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Moduli of supersingular Enriques surfaces

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Contents

Introduction

Motivation

One of the central and classic topics in algebraic geometry is that of the classification of projective algebraic varieties over a base field k. It turns out that this is a very complicated problem, thus has to be approached step by step.

Curves and abelian varieties. There are two kinds of algebraic varieties that are comparatively well understood: algebraic curves, which are the one-dimensional algebraic varieties, and abelian varieties. The latter are varieties which carry the structure of an abelian group.

In the case of curves, we first associate to an algebraic curve C its genus $g(C) = h^1(\mathcal{O}_C)$. In [DM69] it was proved that the moduli problem of smooth algebraic curves of genus g is a smooth algebraic stack $\mathcal{M}_g$ of dimension $3g-3$ and it has a natural compactification $\overline{\mathcal{M}}_g$ which is a separated Deligne-Mumford stack. An even older result, going back to Riemann in the 19th century, is the fact that this moduli problem has a coarse moduli space M_g which is a quasi-projective variety over k [Mum65]. In layman's terms, while $\mathcal{M}_g$ is a sufficiently geometric object and describes all families of smooth algebraic curves of genus g, it is also highly abstract. The space M_g on the other hand is easier to describe and work with explicitly, but only parametrizes curves defined over a point instead of general families of curves.

In the case of abelian varieties, we consider abelian varieties X of a fixed dimension g together with a so-called principal polarization. This is the choice of an isomorphism $p\colon X \xrightarrow{\simeq} X^t$, where X^t is the dual abelian variety of X. The corresponding moduli problem $\mathcal{A}_g$ is a separated Deligne-Mumford stack of dimension $\frac{g(g+1)}{2}$ and has a coarse moduli space A_g which is a quasi-projective variety over the base field k [Mum65].

To an algebraic curve C of genus g we associate its Jacobian variety $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$ which is an abelian variety of dimension g and has a canonical principal polarization λ_C. We therefore obtain an induced map of coarse moduli spaces $T\colon M_g \longrightarrow A_g$. The celebrated Torelli theorem states that this map is an injection on points.

THEOREM. *[Mil86, Theorem 12.1] Two curves C and C' over an algebraically closed field k are isomorphic if and only if the principally polarized Jacobians $(\mathrm{Jac}(C), \lambda_C)$ and $(\mathrm{Jac}(C'), \lambda_{C'})$ are isomorphic.*

The upshot of this result is that we can now understand algebraic curves and their morphisms by looking at linear algebra data associated to them since the moduli space A_g describes abelian varieties in terms of linear algebra data.

K3 surfaces and Enriques surfaces. After looking at curves and abelian varieties, we might ask which other classes of algebraic varieties there exist for which we have a good chance of finding reasonably well-behaved moduli problems. In this thesis, we discuss K3 surfaces and Enriques surfaces. A K3 surface is a two-dimensional algebraic variety with trivial canonical bundle and irregularity zero. Closely related are Enriques surfaces, which can be characterized as quotients of K3 surfaces by a fixed-point free involution.

A feature of K3 surfaces is that they are strongly related to abelian varieties in two ways. On one hand, the prime examples of K3 surfaces are the so-called Kummer surfaces, which are constructed from abelian surfaces. Over the complex numbers, it is even true that Kummer surfaces are dense in the period space of all K3 surfaces [Huy16, Remark 13.3.24]. On the other hand, there exists the Kuga-Satake construction, which associates to a K3 surface X an abelian variety $\mathrm{KS}(X)$ by using the Hodge cohomology associated to X, and this construction is, again, faithful. The original work in characteristic zero was [KS67], and for more details we refer to [Huy16, Chapter §4], for the precise result in characteristic $p \geq 3$, where the construction is not explicit, we refer to [MP15]. It is therefore not surprising that, at least in characteristic zero, there is a Torelli theorem for K3 surfaces [PSS71], [BR75].

THEOREM. *[PSS71] [BR75] Two complex K3 surfaces X and X' are isomorphic if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(X',\mathbb{Z})$. For any $\psi\colon H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(X',\mathbb{Z})$ mapping the ample cone of X to the ample cone of X', there exists a unique $f\colon X \xrightarrow{\simeq} X'$ with $f^* = \psi$.*

This allows us to understand complex K3 surfaces in terms of linear algebra data associated with them. The Torelli theorem can be used to show that the separated Deligne-Mumford stack $\mathcal{M}_d$ of K3 surfaces together with a polarization of degree $2d$ [Riz06] has a coarse moduli space M_d, which is a quasi-projective variety of dimension 19 over $\mathbb{C}$ [PSS71].

Over a field of characteristic $p \neq 2$, Enriques surfaces are precisely the quotients of K3 surfaces by fixed-point free involutions. Using this connection between Enriques surfaces and K3 surfaces, Namikawa proved a Torelli theorem for complex Enriques surfaces and showed that there is a 10-dimensional quasi-projective variety that is a coarse moduli space for complex Enriques surfaces [Nam85]. If Y is an Enriques surface, then its Neron-Severi group $\mathrm{NS}(Y)$ is isomorphic to the lattice $\Gamma' = \Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ with $\Gamma = U_2 \oplus E_8(-1)$. By the Torelli theorem for complex K3 surfaces, fixed-point free involutions of a K3 surface X can then be characterized in terms of certain embeddings $\Gamma(2) \hookrightarrow \mathrm{NS}(X)$.

Supersingular K3 surfaces. The situation over base fields of positive characteristic is more complicated, and we turn our focus towards so-called supersingular K3 surfaces.

A basic invariant associated with a K3 surface X is the rank $\rho(X)$ of its Néron-Severi group. Since $\mathrm{Pic}X$ is embedded in $H^2(X)$ via the first Chern map and we always have $\mathrm{rk}(H^2(X)) = 22$, we have the inequality $\rho(X) \leq 22$. A K3 surface X with $\rho(X) = 22$ is called *supersingular*. Over a base field of characteristic zero we even have the stronger inequality $\rho(X) \leq 20$, thus supersingularity is a phenomenon that can only occur in positive characteristic.

For supersingular K3 surfaces over an algebraically closed field of characteristic at least 3, crystalline cohomology plays a role similar to the role of Hodge cohomology in characteristic zero.

Ogus proved a Torelli theorem for supersingular K3 surfaces [Ogu83] that shows supersingular K3 surfaces are determined by their corresponding K3 crystals.

THEOREM. *[Ogu83] Two supersingular K3 surfaces X and X' are isomorphic if and only if there exists an isomorphism of crystals $H^2_{\mathrm{crys}}(X/W) \xrightarrow{\simeq} H^2_{\mathrm{crys}}(X/W)$. Furthermore, for any isomorphism $\psi\colon H^2_{\mathrm{crys}}(X/W) \xrightarrow{\simeq} H^2_{\mathrm{crys}}(X/W)$ mapping $\mathrm{NS}(X)$ to $\mathrm{NS}(X')$ and the ample cone of X to the ample cone of X' there exists a unique $f\colon X \xrightarrow{\simeq} X'$ with $f^* = \psi$.*

For a K3 lattice N, an *N-marking* of a supersingular K3 surface X is an embedding of lattices $\gamma\colon N \hookrightarrow \mathrm{NS}(X)$. Supersingular K3 surfaces are stratified by the Artin invariant σ, where $-p^{2\sigma}$ is the discriminant of $\mathrm{NS}(X)$. We always have $1 \leq \sigma \leq 10$ [Art74].

A version of Ogus' Torelli theorem states that for families of N-marked supersingular K3 surfaces of Artin invariant at most σ, there exists a fine moduli space $\mathcal{S}_\sigma$ that is a smooth scheme of dimension $\sigma - 1$, locally of finite type, but not separated. There is an étale surjective period map $\pi_\sigma\colon \mathcal{S}_\sigma \longrightarrow \mathcal{M}_\sigma$ from $\mathcal{S}_\sigma$ to a period scheme $\mathcal{M}_\sigma$. The latter is smooth and projective of dimension $\sigma - 1$ over $\mathbb{F}_p$ and is a moduli space for marked K3 crystals. The functors represented by $\mathcal{S}_\sigma$ and $\mathcal{M}_\sigma$ have interpretations in terms of so-called *characteristic subspaces* of $pN^\vee/pN$, thus again we may understand N-marked supersingular K3 surfaces in terms of linear algebra associated to them.

Results

This is where our work starts. In particular, little research has been done regarding Enriques quotients of supersingular K3 surfaces so far. We are motivated by the results that have been obtained in characteristic zero by using the Torelli theorem and try to get similar results for supersingular K3 surfaces with the aid of the supersingular Torelli theorem.

On the number of Enriques quotients of a K3 surface. In chapter 1, we are concerned with the number of isomorphism classes of Enriques quotients for a given K3 surface.

If X is a K3 surface over an arbitrary field k and $\iota\colon X \longrightarrow X$ is an involution without fixed points, then the quotient variety $X/\langle\iota\rangle$ is an Enriques surface. For any Enriques surface Y over a field of characteristic $p \neq 2$, there exists (up to isomorphism) a unique K3 surface X such that Y is isomorphic to such a quotient $X/\langle\iota\rangle$. In other words, any Enriques surface has a unique K3 cover. We may now ask, given a K3 surface X, how many isomorphism classes of Enriques surfaces Y there are, such that there exists a fixed-point free involution $\iota\colon X \to X$ and an isomorphism $Y \cong X/\langle\iota\rangle$.

By the Torelli theorem for complex K3 surfaces, fixed-point free involutions of a complex K3 surface X can be characterized in terms of primitive embeddings $\Gamma(2) \hookrightarrow \mathrm{NS}(X)$ without vectors of self-intersection -2 in the complement of $\Gamma(2)$. Denoting the set of all such embeddings by $\mathfrak{M}$, Ohashi [Oha07] used this connection to prove the following formula, which yields an upper bound for the number of isomorphism classes of Enriques quotients of any complex K3 surface and is an equality for generic K3 surfaces.

THEOREM. *[Oha07, Theorem 2.3] Let X be a complex K3 surface. Let $M_1, \ldots, M_k \in \mathfrak{M}$ be a complete set of representatives for the action of $O(\mathrm{NS}(X))$ on $\mathfrak{M}$. For each $j \in \{1, \ldots, k\}$, we* *let*

$$K^{(j)} = \{\psi \in O(\mathrm{NS}(X)) \mid \psi(M_j) = M_j\}$$

be the stabilizer of M_j and $\mathrm{pr}(K^{(j)})$ be its canonical image in $O(q_{\mathrm{NS}(X)})$. Then we have an inequality

$$\#\{\text{Enriques quotients of } X\} \leq \sum_{j=1}^{k} \#\left(O\left(q_{\mathrm{NS}(X)}\right)/\mathrm{pr}\left(K^{(j)}\right)\right).$$

If X is such that the canonical morphism $\psi\colon O(\mathrm{NS}(X)) \to O\left(q_{\mathrm{NS}(X)}\right)$ is surjective and for each automorphism $\theta \in \mathrm{Aut}(X)$ the induced automorphism on the quotient $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$ is either the identity or multiplication by -1, then the inequality above becomes an equality.

In particular, it follows from the theorem above that the number of Enriques quotients of a complex K3 surface is finite.

We now want to understand the situation for K3 surfaces over fields of positive characteristic. Some of our results might already be known to the experts, but we could not find them in the literature. We use results by Lieblich and Maulik [LM11] about Neron-Severi group preserving lifts of K3 surfaces to characteristic zero to prove the following result.

THEOREM (see Theorem 1.3). *Let X be a K3 surface over an algebraically closed field k. If X is of finite height, then the number of isomorphism classes of Enriques quotients of X is finite.*

For many K3 surfaces of finite height, there exist special lifts to characteristic zero that allow to compare their Enriques involutions. In particular, the situation for K3 surfaces of finite height should be very similar to the situation for those in characteristic zero and we refer to Remark 1.4 for details.

In view of these results, we then turn our focus towards Enriques quotients of (Shioda-) supersingular K3 surfaces over fields of characteristic $p \geq 3$. Using Ogus' crystalline Torelli theorem for supersingular K3 surfaces [Ogu83], we prove a formula for an upper bound of Enriques quotients of a supersingular K3 surface analogously to the results of Ohashi in the complex case.

THEOREM (see Theorem 1.16). *Let k be an algebraically closed field of characteristic $p \geq 3$ and let X be a supersingular K3 surface over k. Let $M_1, \ldots, M_l \in \mathfrak{M}$ be a complete set of representatives for the action of $O(\mathrm{NS}(X))$ on $\mathfrak{M}$. For each $j \in \{1, \ldots, l\}$, we let*

$$K^{(j)} = \{\psi \in O(\mathrm{NS}(X)) \mid \psi(M_j) = M_j\}$$

be the stabilizer of M_j and $\mathrm{pr}(K^{(j)})$ be its canonical image in $O(q_{\mathrm{NS}(X)})$. Then we have inequalities

$$l \leq \#\{\text{Enriques quotients of } X\} \leq \sum_{j=1}^{l} \#\left(O\left(q_{\mathrm{NS}(X)}\right)/\mathrm{pr}\left(K^{(j)}\right)\right).$$

If X is such that for each automorphism $\theta \in \mathrm{Aut}(X)$ the induced automorphism on the quotient $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$ is either the identity or multiplication by -1, then the inequality above becomes an equality on the right side.

REMARK. It essentially follows from results of Nygaard [Nyg80] that the formula yields an equality on the right side in the generic case.

We then turn towards applications. The following result is due to Jang [Jan15].

THEOREM. *[Jan15, Corollary 2.4] Let X be a supersingular K3 surface over an algebraically closed field k of characteristic $p \geq 3$. Then X has an Enriques quotient if and only if the Artin invariant σ of X is at most 5.*

The proof of the above proposition uses lifting to characteristic zero. In an earlier article, [Jan13] Jang proved the following weaker version of the theorem above via a lattice theoretic argument.

PROPOSITION. *[Jan13, Theorem 4.5, Proposition 3.5] Let k be an algebraically closed field of characteristic p and let X be a supersingular K3 surface of Artin invariant σ. If $\sigma = 1$, then X has an Enriques involution. If $\sigma \in \{3, 5\}$, and $p = 11$ or $p \geq 19$, then X has an Enriques involution. If $\sigma \in \{2, 4\}$, and $p = 19$ or $p \geq 29$, then X has an Enriques involution. If $\sigma \geq 6$, then X has no Enriques involution.*

The proof of the previous proposition in [Jan13] boils down to the following: if X is a supersingular K3 surface of Artin invariant $\sigma \leq 5$, we need to show that there exists a primitive embedding of lattices $\Gamma(2) \hookrightarrow \mathrm{NS}(X)$ without any vector of self-intersection -2 in the complement of $\Gamma(2)$. Jang proved that such embeddings exist when the characteristic of the base field is large enough, but the same argument does not work over fields of small characteristic. With the help of the algebra software MAGMA, we explicitly show that such embeddings exist in the remaining cases. Hence our results, combined with Jang's, yield a new proof for [Jan15, Corollary 2.4] that does not rely on previous results over fields of characteristic zero.

Having established that the set of isomorphism classes of Enriques quotients of a supersingular K3 surface X of Artin invariant $\sigma \leq 5$ is always nonempty, we are now interested in calculating some explicit numbers. In practice it turns out that this is a hard problem, however when the characteristic p of the ground field is small, we found the following lower bounds with the help of MAGMA.

PROPOSITION (see Proposition 1.22). *For the number of isomorphism classes of Enriques quotients of a supersingular K3 surface of Artin invariant σ over an algebraically closed ground field k of characteristic p we found the following weak lower bounds $\mathrm{Rep}(p, \sigma)$:*

TABLE 1. Some results for the lower bounds $\mathrm{Rep}(p, \sigma)$

p	$\sigma = 1$	$\sigma = 2$	$\sigma = 3$	$\sigma = 4$	$\sigma = 5$
3	2	12	30	20	7
5	10	222	862	302	24
7	42	3565	?	4313	81
11	256	?	?	?	438
13	537	?	?	?	866
17	2298	?	?	?	2974

The situation in the case where $p = 3$ and $\sigma = 1$ is simple enough that calculating the number of isomorphism classes of Enriques quotients of a K3 surface in this case becomes feasible, and we obtain the following result.

THEOREM (see Theorem 1.26). *There are exactly two isomorphism classes of Enriques quotients of the supersingular K3 surface X of Artin invariant 1 over an algebraically closed field k of characteristic 3.*

A fibration on the period space of supersingular K3 surfaces. In chapter 2, we are concerned with the period space of N-marked supersingular K3 surfaces.

It is well known that there is a stratification $\mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \hookrightarrow \mathcal{M}_3 \hookrightarrow \ldots$ on the period space of supersingular K3 surfaces and, in [Lie15b], it was claimed that to this stratification there exist sections $\varpi_\sigma\colon \mathcal{M}_\sigma \to \mathcal{M}_{\sigma-1}$ that turn $\mathcal{M}_\sigma$ into a $\mathbb{P}^1$-bundle over $\mathcal{M}_{\sigma-1}$. There was an error in the proof of this statement, however, and so far it has only been known that the ϖ_σ exist as rational maps [BL18].

When defining the period space $\mathcal{M}_\sigma$ we first have to fix a supersingular K3 lattice N_σ of Artin invariant σ. When defining L_σ to be the quotient $pN_\sigma^\vee/pN_\sigma$, the scheme $\mathcal{M}_\sigma$ represents the functor which associates to an algebra A the set of characteristic generatrices, that means certain isotropic subspaces, in $L_\sigma \otimes A$. After fixing a basis $\{v, \varphi, e_3, \ldots, e_n\}$ of L_σ such that $v^2 = \varphi(v)^2 = 0$, $v \cdot \varphi(v) = 1$, $v \cdot e_i = \varphi \cdot e_i = 0$ for all i and such that the set $\{e_i\}_{i=3,\ldots,n}$ is an orthonormal system, we obtain the following result.

THEOREM (see Theorem 2.1). *For any $3 \leq i \leq n$ there exists a surjective morphism*

$$\tilde{\Phi}_i\colon \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$$

and an open subset $U_i \subseteq \mathcal{M}_\sigma$ such that for any $\mathbb{F}_p$-algebra A and any $G \in U_i(A)$ we have

$$\tilde{\Phi}_i(G) = \left(G \cap \langle e_i\rangle^\perp \otimes A\right) / \left(\langle e_i\rangle \otimes A\right).$$

Here, the definition of the $\tilde{\Phi}_i$ on the open subschemes U_i agrees with the definition of ϖ_σ in [Lie15b]. We prove the theorem via an inductive argument, where we first have to treat the cases $\sigma = 1$ to $\sigma = 4$ explicitly. As a byproduct of our proof, we also obtain explicit descriptions of the moduli schemes $\mathcal{M}_\sigma$ as closed subvarieties of Grassmannians for these σ. For larger σ, morally speaking, we have enough rational maps of the kind $\tilde{\Phi}_i\colon \mathcal{M}_\sigma \dashrightarrow \mathcal{M}_{\sigma-1}$ to control the situation in terms of their images, and we can do induction.

In general, the resulting fibration $\tilde{\Phi}_i$ will not be a $\mathbb{P}^1$-bundle any more. Instead we have the following result on the fibers of the $\tilde{\Phi}_i$.

PROPOSITION (see Proposition 2.10). *Let $G \in \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)$ be a characteristic subspace of L with Artin invariant $\sigma(G) \leq \sigma - 1$. Then, the fiber $\tilde{\Phi}_i^{-1}(G)$ is connected and reduced. If $\sigma(G) = \sigma - 1$, then $\tilde{\Phi}_i^{-1}(G)$ is irreducible. If $\sigma(G) < \sigma - 1$, then $\tilde{\Phi}_i^{-1}(G)$ has $p \cdot (\sigma - 1 - \sigma(G))$ many irreducible components which intersect in a unique common point. In any case, each irreducible component of $\tilde{\Phi}_i^{-1}(G)$ is birationally equivalent to $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$.*

This statement is mostly a consequence of results in [BL18].

A moduli space for supersingular Enriques surfaces. The highlight of our work is chapter 3, where we construct a fine moduli space for marked Enriques surfaces that are quotients of supersingular K3 surfaces.

If X is a supersingular K3 surface over an algebraically closed field of characteristic $p \geq 3$ and $\iota\colon X \to X$ is a fixed-point free involution, we write $G = \langle \iota \rangle$ for the cyclic group of order 2 that is generated by ι.

DEFINITION. A quotient of surfaces $X \to X/G = Y$ defined by such a pair (X, ι) is called a *supersingular Enriques surface* Y. The *Artin invariant* of a supersingular Enriques surface Y is the Artin invariant of the supersingular K3 surface X that universally covers Y.

In this chapter, we construct a fine moduli space for marked supersingular Enriques surfaces. More precisely, writing $\mathcal{A}_{\mathbb{F}_p}$ for the category of algebraic spaces over $\mathbb{F}_p$, N_σ for a fixed K3 lattice of Artin invariant σ and $\Gamma' = \Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ as above, we study the functor

$$\underline{\mathcal{E}}_\sigma\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\mathrm{Sets}),$$

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of } \Gamma'\text{-marked} \\ \text{supersingular Enriques surfaces } (\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S}) \\ \text{such that the canonical K3 cover } \mathcal{X} \to \mathcal{Y} \\ \text{admits an } N_\sigma\text{-marking} \end{array} \right\}.$$

Using Ogus' supersingular Torelli theorem, we attack this moduli problem by starting with the moduli space for N-marked supersingular K3 surfaces. Similar to the construction in the complex case by Namikawa [Nam85], we regard Enriques surfaces as equivalence classes of certain embeddings of $\Gamma(2)$ into the Néron-Severi lattice of a K3 surface. Over the complex numbers this means that Namikawa obtains the moduli space of Enriques surfaces by taking a certain open subscheme of the moduli space of K3 surfaces, and then taking the quotient by a group action.

However, the supersingular case is more complicated than the situation over the complex numbers. One of the main problems we face is the fact that in our situation we are, morally speaking, dealing with several moduli spaces $\mathcal{S}_i$ nested in each other with group actions on these subspaces. We use different techniques from [TT16] and [Ryd13] concerning pushouts of algebraic spaces and quotients of algebraic spaces by group actions, and finally obtain the following result.

THEOREM (see Theorem 3.29). *The functor $\underline{\mathcal{E}}_\sigma$ is represented by a quasi-separated algebraic space $\mathcal{E}_\sigma$ that is locally of finite type over $\mathbb{F}_p$ and there exists a separated $\mathbb{F}_p$-scheme $\mathcal{Q}_\sigma$ of finite type and AF, and a canonical étale surjective morphism $\pi_\sigma^E\colon \mathcal{E}_\sigma \to \mathcal{Q}_\sigma$.*

Here, a scheme X is called AF (affine finie), if every finite subset of X is contained in an affine open subscheme of X.

The geometry of the space $\mathcal{E}_\sigma$ is complicated in general, but we have some results on the number of its connected and irreducible components. In short, these numbers depend on properties of the lattice N_σ and we refer to Section 5 for details.

Since the scheme $\mathcal{Q}_\sigma$ in the theorem above was constructed from the scheme $\mathcal{M}_\sigma$, we also obtain a Torelli theorem for Enriques quotients of supersingular K3 surfaces.

THEOREM (see Theorem 3.53). *Let Y_1 and Y_2 be supersingular Enriques surfaces. Then Y_1 and Y_2 are isomorphic if and only if $\pi_\sigma^E(Y_1) = \pi_\sigma^E(Y_2)$ for some $\sigma \leq 5$.*

The period map π_σ^E is defined in the following way: the scheme $\mathcal{Q}_\sigma$ represents the functor that associates to a smooth scheme S the set of isomorphism classes of families of K3 crystals H over S together with maps $\gamma\colon \Gamma(2) \hookrightarrow T_H \hookrightarrow H$ that are compatible with intersection forms and such that there exists a factorization $\gamma\colon \Gamma(2) \hookrightarrow N_\sigma \hookrightarrow T_H \hookrightarrow H$ without (-2)-vectors in the orthogonal complement $\gamma(\Gamma(2))^\perp \subset N_\sigma$. For a supersingular Enriques surface Y, we can choose a Γ-marking $\gamma\colon \Gamma \to \mathrm{NS}(Y)$, and this induces a point $\pi_\sigma^E(Y,\gamma) \in \mathcal{Q}_\sigma$. We show that $\pi_\sigma(Y,\gamma)$ is independent of the choice of γ and set $\pi_\sigma^E(Y) = \pi_\sigma^E(Y,\gamma)$.

This construction justifies calling $\pi_\sigma^E(Y)$ the *period of* Y, and we call $\mathcal{Q}_\sigma$ the *period space* of supersingular Enriques surfaces of Artin invariant at most σ.

It remains to mention characteristic p=2. Here there are three types of Enriques surfaces, and a moduli space in this case has two components [BM76] [Lie15a]. For the component corresponding to simply connected Enriques surfaces, Ekedahl, Hyland and Shepherd-Barron [EHSB12] constructed a period map and established a Torelli theorem. In their work, however, the K3-like cover is not smooth and the covering is not étale, which is why this theory has a slightly different flavor.

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CHAPTER 0

Basic facts on Grassmannians, lattices and K3 surfaces

We assume that the reader is familiar with linear algebra and the basics of commutative algebra and algebraic geometry, for example the contents of [AM16] and [Har77]. We further assume some basic knowledge about crystalline cohomology. In this chapter, we introduce the concepts necessary to understand the following chapters that exceed the aforementioned requirements. In particular, we recall a construction for charts of Grassmannian varieties, we give a short overview about the lattice theory that we will need later on, and we give an introduction to the theory of K3 surfaces and their Enriques quotients with a focus on supersingular K3 surfaces. Our aim of this discussion is to be concise and only treat the bare minimum required to understand the rest of our work. We refer the interested reader to the sources mentioned during those sections for further information on, and motivation behind, the discussed topics.

1. Charts of Grassmannian varieties

At one point, we will need to do some explicit calculations with subvarieties of Grassmannian varieties. To this end, we fix some notations and recall a definition for charts of Grassmannians. Let k be a field and let $m < n$ be positive integers. We define a functor

$$\begin{aligned} \underline{Gr}_k(m,n)\colon (k\text{-schemes})^{\mathrm{op}} &\longrightarrow (\text{Sets}),\\ S &\longmapsto \left\{ \begin{array}{l} \text{surjections } \mathcal{O}_S^n \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a finite} \\ \text{locally free } \mathcal{O}_S\text{-module of rank } n-m \end{array} \right\}. \end{aligned}$$

It is well known that this functor is representable by a smooth projective k-variety $Gr_k(m,n)$ of dimension $m(n-m)$, the *Grassmannian variety of m-dimensional subspaces in k^n*. Let $B_1 = \{e_1, \dots, e_{n-m}\}$ and $B_2 = \{e'_1, \dots, e'_m\}$ be linearly independent subsets of k^n such that $B_1 \cup B_2$ is a basis of k^n. We denote by U_1 the affine open subset in $Gr_k(m,n)$ that parametrizes the m-dimensional subspaces W of k^n such that $W + \langle B_1 \rangle = k^n$. We can now define a chart

$$c_{B_2}^{B_1}\colon U_1(k) \longrightarrow \mathbb{A}_k^{(n-m)\times m}(k)$$

as follows. We consider the projection $k^n \to k^n/\langle B_1\rangle$ and write $\overline{e}'_i$ for the image of e'_i for each i. If $W \in U_1(k)$ is an m-dimensional subspace of k^n, then $W \hookrightarrow k^n \to k^n/\langle B_1 \rangle$ is an isomorphism. Let $v_j = \sum_{i=1}^{n-m} a_{ij} e_i + \sum_{i=1}^{m} b_{ij} e'_i$ be the unique preimage of $\overline{e}'_j$ under this isomorphism. We then set

$$c_{B_2}^{B_1}(W) = \left(a_{ij}\right)_{\substack{i=1,\dots,n-m \\ j=1,\dots,m}},$$

which is an element of $\mathbb{A}_k^{(n-m)\times m}(k)$. We can check that, running over different choices of B_1 and B_2, we can obtain an atlas for the variety $Gr_k(m,n)$.

2. Lattices

We fix some notation and recall basic definitions and results on lattices from [Nik80].

In the following, by a *lattice* $(L, \langle\cdot,\cdot\rangle)$ we mean a free $\mathbb{Z}$-module L of finite rank together with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle\colon L\times L\to\mathbb{Z}$. A morphism of lattices is a morphism of the underlying $\mathbb{Z}$-modules that is compatible with intersection forms. Trying to simplify notation, we will often talk about the lattice L, omitting the bilinear form. The lattice L is called *even* if $\langle x,x\rangle\in\mathbb{Z}$ is even for each $x\in L$. A lattice is *odd* if it is not even. For $a\in\mathbb{Q}$ and if $a\langle L,L\rangle\subset\mathbb{Z}$, we denote by $L(a)$ the twisted lattice with underlying $\mathbb{Z}$-module L and bilinear form $a\langle\cdot,\cdot\rangle$.

After choosing a basis $\{e_1,\dots,e_n\}$ of L, the *discriminant of* L is defined to be $\mathrm{disc}L = \det\left((e_i\cdot e_j)_{ij}\right)\in\mathbb{Z}$. This definition does not depend on the basis chosen. The lattice L is called *unimodular* if $|\mathrm{disc}L| = 1$. The *dual lattice of* L is the free $\mathbb{Z}$-module $L^\vee = \mathrm{Hom}(L,\mathbb{Z})\subseteq L\otimes\mathbb{Q}$ together with the bilinear form $\langle\cdot,\cdot\rangle_{L^\vee}\colon L^\vee\times L^\vee\to\mathbb{Q}$ induced from $\langle\cdot,\cdot\rangle_{\mathbb{Q}}\colon L_{\mathbb{Q}}\times L_{\mathbb{Q}}\to\mathbb{Q}$. The *discriminant lattice* $A_L = L^\vee/L$ is a finite abelian group and is equipped with a canonical finite quadratic form $q_L\colon A_L\to\mathbb{Q}/2\mathbb{Z}$ induced from $\langle\cdot,\cdot\rangle$. One can show that $\#A_L = \mathrm{disc}L$. Let p be a prime number. If $p\cdot A_L = 0$ we say that the lattice L is p*-elementary*.

We define the *signature of* L to be the signature (l_+, l_-) of the quadratic space $L\otimes\mathbb{Q}$. Likewise, the lattice L is called *positive definite* (respectively *negative definite*) if and only if the quadratic space $L\otimes\mathbb{Q}$ is *positive definite* (respectively *negative definite*). There are two lattices that we will use frequently within this work. Namely, we will write U for the even unimodular lattice of signature $(1,1)$ and E_8 for the even unimodular lattice of signature $(0,8)$. It is well known that by prescribing these invariants the lattices U and E_8 are well defined up to isomorphism.

Let us now turn to morphisms of lattices. It is easy to see that any morphism $\psi\colon L_1\to L_2$ of lattices is automatically injective. We will therefore also use the term *embedding of lattices* when talking about morphisms. A given embedding of lattices $\psi\colon L_1\hookrightarrow L_2$ is called *primitive* if the quotient L_2/L_1 is a free $\mathbb{Z}$-module. On the other hand if the quotient L_2/L_1 is finite, we then call L_2 an *overlattice of* L_1.

To a lattice L we associate its *genus* $[L]$, which is the class consisting of all lattices L' such that $L\otimes\hat{\mathbb{Z}}_p\cong L'\otimes\hat{\mathbb{Z}}_p$ for all primes p and $L\otimes\mathbb{R}\cong L'\otimes\mathbb{R}$. We will use the following characterization of the genus of a lattice due to [Nik80, Corollary 1.9.4].

PROPOSITION 0.1. *Let L be an even lattice. Then the genus $[L]$ is uniquely defined by the signature of L and the discriminant form q_L.*

There is also a version of Proposition 0.1 for the odd case. We will only need the even case in this work, however, and therefore omit the odd version.

3. K3 surfaces and Enriques surfaces

In this section, we fix some notation and recall basic definitions and results on K3 surfaces and Enriques surfaces from [Huy16].

We fix a field k. A *variety over* k is a separated k-scheme $X \to \operatorname{Spec} k$ that is of finite type over k and geometrically integral. We say that a variety X is *smooth* if the cotangent sheaf $\Omega_{X/k}$ is locally free of dimension $\dim X$ over $\mathcal{O}_X$. If X is a variety over k of dimension n then we denote the n-th exterior power of $\Omega_{X/k}$ by $\omega_X = \Omega_X^n$. The sheaf ω_X is a line bundle on X, and called the *canonical bundle of* X. By a *surface over* k, we mean a proper k-variety $X \to \operatorname{Spec} k$ of dimension two. The *irregularity* q *of a surface* X is given via the k-vector space dimension of the first cohomology of the structure sheaf, that is to say $q(X) = h^1(X, \mathcal{O}_X)$. We can now define K3 surfaces.

DEFINITION 0.2. A *K3 surface* over k is a smooth surface $X \to \operatorname{Spec} k$ such that $\omega_X \cong \mathcal{O}_X$ and $q(X) = 0$.

REMARK 0.3. It is a well-known fact that any smooth surface is automatically projective. In particular any K3 surface is projective.

We will need the following facts on the cohomology of a K3 surface.

PROPOSITION 0.4. *Let X be a K3 surface. For the l-adic étale Betti numbers we have*

(1) $b_i(X) = 1$ if and only if $i = 0$ or $i = 4$,
(2) $b_i(X) = 22$ if and only if $i = 2$,
(3) $b_i(X) = 0$ else.

Furthermore, the Frölicher spectral sequence $E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k)$ degenerates, where $H^\bullet_{\mathrm{dR}}(-)$ denotes de Rham cohomology, and $H^n_{\mathrm{cris}}(X/W)$ is a free W-module of rank $b_n(X)$ for all $n \geq 0$, where W is the Witt ring and $H^\bullet_{\mathrm{cris}}(-)$ denotes crystalline cohomology.

Closely related to K3 surfaces are Enriques surfaces.

DEFINITION 0.5. Let k be an algebraically closed field. An *Enriques surface over* k is a smooth surface $Y \to \operatorname{Spec} k$ such that $\omega_Y \equiv \mathcal{O}_Y$ and $b_2(Y) = 10$, where $\equiv$ denotes numerical equivalence and b_i denotes the i-th étale or crystalline Betti number.

For an Enriques surface Y we always have $\chi(\mathcal{O}_Y) = 1$ and $b_1(Y) = 0$. Furthermore, there always exists an isomorphism $\omega_Y^{\otimes 2} \cong \mathcal{O}_Y$, and $\omega_Y \ncong \mathcal{O}_Y$ if and only if $H^1(Y, \mathcal{O}_Y) = 0$. If k is of characteristic $p \neq 2$, we have $H^1(Y, \mathcal{O}_Y) = 0$, but for $p = 2$ only the inequality $h^1(\mathcal{O}_Y) \leq 1$ holds true. Thus when $H^1(Y, \mathcal{O}_Y)$ is non-zero, then the action of the absolute Frobenius F on $H^1(Y, \mathcal{O}_Y)$ is either zero or a bijection. We distinguish three cases.

DEFINITION 0.6. An Enriques surface Y is called

(1) *classical* if $h^1(\mathcal{O}_Y) = 0$.
(2) *singular* if $h^1(\mathcal{O}_Y) = 1$ and F is bijective on $H^1(Y, \mathcal{O}_Y)$.
(3) *supersingular* if $h^1(\mathcal{O}_Y) = 1$ and F is zero on $H^1(Y, \mathcal{O}_Y)$.

We note that every Enriques surface is classical in characteristic unequal two. This work is mainly concerned with a class of K3 surfaces and Enriques surfaces over fields of characteristic $p \geq 3$. To simplify this section we will from now on assume that $\operatorname{char} k \neq 2$. We refer the interested reader to [Lie15a] for details on Enriques surfaces and their moduli spaces in characteristic two. The following characterization of Enriques surfaces in characteristic unequal two is well known.

PROPOSITION 0.7. *Let k be an algebraically closed field of characteristic not two and let $Y \to \operatorname{Spec} k$ be a smooth surface. Then Y is an Enriques surface if and only if $\omega_Y^{\otimes 2} \cong \mathcal{O}_Y$ and $H^1(Y, \mathcal{O}_Y) = 0$.*

For a k-scheme Y, a line bundle $\mathcal{L}$ on Y is called *n-torsion* if $\mathcal{L}^{\otimes n} \cong \mathcal{O}_Y$. If $\mathcal{L}$ on Y is an n-torsion line bundle and $\varphi\colon \mathcal{L}^{\otimes n} \to \mathcal{O}_Y$ is a fixed trivialization, then the $\mathcal{O}_Y$-module $\mathcal{A} := \mathcal{O}_Y \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{\otimes(n-1)}$ has a canonical multiplicative structure induced from φ, making $\mathcal{A}$ into an $\mathcal{O}_Y$-algebra. We obtain a canonical morphism of k-schemes $X = \operatorname{Spec} \mathcal{A} \to Y$ and, if the characteristic of k does not divide n, this map is étale of degree n. In the special case where Y is an Enriques surface over k with characteristic of k unequal two and $\mathcal{L} = \omega_Y$, it turns out that X is a K3 surface and we call it *the universal K3 cover of Y*.

We will need the following fact about the Picard group of an Enriques surface.

PROPOSITION 0.8. *Let Y be an Enriques surface. Then the Néron-Severi lattice $\mathrm{NS}(Y)$ is isomorphic to $U \oplus E_8 \oplus \mathbb{Z}/2\mathbb{Z}$. In particular, the torsion part of the Picard group $\operatorname{Pic}(Y)$ has order 2.*

4. Supersingular K3 surfaces

In this section, we define supersingularity for K3 surfaces, following [Shi74] and [Art74].

Let X be a suface with Néron-Severi group $\mathrm{NS}(X)$. We denote by $\rho(X) = \operatorname{rk}(\mathrm{NS}(X))$ the *Picard rank of X*. The following theorem, due to Igusa [Igu60], follows directly from the existence of the injective Chern map $c_1\colon \mathrm{NS}(X) \to H^2_*(X)$, where $H^\bullet_*$ is étale or crystalline cohomology.

THEOREM 0.9. *Let X be a surface. Then $\rho(X) \leq b_2(X)$.*

In particular, $\rho(X)$ is always finite. The theorem motivates the notion of supersingularity in the sense of Shioda.

DEFINITION 0.10. A surface X is called *Shioda-supersingular* if we have $\rho(X) = b_2(X)$.

In characteristic zero, one can even prove the inequality $\rho(X) \leq h^{1,1}(X)$ and, since for any K3 surface X we have $h^{1,1}(X) = 20 < 22 = b_2(X)$, it follows that Shioda-supersingular K3 surfaces only exist in positive characteristic.

Artin introduced a further notion of supersingularity for K3 surfaces that we want to outline. Given a K3 surface X, we consider the functor

$$\begin{aligned}\mathrm{Br}\colon (\text{Artin Algebras})^{\mathrm{op}} &\to (\text{Abelian Groups}),\\ A &\mapsto \ker\left(H^2(X \times \operatorname{Spec} A, \mathcal{O}^\times_{S \times \operatorname{Spec} A}) \to H^2(S, \mathcal{O}^\times_S)\right)\end{aligned}$$

which is prorepresentable by a one-dimensional formal group law, the *formal Brauer group Φ^2_X of X*. Let k be a perfect field in positive characteristic $p > 0$ and let $W(k)$ be the Witt ring over k, then we denote by $\operatorname{Cart}(k)$ the non-commutative ring $W(k)\langle\langle V\rangle\rangle\langle F\rangle$ of power series in V and polynomials in F modulo the relations

$$FV = p,\ VrF = V(r),\ Fr = \sigma(r)F,\ rV = V\sigma(r) \text{ for all } r \in W(k),$$

where $\sigma(r)$ denotes Frobenius of $W(k)$ and $V(r)$ denotes Verschiebung of $W(k)$. The following theorem is proved in [Mum69, Section 1].

THEOREM 0.11. *Let k be a perfect field of characteristic $p > 0$. Then there exists a covariant equivalence D of categories between*

1. *the category of commutative formal group laws over k, and*
2. *the category of left $\mathrm{Cart}(k)$-modules M such that*
   (a) *$\bigcap_i V^i M = 0$,*
   (b) *V acts injectively on M,*
   (c) *M/VM is a finite-dimensional k-vector space.*

For a K3 surface X there exists an isomorphism of left $\mathrm{Cart}(k)$-modules

$$D\Phi_X^2 \cong H^2(X, W\mathcal{O}_X)$$

and we have the following definition of supersingularity due to Artin.

DEFINITION 0.12. A K3 surface X is called *Artin-supersingular* if its formal Brauer group Φ_X^2 is of infinite height. If the ground field k is algebraically closed, this is equivalent to saying that there exists an isomorphism of left $\mathrm{Cart}(k)$-modules $H^2(X, W\mathcal{O}_X) \cong k[\![x]\!]$, where $\mathrm{Cart}(k)$ acts on $k[\![x]\!]$ via $F = 0$ and $V \cdot x^n = x^{n+1}$.

It follows from the Tate conjecture that, over any perfect field k, a K3 surface is Artin supersingular if and only if it is Shioda supersingular [Mau14]. Charles first proved the Tate conjecture over fields of characteristic at least 5 [Cha13]. Using the Kuga-Satake construction, Madapusi Pera gave a proof of the Tate conjecture over fields of characteristic at least 3 [MP15]. Over fields of characteristic $p = 2$, the Tate conjecture was proved by Kim and Madapusi Pera [KMP16].

DEFINITION 0.13. Let X be a K3 surface over a perfect field k of characteristic $p \geq 2$. We say that X is *supersingular* if one of the following equivalent conditions holds true:

1. X is Shioda-supersingular, or
2. X is Artin-supersingular.

Over fields of characteristic $p \geq 3$, there is yet another characterization of supersingularity.

THEOREM 0.14. *[Lie15b, Theorem 5.3] Let X be a K3 surface over a perfect field k of characteristic $p \geq 3$. Then X is supersingular if and only if X is unirational.*

REMARK 0.15. The version of the previous theorem in [Lie15b] requires $p \geq 5$. However, since a crystalline Torelli theorem has now been proved over fields of characteristic $p = 3$ [BL18, Section 5.1], the theorem also holds over fields of characteristic $p = 3$.

5. Ogus' Crystalline Torelli theorem for supersingular K3 surfaces

Although supersingular K3 surfaces only exist over fields of positive characteristic, they come with a feature that makes them, in a way, similar to K3 surfaces over a field of characteristic zero. Namely, there exist Torelli theorems both for K3 surfaces over the complex numbers (cf. for example [Huy16]) and for supersingular K3 surfaces over fields of characteristic $p \geq 3$. In this section, we will outline some of the results on Torelli theorems and moduli spaces for supersingular K3 surfaces. Most of the following content is due to Ogus [Ogu79][Ogu83]. A strong inspiration behind our treatment in this section and a good source for the interested reader is [Lie16].

5.1. K3 crystals. For the definition of F-crystals and their slopes we refer to [Kat79, Chapter I.1]. Given a supersingular K3 surface X, it turns out that a lot of information is encoded in its second crystalline cohomology. We say that $H^2_{\mathrm{crys}}(X/W)$ is a *supersingular K3 crystal* of rank 22 in the sense of the following definition, due to Ogus [Ogu79].

DEFINITION 0.16. Let k be a perfect field of positive characteristic p and let $W = W(k)$ be its Witt ring with lift of Frobenius $\sigma\colon W \to W$. A *supersingular K3 crystal of rank* n over k is a free W-module H of rank n together with an injective σ-linear map

$$\varphi\colon H \to H,$$

i.e. φ is a morphism of abelian groups and $\varphi(a \cdot m) = \sigma(a) \cdot \varphi(m)$ for all $a \in W$ and $m \in H$, and a symmetric bilinear form

$$\langle -, - \rangle\colon H \times H \to W,$$

such that

(1) $p^2 H \subseteq \mathrm{im}(\varphi)$,
(2) the map $\varphi \otimes_W k$ is of rank 1,
(3) $\langle -, - \rangle$ is a perfect pairing,
(4) $\langle \varphi(x), \varphi(y) \rangle = p^2 \sigma\left(\langle x, y \rangle\right)$, and
(5) the F-crystal (H, φ) is purely of slope 1.

The *Tate module* T_H of a K3 crystal H is the $\mathbb{Z}_p$-module

$$T_H \coloneqq \{x \in H \mid \varphi(x) = px\}.$$

One can show that if $H = H^2_{\mathrm{crys}}(X/W)$ is the second crystalline cohomology of a supersingular K3 surface X, and $c_1\colon \mathrm{Pic}(X) \to H^2_{\mathrm{crys}}(X/W)$ is the first crystalline Chern class map, we have $c_1(\mathrm{Pic}(X)) \subseteq T_H$. If X is defined over a finite field, the Tate conjecture is known (see [Cha13] [MP15]) and it follows that we even have the equality $c_1(\mathrm{NS}(X)) \otimes \hat{\mathbb{Z}}_p = T_H$. The following proposition on the structure of the Tate module of a supersingular K3 crystal is due to Ogus [Ogu79].

PROPOSITION 0.17. *Let $(H, \varphi, \langle -, - \rangle)$ be a supersingular K3 crystal over a field k of characteristic $p > 2$ and let T_H be its Tate module. Then $\mathrm{rk}_W H = \mathrm{rk}_{\hat{\mathbb{Z}}_p} T_H$ and the bilinear form $(H, \langle -, - \rangle)$ induces a non-degenerate form $T_H \times T_H \to \hat{\mathbb{Z}}_p$ via restriction to T_H which is not perfect. More precisely, we find*

(1) $\mathrm{ord}_p(A_{T_H}) = 2\sigma$ for some positive integer σ,
(2) $(T_H, \langle -, - \rangle)$ is determined up to isometry by σ,
(3) $\mathrm{rk}_W H \geq 2\sigma$ and
(4) there exists an orthogonal decomposition

$$(T_H, \langle -, - \rangle) \cong (T_0, p\langle -, - \rangle) \perp (T_1, \langle -, - \rangle),$$

where T_0 and T_1 are $\hat{\mathbb{Z}}_p$-lattices with perfect bilinear forms and of ranks $\mathrm{rk} T_0 = 2\sigma$ and $\mathrm{rk} T_1 = \mathrm{rk}_W H - 2\sigma$.

The positive integer σ is called the *Artin invariant* of the K3 crystal H [Ogu79]. When H is the second crystalline cohomology of a supersingular K3 surface X, we have $1 \leq \sigma(H) \leq 10$.

5.2. K3 lattices. The previous subsection indicates that the Néron-Severi lattice $\mathrm{NS}(X)$ of a supersingular K3 surface X plays an important role in the study of supersingular K3 surfaces via the first Chern class map. We say that $\mathrm{NS}(X)$ is a *supersingular K3 lattice* in the sense of the following definition due to Ogus [Ogu79].

DEFINITION 0.18. A *supersingular K3 lattice* is an even lattice $(N, \langle -, - \rangle)$ of rank 22 such that

(1) the discriminant $d(N \otimes_{\mathbb{Z}} \mathbb{Q})$ is -1 in $\mathbb{Q}^*/\mathbb{Q}^{*2}$,
(2) the signature of N is $(1, 21)$, and
(3) the lattice N is p-elementary for some prime number p.

When N is the Néron-Severi lattice of a supersingular K3 surface X, then the prime number p in the previous definition turns out to be the characteristic of the base field. One can show that if N is a supersingular K3 lattice, then its discriminant is of the form $d(N) = -p^{2\sigma}$ for some integer σ such that $1 \leq \sigma \leq 10$. The integer σ is called the *Artin invariant* of the lattice N. If X is a supersingular K3 surface, we call $\sigma(\mathrm{NS}(X))$ the *Artin invariant* of the supersingular K3 surface X and we find that $\sigma(\mathrm{NS}(X)) = \sigma(H^2_{\mathrm{crys}}(X/W))$. The following theorem is due to Rudakov and Shafarevich [RS81, Section 1].

THEOREM 0.19. *If $p \neq 2$, then the Artin invariant σ determines a supersingular K3 lattice up to isometry.*

5.3. Characteristic subspaces and K3 crystals. In this subsection, we introduce characteristic subspaces. These objects yield another way to describe K3 crystals, a little closer to classic linear algebra in flavor. For this subsection we fix a prime $p > 2$ and a perfect field k of characteristic p with Frobenius $F\colon k \to k$, $x \mapsto x^p$.

DEFINITION 0.20. Let σ be a non-negative integer and let V be a 2σ-dimensional $\mathbb{F}_p$-vector space together with a non-degenerate and non-neutral quadratic form

$$\langle -, - \rangle \colon V \times V \to \mathbb{F}_p.$$

The condition that $\langle -, - \rangle$ is *non-neutral* means that there exists no σ-dimensional isotropic subspace of V. Set $\varphi \coloneqq \mathrm{id}_V \otimes F\colon V \otimes_{\mathbb{F}_p} k \to V \otimes_{\mathbb{F}_p} k$. A k-subspace $G \subset V \otimes_{\mathbb{F}_p} k$ is called *characteristic* if

(1) G is a totally isotropic subspace of dimension σ, and
(2) $G + \varphi(G)$ is of dimension $\sigma + 1$.

A *strictly characteristic subspace* is a characteristic subspace G such that

$$V \otimes_{\mathbb{F}_p} k = \sum_{i=0}^{\infty} \varphi^i(G)$$

holds true.

We can now introduce the categories

$$\mathrm{K3}(k) \coloneqq \left\{ \begin{array}{l} \text{Supersingular K3 crystals} \\ \text{with only isomorphisms as morphisms} \end{array} \right\}$$

and

$$\mathbb{C}3(k) \coloneqq \left\{ \begin{array}{l} \text{Pairs } (T, G)\text{, where } T \text{ is a supersingular} \\ \text{K3 lattice over } \hat{\mathbb{Z}}_p\text{, and } G \subseteq T_0 \otimes_{\hat{\mathbb{Z}}_p} k \\ \text{is a strictly characteristic subspace} \\ \text{with only isomorphisms as morphisms} \end{array} \right\}.$$

It turns out that, over an algebraically closed field, these two categories are equivalent.

THEOREM 0.21. *[Ogu79, Theorem 3.20] Let k be an algebraically closed field of characteristic $p > 0$. Then the functor*

$$\begin{aligned} \mathrm{K3}(k) &\longrightarrow \mathbb{C}3(k), \\ (H, \varphi, \langle -, - \rangle) &\longmapsto \left(T_H, \ker\left(T_H \otimes_{\hat{\mathbb{Z}}_p} k \to H \otimes_{\hat{\mathbb{Z}}_p} k\right) \subset T_0 \otimes_{\hat{\mathbb{Z}}_p} k\right) \end{aligned}$$

defines an equivalence of categories.

If we denote by $\mathbb{C}3(k)_\sigma$ the subcategory of $\mathbb{C}3(k)$ consisting of objects (T, G) where T is a supersingular K3 lattice of Artin invariant σ, then there is a coarse moduli space.

THEOREM 0.22. *[Ogu79, Theorem 3.21] Let k be an algebraically closed field of characteristic $p > 0$. We denote by μ_n the cyclic group of n-th roots of unity. There exists a canonical bijection*

$$(\mathbb{C}3(k)_\sigma / \simeq) \longrightarrow \mathbb{A}_k^{\sigma-1}(k)/\mu_{p^\sigma+1}(k).$$

The previous theorem concerns characteristic subspaces defined on closed points with algebraically closed residue field. Next, we consider families of characteristic subspaces.

DEFINITION 0.23. Let σ be a non-negative integer and let $(V, \langle -, - \rangle)$ be a 2σ-dimensional $\mathbb{F}_p$-vector space together with a non-neutral quadratic form. If A is an $\mathbb{F}_p$-algebra, a direct summand $G \subset V \otimes_{\mathbb{F}_p} A$ is called a *geneatrix* if $\mathrm{rk}(G) = \sigma$ and $\langle -, - \rangle$ vanishes when restricted to G. A *characteristic geneatrix* is a geneatrix G such that $G + F_A(G)$ is a direct summand of rank $\sigma + 1$ in $V \otimes_{\mathbb{F}_p} A$. We write $\underline{M}_V(A)$ for the set of characteristic geneatrices in $V \otimes_{\mathbb{F}_p} A$.

It turns out that there exists a moduli space for characteristic geneatrices.

PROPOSITION 0.24. *[Ogu79, Proposition 4.6] The functor*

$$\begin{aligned} (\mathbb{F}_p\textit{-algebras})^{\mathrm{op}} &\longrightarrow (\textit{Sets}), \\ A &\longmapsto \underline{M}_V \end{aligned}$$

is representable by an $\mathbb{F}_p$-scheme M_V that is smooth, projective and of dimension $\sigma - 1$.

If N is a supersingular K3 lattice with Artin invariant σ, then $N_0 = pN^\vee/pN$ is a 2σ-dimensional $\mathbb{F}_p$-vector space together with a non-degenerate and non-neutral quadratic form induced from the bilinear form on N.

DEFINITION 0.25. We set $\mathcal{M}_\sigma \coloneqq M_{N_0}$ and call this scheme the *moduli space of N-rigidified K3 crystals*.

5.4. Ample cones. Next, we will need to enlarge $\mathcal{M}_\sigma$ by equipping N-rigidified K3 crystals with ample cones. For the rest of this section we fix a prime $p \geq 3$.

DEFINITION 0.26. Let N be a supersingular K3 lattice. The set $\Delta_N \coloneqq \{l \in N \mid l^2 = -2\}$ is called the set of *roots of* N. The *Weyl group* W_N of N is the subgroup of the orthogonal group $O(N)$ generated by all automorphisms of the form $s_l \colon x \mapsto x + \langle x, l\rangle l$ with $l \in \Delta_N$. We denote by $\pm W_N$ the subgroup of $O(N)$ generated by W_N and $\pm\mathrm{id}$. Furthermore, we define

$$V_N \coloneqq \{x \in N \otimes \mathbb{R} \mid x^2 > 0 \text{ and } \langle x, l\rangle \neq 0 \text{ for all } l \in \Delta_N\}.$$

The set V_N is an open subset of $N \otimes \mathbb{R}$, and each of its connected components meets N. The connected components of V_N are called the *ample cones of* N, and we denote by C_N the set of ample cones of N.

REMARK 0.27. The group $\pm W_N$ operates simply and transitively on C_N [Ogu83].

DEFINITION 0.28. Let N be a supersingular K3 lattice of Artin invariant σ and let S be an algebraic space over $\mathbb{F}_p$. For a characteristic geneatrix $G \in \mathcal{M}_\sigma(S)$ and any point $s \in S$ we define

$$\begin{aligned}
\Lambda(s) &\coloneqq N_0 \cap G(s),\\
N(s) &\coloneqq \{x \in N \otimes \mathbb{Q} \mid px \in N \text{ and } \overline{px} \in \Lambda(s)\},\\
\Delta(s) &\coloneqq \{l \in N(s) \mid l^2 = -2\}.
\end{aligned}$$

An *ample cone for* G is an element $\alpha \in \prod_{s \in S} C_{N(s)}$ such that $\alpha(s) \subseteq \alpha(t)$ whenever $s \in \overline{\{t\}}$.

We now consider the functor

$$\begin{aligned}
\underline{\mathcal{P}_\sigma} \colon (\text{Algebraic spaces over } \mathbb{F}_p)^{\mathrm{op}} &\longrightarrow (\text{Sets}),\\
S &\longmapsto \left\{ \begin{array}{l} \text{Characteristic spaces } G \in \mathcal{M}_\sigma(S) \\ \text{equipped with ample cones} \end{array} \right\}.
\end{aligned}$$

Forgetting ample cones yields a natural map of functors $\underline{\mathcal{P}_\sigma} \to \underline{\mathcal{M}_\sigma}$ and we have the following result.

THEOREM 0.29. *[Ogu83, Proposition 1.16] The functor $\underline{\mathcal{P}_\sigma}$ is represented by a scheme $\mathcal{P}_\sigma$ that is locally of finite type, almost proper and smooth of dimension $\sigma - 1$ over $\mathbb{F}_p$. The natural map $\mathcal{P}_\sigma \to \mathcal{M}_\sigma$ is étale, surjective and locally of finite type.*

Here, "almost proper" means that $\mathcal{P}_\sigma$ satisfies the valuative criterion for properness but is neither separated nor of finite type.

5.5. The crystalline Torelli theorem. In this subsection, we explain the connection between characteristic spaces with ample cones and N-marked K3 surfaces following [Ogu83]. Again, we fix a prime $p \geq 3$ and for each integer σ such that $1 \leq \sigma \leq 10$ a K3 lattice N_σ with $\sigma(N_\sigma) = \sigma$.
A *family of supersingular K3 surfaces* is a smooth and proper morphism of algebraic spaces $f \colon \mathcal{X} \to S$ over $\mathbb{F}_p$ such that for each field k and each $\operatorname{Spec} k \to S$ the fiber $\mathcal{X}_k \to \operatorname{Spec} k$ is a projective supersingular K3 surface. An N_σ-marking of such a family $\mathcal{X} \to S$ is a morphism $\psi \colon \underline{N_\sigma} \to \mathrm{Pic}_{\mathcal{X}/S}$ of group objects in the category of algebraic spaces, which is compatible with

intersection forms. There is an obvious notion of morphisms of families N_σ-marked K3 surfaces. We consider the moduli problem

$$\underline{\mathcal{S}}_\sigma \colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\text{Sets})\,,$$
$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of } N_\sigma\text{-marked} \\ \text{K3 surfaces } (f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}) \end{array} \right\}.$$

One can show that $\underline{\mathcal{S}}_\sigma$ is representable by an algebraic space $\mathcal{S}_\sigma$ over $\mathbb{F}_p$ that is locally of finite presentation, locally separated and smooth of dimension $\sigma - 1$ [Ogu83, Theorem 2.7].

There is a canonical morphism of functors $\tilde{\pi}_\sigma \colon \underline{\mathcal{S}}_\sigma \to \underline{\mathcal{P}}_\sigma$. It is the content of Ogus' crystalline Torelli theorem that this is an isomorphism of functors, so a postiori $\mathcal{S}_\sigma$ is an $\mathbb{F}_p$-scheme.

THEOREM 0.30. *[Ogu83, Theorem III'] The morphism* $\tilde{\pi}_\sigma \colon \underline{\mathcal{S}}_\sigma \longrightarrow \underline{\mathcal{P}}_\sigma$ *is an isomorphism.*

CHAPTER 1

On the number of Enriques quotients of a K3 surface

Let k be an algebraically closed field and let X be a K3 surface over k. An involution $\iota\colon X \to X$ of the K3 surface X is called *free* if it has no fixed points. A quotient of X by a free involution is an Enriques surface. We call these the *Enriques quotients of* X. For complex K3 surfaces it is known that the number of Enriques quotients up to isomorphism for a given K3 surface is finite.

In this chapter, we show that most classes of K3 surfaces have only finitely many Enriques quotients up to isomorphism. For supersingular K3 surfaces over fields of characteristic $p \geq 3$, we give a formula that generically yields the number of their Enriques quotients. Via a lattice theoretic argument we reprove a result by Jang [Jan15] that states that supersingular K3 surfaces always have an Enriques quotient. For some small characteristics and some Artin invariants, we explicitly compute lower bounds for the number of Enriques quotients of a supersingular K3 surface. We show that the supersingular K3 surface of Artin invariant 1 over an algebraically closed field of characteristic 3 has exactly two isomorphism classes Enriques quotients.

1. Enriques quotients of K3 surfaces of finite height

Lieblich and Maulik showed in [LM11] that finite height K3 surfaces in positive characteristic admit well-behaved lifts to characteristic zero, and we will use these lifting techniques - and the fact that K3 surfaces over the complex numbers only have finitely many Enriques quotients - to show that the same holds in positive characteristic.

DEFINITION 1.1. Let k be an algebraically closed field and let X be a K3 surface over k with Néron-Severi lattice $\mathrm{NS}(X)$. We denote the *group of isometries of* $\mathrm{NS}(X)$ by $O(\mathrm{NS}(X))$.

The *positive cone* $\mathcal{C}_X$ is the connected component of $\{x \in \mathrm{NS}(X)\otimes\mathbb{R} \mid x^2 > 0\} \subseteq \mathrm{NS}(X)\otimes\mathbb{R}$ that contains an ample divisor. The *ample cone* $\mathcal{A}_X$ is the subcone of $\mathcal{C}_X$ generated as a semigroup by ample divisors multiplied by positive real numbers.

The *Weyl group of* X is the group $W_X = W_{\mathrm{NS}(X)}$. We set

$$O^+(\mathrm{NS}(X)) \coloneqq \{\varphi \in O(\mathrm{NS}(X)) \mid \varphi(\mathcal{A}_X) = \mathcal{A}_X\}$$

to be the group of isometries of $\mathrm{NS}(X)$ that preserve the ample cone. Furthermore, we define

$$O_0(\mathrm{NS}(X)) \coloneqq \ker\left(O(\mathrm{NS}(X)) \to O(q_{\mathrm{NS}(X)})\right)$$

and

$$O_0(\mathrm{NS}(X))^+ \coloneqq O_0(\mathrm{NS}(X)) \cap O^+(\mathrm{NS}(X)).$$

We will need the following easy lemma.

PROPOSITION 1.2. *Let X be a K3 surface over an arbitrary field k and let ι_1 and ι_2 be free involutions on X. Then the Enriques surfaces X/ι_1 and X/ι_2 are isomorphic if and only if there exists some automorphism $g \in \mathrm{Aut}(X)$ such that $g\iota_1 g^{-1} = \iota_2$.*

PROOF. This is [Oha07, Proposition 2.1.]. The proof does not depend on the base field. $\square$

THEOREM 1.3. *Let X be a K3 surface over an algebraically closed field k. If X is of finite height, then the number of isomorphism classes of Enriques quotients of X is finite.*

PROOF. By [LM11, Theorem 2.1, Corollary 4.2] the K3 surface X is the closed fiber of a smooth projective relative K3 surface $\mathcal{X} \to \mathrm{Spec} W$ with generic fiber $\mathcal{X}_1$ such that the specialization morphism $\mathrm{sp}\colon \mathrm{Pic}(\mathcal{X}_1) \to \mathrm{Pic}(X)$ is an isomorphism. Furthermore, also by [LM11, Theorem 2.1], there is an injective homomorphism of groups $\gamma\colon \mathrm{Aut}(\mathcal{X}_1) \hookrightarrow \mathrm{Aut}(X)$ such that sp is γ-equivariant with regard to the natural actions of $\mathrm{Aut}(\mathcal{X}_1)$ on $\mathrm{Pic}(\mathcal{X}_1)$ and $\mathrm{Aut}(X)$ on $\mathrm{Pic}(X)$. In other words, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Aut}(\mathcal{X}_1) \arrow[r, "\tilde{r}"] \arrow[d, "\gamma"'] & O\left(\mathrm{NS}(\mathcal{X}_1)\right) \arrow[d, "\mathrm{sp}^*"] \\ \mathrm{Aut}(X) \arrow[r, "r"] & O\left(\mathrm{NS}(X)\right) \end{tikzcd}$$

where sp^* is an isomorphism and the kernels $\ker(r)$ and $\ker(\tilde{r})$ are finite. The argument is the same as in the proof of [LM11, Theorem 6.1]): fixing a very ample line bundle $\mathcal{L}$ on X, we find that $\ker(r)$ is contained in the automorphism group $\mathrm{Aut}(X, \mathcal{L})$ of the pair. Since this group is contained in some projective linear group, it is of finite type, and since X has no non-trivial vector fields [RS76, Theorem 7.], it is discrete. It thus follows that $\mathrm{Aut}(X, \mathcal{L})$ is finite, and therefore that $\ker(r)$ is also finite. The proof for $\ker(\tilde{r})$ is the same.

By the Lefschetz principle, the K3 surface $\mathcal{X}_1$ has a complex model $\tilde{\mathcal{X}_1}$, i.e. there exists a K3 surface $\tilde{\mathcal{X}_1}$ over $\mathbb{C}$ such that $\mathcal{X}_1 \cong \tilde{\mathcal{X}_1} \times_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\, k$. It follows from the fact that K3 surfaces have no non-trivial vector fields [RS76, Theorem 7], that the automorphism group of any K3 surface is discrete. But this implies that there is a natural bijection between the $\mathbb{C}$-valued points of the functor $\mathrm{Aut}_{\tilde{\mathcal{X}_1}}$ and the k-valued points of $\mathrm{Aut}_{\tilde{\mathcal{X}_1}}$, or in other words that the natural morphism between the automorphism groups of $\mathcal{X}_1$ and $\tilde{\mathcal{X}_1}$ is an isomorphism. In the above diagram, we may therefore assume that $\mathcal{X}_1$ is already defined over $\mathbb{C}$. We have inclusions of subgroups

$$O_0^+(\mathrm{NS}(\mathcal{X}_1)) \subseteq \mathrm{im}(\tilde{r}) \subseteq \mathrm{im}(r) \subseteq O^+(\mathrm{NS}(\mathcal{X}_1)),$$

where the inclusion $O_0^+(\mathrm{NS}(\mathcal{X}_1)) \subseteq \mathrm{im}(\tilde{r})$ is due to the Global Torelli theorem [PSS71], since any $\varphi \in O_0^+(\mathrm{NS}(\mathcal{X}_1))$ preserves the ample cone $\mathcal{A}_X$ and can be extended to an isometry of $H^2(X, \mathbb{Z})$ that acts trivially on the transcendental lattice of X. Hence every element of $O_0^+(\mathrm{NS}(\mathcal{X}_1))$ comes from an automorphism of $\mathcal{X}_1$. Since γ is injective, we have $\mathrm{im}(\tilde{r}) \subseteq \mathrm{im}(r)$ and, since the pullback of an ample line bundle of X under an automorphism of X is ample again, we have $\mathrm{im}(r) \subseteq O^+(\mathrm{NS}(X))$. The isomorphism $\mathrm{sp}\colon \mathrm{Pic}(\mathcal{X}_1) \to \mathrm{Pic}(X)$ preserves the ample cone [LM11, Corollary 2.3], and so we find $\mathrm{im}(r) \subseteq O^+(\mathrm{NS}(\mathcal{X}_1))$.

The index of $O_0^+(\mathrm{NS}(\mathcal{X}_1))$ in $O^+(\mathrm{NS}(\mathcal{X}_1))$ is finite because $O\left(q_{\mathrm{NS}(\mathcal{X}_1)}\right)$ is a finite group. Thus the indices $[\mathrm{im}(r)\colon O_0^+(\mathrm{NS}(\mathcal{X}_1))]$ and $[O^+(\mathrm{NS}(\mathcal{X}_1))\colon \mathrm{im}(r)]$ are also finite. It follows from [Oha07, Lemma 1.4. (c)], and the fact that there are only finitely many conjugacy classes of finite subgroups in $O^+(\mathrm{NS}(\mathcal{X}_1))$ [Oha07, proof of Theorem 1.5], that there are only finitely many conjugacy classes of finite subgroups in $\mathrm{im}(r)$. Since $\ker(r)$ is finite, it follows from [Oha07, Lemma 1.4 (a)] that $\mathrm{Aut}(X)$ contains only finitely many conjugacy classes of finite subgroups. In particular, X has only finitely many isomorphism classes of Enriques quotients. $\square$

REMARK 1.4. The theory of Enriques quotients of K3 surfaces of finite height is closely related to the characteristic zero situation. In many cases, given a finite height K3 surface X, we can choose a Neron-Severi group preserving lift $\mathcal{X}_1$ of X such that the specialization morphism $\gamma\colon \mathrm{Aut}(\mathcal{X}_1) \longrightarrow \mathrm{Aut}(X)$ is an isomorphism.

This is possible, for example, when X is *ordinary*, that means X is of height 1 [Nyg83] [Sri19, Theorem 4.11] [LT19, Proposition 2.3]. Another class for which such lifts exist are the so-called *weakly tame* K3 surfaces over fields of characteristic $p \geq 3$. In particular, every K3 surface of finite height over a field k of characteristic $p \geq 23$ is weakly tame. For definitions and details we refer to [Jan17].

In these situations we can then use the results from [Oha07] to obtain the number of isomorphism classes of Enriques quotients of X.

2. The supersingular case

Let X be a supersingular K3 surface over an algebraically closed field k of characteristic $p \geq 3$. The following proposition shows that X has only finitely many isomorphism classes of Enriques quotients.

PROPOSITION 1.5. *Let X be a supersingular K3 surface over an algebraically closed field k of characteristic $p \geq 3$. The number of isomorphism classes of Enriques quotients of X is finite.*

PROOF. There are only finitely many conjugacy classes of finite subgroups in $O^+(\mathrm{NS}(X))$ by [PR94, Theorem 4.3]. Furthermore, the semidirect product $\mathrm{Aut}(X) \ltimes W_X$ is a subgroup of finite index in $O^+(\mathrm{NS}(X))$ by [LM11, Proposition 5.2.]. Thus, [Oha07, Lemma 1.4 (b), (c)] implies that there are only finitely many conjugacy classes of finite subgroups in $\mathrm{Aut}(X)$. In particular, X has only finitely many isomorphism classes of Enriques quotients. $\square$

Our goal for the rest of this section is to find a formula for the number of Enriques quotients of X in the style of [Oha07, Theorem 2.3]. The argumentation does not rely on the previous proposition.

If Y is an Enriques surface, then its Neron-Severi group $\mathrm{NS}(Y)$ is isomorphic to the lattice $\Gamma = U_2 \oplus E_8(-1)$, which is up to isomorphism the unique unimodular, even lattice of signature $(1,9)$. Following [Oha07], if X is a supersingular K3 surface over a field of characteristic $p \geq 3$, we define

$$\mathfrak{M} := \left\{ \begin{array}{c|l} N \subseteq \mathrm{NS}(X) & \text{primitive sublattices satisfying} \\ & (A)\colon N \cong \Gamma(2) \\ & (B)\colon \text{No vector of square } -2 \text{ in } \mathrm{NS}(X) \text{ is orthogonal to } N \end{array} \right\}$$

and

$$\mathfrak{M}^* := \{N \in \mathfrak{M} \mid N \text{ contains an ample divisor}\}\,.$$

The following proposition describes free involutions on a supersingular K3 surface in terms of embeddings of lattices.

PROPOSITION 1.6. *[Jan13, Theorem 4.1] Let k be a an algebraically closed field of characteristic $p \geq 3$. For a supersingular K3 surface X over k, there is a natural bijection*

$$\mathfrak{M}^* \stackrel{1:1}{\longleftrightarrow} \{\textit{free involutions of } X\}.$$

PROOF. First, let $\iota\colon X \to X$ be a free involution of X and let $f\colon X \to Y$ be the associated Enriques quotient. Since the map f is finite étale of degree 2, we obtain a primitive embedding of lattices

$$U(2) \oplus E_8(2) \cong f^*(\mathrm{NS}(Y)) \hookrightarrow \mathrm{NS}(X).$$

We write $N = f^*(\mathrm{NS}(Y))$ and $M = N^\perp$, such that

$$N = \{v \in \mathrm{NS}(X) \mid \iota^*(v) = v\} \text{ and } M = \{v \in \mathrm{NS}(X) \mid \iota^*(v) = -v\}.$$

Then N has property (B): By the Riemann-Roch theorem, if v is a (-2)-divisor on X, then v or $-v$ is effective. Thus, if $v \in M$ was a (-2)-divisor, then both v and $-v$ are effective, which is absurd. Pullback along finite morphisms preserves ampleness, hence N contains an ample line bundle and we have shown that $N \in \mathfrak{M}^*$.

On the other hand, assume we are given some $N \in \mathfrak{M}^*$ and define

$$\begin{aligned}\psi\colon N \oplus N^\perp &\longrightarrow N \oplus N^\perp,\\ (v,w) &\longmapsto (v,-w).\end{aligned}$$

Then ψ extends to $\mathrm{NS}(X)$ [Jan13, Lemma 4.2.] and by the supersingular Torelli theorem [Ogu83] induces an involution ι on X. We prove in Lemma 1.9 that ι is free, and we readily see that the construction of ι is inverse to the construction of N given above. □

The next two lemmas are needed for the proof of Lemma 1.9.

LEMMA 1.7. *[Jan13, Proposition 2.2] Let X be a supersingular K3 surface. Then there exists a canonical surjection*

$$H^2_{\mathrm{cris}}(X/W)/(\mathrm{NS}(X) \otimes W) \to H^2(X, \mathcal{O}_X).$$

PROOF. We have a natural isomorphism $H^1(X, W\Omega^1_{X/k}) \cong \mathrm{NS}(X) \otimes W$ [Nyg79, Proposition 3.2] and the equality $H^0(X, W\Omega^2_{X/k}) = 0$, because the Newton slopes of $H^2_{\mathrm{cris}}(X/W)$ are all smaller than 2 and the Newton slopes of $H^0(X, W\Omega^2_{X/k})$ would be contained in the intervall $[2,3)$ [Jan13]. From the slope spectral sequence we get a natural isomorphism

$$H^2_{\mathrm{cris}}(X/W)/(\mathrm{NS}(X) \otimes W) \cong \ker(d\colon H^2(X, W\mathcal{O}_X) \to H^2(X, W\Omega^1_{X/k}))$$

thus it follows from [Nyg80, Lemma 1.11.] that

$$H^2_{\mathrm{cris}}(X/W)/(\mathrm{NS}(X) \otimes W) \cong k[\![V]\!]/V^{\sigma_X}.$$

There is an isomorphism $H^2(X, W\mathcal{O}_X) \cong k[\![V]\!]$ [Nyg79, cf. proof of Theorem 3.4], and the short exact sequence

$$0 \longrightarrow W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

yields a natural isomorphism $H^2(X, \mathcal{O}_X) \cong H^2(X, W\mathcal{O}_X)/VH^2(X, W\mathcal{O}_X)$. But we also have the equality $\ker d + VH^2(X, W\mathcal{O}_X) = H^2(X, W\mathcal{O}_X)$ and this implies that there is a canonical surjection as in the lemma. □

LEMMA 1.8. *Let X be a scheme, $\iota \in \operatorname{Aut}(X)$ an automorphism of finite order spanning the subgroup $G \subseteq \operatorname{Aut}(X)$. We write $Y := X/G$. There is a canonical isomorphism of groups*

$$\operatorname{Pic}(Y)_{\text{free}} \xrightarrow{\simeq} \operatorname{Pic}(X)^G_{\text{free}}.$$

PROOF. The method used in the proof is due to [DK04]. We consider the functor

$$\begin{aligned}(\text{abelian } G\text{-sheaves on } X) &\longrightarrow (\mathrm{Ab}),\\ \mathcal{F} &\longmapsto \Gamma(X, \mathcal{F})^G.\end{aligned}$$

We can write this functor as a composition of functors in two ways, namely

$$\begin{aligned}(\text{abelian } G\text{-sheaves on } X) \to (\text{abelian sheaves on } Y) &\to (\mathrm{Ab}),\\ \mathcal{F} \mapsto \pi^G_* \mathcal{F} &\mapsto \Gamma(Y, \pi^G_* \mathcal{F}),\end{aligned}$$

where $\pi^G_* \mathcal{F}(U) = \Gamma(\pi^{-1}(U), \mathcal{F})^G$ or

$$\begin{aligned}(\text{abelian } G\text{-sheaves on } X) \to (\mathrm{Ab}) &\to (\mathrm{Ab}),\\ \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) &\mapsto \Gamma(X, \mathcal{F})^G.\end{aligned}$$

Thus, we obtain an associated Grothendieck spectral sequence

$$E_2^{p,q}(\mathcal{O}_X^*) = H^p(Y, \mathcal{H}^q(G, \mathcal{O}_Y^*)) \Rightarrow H^{p+q}(X, \mathcal{H}^i(G, \mathcal{O}_X^*)).$$

The 5-exact sequence associated to this spectral sequence then yields

$$0 \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(X)^G \longrightarrow H^0(Y, \operatorname{Hom}(G, \mathcal{O}_Y^*)) \longrightarrow \dots$$

where $H^0(Y, \operatorname{Hom}(G, \mathcal{O}_Y^*))$ is torsion, implying the claim. □

LEMMA 1.9. *The involution ι defined in the proof of Proposition 1.6 is free.*

PROOF. The plan of the proof is as follows. We will show that if ι had fixed points, then the quotient $Y := X/\iota$ would be a minimal rational surface. We then show that this is impossible.

We let N be as in the proof of Lemma 1.6 and write $M := N^\perp$. We can then choose some unimodular W-overlattice M' of $M \otimes W$ such that $H^2_{\text{cris}}(X/W) = (N \otimes W) \oplus M'$. Since $N \otimes W$ is unimodular, with this notation there is a natural isomorphism

$$M'/(M \otimes W) \cong H^2_{\text{cris}}(X/W)/(\mathrm{NS}(X) \otimes W).$$

But ι acts as -1 on M and therefore also on M' and on $H^2_{\text{cris}}(X/W)/(\mathrm{NS}(X) \otimes W)$. Using the natural surjection from Lemma 1.7, the morphism ι then also acts as -1 on $H^2(X, \mathcal{O}_X)$. It now follows from Serre duality and the fact that $\omega_X \cong \mathcal{O}_X$ that ι also acts as -1 on $H^0(X,\omega_X) = H^0(X,\Omega_X^2)$ and on Ω_X^2.

Now we assume that $x' \in X$ is a fixed point for ι. Since ι acts via -1 on the determinant of the cotangent bundle, taking an étale neighborhood of x' isomorphic to $\operatorname{Spec}(k[t_1,t_2])$, the morphism ι can locally be given as $t_1 \mapsto t_1$, $t_2 \mapsto -t_2$. This implies that the fixed locus of ι is a smooth divisor in X. Furthermore, we see that the morphism $\pi\colon X \longrightarrow Y$ is flat. Indeed, if $x \in X$ is not a fixed point of ι, then π is locally at x an étale 2-cover and if $x \in X$ is a fixed point of ι, then q is étale locally at x given by the free ring extension $k[t_1,t_2^2] \to k[t_1,t_2]$.

Now Lemma 1.8 above and [Bea96, Proposition I.8 (ii)] imply that $\mathrm{NS}(Y)_{\mathrm{free}} \cong U \oplus E_8$. In particular, the smooth surface Y contains no (-1)-curves and is therefore minimal. We write $C \subseteq Y$ for the ramification locus of π. Then $\pi^*(K_Y + C) = K_X = \mathcal{O}_X$. Using this, the projection formula now yields a canonical isomorphism

$$\pi_*\mathcal{O}_X \otimes \omega_Y \otimes \mathcal{O}_Y(C) \cong \pi_*\mathcal{O}_X.$$

Flatness and finiteness of π, together with the connectedness of Y, imply that π is faithfully flat. Thus we obtain $\omega_Y \cong \mathcal{O}_Y(-C)$, which means that ω_Y^{-1} is effective. This implies that for the Kodaira dimension we have $\kappa(Y) = -\infty$, and therefore the surface Y is ruled or rational.

The Albanese variety $\mathrm{Alb}(X)$ is trivial, since $H^1_{\text{ét}}(X,\mathbb{Q}_l) = 0$. We also have a surjection $\mathrm{Alb}(X) \twoheadrightarrow \mathrm{Alb}(Y)$ and if Y is birational to $C \times \mathbb{P}^1$, then $\dim(\mathrm{Alb}(Y)) = g(C)$ which implies $C \cong \mathbb{P}^1$.

Thus Y is rational and therefore either isomorphic to $\mathbb{P}^2$ or to some Hirzebruch surface $\mathbb{F}_n$, implying that $H^2_{\acute{e}t}(Y,\mathbb{Q}_l) \in \{1,2\}$. On the other hand, we have $\mathrm{NS}(Y) \subseteq H^2_{\acute{e}t}(Y,\mathbb{Q}_l)$ with $\mathrm{rk}(\mathrm{NS}(Y)) = 10$, which is a contradiction. It follows that the involution ι had no fixed points. $\square$

The previous lemma concludes the proof of Proposition 1.6.

If A is a finitely generated abelian group and q a prime number, we denote by $A^{(q)}$ the q-torsion part of A and by $l(A)$ the minimal cardinality among all sets of generators of A.

LEMMA 1.10. *Let k be an algebraically closed field of characteristic $p \geq 3$ and X a supersingular K3 surface over k. The canonical morphism* $\mathrm{pr}\colon O(\mathrm{NS}(X)) \to O(q_{\mathrm{NS}(X)})$ *is surjective.*

PROOF. The Néron-Severi lattice of a supersingular K3 surface X is even, indefinite and non-degenerate with $\mathrm{rk}(\mathrm{NS}(X)) = 22$ and $2 \leq l(A^{(p)}_{\mathrm{NS}(X)}) \leq 20$, $l(A^{(q)}_{\mathrm{NS}(X)}) = 0$ for any prime $q \neq p$. Now the lemma follows from [Nik80, Theorem 1.14.2]. $\square$

We will need the following lemma and proposition. The statement we need to show has already been proved in [Nyg80, Theorem 2.1 and Remark 2.2], but not been stated explicitly. We will therefore give a full proof.

When G is a formal group law, we write DG for the associated Dieudonné module as in [Mum69, Section 1].

LEMMA 1.11. *Let*

$$\psi\colon D\hat{\mathbb{G}}_a \xrightarrow{\cong} D\hat{\mathbb{G}}_a$$

be a continuous automorphism of left $\mathrm{Cart}(k)$-modules such that there exists a non-trivial finite dimensional k-subvector space $U \subset D\hat{\mathbb{G}}_a$ with $\psi(U) \subseteq U$. Then ψ is the multiplication by some element $a \in k^\times$ from the right.

PROOF. We have

$$D\hat{\mathbb{G}}_a = \prod_{i=0}^{\infty} V^i k$$

as a $\mathrm{Cart}(k)$-module with trivial F-action and W-action coming from the projection $W \twoheadrightarrow k$.

We let $\psi\colon D\hat{\mathbb{G}}_a \xrightarrow{\cong} D\hat{\mathbb{G}}_a$ be an automorphism such that $\psi(1) = \sum_{i=0}^{\infty} a_i V^i$ and take an arbitrary element $x = \sum x_j V^j \in D\hat{\mathbb{G}}_a$. Then, since $V a_i = a_i^{\frac{1}{p}}$ it follows by continuity that

$$\psi(x) = \sum a_i^{\frac{1}{p^j}} x_j V^{i+j}.$$

In other words, we can regard ψ as the k-linear automorphism of $k[\![V]\!]$ given by multiplication with $a = \sum a_i V^i \in k[\![V]\!]$ from the right. We want to see that a is an element of k.

Since ψ is an automorphism, we have that $a_0 \neq 0$. When $x = \sum_{i=0}^{\infty} b_i V^i \in D\hat{\mathbb{G}}_a$ is a power series, we write $\mathrm{subdeg}(x) = \min\{i \mid b_i \neq 0\}$. We assume that $a \notin k^\times$ and let $u^{(0)} \in U$. Then

$$u^{(1)} \coloneqq \psi\left(u^{(0)}\right) - a_0^{\left(p^{-\mathrm{subdeg}\left(u^{(0)}\right)}\right)} u^{(0)}$$

is also an element of U and we have $\mathrm{subdeg}(u^{(1)}) > \mathrm{subdeg}(u^{(0)})$. Inductively, taking

$$u^{(n+1)} \coloneqq \psi\left(u^{(n)}\right) - a_0^{\left(p^{-\mathrm{subdeg}\left(u^{(n)}\right)}\right)} u^{(n)},$$

we find that $u^{(n+1)}$ is an element of U with $\mathrm{subdeg}(u^{(n+1)}) > \mathrm{subdeg}(u^{(n)})$. This is a contradiction to the finiteness of the dimension of U and hence concludes the proof of the lemma. $\square$

With the use of the technical Lemma 1.11 we can prove the following nice observation.

PROPOSITION 1.12. *Let k be an algebraically closed field of characteristic $p \geq 3$ and let X be a supersingular K3 surface of Artin invariant σ_X over k such that the point corresponding to X in the moduli space of supersingular K3 crystals $\mathbb{A}_k^{\sigma_X - 1}/\mu_{p^{\sigma_X}+1}$ has coordinates $(b_1, \ldots, b_{\sigma_X - 1})$ with $b_1 \neq 0$. Let $\theta \in \mathrm{Aut}(X)$ be an automorphism of X. Then the induced automorphism $\theta^* \in O(q_{\mathrm{NS}(X)})$ of $A_{\mathrm{NS}(X)}$ is the identity or multiplication with -1.*

PROOF. To simplify notation, we write $\mathrm{NS} = \mathrm{NS}(X)$ and $\sigma = \sigma_X$. Since there exists a natural isomorphism of lattices $A_{\mathrm{NS}} \otimes k \cong T_0 \otimes k$, it follows from [Nyg80, Theorem 1.12] that there exists a functorial embedding $A_{\mathrm{NS}} \otimes k \hookrightarrow H^2(X, W\mathcal{O}_X)$.

More precisely, from [Nyg80, Lemma 1.11] it follows that the image of the quadratic space $A_{\mathrm{NS}} \otimes k$ in $H^2(X, W\mathcal{O}_X) \cong D\Phi_X^2 = k[\![V]\!]$ has basis $\{1, \ldots, V^{2\sigma - 1}\}$. Furthermore, the embedding

$$H^2_{\mathrm{cris}}(X/W)/(\mathrm{NS} \otimes W) \hookrightarrow H^2(X, W\mathcal{O}_X)$$

identifies $H^2_{\mathrm{cris}}(X/W)/(\mathrm{NS}\otimes W)$ with the subspace of $A_{\mathrm{NS}}\otimes k$ with basis $\{1,\ldots,V^{\sigma-1}\}$ and it follows from [Ogu83, Proposition 2.12] that this is a strictly characteristic subspace.

We write $\langle -,-\rangle$ for the bilinear form on $A_{\mathrm{NS}}\otimes k$ and we claim that $\langle V^{\sigma-1},V^{2\sigma-1}\rangle\neq 0$. Indeed, we have that $\mathrm{span}(1,\ldots,V^{\sigma-1})$ is a maximal isotropic subspace in $A_{\mathrm{NS}}\otimes k$. We assume that we have $\langle V^{\sigma-1},V^{2\sigma-1}\rangle = 0$. We write $\varphi\colon A_{\mathrm{NS}}\otimes k\to A_{\mathrm{NS}}\otimes k$ for the action of the Frobenius. For $1<n\leq\sigma$ we find

$$\langle V^{\sigma-n},V^{2\sigma-1}\rangle = \langle\varphi^{n-1}(V^{\sigma-1}),\varphi^{n-1}(V^{n-2})\rangle = \langle V^{\sigma-1},V^{n-2}\rangle = 0.$$

Thus the space $\mathrm{span}(1,\ldots,V^{\sigma-1})+\langle V^{2\sigma-1}\rangle$ would be isotropic. This yields a contradiction.

Now let $\theta\colon X\to X$ be an automorphism. Then the induced $\theta^*\colon A_{\mathrm{NS}}\otimes k\to A_{\mathrm{NS}}\otimes k$ is an automorphism of lattices, and it follows from Lemma 1.11 that $\theta^*(V^i) = a^{\frac{1}{p^i}}$ for some $a\in k^\times$ and all $i\in\mathbb{N}$. Thus we find

$$\begin{aligned}\langle V^{\sigma-1},V^{2\sigma-1}\rangle &= \langle\theta^*(V^{\sigma-1}),\theta^*(V^{2\sigma}-1)\rangle\\ &= \langle a^{\frac{1}{p^{\sigma-1}}}V^{\sigma-1},a^{\frac{1}{p^{2\sigma-1}}}V^{2\sigma-1}\rangle\\ &= a^{\frac{1+p^\sigma}{p^{2\sigma-1}}}\langle V^{\sigma-1},V^{2\sigma-1}\rangle\end{aligned}$$

and it follows that $a^{p^\sigma+1}=1$.

On the other hand, from [Nyg80, Proposition 1.18] we get that

$$b_1 = \langle V^{\sigma-2},V^{2\sigma-1}\rangle.$$

Since $b_1\neq 0$, it follows from

$$\begin{aligned}b_1 &= \langle V^{\sigma-2},V^{2\sigma-1}\rangle\\ &= \langle\theta^*(V^{\sigma-2}),\theta^*(V^{2\sigma-1})\rangle\\ &= a^{\frac{p^{\sigma+1}+1}{p^{2\sigma-1}}}\langle V^{\sigma-2},V^{2\sigma-1}\rangle\end{aligned}$$

that we have $a^{p^{\sigma+1}+1}=1$. Thus we find

$$1 = \frac{a^{p^{\sigma+1}+1}}{a^{p^\sigma+1}} = (a^{p-1})^{p^\sigma}$$

and therefore also

$$1 = a^{p-1}.$$

In other words, we have that $a\in\mathbb{F}_p$. But then the morphism $\theta^*\colon A_{\mathrm{NS}}\otimes k\to A_{\mathrm{NS}}\otimes k$ is just multiplication by a and, from the equality $a^{p^\sigma+1}=1$, it follows that $a^2=1$. $\square$

REMARK 1.13. Of course, the subset of $\mathbb{A}_k^{\sigma_X-1}/\mu_{p^{\sigma_X}+1}$ consisting of points $(b_1,\ldots,b_{\sigma_X-1})$ with $b_1\neq 0$ is open. If $\sigma>1$, then this subset is also dense in $\mathbb{A}_k^{\sigma_X-1}/\mu_{p^{\sigma_X}+1}$. It follows from [Ogu79, Proposition 4.10] that, in this case, the corresponding subset in the period space of supersingular K3 surfaces $\mathcal{M}_\sigma$ is also dense.

REMARK 1.14. There are also supersingular K3 surfaces X with $b_1 = 0$ such that each automorphism of X induces either the identity or multiplication by -1 on the transcendental lattice. For example, let X be with $\sigma_X = 4$ and such that $b_1 = 0$ and $b_2 = 1$. Going back to the argument in the proof of Proposition 1.12, we then find

$$1 = \langle V^{\sigma-3}, V^{2\sigma-1}\rangle = a^{\frac{p^{\sigma+2}+1}{p^{2\sigma-1}}}\langle V^{\sigma-3}, V^{2\sigma-1}\rangle,$$

and it thus follows that $a^{p^{\sigma+2}+1} = 1 = a^{p^{\sigma}+1}$. Hence, it is $(a^{p^2-1})^{p^{\sigma}} = 1$ and we find $a \in \mathbb{F}_{p^2}$. But then, using that $\sigma = 4$, we have

$$1 = a^{p^4+1} = (a^{p^2})^{p^2} \cdot a = a^2$$

and we can conclude as in the proof of Proposition 1.12.

REMARK 1.15. On the other hand, there also exist examples of supersingular K3 surfaces X and automorphisms $\theta \in \mathrm{Aut}(X)$ such that the induced morphism on $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$ is not the identity or multiplication by -1. For example if $\sigma_X = 1$, then the image of the canonical map $\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{NS}(X)^\vee/\mathrm{NS}(X))$ is known to be a cyclic group of order $p+1$ [Jan16, Remark 3.4].

The following theorem is the supersingular version of a characteristic zero theorem by Ohashi [Oha07, Theorem 2.3.]. Similar to the situation in characteristic zero, we only obtain an inequality in general. In characteristic zero there are two conditions on a K3 surface X that have to be fullfilled in order to obtain an equality. One of these is the surjectivity of the canonical morphism $\mathrm{pr}\colon O(\mathrm{NS}(X)) \to O(q_{\mathrm{NS}(X)})$. This is always true for supersingular K3 surfaces by Lemma 1.10. The other condition is that each automorphism of X induces $\pm\mathrm{id}$ on the transcendental lattice of X. We gave a sufficient criterion under which this is always true in Proposition 1.12.

THEOREM 1.16. *Let k be an algebraically closed field of characteristic $p \geq 3$ and let X be a supersingular K3 surface over k. Let $M_1, \ldots, M_l \in \mathfrak{M}$ be a complete set of representatives for the action of $O(\mathrm{NS}(X))$ on $\mathfrak{M}$. For each $j \in \{1, \ldots, l\}$, we let*

$$K^{(j)} = \{\psi \in O(\mathrm{NS}(X)) \mid \psi(M_j) = M_j\}$$

be the stabilizer of M_j and $\mathrm{pr}(K^{(j)})$ be its canonical image in $O(q_{\mathrm{NS}(X)})$. Then we have inequalities

$$l \leq \#\left\{\textit{Enriques quotients of } X\right\} \leq \sum_{j=1}^{l} \#\left(O\left(q_{\mathrm{NS}(X)}\right)/\mathrm{pr}\left(K^{(j)}\right)\right).$$

If X is such that for each automorphism $\theta \in \mathrm{Aut}(X)$ the induced automorphism on the quotient $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$ is either the identity or multiplication by -1, then the inequality above becomes an equality on the right side.

PROOF. It follows from [Nik80, Proposition 1.15.1.] that the number of representatives M_j is indeed finite. Therefore, using Proposition 1.6 and Lemma 1.10, the proof goes word by word as the proof of [Oha07, Theorem 2.3.]. □

REMARK 1.17. It follows from Remark 1.13 that for a generic supersingular K3 surface X of Artin invariant $\sigma > 1$ the inequality on the right hand side in Theorem 1.16 is an equality.

3. Existence of Enriques quotients for supersingular K3 surfaces

In the previous section, in Theorem 1.16 we gave a formula that computes the number of Enriques quotients for a generic supersingular K3 surface X. However, it turns out that explicitly calculating this number is difficult. A priori it is not even clear that this number is non-zero, or in other words that for a given supersingular K3 surface X the corresponding set of lattices $\mathfrak{M}$ is non-empty. The following result is due to J. Jang.

PROPOSITION 1.18. *[Jan13, Theorem 4.5, Proposition 3.5] Let k be an algebraically closed field of characteristic p and let X be a supersingular K3 surface of Artin invariant σ. If $\sigma = 1$, then X has an Enriques involution. If $\sigma \in \{3, 5\}$, and $p = 11$ or $p \geq 19$, then X has an Enriques involution. If $\sigma \in \{2, 4\}$, and $p = 19$ or $p \geq 29$, then X has an Enriques involution. If $\sigma \geq 6$, then X has no Enriques involution.*

The idea of the proof is as follows. Associated to a supersingular K3 surface X of Artin invariant σ over a field k of characteristic p, one constructs a K3 surface $X_{\sigma,d}$ over $\mathbb{C}$ such that the transcendental lattice $T(X_{\sigma,d})$ is isomorphic to a lattice $U(2) \oplus M_{\sigma,d}$, where $M_{\sigma,d}$ is a certain lattice that admits an embedding into $\Gamma(2)$ such that its orthogonal complement does not contain any (-2)-vectors. For large enough characteristic p as in the statement of the proposition one can choose d such that we find a chain of primitive embeddings $\Gamma(2) \hookrightarrow \mathrm{NS}(X_{\sigma,d}) \hookrightarrow \mathrm{NS}(X)$. In this situation one can show that the orthogonal complement of $U(2) \oplus E_8(2)$ in $\mathrm{NS}(X)$ does not contain any (-2)-vectors. However, this method is not applicable for small p. We note that there are only 24 cases left to work out and we can try to show the existence of an Enriques quotient in those remaining cases by hand.

THEOREM 1.19. *Let k be an algebraically closed field of characteristic p where $p \geq 3$ and let X be a supersingular K3 surface of Artin invariant σ. Then X has an Enriques involution if and only if $\sigma \leq 5$.*

This result has already been shown by Jang in a later paper [Jan15] via lifting techniques, but we want to reprove it using the lattice argument that we described above.

3.1. Computational approach. Let X be a supersingular K3 surface of Artin invariant σ over an algebraically closed field k with characteristic $p \geq 3$. By the results in the previous section, it suffices to show that there exists a primitive embedding of the lattice $\Gamma(2)$ into $\mathrm{NS}(X)$ such that the orthogonal complement of $\Gamma(2)$ in $\mathrm{NS}(X)$ does not contain any vector of self-intersection -2. We denote by $A_{S_{p,\sigma}}$ the discriminant lattice of $\mathrm{NS}(X)$ and by $q_{S_{p,\sigma}}$ the quadratic form on $A_{S_{p,\sigma}}$. Similarly we write $A_{\Gamma(2)}$ for the discriminant lattice of $\Gamma(2)$ and $q_{\Gamma(2)}$ for the quadratic form on $A_{\Gamma(2)}$.

REMARK 1.20. The lattice $\mathrm{NS}(X)$ is the unique lattice up to isomorphism in its genus [RS81, Section 1], so by [Nik80, Proposition 1.15.1] the datum of a primitive embedding $\Gamma(2) \hookrightarrow \mathrm{NS}(X)$ with orthogonal complement L is equivalent to the datum of an even lattice L with invariants $(0, 12, \delta_{p,\sigma})$, where $\delta_{p,\sigma}$ is $-q_{S_{p,\sigma}} \oplus q_{\Gamma(2)}$ acting on $A_{S_{p,\sigma}} \oplus A_{\Gamma(2)}$ and $(0, 12)$ is the signature of

L. To see this, observe that in our case $\#A_{S_{p,\sigma}} = p^{2\sigma}$ and $\#A_{\Gamma(2)} = 2^{10}$ are coprime, and so the isomorphism of subgroups γ in the cited proposition has to be the zero-morphism.

It follows from the previous remark that to prove Theorem 1.19, we have to construct lattices $L_{p,\sigma}$ of genus $(0, 12, \delta_{p,\sigma})$ such that the $L_{p,\sigma}$ do not contain any vectors of self-intersection -2. Using the computer algebra program MAGMA, we constructed the lattices $L_{p,\sigma}$ in the missing cases. I am indebted to Markus Kirschmer for helping me to use the program and writing code to automatize step 1 of the following method:

- Step 1. Construct an arbitrary lattice L of genus $(0, 12, \delta_{p,\sigma})$. This can be done, for example, in the following way. Using [RS81, Chapter 1.] we can construct the lattice $\mathrm{NS}(X)$ explicitly. Then we choose an arbitrary primitive embedding $N \hookrightarrow \mathrm{NS}(X)$ and take L to be the orthogonal complement under this embedding. We remark that in general the lattice L may contain vectors of self-intersection -2.
- Step 2. Apply Kneser's neighbor method [Kne57], which has been implemented for MAGMA, to the positive definite lattice $-L$. This generates a list of further candidate lattices in the same genus as $-L$. Using the "Minimum();" function in MAGMA we can test for the minimum length of vectors in those candidate lattices until we find a candidate that does not contain any vectors of length 2.

Note that we might have to iterate the neighbor method.

Applying the above method, we found the following lattices $L_{p,q}$ of genus $(0, 12, \delta_{p,q})$ that do not contain any vectors of self-intersection -2. We represent these lattices via their Gram matrix. Their existence in conjuction with the results from [Jan13] imply Theorem 1.19.

$$L_{(p=3,\sigma=2)} = \begin{pmatrix} -4 & 2 & 2 & 0 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & 0 \\ 2 & -4 & -2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 2 & -2 & -4 & -1 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & -4 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ -2 & 2 & 2 & 0 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & -2 & -4 & -1 & 0 & -1 & 0 & 1 & -1 \\ -2 & 0 & 0 & 1 & 0 & -1 & -4 & -2 & -2 & 2 & 1 & 1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -2 & -4 & -2 & 2 & 2 & 2 \\ -2 & 0 & 0 & -1 & 0 & -1 & -2 & -2 & -4 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & -4 & -2 & 0 \\ 2 & -1 & -1 & 1 & 0 & 1 & 1 & 2 & 1 & -2 & -6 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & 1 & 2 & 1 & 0 & 0 & -6 \end{pmatrix}$$

$$L_{(p=5,\sigma=2)} = \begin{pmatrix}
-6 & -2 & -2 & 2 & 2 & -2 & 1 & -1 & 2 & 6 & -2 & -6 \\
-2 & -4 & -2 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & -2 \\
-2 & -2 & -4 & 0 & 0 & 2 & -2 & 2 & -2 & 0 & 0 & -2 \\
2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & -4 & 2 & 0 & 0 & 0 & -2 & -2 & 2 \\
-2 & 2 & 2 & 0 & 2 & -6 & 3 & -3 & 4 & 4 & 0 & -2 \\
1 & 0 & -2 & 0 & 0 & 3 & -8 & 3 & -3 & -3 & 1 & -1 \\
-1 & 0 & 2 & 0 & 0 & -3 & 3 & -8 & 3 & 3 & -1 & 1 \\
2 & -2 & -2 & 0 & 0 & 4 & -3 & 3 & -10 & -8 & 6 & 2 \\
6 & 0 & 0 & 0 & -2 & 4 & -3 & 3 & -8 & -14 & 6 & 10 \\
-2 & 0 & 0 & 0 & -2 & 0 & 1 & -1 & 6 & 6 & -10 & -2 \\
-6 & -2 & -2 & 0 & 2 & -2 & -1 & 1 & 2 & 10 & -2 & -14
\end{pmatrix}$$

$$L_{(p=7,\sigma=2)} = \begin{pmatrix}
-4 & -2 & 0 & -2 & 0 & -2 & 2 & 0 & -2 & 2 & -2 & -4 \\
-2 & -4 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 \\
0 & 0 & -4 & 0 & -2 & 1 & -1 & -1 & -1 & 1 & -3 & 0 \\
-2 & 0 & 0 & -4 & 0 & 0 & 2 & 0 & -2 & 4 & -2 & -4 \\
0 & 0 & -2 & 0 & -4 & 1 & 1 & -1 & -1 & -1 & -3 & 0 \\
-2 & 0 & 1 & 0 & 1 & -6 & 0 & 2 & 0 & 2 & 0 & -3 \\
2 & 0 & -1 & 2 & 1 & 0 & -6 & 2 & 4 & -2 & 4 & 5 \\
0 & 2 & -1 & 0 & -1 & 2 & 2 & -6 & -6 & -2 & -6 & -1 \\
-2 & 2 & -1 & -2 & -1 & 0 & 4 & -6 & -18 & -2 & -16 & -11 \\
2 & 0 & 1 & 4 & -1 & 2 & -2 & -2 & -2 & -14 & 0 & 7 \\
-2 & 2 & -3 & -2 & -3 & 0 & 4 & -6 & -16 & 0 & -18 & -11 \\
-4 & 0 & 0 & -4 & 0 & -3 & 5 & -1 & -11 & 7 & -11 & -16
\end{pmatrix}$$

$$L_{(p=11,\sigma=2)} = \begin{pmatrix}
-4 & 0 & 0 & 1 & -1 & -1 & 2 & 0 & -2 & -1 & -1 & 0 \\
0 & -4 & 0 & -1 & 1 & 1 & -2 & 0 & 2 & -1 & 1 & 0 \\
0 & 0 & -4 & 0 & -2 & 2 & -2 & 0 & 0 & 2 & 0 & 0 \\
1 & -1 & 0 & -6 & -2 & -2 & -2 & -2 & 3 & 1 & -1 & -2 \\
-1 & 1 & -2 & -2 & -6 & -2 & -2 & 0 & 1 & 1 & -1 & 0 \\
-1 & 1 & 2 & -2 & -2 & -6 & 2 & 0 & -1 & -1 & -1 & 0 \\
2 & -2 & -2 & -2 & -2 & 2 & -8 & -2 & 4 & 2 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & -2 & -4 & 2 & 2 & 0 & 0 \\
-2 & 2 & 0 & 3 & 1 & -1 & 4 & 2 & -8 & -1 & 1 & 2 \\
-1 & -1 & 2 & 1 & 1 & -1 & 2 & 2 & -1 & -8 & 0 & 0 \\
-1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & -8 & -2 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & -8
\end{pmatrix}$$

$$L_{(p=13,\sigma=2)} = \begin{pmatrix} -10 & 0 & 4 & 2 & 0 & 2 & -2 & -6 & -2 & -4 & -1 & 11 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 2 & -2 \\ 4 & 0 & -4 & -2 & 0 & -2 & 2 & 2 & 4 & 4 & 4 & -6 \\ 2 & 0 & -2 & -4 & 0 & -2 & 2 & 0 & 2 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 2 & 0 & -2 & -2 & 0 & -4 & 2 & 2 & 2 & 2 & 2 & -2 \\ -2 & 0 & 2 & 2 & 0 & 2 & -4 & -2 & -4 & -4 & -2 & 2 \\ -6 & 0 & 2 & 0 & 0 & 2 & -2 & -10 & -2 & -6 & -5 & 11 \\ -2 & -2 & 4 & 2 & 0 & 2 & -4 & -2 & -14 & -6 & -7 & 1 \\ -4 & 2 & 4 & 2 & -2 & 2 & -4 & -6 & -6 & -16 & -10 & 8 \\ -1 & 2 & 4 & 2 & 0 & 2 & -2 & -5 & -7 & -10 & -20 & 14 \\ 11 & -2 & -6 & -2 & 0 & -2 & 2 & 11 & 1 & 8 & 14 & -28 \end{pmatrix}$$

$$L_{(p=17,\sigma=2)} = \begin{pmatrix} -4 & -2 & 2 & 2 & -2 & -2 & 2 & 0 & 2 & -2 & -2 & -2 \\ -2 & -4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 2 & 2 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & -4 & -2 & 2 & 0 & 2 & 0 & -2 & 0 \\ -2 & 0 & 0 & 0 & -2 & -4 & 2 & 0 & 2 & 0 & -2 & -2 \\ 2 & 0 & 0 & 0 & 2 & 2 & -4 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -3 & 3 & 2 & 2 \\ 2 & 0 & 0 & 0 & 2 & 2 & -2 & -3 & -16 & 6 & 7 & 2 \\ -2 & -2 & 2 & 2 & 0 & 0 & 0 & 3 & 6 & -16 & 3 & -2 \\ -2 & 0 & 0 & 0 & -2 & -2 & 2 & 2 & 7 & 3 & -18 & 4 \\ -2 & 0 & 2 & 0 & 0 & -2 & 0 & 2 & 2 & -2 & 4 & -16 \end{pmatrix}$$

$$L_{(p=23,\sigma=2)} = \begin{pmatrix} -4 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & -2 & 2 \\ 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & -2 & -2 & 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & -4 & 0 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & -2 & 0 & -2 & 0 & 0 & -8 & 0 & 2 & 0 \\ 2 & -2 & 0 & -2 & -2 & 0 & -2 & 0 & 0 & -12 & 4 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 4 & -16 & -6 \\ 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -6 & -28 \end{pmatrix}$$

$$L_{(p=3,\sigma=3)} = \begin{pmatrix}
-6 & 3 & 2 & -3 & 0 & 1 & -1 & -1 & 0 & -2 & 0 & -2 \\
3 & -8 & -3 & 2 & 3 & 1 & -1 & 1 & 2 & 0 & 0 & -1 \\
2 & -3 & -6 & 3 & 0 & -1 & -1 & -1 & 0 & 2 & 0 & -2 \\
-3 & 2 & 3 & -8 & -1 & 1 & -1 & 1 & 2 & 0 & -2 & -1 \\
0 & 3 & 0 & -1 & -6 & 1 & -1 & 1 & 0 & 0 & -2 & 0 \\
1 & 1 & -1 & 1 & 1 & -4 & 0 & 0 & -2 & 2 & 0 & 1 \\
-1 & -1 & -1 & -1 & -1 & 0 & -4 & 0 & 0 & 0 & -2 & -1 \\
-1 & 1 & -1 & 1 & 1 & 0 & 0 & -4 & 0 & 0 & 2 & -1 \\
0 & 2 & 0 & 2 & 0 & -2 & 0 & 0 & -4 & 0 & 0 & 2 \\
-2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & -2 & -2 & 0 & -2 & 2 & 0 & 0 & -4 & 0 \\
-2 & -1 & -2 & -1 & 0 & 1 & -1 & -1 & 2 & 0 & 0 & -6
\end{pmatrix}$$

$$L_{(p=5,\sigma=3)} = \begin{pmatrix}
-6 & -2 & 0 & 0 & 2 & -2 & 2 & 1 & 1 & -2 & -4 & 0 \\
-2 & -4 & 0 & 2 & -2 & -2 & 0 & 2 & 2 & 0 & -2 & 2 \\
0 & 0 & -4 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 2 & 0 & -6 & 2 & 2 & 0 & -3 & -1 & 0 & 0 & 0 \\
2 & -2 & 0 & 2 & -6 & 2 & 0 & 3 & 3 & 2 & 4 & 0 \\
-2 & -2 & 2 & 2 & 2 & -8 & -2 & 2 & 0 & 0 & -6 & 2 \\
2 & 0 & 0 & 0 & 0 & -2 & -6 & 1 & -1 & 0 & 0 & 0 \\
1 & 2 & 0 & -3 & 3 & 2 & 1 & -8 & -4 & -2 & 1 & -1 \\
1 & 2 & 2 & -1 & 3 & 0 & -1 & -4 & -8 & 0 & -1 & 1 \\
-2 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & -8 & 0 & -2 \\
-4 & -2 & 0 & 0 & 4 & -6 & 0 & 1 & -1 & 0 & -14 & 8 \\
0 & 2 & 0 & 0 & 0 & 2 & 0 & -1 & 1 & -2 & 8 & -10
\end{pmatrix}$$

$$L_{(p=7,\sigma=3)} = \begin{pmatrix}
-4 & -2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
-2 & -6 & -2 & 2 & 2 & -1 & 1 & 1 & 0 & 1 & 2 & 1 \\
0 & -2 & -6 & 0 & 0 & -3 & -1 & -3 & -2 & 1 & 0 & 1 \\
0 & 2 & 0 & -6 & -2 & -1 & -1 & -1 & 2 & -1 & -2 & -1 \\
2 & 2 & 0 & -2 & -6 & 1 & 1 & -3 & 0 & 1 & -2 & -1 \\
2 & -1 & -3 & -1 & 1 & -8 & -1 & -2 & 0 & -2 & 3 & -2 \\
2 & 1 & -1 & -1 & 1 & -1 & -8 & 1 & 0 & -1 & -3 & 1 \\
2 & 1 & -3 & -1 & -3 & -2 & 1 & -8 & -2 & 2 & -1 & 0 \\
0 & 0 & -2 & 2 & 0 & 0 & 0 & -2 & -8 & 2 & 2 & -2 \\
0 & 1 & 1 & -1 & 1 & -2 & -1 & 2 & 2 & -8 & 3 & -4 \\
0 & 2 & 0 & -2 & -2 & 3 & -3 & -1 & 2 & 3 & -10 & 3 \\
0 & 1 & 1 & -1 & -1 & -2 & 1 & 0 & -2 & -4 & 3 & -12
\end{pmatrix}$$

$$L_{(p=13,\sigma=3)} = \begin{pmatrix} -4 & -2 & -2 & 2 & 2 & 0 & -2 & -2 & -4 & -4 & -4 & 2 \\ -2 & -4 & 0 & 1 & 2 & 1 & 0 & 0 & -2 & -2 & -2 & 0 \\ -2 & 0 & -4 & 0 & -2 & -2 & 0 & 0 & -2 & -2 & -2 & 2 \\ 2 & 1 & 0 & -6 & -3 & 0 & 3 & 1 & 2 & 4 & 4 & 0 \\ 2 & 2 & -2 & -3 & -8 & -3 & 4 & 4 & 4 & 2 & 2 & 0 \\ 0 & 1 & -2 & 0 & -3 & -14 & -1 & -3 & -2 & 6 & 6 & 2 \\ -2 & 0 & 0 & 3 & 4 & -1 & -8 & -4 & -4 & -2 & -2 & 0 \\ -2 & 0 & 0 & 1 & 4 & -3 & -4 & -16 & -2 & 2 & 2 & -2 \\ -4 & -2 & -2 & 2 & 4 & -2 & -4 & -2 & -24 & -6 & -6 & -2 \\ -4 & -2 & -2 & 4 & 2 & 6 & -2 & 2 & -6 & -28 & -2 & -2 \\ -4 & -2 & -2 & 4 & 2 & 6 & -2 & 2 & -6 & -2 & -28 & -2 \\ 2 & 0 & 2 & 0 & 0 & 2 & 0 & -2 & -2 & -2 & -2 & -8 \end{pmatrix}$$

$$L_{(p=17,\sigma=3)} = \begin{pmatrix} -4 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & -2 & 2 & 2 & -2 \\ 0 & -4 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -4 & 0 & 2 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 2 & 0 & 2 & -2 & -2 & 0 & 2 & 0 \\ 2 & -2 & 2 & 2 & -8 & 0 & 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & -2 & 0 & 0 & -2 & 2 & 3 \\ -2 & -2 & -2 & 2 & 0 & -2 & -16 & 2 & 0 & 6 & -4 & 0 \\ 0 & 0 & 2 & -2 & -2 & 0 & 2 & -20 & 0 & -8 & -8 & 8 \\ -2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 & -20 & -8 & -6 & 8 \\ 2 & 0 & 0 & 0 & 0 & -2 & 6 & -8 & -8 & -26 & -10 & 15 \\ 2 & 2 & 0 & 2 & -2 & 2 & -4 & -8 & -6 & -10 & -28 & 4 \\ -2 & 0 & 0 & 0 & 0 & 3 & 0 & 8 & 8 & 15 & 4 & -24 \end{pmatrix}$$

$$L_{(p=3,\sigma=4)} = \begin{pmatrix} -4 & 1 & 1 & -1 & 0 & 0 & -2 & -1 & -1 & 2 & -2 & -2 \\ 1 & -4 & 1 & -1 & 1 & 0 & 2 & 0 & 0 & 1 & -1 & 2 \\ 1 & 1 & -6 & -2 & 1 & 2 & 2 & 3 & 3 & -1 & 3 & -2 \\ -1 & -1 & -2 & -6 & -1 & 2 & 0 & 3 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & -4 & 0 & -2 & 1 & -1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & -4 & -2 & 0 & -2 & 2 & -2 & 2 \\ -2 & 2 & 2 & 0 & -2 & -2 & -8 & 0 & -4 & 2 & -2 & 0 \\ -1 & 0 & 3 & 3 & 1 & 0 & 0 & -8 & 0 & 1 & 1 & -2 \\ -1 & 0 & 3 & -1 & -1 & -2 & -4 & 0 & -8 & -1 & -3 & 2 \\ 2 & 1 & -1 & -1 & -2 & 2 & 2 & 1 & -1 & -8 & 2 & 0 \\ -2 & -1 & 3 & -1 & 0 & -2 & -2 & 1 & -3 & 2 & -8 & 0 \\ -2 & 2 & -2 & 0 & 0 & 2 & 0 & -2 & 2 & 0 & 0 & -8 \end{pmatrix}$$

$$L_{(p=5,\sigma=4)} = \begin{pmatrix} -8 & 2 & -2 & 2 & 4 & 0 & -2 & -2 & -2 & -4 & -2 & 0 \\ 2 & -8 & -2 & 2 & 4 & 0 & -2 & -2 & -2 & -4 & -2 & 0 \\ -2 & -2 & -4 & 2 & 2 & 1 & 0 & 0 & -1 & -3 & -2 & -4 \\ 2 & 2 & 2 & -8 & -4 & 2 & 4 & 4 & 0 & 6 & 4 & 0 \\ 4 & 4 & 2 & -4 & -8 & 0 & 4 & 4 & 2 & 8 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & -10 & -5 & 0 & -2 & -1 & -1 & 8 \\ -2 & -2 & 0 & 4 & 4 & -5 & -12 & -2 & -3 & -5 & -2 & 8 \\ -2 & -2 & 0 & 4 & 4 & 0 & -2 & -12 & 2 & 0 & -2 & -2 \\ -2 & -2 & -1 & 0 & 2 & -2 & -3 & 2 & -6 & -3 & -1 & 2 \\ -4 & -4 & -3 & 6 & 8 & -1 & -5 & 0 & -3 & -16 & -3 & 6 \\ -2 & -2 & -2 & 4 & 2 & -1 & -2 & -2 & -1 & -3 & -8 & 0 \\ 0 & 0 & -4 & 0 & 0 & 8 & 8 & -2 & 2 & 6 & 0 & -24 \end{pmatrix}$$

$$L_{(p=7,\sigma=4)} = \begin{pmatrix} -4 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & -2 & -1 & 0 & -2 \\ 1 & -8 & 0 & -3 & 1 & -2 & -2 & -2 & -5 & -4 & -5 & -4 \\ -1 & 0 & -8 & -1 & 3 & 0 & -4 & -4 & -1 & -2 & -5 & -8 \\ 1 & -3 & -1 & -10 & -2 & -2 & -1 & -5 & -1 & -7 & -5 & -6 \\ 1 & 1 & 3 & -2 & -10 & -4 & -1 & 3 & 1 & 7 & 5 & 2 \\ 0 & -2 & 0 & -2 & -4 & -12 & -2 & 0 & -4 & 4 & 0 & -2 \\ -1 & -2 & -4 & -1 & -1 & -2 & -16 & 0 & 1 & 4 & 1 & -2 \\ -1 & -2 & -4 & -5 & 3 & 0 & 0 & -8 & -3 & -8 & -7 & -8 \\ -2 & -5 & -1 & -1 & 1 & -4 & 1 & -3 & -12 & -5 & -8 & -2 \\ -1 & -4 & -2 & -7 & 7 & 4 & 4 & -8 & -5 & -20 & -13 & -4 \\ 0 & -5 & -5 & -5 & 5 & 0 & 1 & -7 & -8 & -13 & -20 & -6 \\ -2 & -4 & -8 & -6 & 2 & -2 & -2 & -8 & -2 & -4 & -6 & -24 \end{pmatrix}$$

$$L_{(p=11,\sigma=4)} = \begin{pmatrix} -12 & -1 & 2 & 0 & 2 & -1 & 1 & 0 & -1 & 6 & -6 & -6 \\ -1 & -10 & 2 & 2 & 2 & 2 & 2 & -4 & -1 & -3 & 1 & -3 \\ 2 & 2 & -8 & 0 & 0 & 2 & -2 & 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 0 & -8 & 0 & -2 & -4 & 4 & 0 & 4 & -4 & 4 \\ 2 & 2 & 0 & 0 & -8 & 2 & -2 & 0 & 2 & -4 & 0 & 2 \\ -1 & 2 & 2 & -2 & 2 & -10 & 0 & -2 & -1 & -1 & -1 & -1 \\ 1 & 2 & -2 & -4 & -2 & 0 & -14 & 2 & 1 & 7 & -7 & -3 \\ 0 & -4 & 0 & 4 & 0 & -2 & 2 & -16 & 0 & -10 & 8 & 2 \\ -1 & -1 & 2 & 0 & 2 & -1 & 1 & 0 & -12 & -5 & 5 & 5 \\ 6 & -3 & 2 & 4 & -4 & -1 & 7 & -10 & -5 & -24 & 12 & 6 \\ -6 & 1 & 2 & -4 & 0 & -1 & -7 & 8 & 5 & 12 & -20 & -6 \\ -6 & -3 & 0 & 4 & 2 & -1 & -3 & 2 & 5 & 6 & -6 & -28 \end{pmatrix}$$

$$L_{(p=13,\sigma=4)} = \begin{pmatrix}
-8 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & -6 & -2 & -2 & 2 \\
2 & -12 & 1 & 0 & 0 & 2 & -7 & 1 & 0 & -2 & -2 & -3 \\
0 & 1 & -20 & 5 & -5 & 2 & 8 & 1 & 7 & 0 & 0 & -3 \\
0 & 0 & 5 & -16 & 12 & -14 & -13 & -7 & -10 & -10 & -8 & 15 \\
0 & 0 & -5 & 12 & -16 & 14 & 13 & 7 & 14 & 8 & 10 & -15 \\
0 & 2 & 2 & -14 & 14 & -20 & -10 & -10 & -14 & -8 & -8 & 16 \\
-2 & -7 & 8 & -13 & 13 & -10 & -28 & -5 & -7 & -14 & -14 & 13 \\
0 & 1 & 1 & -7 & 7 & -10 & -5 & -18 & -7 & -4 & -4 & 8 \\
-6 & 0 & 7 & -10 & 14 & -14 & -7 & -7 & -32 & -2 & -4 & 15 \\
-2 & -2 & 0 & -10 & 8 & -8 & -14 & -4 & -2 & -28 & -14 & 14 \\
-2 & -2 & 0 & -8 & 10 & -8 & -14 & -4 & -4 & -14 & -28 & 14 \\
2 & -3 & -3 & 15 & -15 & 16 & 13 & 8 & 15 & 14 & 14 & -22
\end{pmatrix}$$

$$L_{(p=17,\sigma=4)} = \begin{pmatrix}
-8 & -4 & 0 & -4 & 0 & -2 & 0 & -2 & 0 & 4 & -4 & -2 \\
-4 & -8 & 0 & 0 & 0 & -4 & 2 & 2 & -2 & 0 & -4 & -2 \\
0 & 0 & -6 & 0 & -1 & 0 & -3 & 0 & 3 & 0 & -2 & 0 \\
-4 & 0 & 0 & -8 & 0 & 0 & 0 & -2 & 0 & 4 & -4 & 2 \\
0 & 0 & -1 & 0 & -20 & 0 & -9 & 0 & 9 & 0 & -6 & 0 \\
-2 & -4 & 0 & 0 & 0 & -36 & 18 & 18 & 16 & 0 & -2 & 16 \\
0 & 2 & -3 & 0 & -9 & 18 & -32 & -18 & -2 & -8 & 0 & 0 \\
-2 & 2 & 0 & -2 & 0 & 18 & -18 & -36 & -16 & 2 & 0 & 0 \\
0 & -2 & 3 & 0 & 9 & 16 & -2 & -16 & -32 & 8 & 0 & 0 \\
4 & 0 & 0 & 4 & 0 & 0 & -8 & 2 & 8 & -20 & 2 & 0 \\
-4 & -4 & -2 & -4 & -6 & -2 & 0 & 0 & 0 & 2 & -16 & 0 \\
-2 & -2 & 0 & 2 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & -36
\end{pmatrix}$$

$$L_{(p=23,\sigma=4)} = \begin{pmatrix}
-4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8 & 0 & -4 & 4 & -4 & 0 & 0 & 2 & -2 & 0 \\
0 & 0 & 0 & -8 & 0 & -4 & 0 & -2 & -4 & 4 & -4 & -2 \\
0 & 0 & -4 & 0 & -16 & 0 & -6 & 2 & 4 & 4 & 2 & -2 \\
0 & 0 & 4 & -4 & 0 & -24 & 8 & -4 & -8 & 8 & 6 & 2 \\
0 & 0 & -4 & 0 & -6 & 8 & -36 & -6 & -12 & -8 & -10 & 6 \\
0 & 0 & 0 & -2 & 2 & -4 & -6 & -32 & -18 & -6 & -8 & 8 \\
0 & 0 & 0 & -4 & 4 & -8 & -12 & -18 & -36 & -12 & -16 & 16 \\
0 & 0 & 2 & 4 & 4 & 8 & -8 & -6 & -12 & -36 & -8 & 8 \\
0 & 0 & -2 & -4 & 2 & 6 & -10 & -8 & -16 & -8 & -36 & 6 \\
0 & 0 & 0 & -2 & -2 & 2 & 6 & 8 & 16 & 8 & 6 & -32
\end{pmatrix}$$

$$L_{(p=3,\sigma=5)} = \begin{pmatrix} -4 & 0 & 0 & -2 & -2 & 2 & -2 & -2 & 2 & -2 & 2 & -2 \\ 0 & -4 & -1 & 2 & 2 & -2 & 2 & 2 & -2 & -2 & 2 & -2 \\ 0 & -1 & -4 & -1 & 2 & 1 & 2 & -1 & -2 & 1 & 2 & -2 \\ -2 & 2 & -1 & -8 & -2 & 2 & -2 & -2 & 2 & 0 & 0 & 0 \\ -2 & 2 & 2 & -2 & -8 & 2 & -2 & -2 & 2 & 0 & 0 & 0 \\ 2 & -2 & 1 & 2 & 2 & -8 & 2 & 2 & -2 & 0 & 0 & 0 \\ -2 & 2 & 2 & -2 & -2 & 2 & -8 & -2 & 2 & 0 & 0 & 0 \\ -2 & 2 & -1 & -2 & -2 & 2 & -2 & -8 & 2 & 0 & 0 & 0 \\ 2 & -2 & -2 & 2 & 2 & -2 & 2 & 2 & -8 & 0 & 0 & 0 \\ -2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 2 & -2 \\ 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -8 & 2 \\ -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & -8 \end{pmatrix}$$

$$L_{(p=5,\sigma=5)} = \begin{pmatrix} -8 & 1 & -4 & 1 & -2 & 2 & 1 & -4 & -1 & -2 & -2 & -1 \\ 1 & -6 & 1 & 5 & -2 & -3 & 3 & -1 & -4 & -3 & -3 & 0 \\ -4 & 1 & -8 & -1 & -4 & 4 & 3 & -4 & -1 & -2 & -2 & 1 \\ 1 & 5 & -1 & -8 & 2 & 3 & -2 & 5 & 5 & 5 & 5 & 3 \\ -2 & -2 & -4 & 2 & -12 & 2 & 4 & -2 & -8 & -6 & -6 & -2 \\ 2 & -3 & 4 & 3 & 2 & -12 & 1 & 2 & -7 & -4 & -4 & -3 \\ 1 & 3 & 3 & -2 & 4 & 1 & -12 & 3 & 7 & 9 & 9 & -3 \\ -4 & -1 & -4 & 5 & -2 & 2 & 3 & -16 & -9 & -8 & -8 & 5 \\ -1 & -4 & -1 & 5 & -8 & -7 & 7 & -9 & -26 & -17 & -17 & 0 \\ -2 & -3 & -2 & 5 & -6 & -4 & 9 & -8 & -17 & -24 & -14 & 5 \\ -2 & -3 & -2 & 5 & -6 & -4 & 9 & -8 & -17 & -14 & -24 & 5 \\ -1 & 0 & 1 & 3 & -2 & -3 & -3 & 5 & 0 & 5 & 5 & -18 \end{pmatrix}$$

$$L_{(p=7,\sigma=5)} = \begin{pmatrix} -8 & 0 & 0 & 0 & 0 & 0 & -4 & -4 & -4 & 6 & 2 & 0 \\ 0 & -8 & 4 & -4 & 4 & 4 & 8 & 0 & 0 & -4 & 4 & 5 \\ 0 & 4 & -16 & 2 & -2 & -2 & -4 & 0 & 0 & 2 & -2 & -6 \\ 0 & -4 & 2 & -16 & 2 & 2 & 4 & 0 & 0 & -2 & 2 & -1 \\ 0 & 4 & -2 & 2 & -16 & -2 & -4 & 0 & 0 & 2 & -2 & -6 \\ 0 & 4 & -2 & 2 & -2 & -16 & -4 & 0 & 0 & 2 & -2 & -6 \\ -4 & 8 & -4 & 4 & -4 & -4 & -24 & -2 & -2 & 14 & 4 & -19 \\ -4 & 0 & 0 & 0 & 0 & 0 & -2 & -16 & -2 & 10 & 8 & 7 \\ -4 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & -16 & 10 & 8 & -7 \\ 6 & -4 & 2 & -2 & 2 & 2 & 14 & 10 & 10 & -24 & -10 & 6 \\ 2 & 4 & -2 & 2 & -2 & -2 & 4 & 8 & 8 & -10 & -20 & 8 \\ 0 & 5 & -6 & -1 & -6 & -6 & -19 & 7 & -7 & 6 & 8 & -32 \end{pmatrix}$$

$$L_{(p=13,\sigma=5)} = \begin{pmatrix} -8 & 0 & 0 & 0 & 0 & 0 & -4 & -4 & -4 & 6 & 2 & 0 \\ 0 & -8 & 4 & -4 & 4 & 4 & 8 & 0 & 0 & -4 & 4 & 5 \\ 0 & 4 & -16 & 2 & -2 & -2 & -4 & 0 & 0 & 2 & -2 & -6 \\ 0 & -4 & 2 & -16 & 2 & 2 & 4 & 0 & 0 & -2 & 2 & -1 \\ 0 & 4 & -2 & 2 & -16 & -2 & -4 & 0 & 0 & 2 & -2 & -6 \\ 0 & 4 & -2 & 2 & -2 & -16 & -4 & 0 & 0 & 2 & -2 & -6 \\ -4 & 8 & -4 & 4 & -4 & -4 & -24 & -2 & -2 & 14 & 4 & -19 \\ -4 & 0 & 0 & 0 & 0 & 0 & -2 & -16 & -2 & 10 & 8 & 7 \\ -4 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & -16 & 10 & 8 & -7 \\ 6 & -4 & 2 & -2 & 2 & 2 & 14 & 10 & 10 & -24 & -10 & 6 \\ 2 & 4 & -2 & 2 & -2 & -2 & 4 & 8 & 8 & -10 & -20 & 8 \\ 0 & 5 & -6 & -1 & -6 & -6 & -19 & 7 & -7 & 6 & 8 & -32 \end{pmatrix}$$

$$L_{(p=17,\sigma=5)} = \begin{pmatrix} -6 & 1 & 2 & -2 & 1 & 3 & -3 & -3 & 3 & 3 & -3 & -3 \\ 1 & -20 & -6 & 6 & -3 & 8 & -8 & -8 & -9 & 8 & -8 & -8 \\ 2 & -6 & -16 & 8 & -6 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 6 & 8 & -16 & 6 & 0 & -2 & 0 & 2 & 2 & -2 & -2 \\ 1 & -3 & -6 & 6 & -20 & -9 & 9 & 9 & 8 & -9 & 9 & 9 \\ 3 & 8 & -2 & 0 & -9 & -40 & 14 & 6 & 3 & -14 & 14 & 14 \\ -3 & -8 & 0 & -2 & 9 & 14 & -40 & -14 & -11 & 6 & -6 & -6 \\ -3 & -8 & 2 & 0 & 9 & 6 & -14 & -40 & -3 & 14 & -14 & -14 \\ 3 & -9 & 0 & 2 & 8 & 3 & -11 & -3 & -40 & 11 & -11 & -11 \\ 3 & 8 & 0 & 2 & -9 & -14 & 6 & 14 & 11 & -40 & 6 & 6 \\ -3 & -8 & 0 & -2 & 9 & 14 & -6 & -14 & -11 & 6 & -40 & -6 \\ -3 & -8 & 0 & -2 & 9 & 14 & -6 & -14 & -11 & 6 & -6 & -40 \end{pmatrix}$$

REMARK 1.21. In theory, with the presented approach, it should be possible to explicitly compute the generic number of isomorphism classes of Enriques quotients of a supersingular K3 surface X with given characteristic p of the ground field k and Artin invariant σ.

Namely, in Theorem 1.16 the M_i are members of isometry classes of lattices in the genus $(0, 12, \delta_{p,\sigma})$ that contain no (-2)-vectors. Two different isometry classes in particular yield two different orbits for the action of $O(\mathrm{NS}(X))$.

The MAGMA-command *Representatives(G);* computes a representative for every isometry class in a given genus G. We can then distinguish the isometry classes that contain no (-2)-vectors and compute the orthogonal group of their discriminant lattice, as well as their stabilizer, in $O(\mathrm{NS})$. We note that each of those steps is still very complicated.

3.2. Lower bounds. Using the method from the previous remark, we computed the number $\mathrm{Rep}(p, \sigma)$ of isometry classes of lattices without (-2)-vectors for some genera $(0, 12, \delta_{p,\sigma})$ in small characteristics. This yields a lower bound for the number of Enriques involutions of a supersingular K3 surface in these cases. However, since the groups $O(q_{\mathrm{NS}})$ are already large in these cases, this bound is possibly not optimal. We also note, that already in these comparatively simple cases, computing each of those numbers was very memory intensive.

PROPOSITION 1.22. *For the number of isomorphism classes of Enriques quotients of a supersingular K3 surface of Artin invariant σ over an algebraically closed ground field k of characteristic p we found the following weak lower bounds* $\mathrm{Rep}(p,\sigma)$*:*

TABLE 1. Some results for the lower bounds $\mathrm{Rep}(p,\sigma)$

p	$\sigma=1$	$\sigma=2$	$\sigma=3$	$\sigma=4$	$\sigma=5$
3	2	12	30	20	7
5	10	222	862	302	24
7	42	3565	?	4313	81
11	256	?	?	?	438
13	537	?	?	?	866
17	2298	?	?	?	2974

3.3. Upper bounds. The cardinality of the quotients $O\left(q_{\mathrm{NS}(X)}\right)/\mathrm{pr}(K^{(j)})$ in Theorem 1.16 is difficult to compute, but we can easily find the cardinalities of the groups $O\left(q_{\mathrm{NS}(X)}\right)$. Therefore we can use Theorem 1.16 and Proposition 1.22 to find (weak) upper bounds for the number isomorphism classes of Enriques quotients for small p and σ.

PROPOSITION 1.23. *For the number of isomorphism classes of Enriques quotients of a supersingular K3 surface of Artin invariant σ over an algebraically closed ground field k of characteristic p we found the following weak upper bounds:*

TABLE 2. Some results for the upper bounds

p	$\sigma=1$	$\sigma=2$	$\sigma=3$	$\sigma=4$	$\sigma=5$
3	16	17280	$8\cdot 10^{8}$	$9\cdot 10^{14}$	$4\cdot 10^{22}$
5	120	$7\cdot 10^{6}$	$6\cdot 10^{13}$	$3\cdot 10^{22}$	$2\cdot 10^{33}$
7	672	$9\cdot 10^{8}$	?	$4\cdot 10^{27}$	$2\cdot 10^{40}$
11	6144	?	?	?	$7\cdot 10^{49}$
13	15036	?	?	?	$3\cdot 10^{53}$
17	82728	?	?	?	$2\cdot 10^{59}$

PROOF. Using the formula (2.4) for quadratic forms of type IV from [Sol65], we can directly compute the cardinality of $O\left(q_{\mathrm{NS}(X)}\right)$ for a supersingular K3 surface X. It follows from Theorem 1.16 that multiplying these cardinalities with the lower bounds from Proposition 1.22 yields upper bounds for the numbers of isomorphism classes of Enriques quotients. □

3.4. The case $p=3$ and $\sigma=1$. The number $\mathrm{Rep}(3,1)=2$ is small enough that computing the number of all isomorphism classes of Enriques quotients of the supersingular K3 surface X of Artin invariant 1 over an algebraically closed field k of characteristic 3 is a feasible goal. More precisely, using Theorem 1.16, we only need to compute the cardinality of the quotient

$O(q_{\mathrm{NS}(X)})/\mathrm{pr}(K^{(j)})$ for two different sublattices M_1 and M_2 of $\mathrm{NS}(X)$. To this end, let us first describe a better way to understand the subgroups $K^{(j)} \subseteq O(\mathrm{NS}(X))$.

PROPOSITION 1.24. *Let X be a supersingular K3 surface and let $M \in \mathfrak{M}$ be a primitive sublattice of $\mathrm{NS}(X)$. If $\psi' \colon M^\perp \to M^\perp$ is an isometry of $M^\perp$, then there exists an isometry $\psi \colon \mathrm{NS}(X) \to \mathrm{NS}(X)$ of $\mathrm{NS}(X)$ such that $\psi|_{M^\perp} = \psi'$. In particular, we have $\psi(M) = M$. Furthermore, the image of ψ in $O\left(q_{\mathrm{NS}(X)}\right)$ only depends on ψ'.*

PROOF. It follows from [Nik80, Theorem 1.14.2] that the canonical morphism of orthogonal groups $O\left(\Gamma(2)\right) \to O\left(q_{\Gamma(2)}\right)$ is surjective. Since M is isomorphic to $\Gamma(2)$ it thus follows from [Nik80, Corollary 1.5.2] that for any automorphism $\psi' \colon M^\perp \to M^\perp$ we can choose an automorphism $\varphi' \colon M \to M$ such that $\psi' \oplus \varphi'$ extends to an automorphism ψ of $\mathrm{NS}(X)$.

Since we have an isomorphism $O(q_{M^\perp}) \cong O\left(q_M\right) \oplus O\left(q_{\mathrm{NS}(X)}\right)$, we also have natural maps

$$\{\psi \in O\left(\mathrm{NS}(X)\right) \mid \psi(M) = M\} \to O(M^\perp) \to O(q_{M^\perp}) \to O\left(q_{\mathrm{NS}(X)}\right)$$

and the second statement of the proposition follows. □

In other words, in Theorem 1.16 the subgroup $\mathrm{pr}\left(K^{(j)}\right)$ of $O\left(q_{\mathrm{NS}(X)}\right)$ is the image of the group $O\left(M_j^\perp\right)$ in $O\left(q_{\mathrm{NS}(X)}\right)$. We thus have the following corollary.

COROLLARY 1.25. *Let k be an algebraically closed field of characteristic $p \geq 3$ and let X be a supersingular K3 surface over k. Let $M_1, \ldots, M_l \in \mathfrak{M}$ be a complete set of representatives for the action of $O(\mathrm{NS}(X))$ on $\mathfrak{M}$. For each $j \in \{1, \ldots, l\}$, we write $\mathrm{im}\left(O(M_j^\perp)\right)$ for the image of $O(M_j^\perp)$ in $O\left(q_{\mathrm{NS}(X)}\right)$ under the natural map $O(M_j^\perp) \to O(q_{M^\perp}) \to O(q_{\mathrm{NS}(X)})$. Then we have inequalities*

$$l \leq \#\left\{\textit{Enriques quotients of } X\right\} \leq \sum_{j=1}^{l} \#\left(O\left(q_{\mathrm{NS}(X)}\right)/\mathrm{im}\left(O(M_j^\perp)\right)\right).$$

If X is such that for each automorphism $\theta \in \mathrm{Aut}(X)$ the induced automorphism on the quotient $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$ is either the identity or multiplication by -1, then the inequality above becomes an equality on the right side.

We use these results to prove the following theorem.

THEOREM 1.26. *There are exactly two isomorphism classes of Enriques quotients of the supersingular K3 surface X of Artin invariant 1 over an algebraically closed field k of characteristic 3.*

PROOF. Since we computed $\mathrm{Rep}(3,1) = 2$, there are at least two isomorphism classes of Enriques quotients of X. Namely, our MAGMA results show that exactly the two lattices $M_1^\perp$ and $M_2^\perp$ show up as orthogonal complements for primitive embeddings $M \hookrightarrow \mathrm{NS}(X)$ with $M \in \mathfrak{M}$, where there are bases B_i of $M_i^\perp(-1)$ such that we find as Gram matrices with regard to these bases the matrices

$$M_1^{\perp}(-1) = \begin{pmatrix} 4 & -2 & 0 & 2 & 2 & -2 & -1 & -2 & 2 & -1 & 0 & 1 \\ -2 & 4 & 2 & -2 & -2 & 2 & 1 & 2 & -2 & 1 & 0 & -1 \\ 0 & 2 & 4 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 4 & 2 & 0 & -1 & -2 & 2 & -1 & 0 & 1 \\ 2 & -2 & -2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & 0 & -2 & 4 & 1 & 2 & 0 & 1 & 0 & -1 \\ -1 & 1 & 2 & -1 & -2 & 1 & 4 & 2 & 0 & 0 & -1 & 1 \\ -2 & 2 & 2 & -2 & -2 & 2 & 2 & 4 & -2 & 2 & 0 & -2 \\ 2 & -2 & 0 & 2 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 2 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & 2 & -2 & 4 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 4 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 & -2 & 2 & -3 & -1 & 6 \end{pmatrix}$$

and

$$M_2^{\perp}(-1) = \begin{pmatrix} 4 & -2 & -2 & -2 & -2 & 2 & 2 & 0 & -2 & -2 & -2 & 0 \\ -2 & 4 & 2 & 2 & 0 & -2 & -2 & -2 & 2 & 2 & 2 & 0 \\ -2 & 2 & 4 & 2 & 2 & -2 & -2 & -1 & 2 & 2 & 2 & -1 \\ -2 & 2 & 2 & 4 & 2 & -2 & -2 & -2 & 2 & 2 & 2 & 0 \\ -2 & 0 & 2 & 2 & 4 & -2 & -2 & 0 & 2 & 2 & 2 & 0 \\ 2 & -2 & -2 & -2 & -2 & 4 & 2 & 1 & -2 & -2 & -2 & 1 \\ 2 & -2 & -2 & -2 & -2 & 2 & 4 & 0 & -2 & -2 & -2 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 & 0 & 4 & -1 & -1 & -1 & 0 \\ -2 & 2 & 2 & 2 & 2 & -2 & -2 & -1 & 4 & 2 & 2 & -1 \\ -2 & 2 & 2 & 2 & 2 & -2 & -2 & -1 & 2 & 4 & 2 & -1 \\ -2 & 2 & 2 & 2 & 2 & -2 & -2 & -1 & 2 & 2 & 4 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 4 \end{pmatrix}.$$

Note that we use the lattices $M_i^{\perp}(-1)$ instead of $M_i^{\perp}$, as MAGMA can only work with positive definite lattices.

To prove the proposition, we need to show that the canonical morphisms of orthogonal groups $O\left(M_i^{\perp}\right) \to O\left(q_{\mathrm{NS}(X)}\right)$ or equivalently $O\left(M_i^{\perp}(-1)\right) \to O\left(q_{\mathrm{NS}(X)(-1)}\right)$ are surjective. We start with $M_1^{\perp}(-1)$. Using the command "AutomorphismGroup();" in MAGMA, we find $O\left(M_1^{\perp}(-1)\right)$. With regard to the basis B_1, it is the multiplicative group of 12×12 matrices over the integers generated by the 5 matrices

$$\psi_{1,1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{1,2} = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\psi_{1,3} = \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{1,4} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\psi_{1,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

Writing $B_1 = (v_1, \ldots, v_{12})$, we find that the images $\overline{e}_1$ and $\overline{e}_2$ of the vectors

$$e_1 = \frac{1}{3}v_7 - \frac{1}{3}v_8 + \frac{2}{3}v_{10} + \frac{1}{3}v_{12}$$

and

$$e_2 = \frac{2}{3}v_1 + v_2 - \frac{2}{3}v_3 - \frac{1}{6}v_6 + \frac{2}{3}v_8 + \frac{1}{3}v_9 + \frac{1}{3}v_{11} + \frac{1}{3}v_{12}$$

in $q_{\mathrm{NS}(X)(-1)} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ generate the lattice $q_{\mathrm{NS}(X)(-1)}$. We find that the $\psi_{1,i}$ act in the following way on $q_{\mathrm{NS}(X)(-1)}$:

$$\begin{aligned} &\overline{\psi}_{1,1} = \mathrm{id}_{q_{\mathrm{NS}(X)}}, \\ &\overline{\psi}_{1,2} \colon \overline{e}_1 \mapsto \overline{e}_1 - \overline{e}_2;\ \overline{e}_2 \mapsto -\overline{e}_1 - \overline{e}_2, \\ &\overline{\psi}_{1,3} \colon \overline{e}_1 \mapsto \overline{e}_1;\ \overline{e}_2 \mapsto -\overline{e}_1 - \overline{e}_2, \\ &\overline{\psi}_{1,4} = \mathrm{id}_{q_{\mathrm{NS}(X)}}, \\ &\overline{\psi}_{1,5} \colon \overline{e}_1 \mapsto \overline{e}_1 - \overline{e}_2;\ \overline{e}_2 \mapsto -\overline{e}_2. \end{aligned}$$

It is straightforward to show that those morphisms generate $O\left(q_{\mathrm{NS}(X)(-1)}\right)$, thus we have

$$\#\left(O\left(q_{\mathrm{NS}(X)}\right)/\mathrm{pr}\left(K^{(1)}\right)\right)=1.$$

Let us now turn towards $M_2^{\perp}(-1)$. With regard to the basis B_2, the group $O\left(M_2^{\perp}(-1)\right)$ is the multiplicative group of 12×12 matrices generated by the 3 matrices

$$\psi_{2,1}=\begin{pmatrix}
-1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 1
\end{pmatrix},$$

$$\psi_{2,2}=\begin{pmatrix}
0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
-1 & -2 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and

$$\psi_{2,3} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 2 & 1 & 0 & 0 & -1 & -1 & 0 & -1 \end{pmatrix}.$$

Writing $B_2 = (w_1, \dots, w_{12})$, we find that the images $\overline{f}_1$ and $\overline{f}_2$ of the vectors

$$f_1 = \frac{1}{3}w_1 + \frac{1}{3}w_3 + \frac{1}{3}w_4 - \frac{1}{3}w_5 - \frac{1}{3}w_6 + \frac{1}{3}w_7 + \frac{2}{3}w_8 + \frac{1}{3}w_9 + \frac{1}{3}w_{10} - \frac{1}{3}w_{11} + \frac{2}{3}w_{12}$$

and

$$f_2 = -w_2 + \frac{2}{3}w_3 - w_5 + \frac{1}{3}w_6 + \frac{2}{3}w_9 + \frac{2}{3}w_{10} + \frac{1}{3}w_{11} + \frac{1}{3}w_{12}$$

in $q_{\mathrm{NS}(X)(-1)} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ generate the lattice $q_{\mathrm{NS}(X)(-1)}$. We then find that the $\psi_{2,i}$ act in the following way on $q_{\mathrm{NS}(X)(-1)}$:

$$\begin{aligned} \overline{\psi}_{2,1} &\colon \overline{f}_1 \mapsto -\overline{f}_1 + \overline{f}_2;\ \overline{f}_2 \mapsto \overline{f}_2, \\ \overline{\psi}_{2,2} &\colon \overline{f}_1 \mapsto -\overline{f}_1;\ \overline{f}_2 \mapsto \overline{f}_1 + \overline{f}_2, \\ \overline{\psi}_{2,3} &\colon \overline{f}_1 \mapsto \overline{f}_1 - \overline{f}_2;\ \overline{f}_2 \mapsto -\overline{f}_2. \end{aligned}$$

It is again straightforward to show that these morphisms generate $O\left(q_{\mathrm{NS}(X)(-1)}\right)$, and it follows that also

$$\#\left(O\left(q_{\mathrm{NS}(X)}\right)/\mathrm{pr}\left(K^{(2)}\right)\right) = 1.$$

Using these results, Theorem 1.16 implies there are exactly two isomorphism classes of Enriques quotients of X. □

In [Mar19], Enriques surfaces with finite automorphism groups are classified and fall into seven types. We thank Gebhard Martin for communicating the following result to us.

PROPOSITION 1.27. *Let k be an algebraically closed field of characteristic $p = 3$ and let Y be the unique Enriques surface with finite automorphism group of type III (respectively of type IV) over k, following the classification in [Mar19]. Then, the K3-cover of Y is the supersingular K3 surface X with Artin invariant $\sigma = 1$.*

PROOF. Let Y be the unique Enriques surface with finite automorphism group of type III (respectively of type IV) in the sense of [Mar19]. It follows from [Mar19, Lemma 11.1] that Y has a complex model $\mathcal{Y}$ of type III (respectively of type IV) in the sense of [Kon86]. From [Kon86, Proposition 3.3.2] (respectively from [Kon86, Proposition 3.4.2]) it follows that the universal K3 cover $\mathcal{X}$ of $\mathcal{Y}$ is the Kummer surface $\mathrm{Km}(\mathcal{E} \times \mathcal{E})$, where $\mathcal{E}$ is the complex elliptic curve of j-invariant $j = 1728$. Thus, the universal K3 cover X of Y is the Kummer surface $\mathrm{Km}(E \times E)$ where E is the elliptic curve of j-invariant $j = 1728$ over k, which is a supersingular elliptic curve in characteristic $p = 3$. □

As a corollary we can identify the two surfaces from Theorem 1.26.

COROLLARY 1.28. *The two Enriques quotients of the supersingular K3 surface of Artin invariant $\sigma = 1$ over an algebraically closed field of characteristic 3 are the unique Enriques surfaces of type III and type IV following the classification in [Mar19].*

OPEN QUESTION 1.29. *When X is the supersingular K3 surface of Artin invariant $\sigma = 1$ over an algebraically closed field of characteristic $p = 3$, it follows from the previous theorem that $\mathrm{pr}\left(K^{(j)}\right) = O\left(q_{\mathrm{NS}(X)}\right)$ for all j, using the notation from Theorem 1.16. If this eqality of lattices held in any characteristic, we would have*

$$\#\{\textit{Enriques quotients of } X\} = k$$

in Theorem 1.16. However, we do not know if we should expect such an equality in general (or maybe whenever $\sigma = 1$). The only general result we know about regarding the surjectivity of maps of the form $O\left(L\right) \to O\left(q_L\right)$, where L is some lattice, is [Nik80, Theorem 1.14.2] which is concerned with indefinite lattices. We would need a similar statement for (a subclass of) definite lattices.

CHAPTER 2

A fibration on the period space of supersingular K3 surfaces

In this chapter, we discuss a result on the stratification of the period space of supersingular K3 crystals that was falsely stated in [Lie15b]. There it was claimed that the strata are subsequently $\mathbb{P}^1$-bundles over each other. We obtain a new, correct version of the result. There are still fibrations between the strata, but the fibers turn out to be reducible in general.

1. Background

Let $\sigma > 0$ be an integer and $p \geq 3$ be a prime number. For a vector space L of dimension 2σ over $\mathbb{F}_p$ together with a symmetric and non-degenerate bilinear form $\langle -, - \rangle$, we denote by Gen_L the *functor of generatrices of L*. That means for any $\mathbb{F}_p$-algebra A, we define

$$\underline{\mathrm{Gen}}_L(A) = \{\text{totally isotropic subspaces } G \text{ of dimension } \sigma \text{ in } L \otimes_{\mathbb{F}_p} A\}.$$

There is a unique sheaf

$$\underline{\mathrm{Gen}}_L \colon (\text{Algebraic spaces over } \mathbb{F}_p)^{\mathrm{op}} \longrightarrow (\text{Sets})$$

that extends the functor $\underline{\mathrm{Gen}}_L$ from the category of $\mathbb{F}_p$-algebras to the category of algebraic spaces over $\mathbb{F}_p$. The functor $\underline{\mathrm{Gen}}_L$ can then be represented by a smooth projective variety that we will denote by Gen_L [Del73, §XII, Proposition 2.8]. There exists a subfunctor $\underline{\mathcal{M}}_L \subset \underline{\mathrm{Gen}}_L$ given on $\mathbb{F}_p$-algebras via the assignment

$$\underline{\mathcal{M}}_L(A) = \{G \in \underline{\mathrm{Gen}}_L(A) \mid \dim\left(G + \varphi(G)\right) = \sigma + 1\},$$

where $\varphi \colon L \otimes_{\mathbb{F}_p} A \to L \otimes_{\mathbb{F}_p} A$ is the map induced from the Frobenius morphism $A \to A, x \mapsto x^p$. Then $\underline{\mathcal{M}}_L$ can also be represented by a projective variety, which we denote $\mathcal{M}_L$ and call *the moduli space of characteristic generatrices in L* [Ogu79, Proposition 4.6].

Now let L and L_+ be the $\mathbb{F}_p$-vector spaces together with bilinear forms that are associated to the supersingular K3 lattices N and N_+ in characteristic p of Artin-invariants $\sigma - 1$ and σ respectively. More precisely, we set $L = pN^\vee/pN$ and $L_+ = pN_+^\vee/pN_+$.

To simplify notation, we will write $2\sigma = n$ and let $\{v, \varphi(v), e_3, \ldots, e_n\}$ be a basis for the n-dimensional $\mathbb{F}_{p^2}$-vector space $L_+ \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}$ such that

$$\begin{aligned} v^2 = \varphi(v)^2 &= 0, \\ v \cdot \varphi(v) &= 1, \\ e_i \cdot e_{i+1} &= 1 \text{ for } i \text{ odd and} \\ e_i \cdot e_j &= 0 \text{ else.} \end{aligned}$$

Furthermore, we use the notations $\mathcal{M}_{\sigma-1} = \mathcal{M}_L$ and $\mathcal{M}_\sigma = \mathcal{M}_{L_+}$ respectively.

In [Lie15b, Theorem 4.3], Christian Liedtke claims that there exists a surjective morphism of schemes $\varpi_\sigma\colon \mathcal{M}_\sigma \to \mathcal{M}_{\sigma-1}$, together with a section $s_\sigma\colon \mathcal{M}_{\sigma-1} \to \mathcal{M}_\sigma$, which turns $\mathcal{M}_\sigma$ into a $\mathbb{P}_1$-bundle over $\mathcal{M}_{\sigma-1}$.

The definition of the morphism $s_\sigma\colon \mathcal{M}_{\sigma-1} \to \mathcal{M}_\sigma$ is standard. Namely, by [Ogu79, Proposition 4.3.], the datum of a characteristic subspace of $L \otimes \overline{\mathbb{F}}_p$ is equivalent to the datum of a marking $(\psi\colon N \to T_H, H)$ for a K3 crystal H. After fixing an embedding of lattices $\iota\colon N_+ \hookrightarrow N$, we can associate to the characteristic subspace of $L \otimes \overline{\mathbb{F}}_p$ corresponding to the marking $(\psi\colon N \to T_H, H)$ the characteristic subspace of $L_+ \otimes \overline{\mathbb{F}}_p$ corresponding to the marking $(\psi \circ \iota\colon N_+ \to T_H, H)$. This construction extends to families, yielding the map of functors $s_\sigma\colon \underline{\mathcal{M}}_{\sigma-1} \to \underline{\mathcal{M}}_\sigma$.

Next, Liedtke wants to define the morphism $\varpi_\sigma\colon \mathcal{M}_\sigma \to \mathcal{M}_{\sigma-1}$ by fixing a basis vector e_i in $L_+ \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}$ and associating to any characteristic generatrix $G \subset L_+ \otimes_{\mathbb{F}_p} A$ the isotropic subspace

$$\varpi_\sigma(G) = \Big(G \cap \langle e_i\rangle^\perp \otimes A\Big) \,/\, (\langle e_i\rangle \otimes A) \subset \langle e_i\rangle^\perp / \langle e_i\rangle \otimes A \cong L \otimes A.$$

However, using the above definition, the subspace $\varpi_\sigma(G)$ will, in general, not be a characteristic generatrix. For example, let $A = \mathbb{F}_{p^2}[t]$ and consider $G = \langle t\varphi(v) + e_3, -v + te_4\rangle \in \mathcal{M}_2(A)$. Then the subspace G in $L_2 \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}[t]$ is indeed a generatrix: clearly G is an isotropic direct summand of rank 2. Furthermore, we have the equality $G + \varphi(G) = \langle v - \varphi(v), te_4 - \varphi(v), e_3 + t_v\rangle$ and therefore also the equality $G + \varphi(G) + \langle e_4\rangle = L_2 \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}[t]$, proving that $G + \varphi(G)$ is a direct summand of rank 3.

We can now consider the mapping ϖ_2 associated with $e_3 \in L_2$. Then we find

$$\varpi_2(G) = \langle t\varphi(v)\rangle,$$

but this is not a direct summand in $L_1 \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}[t]$.

In other words, the morphism $\varpi_\sigma\colon \mathcal{M}_\sigma \to \mathcal{M}_{\sigma-1}$ does not exist as constructed in the proof of [Lie15b, Theorem 4.3]. The aim of this section is to repair the construction of ϖ_σ and to prove the following statement.

THEOREM 2.1. *For any $3 \le i \le n$ there exists a surjective morphism*

$$\tilde{\Phi}_i\colon \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$$

and an open subset $U_i \subseteq \mathcal{M}_\sigma$ such that for any $\mathbb{F}_p$-algebra A and any $G \in U_i(A)$ we have

$$\tilde{\Phi}_i(G) = \Big(G \cap \langle e_i\rangle^\perp \otimes A\Big) \,/\, (\langle e_i\rangle \otimes A)\,.$$

2. Existence of a rational map $\mathcal{M}_\sigma \dashrightarrow \mathcal{M}_{\sigma-1}$

We show that the construction of the morphisms ϖ_σ, as given in the proof of [Lie15b, Theorem 4.3] and described above, yields a well-defined morphism at least when we restrict ϖ_σ to certain open subschemes U_i of $\mathcal{M}_\sigma$. We should mention that the contents of this subsection have essentially already been treated by Daniel Bragg in [BL18, Section 3.1]. Let us first show that the construction is well-defined on points of $\mathcal{M}_\sigma$. The following lemma and its proof are taken from unpublished notes of Christian Liedtke.

LEMMA 2.2. *Let $(V, \langle\cdot,\cdot\rangle)$ be a 2σ-dimensional $\mathbb{F}_p$-vector space with a non-neutral form, let $W \subset V$ be an isotropic subspace, and let k be a field of characteristic p.*

- *The form $\langle \cdot, \cdot \rangle$ induces a non-neutral form on the $\mathbb{F}_p$-vector space $W^\perp/W$ and we have* $\dim W^\perp/W = 2\,(\sigma - \dim W)$.
- *If $G \subset V \otimes k$ is a characteristic subspace, then*

$$\varpi_W(G) = \left(G \cap W^\perp \otimes k\right) / \left(W \otimes k\right) \subset W^\perp/W \otimes k$$

is also a characteristic subspace.

PROOF. Using $W \subset W^\perp$ and $\dim W^\perp = \dim V - \dim W$, we find

$$\dim W^\perp/W = \dim W^\perp - \dim W = \dim V - 2\dim W = 2\,(\sigma - \dim W)\,.$$

Clearly, $\langle \cdot, \cdot \rangle$ induces a non-degenerate form on $W^\perp/W$. Suppose it was neutral, which means suppose there was an isotropic subspace $L \subset W^\perp/W$ of dimension $\sigma - \dim W$. Then the preimage of L in $W^\perp$ would be an isotropic subspace of dimension σ inside V, contradicting the non-neutrality of $\langle \cdot, \cdot \rangle$. This establishes the first part of the lemma.

We will only prove the second part in the case where $\dim W = 1$, since this suffices for our purposes. The general case follows from a simple induction argument. Let $G \subset V \otimes k$ be a characteristic subspace. Being a subquotient of G, the subspace $\varpi_W(G)$ of $W^\perp/W \otimes k$ is isotropic. If $W \otimes k \subset G$, then $G \subset W^\perp$ is the preimage of $\varpi_W(G)$ under $W^\perp \longrightarrow W^\perp/W$, and similarly for $G + \varphi(G)$. It follows

$$\dim \varpi_W(G) = \dim G - 1 = \sigma - 1$$

and

$$\dim\left(\varpi_W(G) + \varphi\left(\varpi_W(G)\right)\right) = \dim\left(G + \varphi(G)\right) - 1 = \sigma.$$

Otherwise, still assuming W to be one-dimensional, we have $(W \otimes k) \cap G = 0$. Since $W^\perp$ is $(2\sigma - 1)$-dimensional, the space $G \cap \left(W^\perp \otimes k\right)$ is of dimension $\sigma - 1$ or σ. In the latter case, $\varpi_W(G)$ would be a σ-dimensional isotropic subspace of the $2(\sigma - 1)$-dimensional space $W^\perp/W \otimes k$ which is impossible. Thus we have $\dim \varpi_W(G) = \sigma - 1$. Since the quadratic form on $W^\perp/W$ is non-neutral, $\varpi_W(G)$ cannot be φ-stable, thus $\varpi_W(G) + \varphi\left(\varpi_W(G)\right)$ is at least σ-dimensional. Since $\varpi_W(G)$ is of dimension $\sigma - 1$ and $(W \otimes k) \cap G = 0$, there exists an element $x \in G$ with $x \notin W^\perp \otimes k$. In particular, since $G + \varphi(G)$ is $(\sigma + 1)$-dimensional, it follows that $\varpi_W\left(G + \varphi(G)\right)$ is at most σ-dimensional. Since $W \otimes k$ and $W^\perp \otimes k$ are φ-stable, we have $\varpi_W\left(\varphi(G)\right) = \varphi\left(\varpi_W(G)\right)$, which implies the inclusions

$$\varpi_W(G) \subset \varpi_W\left(G + \varphi(G)\right)$$

and

$$\varphi\left(\varpi_W(G)\right) \subset \varpi_W\left(G + \varphi(G)\right).$$

It therefore follows that

$$\varpi_W(G) + \varphi\left(\varpi_W(G)\right) \subseteq \varpi_W\left(G + \varphi(G)\right),$$

and since the latter is σ-dimensional, we find with the other dimension estimate established before, that $\varpi_W(G) + \varphi\left(\varpi_W(G)\right)$ is σ-dimensional. $\square$

Next, we want to define the open subset U_i in $\mathcal{M}_\sigma$ so that ϖ_σ yields a well-defined morphism of schemes from U_i to $\mathcal{M}_{\sigma-1}$.

DEFINITION 2.3. We define $\underline{U}_i \subset \underline{\mathcal{M}}_\sigma$ for $i = 3, \ldots, n$ to be the subfunctor mapping any $\mathbb{F}_p$-algebra A to the set

$$\underline{U}_i(A) = \{G \in \mathcal{M}_\sigma(A) \mid G + \langle e_i \rangle \text{ is a direct summand of rank } \sigma + 1 \text{ in } L_+\}.$$

REMARK 2.4. The $\underline{U}_i$ defined above are described by unions of non-vanishing loci of determinants and hence open subfunctors of $\underline{\mathcal{M}}_\sigma$. We denote the corresponding open subschemes of $\mathcal{M}_\sigma$ by U_i.

We will need the following technical lemma.

LEMMA 2.5. *Let G be a generatrix of $L_+ \otimes A$ such that $G + (\langle e_i \rangle \otimes A)$ is a direct summand of rank $\sigma + 1$. Then $G + (\langle e_i \rangle^\perp \otimes A) = L_+ \otimes A$.*

PROOF. We may without loss of generality assume that $i = n - 1$. Let G be as in the lemma and write

$$G = \begin{pmatrix} a_{11} & \ldots & a_{\sigma 1} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{\sigma n} \end{pmatrix},$$

where the columns are basis vectors for G in coordinates of the basis $\{v, \varphi(v), e_3, \ldots, e_n\}$. We assume that $G + (\langle e_{n-1} \rangle^\perp \otimes A) \neq L_+ \otimes A$. This means that there exists a maximal ideal $\mathfrak{m}$ of A such that $a_{1n}, \ldots, a_{\sigma n} = 0$ in $A/\mathfrak{m}$. It follows that the subspace

$$G' = \begin{pmatrix} a_{11} & \ldots & a_{\sigma 1} \\ \vdots & & \vdots \\ a_{1n-2} & \ldots & a_{\sigma n-2} \end{pmatrix}$$

is a generatrix in a quadratic space that is isomorphic to $L \otimes A/\mathfrak{m}$. This implies that the columns of the matrix

$$\begin{pmatrix} a_{11} & \ldots & a_{\sigma 1} \\ \vdots & & \vdots \\ a_{1n-2} & \ldots & a_{\sigma n-2} \end{pmatrix}$$

are linearly dependent over $A/\mathfrak{m}$, since an isotropic subspace of $L \otimes A/\mathfrak{m}$ is at most of dimension $\sigma - 1$. But this yields a contradiction to $G + (\langle e_{n-1} \rangle \otimes A)$ being a direct summand of rank $\sigma + 1$, and we find $G + (\langle e_{n-1} \rangle^\perp \otimes A) = L_+ \otimes A$. $\square$

REMARK 2.6. It follows from Lemma 2.2 that there exists an isomorphism of vector spaces with bilinear form $\langle e_i \rangle^\perp / \langle e_i \rangle \cong L$ for any $i = 3, \ldots, n$, and subsequently there also exists an induced isomorphism of $\mathbb{F}_p$-schemes $\mathcal{M}_{\langle e_i \rangle^\perp / \langle e_i \rangle} \cong \mathcal{M}_{\sigma-1}$. From now on we fix such isomorphisms.

We can now prove that the pointwise construction of $\varpi_{\langle e_i \rangle}$ from Lemma 2.2 yields a well-defined morphism $\Phi_i \colon U_i \longrightarrow \mathcal{M}_{\sigma-1}$.

PROPOSITION 2.7. *For every $i = 3, \ldots, n$ there exists a morphism*

$$\Phi_i \colon U_i \longrightarrow \mathcal{M}_{\sigma-1}$$

which is given on A-valued points via

$$G \longmapsto \Big(G \cap (\langle e_i\rangle^\perp \otimes A)\Big) \,/\, (\langle e_i\rangle \otimes A)\,.$$

PROOF. Clearly, the association $G \mapsto \big(G \cap (\langle e_i\rangle^\perp \otimes A\big) \,/\, (\langle e_i\rangle \otimes A))$ is functorial in A. The schemes U_i and $\mathcal{M}_{\sigma-1}$ are reduced and of finite type over $\mathbb{F}_p$. Thus, by the Yoneda lemma it remains to show that for any reduced finite type $\mathbb{F}_p$-algebra A and any characteristic generatrix $G \subset L_+ \otimes A$ such that $G + \langle e_i\rangle$ is a direct summand of rank $\sigma + 1$ in $L_+ \otimes A$, the A-submodule $\Phi_i(G) = \big(G \cap (\langle e_i\rangle^\perp \otimes A)\big) \,/\, (\langle e_i\rangle \otimes A)$ is indeed a characteristic generatrix of the A-module $\big(\langle e_i\rangle^\perp/\langle e_i\rangle\big) \otimes A$.

By the same argument as in the proof of Lemma 2.2, it follows that the A-submodule $\Phi_i(G)$ in $\big(\langle e_i\rangle^\perp/\langle e_i\rangle\big) \otimes A$ is isotropic. Next, let us prove that $\Phi_i(G)$ is a direct summand of rank $\sigma - 1$ in $\big(\langle e_i\rangle^\perp/\langle e_i\rangle\big) \otimes A$. To this end, we let $\mathfrak{m} \subset A$ be a maximal ideal. It follows from Lemma 2.5 that there are short exact sequences

$$0 \longrightarrow G \cap (\langle e_i\rangle^\perp \otimes A) \longrightarrow G \oplus (\langle e_i\rangle^\perp \otimes A) \longrightarrow L_+ \otimes A \longrightarrow 0$$

and

$$0 \to (G \otimes A/\mathfrak{m}) \cap (\langle e_i\rangle \otimes A/\mathfrak{m})^\perp \to (G \otimes A/\mathfrak{m}) \oplus (\langle e_i\rangle \otimes A/\mathfrak{m})^\perp \to L_+ \otimes A/\mathfrak{m} \to 0.$$

It follows that there exists a natural isomorphism

$$\Big(G \cap (\langle e_i\rangle^\perp \otimes A)\Big) \otimes_A A/\mathfrak{m} \cong (G \otimes A/\mathfrak{m}) \cap (\langle e_i\rangle \otimes A/\mathfrak{m})^\perp,$$

and consequently also a natural isomorphism of embeddings

$$\Big(\Phi_i(G) \otimes_A A/\mathfrak{m} \hookrightarrow \Big(\langle e_i\rangle^\perp/\langle e_i\rangle\Big) \otimes_A A/\mathfrak{m}\Big) \cong \Big(\Phi_i(G \otimes_A A/\mathfrak{m}) \hookrightarrow \Big(\langle e_i\rangle^\perp/\langle e_i\rangle\Big) \otimes_A A/\mathfrak{m}\Big)\,.$$

It follows from Lemma 2.2 that the vector space $\Phi_i(G \otimes_A A/\mathfrak{m})$ has dimension $\sigma - 1$ over $A/\mathfrak{m}$. Since this is true for all maximal ideals $\mathfrak{m}$ of A, using the isomorphism of embeddings above this implies that the submodule $\Phi_i(G) \hookrightarrow \big(\langle e_i\rangle^\perp/\langle e_i\rangle\big) \otimes A$ is locally free of rank $\sigma - 1$, and the cokernel in the short exact sequence

$$0 \longrightarrow \Phi_i(G) \longrightarrow \Big(\langle e_i\rangle^\perp/\langle e_i\rangle\Big) \otimes A \longrightarrow \operatorname{coker} \longrightarrow 0$$

is locally free of rank $\sigma - 1$ as well. In particular, the sequence splits and $\Phi_i(G)$ is a direct summand in $\big(\langle e_i\rangle^\perp/\langle e_i\rangle\big) \otimes A$.

Likewise, for any maximal ideal $\mathfrak{m} \subset A$, there is a natural isomorphism

$$(\Phi_i(G) + \varphi(\Phi_i(G))) \otimes_A A/\mathfrak{m} \cong \Phi_i(G \otimes_A A/\mathfrak{m}) + \varphi(\Phi_i(G \otimes_A A/\mathfrak{m}))$$

and since by Lemma 2.2, $\Phi_i(G \otimes_A A/\mathfrak{m}) + \varphi(\Phi_i(G \otimes_A A/\mathfrak{m}))$ is of dimension σ, it follows that $\Phi_i(G) + \varphi(\Phi_i(G))$ is a direct summand of rank σ in $\big(\langle e_i\rangle^\perp/\langle e_i\rangle\big) \otimes A$. $\square$

3. Extending the rational map to a morphism $\mathcal{M}_\sigma \to \mathcal{M}_{\sigma-1}$

In this section, we prove that the rational maps $\Phi_i \colon \mathcal{M}_\sigma \dashrightarrow \mathcal{M}_{\sigma-1}$ from Proposition 2.7 can be extended to morphisms. From now on we will denote submodules of lattices of the form $L \otimes A$ or $L_+ \otimes A$ via matrices, where the columns of the matrix are vectors spanning the submodule in the coordinates $\{v, \varphi(v), e_3, \ldots\}$.

Let $G \in \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)$ be a characteristic subspace of $L \otimes \overline{\mathbb{F}}_p$. For convenience of notation, we will consider the case $i = n$. It follows directly from the construction of Φ_n that a point $G' \in \mathcal{M}_\sigma(\overline{\mathbb{F}}_p)$ lies in the fiber $\Phi_n^{-1}(G) \subset U_n$ if and only if G' is of the form

$$G' = \begin{pmatrix} a_1 & & & & \\ & & G & & \\ a_{\sigma-1} & & & & \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & -a_1 & \ldots & \ldots & -a_{\sigma-1} \end{pmatrix}$$

for an adequate choice of a matrix representing G and some $a_i \in \overline{\mathbb{F}}_p$. We will assume the following lemma which we will prove later.

LEMMA 2.8. *Let $G \in \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)$ be a characteristic subspace of $L \otimes \overline{\mathbb{F}}_p$ and let*

$$C_G = \left\{ \textit{characteristic generatrices} \begin{pmatrix} a_1 & & & & \\ & & G & & \\ a_{\sigma-1} & & & & \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & -a_1 & \ldots & \ldots & -a_{\sigma-1} \end{pmatrix} \right\} \subset U_{n,\overline{\mathbb{F}}_p}$$

be the fiber $\Phi_n^{-1}(G)$ in $U_{n,\overline{\mathbb{F}}_p}$. Then the closure $\tilde{C}_G$ of C_G in $\mathcal{M}_\sigma$ is

$$\tilde{C}_G = C_G \cup \left\{ \begin{pmatrix} 0 & & & & \\ & & \varphi^{-1}(G) & & \\ 0 & & & & \\ 0 & 0 & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix} \right\}.$$

Furthermore, the scheme $\tilde{C}_G$ is connected.

We will also need the following consequence of Zariski's Main Theorem that can be deduced directly from [GW10, Corollary 12.88].

LEMMA 2.9. *Let $f \colon X \to Y$ be a birational, bijective, proper morphism of noetherian, integral schemes. Suppose Y is normal. Then f is an isomorphism.*

Using Lemma 2.8 and Lemma 2.9 we can now prove Theorem 2.1.

Proof of Theorem 2.1. We still consider without loss of generality the case $i = n$. We will show that the extension

$$\tilde{\Phi}_n \colon \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$$

of Φ_n from U_n to $\mathcal{M}_\sigma$ is given via

$$\begin{pmatrix} a_1 & & & & \\ & & G & & \\ a_{\sigma-1} & & & & \\ 1 & 0 & \dots & \dots & 0 \\ 0 & -a_1 & \dots & \dots & -a_{\sigma-1} \end{pmatrix} \longmapsto G,$$

$$\begin{pmatrix} 0 & & & & \\ & & G & & \\ 0 & & & & \\ 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} \longmapsto \varphi(G)$$

on $\overline{\mathbb{F}}_p$-valued points. Note that each $\overline{\mathbb{F}}_p$-valued point of $\mathcal{M}_\sigma$ can be written in a form as above. We let Γ be the scheme-theoretic graph of Φ_n embedded in $\mathcal{M}_\sigma \times \mathcal{M}_{\sigma-1}$ with Zariski closure $\tilde{\Gamma} \subseteq \mathcal{M}_\sigma \times \mathcal{M}_{\sigma-1}$. We consider the projection onto its second component $p_2 \colon \tilde{\Gamma} \longrightarrow \mathcal{M}_{\sigma-1}$. The morphism p_2 has closed fibers, thus it follows from Lemma 2.8 that the union

$$\tilde{\Gamma}' := \Gamma_{\overline{\mathbb{F}}_p} \cup \left(\begin{pmatrix} 0 & & & & \\ & & \varphi^{-1}(G) & & \\ 0 & & & & \\ 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, G \right)_{G \in \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)}$$

is a subscheme of $\tilde{\Gamma}_{\overline{\mathbb{F}}_p}$.

The morphism $p_2|_{\tilde{\Gamma}'} \colon \tilde{\Gamma}' \longrightarrow \mathcal{M}_{\sigma-1,\overline{\mathbb{F}}_p}$ has a section. For example, we can take the morphism

$$s \colon \mathcal{M}_{\sigma-1,\overline{\mathbb{F}}_p} \longrightarrow \tilde{\Gamma}'$$

which is on $\overline{\mathbb{F}}_p$-valued points defined via

$$G \longmapsto \left(\begin{pmatrix} 0 & & & & \\ & & \varphi^{-1}(G) & & \\ 0 & & & & \\ 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, G \right).$$

It follows that $p_2|_{\tilde{\Gamma}'}$ is universally submersive in the language of [Gro66, Definition 15.7.8]. Furthermore, this morphism has geometrically connected proper fibers, hence it follows from [Gro66, Corollary 15.7.10] that the morphism $p_2|_{\tilde{\Gamma}'}$ is proper. We conclude that $\tilde{\Gamma}'$ is a closed subscheme of $\mathcal{M}_{\sigma,\overline{\mathbb{F}}_p}$, and so we find $\tilde{\Gamma}' = \tilde{\Gamma}_{\overline{\mathbb{F}}_p}$.

Using Lemma 2.9 and the equality that we have just shown, we find that the projection onto the first component $p_1|_{\tilde{\Gamma}_{\overline{\mathbb{F}}_p}} : \tilde{\Gamma}_{\overline{\mathbb{F}}_p} \longrightarrow \mathcal{M}_{\sigma,\overline{\mathbb{F}}_p}$ is an isomorphism. Hence $p_1|_{\tilde{\Gamma}} : \tilde{\Gamma} \longrightarrow \mathcal{M}_\sigma$ is also an isomorphism, and therefore $p_2 \circ \left(p_1|_{\tilde{\Gamma}}\right)^{-1} : \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$ yields the desired morphism.

4. The proof of Lemma 2.8

To finish the proof of Theorem 2.1, it only remains to prove Lemma 2.8 now. The proof uses a rather involved induction argument, where the cases $\sigma \in \{2,3,4\}$ have to be done explicitly. The treatment of these cases can also be seen as explicit descriptions of the moduli spaces $\mathcal{M}_2, \ldots, \mathcal{M}_4$ as subspaces of certain Grassmannian varieties. Before we start with the actual proof, let us note that for $G \in \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)$, using the notation from Lemma 2.8, the scheme C_G is a closed subscheme of dimension 1 in $\mathbb{A}^{n\times n}_{\overline{\mathbb{F}}_p}$. In particular, C_G is an affine scheme. The scheme $\tilde{C}_G$, however, is proper. It thus follows that the difference $\tilde{C}_G \backslash C_G$ is non-empty. We will use this implicitly later on. Further, assuming we have already shown the equality

$$\tilde{C}_G = C_G \cup \begin{pmatrix} 0 & & & & \\ & & \varphi^{-1}(G) & & \\ 0 & & & & \\ 0 & 0 & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix},$$

it directly follows that the scheme $\tilde{C}_G$ is connected. Namely, since each connected component of C_G is affine, the unique point in $\tilde{C}_G \backslash C_G$ has to lie in the closure of each of the connected components of C_G.

4.1. The case $\sigma = 2$.

We have $\mathcal{M}_2(\overline{\mathbb{F}}_p) = \mathbb{P}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p) \amalg \mathbb{P}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p) \hookrightarrow Gr_{\overline{\mathbb{F}}_p}(2,4)(\overline{\mathbb{F}}_p)$ via the embedding

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_1 \mapsto \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_1 & 0 \\ \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_2 \mapsto \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \\ \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}.$$

It is easy to see that there exists an isomorphism of $\mathbb{F}_p$-schemes

$$\mathcal{M}_1 \xrightarrow{\cong} \operatorname{Spec} \mathbb{F}_{p^2}$$

and similarly the $\overline{\mathbb{F}}_p$-valued points of $\mathcal{M}_1$ are given as $\mathcal{M}_1(\overline{\mathbb{F}}_p) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

For $G = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{M}_1(\overline{\mathbb{F}}_p)$ with $\varphi(G) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have the equality

$$C_G = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \\ 1 & 0 \\ 0 & \lambda \end{pmatrix}_{\lambda \in \overline{\mathbb{F}}_p} = \operatorname{im} \left(\begin{pmatrix} \lambda \\ 1 \end{pmatrix}_{1_{\lambda \in \overline{\mathbb{F}}_p}} \right),$$

and it follows that for the closure $\tilde{C}_G$ of C_G we find the equality

$$\tilde{C}_G = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \\ 1 & 0 \\ 0 & \lambda \end{pmatrix}_{\lambda \in \overline{\mathbb{F}}_p} \cup \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

as desired. The argument for $G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ goes the same and we will thus not repeat it.

4.2. The case $\sigma = 3$. We fix a characteristic subspace

$$G = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \\ 1 & 0 \\ 0 & \lambda \end{pmatrix} \in \mathcal{M}_2(\overline{\mathbb{F}}_p)$$

and investigate the fiber $\Phi_6^{-1}(G) \subseteq U_6(\overline{\mathbb{F}}_p)$. For notational reasons we will treat the case $\lambda = 0$ separately. We first consider the case where $\lambda \neq 0$. Then a subspace $G_+ \subset N_3$ is an element of $\Phi_6^{-1}(G)$ if and only if G_+ is of the form

$$G_+ = \begin{pmatrix} 0 & 0 & 1 \\ -c\lambda & -\lambda & 0 \\ 0 & 1 & 0 \\ -b & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & b & c\lambda \end{pmatrix}$$

for some $b, c \in \overline{\mathbb{F}}_p$ such that $\dim(G_+ + \varphi(G_+)) = 4$. The condition $\dim(G_+ + \varphi(G_+)) = 4$ can be translated into equations in b, c and λ in the following way. We have that

$$\varphi(G_+) = \begin{pmatrix} -c^p\lambda^p & -\lambda^p & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ -b^p & 0 & \lambda^p \\ 1 & 0 & 0 \\ 0 & b^p & c^p\lambda^p \end{pmatrix}$$

and the subvector space $G_+ + \varphi(G_+)$ in N_3 is therefore described by the matrix

$$G_+ + \varphi(G_+) = \begin{pmatrix} 0 & 0 & 1 & -c^p\lambda^p & -\lambda^p & 0 \\ -c\lambda & -\lambda & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -b & 0 & \lambda & -b^p & 0 & \lambda^p \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & b & c\lambda & 0 & b^p & c^p\lambda^p \end{pmatrix}.$$

After applying a straightforward Gauß transformation, we obtain the equality

$$\dim(G_+ + \varphi(G_+)) = \mathrm{rk} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -c^p\lambda^p & -\lambda^p & 0 \\ 0 & 0 & 0 & c\lambda & \lambda & 1 \\ 0 & 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & -T & 0 \end{pmatrix}$$

where T is given via $T = b - b^p + (c^p - c)\lambda^{p+1}$. It thus follows that the subspace G_+ in N_3 is a characteristic subspace if and only if $b - b^p + (c^p - c)\lambda^{p+1} = 0$. Hence, we are interested in the closure $\tilde{C}_G$ of the affine curve

$$C_G = \left\{ C_G(b,c) = \begin{pmatrix} 0 & 0 & 1 \\ -c\lambda & -\lambda & 0 \\ 0 & 1 & 0 \\ -b & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & b & c\lambda \end{pmatrix} \mid b - b^p + (c^p - c)\lambda^{p+1} = 0 \right\}$$

in the moduli variety $\mathcal{M}_3$ or equivalently in the Grassmannian variety $Gr_{\overline{\mathbb{F}}_p}(3,6)$.

There exists an isomorphism of affine curves

$$C_G \cong \mathrm{Spec}\left(\overline{\mathbb{F}}_p[b,c]/(b - b^p + (c^p - c)\lambda^{p+1})\right)$$

and consequently we also have an isomorphism of projective curves

$$\tilde{C}_G \cong \mathrm{Proj}\left(\overline{\mathbb{F}}_p[z,b,c]/(z^{p-1} \cdot b - b^p + (c^p - z^{p-1} \cdot c)\lambda^{p+1})\right).$$

We want to understand the locus where $z = 0$. If $z = 0$, then we find $b \neq 0$, or else we would have $b = z = c = 0$, which is impossible. Using the notation from section 1, we define $B_1 =$

$\{\varphi(v), e_3, e_5\}$ and $B_2 = \{v, e_4, e_6\}$ and consider the chart $c_{B_2}^{B_1}\colon Gr_{\overline{\mathbb{F}}_p}(3,6) \dashrightarrow \mathbb{A}^9_{\overline{\mathbb{F}}_p}$. We therefore obtain a description of C_G in charts via

$$c_{B_2}^{B_1}\left(C_G(b,c)\right) = \begin{pmatrix} 0 & \frac{c}{b}\cdot\lambda & -\frac{\lambda}{b} \\ -\frac{c}{b}\cdot\lambda & 0 & \frac{1}{b} \\ \frac{\lambda}{b} & -\frac{1}{b} & 0 \end{pmatrix}$$

and for its projectivization $\tilde{C}_G$ we analogously find the description

$$c_{B_2}^{B_1}\left(\tilde{C}_G(b:c:z)\right) = \begin{pmatrix} 0 & \frac{c}{b}\cdot\lambda & -\frac{\lambda\cdot z}{b} \\ -\frac{c}{b}\cdot\lambda & 0 & \frac{z}{b} \\ \frac{\lambda\cdot z}{b} & -\frac{z}{b} & 0 \end{pmatrix}.$$

Setting $z = 0$, we have the equality $\frac{c}{b}\cdot\lambda = \left(\frac{1}{\lambda}\right)^{\frac{1}{p}}$ and it follows that

$$c_{B_2}^{B_1}\left(\tilde{C}_G(b:c:0)\right) = \begin{pmatrix} 0 & \left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & 0 \\ -\left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By taking preimages we obtain a description of the locus $\tilde{C}_G \backslash C_G$ as

$$\tilde{C}_G(b:c:0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & 0 \\ -\left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda^{\frac{1}{p}} & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & \lambda^{\frac{1}{p}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This solves the case $\lambda \neq 0$. When $\lambda = 0$, we see that a subspace $G_+ \subset N_3$ is an element of the fiber $\Phi_6^{-1}(G) \subset U_6(\overline{\mathbb{F}}_p)$ if and only if G_+ is of the form

$$G_+ = \begin{pmatrix} 0 & 0 & 1 \\ -c & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 0 \\ 1 & 0 & 0 \\ 0 & b & c \end{pmatrix}$$

with $\dim(G_+ \cap \varphi(G_+)) = 2$. The latter condition is easily seen to be equivalent to $b \in \mathbb{F}_p$. We thus have the equality

$$C_G = \left\{ C_G(b,c) = \begin{pmatrix} 0 & 0 & 1 \\ -c & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 0 \\ 1 & 0 & 0 \\ 0 & b & c \end{pmatrix} \mid b \in \mathbb{F}_p \right\}$$

and by an argument analogous to the case $\lambda \neq 0$ we find the desired result

$$\tilde{C}_G \backslash C_G = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The case where G lies in the other connected component of $\mathcal{M}_2 \times_{\mathbb{F}_p} \operatorname{Spec} \overline{\mathbb{F}}_p$ goes analogous.

4.3. The case $\sigma = 4$. Again, we fix a characteristic subspace

$$G = \begin{pmatrix} 0 & 0 & 1 \\ -c & -\lambda & 0 \\ 0 & 1 & 0 \\ -b & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & b & c \end{pmatrix} \in \mathcal{M}_3(\overline{\mathbb{F}}_p)$$

and investigate the fiber $\Phi_8^{-1}(G) \subseteq U_8(\overline{\mathbb{F}}_p)$. Then a subspace $G_+ \subset N_4$ is an element of $\Phi_8^{-1}(G)$ if and only if G_+ can be written in the form

$$G_+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -f & -c & -\lambda & 0 \\ 0 & 0 & 1 & 0 \\ -e & -b & 0 & \lambda \\ -d & 0 & b & c \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & d & e & f \end{pmatrix}$$

where $d, e, f \in \overline{\mathbb{F}}_p$ are such that $\dim(G_+ + \varphi(G_+)) = 5$. The latter condition holds if and only if the matrix

$$G_+ + \varphi(G_+) = \begin{pmatrix} 0 & 0 & 0 & 1 & -f^p & -c^p & -\lambda^p & 0 \\ -f & -c & -\lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -e & -b & 0 & \lambda & -e^p & -b^p & 0 & \lambda^p \\ -d & 0 & b & c & -d^p & 0 & b^p & c^p \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d & e & f & 0 & d^p & e^p & f^p \end{pmatrix}$$

has rank equal to 5. After applying a straightforward Gauß transformation, we obtain

$$\dim(G_+ + \varphi(G_+)) = \operatorname{rk} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -f^p & -c^p & -\lambda^p & 0 \\ 0 & 0 & 0 & 0 & f & c & \lambda & 1 \\ 0 & 0 & 0 & 0 & T_1 & T_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_3 & 0 & -T_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -T_3 & -T_1 & 0 \end{pmatrix}$$

with $T_1 = e - e^p + \lambda f^p - \lambda^p f$, $T_2 = b - b^p + \lambda c^p - c\lambda^p$ and $T_3 = d - d^p + cf^p - c^p f$. It follows that G_+ is characteristic if and only if $T_1 = T_2 = T_3 = 0$. Note that the equality $T_2 = 0$ is automatic because G was a characteristic subspace of N_3. Thus we are interested in the closure $\tilde{C}_G$ of the affine curve

$$C_G = \left\{ C_G(d,e,f) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -f & -c & -\lambda & 0 \\ 0 & 0 & 1 & 0 \\ -e & -b & 0 & \lambda \\ -d & 0 & b & c \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & d & e & f \end{pmatrix} \mid T_1 = T_3 = 0 \right\}$$

in the moduli variety $\mathcal{M}_4$ or equivalently in the Grassmannian variety $Gr_{\overline{\mathbb{F}}_p}(4,8)$.

There exists an isomorphism of affine varieties

$$C_G \cong \operatorname{Spec}\left(\overline{\mathbb{F}}_p[d,e,f]/(e - e^p + \lambda f^p - \lambda^p f, d - d^p + cf^p - c^p f)\right)$$

and consequently we also have an isomorphism of projective varieties

$$\tilde{C}_G \cong \operatorname{Proj}\left(\overline{\mathbb{F}}_p[z,d,e,f]/(z^{p-1}e - e^p + \lambda f^p - z^{p-1}\lambda^p f, z^{p-1}d - d^p + cf^p - z^{p-1}c^p f)\right).$$

Again, we want to understand the locus where $z = 0$. Let us first consider the situation where $\lambda \neq 0$. If we have $z = 0$, then we also find $e \neq 0$, or else we would have $z = e = d = f = 0$. We set $B_1 = \{\varphi(v), e_3, e_5, e_7\}$ and $B_2 = \{v, e_4, e_6, e_8\}$. Using the notation from section 1, we consider the chart $c_{B_2}^{B_1} \colon Gr_{\overline{\mathbb{F}}_p}(4,8) \dashrightarrow \mathbb{A}^{16}_{\overline{\mathbb{F}}_p}$. For C_G we therefore obtain a description in charts as

$$c_{B_2}^{B_1}(C_G(d,e,f)) = \begin{pmatrix} 0 & \frac{f}{e} & -c + \frac{d\lambda + bf}{e} & -\frac{\lambda}{e} \\ -\frac{f}{e} & 0 & -\frac{d}{e} & \frac{1}{e} \\ c - \frac{d\lambda + bf}{e} & \frac{d}{e} & 0 & \frac{b}{e} \\ \frac{\lambda}{e} & \frac{1}{e} & -\frac{b}{e} & 0 \end{pmatrix}$$

and for its projectivization $\tilde{C}_G$, we have the description

$$c_{B_2}^{B_1}(\tilde{C}_G(d:e:f:z)) = \begin{pmatrix} 0 & \frac{f}{e} & -c+\frac{d\lambda+bf}{e} & -\frac{z\lambda}{e} \\ -\frac{f}{e} & 0 & -\frac{d}{e} & \frac{z}{e} \\ c-\frac{d\lambda+bf}{e} & \frac{d}{e} & 0 & \frac{zb}{e} \\ \frac{z\lambda}{e} & \frac{z}{e} & -\frac{zb}{e} & 0 \end{pmatrix}.$$

Setting $z=0$, we find the equalities $\frac{f}{e} = \left(\frac{1}{\lambda}\right)^{\frac{1}{p}}$ and $\frac{d}{e} = \left(\frac{c}{\lambda}\right)^{\frac{1}{p}}$. It follows that

$$c_{B_2}^{B_1}(\tilde{C}_G(d:e:f:0)) = \begin{pmatrix} 0 & \left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & -c+(\lambda c^{\frac{1}{p}}+b)(\frac{1}{\lambda})^{\frac{1}{p}} & 0 \\ -\left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & 0 & -\left(\frac{c}{\lambda}\right)^{\frac{1}{p}} & 0 \\ c-(\lambda c^{\frac{1}{p}}+b)(\frac{1}{\lambda})^{\frac{1}{p}} & \left(\frac{c}{\lambda}\right)^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By taking preimages, we obtain for the locus $\tilde{C}_G \backslash C_G$ the description

$$\tilde{C}_G(d:e:f:0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & -c+(\lambda c^{\frac{1}{p}}+b)(\frac{1}{\lambda})^{\frac{1}{p}} & 0 \\ -\left(\frac{1}{\lambda}\right)^{\frac{1}{p}} & 0 & -\left(\frac{c}{\lambda}\right)^{\frac{1}{p}} & 0 \\ 0 & 1 & 0 & 0 \\ c-(\lambda c^{\frac{1}{p}}+b)(\frac{1}{\lambda})^{\frac{1}{p}} & \left(\frac{c}{\lambda}\right)^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By using the equality $c\lambda^{\frac{1}{p}} - \lambda c^{\frac{1}{p}} - b = -b^{\frac{1}{p}}$ and the resulting equality

$$\begin{pmatrix} 0 \\ -c+(\lambda c^{\frac{1}{p}}+b)(\frac{1}{\lambda})^{\frac{1}{p}} \\ -\left(\frac{c}{\lambda}\right)^{\frac{1}{p}} \\ 0 \\ 0 \\ 1 \end{pmatrix} - c^{\frac{1}{p}} \begin{pmatrix} 1 \\ 0 \\ -\left(\frac{1}{\lambda}\right)^{\frac{1}{p}} \\ 0 \\ c-(\lambda c^{\frac{1}{p}}+b)(\frac{1}{\lambda})^{\frac{1}{p}} \\ 0 \end{pmatrix} - b^{\frac{1}{p}} \begin{pmatrix} 0 \\ \left(\frac{1}{\lambda}\right)^{\frac{1}{p}} \\ 0 \\ 1 \\ \left(\frac{c}{\lambda}\right)^{\frac{1}{p}} \\ 0 \end{pmatrix} = \begin{pmatrix} -c^{\frac{1}{p}} \\ 0 \\ 0 \\ -b^{\frac{1}{p}} \\ 0 \\ 1 \end{pmatrix},$$

we then obtain the desired description for the locus $\tilde{C}_G \backslash C_G$ as

$$\tilde{C}_G(d:e:f:0) = \begin{pmatrix} -\lambda^{\frac{1}{p}} & 0 & -c^{\frac{1}{p}} & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \lambda^{\frac{1}{p}} & -b^{\frac{1}{p}} & 0 \\ b^{\frac{1}{p}} & c^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us now treat the situation where $\lambda = 0$. If we set $z = 0$, then we also have $e = 0$ and hence we find $f \neq 0$. Taking $B_1 = \{v, e_4, e_5, e_7\}$ and $B_2 = \{\varphi(v), e_3, e_6, e_8\}$ we find that

$$c_{B_2}^{B_1}(\tilde{C}_G(d:e:f:0)) = c_{B_2}^{B_1}(\tilde{C}_G(d:0:f:0)) = \begin{pmatrix} 0 & 0 & -\frac{d}{f} & 0 \\ 0 & 0 & -b & 0 \\ \frac{d}{f} & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and using the equalities $\frac{d}{f} = c^{\frac{1}{p}}$ and $b = b^{\frac{1}{p}}$, we obtain the description

$$\tilde{C}_G(d:e:f:0) = \begin{pmatrix} 0 & 0 & c^{\frac{1}{p}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -b^{\frac{1}{p}} & 0 \\ c^{\frac{1}{p}} & b^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again, the case where G lies in the other connected component of $\mathcal{M}_3$ goes analogous.

4.4. The case $\sigma \geq 5$**.** We will now do induction over σ. We assume that Lemma 2.8 has already been shown for all $\sigma' \leq \sigma - 1$ with $\sigma - 1$ at least 4. We want to show that the lemma also holds for σ.

To this end, let $G \in \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)$ and let $C_G \subseteq \mathcal{M}_\sigma$ be as in the statement of the lemma. We first consider the generic case, where G is an element of $U_i^{\sigma-1}(\overline{\mathbb{F}}_p) \subset \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)$ for each $i = 3, \ldots, 2(\sigma-1)$. In this case, we also have $C_G \subseteq U_i^\sigma \subset \mathcal{M}_\sigma$ for each $i \in \{3, \ldots, 2(\sigma-1)\}$.

Let $x \in \tilde{C}_G \backslash C_G$ be an $\overline{\mathbb{F}}_p$-valued point in the closure of C_G that is not in C_G. The subscheme C_G is closed in $U_{2\sigma}^\sigma$, since it is the preimage of the closed point G under the morphism

$\Phi_{2\sigma}\colon U_{2\sigma}^{\sigma} \to \mathcal{M}_{\sigma-1}$. Therefore x is of the form

$$x = \begin{pmatrix} 0 & & & & \\ & & G' & & \\ 0 & & & & \\ 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}$$

for some G'. Furthermore, for every $i \in \{3, 5, \dots, 2\sigma - 3\}$ we have $x \in U_i^{\sigma}$ or $x \in U_{i+1}^{\sigma}$. Let us assume without loss of generality that $x \in U_{2\sigma-6} \cap U_{2\sigma-4} \cap U_{2\sigma-2}$. Applying Φ_i to C_G for $i \in \{2\sigma - 6, 2\sigma - 4, 2\sigma - 2\}$, we get by continuity that

$$\Phi_i(x) = \begin{pmatrix} 0 & & & & \\ & & \Phi_i(\varphi^{-1}(G)) & & \\ 0 & & & & \\ 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

We want to show that $G' = \varphi^{-1}(G)$. To this end, we write

$$G' = \begin{pmatrix} v_1' & v_2' & v_3' & \dots & v_{\sigma-1}' \\ 0 & 1 & 0 & \dots & 0 \\ b_1' & 0 & b_3' & \dots & b_{\sigma-1}' \\ 1 & 0 & 0 & \dots & 0 \\ 0 & a_2' & a_3' & \dots & a_{\sigma-1}' \end{pmatrix}$$

for some $v_i' \in \overline{\mathbb{F}}_p^{2\sigma-6}$ and $a_i', b_i' \in \overline{\mathbb{F}}_p$ as well as

$$\varphi^{-1}(G) = \begin{pmatrix} v_1 & v_2 & v_3 & \dots & v_{\sigma-1} \\ 0 & 1 & 0 & \dots & 0 \\ b_1 & 0 & b_3 & \dots & b_{\sigma-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_{\sigma-1} \end{pmatrix}$$

for some $v_i \in \overline{\mathbb{F}}_p^{2\sigma-6}$ and $a_i, b_i \in \overline{\mathbb{F}}_p$, such that

$$\begin{pmatrix} v_1 & v_2 & v_3 & \dots & v_{\sigma-1} \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & \dots & w_{\sigma-1} \\ 0 & 0 & 1 & 0 & \dots & 0 \\ c_1 & c_2 & 0 & c_4 & \dots & c_{\sigma-1} \end{pmatrix}$$

for some $w_i \in \overline{\mathbb{F}}_p^{2\sigma-8}$ and $c_i \in \overline{\mathbb{F}}_p$. Applying $\Phi_{2\sigma-2}$ to G' yields the equality

$$\begin{pmatrix} v_2' & v_3' & \dots & v_{\sigma-1}' \\ 1 & 0 & \dots & 0 \\ 0 & b_3' & \dots & b_{\sigma-1}' \end{pmatrix} = \begin{pmatrix} v_2 & v_3 & \dots & v_{\sigma-1} \\ 1 & 0 & \dots & 0 \\ 0 & b_3 & \dots & b_{\sigma-1} \end{pmatrix},$$

so we may write G' in the form

$$G' = \begin{pmatrix} v_1' & v_2 & v_3 & \dots & v_{\sigma-1} \\ 0 & 1 & 0 & \dots & 0 \\ b_1' & 0 & b_3 & \dots & b_{\sigma-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \tilde{a}_2 & \tilde{a}_3 & \dots & \tilde{a}_{\sigma-1} \end{pmatrix}$$

for some $\tilde{a}_i \in \overline{\mathbb{F}}_p$. Next, applying $\Phi_{2\sigma-4}$ to G' yields the equality

$$\begin{pmatrix} v_1' & v_3 & \dots & v_{\sigma-1} \\ 1 & 0 & \dots & 0 \\ 0 & \tilde{a}_3 & \dots & \tilde{a}_{\sigma-1} \end{pmatrix} = \begin{pmatrix} v_1 & v_3 & \dots & v_{\sigma-1} \\ 1 & 0 & \dots & 0 \\ 0 & a_3 & \dots & a_{\sigma-1} \end{pmatrix}.$$

The set $\{v_3, \dots, v_{\sigma-1}\}$ is linearly independent, or else we would be able to represent a vector v of the form

$$v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{b} \\ 0 \\ \tilde{a} \end{pmatrix}$$

as a non-trivial linear combination of the columns in G'. But by isotropy it follows that then we would have $\tilde{b} = \tilde{a} = 0$, which is not possible since the columns of G' are linearly independent. So, we may write

$$G' = \begin{pmatrix} v_1 & v_2 & v_3 & \dots & v_{\sigma-1} \\ 0 & 1 & 0 & \dots & 0 \\ \tilde{b}_1 & 0 & b_3 & \dots & b_{\sigma-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \tilde{a}_2 & a_3 & \dots & a_{\sigma-1} \end{pmatrix}$$

for some $\tilde{b}_0 \in \overline{\mathbb{F}}_p$. Now, applying $\Phi_{2\sigma-6}$ to G' yields

$$\begin{pmatrix} w_1 & w_2 & w_4 & \dots & w_{\sigma-1} \\ 0 & 1 & 0 & \dots & 0 \\ \tilde{b}_1 & 0 & b_4 & \dots & b_{\sigma-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \tilde{a}_2 & a_4 & \dots & a_{\sigma-1} \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_4 & \dots & w_{\sigma-1} \\ 0 & 1 & 0 & \dots & 0 \\ b_1 & 0 & b_4 & \dots & b_{\sigma-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_4 & \dots & a_{\sigma-1} \end{pmatrix}.$$

But again, the set $\{w_4, \dots, w_{\sigma-1}\}$ is linearly independent, which shows the equalities $\tilde{b}_1 = b_1$ and $\tilde{a}_2 = a_2$. Thus, we have $G' = \varphi^{-1}(G)$, which proves the lemma in the generic case.

Next, we consider the case where there exists some $i \in \{3, \dots, 2\sigma - 2\}$ such that $G \notin U_i^{\sigma-1}$. Say, without loss of generality we have $i = 2\sigma - 2$. Then G is of the form

$$G = \begin{pmatrix} 0 & v_2 & \dots & v_{\sigma-1} \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

for some $v_i \in \overline{\mathbb{F}}_p^{2\sigma-4}$ and for C_G we find

$$C_G(a_i) = \begin{pmatrix} a & 0 & v_2 & \dots & v_{\sigma-1} \\ a_1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & -a_1 & -a_2 & \dots & -a_{\sigma-1} \end{pmatrix},$$

for some $a_i \in \overline{\mathbb{F}}_p$ and $a \in \overline{\mathbb{F}}_p^{2\sigma-4}$. Let us write

$$G_{a_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -a_1 \end{pmatrix}$$

and

$$G_{(a_1,\dots,a_{\sigma-1})} = \begin{pmatrix} a & v_2 & \dots & v_{\sigma-1} \\ a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & -a_2 & \dots & -a_{\sigma-1} \end{pmatrix}.$$

It is clear that there is an equality

$$\varphi(C_G(a_i)) + C_G(a_i) = (\varphi(G_{a_1}) + G_{a_1}) \oplus \left(\varphi(G_{(a_1,\dots,a_{\sigma-1})}) + G_{(a_1,\dots,a_{\sigma-1})}\right),$$

and we further have the inequality $\dim\left(\varphi(G_{(a_1,\dots,a_{\sigma-1})}) + G_{(a_1,\dots,a_{\sigma-1})}\right) \geq \sigma$ and the equality $\dim\left(\varphi(C_G(a_i)) + C_G(a_i)\right) = \sigma + 1$. It follows that a_1 is an element of $\mathbb{F}_p$. In other words, the morphism

$$\Phi_{2\sigma-3|C_G} \colon C_G \longrightarrow C_{\Phi_{2\sigma-3}(G)}$$

is just the canonical cover of $C_{\Phi_{2\sigma-3}(G)}$ by p disjoint copies of itself. By induction we obtain

$$\tilde{C}_G \backslash C_G = \bigcup_{a_1 \in \mathbb{F}_p} \begin{pmatrix} 0 & 0 & \varphi^{-1}(v_1) & \dots & \varphi^{-1}(v_{\sigma-1}) \\ d_0 & 0 & d_2 & \dots & d_{\sigma-1} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 1 & -a_1 & 0 & \dots & 0 \end{pmatrix}$$

for some $d_i \in \overline{\mathbb{F}}_p$. But by isotropy we have $d_i = 0$, hence we find

$$\tilde{C}_G \backslash C_G = \begin{pmatrix} 0 & & & & \\ & & \varphi^{-1}(G) & & \\ 0 & & & & \\ 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

This is what we wanted to show and finishes the proof of Lemma 2.8.

5. The fibers of the morphisms Φ_i

We conclude this chapter with an observation on the structure of the fibers of the morphisms $\tilde{\Phi}_i \colon \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$, which is a direct consequence of results in [BL18] on the morphisms $\Phi_i \colon U_i^\sigma \to \mathcal{M}_{\sigma-1}$.

PROPOSITION 2.10. *Let $\tilde{\Phi}_i \colon \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$ be as in Theorem 2.1, and let $G \in \mathcal{M}_{\sigma-1}(\overline{\mathbb{F}}_p)$ be a characteristic subspace of L with Artin invariant $\sigma(G) \leq \sigma - 1$. Then, the fiber $\tilde{\Phi}_i^{-1}(G)$ is connected and reduced. If $\sigma(G) = \sigma - 1$, then $\tilde{\Phi}_i^{-1}(G)$ is irreducible. If $\sigma(G) < \sigma - 1$, then $\tilde{\Phi}_i^{-1}(G)$ has $p \cdot (\sigma - 1 - \sigma(G))$ many irreducible components and these intersect in a unique common point. In any case, each irreducible component of $\tilde{\Phi}_i^{-1}(G)$ is birationally equivalent to $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$.*

PROOF. This is a direct consequence of [BL18, Lemma 3.1.15], which is the corresponding result for the morphisms $\Phi_i \colon U_i^\sigma \longrightarrow \mathcal{M}_{\sigma-1}$, and the proof of Lemma 2.8. $\square$

REMARK 2.11. The morphisms $\tilde{\Phi}_i \colon \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ are smooth. However, it follows directly from the explicit equations we computed for the morphisms $\tilde{\Phi}_i \colon \mathcal{M}_3 \longrightarrow \mathcal{M}_2$ and $\tilde{\Phi}_i \colon \mathcal{M}_4 \longrightarrow \mathcal{M}_3$ that their fibers have a singularity "at infinity". We believe that this should hold true for all of the morphisms $\tilde{\Phi}_i \colon \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$ with $\sigma \geq 3$.

More precisely, Daniel Bragg communicated notes to us that suggest any irreducible fiber of a morphism $\tilde{\Phi}_i \colon \mathcal{M}_\sigma \longrightarrow \mathcal{M}_{\sigma-1}$ should be isomorphic to the singular projective curve given by $V(x_0^p - x_1, x_1^p - x_2, \dots, x_{\sigma-2}^p - x_{\sigma-1}) \subseteq \mathbb{P}^\sigma$.

CHAPTER 3

A moduli space for supersingular Enriques surfaces

In this chapter, we construct a moduli space of adequately marked Enriques surfaces that have a supersingular K3 cover over fields of characteristic $p \geq 3$. We show that this moduli space exists as a quasi-separated algebraic space locally of finite type over $\mathbb{F}_p$. Moreover, there exists a period map from this moduli space to a period scheme, and we obtain a Torelli theorem for supersingular Enriques surfaces. The idea of the proof is to start with Ogus' moduli space of K3-lattice-marked supersingular K3 spaces and then manipulate it by taking adequate subspaces and quotients by group actions.

1. Moduli spaces of N_σ-marked supersingular K3 surfaces

This section discusses the moduli spaces for lattice-marked K3 surfaces that were introduced in [Ogu83].

We fix a prime $p \geq 3$ and for each integer σ with $1 \leq \sigma \leq 10$ a representative N_σ for the unique isomorphism class of K3 lattices with $\sigma(N_\sigma) = \sigma$. A *family of supersingular K3 surfaces* is a smooth and proper morphism $f\colon \mathcal{X} \to S$ of algebraic spaces over $\mathbb{F}_p$ such that for each field k and each k-valued point $\operatorname{Spec} k \to S$ the fiber $\mathcal{X}_k \to \operatorname{Spec} k$ is a projective supersingular K3 surface. By [Riz06, Theorem 3.1.1] the relative Picard functor $\underline{\mathrm{Pic}}_{\mathcal{X}/S}$ is representable by a separated algebraic space $\mathrm{Pic}_{\mathcal{X}/S}$ over S. An *N_σ-marking of a family of supersingular K3 surfaces* $f\colon \mathcal{X} \to S$ is a morphism $\psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}$ of group objects in the category of algebraic spaces that is compatible with intersection forms. There is an obvious notion of morphisms of families N_σ-marked K3 surfaces. From now on we will write $\mathbb{A}_{\mathbb{F}_p}$ for the category of algebraic spaces over $\mathbb{F}_p$. We consider the moduli problem

$$\underline{\mathcal{S}}_\sigma\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\mathrm{Sets}),$$
$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of } N_\sigma\text{-marked} \\ \text{supersingular K3 surfaces } (f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}) \end{array} \right\}.$$

It is a classical result of Ogus that the functor $\underline{\mathcal{S}}_\sigma$ is representable by an $\mathbb{F}_p$-scheme $\mathcal{S}_\sigma$ that is smooth of dimension $\sigma - 1$ and locally of finite type over $\mathbb{F}_p$ [Ogu83]. Furthermore, $\mathcal{S}_\sigma$ satisfies the existence part of the valuative criterion for properness. However, $\mathcal{S}_\sigma$ is in general neither quasi-compact nor separated.

Via the period map, the functor $\underline{\mathcal{S}}_\sigma$ is canonically isomorphic to a functor $\underline{\mathcal{P}}_\sigma$ [Ogu83] that is defined to be

$$\begin{aligned}\underline{\mathcal{P}}_\sigma\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} &\longrightarrow (\mathrm{Sets})\,,\\ S &\longmapsto \left\{\begin{array}{l}\text{characteristic generatrices } K\subseteq N_\sigma\otimes_{\mathbb{F}_p}\mathcal{O}_S\\ \text{together with an ample cone}\end{array}\right\}.\end{aligned}$$

Ogus originally proved that the period morphism $\pi\colon \mathcal{S}_\sigma \longrightarrow \mathcal{P}_\sigma$ is an isomorphism over fields of characteristic at least 5, but Bragg and Lieblich recently showed that Ogus' results also hold true in characteristic 3 [BL18, Section 5.1].

If we consider the functor

$$\begin{aligned}\underline{\mathcal{M}}_\sigma\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} &\longrightarrow (\mathrm{Sets})\,,\\ S &\longmapsto \left\{\text{characteristic generatrices } G\subseteq N_\sigma\otimes_{\mathbb{F}_p}\mathcal{O}_S\right\},\end{aligned}$$

then there is a canonical surjection of functors $\pi_\sigma\colon \underline{\mathcal{S}}_\sigma \to \underline{\mathcal{M}}_\sigma$ that is given by forgetting the choice of an ample cone. The functor $\underline{\mathcal{M}}_\sigma$ is representable by a smooth connected projective scheme $\mathcal{M}_\sigma$ of dimension $\sigma-1$ and the morphism of schemes π_σ is étale. For further details on the functor $\underline{\mathcal{M}}_\sigma$, we refer the interested reader to [Ogu79], and for further details on the functor $\underline{\mathcal{S}}_\sigma$ we refer to [Ogu83].

Now let $\sigma' < \sigma$ be positive integers with $\sigma \leq 10$. In our construction of the moduli space of marked Enriques surfaces we will use an inductive argument. Therefore we begin with an observation on the relation between the schemes $\mathcal{S}_\sigma$ and $\mathcal{S}_{\sigma'}$. There exists an embedding of lattices $j\colon N_\sigma \hookrightarrow N_{\sigma'}$ that makes $N_{\sigma'}$ into an overlattice of N_σ. We say that two such embeddings j and j' are *isomorphic* embeddings if there exists an automorphism $\alpha\colon N_{\sigma'} \to N_{\sigma'}$ such that $\alpha\circ j = j'$.

By [Nik80, Proposition 1.4.1] there are only finitely many isomorphism classes of such embeddings $j\colon N_\sigma \hookrightarrow N_{\sigma'}$. For each isomorphism class, we choose a representative j and denote by $\mathcal{R}_{\sigma',\sigma}$ the set of these representatives. An embedding $j\colon N_\sigma \hookrightarrow N_{\sigma'}$ induces a morphism of $\mathbb{F}_p$-schemes

$$\Phi_j\colon \mathcal{S}_{\sigma'} \longrightarrow \mathcal{S}_\sigma$$

by mapping

$$(f\colon \mathcal{X}\to S, \psi\colon \underline{N}_{\sigma'} \to \mathrm{Pic}_{\mathcal{X}/S}) \longmapsto (f\colon \mathcal{X}\to S, \psi\circ j\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S})$$

on S-valued points. Similarly, we also obtain a morphism $\Psi_j\colon \mathcal{M}_{\sigma'} \to \mathcal{M}_\sigma$. It follows from [Ogu79, Remark 4.8] that the Ψ_j are closed immersions. Analogously, we see that the finite union $\mathcal{M}_\sigma^{\sigma'} = \bigcup_{j\in R_{\sigma',\sigma}} \Psi_j(\mathcal{M}_{\sigma'})$ is the closed subscheme in $\mathcal{M}_\sigma$ corresponding to characteristic subspaces G of N_σ with Artin invariant $\sigma(G)\leq\sigma'$. We now want to show that the morphisms Φ_j are also closed immersions.

LEMMA 3.1. *The commutative diagrams*

$$\begin{tikzcd} \mathcal{S}_{\sigma'} \arrow[r, "\Phi_j"] \arrow[d, "\pi_{\sigma'}"'] & \mathcal{S}_\sigma \arrow[d, "\pi_\sigma"] \\ \mathcal{M}_{\sigma'} \arrow[r, "\Psi_j"'] & \mathcal{M}_\sigma \end{tikzcd}$$

are cartesian.

PROOF. It is easy to see that the Φ_j are monomorphisms of functors. So we only need to check the existence part in the definition of fiber products. To this end, we claim that there is an equality $\Phi_j(\mathcal{S}_{\sigma'}) = \pi_{\sigma'}^{-1}(\Psi_j(\mathcal{M}_{\sigma'}))$. Indeed, the inclusion $\Phi_j(\mathcal{S}_{\sigma'}) \subseteq \pi_{\sigma'}^{-1}(\Psi_j(\mathcal{M}_{\sigma'}))$ is clear by definition and we easily see that the two subschemes have the same underlying topological space. That means, we have an equality of sets $\{x \in \pi_\sigma^{-1}(\Psi_j(\mathcal{M}_\sigma))\} = \{x \in \Phi_j(\mathcal{S}_{\sigma'})\}$. The scheme $\pi_{\sigma'}^{-1}(\Psi_j(\mathcal{M}_{\sigma'}))$ is reduced because $\Psi_j(\mathcal{M}_{\sigma'})$ is reduced. Since π is an étale morphism, we obtain the desired equality of subschemes.

Thus, given an $\mathbb{F}_p$-scheme S and S-valued points $y \in \mathcal{M}_{\sigma'}(S)$ and $z \in \mathcal{S}_\sigma(S)$ such that $\Psi_j(y) = \pi_\sigma(z)$, we find that $z \in \Phi_j(\mathcal{S}_{\sigma'}(S))$. If we let x be the preimage of z under $\Phi_j(S)$, then $\Phi_j(x) = z$ and $\pi_{\sigma'}(x) = y$ which shows the claim. □

PROPOSITION 3.2. *The morphisms of functors $\Phi_j\colon \underline{\mathcal{S}}_{\sigma'} \to \underline{\mathcal{S}}_\sigma$ are closed immersions of schemes and the subfunctor $\underline{\mathcal{S}}_\sigma^{\sigma'} \hookrightarrow \underline{\mathcal{S}}_\sigma$ which is defined to be*

$$\underline{\mathcal{S}}_\sigma^{\sigma'} : \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\textit{Sets}),$$

$$S \longmapsto \left\{ \begin{array}{l} \textit{Isomorphism classes of families of } N_\sigma\textit{-marked} \\ \textit{supersingular K3 surfaces } (f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}) \\ \textit{such that each fiber } \mathcal{X}_s \textit{ has } \sigma(\mathcal{X}_s) \leq \sigma' \end{array} \right\}$$

is representable by the closed subscheme $\mathcal{S}_\sigma^{\sigma'} = \bigcup_{j \in R_{\sigma',\sigma}} \Phi_j(\mathcal{S}_{\sigma'}) \subseteq \mathcal{S}_\sigma$.

PROOF. We already mentioned that the morphisms Ψ_j are closed immersions, thus Lemma 3.1 implies that the morphisms Φ_j are closed immersions as well. The assertion on the functor represented by the union $\bigcup_{j \in R_{\sigma',\sigma}} \Phi_j(\mathcal{S}_{\sigma'})$ is a consequence of the equality

$$\bigcup_{j \in R_{\sigma',\sigma}} \Phi_j(\mathcal{S}_{\sigma'}) = \pi^{-1}\left(\bigcup_{j \in R_{\sigma',\sigma}} \Psi_j(\mathcal{M}_{\sigma'}) \right),$$

which follows from the proof of Lemma 3.1. □

2. Auxiliary functors and moduli spaces

In this section, we will introduce some auxiliary functors which we will then use to construct the main functor in the subsequent section.

In the following, we fix a positive integer $\sigma \leq 10$. We consider the lattice $\Gamma = U_2 \oplus E_8(-1)$, which is up to isomorphism the unique unimodular, even lattice of signature $(1, 9)$. The Picard group of any Enriques surface is isomorphic to $\Gamma \oplus \mathbb{Z}/2\mathbb{Z}$. Our idea is as follows: if Y is an Enriques surface with a supersingular covering K3 surface X, then we can see the quotient map $f\colon X \to Y$ as a primitive embedding of lattices $\gamma\colon \Gamma(2) \hookrightarrow \mathrm{NS}(X)$ such that $\Gamma(2)$ contains an ample divisor and such that there is no (-2)-vector in $\gamma(\Gamma(2))^{\perp} \subseteq \mathrm{NS}(X)$ [Jan13]. If we also admit embeddings $\gamma\colon \Gamma(2) \hookrightarrow \mathrm{NS}(X)$ such that there is a (-2)-vector in $\gamma(\Gamma(2))^{\perp} \subset \mathrm{NS}(X)$, then we talk about quotients $X \to Y'$ of X by an involution that maybe has a non-trivial fixed-point locus.

We will therefore define various functors of $\Gamma(2)$-marked K3 surfaces and, in Section 4, we then show that the main functor $\tilde{\mathcal{E}}_\sigma$ of $\Gamma(2)$-marked K3 surfaces from Section 3 is isomorphic to a functor of Γ-marked Enriques surfaces.

By Corollary 2.4 in [Jan15], there exists a primitive embedding of lattices $\gamma\colon \Gamma(2) \hookrightarrow N_\sigma$ such that $\gamma(\Gamma(2))^{\perp} \subset N_\sigma$ contains no vector of self-intersection number -2 and, further, there are only finitely many isomorphism classes $[\gamma\colon \Gamma(2) \hookrightarrow N_\sigma]$ of such embeddings. We fix for each such isomorphism class a representative γ and denote by $\mathcal{R}_\sigma$ the finite set formed by these elements. For an embedding $\gamma \in R_\sigma$ we consider the subfunctor $\underline{\mathcal{S}}'_\gamma \subset \underline{\mathcal{S}}_\sigma$ that is defined to be

$$\underline{\mathcal{S}}'_\gamma\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\mathrm{Sets}),$$
$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of } N_\sigma\text{-marked} \\ \text{supersingular K3 surfaces } (f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}) \\ \text{such that for each geometric fiber } s \in S \\ \text{the sublattice } \gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \\ \text{contains an ample line bundle} \end{array} \right\}.$$

The following lattice-theoretic lemma implies that the induced embedding of lattices $\gamma_s\colon \Gamma(2) \hookrightarrow \mathrm{NS}(\mathcal{X}_s)$ is primitive even on the locus where the N_σ-marking $\psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}$ is not an isomorphism.

LEMMA 3.3. *Let $\gamma\colon \Gamma(2) \hookrightarrow N_\sigma$ be a primitive embedding and let $j\colon N_\sigma \hookrightarrow N_{\sigma-1}$ be an embedding of K3 lattices. Then the composition $j \circ \gamma\colon \Gamma(2) \hookrightarrow N_{\sigma-1}$ is a primitive embedding.*

PROOF. We write $\Gamma(2)^{\mathrm{sat}}$ for the saturation of $\Gamma(2)$ in $N_{\sigma-1}$. Then we have an inclusion $2 \cdot \Gamma(2)^{\mathrm{sat}} \subset \Gamma(2)$, because the lattice $\Gamma(2)$ is 2-elementary. On the other hand, we find that $N_\sigma + \Gamma(2)^{\mathrm{sat}}$ is an overlattice of N_σ with $2 \cdot (N_\sigma + \Gamma(2)^{\mathrm{sat}}) \subset N_\sigma$. Since the lattice N_σ is p-elementary and we have $p \neq 2$, it follows that $N_\sigma + \Gamma(2)^{\mathrm{sat}} = N_\sigma$. Thus we have an equality $\Gamma(2) = \Gamma(2)^{\mathrm{sat}}$. □

For the rest of the discussion, we will always assume an embedding of $\Gamma(2)$ into some lattice to be primitive. The next thing we are interested in is the representability of the functor $\underline{\mathcal{S}}'_\gamma$ for some fixed $\gamma \in R_\sigma$.

PROPOSITION 3.4. *The functor $\underline{\mathcal{S}}'_\gamma$ is an open subfunctor of $\underline{\mathcal{S}}_\sigma$.*

PROOF. By definition, we have to show that for any $\mathbb{F}_p$-scheme S and any isomorphism class $x = (f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S}) \in \underline{\mathcal{S}}_\sigma(S)$ the locus $S_x \subseteq S$ such that $\gamma_s(\Gamma(2))$ contains an ample line bundle for all geometric points $s \in S_x$ is an open subscheme of S.

Given an $\mathbb{F}_p$-scheme S and an S-valued point $x = (f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S}) \in \underline{\mathcal{S}}_\sigma(S)$, using Lemma 3.3, we obtain a unique involution $\iota_\gamma^*\colon \mathrm{Pic}_{\mathcal{X}/S} \to \mathrm{Pic}_{\mathcal{X}/S}$ that is induced from $\iota_\gamma^*|_{\Gamma(2)} = \mathrm{id}_{\Gamma(2)}$ and $\iota_\gamma^*|_{\Gamma(2)^\perp} = -\mathrm{id}_{\Gamma(2)^\perp}$, cf. [Shi09, Proposition 2.1.1.]. By Ogus' Torelli theorem [Ogu83] and the argument in [Jan13, Lemma 4.3.], the automorphism ι_γ^* is induced from an automorphism of S-algebraic spaces $\iota_\gamma\colon \mathcal{X} \to \mathcal{X}$ if and only if $\gamma(\Gamma(2)) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S}$ intersects the ample cone in $\mathrm{NS}(\mathcal{X}_s)$ for all points $s \in S$.

Now, if there is no point $s \in S$ such that $\gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s)$ contains an ample line bundle, then $S_x = \emptyset$ is the empty scheme, which is an open subscheme of S. Otherwise, let $s \in S$ be a point such that $\gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s)$ contains an ample line bundle. Let $\mathcal{O}_{S,s}$ be the local ring of S at s, then $(f\colon \mathcal{X}_{\mathrm{Spec}\,\mathcal{O}_{S,s}} \to \mathrm{Spec}\,\mathcal{O}_{S,s}, \psi\colon N_\sigma \hookrightarrow \mathrm{Pic}_{\mathcal{X}_{\mathrm{Spec}\,\mathcal{O}_{S,s}}/\mathrm{Spec}\,\mathcal{O}_{S,s}})$ is also an an element of $\underline{\mathcal{S}}'_\gamma(\mathrm{Spec}\,\mathcal{O}_{S,s})$ by the discussion in [Ogu83, pages 373-374]. If $\{U_i\}_{i\in I}$ is the directed system of all open subschemes of S such that $s \in U_i$, then we have $\mathrm{Spec}\,\mathcal{O}_{S,s} = \lim U_i$ and we consider the commutative diagram

$$\begin{tikzcd}
\operatorname{colim}\left(\mathrm{Aut}_{U_i}(\mathcal{X}_{U_i})\right) \arrow[r] \arrow[d, "\cong"'] & \operatorname{colim}\Big(\mathrm{Aut}(\mathrm{Pic}_{\mathcal{X}_{U_i}/U_i})\Big) \arrow[d, "\cong"'] \\
\mathrm{Aut}_{\mathrm{Spec}\,\mathcal{O}_{S,s}}(\mathcal{X}_{\mathrm{Spec}\,\mathcal{O}_{S,s}}) \arrow[r] & \mathrm{Aut}(\mathrm{Pic}_{\mathcal{X}_{\mathrm{Spec}\,\mathcal{O}_{S,s}}/\mathrm{Spec}\,\mathcal{O}_{S,s}}).
\end{tikzcd}$$

The morphisms $\mathcal{X} \to S$ and $\mathrm{Pic}_{\mathcal{X}/S} \to S$ are locally of finite presentation, and it follows from [Sta19, Proposition 31.6.1.] that the vertical arrows in the diagram are isomorphisms. Furthermore, the horizontal arrows are injective by the Torelli theorem [Ogu83] and the fact that filtered colimits of sets preserve injections. Since the automorphism $\iota_\gamma^*|_{\mathrm{Spec}\,\mathcal{O}_{S,s}}$ is induced from an automorphism $\iota \in \mathrm{Aut}_{\mathrm{Spec}\,\mathcal{O}_{S,s}}(\mathcal{X}_{\mathrm{Spec}\,\mathcal{O}_{S,s}})$ it follows that there exists an open neighborhood $U(s)$ of s such that $\iota_\gamma^*|_{U(s)}$ is induced from an automorphism $\iota \in \mathrm{Aut}(\mathrm{Pic}_{\mathcal{X}_{Us}/U(s)})$.

Therefore, the sublattice $\gamma_{\tilde{s}}(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_{\tilde{s}})$ contains an ample line bundle for all $\tilde{s} \in U(s)$. Now let A be the set of all $s \in S$ such that $\gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s)$ contains an ample line bundle. Then $S_x = \bigcup_{s\in A} U(s)$, which is an open subscheme of S. □

COROLLARY 3.5. *The functor $\underline{\mathcal{S}}'_\gamma$ is representable by an open subscheme $\mathcal{S}'_\gamma$ of $\mathcal{S}_\sigma$ and the induced morphism $\pi'_\gamma\colon \mathcal{S}'_\gamma \to \mathcal{M}_\sigma$ is étale and surjective.*

PROOF. The representability is a direct consequence of Proposition 3.4. The morphism π'_γ is étale because being étale is local on the source.

Now, if $s \in \mathcal{M}_\sigma(k)$ represents a characteristic generatrix G in $pN_\sigma^\vee/pN_\sigma \otimes_{\mathbb{F}_p} k$, we can choose an ample cone α for G, such that $\gamma_s(\Gamma(2)) \cap \alpha \neq \emptyset$. Using the period isomorphism $\mathcal{S}_\sigma \xrightarrow{\sim} \mathcal{P}_\sigma$, we find a preimage of s in $\mathcal{S}'_\sigma(k)$ from $(G, \alpha) \in \mathcal{P}_\sigma(k)$. Hence π'_γ is surjective. □

Next, we want to be able to forget about the choice of a basis for N_σ in the definition of $\underline{\mathcal{S}}'_\gamma$. To do so, we consider the functor

$$\tilde{\underline{\mathcal{S}}}'_\gamma\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\mathrm{Sets}),$$

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of supersingular} \\ \text{K3 surfaces } f\colon \mathcal{X} \to S \text{ together with a sublattice} \\ \underline{\mathcal{R}} \subseteq \mathrm{Pic}_{\mathcal{X}/S} \text{ and an embedding } \gamma'\colon \underline{\Gamma}(2) \hookrightarrow \underline{\mathcal{R}} \\ \text{such that } (\gamma\colon \underline{\Gamma}(2) \hookrightarrow \underline{N}_\sigma) \cong (\gamma'\colon \underline{\Gamma}(2) \hookrightarrow \underline{\mathcal{R}}) \text{ and} \\ \text{such that for each geometric fiber } s \in S \\ \text{the sublattice } \gamma'_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \\ \text{contains an ample line bundle} \end{array} \right\}.$$

We are once again interested in the representability of the functor $\tilde{\underline{\mathcal{S}}}'_\gamma$. We will see in the proof of the following proposition that $\tilde{\underline{\mathcal{S}}}'_\gamma$ is in fact a quotient of $\underline{\mathcal{S}}'_\gamma$ by a finite group action.

PROPOSITION 3.6. *The functor $\tilde{\underline{\mathcal{S}}}'_\gamma$ is representable by a quasi-separated algebraic space $\tilde{\mathcal{S}}'_\gamma$ that is locally of finite type over $\mathbb{F}_p$ and there exists a canonical finite surjective morphism of algebraic spaces $q\colon \mathcal{S}'_\gamma \to \tilde{\mathcal{S}}'_\gamma$.*

PROOF. Consider the group $O(N_\sigma, \gamma) = \{\varphi \in O(N_\sigma) \mid \varphi \circ \gamma = \gamma \circ \varphi\}$ of isometries of N_σ that preserve the embedding γ. The group $O(N_\sigma, \gamma)$ is a subgroup of $O(\gamma(\Gamma(2))^\perp)$, and the latter group is finite because the lattice $\gamma(\Gamma(2))^\perp$ is negative definite. Hence it follows that $O(N_\sigma, \gamma)$ is a finite group. There is a group action of $O(N_\sigma, \gamma)$ on the functor $\underline{\mathcal{S}}'_\gamma$ that is given on S-valued points for connected schemes S via

$$\varphi \cdot (f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}) = (f\colon \mathcal{X} \to S, \psi \circ \varphi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}).$$

The rest of the proof is separated into two steps.

Step 1: There is a canonical isomorphism of functors $F\colon \underline{\mathcal{S}}'_\gamma / O(N_\sigma, \gamma) \to \tilde{\underline{\mathcal{S}}}'_\gamma$.

There is a canonical morphism of functors $F'\colon \underline{\mathcal{S}}'_\gamma \to \tilde{\underline{\mathcal{S}}}'_\gamma$ which is given on S-valued points via

$$(f\colon \mathcal{X} \to S, \psi\colon \underline{N}_\sigma \to \mathrm{Pic}_{\mathcal{X}/S}) \longmapsto (f\colon \mathcal{X} \to S, \psi(\underline{N}_\sigma) \subseteq \mathrm{Pic}_{\mathcal{X}/S}, \psi \circ \gamma\colon \Gamma(2) \hookrightarrow \psi(\underline{N}_\sigma)).$$

This morphism is invariant under the action of $O(N_\sigma, \gamma)$ on $\underline{\mathcal{S}}'_\gamma$ and it therefore descends to a morphism of functors $F\colon \underline{\mathcal{S}}'_\gamma / O(N_\sigma, \gamma) \to \tilde{\underline{\mathcal{S}}}'_\gamma$. We want to show that F is an isomorphism of functors by checking that, for any $\mathbb{F}_p$-scheme S, the induced map of sets $F(S)$ is a bijection.

a) Surjectivity: It suffices to show that the map $F'(S)\colon \underline{\mathcal{S}}'_\gamma(S) \to \tilde{\underline{\mathcal{S}}}'_\gamma(S)$ is surjective. To this end, we consider an element $s = (f, \underline{\mathcal{R}}, \gamma') \in \tilde{\underline{\mathcal{S}}}'_\gamma(S)$ and we choose an isomorphism of lattice embeddings $\psi\colon (\gamma\colon \underline{\Gamma}(2) \hookrightarrow \underline{N}_\sigma) \xrightarrow{\sim} (\gamma'\colon \underline{\Gamma}(2) \hookrightarrow \underline{\mathcal{R}})$. Then the pair $s' = (f, \psi) \in \underline{\mathcal{S}}'_\gamma(S)$ is a preimage of s under F'.

b) Injectivity: For an element $s = (f, \underline{\mathcal{R}}, \gamma') \in \tilde{\underline{\mathcal{S}}}'_\gamma(S)$ we have to show that any two preimages s' and s'' in $\underline{\mathcal{S}}_\sigma(S)$ only differ by some isometry $\varphi \in O(N_\sigma, \gamma)$. To this end, we write $s' =$

(f, ψ') and $s'' = (f, \psi'')$. We find that $\psi'^{-1}|_{\underline{\mathcal{R}}} \circ \psi'' \in O(N_\sigma, \gamma)$ and we obtain the equality $(\psi'^{-1}|_{\underline{\mathcal{R}}} \circ \psi'') \cdot s' = s''$. This concludes Step 1.

Step 2: The functor $\underline{\mathcal{S}}'_\gamma / O(N_\sigma, \gamma)$ is representable by an algebraic space $\mathcal{S}'_\gamma / O(N_\sigma, \gamma)$ which is quasi-separated and locally of finite type over $\mathbb{F}_p$ and the corresponding quotient morphism $q \colon \mathcal{S}'_\gamma \longrightarrow \mathcal{S}'_\gamma / O(N_\sigma, \gamma)$ is finite.

Analogously to the action of the group $O(N_\sigma, \gamma)$ on $\mathcal{S}'_\gamma$, we obtain an action of $O(N_\sigma, \gamma)$ on the scheme $\mathcal{M}_\sigma$. Using the period map $\underline{\mathcal{S}}_\sigma \xrightarrow{\sim} \underline{\mathcal{P}}_\sigma$, it is clear that the $O(N_\sigma, \gamma)$-action on the open subscheme $\mathcal{S}'_\gamma$ of $\mathcal{S}_\sigma$ is the pullback of the $O(N_\sigma, \gamma)$-action on $\mathcal{M}_\sigma$ under the morphism $\pi'_\gamma \colon \mathcal{S}'_\gamma \to \mathcal{M}_\sigma$.

Next, we claim that the morphism π'_γ is *fixed-point reflecting* in the sense of [Ryd13]. That means for each $x \in \mathcal{S}'_\gamma$ and $\varphi \in O(N_\sigma, \gamma)$, we have that $\varphi \cdot x = x$ if and only if $\varphi \cdot \pi'_\gamma(x) = \pi'_\gamma(x)$.

Indeed, let $x \in \mathcal{S}'_\gamma(k)$ and $\varphi \in O(N_\sigma, \gamma)$ such that x corresponds to a tuple $(G, \alpha) \in \mathcal{P}_\sigma(k)$ where G is a characteristic subspace and α is an ample cone. We need to verify that if $\varphi \cdot G = G$, then we also have $\varphi \cdot (G, \alpha) = (G, \alpha)$. The characteristic subspace G arises as the kernel of the induced morphism $\psi \colon N_\sigma \otimes k \to H^2_{\mathrm{dR}}(X/k)$. The equality $\varphi \cdot G = G$ just means that for the automorphism $\varphi \colon N_\sigma \longrightarrow N_\sigma$ we have the equalities $\ker \psi = G = \ker(\psi \circ \varphi)$. The ample cone on $\ker \psi$ is induced from the preimage $\psi^{-1}(\mathcal{C}_{\mathrm{NS}(X)})$, while the ample cone on $\ker(\psi \circ \varphi)$ is induced from the preimage $(\psi \circ \varphi)^{-1}(\mathcal{C}_{\mathrm{NS}(X)})$. Since the sublattice $\gamma(\Gamma(2)) \hookrightarrow \mathrm{NS}(X)$ contains an ample line bundle and φ commutes with γ, it follows that φ preserves the ample cone in $\mathrm{NS}(X)$, and hence we find $\psi^{-1}(\mathcal{C}_{\mathrm{NS}(X)}) = (\psi \circ \varphi)^{-1}(\mathcal{C}_{\mathrm{NS}(X)})$. Thus we also have $\varphi \circ (G, \alpha) = (G, \alpha)$, and it follows that π'_γ is fixed-point reflecting with respect to the action of $O(N_\sigma, \gamma)$.

The quotient $\mathcal{M}_\sigma / O(N_\sigma, \gamma)$ exists as an algebraic space and is a strongly geometric quotient in the sense of [Ryd13, Definition 2.2] by [Ryd13, Corollary 5.4]. Furthermore, the quotient morphism $\mathcal{M}_\sigma \to \mathcal{M}_\sigma / O(N_\sigma, \gamma)$ is finite and $\mathcal{M}_\sigma / O(N_\sigma, \gamma) \to \operatorname{Spec} \mathbb{F}_p$ is proper and of finite type by [Ryd13, Proposition 4.7]. By [Ryd13, Theorem 3.15] the quotient $\mathcal{M}_\sigma \to \mathcal{M}_\sigma / O(N_\sigma, \gamma)$ satisfies the *descent condition* in the sense of [Ryd13, Definition 3.6] and it follows that the quotient $q \colon \mathcal{S}'_\gamma \to \mathcal{S}'_\gamma / O(N_\sigma, \gamma)$ exists as an algebraic space and is a topological quotient, the morphism q is finite and the morphism $\mathcal{S}'_\gamma / O(N_\sigma, \gamma) \to \mathcal{M}_\sigma / O(N_\sigma, \gamma)$ is étale. □

REMARK 3.7. We do not expect $\tilde{\mathcal{S}}'_\gamma$ to be a scheme in general. A sufficient and necessary condition for $\tilde{\mathcal{S}}'_\gamma$ to be a scheme is that every orbit of the $O(N_\sigma, \gamma)$-action on $\mathcal{S}'_\gamma$ is contained in an affine open subscheme of $\mathcal{S}'_\gamma$ [Ryd13, Theorem 4.4]. Since $\mathcal{S}'_\gamma$ is non-separated, we generally expect this condition to fail.

However, it turns out that the corresponding quotient of $\mathcal{M}_\sigma$, which lies under $\tilde{\mathcal{S}}'_\gamma$, is still a scheme.

PROPOSITION 3.8. *There exist a projective $\mathbb{F}_p$-scheme $\widetilde{\mathcal{M}}'_\gamma$ and a canonical étale surjective morphism of algebraic spaces $\tilde{\pi}'_\gamma \colon \tilde{\mathcal{S}}'_\gamma \to \widetilde{\mathcal{M}}'_\gamma$.*

PROOF. We can take the quotient $\widetilde{\mathcal{M}}'_\gamma = \mathcal{M}_\sigma / O(N_\sigma, \gamma)$. This quotient is indeed a scheme because $\mathcal{M}_\sigma$ is projective and in particular it has the property from Remark 3.7. Furthermore, the scheme $\mathcal{M}_\sigma / O(N_\sigma, \gamma)$ is projective by [Ryd13, Proposition 4.7.]. The other assertions have already been shown in the proof of Proposition 3.6. □

We will use the scheme $\widetilde{\mathcal{M}}'_\gamma$ later to construct the period scheme of supersingular Enriques surfaces.

We will now consider the subfunctor $\underline{\tilde{\mathcal{S}}}''_\gamma$ of $\underline{\tilde{\mathcal{S}}}'_\gamma$ that only allows $\Gamma(2)$-markings without vectors of self-intersection -2 in the complement, which is defined to be

$$\underline{\tilde{\mathcal{S}}}''_\gamma \colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\mathrm{Sets}),$$
$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of supersingular} \\ \text{K3 surfaces } f\colon \mathcal{X} \to S \text{ together with a sublattice} \\ \underline{\mathcal{R}} \subseteq \mathrm{Pic}_{\mathcal{X}/S} \text{ and an embedding } \gamma'\colon \underline{\Gamma}(2) \hookrightarrow \underline{\mathcal{R}} \\ \text{such that } (\gamma\colon \underline{\Gamma}(2) \hookrightarrow \underline{N}_\sigma) \cong (\gamma'\colon \underline{\Gamma}(2) \hookrightarrow \underline{\mathcal{R}}) \text{ and} \\ \text{such that for each geometric fiber } s \in S \\ \text{the sublattice } \gamma'_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \\ \text{contains an ample line bundle} \\ \text{and } \gamma'_s(\Gamma(2))^\perp \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \text{ contains no } (-2)\text{-vector} \end{array} \right\}.$$

The points of $\tilde{\mathcal{S}}''_\gamma$ should be seen as quotients of supersingular K3 surfaces by a fixed-point free involution. For an explanation we refer to the proof of Theorem 4.1. in [Jan13]. We are again interested in the representability of the functor $\underline{\tilde{\mathcal{S}}}''_\gamma$.

PROPOSITION 3.9. *The functor $\underline{\tilde{\mathcal{S}}}''_\gamma$ is representable by an open algebraic subspace $\tilde{\mathcal{S}}''_\gamma$ of $\tilde{\mathcal{S}}'_\gamma$.*

PROOF. We consider the set R' of representatives of all isomorphism classes of embeddings $j\colon N_\sigma \hookrightarrow N_{\sigma'}$ such that $j\left(\gamma(\Gamma(2))\right)^\perp \subseteq N_{\sigma'}$ contains a (-2)-vector. Then the set R' is a subset of the finite set $\bigcup_{\sigma'<\sigma} R_{\sigma',\sigma}$. For each j, the algebraic subspace $q(\Phi_j(\mathcal{S}_{\sigma'}) \cap \mathcal{S}'_\gamma) \subseteq \tilde{\mathcal{S}}'_\gamma$ is closed, and it is clear that the open algebraic subspace

$$\tilde{\mathcal{S}}''_\gamma = \tilde{\mathcal{S}}'_\gamma \setminus \left(\bigcup_{j\in R'} q\left((\Phi_j(\mathcal{S}_{\sigma'})) \cap \mathcal{S}'_\gamma\right) \right)$$

represents the functor $\underline{\tilde{\mathcal{S}}}''_\gamma$. □

We also find an open subscheme of $\widetilde{\mathcal{M}}'_\gamma$ that lies under $\tilde{\mathcal{S}}''_\gamma$.

PROPOSITION 3.10. *There exist a quasi-projective $\mathbb{F}_p$-scheme $\widetilde{\mathcal{M}}''_\gamma$ and a canonical étale surjective morphism of algebraic spaces $\tilde{\pi}''_\gamma\colon \tilde{\mathcal{S}}''_\gamma \to \widetilde{\mathcal{M}}''_\gamma$.*

PROOF. The morphism $\tilde{\pi}'_\gamma\colon \tilde{\mathcal{S}}'_\gamma \to \widetilde{\mathcal{M}}'_\gamma$ is universally open. Hence we may take $\widetilde{\mathcal{M}}''_\gamma$ to be the image of $\tilde{\mathcal{S}}''_\gamma$ under $\tilde{\pi}'_\gamma$ and $\tilde{\pi}''_\gamma$ to be the restriction of $\tilde{\pi}'_\gamma$ to $\tilde{\mathcal{S}}''_\gamma$. □

REMARK 3.11. It is not clear to us whether the functors $\underline{\tilde{\mathcal{S}}}'_\gamma$ and $\underline{\tilde{\mathcal{S}}}''_\gamma$ are equal in general. However, we think this should not be true. The lattice-theoretic question we have to answer is

Do there exist embeddings $j\colon N_\sigma \hookrightarrow N_{\sigma'}$ and $\gamma\colon \Gamma(2) \hookrightarrow N_\sigma$ such that $\gamma(\Gamma(2))^\perp$ contains no (-2)-vector, but $j(\gamma(\Gamma(2)))^\perp$ contains a (-2)-vector?

Assuming the answer to this question is *yes*, we could see Proposition 3.9 as a supersingular analogue to the fact that the period map of Enriques surfaces in characteristic zero maps to a quotient of the moduli space of K3 surfaces minus a divisor [Nam85, Theorem 1.14]. We removed a divisor or the empty set in each sub moduli space $\mathcal{S}_{\sigma'} \subseteq \mathcal{S}_\sigma$.

3. Moduli spaces of $\Gamma(2)$-marked supersingular K3 surfaces

Next, we want to get rid of having to make a choice of a sublattice $\underline{\mathcal{R}}$ in $\mathrm{Pic}_{\mathcal{X}/S}$. The idea is that, on an open dense subset of the moduli space $\tilde{\mathcal{S}}''_\gamma$, we do not have a choice anyway, and the closed complement of this open subspace can be contracted to the corresponding moduli space for Artin invariant $\sigma - 1$ by forgetting about the sublattice $\underline{\mathcal{R}}$.

We now introduce the functor

$$\underline{\tilde{\mathcal{E}}}_\sigma \colon \mathcal{A}^{\mathrm{op}}_{\mathbb{F}_p} \longrightarrow (\mathrm{Sets})\,,$$
$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of supersingular} \\ \text{K3 surfaces } f\colon \mathcal{X} \to S \text{ that admit an } N_\sigma\text{-marking} \\ \text{together with an embedding } \gamma\colon \underline{\Gamma}(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S} \\ \text{such that for each geometric fiber } s \in S \\ \text{the sublattice } \gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \\ \text{contains an ample line bundle} \\ \text{and } \gamma_s(\Gamma(2))^\perp \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \text{ contains no } (-2)\text{-vector} \end{array} \right\}.$$

We are again interested in an object $\tilde{\mathcal{E}}_\sigma$ that represents the functor $\underline{\tilde{\mathcal{E}}}_\sigma$. The discussion will use an inductive argument, so we start by discussing the case $\sigma = 1$.

PROPOSITION 3.12. *The functor $\underline{\tilde{\mathcal{E}}}_1$ is representable by a zero-dimensional quasi-separated algebraic space $\tilde{\mathcal{E}}_1$ locally of finite type over $\mathbb{F}_p$ that has finitely many connected components and each of these components is irreducible.*

PROOF. For each $\gamma \in R_1$ there is a canonical morphism of functors $\underline{\tilde{\mathcal{S}}}''_\gamma \to \underline{\tilde{\mathcal{E}}}_1$ that is given on S-valued points by forgetting about the choice of a sublattice $\underline{\mathcal{R}} \subseteq \mathrm{Pic}_{\mathcal{X}/S}$. Since any such sublattice $\underline{\mathcal{R}} \subseteq \mathrm{Pic}_{\mathcal{X}/S}$ is already equal to $\mathrm{Pic}_{\mathcal{X}/S}$, we see that this morphism is injective on S-valued points, and it follows that $\coprod_{\gamma \in R_1} \underline{\tilde{\mathcal{S}}}''_\gamma \longrightarrow \underline{\tilde{\mathcal{E}}}_1$ is an isomorphism of functors. Hence, the functor $\underline{\tilde{\mathcal{E}}}_1$ is represented by the algebraic space $\coprod_{\gamma \in R_1} \tilde{\mathcal{S}}''_\gamma$. □

REMARK 3.13. More precisely, since $\mathcal{S}_1$ is isomorphic to a disjoint union of finitely many copies of $\mathrm{Spec}\,\mathbb{F}_{p^2}$ and $\tilde{\mathcal{S}}''_\gamma$ is just an open subscheme of a quotient of an open subscheme of $\mathcal{S}_\sigma$, we easily see that $\tilde{\mathcal{E}}_1$ is just a disjoint union of finitely many copies of $\mathrm{Spec}\,\mathbb{F}_{p^2}$ as well.

We will need the following lemma, which might be well known, but we did not find it in the literature in full generality. That is to say, we do not require any assumptions on our objects being schemes, being noetherian or being separated.

LEMMA 3.14. *Let X, Y and Z be algebraic spaces that are locally of finite type over a base scheme S together with S-morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ such that $g \circ f$ is proper (respectively finite) and f is proper (respectively finite) and surjective. Then g is proper (respectively finite).*

PROOF. We prove that g is finite when f and $g \circ f$ are finite. We leave the proper case to the reader. Since Y and Z are locally of finite type, the morphism g is locally of finite type [Sta19, Lemma 61.23.6]. It is clear that g has finite discrete fibers, because the fibers of $g \circ f$ surject onto the fibers of g. Furthermore, the morphism g is quasi-compact [Sta19, Lemma 61.8.6]. It follows that g is quasi-finite. Furthermore, if $T \to Z$ is any morphism and $Q \subseteq Y_T$ is a closed subscheme, then the subscheme $g_T(Q) = g_T \circ f_T(f_T^{-1}(Q))$ is closed. This shows that g is universally closed. Furthermore, the fact that g is affine follows from a version of Chevalley's theorem [Ryd15, Theorem 8.1]. All these properties together imply that g is finite. □

Since every family of supersingular K3 surfaces that admits an $N_{\sigma-1}$-marking also admits an N_σ-marking, the functor $\underline{\tilde{\mathcal{E}}}_{\sigma-1}$ is a subfunctor of $\underline{\tilde{\mathcal{E}}}_\sigma$. For each positive integer $\sigma \leq 10$ there is a canonical morphism of functors

$$p_\sigma : \coprod_{\gamma \in R_\sigma} \underline{\tilde{\mathcal{S}}}''_\gamma \to \underline{\tilde{\mathcal{E}}}_\sigma$$

that is given on S-valued points by forgetting about the sublattice $\underline{\mathcal{R}} \subseteq \mathrm{Pic}_{\mathcal{X}/S}$. Then the preimage of the subfunctor $\underline{\tilde{\mathcal{E}}}_{\sigma-1} \hookrightarrow \underline{\tilde{\mathcal{E}}}_\sigma$ under p_σ is given by the closed algebraic subspace

$$p_\sigma^{-1}(\underline{\tilde{\mathcal{E}}}_{\sigma-1}) = \coprod_{\gamma \in R_\sigma} \left(\left(\bigcup_{j \in R_{\sigma-1,\sigma}} q\left(\Phi_j(\mathcal{S}_{\sigma-1}) \cap \mathcal{S}'_\gamma\right) \right) \setminus \left(\bigcup_{j \in R'} q\left(\Phi_j(\mathcal{S}_{\sigma'}) \cap \mathcal{S}'_\gamma\right) \right) \right)$$

of $\coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma$.

DEFINITION 3.15. For $\gamma \in R_\sigma$ and $j \in R_{\sigma-1,\sigma}$, we write W_j^γ for the locally closed subspace of $\mathcal{S}_\sigma$ defined to be

$$W_j^\gamma = \left(\Phi_j(\mathcal{S}_{\sigma-1}) \cap \mathcal{S}'_\gamma\right) \setminus \left(\bigcup_{j' \in R'} \Phi_j(\mathcal{S}_{\sigma'}) \cap \mathcal{S}'_\gamma \right).$$

REMARK 3.16. The image of W_j^γ under $q\colon \mathcal{S}'_\gamma \longrightarrow \tilde{\mathcal{S}}'_\gamma$ is contained in $\tilde{\mathcal{S}}''_\gamma$. In fact, we have the equality $\bigcup_{\gamma \in R_\sigma, j \in R_{\sigma-1,\sigma}} q(W_j^\gamma) = p_\sigma^{-1}(\underline{\tilde{\mathcal{E}}}_{\sigma-1})$. Moreover, since W_j^γ is a closed subspace of $\mathcal{S}'_\gamma$, it follows from Proposition 3.6 that the morphism $q|_{W_j^\gamma}\colon W_j^\gamma \longrightarrow \tilde{\mathcal{S}}''_\gamma$ is finite.

Furthermore, since $\Phi_j(\mathcal{S}_{\sigma-1}) \cap \mathcal{S}'_\gamma$ is canonically isomorphic to the open subscheme $\mathcal{S}'_{j\circ\gamma}$ of $\mathcal{S}_{\sigma-1}$, we also have a natural finite morphism $q\colon W_j^\gamma \longrightarrow \tilde{\mathcal{S}}''_{j\circ\gamma}$.

LEMMA 3.17. *Assume that $\underline{\tilde{\mathcal{E}}}_{\sigma-1}$ is representable by an algebraic space $\tilde{\mathcal{E}}_{\sigma-1}$ that is locally of finite type over $\mathbb{F}_p$ and that the canonical morphism $\coprod_{\gamma \in R_\sigma, j \in R_{\sigma-1,\sigma}} W_j^\gamma \to \tilde{\mathcal{E}}_{\sigma-1}$ is finite. Then the restriction of p_σ to $p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1})$ is a finite morphism .*

PROOF. This is a direct consequence of Lemma 3.14, and the previous remark. □

THEOREM 3.18. *Let $\sigma \leq 10$ be a positive integer.*

(1) The functor $\underline{\tilde{\mathcal{E}}}_\sigma$ is representable by an algebraic space $\tilde{\mathcal{E}}_\sigma$ that is locally of finite type over $\mathbb{F}_p$ and quasi-separated.

(2) For each isomorphism class of primitive embeddings $\gamma\colon \Gamma(2) \hookrightarrow N_{\sigma+1}$ such that there is no (-2)-vector in $\gamma(\Gamma(2))^\perp \subset N_{\sigma+1}$ and each embedding of lattices $j\colon N_{\sigma+1} \hookrightarrow N_\sigma$ such that there is no (-2)-vector in $j(\gamma(\Gamma(2)))^\perp \subset N_\sigma$, there is a canonical finite morphism $W_j^\gamma \to \tilde{\mathcal{E}}_\sigma$.

PROOF. We do induction over σ. For $\sigma = 1$, the theorem follows from Proposition 3.12 and its proof.

We will now assume that the theorem holds for $\sigma - 1$. We consider the pushout diagram

$$\begin{tikzcd} p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \arrow[r, "\iota"'] \arrow[d, "p_\sigma"'] \arrow[dr, phantom, "\ulcorner"] & \coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma \arrow[d] \\ \tilde{\mathcal{E}}_{\sigma-1} \arrow[r] & \mathcal{P}. \end{tikzcd}$$

By Lemma 3.17 the morphism $p_\sigma\colon p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \to \tilde{\mathcal{E}}_{\sigma-1}$ is finite, hence the Ferrand pushout datum $\tilde{\mathcal{E}}_{\sigma-1} \xleftarrow{p_\sigma} p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \xrightarrow{\iota} \coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma$ is effective by [TT16, Theorem 6.2] and the pushout $\mathcal{P}$ exists as an algebraic space over $\mathbb{F}_p$. Furthermore, the morphism of algebraic spaces $\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma \longrightarrow \mathcal{P}$ is finite by [TT16, Theorem 6.6] and $\mathcal{P} \to \operatorname{Spec}\mathbb{F}_p$ is quasi-separated by [TT16, Theorem 6.8].

We obtain from [TT16, Theorem 4.8] that the topological space underlying $\mathcal{P}$ is just the pushout in the category of topological spaces, there exists a natural isomorphism of algebraic spaces $p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \cong \tilde{\mathcal{E}}_{\sigma-1} \times_{\mathcal{P}} \coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma$, the morphism $\tilde{\mathcal{E}}_{\sigma-1} \to \mathcal{P}$ is a closed immersion of algebraic spaces, the morphism $\left(\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma\right) \setminus \left(p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1})\right) = U \to \mathcal{P}$ is an open immersion of algebraic spaces and we have an equality of sets $|\mathcal{P}| = \left|\tilde{\mathcal{E}}_{\sigma-1}\right| \amalg |U|$.

The finite morphism $\tilde{\mathcal{E}}_{\sigma-1} \amalg \coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma \to \mathcal{P}$ is surjective as a map of topological spaces, and it follows from [AM69, Proposition 7.8] that $\mathcal{P}$ is locally of finite type over $\mathbb{F}_p$.

We now show that the algebraic space $\mathcal{P}$ represents the functor $\underline{\tilde{\mathcal{E}}}_\sigma$ and that the morphism of algebraic spaces $\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma \to \mathcal{P}$ represents the canonical morphism $\coprod_{\gamma\in R_\sigma} \underline{\tilde{\mathcal{S}}}''_\gamma \to \underline{\tilde{\mathcal{E}}}_\sigma$.

Step 1: We define a morphism of presheaves $F\colon \underline{\tilde{\mathcal{E}}}_\sigma \to \underline{\mathcal{P}}$.

If S is an irreducible and reduced $\mathbb{F}_p$-scheme, we define the map $F(S)\colon \underline{\tilde{\mathcal{E}}}_\sigma(S) \to \mathcal{P}(S)$ in the following way. If $x = (f\colon \mathcal{X} \to S, \gamma\colon \Gamma(2) \hookrightarrow \operatorname{Pic}_{\mathcal{X}/S}) \in \underline{\tilde{\mathcal{E}}}_\sigma(S)$ is such that for every $s \in S$ the fiber $\mathcal{X}_s$ has Artin invariant $\sigma(\mathrm{NS}(\mathcal{X}_s)) \leq \sigma - 1$, then x is an element of the subset $\tilde{\mathcal{E}}_{\sigma-1}(S) \subset \underline{\tilde{\mathcal{E}}}_\sigma(S)$. In this case, we set $F(S)(x)$ to be the image of x under the canonical map $\tilde{\mathcal{E}}_{\sigma-1}(S) \to \mathcal{P}(S)$. Note that by the commutativity of the pushout diagram, if x lies in the image of p_σ, we equivalently could have chosen a preimage x' of x in $\tilde{\mathcal{S}}''_\gamma(S)$ for some γ' and set $F(S)(x)$ to be the image of x' under the canonical map $\tilde{\mathcal{S}}''_\gamma(S) \to \mathcal{P}(S)$.

If, on the other hand, x is such that there exists an $s \in S$ with $\sigma(\mathrm{NS}(\mathcal{X}_s)) = \sigma$, then the subset $U \subseteq S$ where $\mathcal{X}_s$ has Artin invariant σ is open. We choose an arbitrary lift $x' = (f, \mathcal{R}', \gamma')$ of x to $\tilde{\mathcal{S}}''_\gamma(S)$. We claim that this lift is unique. Indeed, let $x'' = (f, \mathcal{R}'', \gamma'')$ be another such lift. We take preimages $\tilde{x}' = (f, \psi')$ and $\tilde{x}'' = (f, \psi'')$ in $\mathcal{S}'_\gamma(S)$ and after applying an automorphism of N_σ that preserves the embedding $\gamma\colon \Gamma(2) \hookrightarrow N_\sigma$, we may assume that $\psi'_U = \psi''_U$. But by [Riz06, Theorem 3.1.1.] the morphism of algebraic spaces $\mathrm{Pic}_{\mathcal{X}/S} \to S$ is separated and it therefore follows that $\psi' = \psi''$. Thus we have an isomorphism $x' \cong x''$.

We set $F(S)(x)$ to be the image of $x' \in \mathcal{S}'_\gamma(S)$ under the canonical map $\mathcal{S}'_\gamma(S) \to \mathcal{P}(S)$. It is clear from the construction that the class of maps $F(S)$ yields a morphism of functors.

Step 2: We define a morphism of presheaves $G\colon \underline{\mathcal{P}} \to \underline{\tilde{\mathcal{E}}}_\sigma$ which is an inverse to F.

Using the induction hypothesis, we write

$$\mathcal{X}_{\tilde{\mathcal{E}}_{\sigma-1}} \longrightarrow \tilde{\mathcal{E}}_{\sigma-1}$$

and

$$\mathcal{X}_{\coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma} \longrightarrow \coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma$$

for the universal elements of the functors $\underline{\tilde{\mathcal{E}}}_{\sigma-1}$ and $\coprod_{\gamma\in R_\sigma}\underline{\tilde{\mathcal{S}}''_\gamma}$. Since the scheme corresponding to an S-valued point of $\coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma$ maps to the same scheme corresponding to an S-valued point of $\tilde{\mathcal{E}}_{\sigma-1}$ under p_σ, and we are only forgetting about additional structure, there exists a unique isomorphism

$$p_\sigma^*\mathcal{X}_{\tilde{\mathcal{E}}_{\sigma-1}} \xrightarrow{\simeq} \iota^*\mathcal{X}_{\coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma}.$$

We choose a representative for this pullback of algebraic spaces and denote it by $\mathcal{X}_{p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1})}$.

We find that the Ferrand pushout datum $\mathcal{X}_{\tilde{\mathcal{E}}_{\sigma-1}} \leftarrow \mathcal{X}_{p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1})} \to \mathcal{X}_{\coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma}$ is effective using the same argument as above, and we choose a pushout $\mathcal{X}_\mathcal{P}$ for this datum. The canonical morphism

$$\left(\mathcal{X}_{\tilde{\mathcal{E}}_{\sigma-1}} \leftarrow \mathcal{X}_{p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1})} \to \mathcal{X}_{\coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma}\right) \longrightarrow \left(\tilde{\mathcal{E}}_{\sigma-1} \xleftarrow{p_\sigma} p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \xrightarrow{\iota} \coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma\right)$$

is a flat morphism of pushout data in the sense of [TT16, Chapter 2.2]. Hence, the induced morphism $\mathcal{X}_\mathcal{P} \to \mathcal{P}$ is smooth by [TT16, Theorem 6.3.2,(ii)] and proper by Lemma 3.14. Moreover, the morphism

$$\mathcal{X}_{\coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma} \longrightarrow \coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma$$

is just the pullback of $\mathcal{X}_\mathcal{P}$ along the morphism

$$\coprod_{\gamma\in R_\sigma}\tilde{\mathcal{S}}''_\gamma \longrightarrow \mathcal{P}$$

and the morphism

$$\mathcal{X}_{\tilde{\mathcal{E}}_{\sigma-1}} \longrightarrow \tilde{\mathcal{E}}_{\sigma-1}$$

is just the pullback of $\mathcal{X}_{\mathcal{P}}$ along

$$\tilde{\mathcal{E}}_{\sigma-1} \longrightarrow \mathcal{P}$$

by [TT16, Theorem 6.3.2,(i)]. Since $\mathcal{P}$ is set-theoretically covered by $\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma$ and $\tilde{\mathcal{E}}_{\sigma-1}$, and the geometric fibers of these algebraic spaces are projective supersingular K3 surfaces, it follows that the geometric fibers of $\mathcal{X}_{\mathcal{P}} \to \mathcal{P}$ are projective supersingular K3 surfaces as well. Hence $\mathcal{X}_{\mathcal{P}} \to \mathcal{P}$ is a family of supersingular K3 surfaces. The construction of the relative Picard functor is compatible with base change. Therefore we obtain a morphism of algebraic group spaces compatible with intersection forms

$$\mathrm{Pic}_{\mathcal{X}_{\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma}/\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma} \longrightarrow \mathrm{Pic}_{\mathcal{X}_{\mathcal{P}}/\mathcal{P}}$$

that induces a $\Gamma(2)$-marking $\gamma\colon \Gamma(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}_{\mathcal{P}}/\mathcal{P}}$. If S is an $\mathbb{F}_p$-scheme and $y\colon S \to \mathcal{P}$ is a morphism of $\mathbb{F}_p$-schemes, we define $G(S)(y) \in \underline{\tilde{\mathcal{E}}}_\sigma(S)$ to be the pullback of $\mathcal{X}_{\mathcal{P}}$ under y.

A straightforward computation shows that the morphisms F and G are mutually inverse to each other.

Since we have shown that the canonical morphism $\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma \longrightarrow \mathcal{E}_\sigma$ is finite, it follows from Remark 3.16 that for each $\gamma \in R_{\sigma+1}$ and $j \in R_{\sigma,\sigma+1}$ the canonical morphism $W_j^\gamma \longrightarrow \tilde{\mathcal{E}}_\sigma$ is finite. □

Again, there exists a nice scheme for which $\tilde{\mathcal{E}}_\sigma$ is an étale cover. However, this scheme may not be quasi-projective anymore, and we introduce the following slightly weaker finiteness property.

DEFINITION 3.19. [Ryd13, Definition B.1] A scheme X is called an *AF scheme* if for every finite subset $\{x_i\}$ of X there exists an affine open subscheme U in X such that $\{x_i\}$ is contained in U.

REMARK 3.20. Any quasi-projective scheme over a field k is AF. Furthermore, if X is an AF scheme and G is a finite group acting on X, then the quotient X/G always exists as a scheme, see Remark 3.7.

REMARK 3.21. To our knowledge, the term *AF scheme* was first used in [Ryd13]. However, schemes with this property have been studied before [Art+63, Exp. V], [Art71, §4],[Fer03]. For more facts on AF schemes see [Ryd13, Appendix B].

PROPOSITION 3.22. *There exists a separated $\mathbb{F}_p$-scheme $\mathcal{Q}_\sigma$ that is of finite type and AF, and a canonical étale surjective morphism $\tilde{\mathcal{E}}_\sigma \to \mathcal{Q}_\sigma$.*

PROOF. For $\sigma = 1$, we can take the quasi-projective scheme $\mathcal{Q}_\sigma = \coprod_{\gamma\in R_1} \widetilde{\mathcal{M}}''_\gamma$. This proves the assertion in that case.

We now do induction over σ and assume that the assertion is true for $\sigma - 1$. The pushout diagram of $\mathbb{F}_p$-algebraic spaces

$$\begin{tikzcd} p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \arrow[r, "\iota"'] \arrow[d, "p_\sigma"'] \arrow[dr, phantom, "\ulcorner"] & \coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma \arrow[d] \\ \tilde{\mathcal{E}}_{\sigma-1} \arrow[r] & \tilde{\mathcal{E}}_\sigma \end{tikzcd}$$

induces a pushout diagram of separated $\mathbb{F}_p$-schemes of finite type and AF

$$\begin{tikzcd} p_\sigma^{-1}(\mathcal{Q}_{\sigma-1}) \arrow[r, "\iota"'] \arrow[d, "p_\sigma"'] \arrow[dr, phantom, "\ulcorner"] & \coprod_{\gamma \in R_\sigma} \widetilde{\mathcal{M}}''_\gamma \arrow[d] \\ \mathcal{Q}_{\sigma-1} \arrow[r] & \mathcal{Q}_\sigma \end{tikzcd}$$

together with an étale and surjective morphism of pushout data

$$\left(\tilde{\mathcal{E}}_{\sigma-1} \leftarrow p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \rightarrow \coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma \right) \longrightarrow \left(\mathcal{Q}_{\sigma-1} \leftarrow p_\sigma^{-1}(\mathcal{Q}_{\sigma-1}) \rightarrow \coprod_{\gamma \in R_\sigma} \widetilde{\mathcal{M}}''_\gamma \right).$$

It follows from the previous discussion of the schemes $\mathcal{M}_\sigma$ that ι is a closed immersion and that we may inductively assume that p_σ is finite. By [Fer03, Théorème 5.4.], the pushout $\mathcal{Q}_\sigma$ exists as an AF scheme and the induced morphism $\mathcal{Q}_{\sigma-1} \amalg \coprod_{\gamma \in R_\sigma} \widetilde{\mathcal{M}}''_\gamma \to \mathcal{Q}_\sigma$ is finite surjective. Since $\coprod_{\gamma \in R_\sigma} \widetilde{\mathcal{M}}''_\gamma$ is of finite type over $\mathbb{F}_p$ and we may inductively assume that $\mathcal{Q}_{\sigma-1}$ is of finite type over $\mathbb{F}_p$ as well, it follows that $\mathcal{Q}_\sigma$ is of finite type over $\mathbb{F}_p$. That $\mathcal{Q}_\sigma$ is separated follows from [TT16, Theorem 6.8.] and by [TT16, Theorem 6.4.] the induced morphism $\tilde{\mathcal{E}}_\sigma \to \mathcal{Q}_\sigma$ is étale and surjective. $\square$

REMARK 3.23. We will prove in Section 6 that the scheme $\mathcal{Q}_\sigma$ constructed in the proof of Proposition 3.22 is a coarse moduli scheme for supersingular Enriques surfaces.

4. From $\Gamma(2)$-marked K3 surfaces to Γ'-marked Enriques surfaces

Although we want to construct a moduli space for Enriques surfaces, we have only discussed K3 surfaces so far. In this section, we establish the connection between $\Gamma(2)$-marked supersingular K3 surfaces and Γ'-marked Enriques surfaces that are quotients of supersingular K3 surfaces.

DEFINITION 3.24. If X is a supersingular K3 surface and $\iota\colon X \to X$ is a fixed-point free involution, we write $G = \langle \iota \rangle$ for the cyclic group of order 2 generated by ι. A quotient of surfaces $X \to X/G = Y$ defined by such a pair (X, ι) is called a *supersingular Enriques surface* Y. The *Artin invariant* of a supersingular Enriques surface Y is the Artin invariant of the supersingular K3 surface X that universally covers Y. A *family of supersingular Enriques surfaces* is a smooth and proper morphism of algebraic spaces $f\colon \mathcal{Y} \to S$ over $\mathbb{F}_p$ such that for each field k and each $s\colon \operatorname{Spec} k \to S$ the fiber $f_s\colon \mathcal{Y}_s \to \operatorname{Spec} k$ is a supersingular Enriques surface.

Recall from Section 2 that we defined Γ to be the lattice $\Gamma = U_2 \oplus E_8(-1)$. If Y is a supersingular Enriques surface, then there exists an isomorphism of lattices $\mathrm{Pic}(Y) \cong \Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ and we denote the latter lattice by Γ'. In arbitrary characteristic, by [Lie15a, Proposition 4.4], if $\mathcal{Y} \to S$ is a family of supersingular Enriques surfaces, then the torsion part $\mathrm{Pic}^{\tau}_{\mathcal{Y}/S}$ of the Picard scheme is a finite, flat group scheme of length 2 over S. In particular, when $p \geq 3$ we have an equality of sheaves of groups $\mathrm{Pic}^{\tau}_{\mathcal{Y}/S} = \underline{\mathbb{Z}/2\mathbb{Z}}$ with generator $\omega_{\mathcal{Y}/S}$. Furthermore, in arbitrary characteristic, the quotient $\mathrm{Pic}_{\mathcal{Y}/S}/\mathrm{Pic}^{\tau}_{\mathcal{Y}/S}$ is a locally constant sheaf of torsion-free finitely generated abelian groups. In characteristic $p \geq 3$ this implies that there exists an étale covering $\{U_i \to S\}_{i \in I}$ such that we have an isomorphism $\mathrm{Pic}_{\mathcal{Y}_{U_i}/U_i} \cong \underline{\Gamma \oplus \mathbb{Z}/2\mathbb{Z}}$ for each $i \in I$.

DEFINITION 3.25. A Γ-*marking* of a family $f\colon \mathcal{Y} \to S$ of supersingular Enriques surfaces is the choice of a morphism $\tilde{\gamma}\colon \underline{\Gamma} \to \mathrm{Pic}_{\mathcal{Y}/S}$ of group objects in the category of algebraic spaces compatible with the intersection forms. Analogously we define the notion of a Γ'-*marking*. There are obvious notions of morphisms of families of marked supersingular Enriques surfaces.

As before, we will in the following always assume that $p \neq 2$. We first show that, for any family of Γ'-marked supersingular Enriques surfaces, there exists a canonical family of supersingular K3 surfaces that covers it.

PROPOSITION 3.26 (and Definition). *Given a family of* Γ'*-marked supersingular Enriques surfaces* $(\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}\colon \underline{\Gamma}' \to \mathrm{Pic}_{\mathcal{Y}/S})$ *there exists a family of supersingular K3 surfaces* $f\colon \mathcal{X} \to S$ *together with a morphism* $\mathcal{X} \to \mathcal{Y}$ *that makes* $\mathcal{X}$ *into a* $\underline{\mathbb{Z}/2\mathbb{Z}}$*-torsor over* $\mathcal{Y}$*. Furthermore, this family carries a canonical* $\Gamma(2)$*-marking* $\gamma\colon \underline{\Gamma}(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S}$ *induced from the* Γ'*-marking on* $\mathcal{Y}$ *and the tuple* $(f\colon \mathcal{X} \to S, \gamma\colon \underline{\Gamma}(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S})$ *is unique up to isomorphism. We call* $\mathcal{X} \to \mathcal{Y}$ *the* canonical K3 cover of $\mathcal{Y}$.

PROOF. Note that we always assume characteristic $p \neq 2$ thus we get the equality $\underline{\mathbb{Z}/2\mathbb{Z}}^D = \underline{\mathbb{Z}/2\mathbb{Z}}$ for the Cartier dual. Let $(\tilde{f}\colon \mathcal{Y} \to S, \tilde{\psi}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S})$ be a family of Γ'-marked supersingular Enriques surfaces. There is a unique isomorphism $\underline{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\simeq} \mathrm{Pic}^{\tau}_{\mathcal{Y}/S}$ that corresponds to the unique $\underline{\mathbb{Z}/2\mathbb{Z}}$-torsor $\mathcal{X} \to \mathcal{Y}$ of algebraic spaces over S, cf. [Ray70, Proposition 6.2.1.]. The morphism $\mathcal{X} \to \mathcal{Y}$ is finite and étale, thus it follows that $\mathcal{X} \to S$ is proper and smooth. Furthermore, every fiber $\mathcal{X}_s \to \mathcal{Y}_s$ is just the universal K3 cover of the Enriques surface $\mathcal{Y}_s$, and it follows that $\mathcal{X} \to S$ is a family of supersingular K3 surfaces.

Pullback of line bundles induces a morphism $\mathrm{Pic}_{\mathcal{Y}/S} \to \mathrm{Pic}_{\mathcal{X}/S}$ of group objects in the category of algebraic spaces over S, and because the morphism $\mathcal{X} \to \mathcal{Y}$ is unramified and 2-to-1, the intersection form under this morphism gets multiplied by 2. In other words, after twisting the intersection form of $\mathrm{Pic}_{\mathcal{Y}/S}$ by the factor 2, we obtain a morphism $\mathrm{Pic}_{\mathcal{Y}/S}(2) \to \mathrm{Pic}_{\mathcal{X}/S}$ of group objects in the category of algebraic spaces over S compatible with intersection forms. Now precomposing with the marking $\tilde{\psi}_{|\Gamma}(2)\colon \underline{\Gamma}(2) \to \mathrm{Pic}_{\mathcal{Y}/S}(2)$ yields an embedding $\gamma\colon \underline{\Gamma}(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S}$. $\square$

Next, we show that any Γ-marking on a family of supersingular Enriques surfaces extends in a unique way to a Γ'-marking.

LEMMA 3.27. *Let S be an algebraic space over $\mathbb{F}_p$. The forgetful functor*

$$\begin{pmatrix} \textit{Families of } \Gamma'\textit{-marked} \\ \textit{supersingular Enriques surfaces} \\ (\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S}) \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{Families of } \Gamma\textit{-marked} \\ \textit{supersingular Enriques surfaces} \\ (\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}_{|\Gamma}\colon \Gamma \to \mathrm{Pic}_{\mathcal{Y}/S}) \end{pmatrix}$$

is an equivalence of categories.

PROOF. The automorphism group of the constant group scheme $\underline{\mathbb{Z}/2\mathbb{Z}}$ is trivial. Thus every Γ-marking extends étale locally in a unique way to a Γ'-marking and by uniqueness to a global Γ'-marking. □

We now consider the functor

$$\underline{\mathcal{E}}_\sigma\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\mathrm{Sets}),$$
$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of } \Gamma'\text{-marked} \\ \text{supersingular Enriques surfaces} \\ (\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S}) \\ \text{such that the canonical K3 cover } \mathcal{X} \to \mathcal{Y} \\ \text{admits an } N_\sigma\text{-marking} \end{array} \right\}.$$

We are interested in the representability of the moduli functor $\underline{\mathcal{E}}_\sigma$. In the following proposition we show that the functor $\underline{\mathcal{E}}_\sigma$ is isomorphic to the functor $\underline{\tilde{\mathcal{E}}}_\sigma$ from Section 3.

PROPOSITION 3.28. *There exists an isomorphism of functors* $\mathrm{cov}\colon \underline{\mathcal{E}}_\sigma \to \underline{\tilde{\mathcal{E}}}_\sigma$.

PROOF. We first define the morphism $\mathrm{cov}\colon \underline{\mathcal{E}}_\sigma \to \underline{\tilde{\mathcal{E}}}_\sigma$. To this end, we consider a family of Γ'-marked supersingular Enriques surfaces $y = (\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S}) \in \underline{\mathcal{E}}_\sigma(S)$ that has the canonical K3 cover $(f\colon \mathcal{X} \to S, \gamma\colon \Gamma(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S})$.

If $s\colon \mathrm{Spec}\,\overline{k} \to S$ is a geometric point, then the orthogonal complement of $\gamma_s(\Gamma(2))$ in $\mathrm{NS}(\mathcal{X}_s)$ contains no (-2)-vector. Since the fiber $\mathcal{Y}_s$ is projective, it has an ample divisor. Pullback along finite morphisms preserves ampleness of divisors, so the sublattice $\gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s)$ also contains an ample divisor. We can thus define $\mathrm{cov}(S)(y) = (f\colon \mathcal{X} \to S, \gamma\colon \underline{\Gamma}(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S})$ and this clearly yields a morphism of functors.

We will now define another morphism of functors $\mathrm{quot}\colon \underline{\tilde{\mathcal{E}}}_\sigma \to \underline{\mathcal{E}}_\sigma$ such that the morphisms quot and cov are mutually inverse to each other. To this end, we let S be a scheme and let $x = (f\colon \mathcal{X} \to S, \gamma\colon \Gamma(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S}) \in \underline{\tilde{\mathcal{E}}}_\sigma(S)$. We consider the involution $\iota_\gamma\colon \mathcal{X} \to \mathcal{X}$ from the proof of Proposition 3.4. Then ι_γ induces a free $\langle \iota_\gamma \rangle$-action on $\mathcal{X}$ and so the quotient $\mathcal{Y} = \mathcal{X}/\langle \iota_\gamma \rangle$ exists as an algebraic space over S, and the morphism $\mathcal{X} \to \mathcal{Y}$ makes $\mathcal{X}$ into a $\mathbb{Z}/2\mathbb{Z}$-torsor over $\mathcal{Y}$. Thus, for every $s \in S$, $\mathcal{X}_s$ is a $\mathbb{Z}/2\mathbb{Z}$-torsor over $\mathcal{Y}_s$, and it follows that $\mathcal{Y}_s$ is a supersingular Enriques surface for each $s \in S$. Furthermore, the canonical morphism $\mathrm{Pic}_{\mathcal{Y}/S} \to \mathrm{Pic}_{\mathcal{X}/S}$ induces an isomorphism $\psi\colon \mathrm{Pic}_{\mathcal{Y}/S}(2) \longrightarrow \gamma(\underline{\Gamma}(2))$. We define $\tilde{\gamma}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S}$ to be the unique Γ'-marking of $\mathrm{Pic}_{\mathcal{Y}/S}$ which is induced from ψ^{-1} using Lemma 3.27. Now setting $\mathrm{quot}(S)(x) = (\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S})$ yields the desired inverse. □

The following theorem, which is one of the main results of this work, can be seen as a supersingular version of the results on complex Enriques surfaces in [Nam85] or as a version for Enriques surfaces of the results on supersingular K3 surfaces in [Ogu83].

THEOREM 3.29. *The functor $\underline{\mathcal{E}}_\sigma$ is represented by a quasi-separated algebraic space $\mathcal{E}_\sigma$ that is locally of finite type over $\mathbb{F}_p$ and there exists a separated $\mathbb{F}_p$-scheme $\mathcal{Q}_\sigma$ of finite type and AF, and a canonical étale surjective morphism $\pi_\sigma^E\colon \mathcal{E}_\sigma \to \mathcal{Q}_\sigma$.*

PROOF. This follows directly from Theorem 3.18, Proposition 3.22 and Proposition 3.28. □

REMARK 3.30. It follows from [Jan13, Proposition 3.5] that for any $\sigma \geq 5$ we have a canonical isomorphism $\underline{\mathcal{E}}_\sigma \xrightarrow{\sim} \underline{\mathcal{E}}_5$.

The previous remark motivates the following definition.

DEFINITION 3.31. We call $\mathcal{E}_5$ the *moduli space* of Γ'-marked supersingular Enriques surfaces and $\mathcal{Q}_5$ the *period space* of Γ'-marked supersingular Enriques surfaces.

REMARK 3.32. From the constructions it follows directly that, similar to the case of marked supersingular K3 surfaces, there are canonical stratifications $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \mathcal{E}_3 \hookrightarrow \mathcal{E}_4 \hookrightarrow \mathcal{E}_5$ and $\mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2 \hookrightarrow \mathcal{Q}_3 \hookrightarrow \mathcal{Q}_4 \hookrightarrow \mathcal{Q}_5$ via closed immersions. However, the latter are not sections to fibrations of the form $\mathcal{Q}_\sigma \to \mathcal{Q}_{\sigma-1}$. The main difference to the situation for marked supersingular K3 surfaces, and therefore the reason why such a fibration does not exist, is the following. While the embedding $\mathcal{M}_{\sigma-1} \hookrightarrow \mathcal{M}_\sigma$ depends on the choice of an embedding $j\colon N_\sigma \hookrightarrow N_{\sigma-1}$, the embedding $\mathcal{Q}_{\sigma-1} \hookrightarrow \mathcal{Q}_\sigma$ corresponds to the union over all images of such embeddings $\mathcal{M}_{\sigma-1} \hookrightarrow \mathcal{M}_\sigma$, but the inclusion $\bigcup_{j\in R_{\sigma-1,\sigma}} \Phi_j(\mathcal{M}_{\sigma-1}) \hookrightarrow \mathcal{M}_\sigma$ does not have an inverse.

REMARK 3.33. The period spaces $\mathcal{Q}_\sigma$ come with canonical compactifications which we denote $\mathcal{Q}_\sigma^\dagger$. Namely, we consider the functor

$$\underline{\tilde{\mathcal{E}}}_\sigma^\dagger\colon \mathcal{A}_{\mathbb{F}_p}^{\mathrm{op}} \longrightarrow (\mathrm{Sets}),$$

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of supersingular} \\ \text{K3 surfaces } f\colon \mathcal{X} \to S \text{ admitting an } N_\sigma\text{-marking} \\ \text{together with an embedding } \gamma\colon \underline{\Gamma}(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S} \\ \text{such that for each geometric fiber } s \in S \\ \text{the sublattice } \gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \\ \text{contains an ample line bundle} \end{array} \right\}.$$

By an argument analogous to the proof of Theorem 3.18, it follows that the functor $\underline{\tilde{\mathcal{E}}}_\sigma^\dagger$ is representable by a quasi-separated algebraic space $\tilde{\mathcal{E}}_\sigma^\dagger$ that is locally of finite type over $\mathbb{F}_p$. Furthermore, there exists a proper $\mathbb{F}_p$-scheme $\mathcal{Q}_\sigma^\dagger$ and a canonical étale surjective morphism $\tilde{\mathcal{E}}_\sigma^\dagger \to \mathcal{Q}_\sigma^\dagger$ by an argument analogous to the one in the proof of Proposition 3.22.

The scheme $\mathcal{Q}_\sigma^\dagger$ is indeed proper because inductively there exists a finite surjection of the proper $\mathbb{F}_p$-scheme $\mathcal{Q}_{\sigma-1}^\dagger \amalg \coprod_{\gamma\in R_\sigma} \widetilde{\mathcal{M}'_\gamma}$ onto $\mathcal{Q}_\sigma^\dagger$. The canonical morphism of schemes $\mathcal{Q}_\sigma \to \mathcal{Q}_\sigma^\dagger$ is an open immersion and a subscheme of the closed locus $\mathcal{Q}_\sigma^\dagger \backslash \mathcal{Q}_\sigma$ corresponds to quotients of K3 surfaces by involutions that fix a divisor. This is an analogue to the so-called Coble locus in the characteristic zero setting, see [DK13].

5. Some remarks about the geometry of the moduli space $\mathcal{E}_\sigma$

The geometry of $\mathcal{E}_\sigma$ is quite complicated. However, it is clear that the algebraic space $\mathcal{E}_\sigma$ is reduced, but in general it will not be connected, as in the case $\sigma = 1$ it already has multiple connected components.

Moreover we can not expect the connected components of $\mathcal{E}_\sigma$ to be irreducible, since they are glued together from the algebraic spaces $\tilde{\mathcal{S}}''_\gamma$ with $\gamma \in R_\sigma$, and we can not expect the irreducible components to be smooth; a priori the action of $O(N_\sigma, \gamma)$ on $\mathcal{S}'_\gamma$, which we took the quotient by, is not free, and we do not expect it to factorize over a free action.

Furthermore, when taking the pushout in the proof of Theorem 3.18, we expect more singularities to show up. However, there are some simple general observations on the geometry of the algebraic space $\mathcal{E}_\sigma$.

We will first introduce a subfunctor $\underline{\tilde{\mathcal{E}}}'_\sigma$ of $\underline{\tilde{\mathcal{E}}}_\sigma$ to help us understand the geometry of the algebraic space $\mathcal{E}_\sigma \cong \tilde{\mathcal{E}}_\sigma$. We define

$$\underline{\tilde{\mathcal{E}}}'_\sigma \colon \mathcal{A}^{\mathrm{op}}_{\mathbb{F}_p} \longrightarrow (\text{Sets}),$$

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of} \\ \text{supersingular K3 surfaces } f\colon \mathcal{X} \to S \\ \text{together with a marking } \gamma\colon \underline{\Gamma}(2) \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S} \\ \text{such that there exists an embedding} \\ \psi\colon \underline{N}_\sigma \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S} \text{ with } \gamma(\Gamma(2)) \subset \underline{N}_\sigma \text{ and} \\ \text{such that for each geometric fiber } s \in S \\ \text{the sublattice } \gamma_s(\Gamma(2)) \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \\ \text{contains an ample line bundle and} \\ \gamma_s(\Gamma(2))^\perp \hookrightarrow \mathrm{NS}(\mathcal{X}_s) \text{ contains no } (-2)\text{-vector} \end{array} \right\}.$$

The proof of the following proposition goes similarly to the proof of Theorem 3.18. We therefore only highlight the main differences in the proof.

PROPOSITION 3.34. *The functor $\underline{\tilde{\mathcal{E}}}'_\sigma$ is representable by a closed algebraic subspace $\tilde{\mathcal{E}}'_\sigma$ of $\tilde{\mathcal{E}}_\sigma$.*

PROOF. We do induction over σ. The case $\sigma = 1$ is clear, because in this case we have $\underline{\tilde{\mathcal{E}}}_1 = \underline{\tilde{\mathcal{E}}}'_1$.

We write $\underline{\tilde{\mathcal{E}}}'^s_{\sigma-1}$ for the subfunctor of $\underline{\tilde{\mathcal{E}}}'_{\sigma-1}$ which is defined to be as follows: the S-valued points of $\underline{\tilde{\mathcal{E}}}'^s_{\sigma-1}$ are the families $f\colon \mathcal{X} \to S$ in $\underline{\tilde{\mathcal{E}}}'_{\sigma-1}(S)$ that admit markings of the form $\gamma\colon \underline{\Gamma}(2) \hookrightarrow N_{\sigma-1} \hookrightarrow \mathrm{Pic}_{\mathcal{X}/S}$ such that there is a factorization $\gamma\colon \underline{\Gamma}(2) \hookrightarrow N_\sigma \hookrightarrow N_{\sigma-1}$.

Then $\underline{\tilde{\mathcal{E}}}'^s_{\sigma-1} \subset \underline{\tilde{\mathcal{E}}}'_{\sigma-1}$ is a closed subfunctor, since $\underline{\tilde{\mathcal{E}}}'^s_{\sigma-1}$ is representable by the image of the finite morphism $p_\sigma\colon p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \to \tilde{\mathcal{E}}_{\sigma-1}$. We consider the pushout diagram

$$\begin{tikzcd} p_\sigma^{-1}(\tilde{\mathcal{E}}'^s_{\sigma-1}) \arrow[r, "\iota"'] \arrow[d, "p_\sigma"'] & \coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma \arrow[d] \\ \tilde{\mathcal{E}}'^s_{\sigma-1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{P}. \end{tikzcd}$$

We note that $p_\sigma\colon p_\sigma^{-1}(\tilde{\mathcal{E}}'^s_{\sigma-1}) \to \tilde{\mathcal{E}}'^s_{\sigma-1}$ is finite surjective and therefore also $\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma \to \mathcal{P}$ is finite surjective. Analogously to the proof of Theorem 3.18, we can show that $\mathcal{P}$ exists as an algebraic space and represents the functor $\underline{\tilde{\mathcal{E}}}'_\sigma$. Thus we set $\tilde{\mathcal{E}}'_\sigma = \mathcal{P}$. Since $\tilde{\mathcal{E}}'^s_{\sigma-1}$ is closed in $\tilde{\mathcal{E}}_{\sigma-1}$ it follows from the construction of the algebraic space $\tilde{\mathcal{E}}_\sigma$ that $\tilde{\mathcal{E}}'_\sigma$ is a closed subspace of $\tilde{\mathcal{E}}_\sigma$. □

Again, the functor $\underline{\tilde{\mathcal{E}}}'_\sigma$ has a description in terms of Enriques surfaces. Namely, we define

$$\underline{\mathcal{E}}'_\sigma\colon \mathcal{A}^{\mathrm{op}}_{\mathbb{F}_p} \longrightarrow (\mathrm{Sets}),$$

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of } \Gamma'\text{-marked} \\ \text{supersingular Enriques surfaces } (\tilde{f}\colon \mathcal{Y} \to S, \tilde{\gamma}\colon \Gamma' \to \mathrm{Pic}_{\mathcal{Y}/S}) \\ \text{such that the canonical K3 cover } \mathcal{X} \to \mathcal{Y} \text{ admits} \\ \text{an } N_\sigma\text{-marking such that the induced map} \\ \Gamma(2) \to \mathrm{Pic}_{\mathcal{X}/S} \text{ factorizes through } \underline{N}_\sigma \end{array} \right\}.$$

The proof of the following proposition goes completely analogously to the proof of Proposition 3.28 and we therefore leave it to the reader.

PROPOSITION 3.35. *There exists an isomorphism of functors* $\mathrm{cov}\colon \underline{\mathcal{E}}'_\sigma \to \underline{\tilde{\mathcal{E}}}'_\sigma$.

We will write $\mathcal{E}'_\sigma$ for the algebraic space representing the functor $\underline{\mathcal{E}}'_\sigma$. Coming back to the discussion of the geometry of the space $\mathcal{E}_\sigma$, we note that the space $\mathcal{E}_\sigma$ is of dimension $\sigma - 1$, but its irreducible components might not be equidimensional in general. The upshot of constructing the functor $\underline{\tilde{\mathcal{E}}}'_\sigma$ lies in the following result.

PROPOSITION 3.36. *For any $\sigma' \leq \sigma$, the algebraic space $\mathcal{E}'_{\sigma'}$ is a closed subspace of $\mathcal{E}_\sigma$ and we have the equality*

$$\bigcup_{\sigma'\leq\sigma} \mathcal{E}'_{\sigma'} = \mathcal{E}_\sigma.$$

Furthermore, $\mathcal{E}'_\sigma$ is the maximal closed subspace in $\mathcal{E}_\sigma$ with the property that all of its irreducible components are of dimension $\sigma - 1$.

PROOF. The first statement follows from the construction of the space $\mathcal{E}_\sigma$ via induction over σ and the second statement follows directly from the construction of $\mathcal{E}_\sigma$ and $\mathcal{E}'_\sigma$ and the fact that the morphism $\coprod_{\gamma\in R_\sigma} \mathcal{S}''_\gamma \to \mathcal{E}'_\sigma$ is a finite surjection. □

REMARK 3.37. We do not know if the functors $\underline{\mathcal{E}}_\sigma$ and $\underline{\mathcal{E}}'_\sigma$ are unequal in general. This boils down to asking whether there exist embeddings $\Gamma(2) \hookrightarrow N_{\sigma-1}$ that do not factorize over an embedding $j\colon N_\sigma \hookrightarrow N_{\sigma-1}$. However, we suspect that such embeddings may exist and that for $\sigma > 1$ we should have $\underline{\mathcal{E}}_\sigma \neq \underline{\mathcal{E}}'_\sigma$.

There exists a scheme lying under $\mathcal{E}'_\sigma$ in analogy to Proposition 3.22.

PROPOSITION 3.38. *There exists a separated $\mathbb{F}_p$-scheme $\mathcal{Q}'_\sigma$ that is a closed subscheme of $\mathcal{Q}_\sigma$ and a canonical étale surjective morphism $\tilde{\mathcal{E}}'_\sigma \to \mathcal{Q}'_\sigma$.*

PROOF. The proof goes analogously to the proof of Proposition 3.22 by replacing $\mathcal{Q}_{\sigma-1}$ with the image of $p_\sigma^{-1}(\mathcal{Q}_{\sigma-1})$ in $\mathcal{Q}_{\sigma-1}$ in the pushout construction. □

The following proposition is an analogue to Proposition 3.36.

PROPOSITION 3.39. *For any $\sigma' \leq \sigma$, the scheme $\mathcal{Q}'_\sigma$ is a closed subscheme of $\mathcal{Q}_\sigma$ and we have an equality*

$$\bigcup_{\sigma' \leq \sigma} \mathcal{Q}'_{\sigma'} = \mathcal{Q}_\sigma.$$

Furthermore, $\mathcal{Q}'_\sigma$ is the maximal closed subscheme in $\mathcal{Q}_\sigma$ whose irreducible components are all of dimension $\sigma - 1$.

In the following, we give some results on the geometry of the spaces $\mathcal{E}'_\sigma$ and $\mathcal{Q}'_\sigma$. It follows from Proposition 3.36 and Proposition 3.39 that the geometry of these spaces is intimately related to the geometry of the spaces $\mathcal{E}_\sigma$ and $\mathcal{Q}_\sigma$.

DEFINITION 3.40. We write ε_σ for the number of irreducible components of $\mathcal{E}'_\sigma$.

REMARK 3.41. We recall from Section 1 that the $\mathbb{F}_p$-scheme $\mathcal{S}_\sigma$ is smooth. In particular, each of its connected components is irreducible. From its description as the moduli space of characteristic subspaces together with ample cones, it is clear that $\mathcal{S}_\sigma$ only has finitely many connected components.

PROPOSITION 3.42. *The morphism $p_\sigma \colon \coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma \to \mathcal{E}'_\sigma$ induces a bijection between the sets of irreducible components of $\coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma$ and $\mathcal{E}'_\sigma$. If we write τ_σ for the number of connected components of $\mathcal{S}_\sigma$, we obtain the inequality*

$$\varepsilon_\sigma \leq \tau_\sigma \cdot |R_\sigma|.$$

PROOF. For $\gamma \in R_\sigma$, each irreducible component of the algebraic space $\tilde{\mathcal{S}}''_\gamma$ over $\mathbb{F}_p$ is of dimension $\sigma - 1$. Since there exists a dense open subspace $U \subset \coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma$ such that the restriction $p_\sigma|_U \colon U \to \mathcal{E}'_\sigma$ is an open immersion, it follows that if $E_1, E_2 \subset \coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma$ are two different irreducible components, then the intersection $p_\sigma(E_1) \cap p_\sigma(E_2)$ is at least of codimension 1. Thus, the morphism p_σ induces a bijection between the sets of irreducible components of $\coprod_{\gamma \in R_\sigma} \tilde{\mathcal{S}}''_\gamma$ and $\mathcal{E}'_\sigma$. The inequality follows from the fact that the open subscheme $\mathcal{S}''_\gamma \subset \mathcal{S}_\sigma$ surjects onto $\tilde{\mathcal{S}}''_\gamma$ and each connected component of $\mathcal{S}_\sigma$ is irreducible. $\square$

PROPOSITION 3.43. *There is an equality*

$$\#\{\textit{irreducible components of } \mathcal{Q}'_\sigma\} = |R_\sigma|.$$

PROOF. This follows since the schemes $\widetilde{\mathcal{M}}''_\gamma$ are irreducible and there is a dense open subscheme of $\coprod_{\gamma \in R_\sigma} \widetilde{\mathcal{M}}''_\gamma$ that is isomorphic to a dense open subscheme of $\mathcal{Q}'_\sigma$. $\square$

DEFINITION 3.44. On the set R_σ of isomorphism classes $[\gamma \colon \Gamma(2) \hookrightarrow N_\sigma]$ of embeddings of lattices we define an equivalence relation via

$$[\gamma \colon \Gamma(2) \hookrightarrow N_\sigma] \sim [\gamma' \colon \Gamma(2) \hookrightarrow N_\sigma]$$

if and only if there exists a positive integer $\sigma' \leq \sigma$ and embeddings $j\colon N_\sigma \hookrightarrow N_{\sigma'}$ and $j'\colon N_\sigma \hookrightarrow N_{\sigma'}$, such that the sublattice $j(\gamma(\Gamma(2)))^\perp \subset N_{\sigma'}$ contains no (-2)-vectors and such that there is an equality

$$[j \circ \gamma\colon \Gamma(2) \hookrightarrow N_{\sigma'}] = [j' \circ \gamma'\colon \Gamma(2) \hookrightarrow N_{\sigma'}]$$

of elements in $R_{\sigma'}$.

Using this equivalence relation we obtain the following results.

PROPOSITION 3.45. *There is an equality*

$$\#\{\textit{connected components of } \mathcal{Q}'_\sigma\} = |R_\sigma/\sim|\,.$$

PROOF. It follows from the construction in the proof of Proposition 3.39 that under the surjection of schemes $\coprod_{\gamma\in R_\sigma} \widetilde{\mathcal{M}}''_\gamma \to \mathcal{Q}'_\sigma$ two connected components $\widetilde{\mathcal{M}}''_{\gamma_1}$ and $\widetilde{\mathcal{M}}''_{\gamma_2}$ map to the same connected component of $\mathcal{Q}'_\sigma$ if and only if $\gamma_1 \sim \gamma_2$. □

PROPOSITION 3.46. *We write τ_σ for the number of connected components of $\mathcal{S}_\sigma$ and ε^c_σ for the number of connected components of $\mathcal{E}'_\sigma$. There is an inequality*

$$\varepsilon^c_\sigma \leq \tau_\sigma \cdot |R_\sigma/\sim|\,.$$

PROOF. We consider the surjection of algebraic spaces $\coprod_{\gamma\in R_\sigma} \tilde{\mathcal{S}}''_\gamma \to \mathcal{E}'_\sigma$. For each $\gamma \in R_\sigma$ the algebraic space $\tilde{\mathcal{S}}''_\gamma$ has at most τ_σ many connected components. If $\gamma_1 \sim \gamma_2$, say with $[j_1 \circ \gamma_1] = [j_2 \circ \gamma_2]$, then $\tilde{\mathcal{S}}''_{j_1\circ\gamma_1} \cong \tilde{\mathcal{S}}''_{j_2\circ\gamma_2}$ is a subspace of both $\tilde{\mathcal{S}}''_{\gamma_1}$ and $\tilde{\mathcal{S}}''_{\gamma_2}$, which touches each of the connected components of the $\tilde{\mathcal{S}}''_{\gamma_i}$. Therefore the image of $\tilde{\mathcal{S}}''_{\gamma_1} \amalg \tilde{\mathcal{S}}''_{\gamma_2}$ in $\mathcal{E}'_\sigma$ has at most τ_σ many connected components. This implies the statement of the proposition. □

PROPOSITION 3.47. *We denote by α_σ the number of isomorphism classes $[\gamma\colon \Gamma(2) \hookrightarrow N_\sigma]$ in R_σ such that that for each positive integer $\sigma' < \sigma$ and each embedding of lattices $j\colon N_\sigma \hookrightarrow N_{\sigma'}$ there is a (-2)-vector in the sublattice $j(\gamma(\Gamma(2)))^\perp \subset N_{\sigma'}$. Then we have an inequality*

$$\alpha_\sigma \leq \varepsilon^c_\sigma \leq \tau_\sigma \cdot (\alpha_\sigma + \varepsilon^c_{\sigma-1}).$$

PROOF. The lower bound is a very weak estimate: if γ is such that for each positive integer $\sigma' < \sigma$ and each $j\colon N_\sigma \hookrightarrow N_{\sigma'}$ there is a (-2)-vector in the sublattice $j(\gamma(\Gamma(2)))^\perp \subset N_{\sigma'}$, then $[\gamma]$ is the only element in its equivalence class of $\sim$. Hence the image of $\tilde{\mathcal{S}}''_\gamma$ in $\mathcal{E}'_\sigma$ is disjoint from the image of any $\tilde{\mathcal{S}}''_{\gamma'}$ in $\mathcal{E}'_\sigma$ for all $\gamma' \neq \gamma$.

For the upper bound, we remark that each $\gamma \in R_\sigma$ is either as above, or there exists a positive integer $\sigma' < \sigma$ and an element $\gamma' \in R_{\sigma'}$ such that the images of $\tilde{\mathcal{S}}''_{\gamma'}$ in $\mathcal{E}'_{\sigma'} \subset \mathcal{E}'_\sigma$ and $\tilde{\mathcal{S}}''_\gamma$ in $\mathcal{E}'_\sigma$ intersect non-trivially. □

Analogously to the compactification $\mathcal{E}^\dagger_\sigma$ of $\mathcal{E}_\sigma$, we can construct a compactification $\mathcal{E}'^\dagger_\sigma$ of $\mathcal{E}'_\sigma$. In analogy to Proposition 3.36 we have the following proposition.

PROPOSITION 3.48. *For any $\sigma' \leq \sigma$, the algebraic space ${\mathcal{E}'}^\dagger_{\sigma'}$ is a closed subspace of $\mathcal{E}^\dagger_\sigma$ and we have the equality*

$$\bigcup_{\sigma'\leq\sigma} {\mathcal{E}'}^\dagger_{\sigma'} = \mathcal{E}^\dagger_\sigma.$$

Furthermore, ${\mathcal{E}'}^{\dagger}_{\sigma}$ is the maximal closed subspace in $\mathcal{E}^{\dagger}_{\sigma}$ with the property that all of its irreducible components are of dimension $\sigma - 1$.

We leave the proof to the reader and obtain the following result.

PROPOSITION 3.49. *There are inequalities*

$$\#\{\textit{connected components of } \mathcal{E}'^{\dagger}_{\sigma}\} \leq \#\{\textit{connected components of } \mathcal{E}'^{\dagger}_{\sigma-1}\}$$

and

$$\#\{\textit{irreducible components of } \mathcal{E}'_{\sigma}\} \leq \#\{\textit{irreducible components of } \mathcal{E}'^{\dagger}_{\sigma}\}.$$

PROOF. The proof of the first inequality goes analogously to the proof of the upper bound in the previous proposition. The second inequality is clear since $\mathcal{E}'_{\sigma}$ is an open algebraic subspace in $\mathcal{E}'^{\dagger}_{\sigma}$. □

6. Torelli theorems for supersingular Enriques surfaces

The algebraic spaces $\mathcal{E}_{\sigma}$ are fine moduli spaces for Γ'-marked supersingular Enriques surfaces with Artin invariant at most σ, but their geometry is very complicated. However, it turns out that the much nicer schemes $\mathcal{Q}_{\sigma}$ from Proposition 3.22 are coarse moduli spaces for this moduli problem. The next proposition is a direct consequence of the Torelli theorem for supersingular K3 surfaces [Ogu83] and does not use any of our prior results.

PROPOSITION 3.50. *Let Y and Y' be supersingular Enriques surfaces over an algebraically closed field k of characteristic $p \geq 3$ which have universal K3 covers X and X' respectively. Let $\tilde{\phi}\colon \mathrm{NS}(Y) \to \mathrm{NS}(Y')$ be a morphism of lattices that maps the ample cone of Y to the ample cone of Y' and such that the induced morphism of lattices $\phi\colon \mathrm{NS}(X) \to \mathrm{NS}(X')$ extends via the first Chern map to an isomorphism $H^2_{\mathrm{crys}}(X/W) \to H^2_{\mathrm{crys}}(X'/W)$. Then $\tilde{\phi}$ is induced from an isomorphism $\tilde{\Phi}\colon Y \to Y'$ of supersingular Enriques surfaces.*

PROOF. This follows immediately from a version of the Torelli theorem for supersingular K3 surfaces [Ogu83, cf. Theorem II] and the fact that pullback along finite morphisms preserves ampleness of divisors. □

Next, we want to show that the schemes $\mathcal{Q}_{\sigma}$ are coarse moduli spaces for Enriques surfaces in the sense that their points parametrize isomorphism classes of Enriques surfaces without having to choose any kind of marking.

DEFINITION 3.51. Recall from Theorem 3.29 that there is a canonical étale surjective morphism $\pi^E_{\sigma}\colon \mathcal{E}_{\sigma} \to \mathcal{Q}_{\sigma}$. If Y is a supersingular Enriques surface of Artin invariant $\sigma' \leq \sigma$ over an algebraically closed field k of characteristic $p \geq 3$, we define the *period π^E_{σ} of Y in $\mathcal{Q}_{\sigma}$* to be $\pi^E_{\sigma}(Y) = \pi^E_{\sigma}(Y, \gamma)$, where γ is any Γ-marking of Y.

The following proposition shows that π^E_{σ} is well defined and does not depend on the chosen marking.

PROPOSITION 3.52. *Let k be an algebraically closed field of characteristic $p \geq 3$, let $\sigma \leq 5$ be a positive integer and let Y be a supersingular Enriques surface of Artin invariant at most σ over k. For any choice of markings $\tilde{\gamma}_1 \colon \Gamma \to \mathrm{NS}(Y)$ and $\tilde{\gamma}_2 \colon \Gamma \to \mathrm{NS}(Y)$ we have an equality $\pi^E_\sigma(Y, \tilde{\gamma}_1) = \pi^E_\sigma(Y, \tilde{\gamma}_2)$. In other words, the period of Y in $\mathcal{Q}_\sigma$ is independent of the choice of a marking.*

PROOF. From the construction of $\mathcal{Q}_\sigma$ in Proposition 3.22 and the discussion in [Ogu79, §4 and §5], it follows that the scheme $\mathcal{Q}_\sigma$ represents the functor that associates to a smooth scheme S the set of isomorphism classes of families of K3 crystals H over S, together with maps $\gamma \colon \Gamma(2) \hookrightarrow T_H \hookrightarrow H$ that are compatible with intersection forms, and such that there exists a factorization $\gamma \colon \Gamma(2) \hookrightarrow N_\sigma \hookrightarrow T_H \hookrightarrow H$ without (-2)-vectors in the orthogonal complement $\gamma(\Gamma(2))^\perp \subset N_\sigma$.

Now we let Y be a supersingular Enriques surface that has the universal K3 covering $X \to Y$, and we let $\tilde{\gamma}_1 \colon \Gamma \to \mathrm{NS}(Y)$ and $\tilde{\gamma}_2 \colon \Gamma \to \mathrm{NS}(Y)$ be two choices of markings. We consider the period points

$$\pi^E_\sigma(Y, \tilde{\gamma}_1) = [\gamma_1 \colon \Gamma(2) \hookrightarrow T_{H^2_{\mathrm{crys}}(X/W)} \hookrightarrow H^2_{\mathrm{crys}}(X/W)]$$

and

$$\pi^E_\sigma(Y, \tilde{\gamma}_2) = [\gamma_2 \colon \Gamma(2) \hookrightarrow T_{H^2_{\mathrm{crys}}(X/W)} \hookrightarrow H^2_{\mathrm{crys}}(X/W)].$$

We have that $\mathrm{disc}(\Gamma(2)) = -2^{10}$, therefore $\gamma_1(\Gamma(2)) \otimes W = \gamma_2(\Gamma(2)) \otimes W \subset H^2_{\mathrm{crys}}(X/W)$ is a unimodular W-sublattice, since 2 is a unit in W, and we can write $H^2_{\mathrm{crys}}(X/W) = K \oplus L$ for some sublattice $L \subset H^2_{\mathrm{crys}}(X/W)$ and $K = \gamma_i(\Gamma(2)) \otimes W$. Since the sublattice K is contained in $T_{H^2_{\mathrm{crys}}(X/W)}$, it follows that K is closed under the Frobenius action on $H^2_{\mathrm{crys}}(X/W)$ and therefore its orthogonal complement $L = K^\perp$ is also closed under this action. Thus, the automorphism of the K3 crystal $H^2_{\mathrm{crys}}(X/W)$ given by $(\gamma_2 \circ \gamma_1^{-1}, \mathrm{id}_L) \colon K \oplus L \to K \oplus L$ induces an isomorphism

$$(\gamma_1 \colon \Gamma(2) \hookrightarrow T_{H^2_{\mathrm{crys}}(X/W)} \hookrightarrow H^2_{\mathrm{crys}}(X/W)) \xrightarrow{\cong} (\gamma_2 \colon \Gamma(2) \hookrightarrow T_{H^2_{\mathrm{crys}}(X/W)} \hookrightarrow H^2_{\mathrm{crys}}(X/W))$$

of $\Gamma(2)$-structures on $H^2_{\mathrm{crys}}(X/W)$, and it follows that $\pi^E_\sigma(Y, \tilde{\gamma}_1) = \pi^E_\sigma(Y, \tilde{\gamma}_2)$. □

THEOREM 3.53. *Let Y_1 and Y_2 be supersingular Enriques surfaces. Then Y_1 and Y_2 are isomorphic if and only if $\pi^E_\sigma(Y_1) = \pi^E_\sigma(Y_2)$ for some $\sigma \leq 5$.*

PROOF. It follows from Proposition 3.52 that writing $\pi^E_\sigma(Y)$ makes sense since the period of Y does not depend on the choice of the marking. We also directly obtain the "only if" part of the theorem as a consequence of Proposition 3.52. We now let Y_1 and Y_2 be supersingular Enriques surfaces with the same period point and let $X_1 \to Y_1$ and $X_2 \to Y_2$ be their canonical K3 covers. We choose two markings $\tilde{\gamma}_1 \colon \Gamma \to \mathrm{NS}(Y_1)$ and $\tilde{\gamma}_2 \colon \Gamma \to \mathrm{NS}(Y_2)$. These induce $\Gamma(2)$-markings $\gamma_1 \colon \Gamma(2) \hookrightarrow \mathrm{NS}(X_1)$ and $\gamma_2 \colon \Gamma(2) \hookrightarrow \mathrm{NS}(X_2)$, and we may choose extensions of the morphisms γ_i that are N_σ-markings $\eta_1 \colon N_\sigma \to \mathrm{NS}(X_1)$ and $\eta_2 \colon N_\sigma \to \mathrm{NS}(X_2)$. From the construction of $\mathcal{Q}_\sigma$ in Proposition 3.22 it follows that the markings $\gamma_1 \colon \Gamma(2) \hookrightarrow \mathrm{NS}(X_1)$ and $\gamma_2 \colon \Gamma(2) \hookrightarrow \mathrm{NS}(X_2)$ are isomorphic embeddings, say $[\gamma_1] = [\gamma_2] = [\gamma] \in R_\sigma$, and after applying some isometry $\varphi \in O(N_\sigma, \gamma)$ we may assume that the marked K3 surfaces (X_1, η_1) and (X_2, η_2) have the same period in $\mathcal{M}_\sigma$. Hence, there exists an isomorphism of K3 crystals $\psi \colon H^2_{\mathrm{crys}}(X_1) \longrightarrow H^2_{\mathrm{crys}}(X_2)$

and a commutative diagram

$$\begin{tikzcd}
\Gamma(2) \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}"] & N_\sigma \arrow[r, "\eta_1"] \arrow[d, "\mathrm{id}"] & \mathrm{NS}(X_1) \arrow[r] \arrow[d, "\psi"] & H^2_{\mathrm{crys}}(X_1) \arrow[d, "\psi"] \\
\Gamma(2) \arrow[r, "\gamma"] & N_\sigma \arrow[r, "\eta_2"] & \mathrm{NS}(X_2) \arrow[r] & H^2_{\mathrm{crys}}(X_2).
\end{tikzcd}$$

By a version of the Torelli theorem [Ogu83, cf. Theorem II] the isomorphism ψ is induced by some isomorphism of K3 surfaces $\Psi\colon X_1 \to X_2$. Since $\psi(\gamma_1(\Gamma(2))) = \gamma_2(\Gamma(2))$, if $\iota_1\colon X_1 \to X_1$ and $\iota_2\colon X_2 \to X_2$ are the involutions induced by the γ_i, we have that $\Psi \circ \iota_1 = \iota_2 \circ \Psi$ and it follows that the morphism Ψ descends to an isomorphism of the Enriques quotients $\tilde{\Psi}\colon Y_1 \to Y_2$. $\square$

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