

Interplay Between Homophily-Based Appraisal Dynamics and Influence-Based Opinion Dynamics: Modeling and Analysis

Fangzhou Liu¹, Shaoxuan Cui¹, Wenjun Mei^{2*}, Florian Dörfler² and Martin Buss¹

Abstract—In social systems, the evolution of interpersonal appraisals and individual opinions are not independent processes but intertwine with each other. Despite extensive studies on both opinion dynamics and appraisal dynamics separately, no previous work has ever combined these two processes together. In this paper, we propose a novel and intuitive model on the interplay between homophily-based appraisal dynamics and influence-based opinion dynamics. We assume that individuals’ opinions are updated via the influence network constructed from their interpersonal appraisals, which are in turn updated based on the individual opinions via the homophily mechanism. By theoretical analysis, we characterize the set of equilibria and some transient behavior of our model. Moreover, we establish the equivalence among the convergence of the appraisal network to social balance, the modulus consensus of individual opinions, and the non-vanishing appraisals. Monte Carlo validations further show that the non-vanishing appraisals condition holds for generic initial conditions. Compared with previous works that explain the emergence of social balance via person-to-person homophily mechanism, our model provides an alternative explanation in terms of the person-to-entity homophily mechanism. In addition, our model also describes how individuals’ opinions on multiple irrelevant issues become correlated and converge to modulus consensus over time-varying influence networks.

Index Terms—structural balance, appraisal dynamics, opinion dynamics, network multi-agent systems

I. INTRODUCTION

a) Motivation and problem description: In social sciences, it has been extensively studied, as opinion dynamics, how social influences shape individual opinions. However, few mathematical models have been established to explain how individual opinions react to interpersonal relations. In fact, such reactions are frequently observed and even being intentionally made use of. For example, politicians and news media sometimes throw certain issues to the public to generate conflicts and divisions of our society [1].

In this paper, we propose a novel model on the interplay between opinion dynamics and interpersonal relations, and investigate its consequences. We assume that the interpersonal appraisals, i.e., how much people like or dislike each other, are determined by the opinion homophily mechanism. That is, individuals holding similar opinions tend to be friendly to each other, and vice versa. In the meanwhile, the

interpersonal influences are proportional to the appraisals and construct a time-varying signed influence network, on which individual opinions are iterated via opinion dynamics.

b) Literature Review: Opinion dynamics on signed influence networks have drawn considerable attention recently [2]–[6]. Its dynamical behavior depends on whether the signed influence network satisfies *social balance* [2], i.e., whether it can be partitioned into two antagonistic factions [7], where the social links within each faction are all non-negative and the inter-faction social links are all non-positive. According to [2], opinions evolving on a connected influence network satisfying *social balance* converge to *bipartite consensus*. That is, individuals in one faction reach consensus on some opinion a , while individuals in the other faction reach consensus on $-a$. Specially, one faction can be empty, which leads to *consensus* of all individual’s opinions. If the signed influence network does not satisfy *social balance*, everyone’s opinion converges to zero.

Recently, various models have been proposed to explain how the interpersonal appraisal networks evolve to social balance, e.g., see [8]–[13]. Some of them are based on the homophily mechanism, see [12], [13]. However, the homophily mechanism in these papers is the person-to-person homophily, i.e., individuals holding similar appraisals of the others tend to be friendly to each other. Therefore, the models in [12], [13] are self-driven dynamics of the appraisal networks, with no opinion dynamics involved. As pointed out by Heider [7], the person-to-entity homophily could also play a role in shaping the interpersonal appraisals. This partly motivates our paper, where “entity” refers to individuals opinions on certain issues irrelevant to appraisals.

c) Contributions: To the best of our knowledge, our model is the first one that studies the interplay between opinion dynamics and appraisal dynamics. Theoretical analysis shows that our proposed model is well-defined and the interpersonal appraisals and individual opinions enjoy bounded behaviors. We further characterize the set of equilibria of our model and their local stabilities. Moreover, we establish the equivalence among the convergence of the appraisal network to social balance, the modulus consensus of individual opinions, and the non-vanishing appraisals, which means that eventually the magnitudes of the interpersonal appraisals are uniformly strictly lower bounded from zero. Apart from the theoretical results, numerical study shows the validity of the non-vanishing appraisals condition for almost all generic initial conditions. In terms of sociological interpretations, our model explains the emergence of social balance via the person-to-entity homophily combined with the evolution

This work was supported by the joint Sino-German Research Project, which is funded through the German Research Foundation (DFG) and the National Science Foundation China (NSFC) 61761136005.

¹F. Liu, S. Cui, and M. Buss are with the Chair of Automatic Control Engineering, Technical University of Munich, Munich, 80333 Germany {fangzhou.liu, shaoxuan.cui, mb}@tum.de

²W. Mei and F. Dörfler are with the Automatic Control Laboratory, ETH, 8092 Zurich, Switzerland {wmei, dorfler}@ethz.ch

TABLE I: Notations frequently used in this paper

$\mathbf{1}_n$	The all ones $n \times 1$ vector
$\mathbb{R} (\mathbb{Z}_{\geq 0})$	Set of real numbers (non-negative integers)
$\mathbf{0}_{m \times n}$	A zero matrix with $m \times n$ dimensions
$ X $	Entry-wise absolute value of matrix X
$\text{sgn}(X)$	Entry-wise sign of X , whose entry at i th row and j th column is $\text{sgn}(X_{ij})$
$ X _{\max} (X _{\min})$	$\max_{i,j} X_{ij} $ ($\min_{i,j} X_{ij} $)
$X_{i*} (X_{*i})$	The i th row (column) vector of X
$A < B$	$A_{ij} < B_{ij}$ for any i and j .
$\text{diag}(x)$	The diagonal matrix, whose diagonal elements are the elements of the vector x .

of individual opinions. Our model also describes how individuals' opinions on irrelevant issues eventually become correlated due to the formation of antagonistic factions.

d) Organization: The remainder of this paper is organized as follows. We introduce notations and necessary definitions in Section II as well as the model description. The main theoretical analysis are presented in Section III. Finally, we conduct the numerical experiments in Section IV to validate our model and theoretical contribution.

II. NOTATIONS, DEFINITIONS, AND MODEL SET-UP

Notations: Notations frequently used are defined in Table I and adhere closely to those in [13]. In this paper, we denote by $X = (X_{ij})_{n \times n}$ appraisal matrix for a group of n individuals. Here X_{ij} denotes individual i 's appraisal of j , i.e., $X_{ij} > 0$ ($X_{ij} < 0$ resp.) if i likes (dislikes resp.) j . $X_{ij} = 0$ is i does not know j or holds a neutral attitude towards j . The appraisal matrix X defines a weighted and directed graph $G(X)$, referred to as the *appraisal network*.

Definition 1 (Social balance [7]): An appraisal network $G(X)$ satisfies social balance if $X_{ii} > 0$ and $\text{sgn}(X_{ij}) \text{sgn}(X_{jk}) \text{sgn}(X_{ki}) = 1, \forall i, j, k \in \{1, \dots, n\}$.

Lemma 1 ([13]): For any $X \in \mathbb{R}^{n \times n}$ such that all of its entries are non-zero, $G(X)$ satisfies social balance if and only if it satisfies $X_{ii} > 0$ for any $i \in \{1, \dots, n\}$ and $\text{sgn}(X_{i*}) = \pm \text{sgn}(X_{j*})$, for all $i, j \in \{1, \dots, n\}$.

A. Model Description

Our novel model of the interplay between homophily-based appraisal dynamics and influence-based opinion dynamics is formally presented as follows.

Definition 2 (The interplay model): Let $Y(t) \in \mathbb{R}^{n \times m}$ be the opinion matrix of n agents towards m issues at time t and its entry $Y_{ij}(t)$ represents agent i 's opinion on issue j . Let $X(t)$ be the appraisal matrix at time t . Given the initial condition $Y(0) = Y_0 \in \mathcal{Y}$, where the set $\mathcal{Y} = \{Y | Y(t) \in \mathcal{S}_{nz\text{-row}}, \text{ for any } t \geq 0, \text{ with } Y(0) = Y\}$, the interplay between the appraisal matrix $X(t)$ and the opinion matrix $Y(t)$ is given by the following dynamics:

$$X(t+1) = \text{diag}(|Y(t)|\mathbf{1}_m)^{-1}Y(t)Y^\top(t), \quad (1)$$

$$W(t+1) = \text{diag}(|X(t+1)|\mathbf{1}_n)^{-1}X(t+1), \quad (2)$$

$$Y(t+1) = W(t+1)Y(t), \quad (3)$$

or, equivalently, in the entry-wise form:

$$X(t+1)_{ij} = \frac{\sum_{k=1}^m Y_{ik}(t)Y_{jk}(t)}{\|Y_{i*}(t)\|_1}, \quad (4)$$

$$W(t+1)_{ij} = \frac{X_{ik}(t+1)}{\|X_{i*}^+(t)\|_1}, \quad (5)$$

$$Y(t+1)_{ij} = \sum_{k=1}^n W_{ik}(t+1)Y_{kj}(t). \quad (6)$$

Equations (1)-(3) can be understood as the following iteration process: At each time $t+1$, via the person-to-opinion homophily mechanism, individuals form their interpersonal appraisals $X(t+1)$ based on their previous opinions $Y(t)$. Then a signed influence matrix $W(t+1)$ is constructed proportionally to the appraisal matrix $X(t+1)$. Finally, individuals update their opinions via the signed influence matrix $W(t+1)$, obeying the Altafini model [2]. Note that $|\text{diag}(|Y(t)|\mathbf{1}_m)^{-1}Y(t)|$ and $|\text{diag}(|X(t+1)|\mathbf{1}_n)^{-1}X(t+1)|$ are both row stochastic. This kind of normalization terms have been widely adopted in both appraisal dynamics [13] and opinion dynamics with antagonistic relations [14]–[16]. In addition, the assumption $Y(0) \in \mathcal{Y}$ implies that, starting with the initial condition $Y(0)$, at any time t , every agent i has an non-zero opinion on at least one of the m issues.

By combining (1)-(3), we can obtain the following model in the form of the opinion dynamics:

$$Y(t+1) = \text{diag}(|Y(t)Y^\top(t)|\mathbf{1}_n)^{-1}Y(t)Y^\top(t)Y(t). \quad (7)$$

The system (7) concludes the interplay between homophily-based appraisal dynamics (1) and influence-based opinion dynamics (3). Obviously, \mathcal{Y} is the domain of our model in (7). For the convenience of presentation, the time step t can be omitted in case of no ambiguity, i.e., $X(t+1)$, $Y(t)$, and $Y(t+1)$ are denoted as X^+ , Y , and Y^+ , respectively.

III. THEORETICAL ANALYSIS

In this section, we provide theoretical analysis regarding the interplay model (1)-(3). The following lemma presents some finite-time properties of the interplay model.

Lemma 2 (Finite-time behavior): Consider system (7) and define $f(Y) = \text{diag}(|YY^\top|\mathbf{1}_n)^{-1}YY^\top Y$. For any $Y_0 \in \mathcal{Y}$, the following statements hold:

- i) the map f is well-defined for any $Y_0 \in \mathcal{Y}$;
- ii) the solution $Y(t)$, $t \in \mathbb{Z}_{\geq 0}$, to (7) with the initial condition $Y(0) = Y_0$ exists and is unique;
- iii) the max norm of $Y(t)$ satisfies $|Y(t+1)|_{\max} \leq |Y(t)|_{\max} \leq |Y(0)|_{\max}$;
- iv) for any $c > 0$, the trajectory $cY(t)$ is the solution to (7) from initial condition $Y(0) = cY_0$.
- v) $|X(t+1)|$ is upper bounded by $|Y(t)|_{\max}$ for any $t \in \mathbb{Z}_{\geq 0}$.

Proof: According to the equation (5) and (6), $Y^+ = f(Y)$ is well-defined as long as $X^+ \in \mathcal{S}_{nz\text{-row}}$, which is naturally true by dynamics (1), since $Y(t) \in \mathcal{S}_{nz\text{-row}}$ for any $t \in \mathbb{Z}_{\geq 0}$ by assumption. This fact leads to statement i).

Statement ii) is the direct consequence of i).

By the system equations (2) and (3), we have $|Y_{ij}^+| = \frac{|\sum_{k=1}^n X_{ik}^+ Y_{kj}|}{\|X_{i*}^+\|_1} \leq \frac{\sum_{k=1}^n |X_{ik}^+| |Y_{kj}|}{\|X_{i*}^+\|_1} \leq \max_k |Y_{kj}| \leq |Y(t)|_{\max}$. It implies statement iii) immediately.

Statement iv) is obtained by replacing $Y(t)$ with $cY(t)$ on the right-hand side of (7), i.e., let $\hat{Y}(0) = cY_0$. We obtain

$$\hat{Y}(1) = \text{diag}(|c^2 Y_0 Y_0^\top| \mathbf{1}_n)^{-1} cY_0 (cY_0)^\top cY_0 = cY(1).$$

For any $t \geq 1$, $\hat{Y}(t) = cY(t)$ holds for the same calculation.

Now we prove statement v). According to equation (1),

$$|X_{ij}^+| = \frac{\sum_{k=1}^m |Y_{ik} Y_{jk}|}{\|Y_{i*}\|_1} \leq \frac{\sum_{k=1}^m |Y_{ik}| |Y_{jk}|}{\|Y_{i*}\|_1} \leq \max_k |Y_{jk}|.$$

It yields that $|X_{ij}(t+1)| \leq |Y(t)|_{\max}$ for any $t \in \mathbb{Z}_{\geq 0}$. ■

Remark 1: Specifically, in single-issue case ($m = 1$), \mathcal{S}_{nz-row} is an invariance set of the map f . For multi-issue case ($m > 1$), Monte Carlo validation indicates that, for any Y randomly picked from \mathcal{S}_{nz-row} , $f(Y) \in \mathcal{S}_{nz-row}$ holds almost surely, see Section IV-B for the simulation results.

Before embarking on the main results on the equilibrium and the convergence of the opinion matrix $Y(t)$, we introduce the concept of *non-vanishing* of the appraisal matrix $X(t)$.

Definition 3 ([13]): A time-varying appraisal matrix $X(t)$ satisfies the non-vanishing appraisal condition if $\liminf_{t \rightarrow \infty} \min_{i,j} |X_{ij}(t)| > 0$.

Proposition 1: For any initial condition $Y_0 \in \mathcal{Y}$ such that $X(t)$ satisfies the non-vanishing appraisal condition, define the set $\Upsilon = \{Y(t)\}_{t=0}^\infty$. Υ is the invariance set of the map f . Moreover, f is continuous on Υ .

Proof: Υ is the invariance set of the map f is equivalent to the statement: pick any $y \in \Upsilon$, $f(y) \in \Upsilon$ holds true. Since $X(t)$ satisfies the non-vanishing appraisal condition, $X(t)$ and $Y(t)$ must be defined at any time point $t > 0$. For any $Y(i) \in \Upsilon$, we have $Y^+ = Y(i+1) \in \Upsilon$. Thus, Υ is the invariance set of the map f .

Now, we show that a map is continuous at any isolated point. Let X_0 be any isolated point and h be a map such that $Y_0 = h(X_0)$. For any $\epsilon > 0$ and neighbor set of Y_0 : $B_{Y_0}(\epsilon)$, there always exists $\delta > 0$ and a neighbor set of X_0 : $B_{X_0}(\delta)$ such that $B_{X_0}(\delta) = \{X_0\}$. Thus, $f(B_{X_0}(\delta)) = h(X_0) = Y_0 \in B_{Y_0}(\epsilon)$.

In addition, for any $Y(i) \in \Upsilon$, $Y(i)$ is the isolated point in the set Υ , because there exists a neighborhood of $Y(i)$ which does not contain any other points of Υ . By definition, a map $g: A \rightarrow B$ is continuous iff it is continuous at each point of A . Since any map is continuous at the isolated point and Υ contains only isolated points, f is continuous on Υ . ■

Theorem 1 (Equilibrium set): Given the opinion dynamics in (7), for any $Y^* \in \{Y \mid |X^+| = |\text{diag}(|Y| \mathbf{1}_m)^{-1} Y Y^\top| > \mathbf{0}\}$, Y^* is an equilibrium if and only if Y^* is in the form of $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$, where $a_i \in \mathbb{R}$, $i = 1, \dots, m$, $\rho \in \{\pm 1\}^n$, and $\sum_{i=1}^m a_i^2 \neq 0$.

Proof: By substituting $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$ into (7), we can directly obtain that $Y^+ = Y^*$, which shows $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$ is the equilibrium of system of (7).

Now, we prove Y^* is the equilibrium only if Y^* is in this form for any $Y^* \in \{Y \mid |X^+| = |\text{diag}(|Y| \mathbf{1}_m)^{-1} Y Y^\top| > \mathbf{0}\}$. Let $i = \arg \max_k |Y_{kj}|$, we know that,

$$|Y_{ij}^*| = \frac{|\sum_{k=1}^n X_{ik}^+ Y_{kj}^*|}{\|X_{i*}^+\|_1} \leq \frac{\sum_{k=1}^n |X_{ik}^+| |Y_{kj}^*|}{\|X_{i*}^+\|_1} \leq \max_k |Y_{kj}^*|.$$

Notice that, all entries of X are non-zero under non-vanishing appraisal condition and X is obviously sign-symmetric. In order to make $|Y_{ij}^*| = \max_k |Y_{kj}^*|$ hold, the following conditions must be satisfied:

- $\text{sgn}(X_{i*}^+) = \text{sgn}(X_{*i}^+) = \pm \text{sgn}(Y_{*j}^*)$;
- All entries of Y_{*j}^* have magnitude $\|Y_{*j}^*\|_\infty$.

Therefore, for any k and l , there exist $\delta_k, \delta_l \in \{\pm 1\}$ such that $\text{sgn}(Y_{*k}^*) = \delta_k \text{sgn}(X_{i*}^+)$ and $\text{sgn}(Y_{*l}^*) = \delta_l \text{sgn}(X_{i*}^+)$. Thus we have $\text{sgn}(Y_{*k}^*) = \pm \text{sgn}(Y_{*l}^*)$ and $|Y_{ik}^*| = \|Y_{*k}^*\|_\infty$. It is equivalent that the equilibrium can only be in the form $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$. Clearly, Y^* is not allowed to be a zero matrix. Thus there holds $\sum_{i=1}^m a_i^2 \neq 0$ and hence completes the proof. ■

Remark 2: If the opinion matrix converges to a fixed point $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$, then the appraisal matrix $X(t)$ also converges to some equilibrium X^* such that $X^* = \frac{\sum_{k=1}^m a_k^2}{\sum_{k=1}^m |a_k|} \rho \rho^\top$, which satisfies social balance. Obviously, any $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$ leads to a unique X^* , but not the other way around. That is, some X^* at the steady state can be derived from different equilibria Y^* .

Remark 3: Given an equilibrium $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$, there could exist some i such that $a_i = 0$. For example, the initial condition

$$Y(0) = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -2 & 5 \\ -1 & -2 & 5 \\ 1 & 2 & 5 \end{bmatrix} \text{ leads to the equilibrium}$$

$$Y^* = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } X^* = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}.$$

Specifically, in single-issue case, since all entries of the appraisal matrix $X(t)$ are obviously non-zero, $Y^* = a \rho$ where $a \neq 0$ is the unique equilibrium for any initial condition $Y(0) \in \mathcal{S}_{nz-row}$.

Remark 4: For the case $Y^* \notin \{Y \mid |X^+| = |\text{diag}(|Y| \mathbf{1}_m)^{-1} Y Y^\top| > \mathbf{0}_{n \times n}\}$, Y^* is an equilibrium of the interplay model if $Y^* = P \hat{Y} P^\top$, where P is a permutation matrix and $\hat{Y} = \text{diag}(A_1, \dots, A_l)$ with $A_k = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$ and $l \geq 2$. The corresponding $X^* = P \hat{X} P^\top$, where P is a permutation matrix and \hat{X} is a block diagonal matrix with blocks of the form $\alpha \rho \rho^\top$ and at least 2 blocks. X^* doesn't satisfy social balance and non-vanishing appraisal condition. However, for any generic initial condition, with 99% confidence level, the system does not converge to this equilibrium. This will be validated in the section IV-B by Monte Carlo validation.

Before investigating the property of convergence, we provide the formal definition of modulus sign-consensus and *modulus consensus*.

Definition 4: (Modulus sign-consensus) The opinion dynamics reaches modulus sign-consensus, if there exists a time point $t_0 \geq 0$ such that the non-zero opinion matrix $Y(t)$ satisfies $\text{sgn}(Y(t)_{a*}) = \pm \text{sgn}(Y(t)_{b*})$ for any $a, b \in \{1, 2, \dots, n\}$ and $t \geq t_0$. Moreover, the opinion reaches *sign-consensus*, if $\text{sgn}(Y(t)_{a*}) = \text{sgn}(Y(t)_{b*})$ for any $a, b \in \{1, 2, \dots, n\}$; The opinion reaches *bipartite sign-consensus*, if there exist certain $a, b \in \{1, 2, \dots, n\}$ such that $\text{sgn}(Y(t)_{a*}) = -\text{sgn}(Y(t)_{b*})$.

Definition 5: (Modulus consensus) The opinion dynamics reaches modulus consensus if there exists a time point $t_0 \geq 0$ such that the non-zero opinion matrix $Y(t)$ satisfies $Y(t)_{a*} = \pm Y(t)_{b*}$ for any $a, b \in \{1, 2, \dots, n\}$ and $t \geq t_0$. Moreover, the opinion reaches consensus if $Y(t)_{a*} = Y(t)_{b*}$ for any $a, b \in \{1, 2, \dots, n\}$; The opinion reaches *bipartite consensus* if there exist certain $a, b \in \{1, 2, \dots, n\}$ such that $Y(t)_{a*} = -Y(t)_{b*}$.

Definition 4 shows two possible cases for modulus sign-consensus. The first case is that there are only one faction among all agents, in which all the agents support or are against each issue. The other case is that there are two factions among all agents, i.e., there exist at least two agents a and b such that $\text{sgn}(Y_{a*}) = -\text{sgn}(Y_{b*})$. The agents reach sign-consensus within each faction while they are against others across the factions.

Modulus consensus in Definition 5 is stricter than modulus sign-consensus since the magnitudes of opinions are taken into consideration. Note that modulus consensus in Definition 5 has been already introduced in previous literature [2], [3] for single-issue case. We generalize the definition for the multi-issue case, which better describes agents' attitudes towards a set of issues. For our system, we have confirmed that the equilibrium of opinion matrix in Theorem 1 satisfies modulus consensus, since the condition $Y_{a*}^* = \pm Y_{b*}^*$ for any a and b is apparently fulfilled. We then provide the following proposition to characterize the properties of the opinion dynamics.

Proposition 2 (Properties of modulus sign-consensus):

Consider the system in (7). For any given $t_0 \geq 0$, if $\text{sgn}(Y_{a*}(t_0)) = \pm \text{sgn}(Y_{b*}(t_0))$ for any a and b , the following statements hold true.

- i) the opinion reaches modulus sign-consensus and there holds $\text{sgn}(Y(t)) = \text{sgn}(Y(t_0))$ for all $t \geq t_0$.
- ii) for any $t \geq t_0 \geq 0$, $|Y(t)|_{\min}$ is non-decreasing and lower bounds $|X(t+1)|$.

Proof: By the model (7), there holds

$$\begin{aligned} \text{sgn}(Y_{ij}^+) &= \text{sgn}\left(\frac{\sum_{l=1}^n \sum_{k=1}^m Y_{ik} Y_{lk} Y_{lj}}{\|X_{i*}^+\|_1}\right) \\ &= \text{sgn}\left(\frac{\sum_{k=1}^n X_{ik} Y_{kj}}{\|X_{i*}^+\|_1}\right) = \text{sgn}\left(\sum_{l=1}^n \sum_{k=1}^m Y_{ik} Y_{lk} Y_{lj}\right). \end{aligned} \quad (8)$$

Firstly, we consider the case when $Y_{ij}(t_0) \neq 0$ for all $i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, m\}$. We prove statement i) by induction. Clearly, there holds $\text{sgn}(Y(t)) = \text{sgn}(Y(t_0))$ if $t = t_0$. Suppose $\text{sgn}(Y(s)) = \text{sgn}(Y(t_0))$ for some $s > t_0$.

It follows that $\text{sgn}(Y_{a*}(s)) = \pm \text{sgn}(Y_{b*}(s))$ for any agents a and b . Thus there exist $\delta_{il} = \pm 1$ such that

$$\begin{aligned} \text{sgn}(Y_{ik}(s)Y_{lk}(s)Y_{lj}(s)) \\ = \delta_{il} \text{sgn}(Y_{lk}(s))^2 \text{sgn}(Y_{lj}(s)) = \text{sgn}(Y_{ij}(s)). \end{aligned} \quad (9)$$

In conjugation with (8), it follows that $\text{sgn}(Y_{ij}(s+1)) = \text{sgn}(Y_{ij}(s))$ and hence $\text{sgn}(Y(s+1)) = \text{sgn}(Y(s))$. By induction, one can obtain $\text{sgn}(Y_{ij}(t+1)) = \text{sgn}(Y_{ij}(t_0))$ for all $t \geq t_0$, which is equivalent to statement i).

Then if there exists certain $Y_{ap} = 0$, it yields that $Y_{*p} = \mathbf{0}_{n \times 1}$. Note that the equation (9) still holds since its two sides both equal to zero. In this regard, statement i) is also true.

Now we prove statement ii). Since $\text{sgn}(Y_{a*}(t)) = \pm \text{sgn}(Y_{b*}(t))$, we have

$$\begin{aligned} |X_{ij}^+| &= \frac{|\sum_{k=1}^m Y_{ik} Y_{jk}|}{\|Y_{i*}\|_1} = \frac{\sum_{k=1}^m |Y_{ik}| |Y_{jk}|}{\|Y_{i*}\|_1} \geq \min_k |Y_{jk}| \\ &\geq |Y(t)|_{\min}. \end{aligned}$$

It follows that $\text{sgn}(X_{il}) = \text{sgn}(Y_{i*} Y_{l*}^\top) = \delta_{il}$ and $\text{sgn}(X_{ik}) = \text{sgn}(Y_{i*} Y_{k*}^\top) = \delta_{ik}$, where $\delta_{il}, \delta_{ik} \in \{\pm 1\}$.

Moreover, if all the entries of Y_{*j} are non-zero, $\text{sgn}(Y_{lj}) = \delta_{il} \text{sgn}(Y_{ij})$ and $\text{sgn}(Y_{kj}) = \delta_{ik} \text{sgn}(Y_{ij})$. Since all the index of the above relationships are arbitrary, $\text{sgn}(X_{i*}) = \text{sgn}(Y_{ij}) \text{sgn}(Y_{*j}) = \pm \text{sgn}(Y_{*j})$, which implies

$$\left| \sum_{k=1}^n X_{ik}^+ Y_{kj} \right| = \sum_{k=1}^n |X_{ik}^+| |Y_{kj}|. \quad (10)$$

Note that if all the entries of Y_{*j} are zero, the equation (10) still holds. Thus we can obtain

$$\begin{aligned} |Y_{ij}^+| &= \frac{|\sum_{k=1}^n X_{ik}^+ Y_{kj}|}{\|X_{i*}^+\|_1} = \frac{\sum_{k=1}^n |X_{ik}^+| |Y_{kj}|}{\|X_{i*}^+\|_1} \\ &\geq \min_k |Y_{kj}| \geq |Y(t)|_{\min}. \end{aligned}$$

This completes the proof of statement ii). \blacksquare

Proposition 2 implies that the set of modulus sign-consensus opinion matrices is positively invariant along the dynamics (7). Moreover, if $Y(t_0)$ satisfies modulus sign-consensus, then $Y(t) \in \mathcal{S}_{nz-row}$ and $X(t)$ satisfies the non-vanishing condition for any $t \geq 0$. In addition, as Proposition 2 implies, once the individuals form either one all-friendly faction or two antagonistic factions based on the signs of their opinions, they will stay in their factions from then on.

The following theorem characterizes the relation between social balance of the appraisal network and modulus consensus of the opinions.

Theorem 2 (Social balance and modulus consensus):

Given the model in (7), the following statements hold true

- a) Any equilibrium in the form of $Y^* = [a_1 \rho, a_2 \rho, \dots, a_m \rho]$ where $a_i \neq 0, \forall i \in \{1, 2, \dots, n\}$ and $\rho \in \{\pm 1\}^n$ is locally stable;
- b) For any $Y_0 \in \mathcal{Y}$, the following four statements are equivalent.
 - i) the solution of the appraisal matrix $X(t)$ satisfies the non-vanishing appraisal condition.

- ii) there exists $Y^* = [a_1\rho, a_2\rho, \dots, a_m\rho]$ such that $\lim_{t \rightarrow \infty} Y(t) = Y^*$ and $\lim_{t \rightarrow \infty} X(t) = X^* = \frac{\sum_{k=1}^m a_k^2}{\sum_{k=1}^m |a_k|} \rho \rho^\top$, where $a \in \mathbb{R}$, $\rho \in \{\pm 1\}^n$, and $\sum_{i=1}^m a_i^2 \neq 0$.
- iii) the opinion reaches modulus sign-consensus.
- iv) there exists $t_0 > 0$ such that $G(X(t+1))$ achieves social balance for any $t \geq t_0 > 0$.

Proof: For a), let $\Delta = (\Delta_{ij})_{n \times m}$, where $\max_i |\Delta_{ik}| = \Delta_k < |a_k|$ and $Y(0) = Y^* + \Delta$. For any $t \geq 0$, we have $\text{sgn}(Y(t)) = \text{sgn}(Y(0)) = \text{sgn}(Y^*)$ and $0 \leq |a_k| - \Delta_k \leq \min_i |Y_{ik}(t)| \leq \max_i |Y_{ik}(t)| \leq |a_k| + \Delta_k$. It implies that for any i and j , we have $Y_{ij}(t) = \alpha_{ij} \text{sgn}(Y_{ij}^*)$, where $|a_k| - \Delta_k \leq \alpha_{ij} \leq |a_k| + \Delta_k$. It follows that

$$\begin{aligned} |Y(t) - Y^*|_{\max} &= \max_{ij} |\alpha_{ij} \text{sgn}(Y_{ij}^*) - |a_j| \text{sgn}(Y_{ij}^*)| \\ &= \max_{ij} |\alpha_{ij} - |a_j|| \leq \Delta_k \leq \Delta = \max_k \Delta_k \end{aligned}$$

Accordingly, for any $\epsilon > 0$, there exists $\omega = \min(\max_k |a_k|/2, \epsilon/2)$ such that for any $Y(0)$ satisfying $|Y(0) - Y^*|_{\max} < \omega$, there holds $|Y(t) - Y^*|_{\max} < \epsilon$ for any $t \geq 0$. By definition, this shows that Y^* is locally stable if $a_i \neq 0$ for any i .

Now we prove that statement b) is true. Firstly, we show that statement i) implies ii). By Proposition 1, $f(x)$ is continuous on the set $\Upsilon = \{Y(t)\}_{t=0}^\infty$. Υ is already a compact set, since $0 \leq |Y_{ij}(t)| \leq |Y(0)|_{\max}$. Define $V(Y_{*k}) = \|Y_{*k}\|_\infty$. V is continuous on Υ similar to the proof in Proposition 1. By statement iii) in Lemma 2, $V(Y_{*k}^+) - V(Y_{*k}) \leq 0$ for any $Y \in \Upsilon \subset \mathcal{S}_{nz-row}$. According to the extended LaSalle invariance principle in the article [17], we can obtain that, for any $Y_0 \in \Upsilon$, $Y(t)$ converges to the largest invariant set \mathcal{M} of the set $\mathcal{T} = \{Y \in \Upsilon | V(Y_{*k}^+) - V(Y_{*k}) = 0\}$.

Now we characterize the largest invariant set \mathcal{M} . For any $Y \in \mathcal{M}$, $V(Y_{*j}^+) = V(Y_{*j}) = \|Y_{*j}\|_\infty$. Suppose $|Y_{ij}^+| = \max_l |Y_{lj}^+|$. We have,

$$\begin{aligned} |Y_{ij}^+| &= \frac{|\sum_{k=1}^n X_{ik}^+ Y_{kj}|}{\|X_{i*}^+\|_1} \leq \frac{\sum_{k=1}^n |X_{ik}^+| |Y_{kj}|}{\|X_{i*}^+\|_1} \\ &\leq \max_k |Y_{kj}| = \|Y_{*j}\|_\infty. \end{aligned} \quad (11)$$

Since $X(t)$ satisfies the non-vanishing condition, it implies that $\min_{i,j} |X_{ij}(t)| > 0$ when $t \rightarrow \infty$. Hence, all entries of X are non-zero. Besides, since all the inequalities in (11) must hold as equalities, it requires that $\text{sgn}(X_{i*}^+) = \text{sgn}(X_{*i}^+) = \pm \text{sgn}(Y_{*j})$ and all entries of Y_{*j} must have magnitude $\|Y_{*j}\|_\infty$. Thus we can conclude that $\mathcal{M} = \{Y = Y^* = [a_1\rho, a_2\rho, \dots, a_m\rho]$, where $0 \leq a_i \leq \|Y(0)_{*i}\|_\infty$ and not all a_i are 0, for any i . By inserting Y^* directly into the equation (1), we can obtain that $\lim_{t \rightarrow \infty} X(t) = X^* = \frac{\sum_{k=1}^m a_k^2}{\sum_{k=1}^m |a_k|} \rho \rho^\top$.

Then we illustrate that statement ii) implies iii). Since $\lim_{t \rightarrow \infty} Y(t) = Y^*$ and $\text{sgn}(Y_{a*}(t)) = \pm \text{sgn}(Y_{b*}(t))$ for any a and b after some time point $t_0 > 0$, there exists a neighbor set $\mathcal{U}(Y^*)$ such that $\text{sgn}(Y) = \text{sgn}(Y^*)$ for any $Y \in \mathcal{U}(Y^*)$. It follows that opinion reaches modulus sign-consensus.

Here we prove that statement iii) can imply iv). Since the opinion reaches modulus sign-consensus, there exists

$t_0 > 0$ such that $\text{sgn}(Y_{a*}(t)) = \pm \text{sgn}(Y_{b*}(t))$ for any a, b and $t \geq t_0$. By proposition 2, it holds that $\text{sgn}(Y(t)) = \text{sgn}(Y(t_0))$ for all $t \geq t_0$. We show $G(X(t+1))$ satisfies social balance for any $t \geq t_0 > 0$ by definition. There exist $\delta_{ij}, \delta_{jl}, \delta_{li} \in \{\pm 1\}$ such that $\text{sgn}(X_{ij}^+) = \text{sgn}(\frac{\sum_{k=1}^m Y_{ik} Y_{jk}}{\|Y_{i*}\|_1}) = \delta_{ij}$, $\text{sgn}(X_{jl}^+) = \text{sgn}(\frac{\sum_{k=1}^m Y_{jk} Y_{lk}}{\|Y_{j*}\|_1}) = \delta_{jl}$, and $\text{sgn}(X_{li}^+) = \text{sgn}(\frac{\sum_{k=1}^m Y_{lk} Y_{ik}}{\|Y_{l*}\|_1}) = \delta_{li}$. It yields that $\text{sgn}(X_{ij}^+) \text{sgn}(X_{jl}^+) \text{sgn}(X_{li}^+) = \delta_{ij} \delta_{jl} \delta_{li}$. By listing all possible results of the above product, we obtain $\delta_{ij} \delta_{jl} \delta_{li} = 1$. Hence we have $\text{sgn}(X_{ii}^+) = \frac{\sum_{k=1}^m Y_{ik} Y_{ik}}{\|Y_{i*}\|_1} > 0$. It confirms that $G(X(t+1))$ satisfies social balance for any $t \geq t_0 > 0$.

It remains to show that statement iv) can imply i). Since $G(X(t+1))$ achieves social balance for any $t \geq t_0 > 0$, there holds that $\text{sgn}(X(t+1)_{ij}) \text{sgn}(X(t+1)_{jk}) \text{sgn}(X(t+1)_{ki}) = 1 \forall i, j, k \in \{1, \dots, n\}$ for $t \geq t_0$. It follows that $|\text{sgn}(X(t+1)_{ij})| = 1$ for any i and j . Thus there must exist $\delta > 0$ such that $|X_{ij}(t+1)| > \delta$ for each $t \geq t_0$. Therefore, $\inf \min_{i,j} |X_{ij}(t+1)| \geq \delta > 0$ for each $t \geq t_0$. This completes the proof. ■

Theorem 2 reveals the equivalence among non-vanishing condition of the appraisal matrix, social balance of the appraisal network, and modulus consensus of the opinion dynamics. Specifically, in single-issue case, the opinion matrix $Y(t)$ converges to $Y^* = \frac{\|Y(0)\|_2^2}{\|Y(0)\|_1} \text{sgn}(Y(0))$ and the appraisal matrix $X(t)$ converges to $X^* = \frac{\|Y(0)\|_2^2}{\|Y(0)\|_1} \text{sgn}(Y(0)) \text{sgn}(Y(0))^\top$ in one step for any $Y_0 \in \mathcal{S}_{nz-row}$. This can be proved by substituting the initial value directly into the system equations (1)-(3).

Remark 5: In this paper we do not introduce any assumptions on the logical connections between different issues. However, as a result of the interplay between appraisals and opinions, we find that individuals' opinions on different issues are eventually correlated with each other, and their opinions, are determined by which faction they are in. This feature reflects the so-called *opinion partisanship* phenomenon in political science [18], i.e., the correlation of issue attitudes with party identification.

IV. NUMERICAL EXAMPLES AND VALIDATION

In this section, we illustrate our main results by numerical simulations. Besides, Monte Carlo simulations are conducted to support the validity of the non-vanishing condition.

A. Numerical Examples

We use here the generic initial condition. By generic initial condition, it means that each entry of $Y_0 \in \mathcal{S}_{nz-row}$ is independently randomly generated from the uniform distribution on some support $[-a, a]$. According to the Lemma 2, our model is independent of scaling. In this regard, we decide to set $a = 1$.

We characterize the general case with 9 agents and 6 issues. See Fig. 1 for the visualization of the evolution of the appraisal matrix and the opinion matrix, respectively. It can be observed that in 9 steps, the appraisal matrix $X(t)$ achieves social balance and the opinion matrix $Y(t)$ reaches bipartite consensus and opinion partisanship.

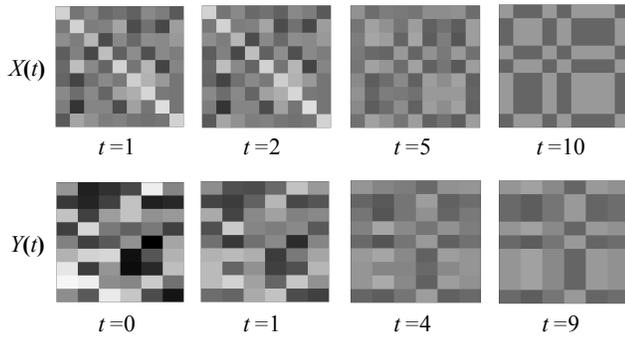


Fig. 1: Visualizations of the emergence of social balance and opinion partisanship in a group of 9 individuals discussing on 6 issues. In the visualized matrices, each grey block is one entry and the lower the entries value, the darker its color is. The first row of this figure shows that the appraisal network evolves to two antagonistic factions. The second row of this figure shows that, together with the appraisal dynamics, the individuals' opinions on different issues are eventually correlated with each other and dependent purely on what factions they are in.

B. Numerical Validation of the Non-Vanishing Appraisal Condition

We conduct Monte Carlo simulations to validate the non-vanishing appraisal condition for generic initial condition. The generic initial conditions are constructed by using the same technique in Section IV-A. For any randomly generated generic initial condition $Y(0)$, the random variable $Z(Y(0))$ is defined by

$$Z(Y(0)) = \begin{cases} 1, & \text{if } \min_{100 \leq t \leq 1000} \min_{i,j} |X_{ij}(t)| \geq 0.001 \\ 0, & \text{otherwise} \end{cases} \quad (12)$$

By running the simulation N times independently, we can obtain samples Z_1, Z_2, \dots, Z_N . It follows that the frequency of the occurrences of non-vanishing appraisal matrix is $\hat{p}_N = \sum_{i=1}^N Z_i / N$. Then the frequency \hat{p} can be utilized to approximate the probability $p = \text{Prob}(Z(Y(0)) = 1)$ with large N . Let $1 - \xi \in (0, 1)$ and $1 - \epsilon \in (0, 1)$ be accuracy and confidence level. The approximation error, i.e., $|\hat{p}_N - p|$, is bounded by ϵ with probability greater than $1 - \xi$, if the Chernoff bound is fulfilled: $N \geq \frac{1}{2\xi^2} \log \frac{2}{\xi}$. In this paper, by setting $\epsilon = \xi = 0.01$ and conducting $N = 27000$ simulations with 9 agents and 4 issues, we find that $\hat{p} = 1$. In this context, we can conclude that, for any generic initial condition with 99% confidence level, there is at least 99% probability that every entry of $|X(t)|$ is greater than a small positive scalar (0.001), for all $t \in [100, 1000]$. Notice that $Y(t) \in \mathcal{S}_{nz\text{-row}}$ for $t = [1, 999]$ holds if $X(1000)$ exists. Therefore, this simulation also validates that for any Y randomly picked from $\mathcal{S}_{nz\text{-row}}$, $f(Y) \in \mathcal{S}_{nz\text{-row}}$ holds almost surely.

V. CONCLUSION

In this paper, we propose a new discrete-time nonlinear model that characterizes the interplay between appraisal

dynamics and opinion dynamics. In particular, the appraisal dynamics is based on person-entity homophily and the opinion dynamics is an influence-based Altafini-like model on cooperative networks. Based on theoretical analysis and numerical experiments, the evolution of this interplay model show that the appraisal matrix reaches social balance and the opinion matrix achieves modulus consensus for almost any initial condition.

REFERENCES

- [1] R. Kirk and S. A. Martin, "The dark power of words: Stratagems of hate in the 2016 presidential campaign," in *The 2016 US Presidential Campaign*. Springer, 2017, pp. 205–229.
- [2] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2012.
- [3] A. Proskurnikov, A. Matveev, and M. Cao, "Opinion dynamics in social networks with hostile camps: Consensus vs. polarization," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1524–1536, 2016.
- [4] J. Liu, X. Chen, T. Başar, and M.-A. Belabbas, "Exponential convergence of the discrete- and continuous-time Altafini models," *IEEE Transactions on Automatic Control*, vol. 62, pp. 6168–6182, 2017.
- [5] F. Liu, D. Xue, S. Hirche, and M. Buss, "Polarizability, consensusability, and neutralizability of opinion dynamics on cooperative networks," *IEEE Transactions on Automatic Control*, vol. 64, no. 8, pp. 3339–3346, 2019.
- [6] G. Shi, C. Altafini, and J. S. Baras, "Dynamics over signed networks," *SIAM Review*, vol. 61, no. 2, pp. 229–257, 2019.
- [7] F. Heider, "Attitudes and cognitive organization," *Readings in attitude theory and measurement*, pp. 39–41, 1967.
- [8] T. Antal, P. L. Krapivsky, and S. Redner, "Dynamics of social balance on networks," *Physical Review E*, vol. 72, no. 3, p. 036121, 2005.
- [9] A. van de Rijdt, "The micro-macro link for the theory of structural balance," *Journal of Mathematical Sociology*, vol. 35, no. 1-3, pp. 94–113, 2011.
- [10] S. A. Marvel, J. Kleinberg, R. D. Kleinberg, and S. H. Strogatz, "Continuous-time model of structural balance," *Proceedings of the National Academy of Sciences*, vol. 108, no. 5, pp. 1771–1776, 2011.
- [11] P. Jia, N. E. Friedkin, and F. Bullo, "The coevolution of appraisal and influence networks leads to structural balance," *IEEE Transactions on Network Science and Engineering*, vol. 3, no. 4, pp. 286–298, 2016.
- [12] V. A. Traag, P. Van Dooren, and P. De Leenheer, "Dynamical models explaining social balance and evolution of cooperation," *PloS one*, vol. 8, no. 4, 2013.
- [13] W. Mei, P. Cisneros-Velarde, G. Chen, N. E. Friedkin, and F. Bullo, "Dynamic social balance and convergent appraisals via homophily and influence mechanisms," *Automatica*, vol. 110, p. 108580, 2019.
- [14] J. M. Hendrickx, "A lifting approach to models of opinion dynamics with antagonisms," in *53rd IEEE Conference on Decision and Control*. IEEE, 2014, pp. 2118–2123.
- [15] A. V. Proskurnikov and M. Cao, "Modulus consensus in discrete-time signed networks and properties of special recurrent inequalities," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, 2017, pp. 2003–2008.
- [16] W. Xia, M. Cao, and K. H. Johansson, "Structural balance and opinion separation in trust–mistrust social networks," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 46–56, 2015.
- [17] W. Mei and F. Bullo, "Lasalle invariance principle for discrete-time dynamical systems: A concise and self-contained tutorial," *arXiv preprint arXiv:1710.03710*, 2017.
- [18] D. Baldassarri and A. Gelman, "Partisans without constraint: Political polarization and trends in american public opinion," *American Journal of Sociology*, vol. 114, no. 2, pp. 408–446, 2008.