldsymbol{x}\right)^{s'/2}. \end{aligned}$$

Now we choose $b = ds'/2$. Thus, the second factor in the equation above becomes

$$\left(\int_\Omega \sigma^d |\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)|^2 d\boldsymbol{x}\right)^{s'/2} = \|\sigma^{d/2}\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|^{s'}_{L^2(\Omega)} \le C(|\ln h|h)^{s'},$$

where one uses Corollary 3.25 or respectively the estimate provided in Section 3.6.

The first factor in the equation can be bounded using [64, Lemma 1.3]:

$$\left(\int_\Omega \sigma^{\frac{-ds'}{2-s'}} d\boldsymbol{x}\right)^{(2-s')/2} = \left(\int_\Omega \sigma^{-d+d\left(1-\frac{s'}{2-s'}\right)} d\boldsymbol{x}\right)^{(2-s')/2} \le Ch^{d\left(1-\frac{s'}{2-s'}\right)\frac{2-s'}{2}} = Ch^{d(1-s')}.$$

These estimates for the two factors then imply

$$\|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^{s'}(\Omega)} \le C|\ln h| h^{1+d(1/s'-1)} = C|\ln h|h^{1-d/s}.$$

Summing up the components estimated for $\varphi$ above and following the same arguments as in Theorem 3.15 for the velocity term, we get a generalized version of the estimate in Theorem 3.15

$$\|\boldsymbol{w}_h\|_{L^\infty(D_1)} \le C|\ln h|\Big(\|\boldsymbol{w}_h\|_{L^\infty(D_2)} + h^{1-d/s}\|\varphi\|_{L^s(D_2)}\Big) + C_\varrho|\ln h|\Big(h\|\boldsymbol{w}\|_{H^1(\Omega)} + \|\boldsymbol{w}\|_{L^2(\Omega)} + h\|\varphi\|_{L^2(\Omega)}\Big).$$

The result of the lemma for $d = 3$ then follows by applying the same arguments as in Corollary 3.17 and Remark 3.18. The two-dimensional result is derived along the same lines, but based on the estimates stated in Section 3.6. □

Similarly, we can prove the following for the global version of Lemma 5.7, based on the arguments in Chapter 3.

**Lemma 5.8** *For* $(\boldsymbol{w}, \varphi) \in (L^\infty(\Omega) \cap H_0^1(\Omega))^d \times L_0^s(\Omega)$ *the solution to* (2.2.1) *and* $(\boldsymbol{w}_h, \varphi_h)$ *the solution to* (2.3.2), *it holds for* $s \in [2, \infty]$ *that*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega)} \le \inf_{(\boldsymbol{v}_h, l_h)\in \boldsymbol{V}_h\times M_h} C|\ln h|\Big(\|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^\infty(\Omega)} + h^{1-d/s}\|\varphi - l_h\|_{L^s(\Omega)}\Big).$$

*Remark* 5.9 The two-dimensional case is not explicitly discussed in Chapter 3 but can be argued as the three-dimensional case, using the estimates stated in Section 3.6.

We continue with the proof of Lemma 5.6.

*Proof of Lemma 5.6.* We set $\boldsymbol{e} = \boldsymbol{w} - \boldsymbol{w}_h$ and thus there holds for

$$\boldsymbol{g}_s(\boldsymbol{x}) = |\boldsymbol{e}(\boldsymbol{x})|^{s-1}\operatorname{sgn}(\boldsymbol{e}(\boldsymbol{x}))$$

that $\boldsymbol{g}_s \in L^{s'}(\Omega)$ and

$$\|\boldsymbol{g}_s\|_{L^{s'}(\Omega)} = \|\boldsymbol{e}\|^{s-1}_{L^s(\Omega)}.$$

We consider the solution $(\tilde{\boldsymbol{w}}, \tilde{\varphi})$ to a dual problem based on (2.2.1) with right-hand side $\boldsymbol{g}_s$. Then, we can write

$$\begin{aligned}
\|\boldsymbol{e}\|_{L^s(\Omega)}^s &= (\boldsymbol{g}_s, \boldsymbol{w} - \boldsymbol{w}_h) \\
&= (\nabla \tilde{\boldsymbol{w}}, \nabla(\boldsymbol{w} - \boldsymbol{w}_h)) - (\tilde{\varphi}, \nabla \cdot (\boldsymbol{w} - \boldsymbol{w}_h)) \\
&= (\nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \nabla(\boldsymbol{w} - \boldsymbol{w}_h)) + (\tilde{\varphi}, \nabla \cdot \boldsymbol{w}_h) + (\nabla \tilde{\boldsymbol{w}}_h, \nabla(\boldsymbol{w} - \boldsymbol{w}_h)) \\
&= (\nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \nabla(\boldsymbol{w} - \boldsymbol{w}_h)) + (\tilde{\varphi}, \nabla \cdot \boldsymbol{w}_h) + (\nabla \cdot \tilde{\boldsymbol{w}}_h, \varphi - \varphi_h) \\
&= (\nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \nabla \boldsymbol{w}) + (\tilde{\varphi}, \nabla \cdot \boldsymbol{w}_h) + (\nabla \cdot \tilde{\boldsymbol{w}}_h, \varphi) - (\nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \nabla \boldsymbol{w}_h) \\
&= (\nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \nabla \boldsymbol{w}) + (\tilde{\varphi}, \nabla \cdot \boldsymbol{w}_h) + (\nabla \cdot \tilde{\boldsymbol{w}}_h, \varphi) - (\tilde{\varphi} - \tilde{\varphi}_h, \nabla \cdot \boldsymbol{w}_h) \\
&= (\nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \nabla \boldsymbol{w}) + (\tilde{\varphi}, \nabla \cdot \boldsymbol{w}_h) + (\nabla \cdot \tilde{\boldsymbol{w}}_h, \varphi) - (\tilde{\varphi}, \nabla \cdot \boldsymbol{w}_h) \\
&= (\nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \nabla \boldsymbol{w}) - (\nabla \cdot (\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h), \varphi) \\
&= (\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h, \boldsymbol{\mu}).
\end{aligned}$$

Here we use the fact that $\tilde{\boldsymbol{w}}$ and $\boldsymbol{w}$ are divergence-free and that $\tilde{\boldsymbol{w}}_h$ and $\boldsymbol{w}_h$ are discretely divergence-free, as well as the weak formulation of the continuous and discrete problem. For the specific steps we refer to the proof of Lemma 4.9. Now we use that $\boldsymbol{\mu}$ is only supported on the domain $\Omega_1$ and apply Lemma 5.7

$$\begin{aligned}
\|\boldsymbol{e}\|_{L^s(\Omega)}^s &= (\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h, \boldsymbol{\mu}) \\
&\leq C \|\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h\|_{C_0(\Omega_1)} \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \\
&\leq C \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \Big( \inf_{(\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h} C |\ln h| \Big( \|\tilde{\boldsymbol{w}} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)} + h^{1-d/s'} \|\tilde{\varphi} - \boldsymbol{l}_h\|_{L^{s'}(\Omega_2)} \Big) \\
&\qquad + C_\varrho |\ln h|^{\hat{r}} \Big( h \|\tilde{\boldsymbol{w}} - \boldsymbol{v}_h\|_{H^1(\Omega)} + \|\tilde{\boldsymbol{w}} - \boldsymbol{v}_h\|_{L^2(\Omega)} + h \|\tilde{\varphi} - \boldsymbol{l}_h\|_{L^2(\Omega)} \Big) \Big).
\end{aligned} \tag{5.5.1}$$

We now apply standard interpolation results that can be found in [21, Sections 4.4, 4.8] and use nodal interpolation for the terms involving $\tilde{\boldsymbol{w}}$ and the Scott-Zhang interpolation operator for the terms involving $\tilde{\varphi}$. Since $s' \geq 2$ the terms in the second line of (5.5.1) have higher orders of convergence in $h$ compared to the first line due to (2.2.3). The limiting terms are clearly in the first line of (5.5.1) such that we henceforth focus on $\|\tilde{\boldsymbol{w}} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)}$ and $\|\tilde{\varphi} - \boldsymbol{q}_h\|_{L^{s'}(\Omega_2)}$.

We now assume $s > 1$ and thus $s' < \infty$. Then, we can apply (2.2.7) to get $W^{2,s'}(\Omega_2)^d \times W^{1,s'}(\Omega_2)$ regularity for $(\tilde{\boldsymbol{w}}, \tilde{\varphi})$ on $\Omega_2$. To that end, consider an enlargement $\Omega_2'$ of $\Omega_2$, such that $\Omega_2 \Subset \Omega_2' \Subset \Omega$. We have $(\tilde{\boldsymbol{w}}, \tilde{\varphi}) \in H^2(\Omega)^d \times H^1(\Omega)$ due to (2.2.3) and $s' \geq 2$ and thus for $2 \leq \tilde{s}' \leq 2d/(d-2)$

$$\begin{aligned}
|\tilde{\boldsymbol{w}}|_{W^{2,\tilde{s}'}(\Omega_2')} + |\tilde{\varphi}|_{W^{1,\tilde{s}'}(\Omega_2')} &\leq C \Big( \|\boldsymbol{g}_s\|_{L^{\tilde{s}'}(\Omega)} + \|\tilde{\boldsymbol{w}}\|_{W^{1,\tilde{s}'}(\Omega)} + \|\tilde{\varphi}\|_{L^{\tilde{s}'}(\Omega)} \Big) \\
&\leq C \Big( \|\boldsymbol{g}_s\|_{L^{\tilde{s}'}(\Omega)} + \|\boldsymbol{g}_s\|_{L^2(\Omega)} \Big),
\end{aligned}$$

due to the Sobolev inequality and (2.2.7). Iterating this argument for $\Omega_2 \Subset \Omega_2'$ with $2 \leq s' < \infty$, we get the estimate

$$\begin{aligned}
|\tilde{\boldsymbol{w}}|_{W^{2,s'}(\Omega_2)} + |\tilde{\varphi}|_{W^{1,s'}(\Omega_2)} &\leq C \Big( \|\boldsymbol{g}_s\|_{L^{s'}(\Omega_2')} + \|\tilde{\boldsymbol{w}}\|_{W^{1,s'}(\Omega_2')} + \|\tilde{\varphi}\|_{L^{s'}(\Omega_2')} \Big) \\
&\leq C \Big( \|\boldsymbol{g}_s\|_{L^{s'}(\Omega)} + \|\boldsymbol{g}_s\|_{L^2(\Omega)} \Big) \leq C \|\boldsymbol{g}_s\|_{L^{s'}(\Omega)},
\end{aligned}$$

since $W^{2,2d/(d-2)}(\Omega_2')^d \times W^{1,2d/(d-2)}(\Omega_2') \subset W^{1,s'}(\Omega_2')^d \times L^{s'}(\Omega_2')$. This implies for the interpolation error due to [21, Corollary 4.4.24]

$$\|\tilde{\boldsymbol{w}} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)} + h^{1-d/s'}\|\tilde{\varphi} - \boldsymbol{q}_h\|_{L^{s'}(\Omega_2)} \leq Ch^{2-d/s'}\|\boldsymbol{g}_s\|_{L^{s'}(\Omega)}.$$

For $s = 1$ (i.e., $s' = \infty$) we again apply estimates for the interpolation error and Proposition 2.2, choose $p = |\ln h|$, and obtain

$$\begin{aligned}\|\tilde{\boldsymbol{w}} - \boldsymbol{v}_h\|_{L^\infty(\Omega_2)} + h\|\tilde{\varphi} - \boldsymbol{q}_h\|_{L^\infty(\Omega_2)} &\leq Cph^{2-d/p}\|\boldsymbol{g}_s\|_{L^\infty(\Omega)} \\ &\leq C|\ln h|h^{2-d/|\ln h|}\|\boldsymbol{g}_s\|_{L^\infty(\Omega)} \\ &\leq C|\ln h|h^2\|\boldsymbol{g}_s\|_{L^\infty(\Omega)}.\end{aligned}$$

Plugging this and the result for $s > 1$ into (5.5.1) we arrive at

$$\begin{aligned}\|\boldsymbol{e}\|_{L^s(\Omega)}^s &\leq C|\ln h|^{1+r}h^{2-d/s'}\|\boldsymbol{\mu}\|_{\mathcal{M}(\Omega)}\|\boldsymbol{g}_s\|_{L^{s'}(\Omega)} \\ &= C|\ln h|^{1+r}h^{2-d/s'}\|\boldsymbol{\mu}\|_{\mathcal{M}(\Omega)}\|\boldsymbol{e}\|_{L^s(\Omega)}^{s-1}\end{aligned}$$

which proves the result. □

We next provide estimates for $\|\boldsymbol{w}_h\|_{L^\infty(\Omega)}$ with right-hand side $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$ in two and three dimensions. In two dimensions our estimate is an improvement of the respective estimate for the Poisson problem in [103] by a logarithmic factor of $|\ln h|^{1/2}$.

**Lemma 5.10** *Let $\boldsymbol{w}_h = \boldsymbol{w}_h(\boldsymbol{\mu})$ be the solution to* (2.3.2) *with right-hand side $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$. Then, we have for $d = 2$*

$$\|\boldsymbol{w}_h\|_{L^\infty(\Omega)} \leq C|\ln h|\|\boldsymbol{\mu}\|_{\mathcal{M}(\Omega)}.$$

*Proof.* By [21, Lemma 4.9.2], the so-called discrete Sobolev inequality, we get

$$\|\boldsymbol{w}_h\|_{L^\infty(\Omega)} \leq C|\ln h|^{1/2}\|\nabla\boldsymbol{w}_h\|_{L^2(\Omega)}.$$

Testing the weak formulation (2.3.2) with $\boldsymbol{w}_h$ gives

$$\|\nabla\boldsymbol{w}_h\|_{L^2(\Omega)}^2 = \langle\boldsymbol{\mu}, \boldsymbol{w}_h\rangle \leq C\|\boldsymbol{\mu}\|_{\mathcal{M}(\Omega)}\|\boldsymbol{w}_h\|_{C_0(\Omega)} = C\|\boldsymbol{\mu}\|_{\mathcal{M}(\Omega)}\|\boldsymbol{w}_h\|_{L^\infty(\Omega)}.$$

Thus, we get

$$\|\boldsymbol{w}_h\|_{L^\infty(\Omega)}^2 \leq C|\ln h|\|\boldsymbol{\mu}\|_{\mathcal{M}(\Omega)}\|\boldsymbol{w}_h\|_{L^\infty(\Omega)}.$$

Dividing by $\|\boldsymbol{w}_h\|_{L^\infty(\Omega)}$ proves the result. □

**Lemma 5.11** *Let $\boldsymbol{w}_h = \boldsymbol{w}_h(\boldsymbol{\mu})$ be the solution to* (2.3.2) *with right-hand side $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$. Then, we have for $d = 3$*

$$\|\boldsymbol{w}_h\|_{L^3(\Omega)} \leq C|\ln h|^{2/3}\|\boldsymbol{\mu}\|_{\mathcal{M}(\Omega)}.$$

*Proof.* We use a duality argument to prove the estimate. Let $(\tilde{\boldsymbol{w}}, \tilde{\varphi}) \in H^2(\Omega)^d \times H^1(\Omega)$ be the solution to

$$\begin{aligned} -\Delta \tilde{\boldsymbol{w}} + \nabla \tilde{\varphi} &= \boldsymbol{w}_h |\boldsymbol{w}_h| && \text{in } \Omega, \\ \nabla \cdot \tilde{\boldsymbol{w}} &= 0 && \text{in } \Omega, \\ \tilde{\boldsymbol{w}} &= \boldsymbol{0} && \text{on } \partial\Omega \end{aligned}$$

and $(\tilde{\boldsymbol{w}}_h, \tilde{\varphi}_h) \in \boldsymbol{V}_h \times M_h$ the respective finite element solution. Then,

$$\begin{aligned} \|\boldsymbol{w}_h\|_{L^d(\Omega)}^d &= (\boldsymbol{w}_h, \boldsymbol{w}_h|\boldsymbol{w}_h|) \\ &= (\nabla \boldsymbol{w}_h, \nabla \tilde{\boldsymbol{w}}_h) \\ &= (\boldsymbol{\mu}, \tilde{\boldsymbol{w}}_h) \le C \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \|\tilde{\boldsymbol{w}}_h\|_{C_0(\Omega)} = C \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \|\tilde{\boldsymbol{w}}_h\|_{L^\infty(\Omega)}. \end{aligned}$$

Now using the $L^\infty/W^{1,p}$ Sobolev inequality from [5, Theorem 10.10] and an inverse inequality, we obtain the following for small $\varepsilon > 0$ and $h < 0$

$$\begin{aligned} \|\tilde{\boldsymbol{w}}_h\|_{L^\infty(\Omega)} &\le \frac{C}{\varepsilon^s} \|\tilde{\boldsymbol{w}}_h\|_{W^{1,3+\varepsilon}(\Omega)} \\ &\le \frac{C}{\varepsilon^s} h^{-3(1/3 - 1/(3+\varepsilon))} \|\tilde{\boldsymbol{w}}_h\|_{W^{1,3}(\Omega)} \\ &\le \frac{C}{\varepsilon^s} h^{-\varepsilon/3} \|\tilde{\boldsymbol{w}}_h\|_{W^{1,3}(\Omega)}, \end{aligned}$$

where $s \sim (1 - 1/(3+\varepsilon))^{-1}$ and thus $\varepsilon^{-s} \le C\varepsilon^{-2/3}$ for $\varepsilon \to 0$. For the behavior of $\varepsilon$, we refer to (4.3.15) and the proof of Theorem 4.16.

Next, we note that $\tilde{\boldsymbol{w}}_h$ is stable in the sense that

$$\|\tilde{\boldsymbol{w}}_h\|_{W^{1,3}(\Omega)} \le C \|\tilde{\boldsymbol{w}}\|_{W^{1,3}(\Omega)} + \|\tilde{\varphi}\|_{L^3(\Omega)},$$

which can be seen, for example, by interpolating the results from [65, 72]. The details can be found in [65, Corollary 6]. Applying regularity result (2.2.4) allows us to bound

$$\begin{aligned} \|\tilde{\boldsymbol{w}}\|_{W^{1,3}(\Omega)} + \|\tilde{\varphi}\|_{L^3(\Omega)} \le C \|\boldsymbol{w}_h|\boldsymbol{w}_h|\|_{W^{-1,3}(\Omega)} &= \sup_{\tilde{\boldsymbol{v}} \in W_0^{1,3/2}(\Omega)^3} \frac{\langle \boldsymbol{w}_h|\boldsymbol{w}_h|, \tilde{\boldsymbol{v}} \rangle}{\|\tilde{\boldsymbol{v}}\|_{W_0^{1,3/2}(\Omega)}} \\ &\le \sup_{\tilde{\boldsymbol{v}} \in W_0^{1,3/2}(\Omega)} \frac{\|\boldsymbol{w}_h|\boldsymbol{w}_h|\|_{L^{3/2}(\Omega)} \|\tilde{\boldsymbol{v}}\|_{L^3(\Omega)}}{\|\tilde{\boldsymbol{v}}\|_{W_0^{1,3/2}(\Omega)}} \\ &\le C \|\boldsymbol{w}_h|\boldsymbol{w}_h|\|_{L^{3/2}(\Omega)} \le C \||\boldsymbol{w}_h|^2\|_{L^{3/2}(\Omega)} \\ &= C \left( \int_\Omega \left( |\boldsymbol{w}_h|^2 \right)^{3/2} d\boldsymbol{x} \right)^{2/3} = C \|\boldsymbol{w}_h\|_{L^3(\Omega)}^2. \end{aligned}$$

Here we used that $W_0^{1,3/2}(\Omega)^3$ embeds into $L^3(\Omega)^3$ by the Sobolev embedding theorem. Thus, we conclude

$$\|\boldsymbol{w}_h\|_{L^3(\Omega)} \le \frac{C}{\varepsilon^s} h^{-\varepsilon/3} \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)}.$$

Now choosing $\varepsilon = 3/|\ln h|$ gives the result. $\square$

## 5.6. Convergence results for the control problem

We first show that the optimal control $\bar{\boldsymbol{q}}_h$ discretized in $\boldsymbol{\mathcal{M}}_h$ actually converges to its counterpart $\bar{\boldsymbol{q}}$. To that end, we require two lemmas, which we were not able to locate in the literature for the Stokes problem. Our approach is motivated by similar techniques for the Poisson problem.

**Lemma 5.12** *Let* $(\boldsymbol{w}, \varphi)$ *solve* (2.2.1) *with right-hand side* $\boldsymbol{f} \in L^2(\Omega)^d$ *and* $(\boldsymbol{w}_h, \varphi_h) \in \boldsymbol{V}_h \times M_h$ *solve the respective finite element problem* (2.3.2)*. Then, it holds that*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega)} \leq C|\ln h| h^{2-d/2} \|\boldsymbol{f}\|_{L^2(\Omega)}.$$

*Proof.* From Lemma 5.8 with $s = 2$ we obtain

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega)} \leq \inf_{(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times M_h} C|\ln h| \Big( \|\boldsymbol{w} - \boldsymbol{v}_h\|_{L^\infty(\Omega)} + h^{1-d/2} \|\varphi - q_h\|_{L^2(\Omega)} \Big).$$

Choosing $(\boldsymbol{v}_h, q_h)$ as the interpolants of $(\boldsymbol{w}, \varphi)$ we see by the interpolation error estimates in [21, Corollary 4.4.24, (4.8.17)] and the regularity results for $(\boldsymbol{w}, \varphi)$ in (2.2.3) that

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^\infty(\Omega)} \leq C|\ln h| h^{2-d/2} \|\boldsymbol{f}\|_{L^2(\Omega)}.$$

□

**Lemma 5.13** *Let* $(\boldsymbol{w}, \varphi)$ *solve* (2.2.1) *with right-hand side* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(\Omega)$ *and* $(\boldsymbol{w}_h, \varphi_h) \in \boldsymbol{V}_h \times M_h$ *solving the respective finite element problem* (2.3.2)*. Then, it holds that*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^2(\Omega)} \leq C|\ln h| h^{2-d/2} \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)}.$$

Lemma 5.13 seems redundant to Lemma 5.6 in our case but it allows us to show convergence of $\bar{\boldsymbol{q}}_h$ also in the case, where $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{q}}_h$ are not compactly supported.

*Proof.* We apply a duality argument. Let $\boldsymbol{f} \in L^2(\Omega)^d$ and $(\tilde{\boldsymbol{w}}, \tilde{\varphi}) \in (H^2(\Omega) \cap H^1_0(\Omega))^d \times H^1(\Omega)$ solve

$$-\Delta \tilde{\boldsymbol{w}} + \nabla \tilde{\varphi} = \boldsymbol{f} \quad \text{in } \Omega, \tag{5.6.1a}$$
$$\nabla \cdot \tilde{\boldsymbol{w}} = 0 \quad \text{in } \Omega, \tag{5.6.1b}$$
$$\tilde{\boldsymbol{w}} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{5.6.1c}$$

Then, integration by parts and the fact that $\boldsymbol{w}$ and $\tilde{\boldsymbol{w}}$ are divergence-free lead to

$$(\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{f}) = (\boldsymbol{w} - \boldsymbol{w}_h, -\Delta \tilde{\boldsymbol{w}} + \nabla \tilde{\varphi}) = (\boldsymbol{\mu}, \tilde{\boldsymbol{w}}) - (\nabla \boldsymbol{w}_h, \nabla \tilde{\boldsymbol{w}}) + (\nabla \cdot \boldsymbol{w}_h, \tilde{\varphi}). \tag{5.6.2}$$

Let $(\tilde{\boldsymbol{w}}_h, \tilde{\varphi}_h)$ be the respective finite element solution to (5.6.1a)–(5.6.1c). Then, there holds that

$$(\nabla \boldsymbol{w}_h, \nabla(\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h)) - (\nabla \cdot \boldsymbol{w}_h, \tilde{\varphi} - \tilde{\varphi}_h) = 0.$$

Using this, we can rewrite (5.6.2) and use the estimate in Lemma 5.12 in

$$\begin{aligned}(\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{f}) &= (\boldsymbol{w} - \boldsymbol{w}_h, -\Delta\tilde{\boldsymbol{w}} + \nabla\tilde{\varphi}) = (\boldsymbol{\mu}, \tilde{\boldsymbol{w}}) - (\nabla \boldsymbol{w}_h, \nabla\tilde{\boldsymbol{w}}_h) + (\nabla \cdot \boldsymbol{w}_h, \tilde{\varphi}_h)\\ &= (\boldsymbol{\mu}, \tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h) \leq \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \|\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h\|_{L^\infty(\Omega)} \leq C|\ln h| h^{2-d/2} \|\boldsymbol{\mu}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)}.\end{aligned}$$

Since $\boldsymbol{f}$ was arbitrary this proves the result. □

This allows us now to prove the following convergence result for $\bar{\boldsymbol{q}}_h$. A similar result for the optimal control problem governed by the Poisson equation is shown in [27].

**Theorem 5.14** *Let $\bar{\boldsymbol{q}}_h \in \boldsymbol{\mathcal{M}}_h$ be the unique solution to* (5.4.1) *and $\bar{\boldsymbol{q}}$ the solution to* (5.1.1)*. Then, for $h \to 0$ we have*

$$\bar{\boldsymbol{q}}_h \overset{*}{\rightharpoonup} \bar{\boldsymbol{q}} \text{ in } \boldsymbol{\mathcal{M}}(\Omega), \tag{5.6.3}$$

$$\|\bar{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)} \to \|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)}, \tag{5.6.4}$$

$$\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{L^2(\Omega)} \to 0, \tag{5.6.5}$$

$$J_h(\bar{\boldsymbol{q}}_h) \to J(\bar{\boldsymbol{q}}). \tag{5.6.6}$$

*Proof.* We proceed as in [27] and prove

$$\boldsymbol{q}_h \overset{*}{\rightharpoonup} \boldsymbol{q} \text{ in } \boldsymbol{\mathcal{M}}(\Omega) \text{ implies } \|\boldsymbol{u}_h(\boldsymbol{q}_h) - \boldsymbol{u}(\boldsymbol{q})\|_{L^2(\Omega)} \to 0, \tag{5.6.7}$$

where $\boldsymbol{u}_h(\boldsymbol{q}_h)$ and $\boldsymbol{u}(\boldsymbol{q})$ are the respective discrete and continuous states with right-hand sides $\boldsymbol{q}_h$ and $\boldsymbol{q}$. The compact embedding $\boldsymbol{\mathcal{M}}(\Omega) \hookrightarrow W^{-1,s}(\Omega)^d$ for $1 \leq s < \frac{d}{d-1}$, implies the strong convergence of $\boldsymbol{q}_h \to \boldsymbol{q}$ in $W^{-1,s}(\Omega)^d$. Now by (2.2.6) we observe strong convergence of $\boldsymbol{u}(\boldsymbol{q}_h) \to \boldsymbol{u}(\boldsymbol{q})$ in $W^{1,s}(\Omega)^d$. We have by Lemma 5.13 that $\|\boldsymbol{u}_h(\boldsymbol{q}_h) - \boldsymbol{u}(\boldsymbol{q}_h)\|_{L^2(\Omega)} \to 0$, which by the triangle inequality implies $\|\boldsymbol{u}_h(\boldsymbol{q}_h) - \boldsymbol{u}(\boldsymbol{q})\|_{L^2(\Omega)} \to 0$.

Next, we prove weak-star convergence of $\bar{\boldsymbol{q}}_h$. We easily see that $\bar{\boldsymbol{q}}_h$ is bounded in $\boldsymbol{\mathcal{M}}(\Omega)$, since

$$\alpha\|\bar{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)} \leq J_h(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h) \leq J_h(\boldsymbol{0}, \boldsymbol{u}_h(\boldsymbol{0})) = \frac{1}{2}\|\boldsymbol{u}_d\|^2_{L^2(\Omega)}.$$

Then, we can take a subsequence for which we have $\bar{\boldsymbol{q}}_h \overset{*}{\rightharpoonup} \boldsymbol{\mu}$ in $\boldsymbol{\mathcal{M}}(\Omega)$. Now, this implies by (5.6.7) and lower semicontinuity of $\|\cdot\|_{\boldsymbol{\mathcal{M}}(\Omega)}$ as well as a weak-* convergence result for $\Lambda_h$ in [27, Theorem 3.1], that

$$\begin{aligned}J(\boldsymbol{\mu}, \boldsymbol{u}(\boldsymbol{\mu})) \leq \liminf_{h\to 0} J_h(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h) \leq \limsup_{h\to 0} J_h(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h) &\leq \limsup_{h\to 0} J_h(\Lambda_h\bar{\boldsymbol{q}}, \boldsymbol{u}_h(\Lambda_h\bar{\boldsymbol{q}}))\\ &= J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}}).\end{aligned}$$

Since $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}})$ is the unique solution of the continuous problem, it follows that $\boldsymbol{\mu} = \bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{q}}_h \overset{*}{\rightharpoonup} \bar{\boldsymbol{q}}$ in $\boldsymbol{\mathcal{M}}(\Omega)$. This proves (5.6.3) and (5.6.6). (5.6.5) follows from (5.6.7). Together (5.6.5) and (5.6.6) imply (5.6.4). □

This allows us to prove the following important corollary to Lemma 5.2 and Lemma 5.4.

**Corollary 5.15** *The optimal solution $\bar{\boldsymbol{q}} \in \boldsymbol{\mathcal{M}}(\Omega)$ to problem* (5.1.1) *and the optimal solution $\bar{\boldsymbol{q}}_h \in \boldsymbol{\mathcal{M}}_h$ to problem* (5.4.1) *are compactly supported in $\Omega$, i.e., there exists $\Omega_0 \Subset \Omega$ such that* $\operatorname{supp} \bar{\boldsymbol{q}}$, $\operatorname{supp} \bar{\boldsymbol{q}}_h \subset \Omega_0$*, and* $\operatorname{dist}(\Omega_0, \partial\Omega) \geq \varrho > 0$.

*Proof.* For $\bar{\boldsymbol{q}}$ this follows directly from the fact that the support of $\bar{\boldsymbol{q}}_j$ is contained in the set where the absolute value of the adjoint state is equal to $\alpha$. Since the adjoint state fulfills homogeneous Dirichlet boundary conditions and $\bar{\boldsymbol{z}} \in H^2(\Omega)^d \hookrightarrow C^{0,\zeta}(\Omega)^d$ one can construct such a set $\Omega_0$.

For the discrete case $\bar{\boldsymbol{q}}_h$ we need to show that the domain on which $|\bar{\boldsymbol{z}}_h| = \alpha$ cannot be arbitrarily close to the boundary for $h \to 0$. We consider $(\hat{\boldsymbol{z}}, \hat{r})$, the solution to

$$\begin{aligned} -\Delta \hat{\boldsymbol{z}} + \nabla \hat{r} &= S_h \bar{\boldsymbol{q}}_h - \boldsymbol{u}_d && \text{in } \Omega,\\ \nabla \cdot \hat{\boldsymbol{z}} &= 0 && \text{in } \Omega,\\ \hat{\boldsymbol{z}} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned}$$

The right-hand side is bounded independently of $h$ in $L^2(\Omega)^d$ since $S\bar{\boldsymbol{q}} \in W^{1,s}(\Omega)^d \hookrightarrow L^2(\Omega)^d$ and $\|\bar{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)} \leq C$ as well as $S_h \bar{\boldsymbol{q}}_h \to S\bar{\boldsymbol{q}}$ in $L^2(\Omega)^d$ for $h \to 0$, which follows by Theorem 5.14 and Lemma 5.13. Thus, there holds that $\hat{\boldsymbol{z}} \in H^2(\Omega)^d \hookrightarrow C^{0,\zeta}(\Omega)^d$ and one can construct a set $\Omega_0$ on which it holds that $|\hat{\boldsymbol{z}}| < \alpha$ close to the boundary, which is independent of $h$, for $h$ small enough, since $S_h \bar{\boldsymbol{q}}_h \to S\bar{\boldsymbol{q}}$.

To now show that $\bar{\boldsymbol{z}}_h$ also has this property we need an estimate like $\|\bar{\boldsymbol{z}}_h - \hat{\boldsymbol{z}}\|_{L^\infty(\Omega)} \to 0$. But this follows from Lemma 5.12. □

We continue with convergence results for the cost functional, for which we need the following assumptions.

**Assumption 5.16** *We assume*

$$\boldsymbol{u}_d \in \begin{cases} L^\infty(\Omega)^2 & \textit{for } d = 2,\\ L^3(\Omega)^3 & \textit{for } d = 3. \end{cases}$$

**Theorem 5.17** *Let Assumption 5.16 be fulfilled. For $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}})$ the solution to* (5.1.1) *and $(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h) \in \boldsymbol{\mathcal{M}}_h \times \boldsymbol{V}_h$ the solution to the discrete problem* (5.4.1)*, there holds that*

$$|J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}}) - J(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h)| \leq C |\ln h|^{2+r} h^{4-d},$$

*with $r = -1/3$ in the case $d = 3$ and $r = 1$ in the case $d = 2$.*

With the results we have derived up to now, the theorem can be proved in the same way as the result for the elliptic case in [103, Theorem 4.2].

*Proof.* Since $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}})$ and $(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h)$ are optimal for their respective problems, we have

$$J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}}) \leq J(\bar{\boldsymbol{q}}_h, \boldsymbol{u}(\bar{\boldsymbol{q}}_h)) \quad \text{and} \quad J(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h) \leq J(\bar{\boldsymbol{q}}, \boldsymbol{u}_h(\bar{\boldsymbol{q}}))$$

and therefore

$$J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}}) - J(\bar{\boldsymbol{q}}, \boldsymbol{u}_h(\bar{\boldsymbol{q}})) \leq J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}}) - J(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h) \leq J(\bar{\boldsymbol{q}}_h, \boldsymbol{u}(\bar{\boldsymbol{q}}_h)) - J(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h).$$

Thus, to bound $|J(\bar{\boldsymbol{q}}, \bar{\boldsymbol{u}}) - J(\bar{\boldsymbol{q}}_h, \bar{\boldsymbol{u}}_h)|$ we need to estimate the error for $\boldsymbol{q} \in \{\bar{\boldsymbol{q}}, \bar{\boldsymbol{q}}_h\}$ of

$$|J(\boldsymbol{q}, \boldsymbol{u}(\boldsymbol{q})) - J(\boldsymbol{q}, \boldsymbol{u}_h(\boldsymbol{q}))| = \frac{1}{2}\Big|\|\boldsymbol{u}(\boldsymbol{q}) - \boldsymbol{u}_d\|^2_{L^2(\Omega)} - \|\boldsymbol{u}_h(\boldsymbol{q}) - \boldsymbol{u}_d\|^2_{L^2(\Omega)}\Big|.$$

Here we will use the property of $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{q}}_h$ being supported only away from the boundary. In the following we define $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{q})$ and $\boldsymbol{u}_h = \boldsymbol{u}_h(\boldsymbol{q})$. Then,

$$\begin{aligned}
J(\boldsymbol{q}, \boldsymbol{u}) - J(\boldsymbol{q}, \boldsymbol{u}_h) &= \frac{1}{2}\Big(\|\boldsymbol{u} - \boldsymbol{u}_d\|^2_{L^2(\Omega)} - \|\boldsymbol{u}_h - \boldsymbol{u}_d\|^2_{L^2(\Omega)}\Big) \\
&= \frac{1}{2}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} + \boldsymbol{u}_h - 2\boldsymbol{u}_d) \\
&= -(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_d) + \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(\Omega)} + (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h).
\end{aligned}$$

Now, because of the compact support of $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{q}}_h$, shown in Corollary 5.15, we may apply Lemma 5.6 directly to the second term, resulting in

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(\Omega)} \leq C|\ln h|^2 h^{4-d}\|\boldsymbol{q}\|^2_{\boldsymbol{\mathcal{M}}(\Omega)}.$$

For the remaining terms we also apply Lemma 5.6 but consider different cases depending on the dimension.

**Case $d = 2$**

$$\begin{aligned}
(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_d) &\leq \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^1(\Omega)}\|\boldsymbol{u}_d\|_{L^\infty(\Omega)} \leq C|\ln h|^2 h^2\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)}, \\
(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h) &\leq \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^1(\Omega)}\|\boldsymbol{u}_h\|_{L^\infty(\Omega)} \leq C|\ln h|^3 h^2\|\boldsymbol{q}\|^2_{\boldsymbol{\mathcal{M}}(\Omega)},
\end{aligned}$$

where we used Lemma 5.10 in the second line.

**Case $d = 3$** Similarly, to before we apply Lemma 5.6 and Lemma 5.11

$$\begin{aligned}
(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_d) &\leq \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^{3/2}(\Omega)}\|\boldsymbol{u}_d\|_{L^3(\Omega)} \leq C|\ln h| h\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)}, \\
(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h) &\leq \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^{3/2}(\Omega)}\|\boldsymbol{u}_h\|_{L^3(\Omega)} \leq C|\ln h|^{1+2/3} h\|\boldsymbol{q}\|^2_{\boldsymbol{\mathcal{M}}(\Omega)}.
\end{aligned}$$

This completes the proof. □

A respective result for the solution to the state equation follows as in [103].

**Theorem 5.18** *Let the assumptions of Theorem 5.17 apply. Then, it holds for $r$ as above*

$$\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{L^2(\Omega)} \leq C|\ln h|^{1+r/2} h^{2-d/2}.$$

*Proof.* We apply the optimality condition (5.3.2) for the choice $\boldsymbol{q} = \bar{\boldsymbol{q}}_h$ and the discrete optimality condition (5.4.3) for $\boldsymbol{q} = \bar{\boldsymbol{q}}$ which leads to

$$\begin{aligned}
(\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} - \boldsymbol{u}_d) + \alpha\big(\|\bar{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)} - \|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)}\big) &\geq 0,\\
(\boldsymbol{u}_h(\bar{\boldsymbol{q}}) - \bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_d) + \alpha\Big(\|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)} - \|\bar{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)}\Big) &\geq 0.
\end{aligned}$$

Adding up we see

$$(\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} - \boldsymbol{u}_d) + (\boldsymbol{u}_h(\bar{\boldsymbol{q}}) - \bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_d) \geq 0.$$

Expanding this result gives

$$(\bar{\boldsymbol{u}}_h - \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} - \boldsymbol{u}_d) + (\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}} - \boldsymbol{u}_d) + (\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_d) + (\boldsymbol{u}_h(\bar{\boldsymbol{q}}) - \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_d) \geq 0$$

and thus after adding the first and the third term

$$\begin{aligned}
\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|^2_{L^2(\Omega)} &\leq (\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}} - \boldsymbol{u}_d) + (\boldsymbol{u}_h(\bar{\boldsymbol{q}}) - \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_d)\\
&= (\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}} - \boldsymbol{u}_h(\bar{\boldsymbol{q}})) + (\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}_h, \boldsymbol{u}_h(\bar{\boldsymbol{q}}) - \boldsymbol{u}_d) + (\boldsymbol{u}_h(\bar{\boldsymbol{q}}) - \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_d).
\end{aligned}$$

The first term on the right-hand side can be bounded using Lemma 5.6 for $s = 2$

$$(\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}} - \boldsymbol{u}_h(\bar{\boldsymbol{q}})) \leq \|\boldsymbol{u}(\bar{\boldsymbol{q}}_h) - \bar{\boldsymbol{u}}_h\|_{L^2(\Omega)} \|\bar{\boldsymbol{u}} - \boldsymbol{u}_h(\bar{\boldsymbol{q}})\|_{L^2(\Omega)} \leq C|\ln h|^2 h^{4-d} \|\bar{\boldsymbol{q}}\|_{\boldsymbol{\mathcal{M}}(\Omega)} \|\bar{\boldsymbol{q}}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)}.$$

The remaining parts can be handled as in Theorem 5.17, resulting in

$$\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|^2_{L^2(\Omega)} \leq C|\ln h|^{2+r} h^{4-d},$$

which completes the proof. □

## 5.7. Numerical experiments

We conduct numerical experiments to support the results in Section 5.6. The optimal control problems are solved by the optimization library RoDoBo [106] and the finite element toolkit Gascoigne [60]. The empirical convergence rates are computed by comparing values with the respective value computed on a mesh twice as fine as the finest mesh which we compare.

### 5.7.1. Computational aspects

For our numerical experiments we consider a slightly different setting compared to that introduced in Section 5.4, using local projection stabilization finite element methods on a discretization with square or cube shaped cells, instead of Taylor-Hood finite elements on a triangulation.

Since it is difficult working directly in the measure space, we consider in the following respective regularized problems. First we quickly touch base with regularization in the continuous case, afterwards we introduce the, for the computation more relevant, regularized discrete problem. In the continuous setting, an $L^2$ regularized problem may be given by:

$$\text{Minimize } J(\boldsymbol{u}, \boldsymbol{q}) = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_d\|^2_{L^2(\Omega)} + \alpha\|\boldsymbol{q}\|_{\boldsymbol{\mathcal{M}}(\Omega)} + \frac{\varepsilon}{2}\|\boldsymbol{q}\|^2_{L^2(\Omega)} \tag{5.7.1a}$$

$$\begin{aligned} \text{subject to} \quad -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{q} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned}$$

For a discussion of the behavior of Problem (5.7.1) for $\varepsilon \to 0$ in the elliptic case, we refer the reader to [36], where the authors analyze sparse optimal control problems governed by general elliptic equations. We note that the Stokes operator is self-adjoint and fulfills the norm requirement in the introduction of [36] (cf. [71, (1.9)]) and even though it has a different domain with $(H^2(\Omega)^d \cap \{\boldsymbol{v} \in H^1_0(\Omega)^d \colon \nabla \cdot \boldsymbol{v} = 0\})^d$, the results in [36] can potentially be extended to the Stokes operator.

From the regularization in the continuous case it is not immediately clear how to proceed for the discrete case, since $\boldsymbol{\mathcal{M}}_h$ consists of Dirac measures at the degrees of freedom such that an $L^2$ regularization is not possible. A solution is offered in [29, 102, 103] with an approach based on evaluating scalar products and their respective induced norms only at the nodal values, weighted with the diagonal of the so-called lumped mass matrix. In particular, one considers the following modified discrete problem

$$\text{Minimize } J(\boldsymbol{u}_h, \boldsymbol{q}_h) = \frac{1}{2}\|\boldsymbol{u}_h - \boldsymbol{u}_d\|^2_{L^2(\Omega)} + \alpha\|\boldsymbol{q}_h\|_{\boldsymbol{\mathcal{M}}(\Omega)} + \frac{\varepsilon}{2}\|\boldsymbol{q}_h\|^2_{L^2_h(\Omega)} \tag{5.7.2a}$$

$$\text{subject to} \qquad a((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, l_h)) = \langle \boldsymbol{q}_h, \boldsymbol{v}_h \rangle \qquad \forall (\boldsymbol{v}_h, l_h) \in \boldsymbol{V}_h \times M_h. \tag{5.7.2b}$$

Here the norm $\|\cdot\|_{L^2_h(\Omega)}$ is given as

$$\|\boldsymbol{q}_h\|^2_{L^2_h(\Omega)} = \sum_{i=1}^{N(h)} d_i^{-1} |\boldsymbol{\lambda}_i|^2,$$

where $\boldsymbol{\lambda}_i$ is the coefficient vector of the control $\boldsymbol{q}_h \in \boldsymbol{\mathcal{M}}_h$ at the nodal Dirac measure $\delta_{\boldsymbol{x}_i}$ and $(d_i)_{1 \le i \le N(h)}$ is the diagonal of the lumped mass matrix.

For more details we refer to [102, p. 99ff.]. The author argues that (in the elliptic case) the solution to such a regularized discrete problem is equivalent for $\varepsilon \to 0$ to the solution of the not regularized discrete problem (cf. [102, Proposition 4.28]). The proof does not depend on any properties of the elliptic problem such it may be extended to our situation, i.e., the solution of (5.7.2) is equivalent to the solution of (5.4.1) for $\varepsilon \to 0$.

In our computations we realize the control $\boldsymbol{q}_h \in \{\boldsymbol{v}_h \in C_0(\Omega)^d \colon \boldsymbol{v}_h|_k \in \mathcal{P}_1(T)^d \;\forall T \in \mathcal{T}_h\}$ and do not calculate the lumped mass matrix explicitly but use the fact that, for the trapezoidal quadrature rule, we can forgo this step, because the norms and scalar product based on the trapezoidal quadrature rule are equivalent to the norms and scalar product using the diagonal of the lumped mass matrix (cf. [102, below (4.32)]). Note that in our case such an approach is only possible since we are not using Taylor-Hood elements of higher degree, but instead cell-wise linear functions and the local projection stabilization. The alternative is working directly with the diagonal of the lumped mass matrix.

We choose $\varepsilon$ such that

$$\frac{\varepsilon}{2}\|\boldsymbol{q}\|^2_{L^2_h(\Omega)} \le C_{reg} h^2,$$

where $C_{reg}$ is chosen heuristically, to ensure the impact of $\varepsilon/2\|\boldsymbol{q}\|^2_{L^2_h(\Omega)}$ is not greater than the error introduced by the finite element discretization.

### 5.7.2. Two-dimensional domain

We consider the problem on the unit square $\Omega = (0,1)^2$ which is in part inspired by the examples in [57, 103]. Let $r$ be defined as $r = \max(\rho, |\boldsymbol{x} - \hat{\boldsymbol{x}}|)$, where $\hat{\boldsymbol{x}} = \left(\frac{1}{2}\;\frac{1}{2}\right)^T$ and $\rho \in \left\{0, \frac{1}{4}\right\}$. Then, we set the desired state as

$$\boldsymbol{u}_d = \begin{pmatrix} -\ln r + \frac{1}{r^2}\left((\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)^2 + (\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)(\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2)\right) \\ -\ln r + \frac{1}{r^2}\left((\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2)^2 + (\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)(\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2)\right) \end{pmatrix}.$$

Furthermore, we choose $\alpha = 3 \cdot 10^{-4}$ and $C_{reg} = 0.5 \cdot 10^{-2}$.

A visualization of the solutions for control and state for the fourth refinement level is given in Figure 5.1 for $\rho = 1/4$. We have chosen to plot $\bar{\boldsymbol{q}}_h$ using linear interpolation between the values at the nodes rather than simple points for the Dirac measures to better visualize the behavior of the weights for each Dirac. The small support of the control shows the sparsity properties that we expect from the problem formulation.

While in [102] the authors construct an exact solution, we here only compare with an approximate solution on the finest mesh. This is because of the different structure of the Stokes problem, which leads to difficulties in constructing such an exact solution.

A fundamental solution to the Stokes problem is available, e.g., in [58, Section IV] and exact solutions have been constructed in the case of singular or multiple Dirac measures on the right-hand side in [4]. Unfortunately such a construction is not easily available for *homogeneous* Dirichlet boundary conditions. The use of homogeneous boundary conditions is relevant to the argumentation in Corollary 5.15, where we show that $\bar{\boldsymbol{q}}$, $\bar{\boldsymbol{q}}_h$ are only supported in the interior of the domain. Furthermore, results are already available on the numerical behavior of the problem with singular sources as the control on the right-hand side in [57]. The authors observe similar convergence rates in the state as shown here.

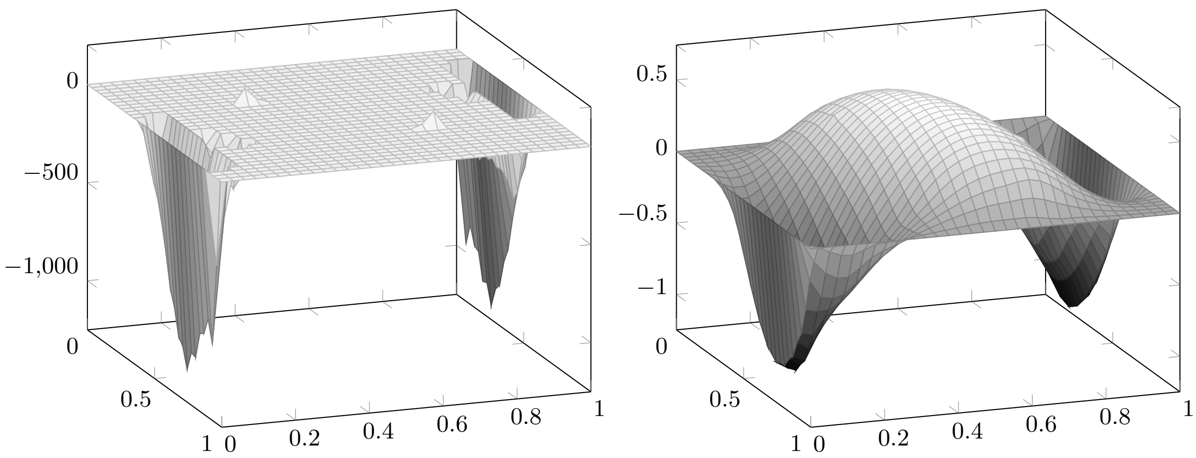


Figure 5.1.: Surface plots of the first component of control and state solutions for the two-dimensional example.

Therefore, with our choice of $\boldsymbol{u}_d$, we here consider an example without an exact solution but for which the resulting control is more than a single Dirac measure. The empirical convergence rates, computed by using the discrete solutions $\tilde{\boldsymbol{q}}_h$ and $\tilde{\boldsymbol{u}}_h$ on a finer mesh, for $\rho = \frac{1}{4}$ are given in Figure 5.2.

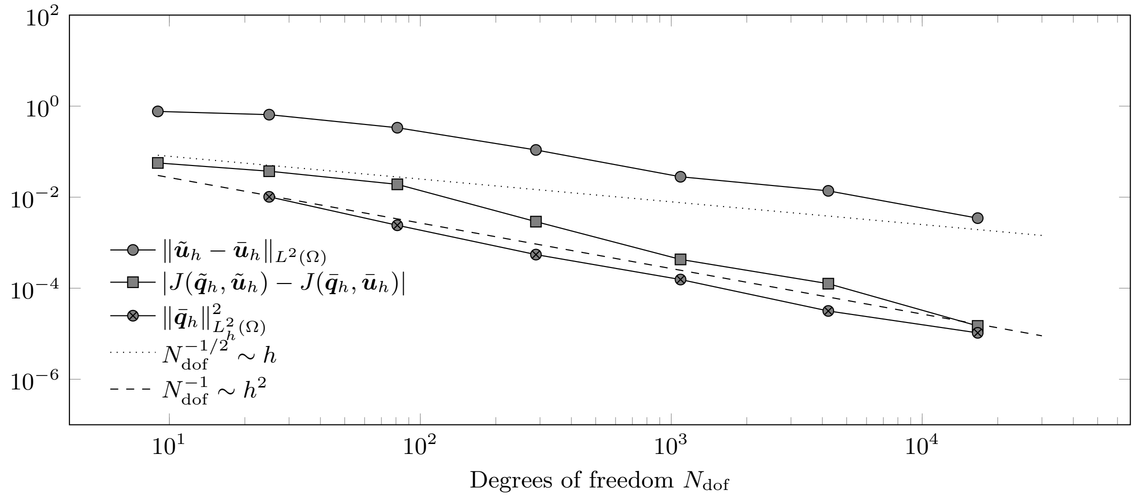


Figure 5.2.: Convergence rates for the two-dimensional example ($\rho = \frac{1}{4}$).

When considering $\rho = 0$, we obtain the following convergence rates in Figure 5.3.

The convergence rates support our theoretical results, which were derived in the previous section. We observe, after faster convergence in the beginning, an empirical convergence rate of $O(h)$ for the state and $O(h^2)$ for the cost functional for both cases of $\rho$. For $\rho = 1/4$ this confirms our results from Section 5.6 with Assumption 5.16 active. Figure 5.3 indicates that at least for this example we have similar convergence rates for unconstrained $\boldsymbol{u}_d$.

The dip in the final measurements, i.e., for $N_{\mathrm{dof}} \geq 10^4$, can be explained by the fact that we

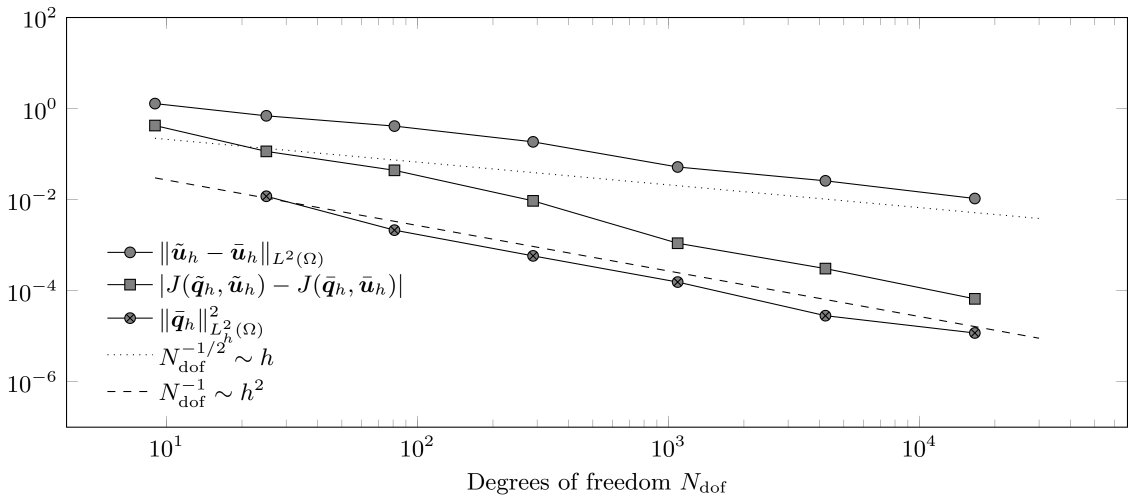


Figure 5.3.: Convergence rates for the two-dimensional example ($\rho = 0$).

here only calculate the empirical error.

### 5.7.3. Three-dimensional domain

Let $\Omega = (0,1)^3$ and $r$ be defined as $r = \max(\rho, |\boldsymbol{x} - \hat{\boldsymbol{x}}|)$, where $\hat{\boldsymbol{x}} = \left(\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\right)^T$,

$$\boldsymbol{u}_d = \begin{pmatrix} -\frac{1}{r} + \frac{1}{r^3}\left((\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)^2 + (\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)(\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2) + (\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)(\boldsymbol{x}_3 - \hat{\boldsymbol{x}}_3)\right) \\ -\frac{1}{r} + \frac{1}{r^3}\left((\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2)^2 + (\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)(\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2) + (\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2)(\boldsymbol{x}_3 - \hat{\boldsymbol{x}}_3)\right) \\ -\frac{1}{r} + \frac{1}{r^3}\left((\boldsymbol{x}_3 - \hat{\boldsymbol{x}}_3)^2 + (\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1)(\boldsymbol{x}_3 - \hat{\boldsymbol{x}}_3) + (\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2)(\boldsymbol{x}_3 - \hat{\boldsymbol{x}}_3)\right) \end{pmatrix},$$

and $\rho = \left\{0, \frac{1}{4}\right\}$. In the case $\rho = \frac{1}{4}$ we choose $\alpha = 3 \cdot 10^{-3}$, $C_{reg} = 10^{-2}$. For $\rho = 0$ we choose $\alpha = 3 \cdot 10^{-2}$, $C_{reg} = 10^{-2}$. We chose different parameters here to improve the convergence speed of the implementation of our examples while still ensuring that $\varepsilon/2\|\boldsymbol{q}\|^2_{L^2_h(\Omega)}$ is small enough.

The empirical convergence rates $O(h^{1/2})$ for the state and $O(h)$ observed in Figure 5.5 support our theoretical results in the case $\rho = 0$. Note that in this case, for a right-hand side $\boldsymbol{u}_d$, the resulting optimal control is very close to a single Dirac in the middle of the domain. It is not exactly the expected Dirac because we do not choose appropriate boundary conditions. Nevertheless, $\boldsymbol{u}_d \in W^{1,3/2-\tilde{\varepsilon}}(\Omega)^3$ for $\tilde{\varepsilon} > 0$ and thus $\boldsymbol{u}_d$ is only in $L^{3-\hat{\varepsilon}}(\Omega)^3$ for $\hat{\varepsilon} > 0$, so we are very close to the requirement in Assumption 5.16.

In Figure 5.4 we observe a convergence rate as above for the cost functional but a significantly better convergence rate for the state with $O(h)$. This is likely due to the fact that for $\rho = 1/4$ we have $\boldsymbol{u}_d \in L^\infty(\Omega)^3$. For $\boldsymbol{u}_d \in L^\infty(\Omega)^3$ an improved convergence rate for the state has been shown for the Poisson constrained problem in [103]. The techniques used there are based on a maximum principle that is not available in this form for the Stokes problem but our numerical results indicate that a similar convergence result might hold here.

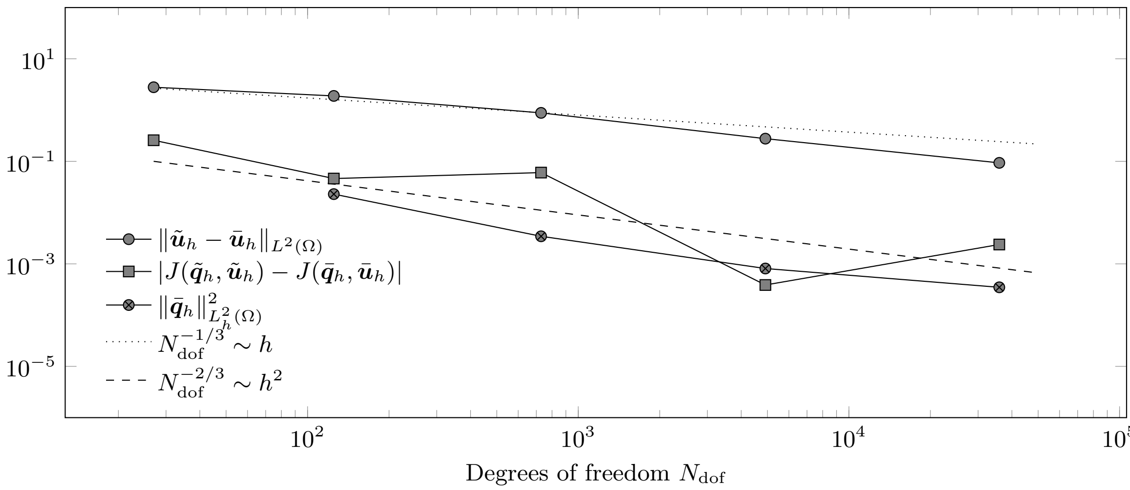


Figure 5.4.: Convergence rates for the three-dimensional example ($\rho = \frac{1}{4}$).

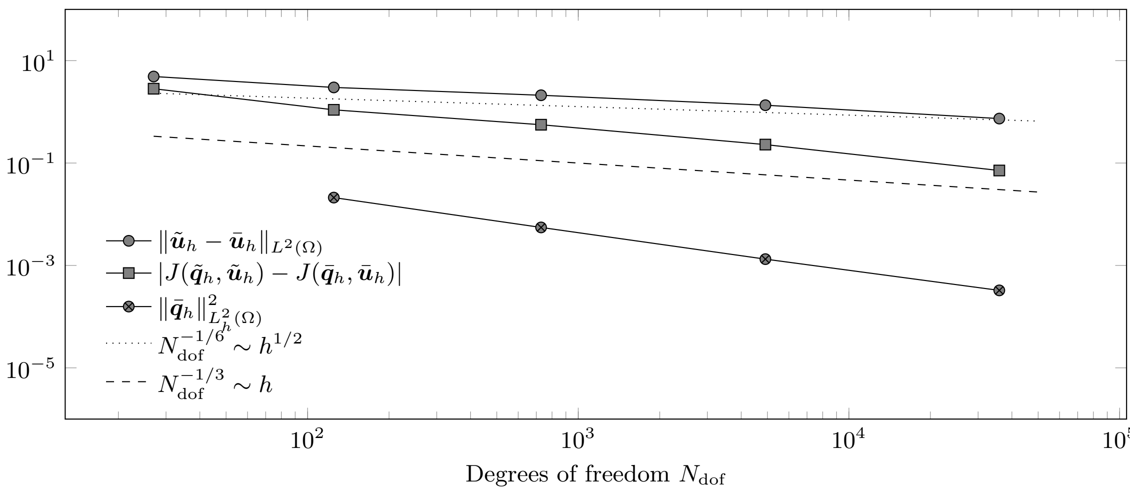


Figure 5.5.: Convergence rates for the three-dimensional example ($\rho = 0$).

# Chapter 6.

# The instationary Stokes problem and (semi-)discrete maximal regularity

## 6.1. Introduction

In this chapter we discuss properties of the instationary Stokes problem

$$\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } I \times \Omega, \tag{6.1.1a}$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } I \times \Omega, \tag{6.1.1b}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } I \times \partial\Omega, \tag{6.1.1c}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{6.1.1d}$$

Here $\Omega \subset \mathbb{R}^d$ is either a convex polyhedron or is star-shaped and has a smooth boundary and we consider the time domain $I = [0, \mathscr{T}]$ with $\mathscr{T} > 0$. We assume a right-hand side $\boldsymbol{f} \in L^s(I; L^p(\Omega)^d)$, for $1 \leq s, p \leq \infty$, and $\boldsymbol{u}_0 \in W_0^{1,p}(\Omega)^d$ with $\nabla \cdot \boldsymbol{u}_0 = 0$. The symbol $L^s(I; L^p(\Omega)^d)$ denotes a Bochner space and its respective norm is defined as

$$\|\boldsymbol{f}\|_{L^s(I;L^p(\Omega)^d)} = \left( \int_I \|\boldsymbol{f}(t)\|_{L^p(\Omega)}^s \right)^{1/s}, \quad \|\boldsymbol{f}\|_{L^\infty(I;L^p(\Omega))} = \text{ess sup}_{t \in I} \|\boldsymbol{f}(t)\|_{L^p(\Omega)}.$$

We would like to see best-approximation results as in Chapter 3 also for the instationary case, discretized in time and space. In the case of the heat equation, such estimates have been derived in [49, 86, 110] for the maximum norm of the diffusion and for the maximum norm of the gradient of the diffusion in [88, 89, 122]. Further results are also available in case of discretization solely in space, for an overview over respective references we refer to [86, 88].

We are not aware of any best-approximation max-norm estimates in time *and* space for the instationary Stokes problem (6.1.1a)–(6.1.1d) in the literature. $L^\infty$ error estimates for the instationary Stokes equation in space on two-dimensional domains can be found in [113]. A result for the fully discrete problem in form of $L^\infty/L^2$ estimates based on discontinuous Galerkin methods is provided by Chrysafinos and Walkington in [35], including an overview over related results for (semi-)discrete problems based on other discretization approaches. Recently the numerical behavior of a stabilized discontinuous Galerkin scheme for the Stokes problem has been analyzed in [3]. Furthermore, there are results for the fully discrete Navier-Stokes problem under moderate regularity assumptions by Heywood and Rannacher in [77]. In this chapter, we focus on an approach via a discontinuous Galerkin time stepping scheme similar to the approach in [35, 85].

During the work on this problem we encountered difficulties which do not appear for the parabolic problem and which stopped us short of proving maximum norm best-approximation results for Problem (6.1.1a)–(6.1.1d).

To show pointwise best-approximation results as in [86, 88] we pursued the following approach: Pointwise best-approximation results would be based on maximal regularity and pointwise smoothing estimates for the (semi-)discrete Stokes system. In turn, to show these estimates, one needs to derive a respective estimate for the discrete Stokes resolvent problem.

Unfortunately, the theory on resolvent estimates for the Stokes problem is not yet complete in the literature and we were not able to supplement it here. Thus, this program has not been concluded successfully yet. In this chapter, we discuss the difficulties encountered in our approach with focus on the resolvent estimate. Furthermore, we derive maximal regularity and approximation results based on our current progress.

In the next section we introduce the Stokes resolvent problem and discuss the appearing difficulties. Afterwards we give an overview over current state of the art results for the continuous and discrete resolvent problem and finally we derive approximation results based on available resolvent estimates.

## 6.2. Stokes resolvent estimates in $L^p(\Omega)$

We consider the Stokes resolvent problem in a bounded domain $\Omega \subset \mathbb{R}^d$

$$z\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{6.2.1a}$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{6.2.1b}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{6.2.1c}$$

Here $z \in \Sigma_{\theta,\bar{\omega}}$, which is defined as

$$\Sigma_{\theta,\bar{\omega}} = \{c \in \mathbb{C} \colon c \neq \bar{\omega} \text{ and } |\arg(c - \bar{\omega})| < \theta\}$$

and $\theta \in (\pi/2, \pi)$, $\bar{\omega} \leq 0$ since we want to apply results from [50] for which estimates of the solution to (6.2.1a)–(6.2.1c) with $z \in \Sigma_{\theta,\bar{\omega}}$ are a key requirement.

The solution $(\boldsymbol{u}, p)$ is a complex-valued function in $H_0^1(\Omega)^d \times L^2(\Omega)$, assuming sufficient regularity of the boundary. In this section we understand $H_0^1(\Omega)^d \times L^2(\Omega)$ as complex-valued function spaces with a Hermitian inner product.

To show maximal regularity, our approach requires the following resolvent estimate for $z \in \Sigma_{\theta,\bar{\omega}}$ in the continuous case

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leq \frac{C}{|z-\bar{\omega}|}\|\boldsymbol{f}\|_{L^p(\Omega)} \tag{6.2.2}$$

and for the discrete case

$$\|\boldsymbol{u}_h\|_{L^p(\Omega)} \leq \frac{C_h}{|z-\bar{\omega}|}\|\boldsymbol{f}_h\|_{L^p(\Omega)} \tag{6.2.3}$$

for the velocity solution $\boldsymbol{u}_h$ of the respective discrete problem with right-hand side $\boldsymbol{f}_h \in \boldsymbol{V}_h$ and $1 \leq p \leq \infty$. In practice it is enough to show the result for $p = \infty$ and $p = 2$ in the discrete case. $C_h$ may linearly depend on $|\ln h|$. The extension to the full range follows by interpolation and duality. Note that because of the complex nature of $z$ we need corresponding finite element spaces which we define below.

When approaching maximal regularity estimates, one can characterize the solution operator to the instationary problem (6.1.1a)–(6.1.1d) as an analytic semi-group in the time variable $t$. This solution operator can be represented, in particular in the discrete case, by a Dunford-Taylor integral in the complex plane (cf. [121, Chap. 9]). Respective resolvent estimates can then be leveraged to show maximal regularity in the continuous and the discrete case.

### 6.2.1. Overview: Continuous resolvent estimates

Before we discuss (6.2.3) we want to give an overview over the state of continuous resolvent estimates (6.2.2), not exclusively, but with emphasis on the behavior on convex polyhedral domains. Apart from the original proofs of the resolvent estimate (6.2.2) on $C^2$ domains in [63, 117], to our knowledge there are mainly two elementary techniques which have been used to show estimate (6.2.2) in recent years. On the one hand, there is the approach of showing the estimate on the whole space and half space and then generalizing the results to perturbed half spaces and bounded domains with smooth boundaries in [53, 55, 56]. On the other hand, there is the potential theoretical approach deployed in [43, 115]. Furthermore, more recent results in $L^\infty(\Omega)$ in [2] use a localization technique based on the estimates in [53].

The downside of the perturbed half space technique is that to transfer the results to bounded domains via a partition of unity, one needs a boundary smooth enough to construct the respective diffeomorphisms. While there are results which extend this technique to Lipschitz domains, e.g., in [59], assuming small Lipschitz constants, this falls short of a resolvent estimate on general convex polyhedral domains.

Using the potential theoretical approach one can derive resolvent estimates also for Lipschitz domains and in three dimensions. This works very well for the heat equation in [114]. But it turns out, via a counter example of Deuring for large $p$ in [44] featuring a reentrant corner, that a similar result cannot be achieved in three dimensions for the Stokes problem. The available result on Lipschitz domains, again by Shen in [115], is limited to a small interval of $p$.

To summarize the results so far, on $C^{1,1}$ domains bounded resolvent estimates have been shown for $1 < p < \infty$ in [53] and have even been extended to weighted spaces in [56]. On three-dimensional Lipschitz domains a resolvent estimate has been shown for

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{6} + \varepsilon,$$

for $\varepsilon > 0$. Further results are available for exterior domains, the whole space, and the half space in [43, 54, 55, 95].

The case $p = \infty$ on bounded $C^3$ domains has been covered in [2] as well as a respective analyticity result for the semi-group in [1]. The authors use a property called admissible domains, which in particular applies to $C^3$ bounded domains, to control the behavior of the pressure. To prove the resolvent result the authors employ a localization technique. The appearing local compressible Stokes resolvent problems are handled using, among other results, the estimate from [53].

We note that the counterexample for Lipschitz domains still leaves open the possibility of an estimate on convex domains. Indeed, resolvent estimates on convex polyhedral domains are an open problem up to date. It even appears in a collection of open problems by Maz'ya in [91, Problem 66]. The available theory on the Stokes equations indicates a positive answer to this problem. It has been shown in [94] that elliptic regularity results also extend to the Stokes problem. Furthermore, it has been shown by Geng and Shen in [62], where they analyze a Neumann problem on a convex domain $\Omega$, that the Helmholtz projection, which projects into a divergence-free subspace, is a bounded operator in $L^p(\Omega)$ for $1 < p < \infty$. The existence of such a projection operator and the underlying Neumann problem are closely connected to maximal regularity estimates and the generation of an analytic semi-group, at least for domains with $C^3$ boundaries. We refer to the discussion in [61].

The existence of respective resolvent estimates on convex polyhedral domains therefore seems to be an interesting problem, in particular since it is not covered by the techniques discussed above. Recently a similar problem has been discussed on wedge domains in [81], considering perfect slip boundary conditions compared to the no-slip boundary conditions discussed in this thesis. Results on wedge domains are a first step towards estimates on convex polygonal domains, since every corner can be represented by a respective wedge domain.

To recapitulate, even in the continuous setting the problem of resolvent estimates on convex polyhedral domains is still open but the theory points to the existence of such an estimate.

### 6.2.2. An attempt at a discrete resolvent estimate

We are unaware of any result regarding discrete resolvent estimates for the Stokes problem (in $L^p(\Omega)^d$, $p \neq 2$) in the literature. This absence is also discussed shortly by Guermond and Pasciak in [71, Remark 5.2], where they remark that a resolvent estimate would provide a more elegant approach to the fractional Sobolev space estimates they prove in [71].

While we were not able to show a discrete resolvent estimate in this work, we want to at least provide insights on where our approach to show a discrete resolvent estimate on convex polyhedral domains failed. To do so, we pursue a technique based on the work in [13, 85, 121].

We focus on the two-dimensional case. Let $\delta_h$ denote the regularized Dirac delta from Chapter 3 and assume that $P_h$ and $r_h$ are operators for which the properties stated in Section 3.2 hold. Furthermore, we will use the weight function $\sigma$ which is also defined in Section 3.2.

In this section we define the $L^2$-inner product as

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \bar{\boldsymbol{v}}(\boldsymbol{x}) d\boldsymbol{x}.$$

Here $\bar{\boldsymbol{v}}$ denotes the convex conjugate of $\boldsymbol{v}$ and we use finite element spaces $\mathbb{V}_h = \boldsymbol{V} + i\boldsymbol{V}_h$ and $\mathbb{M}_h = M_h + iM_h$ with $\boldsymbol{V}_h$ and $M_h$ fulfilling the assumptions of Section 3.2. We denote the $L^2$ projection into $\mathbb{V}_h$ by $\tilde{P}_h$.

The goal is to show (6.2.3) for $p = \infty$. We only consider the case $\bar{\omega} = 0$ here. Using $\delta_h$, we then define a regularized Green's function with velocity $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}, z, \boldsymbol{x}_0)$ and pressure $\lambda = \lambda(\boldsymbol{x}, z, \boldsymbol{x}_0)$ as the solution to

$$z\boldsymbol{g} - \Delta \boldsymbol{g} + \nabla \lambda = \tilde{P}_h(\delta_h \boldsymbol{e}_i) \quad \text{in } \Omega, \tag{6.2.4a}$$
$$\nabla \cdot \boldsymbol{g} = 0 \quad \text{in } \Omega, \tag{6.2.4b}$$
$$\boldsymbol{g} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{6.2.4c}$$

Here $\boldsymbol{e}_i$ denotes the $i$-th unit vector. We also define the discrete analog $(\boldsymbol{g}_h, \lambda_h) \in \mathbb{V}_h \times \mathbb{M}_h$ given as the solution to

$$z(\boldsymbol{v}_h, \boldsymbol{g}_h) + (\nabla \boldsymbol{v}_h, \nabla \boldsymbol{g}_h) - (\nabla \cdot \boldsymbol{v}_h, \lambda_h) = (\boldsymbol{v}_h, \tilde{P}_h(\delta_h \boldsymbol{e}_i)) \quad \forall \boldsymbol{v}_h \in \mathbb{V}_h, \tag{6.2.5a}$$
$$(q_h, \nabla \cdot \boldsymbol{g}_h) = 0 \quad \forall q_h \in \mathbb{M}_h. \tag{6.2.5b}$$

Now, using the properties of $\delta_h$ and $\sigma$ we can bound $\boldsymbol{u}_h$, the solution to the discrete resolvent problem, as follows

$$\begin{aligned} |\boldsymbol{u}_{h,i}(\boldsymbol{x}_0)| = |(\boldsymbol{u}_h, \tilde{P}_h(\delta_h \boldsymbol{e}_i))| = |(\boldsymbol{f}, \boldsymbol{g}_h)| &\leq \|\sigma^{-1}\|_{L^2(\Omega)} \|\boldsymbol{f}\|_{L^\infty(\Omega)} \|\sigma \boldsymbol{g}_h\|_{L^2(\Omega)} \\ &\leq C|\ln h|^{1/2} \|\boldsymbol{f}\|_{L^\infty(\Omega)} \|\sigma \boldsymbol{g}_h\|_{L^2(\Omega)}. \end{aligned}$$

Thus, for a resolvent estimate as in (6.2.3) we need to show an estimate like

$$\|\sigma \boldsymbol{g}_h\|_{L^2(\Omega)} \leq C|\ln h|^{1/2}|z|^{-1}.$$

To this end, consider the expression

$$z\|\sigma \boldsymbol{g}_h\|^2_{L^2(\Omega)} + \|\sigma \nabla \boldsymbol{g}_h\|^2_{L^2(\Omega)} = z(\sigma^2 \boldsymbol{g}_h, \boldsymbol{g}_h) + (\nabla(\sigma^2 \boldsymbol{g}_h), \nabla \boldsymbol{g}_h) - 2(\sigma \nabla \sigma \boldsymbol{g}_h, \nabla \boldsymbol{g}_h). \tag{6.2.6}$$

Next, we test (6.2.5) with $\boldsymbol{v}_h = \tilde{P}_h(\sigma^2 \boldsymbol{g}_h)$, move all terms to the right-hand side, and add it to (6.2.6), to obtain

$$z\|\sigma \boldsymbol{g}_h\|^2_{L^2(\Omega)} + \|\sigma \nabla \boldsymbol{g}_h\|^2_{L^2(\Omega)} = F, \tag{6.2.7}$$

where

$$\begin{aligned} F &= -(\tilde{P}_h(\sigma^2 \boldsymbol{g}_h), \tilde{P}_h(\delta_h \boldsymbol{e}_i)) + (\nabla(\sigma^2 \boldsymbol{g}_h - \tilde{P}_h(\sigma^2 \boldsymbol{g}_h)), \nabla \boldsymbol{g}_h) - 2(\sigma \nabla \sigma \boldsymbol{g}_h, \nabla \boldsymbol{g}_h) - (\nabla \cdot \tilde{P}_h(\sigma^2 \boldsymbol{g}_h), \lambda_h) \\ &= F_1 + F_2 + F_3 + F_4. \end{aligned}$$

By standard arguments, which can be found, e.g., in [15, (3.8)], we can bound (6.2.7) using the angle of the sector $\Sigma_{\theta,\bar{\omega}}$. To that end, note that (6.2.6) is of the form

$$a \exp(i\phi) + b = f,$$

with $a, b \in \mathbb{R}$. Then, multiplying with $\exp(-i\phi/2)$, taking real parts, and the observation $0 \leq \phi \leq \theta < \pi$ due to the choice of $\Sigma_{\theta,\bar{\omega}}$ gives way to the following estimate

$$|z| \|\sigma \boldsymbol{g}_h\|^2_{L^2(\Omega)} + \|\sigma \nabla \boldsymbol{g}_h\|^2_{L^2(\Omega)} \leq C_\theta |F|.$$

Applying the absolute value allows us to discuss the following estimates detached from the fact that we are dealing with function spaces over complex fields. The terms $F_1$, $F_2$, and $F_3$ can be dealt with as in the proof of [121, Theorem 6.5]:

$$|F_1| + |F_2| + |F_3| \leq C\Big(\|\boldsymbol{g}_h\|_{L^2(\Omega)} + |z|^{-1}\Big).$$

The term $\|\boldsymbol{g}_h\|_{L^2(\Omega)}$ is then bounded by $|z|^{-1} |\ln h|^{1/2}$ using a discrete Sobolev inequality (cf. [121, Lemma 6.4]) and testing (6.2.5a) with $\boldsymbol{g}_h$.

It remains to bound the additional pressure term

$$\begin{aligned} |F_4| = |(\nabla \cdot \tilde{P}_h(\sigma^2 \boldsymbol{g}_h), \lambda_h)| &= |(\nabla \cdot (\sigma^2 \boldsymbol{g}_h), \lambda_h) + (\nabla \cdot (\tilde{P}_h(\sigma^2 \boldsymbol{g}_h) - \sigma^2 \boldsymbol{g}_h), \lambda_h)| \\ &= |(2\sigma \boldsymbol{g}_h \cdot \nabla \sigma, \lambda_h) + (\sigma^2 \nabla \cdot \boldsymbol{g}_h, \lambda_h) + (\nabla \cdot (\tilde{P}_h(\sigma^2 \boldsymbol{g}_h) - \sigma^2 \boldsymbol{g}_h), \lambda_h)|. \end{aligned}$$

Note that we only could avoid this term if we used an appropriate divergence-free test function. No such function is known to us. It is also not feasible to modify $\tilde{P}_h$ to be a projection into the space of divergence-free finite element functions which satisfy zero Dirichlet boundary conditions since such a projection is no longer stable in $H^1_0(\Omega)$, see for example the discussion in [71, Remark 3.1].

Assuming that similar super-approximation results as in [85, Lemma 3] hold for $\tilde{P}_h$ and super-approximation results as in Assumption 3.37 hold for $r_h$, we can then further simplify the term above and obtain

$$\begin{aligned} |(\nabla \cdot \tilde{P}_h(\sigma^2 \boldsymbol{g}_h), \lambda_h)| &\leq C\|\lambda_h\|_{L^2(\Omega)} \|\sigma \boldsymbol{g}_h\|_{L^2(\Omega)} + |(\sigma^2 \nabla \cdot \boldsymbol{g}_h, \lambda_h)| \\ &\leq C\|\lambda_h\|_{L^2(\Omega)} \|\sigma \boldsymbol{g}_h\|_{L^2(\Omega)} + \|\sigma \nabla \boldsymbol{g}_h\|_{L^2(\Omega)} \|\sigma^{-1}(\sigma^2 \lambda_h - r_h(\sigma^2 \lambda_h))\|_{L^2(\Omega)} \\ &\leq C\|\lambda_h\|_{L^2(\Omega)} \|\sigma \boldsymbol{g}_h\|_{L^2(\Omega)} + Ch\|\sigma \nabla \boldsymbol{g}_h\|_{L^2(\Omega)} \|\lambda_h\|_{L^2(\Omega)}, \end{aligned}$$

where we used the fact that $\boldsymbol{g}_h$ is discretely divergence-free due to (6.2.5b). The assumption above on $\tilde{P}_h$ is reasonable at least for Taylor-Hood finite elements, see the arguments in [85, Lemma 1, 2 and 3].

Unfortunately this is where viability of this approach seems to end. The term $\|\sigma \boldsymbol{g}_h\|_{L^2(\Omega)}$ can simply be kicked back but the seemingly benign $L^2$ norm estimate of $\lambda_h$ turns out to be problematic. To conclude (6.2.3) for $p = \infty$ we need to show

$$\|\lambda_h\|_{L^2(\Omega)} \leq C, \tag{6.2.8}$$

where $C$ may have a dependency on $h$ of the form $C \sim |\ln h|$.

We first discuss the term $\|\lambda\|_{L^2(\Omega)}$, the continuous counterpart of $\|\lambda_h\|_{L^2(\Omega)}$, to demonstrate that the difficulties regarding $\lambda_h$ are inherently part of the Stokes problem and not caused by the finite element discretization. Afterwards we point out similar issues appearing for the discrete problem.

To begin with, we note that for the resolvent problem the $L^2$ estimate of $\lambda$ is closely related to the $H^{-1}$ estimate of $\boldsymbol{g}$. This is easily seen when invoking a standard $L^2$ regularity result for the pressure, for example in [58, Lemma IV.1.1]. Indeed, it has been shown in [123, Lemma 3.5] that the existence of an $L^2$ estimate for the pressure corresponds to an $H^{-1}$ estimate of the velocity and vice versa.

In [123] the issue of $L^2$ estimates for the pressure is discussed at length. In the case of $|z| \leq 1$ the following proposition (cf. [123, Proposition 3.6]) shows that the $L^2$ norm of the pressure can be bounded by the $H^{-1}$ norm.

**Proposition 6.1** *Let $\Omega$ be a bounded Lipschitz domain and $\theta \in (0, \pi]$. For all $1/4 < \bar{\alpha} \leq 1/2$ there exists a constant $C > 0$ such that for $\boldsymbol{f} \in H^{-1}(\Omega)^d$ and $z \in \Sigma_\theta$ the pressure $p \in L^2_0(\Omega)$ in the solution to* (6.2.1a)–(6.2.1c) *with right-hand side $\boldsymbol{f}$ satisfies*

$$\|p\|_{L^2(\Omega)} \leq C \max\{1, |z|^{\bar{\alpha}}\} \|\boldsymbol{f}\|_{H^{-1}(\Omega)}.$$

That this bound is actually sharp with respect to the lower bound of $\bar{\alpha}$ is then proven by Tolksdorf in [123, Proposition 3.8] on $C^4$ domains.

**Proposition 6.2** *Let $\Omega$ be a bounded domain with $C^4$-boundary, $\theta \in (\pi/2, \pi)$, and $0 \leq \bar{\alpha} < 1/4$. Then for all $n \in \mathbb{N}$ there exists $\boldsymbol{f}_n \in H^{-1}(\Omega)^d$ and $z_n \in \Sigma_\theta$ with $|z_n| \geq 1$ such that the pressure $p_n \in L^2_0(\Omega)$ in the solution to* (6.2.1a)–(6.2.1c) *with right-hand side $\boldsymbol{f}_n$ satisfies*

$$\|p_n\|_{L^2(\Omega)} > n|z_n|^{\bar{\alpha}} \|\boldsymbol{f}_n\|_{H^{-1}(\Omega)}.$$

Note that this does not preclude an estimate for a specific right-hand side like $\delta_h$, but so far no such estimate has emerged for (6.2.8).

The result also holds on more general domains, for which the Helmholtz projection $\hat{\mathbb{P}}\colon L^2(\Omega)^d \to L^2_\sigma(\Omega)^d$, which is a projection from $L^2(\Omega)^d$ onto the space of divergence-free $L^2$ functions ($L^2_\sigma(\Omega)^d$), does not preserve boundary values (see the final step in the proof of [123, Proposition 3.4]).

Furthermore, this result implies that the issue rests with the Dirichlet boundary conditions imposed on the system. Indeed, for a Stokes resolvent problem with Neumann boundary conditions an estimate like (6.2.8) is obtained in the continuous case (cf. [123, Proposition 3.1]).

Next, we consider the problem for $\lambda_h$. To that end, we follow the definitions from [71] and let

$$\mathbb{X}_h = \{\boldsymbol{v}_h \in \mathbb{V}_h \colon (\nabla \cdot \boldsymbol{v}_h, q_h) = 0, \quad \forall q_h \in \mathbb{M}_h\}$$

be the set of discretely divergence-free vectors. Then, we define the discrete Helmholtz projection $\hat{\mathbb{P}}_h \colon L^2(\Omega)^2 \to \mathbb{X}_h$ as the $L^2$ projection onto $\mathbb{X}_h$, in particular $\hat{\mathbb{P}}_h(v)$ then has zero trace. In addition, we consider a solution $\varphi \in H_0^1(\Omega)^2$ to the divergence problem

$$\nabla \cdot \varphi = \lambda_h \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega.$$

By [7, Lemma 3.3] a solution $\varphi$ exists and fulfills

$$\|\varphi\|_{H^1(\Omega)} \leq \|\lambda_h\|_{L^2(\Omega)}. \tag{6.2.9}$$

Then we can rewrite $\|\lambda_h\|^2_{L^2(\Omega)}$ as

$$\begin{aligned}(\lambda_h, \lambda_h) &= (\lambda_h, \nabla \cdot \varphi) = (\lambda_h, \nabla \cdot P_h(\varphi)) \\ &= z(\boldsymbol{g}_h, P_h(\varphi)) + (\nabla \boldsymbol{g}_h, \nabla P_h(\varphi)) + (\delta_h, P_h(\varphi)),\end{aligned}$$

where we assume that the divergence property Assumption 3.5 holds for $P_h$ and use (6.2.5). Applying the Hölder inequality, it is possible to bound the last two terms, using (6.2.9) and $\|\delta_h\|_{H^{-1}(\Omega)} \leq C|\ln h|$. Since $\boldsymbol{g}_h \in \mathbb{X}_h$ it follows for the first term

$$z(\boldsymbol{g}_h, P_h(\varphi)) = z(\boldsymbol{g}_h, \hat{\mathbb{P}}_h P_h(\varphi)) = -(\nabla \boldsymbol{g}_h, \nabla \hat{\mathbb{P}}_h P_h(\varphi)) + (\delta_h, \hat{\mathbb{P}}_h P_h(\varphi))$$

where we tested (6.2.5) with $\mathbb{P}_h P_h(\varphi)$. Thus, after applying the Cauchy-Schwarz inequality we would have to estimate $\|\hat{\mathbb{P}}_h P_h(\varphi)\|_{H^1(\Omega)}$.

In [71, Lemma 3.1] Guermond and Pasciak show $H^s(\Omega)^2$ stability for $\hat{\mathbb{P}}_h$ for $s \in [0, 1/2)$. But as they point out in [71, Remark 3.1], this does not hold for $s > 1/2$ because of the incompatibility of the continuous and discrete Helmholtz projection $\hat{\mathbb{P}}$ and $\hat{\mathbb{P}}_h$ at the boundary.

If one considers the approximation error $\|\hat{\mathbb{P}}\boldsymbol{v} - \hat{\mathbb{P}}_h v\|_{L^2(\Omega)}$ this amounts to a convergence rate of $h^s$ with $0 \leq s < 1/2$. This factor correlates with the findings for $\|\lambda\|_{L^2(\Omega)}$ in Proposition 6.1. Assuming one can make the connection $|z| \sim Ch^{-2}$ we also have a factor of $h^{-1/2}$ for the case $\bar{\alpha} \searrow 1/4$ in Proposition 6.1.

To summarize this discussion, we note that the approach used for the Poisson resolvent problem in [85] opens some doors for the Stokes resolvent problem. But the pressure term cannot easily be eliminated or estimated as in [64, 65], since that would require a $H^{-1}$ estimate of $\boldsymbol{g}_h$, or dealt with directly as in [72] since we have no estimates for the respective Green's function of the continuous problem (estimates for the fundamental solution can be found in [115]).

An alternative approach, trying to transfer the results of [2, 56] to the discrete case via approximation error estimates for the Stokes resolvent problem also seems to be hindered by the pressure error term. Note that this approach also would place additional restrictions on the domain, since [2, 56] assume smooth domains.

## 6.3. (Semi-)Discrete maximal regularity estimates

Having now discussed the impediments still hindering our pursuit of discrete maximal regularity estimates, we use this section to show some results based on an $L^2$ resolvent estimate for the discrete problem on convex polyhedral domains. To that end, we argue that the approach in [87] can be extended directly to the Stokes operator if a respective resolvent estimate is available.

### 6.3.1. Function spaces and Stokes operator

The arguments in [87] and by extension [50] are based on an operator calculus for $-\Delta$ and $-\Delta_h$, so to extend these results to the Stokes problem we next formally introduce the continuous and discrete Stokes operator. But before we come to that, we define the following function spaces

$$\begin{aligned}\mathcal{X} &= \{\boldsymbol{v} \in C_0^\infty(\Omega)^d \colon \nabla\cdot\boldsymbol{v} = 0\},\\ \boldsymbol{X}_{0,p} &= \overline{\mathcal{X}}^{L^p},\\ \boldsymbol{X}_{1,p} &= \overline{\mathcal{X}}^{W^{1,p}},\\ \boldsymbol{X}_{2,p} &= \overline{\mathcal{X}}^{W^{1,p}} \cap W^{2,p}(\Omega)^d.\end{aligned}$$

Here we orient ourselves at the notation introduced in [71]. Note, that since we assume $\Omega$ to be bounded and convex (or respectively star-shaped), it holds $\boldsymbol{X}_{1,p} = \{\boldsymbol{v} \in W_0^{1,p} \mid \nabla\cdot\boldsymbol{v} = 0\}$ by [58, Theorem III.4.1]. For $1 < p < \infty$ the Helmholtz decomposition is given as

$$L^p(\Omega)^d = \boldsymbol{X}_{0,p} \oplus \nabla(W^{1,p}(\Omega) \cap L_0^p(\Omega)). \tag{6.3.1}$$

The existence of such a decomposition has been shown on $C^1$ domains in [116] and has been extended to general convex domains in [62]. By $\mathbb{P}_p \colon L^p(\Omega)^d \to \boldsymbol{X}_{0,p}$ we denote the Helmholtz projection. Then, we define the Stokes operator $A_p \colon \boldsymbol{X}_{2,p} \to \boldsymbol{X}_{0,p}$ as

$$A_p = -\mathbb{P}_p\Delta|_{\boldsymbol{X}_{2,p}}.$$

Here $-\Delta : (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^d \to L^p(\Omega)^d$ is the vector-valued Laplace operator.

Finally, we use the set of discretely divergence-free vectors $\boldsymbol{X}_h$, which we define here as

$$\boldsymbol{X}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h \;:\; (\nabla\cdot\boldsymbol{v}_h, q_h) = 0 \quad \forall q_h \in M_h\}.$$

We define the discrete Helmholtz projection $\mathbb{P}_h$ as the $L^2$ projection into $\boldsymbol{X}_h$, i.e., $(\mathbb{P}_h\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{u}, \boldsymbol{v}_h)\ \forall \boldsymbol{v}_h \in \boldsymbol{X}_h$. Then, we may write the discrete Stokes operator $A_h : \boldsymbol{X}_h \to \boldsymbol{X}_h$ as $A_h = -\mathbb{P}_h\Delta_h$, i.e., as an operator on $\boldsymbol{X}_h$ such that

$$(A_h\boldsymbol{u}_h, \boldsymbol{v}_h) = (\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) \qquad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

The discrete Laplace operator $\Delta_h \colon \boldsymbol{V}_h \to \boldsymbol{V}_h$ used above is defined as

$$(\Delta_h \boldsymbol{z}_h, \boldsymbol{v}_h) = -(\nabla \boldsymbol{z}_h, \nabla \boldsymbol{v}_h) \qquad \forall \boldsymbol{z}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Furthermore, since $\boldsymbol{X}_h \subset \boldsymbol{V}_h$ we obtain for $\boldsymbol{v}_h \in \boldsymbol{X}_h$

$$(A_h \boldsymbol{v}_h, \boldsymbol{v}_h) = (\nabla \boldsymbol{v}_h, \nabla \boldsymbol{v}_h) = (-\Delta_h \boldsymbol{v}_h, \boldsymbol{v}_h) \geq d_0 \|\boldsymbol{v}_h\|^2_{L^2(\Omega)},$$

where $d_0$ is the smallest eigenvalue of $-\Delta$, since the smallest discrete eigenvalue of $-\Delta_h$ can be bounded from below by $d_0$. This implies that the eigenvalues of $A_h$ are also positive and bounded from below by $d_0$.

### 6.3.2. The discontinuous Galerkin method

In this section we introduce some basic facts regarding the discontinuous Galerkin method, based on the definitions in [35, 87]. We use discontinuous finite elements in time and, if we discretize in space, we use continuous finite elements in space (e.g., Taylor-Hood). The resulting scheme we denote by $dG(w)$ or respectively $dG(w)cG(k)$, where $w$ denotes the order of the polynomials used in the finite element method in time and $k$ is the order of the polynomials in the finite element method for the velocity in space.

To approximate the solution of (6.1.1a)–(6.1.1d) in time, we partition $I = [0, \mathscr{T}]$ into subintervals $I_m = (t_{m-1}, t_m]$ of length $\tau_m = t_m - t_{m-1}$, where $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = \mathscr{T}$. The maximal and minimal time steps are denoted by $\tau = \max_m \tau_m$ and $\tau_{\min} = \min_m \tau_m$. The time mesh fulfills the following assumptions:

1. There are constants $C, \beta > 0$ independent on $\tau$ such that
$$\tau_{\min} \geq C\tau^{\beta}.$$
2. There is a constant $\kappa > 0$ independent of $\tau$ such that for all $m = 1, 2, \ldots, M-1$
$$\kappa^{-1} \leq \frac{\tau_m}{\tau_{m+1}} \leq \kappa.$$
3. It holds $\tau \leq \frac{\mathscr{T}}{4}$.

Then, we define the semi-discrete space $\boldsymbol{V}_\tau^w(\mathcal{B})$ of piecewise polynomial functions in time as

$$\boldsymbol{V}_\tau^w(\mathcal{B}) = \{\boldsymbol{v}_\tau \in L^2(I; \mathcal{B}) \colon \boldsymbol{v}_\tau|_{I_m} \in \mathcal{P}_{w,I_m}(\mathcal{B}), m = 1, 2, \ldots, M\},$$

with $\mathcal{P}_{w,I_m}(\mathcal{B})$ the space of polynomial functions of degree $w$ in time with values in a Banach space $\mathcal{B}$, i.e.,

$$\mathcal{P}_{w,I_m}(\mathcal{B}) = \left\{ \boldsymbol{v}_\tau \colon I_m \to \mathcal{B} \colon \boldsymbol{v}_\tau(t) = \sum_{j=0}^{w} \boldsymbol{v}^j \phi_j(t),\ \boldsymbol{v}^j \in \mathcal{B}, j = 0, \ldots, w \right\}. \tag{6.3.2}$$

Here $\phi_j(t)$ are the polynomial basis functions in $t$ with degree less or equal $w$ over $I_m$. We use the following notation for functions in $\boldsymbol{V}_\tau^w(H_0^1(\Omega)^d)$

$$\boldsymbol{u}_m^+ = \lim_{\varepsilon\to 0^+} \boldsymbol{u}(t_m+\varepsilon), \quad \boldsymbol{u}_m^- = \lim_{\varepsilon\to 0^+} \boldsymbol{u}(t_m-\varepsilon), \quad [\boldsymbol{u}]_m = \boldsymbol{u}_m^+ - \boldsymbol{u}_m^-.$$

Then, we consider the following bilinear form

$$\begin{aligned} B((\boldsymbol{u},p),(\boldsymbol{v},q)) = \sum_{m=1}^{M} \langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle_{I_m\times\Omega} + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{I\times\Omega} - (p, \nabla\cdot \boldsymbol{v})_{I\times\Omega} + (\nabla\cdot \boldsymbol{u}, q)_{I\times\Omega} \\ + \sum_{m=2}^{M} ([\boldsymbol{u}]_{m-1}, \boldsymbol{v}_{m-1}^+)_\Omega + (\boldsymbol{u}_0^+, \boldsymbol{v}_0^+)_\Omega. \quad (6.3.3) \end{aligned}$$

Here $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{I_m\times\Omega}$ are the commonly used $L^2$ space and space-time inner-products. $\langle\cdot,\cdot\rangle_{I_m\times\Omega}$ is the duality product between $L^2(I_m;H^{-1}(\Omega)^d)$ and $L^2(I_m;H_0^1(\Omega)^d)$. We use this form to state the time-discrete formulation of (6.1.1a)–(6.1.1d) by testing with $(\boldsymbol{v}_\tau, q_\tau) \in \boldsymbol{V}_\tau^w(H_0^1(\Omega)^d \times L_0^2(\Omega))$

$$B((\boldsymbol{u}_\tau,p_\tau),(\boldsymbol{v}_\tau,q_\tau)) = (\boldsymbol{f}, \boldsymbol{v}_\tau)_{I\times\Omega} + (\boldsymbol{u}_0, \boldsymbol{v}_{\tau,0}^+)_\Omega \qquad \forall (\boldsymbol{v}_\tau, q_\tau) \in \boldsymbol{V}_\tau^w(H_0^1(\Omega)^d \times L_0^2(\Omega)). \quad (6.3.4)$$

Similarly, we can state the fully discrete problem

$$B((\boldsymbol{u}_{\tau h},p_{\tau h}),(\boldsymbol{v}_{\tau h},q_{\tau h})) = (\boldsymbol{f}, \boldsymbol{v}_{\tau h})_{I\times\Omega} + (\boldsymbol{u}_0, \boldsymbol{v}_{\tau h,0}^+)_\Omega \qquad \forall (\boldsymbol{v}_{\tau h}, q_{\tau h}) \in \boldsymbol{V}_\tau^w(\boldsymbol{V}_h \times M_h). \quad (6.3.5)$$

In the following we also consider a dual problem, where we use a dual formulation of $B$

$$\begin{aligned} B((\boldsymbol{u},p),(\boldsymbol{v},q)) = -\sum_{m=1}^{M} \langle \boldsymbol{u}, \partial_t \boldsymbol{v}\rangle_{I_m\times\Omega} + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{I\times\Omega} - (p, \nabla\cdot \boldsymbol{v})_{I\times\Omega} + (\nabla\cdot \boldsymbol{u}, q)_{I\times\Omega} \\ - \sum_{m=1}^{M-1} (\boldsymbol{u}_m^-, [\boldsymbol{v}]_m)_\Omega + (\boldsymbol{u}_M^-, \boldsymbol{v}_M^-)_\Omega, \end{aligned}$$

which is obtained by integration by parts and rearranging the terms in the sum in (6.3.3). If we restrict $\boldsymbol{u}_\tau$ to $\boldsymbol{V}_\tau^w(\boldsymbol{X}_{1,2})$ and test $B$ with $\boldsymbol{v}_\tau \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_{1,2})$ (or respectively $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_h)$), we get the weak representation of the problem (6.1.1a)–(6.1.1d) in operator form. The pressure space terms in (6.3.3) vanish. We denote this form by

$$\mathfrak{B}(\boldsymbol{u},\boldsymbol{v}) = \sum_{m=1}^{M} \langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle_{I_m\times\Omega} + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{I\times\Omega} + \sum_{m=2}^{M} ([\boldsymbol{u}]_{m-1}, \boldsymbol{v}_{m-1}^+)_\Omega + (\boldsymbol{u}_0^+, \boldsymbol{v}_0^+)_\Omega.$$

The respective time-discrete weak problem is then given by

$$\mathfrak{B}(\boldsymbol{u}_\tau, \boldsymbol{v}_\tau) = (\mathbb{P}_2 \boldsymbol{f}, \boldsymbol{v}_\tau)_{I\times\Omega} + (\boldsymbol{u}_0, \boldsymbol{v}_{\tau,0}^+)_\Omega \qquad \forall \boldsymbol{v}_\tau \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_{1,2}). \quad (6.3.6)$$

Again we can state the fully discrete problem as

$$\mathfrak{B}(\boldsymbol{u}_{\tau h}, \boldsymbol{v}_{\tau h}) = (\mathbb{P}_h \boldsymbol{f}, \boldsymbol{v}_{\tau h})_{I\times\Omega} + (\boldsymbol{u}_0, \boldsymbol{v}_{\tau h,0}^+)_\Omega \qquad \forall \boldsymbol{v}_{\tau h} \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_h), \quad (6.3.7)$$

where we assume $\boldsymbol{u}_{\tau h} \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_h)$.

We note that if $\boldsymbol{u}_\tau$ (or $\boldsymbol{u}_{\tau h}$) solves (6.3.4) (or (6.3.5)) then it is also a solution to (6.3.6) (or (6.3.7)).

For the right-hand side we see because of the decomposition (6.3.1), which can be applied to $\boldsymbol{f} \in L^2(\Omega)^d$, that

$$(\mathbb{P}_2 \boldsymbol{f}, \boldsymbol{v}_\tau)_{I\times\Omega} = (\boldsymbol{f}, \boldsymbol{v}_\tau)_{I\times\Omega}$$

when testing (6.3.4) with $\boldsymbol{v}_\tau \in \boldsymbol{X}_{1,2}$. Clearly the term involving the pressure in $B$ vanishes for $\boldsymbol{v}_\tau \in \boldsymbol{X}_{1,2}$. A similar result holds for the discrete problem based on the definition of $\mathbb{P}_h$ as the respective $L^2$ projection into $\boldsymbol{X}_h$ and $\boldsymbol{v}_{\tau h}$ discretely divergence-free.

Furthermore, it is important to confirm that the solution $\boldsymbol{u}_\tau$ to (6.3.4) is divergence-free at every $t \in I$ such it fulfills the requirement $\boldsymbol{u}_\tau \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_{1,2})$ which is necessary for a solution to (6.3.6). Due to the construction of $\mathcal{P}_{w,I_m}(\mathcal{B})$ in (6.3.2) and due to the linearity of the divergence it is sufficient to check this condition for each $\boldsymbol{u}^j$ on an interval $I_m$. On $I_m$ one has $\boldsymbol{u}_\tau = \sum_{j=0}^{w} \boldsymbol{u}^j \phi_j(t)$ with $\boldsymbol{u}^j \in H_0^1(\Omega)^d$. We now focus only on the interval $I_m$ and test (6.3.4) with $(\boldsymbol{v}\phi_i, q\phi_i) \in \boldsymbol{V}_\tau^w(H_0^1(\Omega)^d \times L_0^2(\Omega))$ for $0 \leq i \leq w$ and $(\boldsymbol{v}, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$. Then the fourth term in (6.3.3) implies

$$((\phi_j, \phi_i))_{ij}((\nabla \cdot \boldsymbol{u}^0, q) \quad \cdots \quad (\nabla \cdot \boldsymbol{u}^w, q))^T = \boldsymbol{0}$$

for all $q \in L_0^2(\Omega)$. The matrix $((\phi_j, \phi_i))_{ij}$ is symmetric positive definite (which can be seen directly, e.g., for a basis of Legendre polynomials), which implies $(\nabla \cdot \boldsymbol{u}^j, q) = 0$ for all $j$. Since $q$ has been chosen arbitrarily, we can conclude that $\boldsymbol{u}_\tau$ is divergence-free. Similarly, we may argue for the fully discrete case to prove that the solution $\boldsymbol{u}_{\tau h}$ to (6.3.5) is discretely divergence-free for each $t \in I$. A similar argument in made in [35, Lemma 2.3].

### 6.3.3. Maximal regularity estimates for the Stokes operator

Careful review of the proofs in [87] shows that the arguments continue to hold for a Stokes-type operator $\tilde{A} \in \{A_p, A_h\}$, which satisfies a resolvent estimate of type (6.2.2) or (6.2.3). The operator can be restricted to a subspace of $L^p(\Omega)^d$, in our case $\boldsymbol{X}_{0,p}$ or $\boldsymbol{X}_h$, in [87] that is done for $V_h$. In particular, regarding the use of the resolvent estimate the authors invoke only [50, Theorem 5.1] and the proof thereof. For the arguments in [50, 87] to hold, resolvent estimates like (6.2.2) and (6.2.3) are sufficient. We will show such estimates in the next sections.

We first state the results in operator form. The semi-discrete instationary problem in operator form is given as follows for $\boldsymbol{f} \in L^s(I; L^p(\Omega)^d)$ and $\boldsymbol{u}_0 \in \boldsymbol{X}_{0,p}$

$$\partial_t \tilde{\boldsymbol{u}} + \tilde{A}\tilde{\boldsymbol{u}} = \tilde{\mathbb{P}}\boldsymbol{f} \qquad \forall t \in I, \tag{6.3.8a}$$

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}, 0) = \tilde{\mathbb{P}}\boldsymbol{u}_0. \tag{6.3.8b}$$

In the following $\tilde{\boldsymbol{u}}$ is either an element of $\boldsymbol{X}_{2,p}$ or $\boldsymbol{X}_h$ and $\tilde{\mathbb{P}}$ the respective Helmholtz projection operator. The time discrete solution $\tilde{\boldsymbol{u}}_\tau \in \boldsymbol{V}_\tau^w(\tilde{\boldsymbol{X}})$ ($\tilde{\boldsymbol{X}} \in \{\boldsymbol{X}_{2,p}, \boldsymbol{X}_h\}$) of this reduced problem is then given by

$$\mathfrak{B}(\tilde{\boldsymbol{u}}_\tau, \tilde{\boldsymbol{v}}_\tau) = (\tilde{\mathbb{P}}\boldsymbol{f}, \tilde{\boldsymbol{v}}_\tau)_{I\times\Omega} + (\tilde{\mathbb{P}}\boldsymbol{u}_0, \tilde{\boldsymbol{v}}_{\tau,0}^+)_\Omega \qquad \forall \tilde{\boldsymbol{v}}_\tau \in \boldsymbol{V}_\tau^w(\tilde{\boldsymbol{X}}), \tilde{\boldsymbol{X}} \in \{\boldsymbol{X}_{2,p}, \boldsymbol{X}_h\}. \tag{6.3.9}$$

Then, the following theorems hold as in [87, Theorems 3-8, Theorems 10-11] if we have respective resolvent estimates as in (6.2.2) and (6.2.3).

**Theorem 6.3** (Homogeneous smoothing estimate) *Let $\tilde{\boldsymbol{u}}_\tau$ be the solution of* (6.3.9) *with* $\boldsymbol{f} = \boldsymbol{0}$. *Then, there holds for p in the range for which* (6.2.2) *or* (6.2.3) *are satisfied and* $m = 1, \ldots, M$

$$\|\partial_t \tilde{\boldsymbol{u}}_\tau\|_{L^\infty(I_m;L^p(\Omega))} + \|\tilde{A}\tilde{\boldsymbol{u}}_\tau\|_{L^\infty(I_m;L^p(\Omega))} + \|\tau_m^{-1}[\tilde{\boldsymbol{u}}_\tau]_{m-1}\|_{L^p(\Omega)} \leq \frac{C}{t_m}\|\tilde{\mathbb{P}}\boldsymbol{u}_0\|_{L^p(\Omega)}.$$

*Here we have* $[\tilde{\boldsymbol{u}}_\tau]_0 = \tilde{\boldsymbol{u}}^+_{\tau h,0} - \tilde{\mathbb{P}}\boldsymbol{u}_0$.

*Remark* 6.4 The crucial step in proving (semi-)discrete maximal regularity, where one makes use of the resolvent estimate for the Stokes problem, happens when bounding a representation of the (semi-)discrete solution on $I_m$ in form of the Dunford-Taylor integral, i.e., for example in [50, (5.8)] (or in other forms on [50, p. 1321 and p. 1322])

$$\tilde{A}\tilde{\boldsymbol{u}}^-_{\tau,m} = \frac{1}{2\pi i}\int_\Gamma \prod_{l=1}^m r(\tau_l z)\tilde{A}R(z,\tilde{A})dz\hat{\boldsymbol{u}}_0 \qquad \text{for } \; m = 2, \ldots, M,$$

where $r(z)$ is a rational function, $\Gamma$ a suitable curve enclosing the resolvent set of $\tilde{A}$, and $R(z,\tilde{A})$ the solution operator for the Stokes resolvent problem in operator form, i.e., $R(z,\tilde{A}) = (z-\tilde{A})^{-1}$. In particular, this means that $\hat{\boldsymbol{u}}_0$ needs to be in the appropriate space $\boldsymbol{X}_{0,p}$ or $\boldsymbol{X}_h$ such that the application of $R(z,\tilde{A})$ to $\hat{\boldsymbol{u}}_0$ makes sense. For the homogeneous smoothing estimate $\tilde{\mathbb{P}}\boldsymbol{u}_0$ (cf. (6.3.8b)) takes the role of $\hat{\boldsymbol{u}}_0$ (cf. proof of [87, Theorem 3]), while for the estimate of the inhomogeneous problem below one thinks of the right-hand side, i.e., $\tilde{\mathbb{P}}\boldsymbol{f}$ (cf. (6.3.8a)), at a time $t_m$ as the initial condition $\hat{\boldsymbol{u}}_0$ in the respective integral. This also explains the presence of $\tilde{\mathbb{P}}$ on the right-hand side in Theorems 6.3 and 6.6.

Furthermore, as we have argued before for the (discrete) instationary Stokes problem, we also have that any solution to the full resolvent problem (6.2.1a)–(6.2.1c) (or a discrete variant) with right-hand side $\boldsymbol{f}$ also provides a solution to the expression in operator form $R(z,\tilde{A})\tilde{\mathbb{P}}\boldsymbol{f}$ and thus one can apply resolvent estimates as in (6.2.2) and (6.2.3) also to $R(z,\tilde{A})\tilde{\mathbb{P}}\boldsymbol{f}$. This in turn allows one to successfully apply the arguments in [50, 87] to our problem.

*Remark* 6.5 Note that as an $L^2$ projection, $\mathbb{P}_h$ is stable in $L^2(\Omega)$. In the continuous case a respective bound for $\mathbb{P}_p$ is given in [62, Theorem 1.3].

Furthermore, we have the following result for the inhomogeneous problem.

**Theorem 6.6** (Inhomogeneous problem) *Let $\tilde{\boldsymbol{u}}_\tau$ be the solution of* (6.3.9) *with* $\boldsymbol{u}_0 = 0$. *Then, the following holds for* $1 \leq s < \infty$ *and p in the range for which* (6.2.2) *or* (6.2.3) *are satisfied:*

$$\left(\sum_{m=1}^M \|\partial_t \tilde{\boldsymbol{u}}_\tau\|^s_{L^s(I_m;L^p(\Omega))}\right)^{1/s} + \|\tilde{A}\tilde{\boldsymbol{u}}_\tau\|_{L^s(I;L^p(\Omega))} + \left(\sum_{m=1}^M \tau_m\|\tau_m^{-1}[\tilde{\boldsymbol{u}}_\tau]_{m-1}\|^s_{L^p(\Omega)}\right)^{1/s}$$
$$\leq C\ln\frac{\mathscr{T}}{\tau}\|\tilde{\mathbb{P}}\boldsymbol{f}\|_{L^s(I;L^p(\Omega))}.$$

*For $s = \infty$ the estimate continuous to hold in the fully discrete case and holds for the second and third term in the semi-discrete case (for the excluded term cf. [87, Theorem 8]). Here we have* $[\tilde{\boldsymbol{u}}_\tau]_0 = \tilde{\boldsymbol{u}}^+_{\tau h,0}$ *and obvious notation changes for* $s = \infty$.

This shows the maximal regularity result for the discrete velocity $\boldsymbol{u}_\tau$ of the Stokes problem in operator form. From this, one can infer the estimate for the full problem, as given in (6.3.4) and (6.3.5), by using the properties of $\tilde{\mathbb{P}}$ and the fact that if $\tilde{\boldsymbol{u}}_\tau$ solves (6.3.4) and (6.3.5) it also satisfies (6.3.9).

We discuss this in the next sections for the discrete and semi-discrete problem alongside respective resolvent estimates.

Finally, we remark that the maximal regularity estimates in form of Theorems 6.3 and 6.6 can be extended to weighted spaces, using the same arguments as above, if a respective resolvent estimate holds in the weighted space.

### 6.3.4. Discrete maximal regularity in $L^\infty/L^2$ for the velocity and approximation error estimate

The objective of this section is to show an approximation error estimate for the fully discrete problem in $L^\infty(I; L^2(\Omega)^d)$. A similar approximation result has already been shown in [35, Section 4], we discuss it in more detail later after giving an alternative approach here based on maximal regularity estimates. In the following $\Omega$ is a convex polyhedron.

First we show a resolvent estimate in $L^2(\Omega)^d$ for $\boldsymbol{u}_h$ solving the velocity-pressure formulation of the Stokes problem. The estimate also holds for the solution of the problem in operator form, i.e., $A_h^{-1}\mathbb{P}_h\boldsymbol{f}$. This follows from the fact that $A_h$ is positive definite, thus $A_h^{-1}$ is well-defined, and the fact that there holds $A_h\boldsymbol{u}_h = \mathbb{P}_h\boldsymbol{f}$ for $\boldsymbol{u}_h$ being the solution to the velocity-pressure formulation of the discrete Stokes resolvent problem (cf. the discussion at the end of Section 6.3.2), i.e., $\boldsymbol{u}_h$ is also the unique solution to the problem in operator form.

**Lemma 6.7** *For any $\theta \in (\pi/2, \pi)$ there exists a constant $C = C_\theta$ independent of $z$ such that for any $\nu \in [0, a_0]$ with $a_0 > 0$ being the smallest eigenvalue of $-\Delta_h$ it holds*

$$\|\boldsymbol{u}_h\|_{L^2(\Omega)} \leq \frac{C_\theta}{|z+\nu|}\|\boldsymbol{f}\|_{L^2(\Omega)} \quad \forall z \in \Sigma_{\theta,-\nu},$$

*where $\boldsymbol{u}_h$ is the velocity part of the discrete solution to* (6.2.1a)–(6.2.1c) *with right-hand side $\boldsymbol{f} \in L^2(\Omega)^d$.*

*Remark* 6.8 This corresponds to the condition (6.2.3) with $\bar{\omega} \leq 0$ and thus the assumptions in [50].

*Proof.* Testing the discrete problem with $\boldsymbol{u}_h$, we may expand it to

$$(z+\nu)\|\boldsymbol{u}_h\|^2_{L^2(\Omega)} + ((-\Delta_h - \nu)\boldsymbol{u}_h, \boldsymbol{u}_h) = (\boldsymbol{f}, \boldsymbol{u}_h). \tag{6.3.10}$$

Now, since $-\Delta_h$ is positive definite, we have that $-\Delta_h - \nu$ is still a non-negative operator, and thus $((-\Delta_h - \nu)\boldsymbol{u}_h, \boldsymbol{u}_h) \geq 0$. Since $z$ is restricted to the sector $\Sigma_{\theta,\nu}$, we can rewrite (6.3.10) as

$$|z+\nu|\exp(i\phi)\|\boldsymbol{u}_h\|^2_{L^2(\Omega)} + \delta = (\boldsymbol{f}, \boldsymbol{u}_h),$$

where $\delta \geq 0$ and $|\phi| < \theta$. If we now multiply with $\exp(-i\phi/2)$, take the real part, and use that $\cos(\theta/2) > 0$, we get

$$|z+\nu| \|\boldsymbol{u}_h\|^2_{L^2(\Omega)} \leq \cos(\theta/2)^{-1} |(\boldsymbol{f}, \boldsymbol{u}_h)| = C_\theta |(\boldsymbol{f}, \boldsymbol{u}_h)|$$

which after an application of Cauchy's inequality completes the proof. □

Hence, we conclude that the results from Section 6.3.3 apply to $A_h$ on $L^2(\Omega)^d$. Thus, there hold the following corollaries for the homogeneous and inhomogeneous problem.

**Corollary 6.9** *Let $\boldsymbol{u}_{\tau h}$ be the velocity solution to* (6.3.5) *or solution to* (6.3.7) *with $\boldsymbol{f} = \boldsymbol{0}$. Then, there holds for $m = 1, \ldots, M$*

$$\|\partial_t \boldsymbol{u}_{\tau h}\|_{L^\infty(I_m;L^2(\Omega))} + \|A_h \boldsymbol{u}_{\tau h}\|_{L^\infty(I_m;L^2(\Omega))} + \|\Delta_h \boldsymbol{u}_{\tau h}\|_{L^\infty(I_m;L^2(\Omega))} + \|\tau_m^{-1}[\boldsymbol{u}_{\tau h}]_{m-1}\|_{L^2(\Omega)} \leq \frac{C}{t_m} \|\mathbb{P}_h \boldsymbol{u}_0\|_{L^2(\Omega)}.$$

*Here we have $[\boldsymbol{u}_{\tau h}]_0 = \boldsymbol{u}^+_{\tau h,0} - \mathbb{P}_h \boldsymbol{u}_0$.*

The bound for $\Delta_h \boldsymbol{u}_{\tau h}$ can be inferred from [71, Lemma 4.1]. As above, we also have the following corollary for the inhomogeneous problem.

**Corollary 6.10** *For $1 \leq s \leq \infty$ and $\boldsymbol{u}_{\tau h}$ the velocity solution to* (6.3.5) *or solution to* (6.3.7) *with $\boldsymbol{u}_0 = \boldsymbol{0}$, there holds*

$$\left( \sum_{m=1}^M \|\partial_t \boldsymbol{u}_{\tau h}\|^s_{L^s(I_m;L^2(\Omega))} \right)^{1/s} + \|A_h \boldsymbol{u}_{\tau h}\|_{L^s(I;L^2(\Omega))} + \|\Delta_h \boldsymbol{u}_{\tau h}\|_{L^s(I;L^2(\Omega))} + \left( \sum_{m=1}^M \tau_m \|\tau_m^{-1}[\boldsymbol{u}_{\tau h}]_{m-1}\|^s_{L^2(\Omega)} \right)^{1/s} \leq C \ln \frac{\mathscr{T}}{\tau} \|\mathbb{P}_h \boldsymbol{f}\|_{L^s(I;L^2(\Omega))},$$

*with obvious notation changes for $s = \infty$. Here we have $[\boldsymbol{u}_{\tau h}]_0 = \boldsymbol{u}^+_{\tau h,0}$.*

We note again that from the arguments at the end of Section 6.3.2 these estimates also hold for $\boldsymbol{u}_{\tau h}$ being the velocity part of the solution to (6.3.5).

These results now allow us to show an $L^\infty(\Omega)/L^2(\Omega)$ approximation error estimate for the velocity, after we introduce the analog of the Ritz projection for the Stokes problem given as $(R_h^S \boldsymbol{w}, R_h^{S,p}\varphi) \in \boldsymbol{V}_h \times M_h$ of $(\boldsymbol{w}, \varphi) \in H_0^1(\Omega)^d \times L^2(\Omega)$ through the orthogonality relation

$$(\nabla(\boldsymbol{w} - R_h^S \boldsymbol{w}), \nabla \boldsymbol{v}_h) - (\varphi - R_h^{S,p}\varphi, \nabla \cdot \boldsymbol{v}_h) = \boldsymbol{0} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{6.3.11}$$

$$(\nabla \cdot R_h^S \boldsymbol{w}, q_h) = 0 \qquad \forall q_h \in M_h. \tag{6.3.12}$$

In the following we also assume that $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are such that $\boldsymbol{u} \in C(\bar{I}; L^2(\Omega)^d)$.

**Theorem 6.11** *Let $\boldsymbol{u}$ and $\boldsymbol{u}_{\tau h}$ solve* (6.1.1a)–(6.1.1d) *and the respective finite element problem* (6.3.5)*. Then, there holds*

$$\|\boldsymbol{u}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \le C \ln \frac{\mathscr{T}}{\tau}\Big(\|\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} + \|\boldsymbol{u} - R_h^S \boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))}\Big)$$

We proceed with a proof along the arguments of [86, Theorem 1].

*Proof.* Let $\tilde{t} \in (0, \mathscr{T}]$ and without loss of generality assume $\tilde{t} \in (t_{M-1}, \mathscr{T})$, since the interval boundary is a set of measure zero.

We consider the inhomogeneous dual problem

$$\begin{aligned}
-\partial_t \boldsymbol{g}(t,\boldsymbol{x}) - \Delta \boldsymbol{g}(t,\boldsymbol{x}) + \nabla\lambda(t,\boldsymbol{x}) &= \boldsymbol{u}_{\tau h}(\tilde{t},\boldsymbol{x})\theta(t), && (t,\boldsymbol{x}) \in I \times \Omega,\\
\nabla \cdot \boldsymbol{g}(t,\boldsymbol{x}) &= 0, && (t,\boldsymbol{x}) \in I \times \Omega,\\
\boldsymbol{g}(t,\boldsymbol{x}) &= \boldsymbol{0}, && (t,\boldsymbol{x}) \in I \times \partial\Omega,\\
\boldsymbol{g}(\mathscr{T},x) &= \boldsymbol{0}, && \boldsymbol{x} \in \Omega.
\end{aligned}$$

Here $\theta \in C^1(\bar{I})$ is a regularized Delta function (cf. [112, Appendix A.5]) in time adhering to the following properties:

$$\operatorname{supp}\theta \subset (t_{M-1}, \mathscr{T}), \qquad \|\theta\|_{L^1(I_M)} \le C, \tag{6.3.14}$$

and

$$(\theta, \boldsymbol{v}_\tau)_{I_M} = \boldsymbol{v}_\tau(\tilde{t}) \qquad \forall \boldsymbol{v}_\tau \in \mathcal{P}_w(I_M).$$

Then, we have for the finite element approximation $(\boldsymbol{g}_{\tau h}, \lambda_{\tau h})$ given by

$$B((\boldsymbol{v}_{\tau h}, q_{\tau h}), (\boldsymbol{g} - \boldsymbol{g}_{\tau h}, \lambda - \lambda_{\tau h})) = 0 \qquad \forall (\boldsymbol{v}_{\tau h}, q_{\tau h}) \in \boldsymbol{V}_\tau^w(\boldsymbol{V}_h \times M_h)$$

and again by the consistency of the finite element scheme

$$\begin{aligned}
\|\boldsymbol{u}_{\tau h}(\tilde{t})\|^2_{L^2(\Omega)} &= (\boldsymbol{u}_{\tau h}, \theta(t)\boldsymbol{u}_{\tau h}(\tilde{t})) = B((\boldsymbol{u}_{\tau h}, p_{\tau h}), (\boldsymbol{g}_{\tau h}, \lambda_{\tau h})) = B((\boldsymbol{u}, p), (\boldsymbol{g}_{\tau h}, \lambda_{\tau h}))\\
&= -\sum_{m=1}^M (\boldsymbol{u}, \partial_t \boldsymbol{g}_{\tau h})_{I_m\times\Omega} + (\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{\tau h})_{I\times\Omega} - (p, \nabla\cdot \boldsymbol{g}_{\tau h}) - \sum_{m=1}^M (\boldsymbol{u}_m^-, [\boldsymbol{g}_{\tau h}]_m)_\Omega\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

In the last sum we set $\boldsymbol{g}_{\tau h, M+1} = \boldsymbol{0}$ such that $[\boldsymbol{g}_{\tau h}]_M = -\boldsymbol{g}_{\tau h,M}$. Using Hölder's inequality, we estimate

$$\begin{aligned}
J_1 &\le \sum_{m=1}^M \|\boldsymbol{u}\|_{L^\infty(I_m;L^2(\Omega))} \|\partial_t \boldsymbol{g}_{\tau h}\|_{L^1(I_m;L^2(\Omega))} \le \|\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} \sum_{m=1}^M \|\partial_t \boldsymbol{g}_{\tau h}\|_{L^1(I_m;L^2(\Omega))}\\
J_4 &\le \sum_{m=1}^M \|\boldsymbol{u}_m^-\|_{L^2(\Omega)} \|[\boldsymbol{g}_{\tau h}]_{m-1}\|_{L^2(\Omega)} \le \|\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} \sum_{m=1}^M \|[\boldsymbol{g}_{\tau h}]_m\|_{L^2(\Omega)}.
\end{aligned}$$

For $J_2 + J_3$ we can argue by using $R_h^S$. Then, we have

$$\begin{aligned} J_2 + J_3 &= (\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{\tau h})_{I\times\Omega} - (p, \nabla\cdot \boldsymbol{g}_{\tau h})_{I\times\Omega} \\ &= (\nabla R_h^S \boldsymbol{u}, \nabla \boldsymbol{g}_{\tau h})_{I\times\Omega} - (R_h^{S,p} p, \nabla\cdot \boldsymbol{g}_{\tau h})_{I\times\Omega} \\ &= (\nabla R_h^S \boldsymbol{u}, \nabla \boldsymbol{g}_{\tau h})_{I\times\Omega}, \end{aligned}$$

where the pressure term vanishes, since $\boldsymbol{g}_{\tau h}$ is discretely divergence-free. Now we can apply the definition of the discrete Laplace operator to see

$$\begin{aligned} (\nabla R_h^S \boldsymbol{u}, \nabla \boldsymbol{g}_{\tau h})_{I\times\Omega} &= (R_h^S \boldsymbol{u}, -\Delta_h \boldsymbol{g}_{\tau h})_{I\times\Omega} \\ &\le C\Big(\|\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} + \|\boldsymbol{u} - R_h^S\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))}\Big)\|\Delta_h \boldsymbol{g}_{\tau h}\|_{L^1(I;L^2(\Omega))}. \end{aligned}$$

These estimates and Corollary 6.10 allow us to conclude

$$\begin{aligned} \|\boldsymbol{u}_{\tau h}(\tilde{t})\|^2_{L^2(\Omega)} &= -\sum_{m=1}^{M} (\boldsymbol{u}, \partial_t \boldsymbol{g}_{\tau h})_{I_m\times\Omega} + (\nabla R_h^S \boldsymbol{u}, \nabla \boldsymbol{g}_{\tau h})_{I\times\Omega} - \sum_{m=1}^{M} (\boldsymbol{u}_m^-, [\boldsymbol{g}_{\tau h}]_m)_\Omega \\ &\le C\Big(\|\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} + \|\boldsymbol{u} - R_h^S\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))}\Big) \\ &\quad \left(\sum_{m=1}^{M} \|\partial_t \boldsymbol{g}_{\tau h}\|_{L^1(I_m;L^2(\Omega))} + \|\Delta_h \boldsymbol{g}_{\tau h}\|_{L^1(I;L^2(\Omega))} + \sum_{m=1}^{M} \|[\boldsymbol{g}_{\tau h}]_m\|_{L^2(\Omega)}\right) \\ &\le C \ln\frac{\mathscr{T}}{\tau}\Big(\|\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} + \|\boldsymbol{u} - R_h^S\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))}\Big)\|\boldsymbol{u}_{\tau h}(\tilde{t})\|_{L^2(\Omega)}\|\theta\|_{L^1(I_M)}. \end{aligned}$$

Together with (6.3.14) this completes the proof of the theorem.

□

*Remark* 6.12 Based on Theorem 6.11 one now derives approximation error estimates by considering $\boldsymbol{u} - \boldsymbol{v}_{\tau h}$ and $p - q_{\tau h}$ with arbitrary $(\boldsymbol{v}_{\tau h}, q_{\tau h}) \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_h \times M_h)$. Note that this leaves the term involving $R_h^S$ unchanged since the projection defined in (6.3.11) and (6.3.12) is invariant on $\boldsymbol{X}_h \times M_h$. Using that the finite element solution to the Stokes problem in $cG(w)dG(k)$ is invariant on $\boldsymbol{V}_\tau^w(\boldsymbol{X}_h \times M_h)$, i.e., $B((\boldsymbol{w}_{\tau h} - \tilde{\boldsymbol{w}}_{\tau h}, \varphi_{\tau h} - \tilde{\varphi}_{\tau h}), (\boldsymbol{v}_{\tau h}, q_{\tau h})) = 0$ for all $(\boldsymbol{v}_{\tau h}, q_{\tau h}) \in \boldsymbol{V}_\tau^w(\boldsymbol{V}_h \times M_h)$ implies $(\boldsymbol{w}_{\tau h}, \varphi_{\tau h}) = (\tilde{\boldsymbol{w}}_{\tau h}, \tilde{\varphi}_{\tau h})$, we get

$$\|\boldsymbol{u} - \boldsymbol{u}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \le C\ln\frac{\mathscr{T}}{\tau}\left(\inf_{(\boldsymbol{v}_{\tau h}, q_{\tau h})\in \boldsymbol{V}_\tau^w(\boldsymbol{X}_h\times M_h)} \|\boldsymbol{u} - \boldsymbol{v}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} + \|\boldsymbol{u} - R_h^S\boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))}\right). \tag{6.3.15}$$

*Remark* 6.13 If we would want to derive local stability results, we would require a respective weighted resolvent estimate.

*Remark* 6.14 The pressure $p$ enters the estimate on the right-hand side via the second term, it is independent of the time step. See also the estimate in [35], where the pressure term is independent of the time-step. We state the full estimate from [35] below.

To conclude this section we compare the results from Theorem 6.11 and Remark 6.12 with the results in [35, 77]. As in the literature, we have to assume enough regularity to apply interpolation results in time and space. To achieve optimal convergence with respect to the finite element discretization this means that we require $\boldsymbol{u} \in W^{w+1,\infty}(I; L^2(\Omega)^d)$ for the convergence estimate in time and $\boldsymbol{u} \in L^\infty(I; H^{k+1}(\Omega)^d)$ for the convergence estimate in space. Furthermore, throughout the following comparison we also need respective regularity for the pressure $p \in L^\infty(I; H^k(\Omega))$ which usually follows with the above proposed regularity of $\boldsymbol{u}$. We need the regularity in $p$ to give a bound for the pressure term appearing in the Ritz projection portion of (6.3.15).

We comment quickly on the viability of these assumptions. Standard results, e.g., in [120, Proposition 1.2] only deliver $\boldsymbol{u} \in L^2(I; H^2(\Omega)^d)$ and $\boldsymbol{u} \in H^1(I; L^2(\Omega)^d)$ regularity results based on respective regularity results of the stationary Stokes problem and sufficiently smooth right-hand sides and starting conditions.

The authors of [41] claim that on convex polygonal or polyhedral domains one has for a right-hand side $\boldsymbol{f}$ in $H^1(I; L^2(\Omega)^d)$ and $\boldsymbol{u}_0 \in \boldsymbol{X}_{2,2}$ the regularity $\boldsymbol{u} \in C(\bar{I}; H^2(\Omega)^d)$ and $\boldsymbol{u} \in C^1(\bar{I}; L^2(\Omega)^d)$. Their argument is based on the $H^2$ regularity results for the stationary Stokes problem as stated in [38]. Potentially one can argue with even less regularity in time for a right-hand side $\boldsymbol{f} \in L^\infty(I; L^2(\Omega))$, as done for the heat equation in [19, Proposition 5.34] at the cost of picking up additional logarithms.

We now assume the regularity as stated above for $dG(0)cG(1)$ in time, i.e., $\boldsymbol{u} \in W^{1,\infty}(I; L^2(\Omega)^d)$ $\cap\, L^\infty(I; H^2(\Omega)^d)$, and apply proper interpolation or respective projection operators which are divergence preserving. That means for the time discretization, e.g., a nodal interpolation operator $i_\tau$ on every $I_m$. In space, any inf-sup stable finite elements, which permit the construction of a discrete-divergence preserving Fortin operator $P_h$ (cf. [66]) and also fulfill respective (local) approximation error estimates, can be chosen. Then, we can conclude the following error estimate from (6.3.15) for $dG(0)cG(1)$

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} &\le C \ln \frac{\mathscr{T}}{\tau} \Bigg( \inf_{(\boldsymbol{v}_{\tau h}, q_{\tau h}) \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_h \times M_h)} \|\boldsymbol{u} - \boldsymbol{v}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \\
&\qquad + \|\boldsymbol{u} - R_h^S \boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} \Bigg) \\
&\le C \ln \frac{\mathscr{T}}{\tau} \Big( \|\boldsymbol{u} - i_\tau \boldsymbol{u}\|_{L^\infty(I;L^2(\Omega))} + \|i_\tau(\boldsymbol{u} - P_h \boldsymbol{u})\|_{L^\infty(I;L^2(\Omega))} \\
&\qquad + h^2 \Big( \|\boldsymbol{u}\|_{L^\infty(I;H^2(\Omega))} + \|p\|_{L^\infty(I;H^1(\Omega))} \Big) \Big) \\
&\le C \ln \frac{\mathscr{T}}{\tau} \Big( \tau \|\boldsymbol{u}\|_{W^{1,\infty}(I;L^2(\Omega))} + h^2 \Big( \|\boldsymbol{u}\|_{L^\infty(I;H^2(\Omega))} + \|p\|_{L^\infty(I;H^1(\Omega))} \Big) \Big),
\end{aligned} \tag{6.3.16}$$

where we used standard results for the approximation error of $R_h^S$ in $L^2(\Omega)^d$ which holds among others, e.g., for Taylor-Hood finite elements, the mini element, etc. (cf. [66]).

We here considered the $dG(0)cG(1)$ case since this has the lowest and thus most realistic requirements for regularity in time. Our convergence results are comparable to the ones in

[35, 76] except for the leading constants $C$. In the following we denote the constant from [76] by $C_{HR}$.

The authors of [75] discuss for the full Navier-Stokes problem the case of the backward Euler method in time, i.e., $dG(0)$ in time, in a remark on [76, p. 765] and [76, Proposition 3.1], deriving an estimate of the form

$$\|\boldsymbol{u} - \boldsymbol{u}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \leq C_{HR}(h + \sqrt{\tau})^2, \tag{6.3.17}$$

or alternatively an estimate by choosing the step size relation as $\tau = h^2$. Here $C_{HR}$ represents various regularity constants. We note that the authors require differentiability in time of the right-hand side $\boldsymbol{f}$, which seems to be more than what one typically requires for convergence with $dG(0)$ elements in time. Unfortunately the other assumptions in [75] are not directly comparable with our setting, so we refer for the detailed composition of $C_{HR}$ to [76, Proposition 3.1].

*Remark* 6.15 In the light of our choice of $dG(0)cG(1)$ as the approximation space, we want to highlight [75, Corollary 2.1], where the authors show that bounds for, e.g., $\nabla^3\boldsymbol{u}$, $\partial_{tt}\boldsymbol{u}$ go hand in hand with the need of the data $\boldsymbol{u}_0$, $\boldsymbol{f}$, and initial pressure $p_0$ satisfying a nonlocal compatibility condition for $t \to 0$ at the boundary, which is potentially hard to verify.

The authors of [35] operate in a similar setting as here, discussing the discontinuous Galerkin method for the Stokes problem. For the velocity solution to the Stokes problem $\boldsymbol{u} \in C(\bar{I}; H^2(\Omega)^d) \cap H^1(I; H_0^1(\Omega)^d)$ they derive the following estimate on [35, p. 2152]

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \\ &\quad \leq C\Big(h\Big(\|\boldsymbol{u}\|_{L^2(I;H^2(\Omega))} + h\|\boldsymbol{u}\|_{L^\infty(I;H^2(\Omega))}\Big) + \tau\Big(\|\boldsymbol{u}\|_{H^1(I;H^1(\Omega))} + \|\boldsymbol{u}\|_{W^{1,\infty}(I;L^2(\Omega))}\Big) \\ &\qquad + h\|\boldsymbol{u}_0\|_{H^1(\Omega)} + \|\boldsymbol{u}\|_{C(\bar{I};H^2(\Omega))} \min\left(h^{3/2}/\tau, \sqrt{h/\tau}\right) h^{3/2} + h\|p\|_{L^2(I;H^1(\Omega))}\Big). \end{aligned} \tag{6.3.18}$$

Comparing now our result (6.3.16) with (6.3.17) and (6.3.18) we want to emphasize the following differences. First, apart from the logarithmic term (6.3.16) does not contain any mixed terms of $\tau$ and $h$, which means that time and space discretization are not intertwined in terms of convergence. Furthermore, the (almost) best-approximation form of (6.3.15) allows one to tailor one's approximation spaces with respect to the regularity. Thus, the estimate (6.3.16) hopefully simplifies the application of such error estimates.

We note that the arguments in (6.3.17) and (6.3.18) do not specifically target the Stokes problem but rather provide their estimates as an intermediate step to estimating the Navier-Stokes problem.

### 6.3.5. Discrete maximal regularity in $L^\infty/L^2$ for the pressure

The discrete maximal regularity estimates above, which have so far solely focused on the discrete velocity, can be extended to cover the gradient of the pressure in the case $s = \infty$ for certain kinds of finite element discretizations.

*Remark* 6.16 The case $1 \leq s < \infty$ seems more complicated for our approach, it is not obvious how to deal with the supremum appearing in the following proposition for a time dependent pressure. For $s = \infty$ this can be somewhat decoupled.

There holds the following proposition.

**Proposition 6.17** *Let $\boldsymbol{V}_h \times M_h$ be composed of finite elements based on the mini element or $\boldsymbol{V}_h \times M_h$ be composed of second order Taylor-Hood finite elements, then there holds*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{(\nabla l_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{L^2(\Omega)}} \geq C \|\nabla l_h\|_{L^2(\Omega)} \qquad \forall l_h \in M_h.$$

Note, that the expression $\nabla l_h$ is well-defined for this selection of finite element spaces.

*Proof.* The result follows from a compatibility assumption in [70, (2.1)] which is shown to be valid for the mini element or the Taylor-Hood finite element in [70, Lemma 2.2]. The compatibility assumption is given for $\tilde{P}_h$ the $L^2$ projection into $\boldsymbol{V}_h$ by

$$\|\tilde{P}_h \nabla l_h\|_{L^2(\Omega)} \geq C \|\nabla l_h\|_{L^2(\Omega)} \qquad \forall l_h \in M_h.$$

Using the properties of the $L^2$ projection, the inequality may be rewritten in the form of Proposition 6.17. □

Before we discuss the discrete regularity estimate for $s = \infty$ for the pressure we also need the following estimate for the orthogonal projection $\tilde{P}_\tau$ from $L^2(I, \mathbb{R})$ into $\boldsymbol{V}_\tau^w(\mathbb{R})$ defined as

$$(\tilde{P}_\tau(r), l_\tau)_I = (r, l_\tau)_I \qquad \forall l_\tau \in \boldsymbol{V}_\tau^w(\mathbb{R}), \tag{6.3.19}$$

for $r \in L^2(I; \mathbb{R})$.

**Lemma 6.18** *There holds for $1 \leq s \leq \infty$,*

$$\|\tilde{P}_\tau(r)\|_{L^s(I)} \leq C \|r\|_{L^s(I)},$$

*with $C$ independent of $\tau$.*

*Proof.* Since $\boldsymbol{V}_\tau^w(\mathbb{R})$ contains by definition functions discontinuous in time, it is enough to prove the estimate on an interval $I_m$. Furthermore, because of our assumptions on the time mesh and the nature of $\tilde{P}_{\tau,I_m}$ (the restriction of $\tilde{P}_\tau$ to $I_m$), we may transform $\|\tilde{P}_{\tau,I_m}(r)\|_{L^s(I_m)}$ to a reference interval $\hat{I} = (0, 1)$, show the estimate there, and then transform back to the interval $I_m$, leaving the estimate independent of $\tau$. In the following $\hat{r}$ is the respective transform of $r$ on the reference interval.

This means, it is enough to show that the projection $\tilde{P}_{\tau,\hat{I}}$ into $\mathcal{P}_{w,\hat{I}}(\mathbb{R})$ on $\hat{I}$ is bounded in $L^s(\hat{I})$. To that end we can explicitly calculate the coefficients of $\tilde{P}_{\tau,\hat{I}}(\hat{r})$ by choosing a basis $\{\phi_i\}_{1 \leq i \leq w+1}$ of $\mathcal{P}_{w,\hat{I}}(\mathbb{R})$. Then, (6.3.19) implies

$$(\tilde{P}_{\tau,\hat{I}}(\hat{r}), \phi_i)_{\hat{I}} = \left( \sum_{j=1}^{w+1} a_j \phi_j, \phi_i \right)_{\hat{I}} = (\hat{r}, \phi_i)_{\hat{I}} \qquad \text{for } 1 \leq i \leq w+1, \tag{6.3.20}$$

where $a_j \in \mathbb{R}$ are the coefficients of $\tilde{P}_{\tau,\hat{I}}(\hat{r})$, which we may also write as the coefficient vector $\boldsymbol{a}$. The vector $\boldsymbol{a}$ is then the solution to the linear system given by (6.3.20). Choosing an orthonormal basis of Legendre polynomials we have

$$a_j = (\hat{r}, \phi_j)_{\hat{I}}.$$

Hence, we can estimate

$$\begin{aligned}
\|\tilde{P}_{\tau,\hat{I}}(\hat{r})\|_{L^s(\hat{I})} &\le C(w+1) \max_{1\le j\le w+1} \|\phi_j\|_{L^s(\hat{I})} |(\hat{r}, \phi_j)_{\hat{I}}| \\
&\le C(w+1) \max_{1\le j\le w+1} \|\phi_j\|_{L^s(\hat{I})} \|\phi_j\|_{L^{s'}(\hat{I})} \|\hat{r}\|_{L^s(\hat{I})} \\
&\le C\|\hat{r}\|_{L^s(\hat{I})},
\end{aligned}$$

since $\phi_j$ are bounded continuous functions on a bounded domain. This proves the result on $\hat{I}$ and, after respective transformations, also for $\tilde{P}_\tau(r)$ on $I_m$ and $I$. □

Thus, we are able to state the following discrete regularity and smoothing estimate for the pressure.

**Corollary 6.19** *In the case $s = \infty$ and $p_{\tau h}$ the pressure solution to* (6.3.5) *with $\boldsymbol{u}_0 = \boldsymbol{0}$, there holds*

$$\|\nabla p_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \le C \ln \frac{\mathscr{T}}{\tau} \|\mathbb{P}_h \boldsymbol{f}\|_{L^\infty(I;L^2(\Omega))}.$$

*Proof.* In the following $\mathring{I}_m$ denotes the interior of $I_m$. Let $\tilde{t} \in \mathring{I}_{\tilde{m}}$ (the boundary of $I_{\tilde{m}}$ is a set of measure zero) for $1 \le \tilde{m} \le M$ and let $\theta(t)$ from the proof of Theorem 6.11 be the regularized Dirac function supported in the time interval $\mathring{I}_{\tilde{m}}$ such that for $\tilde{t} \in \mathring{I}_{\tilde{m}}$ it holds by Proposition 6.17

$$\begin{aligned}
\|\nabla p_{\tau h}(\tilde{t})\|_{L^2(\Omega)} &\le C \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{v}_h \ne \boldsymbol{0}} \|\boldsymbol{v}_h\|^{-1}_{L^2(\Omega)} (\nabla p_{\tau h}(\tilde{t}), \boldsymbol{v}_h)_\Omega \\
&= C \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{v}_h \ne \boldsymbol{0}} \|\boldsymbol{v}_h\|^{-1}_{L^2(\Omega)} (\nabla p_{\tau h}, \theta \boldsymbol{v}_h)_{I\times\Omega} \\
&= C \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{v}_h \ne \boldsymbol{0}} \|\boldsymbol{v}_h\|^{-1}_{L^2(\Omega)} (-p_{\tau h}, \theta \nabla \cdot \boldsymbol{v}_h)_{I\times\Omega}. \qquad (6.3.21)
\end{aligned}$$

Clearly $\boldsymbol{v}_h$ is constant in time. Now let $\tilde{\boldsymbol{v}}_{\tau h} := \tilde{P}_\tau(\theta(t))\boldsymbol{v}_h$, where $\tilde{P}_\tau$ is the orthogonal projection into $\boldsymbol{V}_\tau^w(\mathbb{R})$ given by $(r, l_\tau) = (\tilde{P}_\tau(r), l_\tau)$ for $r \in L^2(I)$ and all $l_\tau \in \boldsymbol{V}_\tau^w(\mathbb{R})$ (cf. Lemma 6.18). Furthermore, note $p_{\tau h}(\boldsymbol{x})$ is an element of $\boldsymbol{V}_\tau^w(\mathbb{R})$ and thus we have

$$(p_{\tau h}, \theta \nabla \cdot \boldsymbol{v}_h)_{I\times\Omega} = (p_{\tau h}, \nabla \cdot \tilde{\boldsymbol{v}}_h)_{I\times\Omega}$$

Since $\tilde{P}_\tau(\theta(t)) \in \boldsymbol{V}_\tau^w(\mathbb{R})$ and it is constant in space, as well as $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and it is constant in time, we have $\tilde{\boldsymbol{v}}_{\tau h} \in \boldsymbol{V}_\tau^w(\boldsymbol{V}_h)$. We can now use the weak formulation of the fully discrete Stokes problem in (6.3.5) to see after testing with $(\tilde{\boldsymbol{v}}_h, 0)$

$$\begin{aligned}
(p_{\tau h}, \nabla \cdot \tilde{\boldsymbol{v}}_h)_{I\times\Omega} = \sum_{m=1}^M \langle \partial_t \boldsymbol{u}_{\tau h}, \tilde{\boldsymbol{v}}_{\tau h} \rangle_{I_m \times\Omega} - (\Delta_h \boldsymbol{u}_{\tau h}, \tilde{\boldsymbol{v}}_{\tau h})_{I\times\Omega} \\
+ \sum_{m=2}^M ([\boldsymbol{u}_{\tau h}]_{m-1}, \tilde{\boldsymbol{v}}^+_{\tau h, m-1})_\Omega - (\boldsymbol{f}, \tilde{\boldsymbol{v}}_{\tau h})_{I\times\Omega}.
\end{aligned}$$

Here we have already simplified the terms, using $\boldsymbol{u}_0 = \boldsymbol{0}$ and integrated by parts. We now may use the Hölder inequality so that we end up with the following estimate

$$\begin{aligned}|(p_{\tau h}, \nabla \cdot \tilde{\boldsymbol{v}}_h)_{I\times\Omega}| \leq C\Bigg[ &\max_{1\leq m\leq M} \|\partial_t \boldsymbol{u}_{\tau h}\|_{L^\infty(I_m;L^2(\Omega))} + \|\Delta_h \boldsymbol{u}_{\tau h}\|_{L^\infty(I;L^2(\Omega))} \\ &+ \max_{2\leq m\leq M} \|\tau_m^{-1}[\boldsymbol{u}_{\tau h}]_{m-1}\|_{L^2(\Omega)} + \|\mathbb{P}_h \boldsymbol{f}\|_{L^\infty(I;L^2(\Omega))}\Bigg] \\ &\left[\sum_{m=1}^{M} \tau_m \|\tilde{\boldsymbol{v}}^+_{\tau h,m-1}\|_{L^2(\Omega)} + \|\tilde{\boldsymbol{v}}_{\tau h}\|_{L^1(I;L^2(\Omega))}\right]. \end{aligned} \tag{6.3.22}$$

The first factor is certainly bounded by Corollary 6.10. We now focus on the second factor. For the last term in the second factor we now may apply Lemma 6.18 and get by the construction of $\tilde{\boldsymbol{v}}_h$ and $\theta(t)$

$$\|\tilde{\boldsymbol{v}}_{\tau h}\|_{L^1(I;L^2(\Omega))} \leq C\|\theta\|_{L^1(I)}\|\boldsymbol{v}_h\|_{L^2(\Omega)} \leq C\|\boldsymbol{v}_h\|_{L^2(\Omega)}. \tag{6.3.23}$$

For the first term in the second factor notice that since $\theta$ is only supported in $\mathring{I}_{\tilde{m}}$, it is enough to estimate

$$\tau_{\tilde{m}+1}\|\tilde{\boldsymbol{v}}^+_{\tau h,\tilde{m}}\|_{L^2(\Omega)} \leq C\|\boldsymbol{v}_h\|_{L^2(\Omega)}. \tag{6.3.24}$$

This holds since $\tilde{P}_\tau(\theta)^+_{\tilde{m}} \leq C\|\theta\|_{L^\infty(I_{\tilde{m}})} \leq C\tau_{\tilde{m}}^{-1}$ (cf. [112, (A.2)]) and the second assumption on the time mesh.

We now collect the results in (6.3.21)–(6.3.24) to see

$$\begin{aligned}\|\nabla p_{\tau h}(\tilde{t})\|_{L^2(\Omega)} &\leq C \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_h, \boldsymbol{v}_h\neq\boldsymbol{0}} \|\boldsymbol{v}_h\|^{-1}_{L^2(\Omega)}(-p_{\tau h}, \theta\nabla\cdot\boldsymbol{v}_h)_{I\times\Omega} \\ &\leq C\ln\frac{\mathscr{T}}{\tau} \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_h, \boldsymbol{v}_h\neq\boldsymbol{0}} \|\boldsymbol{v}_h\|^{-1}_{L^2(\Omega)}\|\mathbb{P}_h\boldsymbol{f}\|_{L^\infty(I;L^2(\Omega))} \\ &\qquad \left[\sum_{m=1}^{M} \tau_m \|\tilde{\boldsymbol{v}}^+_{\tau h,m-1}\|_{L^2(\Omega)} + \|\tilde{\boldsymbol{v}}_{\tau h}\|_{L^1(I;L^2(\Omega))}\right] \\ &\leq C\ln\frac{\mathscr{T}}{\tau}\|\mathbb{P}_h\boldsymbol{f}\|_{L^\infty(I;L^2(\Omega))}.\end{aligned}$$

□

**Corollary 6.20** *Let $p_{\tau h}$ be the pressure solution to* (6.3.5) *with $\boldsymbol{f} = \boldsymbol{0}$. Then, there holds for $m = 1, \ldots, M$*

$$\|\nabla p_{\tau h}\|_{L^\infty(I_m;L^2(\Omega))} \leq \frac{C}{t_m}\|\mathbb{P}_h\boldsymbol{u}_0\|_{L^2(\Omega)}.$$

*Proof.* The result follows by the same arguments which we used to show the corollary above. A notable difference is that we only consider $I_m$ here, not the whole domain, and use Corollary 6.9 instead of Corollary 6.10. □

### 6.3.6. Time-discrete maximal regularity for the Stokes problem on smooth domains

As the conclusion to this chapter we adapt resolvent estimates for $\boldsymbol{u}$, the solution of the continuous Stokes resolvent problem on a $C^{1,1}$ domain, and state the respective semi-discrete maximal regularity results based on Section 6.3.3. We begin with an estimate for the $L^2$ resolvent, similar to the discrete case in Lemma 6.7.

**Lemma 6.21** *For any $\theta \in (\pi/2, \pi)$ there exists a constant $C = C_\theta$ independent of $z$ such that for any $\nu \in [0, a_0]$ with $a_0 > 0$ being the smallest eigenvalue of $-\Delta$ it holds*

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq \frac{C_\theta}{|z+\nu|}\|\boldsymbol{f}\|_{L^2(\Omega)} \quad \forall z \in \Sigma_{\theta,-\nu},$$

*where $\boldsymbol{u}$ is the velocity part of the solution to* (6.2.1a)–(6.2.1c) *with right-hand side $\boldsymbol{f} \in L^2(\Omega)^d$.*

*Proof.* For the proof we can argue as in Lemma 6.7 but with the eigenvalues of $-\Delta$. □

Using the $H^2$ regularity of the solution $\boldsymbol{u}$ to the Stokes problem this result can be extended to the case $1 < p < \infty$.

**Lemma 6.22** *For any $\theta \in (\pi/2, \pi)$ there exists a constant $C = C_\theta$ independent of $z$ such that for any $\nu \in [0, a_0]$ with $a_0 > 0$ being the smallest eigenvalue of $\Delta$, it holds for $1 < p < \infty$*

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leq \frac{C_\theta}{|z+\nu|}\|\boldsymbol{f}\|_{L^p(\Omega)} \quad \forall z \in \Sigma_{\theta,-\nu},$$

*where $\boldsymbol{u}$ is the velocity part of the solution to* (6.2.1a)–(6.2.1c) *with right-hand side $\boldsymbol{f} \in L^p(\Omega)^d$.*

*Proof.* We only consider the case $2 < p < \infty$. The case $1 < p < 2$ follows by a duality argument.

The case $\nu = 0$ is available from [53, Theorem 1.2]. We extend this result to $\nu \in [0, a_0]$ by arguing similarly to [14, Lemma 6.1] and consider two cases for the decomposition of the sector $\Sigma_{\theta,-\nu} = D_1 \cup D_2$ into

$$D_1 = \left\{ z \in \Sigma_{\theta,-\nu} \colon |\arg z| \leq \frac{\pi+\theta}{2} \text{ and } |z| \geq \frac{|\nu|}{2} \right\}$$

and

$$D_2 = \left\{ z \in \Sigma_{\theta,-\nu} \colon |\arg z| \geq \frac{\pi+\theta}{2} \right\} \cup \left\{ |z| \leq \frac{|\nu|}{2} \right\}.$$

The choice of $D_1$ allows us to apply [53, Corollary 1.6] in this case. Using $|z+\nu| \leq 3|z|$ on $D_1$ the result follows.

On $D_2$, we note that $\boldsymbol{u}$ can be seen as the solution to (2.2.1) with right-hand side $\boldsymbol{f} - z\boldsymbol{u}$, i.e.,

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} - z\boldsymbol{u} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned}$$

Thus, by $H^2(\Omega)^d$ regularity in (2.2.3) we get

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leq \|\boldsymbol{u}\|_{H^2(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{L^2(\Omega)} + |z|\|\boldsymbol{u}\|_{L^2(\Omega)}\Big)$$

and using Lemma 6.21 we have

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leq C\left(\|\boldsymbol{f}\|_{L^2(\Omega)} + \frac{|z|}{|z+\nu|}\|\boldsymbol{f}\|_{L^2(\Omega)}\right) \leq C(|z+\nu| + |z|)\frac{\|\boldsymbol{f}\|_{L^p(\Omega)}}{|z+\nu|}.$$

Since $z$ is bounded on $D_2$ by a constant depending on $\theta$ and $\nu$ (cf. Figure 6.2) the result follows. □

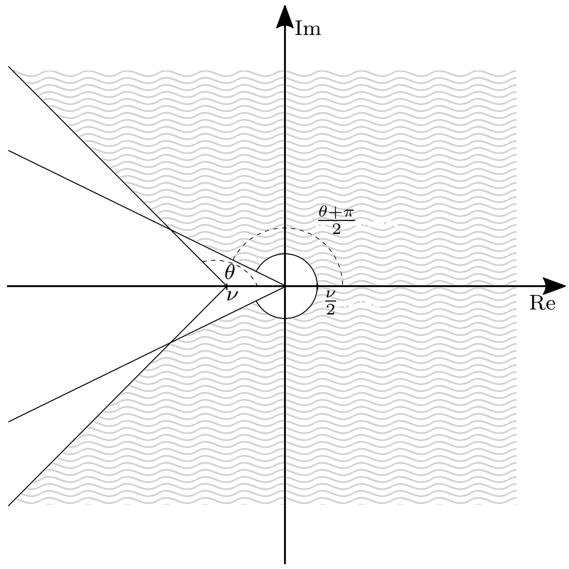


Figure 6.1.: $D_1$.

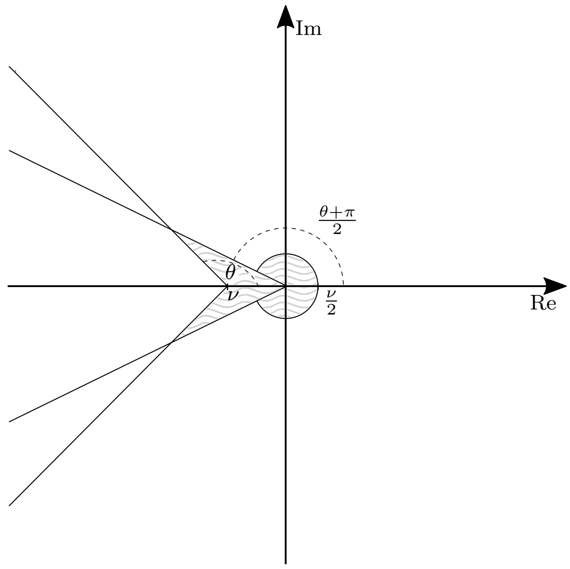


Figure 6.2.: $D_2$.

With these results at hand we can apply the estimates from Section 6.3.3.

We have the following corollary to Theorem 6.6, where we used the fact that $\|\boldsymbol{v}\|_{W^{2,p}(\Omega)} \leq C\|A_p\boldsymbol{v}\|_{L^p}$ (cf. [6, Theorem 4]) for $\boldsymbol{v} \in \boldsymbol{X}_{2,p}$ to obtain an estimate for $\Delta\boldsymbol{u}_\tau$.

**Corollary 6.23** *For $1 \le s < \infty$, $1 < p < \infty$, and $\boldsymbol{u}_\tau$ the velocity solution to* (6.3.4) *or solution to* (6.3.6) *with $\boldsymbol{u}_0 = \boldsymbol{0}$, there holds*

$$\left(\sum_{m=1}^{M} \|\partial_t \boldsymbol{u}_\tau\|^s_{L^s(I_m;L^p(\Omega))}\right)^{1/s} + \|\Delta \boldsymbol{u}_\tau\|_{L^s(I;L^p(\Omega))} + \|A_p \boldsymbol{u}_\tau\|_{L^s(I;L^p(\Omega))} + \left(\sum_{m=1}^{M} \tau_m \|\tau_m^{-1}[\boldsymbol{u}_\tau]_{m-1}\|^s_{L^p(\Omega)}\right)^{1/s} \le C \ln \frac{\mathcal{T}}{\tau} \|\mathbb{P}_p \boldsymbol{f}\|_{L^s(I;L^p(\Omega))}.$$

*For $s = \infty$ one has*

$$\|\Delta \boldsymbol{u}_\tau\|_{L^\infty(I;L^p(\Omega))} + \|A_p \boldsymbol{u}_\tau\|_{L^\infty(I;L^p(\Omega))} + \max_{1 \le m \le M} \|\tau_m^{-1}[\boldsymbol{u}_\tau]_{m-1}\|_{L^p(\Omega)} \le C \ln \frac{\mathcal{T}}{\tau} \|\mathbb{P}_p \boldsymbol{f}\|_{L^\infty(I;L^p(\Omega))}.$$

*Here we have $[\boldsymbol{u}_\tau]_0 = \boldsymbol{u}^+_{\tau,0}$.*

And as a corollary to Theorem 6.3 we have the following.

**Corollary 6.24** *Let $1 < p < \infty$ and $\boldsymbol{u}_\tau$ be the velocity solution to* (6.3.4) *or solution to* (6.3.6) *with $\boldsymbol{f} = \boldsymbol{0}$. Then, there holds for $m = 1, \dots, M$*

$$\|\partial_t \boldsymbol{u}_\tau\|_{L^\infty(I_m;L^p(\Omega))} + \|A_p \boldsymbol{u}_\tau\|_{L^\infty(I_m;L^p(\Omega))} + \|\Delta \boldsymbol{u}_\tau\|_{L^\infty(I_m;L^p(\Omega))} + \|\tau_m^{-1}[\boldsymbol{u}_\tau]_{m-1}\|_{L^p(\Omega)} \le \frac{C}{t_m} \|\boldsymbol{u}_0\|_{L^p(\Omega)}.$$

*Here we have $[\boldsymbol{u}_\tau]_0 = \boldsymbol{u}^+_{\tau,0} - \boldsymbol{u}_0$.*

Using Corollary 6.23 it is then possible to derive an optimal error estimate for the semi-discrete problem. To that end, we define a projection $\pi_\tau$ for $\boldsymbol{u} \in C(\bar{I}, H^1(\Omega)^d)$ with $\pi_\tau(\boldsymbol{u})|_{I_m} \in \mathcal{P}_{w,I_m}(H^1(\Omega)^d)$ for $m = 1, \dots, M$ on each subinterval $I_m$ characterized by

$$(\pi_\tau(\boldsymbol{u}) - \boldsymbol{u}, \varphi)_{I_m \times \Omega} = \boldsymbol{0}, \qquad \forall \varphi \in \mathcal{P}_{w-1,I_m}(H^1(\Omega)^d), w > 0, \tag{6.3.26a}$$
$$(\nabla \cdot (\pi_\tau(\boldsymbol{u}) - \boldsymbol{u}), q)_{I_m \times \Omega} = \boldsymbol{0}, \qquad \forall q \in \mathcal{P}_{w,I_m}(L^2_0(\Omega)), \tag{6.3.26b}$$
$$\pi_\tau(\boldsymbol{u}(t_m^-)) = \boldsymbol{u}(t_m^-). \tag{6.3.26c}$$

Note that the second condition is not a particularly strong restriction. For example, any nodal interpolation of $\boldsymbol{u}$ on $I_m$ would satisfy the condition, since $\nabla \cdot \boldsymbol{u}(t) = 0$ for all $t \in I$.

Then, one derives the following error estimate based on $\pi_\tau$.

**Theorem 6.25** *Let $\boldsymbol{u}$ be the solution to* (6.1.1a)–(6.1.1d) *with $\boldsymbol{u} \in C(\bar{I}, H^1(\Omega)^d \cap L^p(\Omega)^d)$ and $\boldsymbol{u}_\tau$ the respective dG($w$) approximation* (6.3.4) *for $w \ge 0$. Then, there holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_\tau\|_{L^s(I;L^p(\Omega)^d)} \le C \ln \frac{\mathcal{T}}{\tau} \|\boldsymbol{u} - \pi_\tau(\boldsymbol{u})\|_{L^s(I;L^p(\Omega)^d)}, \qquad 1 \le s < \infty, 1 < p < \infty.$$

*Proof.* With condition (6.3.26b) the result follows by a duality argument as in the proof of [87, Theorem 9], since $\pi_\tau(\boldsymbol{u})$ is divergence-free and thus the respective pressure term in $B((\boldsymbol{u} - \pi_\tau \boldsymbol{u}, p - p_\tau), (\boldsymbol{z}_\tau, r_\tau))$ does not make an appearance for $(\boldsymbol{z}_\tau, r_\tau) \in \boldsymbol{V}_\tau^w(\boldsymbol{X}_{1,2} \times L_0^2(\Omega))$, the solution of the respective dual problem. □

Assuming that $\boldsymbol{u}$ is sufficiently smooth and $\pi_\tau$ satisfies a respective convergence estimate, one immediately derives the following convergence error estimate for $\boldsymbol{u}_\tau$.

**Corollary 6.26** *For* $\boldsymbol{u} \in W^{w+1,s}(I; L^p(\Omega)^d) \cap C(\bar{I}, H^1(\Omega)^d)$ *the solution to* (6.1.1a)–(6.1.1d) *and* $\boldsymbol{u}_\tau$ *the respective dG*$(w)$ *approximation* (6.3.4) *for* $w \geq 0$ *there holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_\tau\|_{L^s(I;L^p(\Omega)^d)} \leq C\tau^{w+1} \ln \frac{\mathscr{T}}{\tau} \|\boldsymbol{u}\|_{W^{w+1,s}(I;L^p(\Omega)^d)}, \qquad 1 \leq s < \infty, 1 < p < \infty.$$

These estimates conclude the chapter on the instationary Stokes problem.

# Chapter 7.

# Conclusion and outlook

We were able to show that global and local best-approximation results, which are well-known for elliptic problems, extend to the case of the stationary divergence-free Stokes problem in Chapter 3. To that end, we used weighted $L^2$ estimates combined with a dyadic decomposition technique. While the results are given under a set of assumptions, we have seen that the finite element space based on the Taylor-Hood finite element fulfills these assumptions. Furthermore, we argue in Appendix A that the *mini* element can be treated similarly.

In view of the results in Chapter 3, particularly the local best-approximation results, we were then able to give some new convergence rate estimates for the optimal control problems in Chapters 4 and 5. There, we used that the appearing measures were only locally supported which allowed us to consider the regularity of dual problems away from the non-smooth boundary. Note that compared to the analysis of the Poisson problem in [103], we considered non-smooth domains $\Omega$ in Chapter 5.

Further potential applications of the results in Chapter 3 include pointwise state constrained optimal control problems or best-approximation error estimates for the *instationary* Stokes problem.

Which brings us to the topic of Chapter 6. To prove best-approximation results in $L^\infty(I \times \Omega)$ one wants to prove a discrete resolvent estimate which allows one to apply an established maximal regularity result. Unfortunately, a discrete $L^\infty$ resolvent estimate on a convex non-smooth domain has not been achieved in this work, we note that this is still an open problem also in the continuous case. Nonetheless, we were able to give a new proof to an approximation error estimate in $L^\infty/L^2$, since in the spatial $L^2$ case the pressure does not interfere in the resolvent estimate.

In the introduction we raised the question of how far one can argue similarly to the Poisson problem for the Stokes case. As it was already mentioned and as we have seen in Chapter 3, as long as it is possible to get rid of the pressure, the arguments remain quite similar. When this was not possible, recall the $L^1$ estimate of $\lambda_0$, we there had the key advantage that we had precise Hölder estimates of the respective Green's function which allowed us to argue via a dyadic decomposition. Such an estimate was not available in the resolvent case, making it more difficult to deal with the respective estimates.

In the application to the optimal control problems, we want to highlight Remark 4.12 as a significant difference to the Poisson problem. Also, one cannot argue as in [103] to achieve better convergence rates for the state in Chapter 5 since respective results as the maximum principle and harmonic function theory are not available for the Stokes problem.

While the results in Chapter 3 can be considered for the most part comprehensive, there still remain some open questions. For example, for which finite element methods the assumptions in Section 3.2 hold. In particular, there is the question of extending the result for the Taylor-Hood method for $d = 3$ to the case of polynomials of degree $k = 2$. This likely is possible with a similar technique as in [73].

Regarding the approximation error estimates derived for the pointwise tracking-type problem, the numerical approximation rate indicates that there is still room for improvement, which comes down to better understanding the behavior of the solution to the continuous Stokes problem under the projection $P_{[\boldsymbol{a},\boldsymbol{b}]}$.

Similarly, for the measure controlled problem in Chapter 5 the numerical results indicate better convergence for the state in case of a bounded $\boldsymbol{u}_d$. There is the potential, that this can be addressed by maximum modulus estimates for the Stokes problem (cf. [94, Section 11.6]) but right-now there appears to be no obvious way forward yet.

One future objective is still discrete maximal regularity in $L^\infty$ for the Stokes problem. Up to now it seems like the techniques we focused on so far fail when dealing with the pressure, or equivalently the divergence constraint, in the resolvent problem. That indicates a different approach is required, which either allows one to avoid the pressure term or provides a suitable bound for it. The latter case seems to be limited by the results in [123] though.
A result for the maximal regularity estimate would allow one to make significant entrails into instationary sparse optimal control problems of the Stokes-type. Certainly that perspective warrants further research.

# Appendix A.

# Pointwise approximation error estimates for the mini element

In Chapter 3 we have shown results only for Taylor-Hood finite elements of order greater or equal three in the three-dimensional setting and for elements of order greater or equal two in two dimensions. In this section we extend the results to the “mini” element. Compared to the result for Taylor-Hood finite elements, the mini element is simpler to implement since it essentially only requires linear basis functions for velocity and pressure.

The mini element has been developed by Arnold, Brezzi, and Fortin in [11]. To achieve stability with respect to the inf-sup condition, the velocity space consisting of linear finite elements is extended by bubble functions.

For our discretization, we choose the space of piecewise linear functions

$$V_h = \{v \in C(\bar{\Omega})\colon v_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h\},$$

i.e., the space of Lagrange finite elements of order one, and the space of so-called Bubble functions

$$B_h = \{v \mid v_T \in \mathcal{P}_{d+1}(T) \cap H_0^1(T)\ \forall T \in \mathcal{T}_h\},$$

with $v \in B_h$ on $T$ being of the form $v = \chi(T) \prod_{i=1}^{d+1} \lambda_i$. Here $\lambda_i(\boldsymbol{x})$ are the barycentric coordinates of $T$ (cf. [66, p. 96]) and $\chi(T)$ is the respective coefficient for the cell $T$. Consequently, the pressure space is given as

$$M_h = V_h \cap L_0^2(\Omega)$$

and the velocity space by

$$\boldsymbol{V}_h = (V_h \cap H_0^1(\Omega))^d + B_h^d.$$

Here we seek to construct a projection operator $\Pi_h : C_0(\Omega)^d \to \boldsymbol{V}_h$ such that

$$\int_\Omega q_h \nabla \cdot (\Pi_h \boldsymbol{v} - \boldsymbol{v}) d\boldsymbol{x} = 0 \qquad \forall q_h \in M_h,\ \forall \boldsymbol{v} \in (C_0(\Omega) \cap H_0^1(\Omega))^d, \tag{A.1}$$

$$\Pi_h \boldsymbol{v}_h = \boldsymbol{v}_h \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

and

$$\|\nabla \Pi_h \boldsymbol{v}\|_{L^1(\Omega)} \leq C \|\nabla \boldsymbol{v}\|_{L^1(\Omega)} \qquad \forall \boldsymbol{v} \in (C_0(\Omega) \cap H_0^1(\Omega))^d.$$

For finite element spaces $\boldsymbol{V}_h$ and $M_h$ the condition (A.1) is implied by the property

$$\int_T (\Pi_h \boldsymbol{v} - \boldsymbol{v}) d\boldsymbol{x} = 0 \qquad \forall \boldsymbol{v} \in (C_0(\Omega) \cap H_0^1(\Omega))^d, \tag{A.2}$$

which can be seen after integrating by parts and because of $\nabla q_h$ being a piecewise constant function. This simplified condition allows one to define $\Pi_h$ in the following way:

Let $I_h$ be a nodal interpolation operator from $(C_0(\Omega) \cap H_0^1(\Omega))^d \to (H_0^1(\Omega) \cap V_h)^d$. Then, we define for $\boldsymbol{v} \in (C_0(\Omega) \cap H_0^1(\Omega))^d$

$$\Pi_h \boldsymbol{v} = I_h(\boldsymbol{v}) + \chi(T) \prod_{i=1}^{d+1} \lambda_i \qquad \text{on } T, \tag{A.3}$$

where for each $T \in \mathcal{T}_h$ the scalar $\chi(T)$ is chosen such that the following identity holds

$$\chi(T) \int_T \prod_{i=1}^{d+1} \lambda_i d\boldsymbol{x} = \int_T I_h(\boldsymbol{v}) - \boldsymbol{v} d\boldsymbol{x}, \tag{A.4}$$

which is motivated by (A.2). In the following we write $I_h^b(\boldsymbol{v}) = \chi(T) \prod_{i=1}^{d+1} \lambda_i$. Due to (A.4) $I_h^b$ clearly is a linear operator. Also note, since the bubble function is zero at $\partial T$ for each $T \in \mathcal{T}_h$, we have that $I_h(\Pi_h(\boldsymbol{v})) = I_h(\boldsymbol{v})$ for $\boldsymbol{v} \in (C_0(\Omega) \cap H_0^1(\Omega))^d$ and thus $\Pi_h \boldsymbol{v}_h = \boldsymbol{v}_h$ for $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

We show that $I_h^b$ is stable in $W^{1,1}(T)$. Since the bubble function is zero on $\partial T$ it suffices to estimate $\|\nabla I_h^b(\boldsymbol{v})\|_{L^1(T)}$. Then, we see because of (A.4) and an inverse estimate

$$\begin{aligned}
\|\nabla I_h^b(\boldsymbol{v})\|_{L^1(T)} &\le Ch^{-1}|\chi(T)| \left\| \prod_{i=1}^{d+1} \lambda_i \right\|_{L^1(T)} \le Ch^{d-1}|\chi(T)| \\
&\le Ch^{-1} \left| \int_T I_h(\boldsymbol{v}) - \boldsymbol{v} d\boldsymbol{x} \right| \le Ch^{-1} \|I_h(\boldsymbol{v}) - \boldsymbol{v}\|_{L^1(T)}.
\end{aligned}$$

Now, because of the convergence of the interpolation operator $I_h$ in $L^1$, we can conclude

$$\|\nabla I_h^b(\boldsymbol{v})\|_{L^1(T)} \le C \|\nabla \boldsymbol{v}\|_{L^1(T)}. \tag{A.5}$$

Having collected these basic results, we now come to the question of pointwise approximation error estimates. We note that Assumptions 3.4–3.7 on the finite element space for the Stokes problem in Chapter 3 can be verified as in [12, 64]. Thus, all the auxiliary results in Section 3.3.3 hold, excluding the estimates for the Ritz projection $R_h$, which maps in this case into the wrong space, since here $\boldsymbol{V}_h$ is not a space of Lagrange finite elements. But then, separation of the bubble functions from the linear parts as in (A.3) allows us to still proceed as in Chapter 3.

**Theorem A.1** (Global $L^\infty$ estimate for the velocity, mini element) *Under the assumptions of* (2.3.1) *and Section 3.2.2, for* $(\boldsymbol{u}, p) \in (L^\infty(\Omega)^3 \times L^\infty(\Omega)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ *the solution to* (3.1.1a)–(3.1.1c) *and* $(\boldsymbol{u}_h, p_h)$ *the solution to* (2.3.2) *based on the finite element space* $\boldsymbol{V}_h \times M_h$ *for the mini element, it holds*

$$\|\boldsymbol{u}_h\|_{L^\infty(\Omega)} \le C|\ln h| \Big( |\ln h| \|\boldsymbol{u}\|_{L^\infty(\Omega)} + h\|p\|_{L^\infty(\Omega)} \Big).$$

*Proof.* As in the proof of Theorem 3.14 we obtain, independent of the fact that we are using the mini element, the following identity (cf. (3.3.9))

$$\boldsymbol{u}_{h,i}(\boldsymbol{x}_0) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) - (p, \nabla \cdot \boldsymbol{g}_{0,h}).$$

Note that by assumption, $\boldsymbol{g}_{0,h}$ is also in the finite element space of the mini element. We focus on the first term, since the pressure term can be dealt with as in (3.3.10). In the following we only discuss the details of the terms which do not already appear in the proof of Theorem 3.14.

To simplify the first term, we want to introduce the Ritz projection $R_h$ to apply results already available for the Lagrange finite element approximation of the Poisson problem. In the original proof of Theorem 3.14 we can do this directly, since for Taylor-Hood finite elements one uses standard Lagrange finite elements of order three or higher for the discretization of the velocity. Here we need to deal with the additional bubble function term (cf. (A.3)).

In the following we treat the Lagrange part and the bubble function separately. We use that $I_h$ maps into the space of Lagrange finite elements to see

$$\begin{aligned}(\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) &= (\nabla \boldsymbol{u}, \nabla I_h(\boldsymbol{g}_{0,h})) + (\nabla \boldsymbol{u}, \nabla I_h^b(\boldsymbol{g}_{0,h})) \\ &= (\nabla R_h(\boldsymbol{u}), \nabla I_h(\boldsymbol{g}_{0,h})) + (\nabla \boldsymbol{u}, \nabla I_h^b(\boldsymbol{g}_{0,h})) \\ &= (\nabla R_h(\boldsymbol{u}), \nabla \Pi_h(\boldsymbol{g}_{0,h})) + (\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h})) \\ &= (\nabla R_h(\boldsymbol{u}), \nabla \boldsymbol{g}_{0,h}) + (\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h})). \end{aligned} \tag{A.6}$$

Here we first separated $I_h^b(\boldsymbol{g}_{0,h})$, used the properties of $R_h$, and then added in the term $(\nabla R_h(\boldsymbol{u}), I_h^b(\boldsymbol{g}_{0,h}))$. Since by construction $\boldsymbol{g}_{0,h}$ is a finite element function in $\boldsymbol{V}_h$, we were able to use

$$\boldsymbol{g}_{0,h} = \Pi_h \boldsymbol{g}_{0,h} = I_h(\boldsymbol{g}_{0,h}) + I_h^b(\boldsymbol{g}_{0,h}).$$

Now the first term in (A.6) can be treated as in the proof of Theorem 3.14. We need to further analyze $(\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h}))$. To that end, we integrate by parts on each cell $T \in \mathcal{T}_h$. A similar technique has been used in the derivation of discrete max-norm estimates for the Poisson problem in [85, Lemma 6]. It follows by integration by parts and the continuity of $\boldsymbol{u}$ and $R_h(\boldsymbol{u})$ that

$$\begin{aligned}(\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h}))_T &= (\boldsymbol{u} - R_h(\boldsymbol{u}), [\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!])_{\partial T} + (\boldsymbol{u} - R_h(\boldsymbol{u}), -\Delta I_h^b(\boldsymbol{g}_{0,h}))_T \\ &\leq \|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{L^\infty(T)} \Big( \|[\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!]\|_{L^1(\partial T)} + \|\Delta I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)} \Big) \\ &\leq \Big( \|\boldsymbol{u}\|_{L^\infty(T)} + \|R_h(\boldsymbol{u})\|_{L^\infty(T)} \Big) \Big( \|[\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!]\|_{L^1(\partial T)} \\ &\qquad + \|\Delta I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)} \Big). \end{aligned}$$

Here $[\![ \cdot ]\!]$ denotes the jump across the boundary of a cell $T$. By estimating the element-boundary terms using a trace-type inequality (cf. [74, Proposition 4.11]), we obtain

$$\|[\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!]\|_{L^1(\partial T)} + \|\Delta I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)} \leq C\Big( h^{-1} \|\nabla I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)} + \|\nabla^2 I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)} \Big).$$

We then apply an inverse estimate to $\|\nabla^2 I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)}$ and get

$$\|[\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!]\|_{L^1(\partial T)} + \|\Delta I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)} \leq C h^{-1} \|\nabla I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)}. \tag{A.7}$$

Finally, we insert $I_h^b(\boldsymbol{g}_0) - I_h^b(\boldsymbol{g}_0)$ into (A.7) and use the linearity of $I_h^b$ to see

$$\|[\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!]\|_{L^1(\partial T)} + \|\Delta I_h^b(\boldsymbol{g}_{0,h})\|_{L^1(T)} \leq Ch^{-1}\Big(\|\nabla I_h^b(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(T)} + \|\nabla I_h^b(\boldsymbol{g}_0)\|_{L^1(T)}\Big). \tag{A.8}$$

By the stability result (A.5) the first term can be bounded as

$$\|\nabla I_h^b(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(T)} \leq \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^1(T)}.$$

Summing over all $T \in \mathcal{T}_h$, this term can then be estimated as in the proof of Theorem 3.14 by Lemma 3.24.

It remains to estimate $h^{-1}\|\nabla I_h^b(\boldsymbol{g}_0)\|_{L^1(T)}$. We use (A.4) and an inverse estimate to see

$$\begin{aligned} h^{-1}\|\nabla I_h^b(\boldsymbol{g}_0)\|_{L^1(T)} \leq Ch^{-2}|\chi(T)| \int_T \prod_{i=1}^{d+1} \lambda_i d\boldsymbol{x} &= Ch^{-2}\left|\int_T I_h(\boldsymbol{g}_0) - \boldsymbol{g}_0 d\boldsymbol{x}\right| \\ &\leq Ch^{-2}\|I_h(\boldsymbol{g}_0) - \boldsymbol{g}_0\|_{L^1(T)} \\ &\leq C\|\nabla^2 \boldsymbol{g}_0\|_{L^1(T)}. \end{aligned}$$

Here we used the positivity of $\prod_{i=1}^{d+1} \lambda_i$ and the $H^2(\Omega)^3$ regularity of $\boldsymbol{g}_0$. One can apply Lemma 3.29 to further bound $\sum_{T\in\mathcal{T}_h} \|\nabla^2 \boldsymbol{g}_0\|_{L^1(T)}$.

The result then follows as in Theorem 3.14 after pulling out the maximal $L^\infty(T)$ norm estimate over all $T$ and summing over all $L^1(T)$ terms. Note that since we use linear polynomials for the mini element we get an additional logarithmic factor due to Proposition 3.26 when estimating $R_h(\boldsymbol{u}_h)$ in the $L^\infty$ norm. □

We similarly prove the local version of the result for the mini element.

**Theorem A.2** (Interior $L^\infty$ error estimate for the velocity) *Under the assumptions of* (2.3.1) *and Section 3.2.2, with $D_1 = B_r(\tilde{\boldsymbol{x}}) \cap \Omega$, $D_2 = B_{\tilde{r}}(\tilde{\boldsymbol{x}}) \cap \Omega$, $\tilde{r} > r > \bar{\kappa}h$ (with $\bar{\kappa}$ large enough), $\varrho = \tilde{r} - r \geq \bar{\kappa}h$ and for $(\boldsymbol{u}, p) \in (L^\infty(D_2)^3 \times L^\infty(D_2)) \cap (H_0^1(\Omega)^3 \times L_0^2(\Omega))$ the solution to* (3.1.1a)–(3.1.1c) *and $(\boldsymbol{u}_h, p_h)$ the solution to* (2.3.2) *based on the finite element space $\boldsymbol{V}_h \times M_h$ for the mini element, it holds*

$$\begin{aligned} \|\boldsymbol{u}_h\|_{L^\infty(D_1)} \leq C|\ln h| \Big(|\ln h|\|\boldsymbol{u}\|_{L^\infty(D_2)} + h\|p\|_{L^\infty(D_2)}\Big) \\ + C_\varrho|\ln h| \Big(h\|\boldsymbol{u}\|_{H^1(\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)} + h\|p\|_{L^2(\Omega)}\Big). \end{aligned}$$

*Here, the constant $C_\varrho$ depends on the distance of $B_r(\tilde{\boldsymbol{x}})$ from $\partial B_{\tilde{r}}(\tilde{\boldsymbol{x}})$.*

*Proof.* As in the proof of the global version we start from (3.3.9)

$$\boldsymbol{u}_{h,i}(\boldsymbol{x}_0) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{g}_{0,h}) - (p, \nabla \cdot \boldsymbol{g}_{0,h}).$$

Again the pressure term can be resolved as in the proof for Taylor-Hood finite elements. As in the proof of Theorem 3.15 we need to treat the domains $D_* = B_{r+\varrho/2}(\tilde{\boldsymbol{x}}) \cap \Omega \subset D_2$ and

$\Omega \backslash D_*$ separately. Recalling the global case, the only new term that needs to be estimated when using the mini element is $(\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h}))$ in (A.6), which we split into a $D_*$ and $\Omega \backslash D_*$ part

$$(\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h})) = (\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h}))_{D_*} + (\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h}))_{\Omega \backslash D_*}.$$

Now for the first term we can again integrate by parts on each cell and get for $T \in U = \{T \in \mathcal{T}_h \colon T \subseteq D_*\}$

$$(\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h}))_T = (\boldsymbol{u} - R_h(\boldsymbol{u}), [\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!])_{\partial T} + (\boldsymbol{u} - R_h(\boldsymbol{u}), -\Delta I_h^b(\boldsymbol{g}_{0,h}))_T.$$

Note that if we choose $\bar{\kappa}$ large enough, $D_1 \subset U$. This term can be bounded by the same arguments as in the proof of Theorem A.1.

For the second term on $\Omega \backslash D_*$ we use the Cauchy-Schwarz inequality to again obtain for $T \in U^C$

$$\begin{aligned}
(\nabla(\boldsymbol{u} - R_h(\boldsymbol{u})), \nabla I_h^b(\boldsymbol{g}_{0,h}))_T &= (\boldsymbol{u} - R_h(\boldsymbol{u}), [\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!])_{\partial T} + (\boldsymbol{u} - R_h(\boldsymbol{u}), -\Delta I_h^b(\boldsymbol{g}_{0,h}))_T \\
&= \|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{L^2(\partial T)} \|[\![\partial_n I_h^b(\boldsymbol{g}_{0,h})]\!]\|_{L^2(\partial T)} \\
&\quad\quad + \|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{L^2(T)} \|\Delta I_h^b(\boldsymbol{g}_{0,h})\|_{L^2(T)} \\
&\leq C\Big(h^{-1/2}\|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{L^2(T)} + h^{1/2}\|\boldsymbol{u}\|_{H^1(T)} + h^{1/2}\|R_h(\boldsymbol{u})\|_{H^1(T)}\Big) \\
&\quad\quad h^{-1/2}\|\nabla I_h^b(\boldsymbol{g}_{0,h})\|_{L^2(T)} + \|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{L^2(T)} \|\Delta I_h^b(\boldsymbol{g}_{0,h})\|_{L^2(T)} \\
&\leq Ch^{-1}\Big(\|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{L^2(T)} + h\|\boldsymbol{u}\|_{H^1(T)} + h\|R_h(\boldsymbol{u})\|_{H^1(T)}\Big) \\
&\quad\quad \|\nabla I_h^b(\boldsymbol{g}_{0,h})\|_{L^2(T)}.
\end{aligned}$$

Here we used a cell trace inequality as in [52, Lemma 7.2] in $L^2$ and an inverse estimate for the finite element functions. After we sum up, the factor $\|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{L^2(\Omega)}$ can be treated as in the case of Taylor-Hood finite elements and we have by $H^1(\Omega)^d$ stability of $R_h$ that $\|R_h(\boldsymbol{u})\|_{H^1(\Omega)} \leq C\|\boldsymbol{u}\|_{H^1(\Omega)}$.

For $\|\nabla I_h^b(\boldsymbol{g}_{0,h})\|_{L^2(T)}$ we may proceed as in (A.8) and below to obtain

$$\|\nabla I_h^b(\boldsymbol{g}_{0,h})\|_{L^2(T)} \leq \|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(T)} + h\|\nabla^2 \boldsymbol{g}_0\|_{L^2(T)}. \tag{A.9}$$

The necessary $H^1$ stability estimate for $\|I_h^b(\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0))\|_{L^2(T)}$ and the $H^1$ estimate of $\|\nabla I_h^b(\boldsymbol{g}_0)\|_{L^2(T)}$ can be derived for our choice of $I_h$ as in [11, (2.15), (2.16)]. To finally bound (A.9) note that for $T \in U^C$ the distance $\mathrm{dist}(T, \partial D_1) > 0$ and depends on $\varrho$. Thus we can insert $\sigma^{3/2}$ into (A.9) as in (3.3.6) and get

$$\|\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(T)} + h\|\nabla^2 \boldsymbol{g}_0\|_{L^2(T)} \leq C_\varrho\Big(\|\sigma^{3/2}\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(T)} + h\|\sigma^{3/2}\nabla^2 \boldsymbol{g}_0\|_{L^2(T)}\Big).$$

After summing up, the term $\|\sigma^{3/2}\nabla(\boldsymbol{g}_{0,h} - \boldsymbol{g}_0)\|_{L^2(\Omega)}$ can then be bounded using Corollary 3.25 and $\|\sigma^{3/2}\nabla^2 \boldsymbol{g}_0\|_{L^2(\Omega)}$ can be bounded using Lemma 3.29.

The result then follows as in the proof of Theorem 3.15 after summing over all $T \in \mathcal{T}_h$. Note that we again pick up an additional logarithmic factor because of the behavior of the local Ritz projection $R_h$ for *linear* Lagrange elements (cf. Proposition 3.27). □

*Remark* A.3 The two-dimensional case can be argued similarly using the arguments above and from Section 3.6.

*Remark* A.4 In addition to the Taylor-Hood finite element and the mini element it has been shown in [64] that the Bernardi-Raugel element fulfills some of the Assumptions 3.4–3.7 and for this discretization the velocity $\boldsymbol{u}_h$ is of the form

$$\boldsymbol{u}_h = I_h(\boldsymbol{u}_h) + I_h^{BR}(\boldsymbol{u}_h).$$

The extension of the results in Chapter 3 to the Bernardi-Raugel element is still an open question.

# Appendix B.

# Best-approximation estimate for general $\boldsymbol{v_h}$ and Taylor-Hood finite elements

In this section we discuss details of the proof of the best-approximation result in Remark 3.18 for general $\boldsymbol{v}_h \in \boldsymbol{V}_h$. In particular, for the discretization with Taylor-Hood finite elements we want to prove the following lemma.

**Lemma B.1** *Let $\boldsymbol{u}$ be a divergence-free continuous function in $(H_0^1(\Omega)\times L^\infty(\Omega))^d$, $d \in \{2,3\}$, and $\boldsymbol{v}_h \in \boldsymbol{V}_h$, then there exists $\tilde{\boldsymbol{v}}_h \in \boldsymbol{X}_h$, the space of discretely divergence-free finite element functions, such that*

$$\|\boldsymbol{u} - \tilde{\boldsymbol{v}}_h\|_{L^\infty(\Omega)} \leq C\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^\infty(\Omega)} \tag{B.1}$$

*for $C$ independent of $h$, polynomial degree $k \geq 2$ for $d = 2$, and $k \geq 3$ for $d = 3$.*

*Proof.* The proof will follow the arguments in [65, Section 3.3] for the construction of such a $\tilde{\boldsymbol{v}}_h$. We then argue that (B.1) holds for $\tilde{\boldsymbol{v}}_h$, using the assumption that $\nabla \cdot \boldsymbol{u} = 0$. We discuss the case for elements in $\boldsymbol{V}_h$ in the cases $k \geq 2$ for $d = 2$ and $k \geq 3$ for $d = 3$. The case $k = 2$ for $d = 3$ may be argued similarly but relies on macro-elements consisting of multiple cells $T \in \mathcal{T}_h$, for which the first step in the construction of $\tilde{\boldsymbol{v}}_h$ would need to be modified. For the details in the case $k = 2$, $d = 3$ we refer to [65, Section 3.3].

We begin our construction of $\tilde{\boldsymbol{v}}_h$ with a construction of a $\bar{\boldsymbol{v}}_h$ such that

$$\int_T \nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h) d\boldsymbol{x} = 0. \tag{B.2}$$

To that end we modify $\boldsymbol{v}_h$ via the basis function at a degree of freedom in the interior of each face of $\partial T$ to achieve the following, to (B.2) equivalent, condition

$$\int_{\partial T} (\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\boldsymbol{n} d\boldsymbol{s} = 0, \tag{B.3}$$

where $\boldsymbol{n}$ is the respective normal vector.

For $k \geq 2$ and $d = 2$ as well as for $k \geq 3$ and $d = 3$ such a degree of freedom on a face $\omega_j$, with $1 \leq j \leq K_T$ and $K_T$ the number of faces of $T$, is always available and we denote the function which has value one at this degree of freedom and zero at all others by $b_j$.

If we now choose on $T$

$$\bar{\boldsymbol{v}}_h = \boldsymbol{v}_h + \sum_{j=1}^{K_T} \boldsymbol{c}_j b_j,$$

with

$$\boldsymbol{c}_j = \frac{1}{\int_{\omega_j} b_j d\boldsymbol{s}} \int_{\omega_j} (\boldsymbol{u} - \boldsymbol{v}_h)\boldsymbol{n} d\boldsymbol{s},$$

we have that condition (B.3) holds.

Now, since (B.2) is satisfied for $\boldsymbol{u} - \bar{\boldsymbol{v}}_h$, we can follow the arguments in [67, Theorem 2.1] and construct an operator $\boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h) \in \boldsymbol{V}_h(\mathcal{O}_i)$ on a macro-element $\mathcal{O}_i$ such that

$$\sum_{T \in \mathcal{T}_h} \int_T q_h \nabla \cdot \boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h) d\boldsymbol{x} = \sum_{T \in \mathcal{T}_h} \int_T q_h \nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h) d\boldsymbol{x} \qquad \forall q_h \in \tilde{M}_h.$$

Here $\tilde{M}_h$ denotes the functions $q_h \in M_h$ modified such that they have mean value zero on every $T$. For the precise definition we refer to [67, p. 6]. The existence of $\boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)$ relies on a local inf-sup condition on each $\mathcal{O}_i$

$$\inf_{q_h \in \tilde{M}_h(\mathcal{O}_i)} \sup_{v_h \in \tilde{\boldsymbol{V}}_h(\mathcal{O}_i)} \frac{(q_h, \nabla \cdot \boldsymbol{v}_h)_{\mathcal{O}_i}}{\|\nabla \boldsymbol{v}_h\|_{L^2(\mathcal{O}_i)} \|q_h\|_{L^2(\mathcal{O}_i)}} \geq \tilde{\beta},$$

which holds for Taylor-Hood finite elements in two and three dimensions (cf. [67, Section 3]). The space $\tilde{\boldsymbol{V}}_h(\mathcal{O}_i)$ denotes all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ which are zero on $\partial\mathcal{O}_i$. The definition of $\mathcal{O}_i$ in [67, (2.10)] implies that the maximal number of cells $T$ in $\mathcal{O}_i$ is bounded as well as the maximal number of macro-elements $\mathcal{O}_j$ that may intersect with $\mathcal{O}_i$. We denote an upper bound to these quantities by $\tilde{L}$.

Then, we may define $\tilde{\boldsymbol{v}}_h$ as

$$\tilde{\boldsymbol{v}}_h = \bar{\boldsymbol{v}}_h + \boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)$$

and by construction $\tilde{\boldsymbol{v}}_h$ is then discretely divergence-free, i.e., $\tilde{\boldsymbol{v}}_h \in \boldsymbol{X}_h$.

We now proceed to show (B.1). Expanding $\tilde{\boldsymbol{v}}_h$ we get

$$\|\boldsymbol{u} - \tilde{\boldsymbol{v}}_h\|_{L^\infty(\Omega)} \leq \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^\infty(\Omega)} + \max_{T \in \mathcal{T}_h} \left\| \sum_{j=1}^{K_T} \boldsymbol{c}_j b_j \right\|_{L^\infty(\Omega)} + \|\boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^\infty(\Omega)}.$$

Thus, to see (B.1) it suffices to show

$$\left\| \sum_{j=1}^{K_T} \boldsymbol{c}_j b_j \right\|_{L^\infty(\Omega)} \leq C \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^\infty(\Omega)} \tag{B.4}$$

and

$$\|\boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^\infty(\Omega)} \leq C \|\boldsymbol{u} - \bar{\boldsymbol{v}}_h\|_{L^\infty(\Omega)} \tag{B.5}$$

since (B.5) can be bounded using (B.4).

To derive (B.4) note that the value of $b_j$ is independent of $h$ and $\int_{\omega_j} b_j d\boldsymbol{s}$ is bounded from below by $Ch^{d-1}$. This can be seen, for example, by transforming to the reference triangle. Then, we can estimate

$$\|\boldsymbol{c}_j b_j\|_{L^\infty(\Omega)} \leq C|\boldsymbol{c}_j| \leq Ch^{-d+1} \int_{\omega_j} d\boldsymbol{s} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^\infty(\omega_j)} \leq C\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^\infty(\omega_j)} \leq C\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^\infty(T)}, \tag{B.6}$$

where we used the continuity of $\boldsymbol{u} - \boldsymbol{v}_h$ in the last step.

Now, for $\boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)$ we arrive by following the arguments in the proof of [67, Theorem 2.1, (2.26)] at

$$\begin{aligned} \|\boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^\infty(\Omega)} &= \max_{\mathcal{O}_i} \|\boldsymbol{C}_h(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^\infty(\mathcal{O}_i)} \\ &\leq Ch^{1-d/2} \max_{\mathcal{O}_i} \left( \sum_{T \in D_i} \|\nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|^2_{L^2(T)} \right)^{1/2}. \end{aligned} \tag{B.7}$$

Here $D_i$ contains all cells in $\mathcal{O}_i$ and in the macro elements $\mathcal{O}_j$ which intersect with $\mathcal{O}_i$. The number of these cells can then be bounded by $\tilde{L}^2$.

Thus, showing an estimate for $\|\nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^2(T)}$ is enough to see (B.5). On each element $T$ we have

$$\begin{aligned} \|\nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^2(T)} &= \sup_{q \in L^2(T)} \frac{(\nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h), q)}{\|q\|_{L^2(T)}} = \sup_{q \in L^2(T)} \frac{(-\nabla \cdot \bar{\boldsymbol{v}}_h, q)}{\|q\|_{L^2(T)}} \\ &= \sup_{q \in L^2(T)} \frac{(-\nabla \cdot \bar{\boldsymbol{v}}_h, \tilde{r}_h(q))}{\|q\|_{L^2(T)}} = \sup_{q \in L^2(T)} \frac{(\nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h), \tilde{r}_h(q))}{\|q\|_{L^2(T)}}, \end{aligned}$$

where we used that $\boldsymbol{u}$ is divergence-free and that $\nabla \cdot \bar{\boldsymbol{v}}_h$ is a polynomial on $T$ and thus we can introduce with $\tilde{r}_h$ the respective $L^2$ projection into the space of polynomials on $T$ and add $\nabla \cdot \boldsymbol{u}$ back in. Now, we may integrate by parts and get after applying Hölder's inequality

$$\begin{aligned} \|\nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^2(T)} &= \sup_{q \in L^2(T)} \|q\|^{-1}_{L^2(T)} \left( -(\boldsymbol{u} - \bar{\boldsymbol{v}}_h, \nabla \tilde{r}_h(q))_T + \int_{\partial T} \tilde{r}_h(q)(\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\boldsymbol{n} d\boldsymbol{s} \right) \\ &\leq \sup_{q \in L^2(T)} \|q\|^{-1}_{L^2(T)} \Big( \|\boldsymbol{u} - \bar{\boldsymbol{v}}_h\|_{L^\infty(T)} \|\nabla \tilde{r}_h(q)\|_{L^1(T)} \\ &\qquad + \|\boldsymbol{u} - \bar{\boldsymbol{v}}_h\|_{L^\infty(\partial T)} \|\tilde{r}_h(q)\|_{L^1(\partial T)} \Big) \\ &\leq \sup_{q \in L^2(T)} \|q\|^{-1}_{L^2(T)} \Big( \|\boldsymbol{u} - \bar{\boldsymbol{v}}_h\|_{L^\infty(T)} \|\nabla \tilde{r}_h(q)\|_{L^1(T)} \\ &\qquad + \|\boldsymbol{u} - \bar{\boldsymbol{v}}_h\|_{L^\infty(T)} \Big( h^{-1} \|\tilde{r}_h(q)\|_{L^1(T)} + \|\tilde{r}_h(q)\|_{W^{1,1}(T)} \Big) \Big) \\ &\leq \sup_{q \in L^2(T)} h^{-1} \|q\|^{-1}_{L^2(T)} \|\tilde{r}_h(q)\|_{L^1(T)} \|\boldsymbol{u} - \bar{\boldsymbol{v}}_h\|_{L^\infty(T)}. \end{aligned}$$

Here we used that $\boldsymbol{u} - \bar{\boldsymbol{v}}_h$ is continuous, a trace estimate on $T$ from [74, Proposition 4.11] and for the terms involving derivatives of $\tilde{r}_h(q)$ an inverse estimate on $T$. Finally we note that

$\|\tilde{r}_h(q)\|_{L^1(T)} \leq h^{d/2}\|\tilde{r}_h(q)\|_{L^2(T)}$ due to Hölder's inequality and the fact that $\tilde{r}_h$ is stable in $L^2(\Omega)$ delivers

$$\|\nabla \cdot (\boldsymbol{u} - \bar{\boldsymbol{v}}_h)\|_{L^2(T)} \leq h^{-1+d/2}\|\boldsymbol{u} - \bar{\boldsymbol{v}}_h\|_{L^\infty(T)}.$$

Together with (B.7) this proves (B.5) and thus we can conclude (B.1). □

*Remark* B.2 One can argue similarly as in (B.6) for the mini element.

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