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Variational Analysis of generalized Vortex Models

Interaction and Evolution of Defects

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Contents

Zusammenfassung

Wir behandeln mithilfe von Γ-Konvergenz eine Verallgemeinerung von zwei klassischen Modellen für Spin-Wirbel, und zwar dem planaren XY-Modell und dem Ginzburg-Landau Modell auf zweidimensionalen Riemannschen Mannigfaltigkeit. Das kennzeichnende Merkmal beider Verallgemeinerungen ist die Emergenz von zwei verschiedenen Typen von Singularitäten nach eines entsprechenden coarse-graining Verfahrens: fraktionellen Wirbeln (null-dimensional) vom Grad $\frac{1}{\mathfrak{m}}$ für ein $\mathfrak{m} \in \mathbb{N}$ und Liniendefekten, in deren Umgebung die Winkel der Spins um ein Vielfaches von $\frac{2\pi}{\mathfrak{m}}$ springen. In beiden Fällen, berücksichtigt der Γ-Limes beide Beiträge durch eine renormierte Energie, die von der Wirbel-Konfiguration abhängt, und durch die Oberflächen-Energie des Sprungdefekts. Zuletzt, untersuchen wir einen regularisierten L^2-Gradientenfluss des verallgemeinerten XY-Modells im speziellen Fall von zwei fraktionellen Wirbeln verbunden durch eine immersierte Kruve in $\mathbb{R}^2$. Mithilfe von minimizing movements zeigen wir maximale Existenz des Flusses.

Abstract

We propose and analyze through the language of Γ-convergence a generalization of two classical vortex models, namely the XY model in the plane as well as the Ginzburg-Landau model on a two-dimensional Riemannian manifold. The main feature of both generalizations is the emergence of two different types of singularities after a proper coarse graining procedure: fractional vortices (zero-dimensional) of degree $\frac{1}{\mathfrak{m}}$ for some $\mathfrak{m} \in \mathbb{N}$ and string-like defects (one-dimensional) around which the angles of the spins jump by a multiple of $\frac{2\pi}{\mathfrak{m}}$. In both cases, the Γ-limit takes into account both contributions via a renormalized energy, depending on the vortex configuration, and a surface energy of the jump defects. Our models allow for a simple description of several topological singularities arising in Physics and Materials Science, such as: disclinations and string defects in liquid crystals, fractional vortices and domain walls in micromagnetics, partial dislocations and stacking faults in crystal plasticity. Lastly, we study a regularized L^2 gradient flow of the generalized XY model in the case of two fractional vortices connected by an immersed curve in $\mathbb{R}^2$. Employing minimizing movements we prove maximal-time existence of the flow.

Acknowledgements

First and foremost, I want to thank my advisor Marco Cicalese for introducing me to modern methods in the Calculus of Variations and specifically to many intriguing problems concerning generalized vortex models. He was a great teacher as well as research partner, always finding time whenever I faced problems in and outside of mathematics. During my years as a PhD student I had the great luck of meeting, besides my advisor, many other great mathematicians with whom I could share very fruitful discussions. In this regard, I want to offer my gratidute to Lucia De Luca, Irene Fonseca, Gero Friesecke, Nicola Fusco, Giovanni Leoni, Carlo Mantegazza, Matteo Novaga, and Marcello Ponsiglione. Last, but definitely not least, I am very thankful to Marco Cicalese, Adriana Garroni, and Étienne Sandier for refereeing my thesis and to Daniel Matthes for organizing my defense.

Rufat Badal

Chapter 1

Introduction

Many physical systems present symmetry invariances leading to a number of equivalent ground states. Sometimes two (or more) of such ground states happen to be forced to coexist, so that each of them describes only locally the system, or in other words each of them describes only a certain different portion of the entire system. In this case an additional energy may be required in order to make this new configuration energetically accessible. The least energy required is known as transition energy. The fields describing the system in such a transition configuration are locally similar to those of the ground states, but present defects, making them less regular in certain region of the space (the transition regions). In Materials Science these singularities often appear as irregularities in the microscopic structure of a material and play a fundamental role in explaining a large variety of phenomena, ranging from plastic deformations in metals to superconductivity properties of type-I superconductors.

Despite the large interest in phase separation problems, their rigorous variational analysis has a relatively young life. It can be traced back to the birth of Γ-convergence [33] and to the analysis of the Allen-Cahn phase-field functional done by Modica and Mortola in [53, 54]. The passage from the latter scalar problem to its most significant vectorial analog, the Ginzburg-Landau functional, has required quite some time and the effort of many mathematicians and the development of new techniques (see for instance [58, 45, 47, 59, 2] as well as the books [16, 60]). The overall result has contributed to give solid mathematical basis to many fundamental problems in the theory of superconductors.

In the last years, the mathematical community has shown an increasing interest in the derivation of these effective models starting from fundamental microscopic lattice models (see for instance [13, 3, 57, 4, 6, 7]). The latter results have helped to understand some basic mechanism of formation, interaction and evolution of defects pinpointing the presence of additional energy barriers due to the microscopic length scale and to the essential anisotropic structure of matter.

The purpose of this thesis is to contribute to the mathematical research in this field with particular emphasis to a class of models leading to the presence and interaction of defects of different dimension. More precisely, we will coarse grain atomistic spin systems whose continuous counterpart will be described by a spin field presenting point and line singularities. While the point singularities are associated to fractional vortices, the line singularities represent domain walls. The main goal of our analysis is to understand the interaction between the two different types of singularities, both from a static point of view through Γ-convergence (see [17, 28]) and from a variational evolution point of view through minimizing movements (see [10, 18]).

The first result of this thesis is in Chapter 2 and concerns the analysis via Γ-convergence (see also Section 2.1.1 for the necessary definitions) of a generalization of the classic XY model from condensed matter physics (see [62] for an elementary introduction of this model). A brief mathematical description of the XY model is as follows: Fix an open set $\Omega \subset \mathbb{R}^2$ and consider a spin field u valued in the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and living on a portion of the ε-square lattice belonging to Ω, namely $\varepsilon\mathbb{Z}^2 \cap \Omega$, where $\varepsilon > 0$ is a fixed and usually very small scalar. Admissible spin configurations try to minimize an energy functional that is of nearest-neighbor type. Here, given $i, j \in \varepsilon\mathbb{Z}^2 \cap \Omega$, we say that (i, j) is a nearest-neighbor pair if and only if $|i - j| = \varepsilon$. Then, the energy functional $E_\varepsilon^{(0)}$ modeling the ferromagnetic interaction of the spin system is:

$$E_\varepsilon^{(0)}(u) := -\frac{1}{2}\sum_{\text{n.n.}} \varepsilon^2 \langle u(i), u(j)\rangle, \quad u\colon \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1,$$

where $\sum_{\text{n.n.}}$ is shorthand notation for the sum over all nearest-neighbor pairs, and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^2$. Note that the energy functional above is minimized for $u \equiv \text{const}$, with its global minimum being approximately the negative area $-|\Omega|$ of Ω. The authors of [4] derive a Γ-convergence result whose limit is finite on those u satisfying $|u(x)| \leq 1$ for a.e. $x \in \Omega$ and it is given by $-|\Omega|$. In particular, this result is compatible with ground states configurations obtained by the unconstrained mixing of arbitrary uniform states $\{u = s_1\} := \{x \in \Omega\colon u(x) = s_1\}$ and $\{u = s_2\}$, where $s_1,\ s_2 \in \mathbb{S}^1$, at a mesoscopic scale large enough. To obtain a finer description of the above model, in [4] the energy $E_\varepsilon^{(0)}$ has been studied after subtracting its minimum and rescaling by $\delta_\varepsilon \to 0$:

$$E_\varepsilon^{(1)}(u) := \delta_\varepsilon^{-1} \cdot \frac{1}{2}\sum_{\text{n.n.}} \varepsilon^2 (1 - \langle u(i), u(j)\rangle). \tag{1.1}$$

Note that such a procedure is inspired by the so called development by Γ-convergence introduced in [12] and further developed in [20]. The authors of [4] show that an interesting scaling is given by the choice $\delta_\varepsilon := \varepsilon^2|\log\varepsilon|$, in which case configurations of bounded energy may asymptotically develop a finite amount of vortices. Each vortex of degree $d \in \mathbb{Z}$ contributes by $\pi|d|$ to the Γ-limit of $E_\varepsilon^{(1)}$. We remark at this occasion that the emergence of vortices is a well known result in the statistical physics community, first predicted by J. M. Kosterlitz and D. J. Thouless in their pioneering paper [48]. An intriguing question about the above model concerns the interaction energy between vortices. This is answered in [6] by employing a second-order Γ-convergence analysis. To be more precise, they fix $N \in \mathbb{N}$, which represents the net sum of the unsigned degrees of limit vortices. Subtracting $N\pi$ from the first-order energy functional in (1.1) and rescale with $\tilde{\delta}_\varepsilon := \frac{1}{|\log\varepsilon|}$:

$$\tilde{\delta}_\varepsilon(E_\varepsilon^{(1)} - N\pi) = |\log\varepsilon|(E_\varepsilon^{(1)} - N\pi) = XY_\varepsilon - N\pi|\log\varepsilon|,$$

with $XY_\varepsilon \coloneqq E_\varepsilon^{(1)}|\log\varepsilon|$, that is:

$$XY_\varepsilon(u) = \frac{1}{2}\sum_{\text{n.n.}}(1 - \langle u(i), u(j)\rangle) \tag{1.2}$$

for any $u\colon \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1$. Note that, by a previous result, $N\pi$ is approximately equal to the minimum of $E_\varepsilon^{(1)}$ for ε small. By Γ-convergence, the authors of [6] show that a sequence $\{u_\varepsilon\}$ of spin fields that keep $XY_\varepsilon - N\pi|\log\varepsilon|$ bounded from above, asymptotically develop a system of finitely many vortices, concentrated at points $x_k \in \Omega$ around which they wind $d_k \in \mathbb{Z}$ times (here the sign of d_k corresponds to either clockwise or counter-clockwise rotation of the spin field) and represented by the measure:

$$\mu = \sum_{k=1}^{K} d_k \delta_{x_k},$$

where δ_{x_k} is the Dirac delta function at x_k. Furthermore, the net sum of the unsigned degrees, i.e. the total variation $|\mu| = \sum_{k=1}^{K}|d_k|$ of μ is bounded from above by N. In the case of equality ($|\mu| = N$) it is shown that all vortices must be of degree $d_k = \pm 1$ and that the Γ-limit of $XY_\varepsilon - N\pi|\log\varepsilon|$ (see also Theorem 4.2 in [6]) turns out to be:

$$\mathbb{W}(\mu) + N\gamma, \tag{1.3}$$

where $\mathbb{W}$ is the so-called renormalized energy, inducing an attractive force between vortices of different sign, and a repulsive one between vortices of the same sign as well as any vortex and the boundary $\partial\Omega$, and $\gamma \in \mathbb{R}$ is a scalar independent of μ known as the core energy. In this setting, a slightly stronger Γ-convergence result holds true, that not only keeps track of the vorticity but the whole spin field (identified with a proper piecewise affine interpolation) altogether, in the limit $\varepsilon \to 0$. Denoting the limit spin configuration by u, the Γ-limit of $XY_\varepsilon - N\pi|\log\varepsilon|$ is:

$$\mathcal{W}(u) + N\gamma, \tag{1.4}$$

with $\mathcal{W}$ being an extension of the renormalized energy from vortex measures to continuum spin fields $u\colon \Omega \to \mathbb{S}^1$, qualitatively inducing the same attractive as well as repulsive forces, and γ being the core energy from before. Note that minimizing $\mathcal{W}(u)$ over all spin fields u with vortices given by the measure μ reduces to $\mathbb{W}(\mu)$ from (1.3).

In [15], the author together with M. Cicalese, L. De Luca, and M. Ponsiglione investigate a possible generalization of the XY model.
More precisely, they consider a 2π-periodic potential $f_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ (depending on ε), such as the one depicted in Figure 1.1, and define for $u\colon \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1$:

$$\tilde{XY}_\varepsilon(u) \coloneqq \frac{1}{2}\sum_{\text{n.n.}} f_\varepsilon(\varphi(i) - \varphi(j)), \tag{1.5}$$

where $\varphi\colon \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{R}$ is an angular lift of u (i.e., $e^{\imath\varphi} = u$ where we conveniently identify $\mathbb{R}^2$ with $\mathbb{C}$). Notice that taking $f(t) \coloneqq 1 - \cos(t)$ instead of f_ε in (1.5) results in the classic XY functional defined in (1.2). The main difference

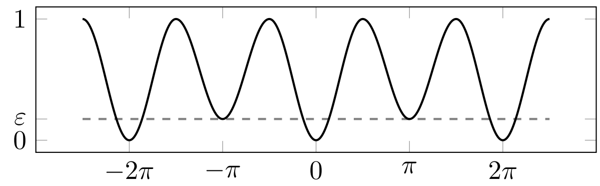


Figure 1.1: Angular potential in the fractional setting.

between XY_ε and $\tilde{XY}_\varepsilon$ is that the latter not only prefers parallel spins whose interaction pays 0 energy, but also antiparallel ones ($u(i) \approx -u(j)$). In fact, each such pair contributes approximately ε to the energy. The preference of $\tilde{XY}_\varepsilon$ for fractional vortices instead of usual ones can be formally explained as follows: A prototypical single vortex configuration such as the one given by $u(x) := \frac{x}{|x|}$ restricted to grid points costs $\pi|\log\varepsilon| + \mathrm{O}(1)$ classical XY_ε energy. As most of the nearest neighbor pairs of u have small difference-angles (of order $\mathrm{O}(\varepsilon)$) and $f_\varepsilon(t) \geq g(2t) \approx \frac{1}{2}(2t)^2 \approx 4g(t)$ for small t it follows that $\tilde{XY}_\varepsilon(u) = 4\pi|\log\varepsilon| + \mathrm{O}(1)$, a four times larger leading order term as before. In Figure 1.2, we can see a spin field v that has the same net vorticity as u but costs asymptotically less energy.

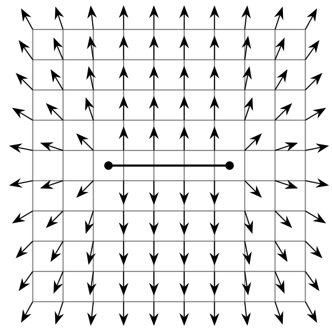


Figure 1.2: Configuration with two fractional vortices each of degree $\frac{1}{2}$ connected by a domain wall.

More precisely, it has two fractional vortices – each of degree $\frac{1}{2}$ and hence contributing $\pi|\log\varepsilon| + \mathrm{O}(1)$ to the final $\tilde{XY}_\varepsilon$ energy – connected by a string-like defect, on which the antiparallel spin pairs accumulate. As the number of antiparallel spin pairs is of order $\mathrm{O}(\frac{1}{\varepsilon})$ their contribution to the energy is $\mathrm{O}(1)$. Consequently, for $\varepsilon > 0$ small, v becomes more efficient from an energetic point of view. In Chapter 2 we illustrate how to make this formal computation rigorous by through a Γ-convergence analysis. In [15] it is show that spin fields whose generalized XY energy is bounded by $N\pi|\log\varepsilon|$ (for some $N \in \mathbb{N}$) asymptotically generate a system of finitely many (possibly fractional) vortices described by the measure:

$$\mu := \sum_{k=1}^{K} \frac{d_k}{2}\delta_{x_k}, \quad d_k \in \mathbb{Z}$$

satisfying $|\mu| \leq N$. In the special case $|\mu| = N$, they further show that

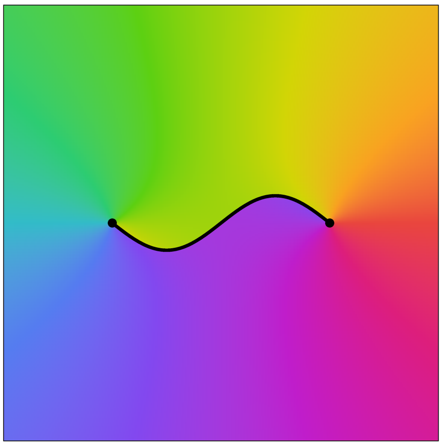


Figure 1.3: Angular lift of a limit spin configuration in the case $N = 2$. The angles jump by exactly π along the sinus shaped curve. Furthermore the spin field develops $\frac{1}{2}$-vortices at both endpoints of the jump defect.

appropriately chosen interpolations of discrete spin fields accumulate at a limit spin configuration u, having N vortex centers each of fractional degree $\pm\frac{1}{2}$ and satisfying $u^+ = -u^-$ at $\mathcal{H}^1$-a.e. point of the jump-set J_u (see also Figure 1.3). Here u^+ and u^- are the approximate jump values of u. The Γ-limit of $\tilde{XY}_\varepsilon - N\pi|\log\varepsilon|$ is then given by:

$$\mathcal{W}(u^2) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + N\gamma. \tag{1.6}$$

In the previous formula, $\mathcal{W}$ and γ are as in (1.4), $\mathcal{H}^1_{\mathrm{cr}}(J_u)$ is the crystalline length of J_u, and in the notation u^2 we identify $\mathbb{R}^2$ with $\mathbb{C}$. At this point, we remark that, in fact, the more general case of fractions $\frac{1}{\mathfrak{m}}$ ($\mathfrak{m} \in \mathbb{N}$) was considered in [15] reducing for $\mathfrak{m} = 2$ to the one we presented above. In this thesis, we will investigate a generalized XY model described by the same energy functional as in (1.5), but with spin fields that are constrained to be equal to $g \in C^\infty(T_\delta(\partial\Omega); \mathbb{S}^1)$ (where $T_\delta(\partial\Omega) \coloneqq \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some $\delta > 0$ small enough) on points of the grid $\varepsilon\mathbb{Z}^2 \cap \Omega$ closest to $\partial\Omega$. As before, we deal with the case of general fractions $\frac{1}{\mathfrak{m}}$ for $\mathfrak{m} \in \mathbb{N}$, but for the sake of simplicity we only state the result for $\mathfrak{m} = 2$. Assuming a nonzero degree $\deg(g, \partial\Omega)$ of g around $\partial\Omega$, we will show that proper interpolations of minimizers of the generalized XY model are asymptotically close to a limit spin configuration u, having $2|\deg(g)|$ fractional vortices with fractional degree equal to $\frac{\operatorname{sgn}(\deg(g))}{2}$, and satisfying $u^+ = -u^-$ at $\mathcal{H}^1$-a.e. point of J_u. As it is typical for Dirichlet constraints on functions that are allowed to jump (more precisely special functions of bounded variation), these may not be preserved in the limit. Nevertheless, we still have that $u^2 = g^2$ on $\partial\Omega$ in the Sobolev sense. This is possible, since, by the chain rule $(u^2)^+ = (u^+)^2 = (-u^-)^2 = (u^2)^-$ for $\mathcal{H}^1$-a.e. point of J_u, which in fact shows that u^2 is Sobolev regular. The Γ-limit of $\tilde{XY}_\varepsilon - 2|\deg(g)||\log\varepsilon|$ is given by (see Theorem 2.16):

$$\mathcal{W}(u^2) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + \mathcal{H}^1_{\mathrm{cr}}(\{x \in \partial\Omega \colon u(x) \neq g(x)\}) + 2|\deg(g, \partial\Omega)|\gamma. \tag{1.7}$$

The main difference between (1.6) and (1.7) is the extra component (third term in the equation above) penalizes the size of the subset of $\partial\Omega$ on which u does not coincide with g (understood in the sense of traces for functions of bounded variation).

As the analysis of the evolution of the generalized XY spin system proves to be a rather large endeavour in its most general formulation, in Chapter 3 we study that of a simplified model of the general case that is (hopefully) retaining most of the core features. In this regard, instead of the full limit spin configuration u, we will work with its singularities (vortex centers encoded by the point measure μ and the jump-set J_u) only. We further assume that $\mu = \frac{1}{2}\delta_{\eta(0)} + \frac{1}{2}\delta_{\eta(1)}$ and $J_u = \mathrm{im}(\eta)$, where $\eta\colon [0,1] \to \Omega$ is a curve in Ω. Note that in order to make this assumption compatible with the boundary condition, we need to assume that the boundary datum g satisfies $\deg(g, \partial\Omega) = 1$. As the renormalized energy diverges to $+\infty$ whenever two vortices of the same sign collide, we can assume that throughout the evolution the last term in (1.7) will remain constant, and hence, can be safely ignored. Minimizing over all spin fields that have exactly this configuration of singularities results in the analysis of the following reduced energy functional:

$$E(\eta) \coloneqq W(\eta(0), \eta(1)) + \mathcal{H}^1_{\mathrm{cr}}(\eta),$$

$W\colon \Omega\times\Omega\setminus\{(x,x)\colon x \in \Omega\} \to \mathbb{R}$ being a Coulomb-type potential (see also (3.17) for further details). In order to stay in a more classical setting, we exchange the crystalline length of γ by the Euclidean one, and consider:

$$E(\eta) \coloneqq W(\eta(0), \eta(1)) + \mathcal{H}^1(\eta).$$

We tackle the problem of studying the motion of η driven by the energy E above from the variational point of view via minimizing movements. This technique was first used (e.g., see aso [9, 49]) in order to model the motion of compact hypersurfaces whose motion is driven by their surface area. Many of such results are formulated in rather weak settings, such as the one of rectifiable currents or sets of finite perimeter, and one cannot a priori exclude phenomena such as instantaneous fattening of the hypersurface. In this work, we will use a parametric approach similar to the one of [36, 56, 37]. In this context, it is necessary to further regularize the energy E by adding a higher-order term:

$$E(\eta) \coloneqq W(\eta(0), \eta(1)) + \mathcal{H}^1(\eta) + \frac{\delta}{2}\int_\eta \kappa_\eta^2 \,\mathrm{d}\mathcal{H}^1, \tag{1.8}$$

where $\delta > 0$ is a positive scalar and κ_η is the curvature of η. Our minimizing movements scheme is set up in the following way: We consider admissible curves in $\mathcal{AC}$ defined by:

$$\mathcal{AC} \coloneqq \big\{\eta \in W^{1,2}([0,1];\mathbb{R}^2)\colon |\eta_x| = L_\eta,\ \eta(0) \neq \eta(1),\ \eta(0) \text{ and } \eta(1) \in \Omega\big\},$$

where x denotes the curve parameter, and $\eta_x \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}\eta$. Starting from an initial curve $\eta_0 \in \mathcal{AC}$ such that its image $\mathrm{im}(\eta_0) \subset \Omega$, we define for fixed $\lambda > 0$ the sequence $\{\eta_n^\lambda\}_n \subset \mathcal{AC}$ through:

$$\begin{cases} \eta_n^\lambda \in \operatorname{argmin}\big\{E(\eta) + \lambda D(\eta, \eta_{n-1}^\lambda)\colon \eta \in \mathcal{AC}\big\} \text{ for all } n \in \mathbb{N}_+, \\ \eta_0^\lambda = \eta_0, \end{cases} \tag{1.9}$$

where $D\colon \mathcal{AC}^2 \to \mathbb{R}$ is an L^2-type dissipation energy given by:

$$D(\eta, \tilde{\eta}) := \frac{1}{4}\int_0^1 \langle \eta - \tilde{\eta}, \tilde{\nu}\rangle^2 \,\mathrm{d}x + \frac{1}{4}\int_0^1 \langle \eta - \tilde{\eta}, \nu\rangle^2 \,\mathrm{d}x + \frac{1}{2}|\eta(0) - \tilde{\eta}(0)|^2 + \frac{1}{2}|\eta(1) - \tilde{\eta}(1)|^2,$$

with $\tilde{\nu}$ and ν denoting the unit normals of $\tilde{\eta}$ and η, respectively. (Using direct methods of the Calculus of Variations, we will, in fact, show that the problem in (1.9) attains at least one minimum.)

Our main result is that, up to extracting a subsequence, the piecewise constant interpolations $\eta^\lambda(t, x) := \eta^\lambda_{\lceil \lambda t\rceil}(x)$ ($\lceil \lambda t\rceil$ being the smallest natural number bigger than λt) converge as $\lambda \to \infty$ towards a limit evolution η. Denoting for $t \in [0, \infty)$ the length of η at time t as $L(t) := \mathcal{H}^1(\eta(t, \cdot))$ and by $s \in [0, L(t)]$ the arc-length parameter we prove that for a.e. $t \in [0, \infty)$ and a.e. $s \in [0, L(t)]$ the following equations needs to be satisfied by η (see also Theorem 3.1): With regard to the geometric evolution of the interior of the curve we have:

$$V^\perp = \kappa - \delta(\kappa_{ss} + \tfrac{1}{2}\kappa^3), \quad V^\perp := \langle \eta_t, \eta_s^\perp\rangle, \tag{1.10}$$

where $V := \eta_t := \frac{\mathrm{d}}{\mathrm{d}t}\eta$ is the velocity of η. This is the classic Willmore flow equation arising when one deals with an L^2-type gradient flow of an energy given by the sum of the perimeter and the squared L^2-norm of the curvature. Furthermore, η satisfies natural boundary conditions $\kappa(t, 0) = \kappa(t, 1) = 0$ at a.e. t, and:

$$V(t, 0) = -\nabla_1 W(\eta(t, 0), \eta(t, 1)) + \eta_s(t, 0) - \delta\kappa_s(t, 0)\eta_s^\perp(t, 0) \tag{1.11}$$

at the first endpoint, where $\nabla_1 W$ is the gradient of W with respect to the first point. A similar equations also holds true at the second endpoint of η. Finally, the following equation in tangential direction (arising from the constant speed constraint in the definition of $\mathcal{AC}$) holds true:

$$V_s^\top = \frac{L'}{L} + \kappa\eta_s^\perp, \quad V^\top := \langle \eta_t, \eta_s^\perp\rangle,\ L' := \frac{\mathrm{d}}{\mathrm{d}t}L. \tag{1.12}$$

Note that the motion of the curve is not restrained to remain inside Ω. But we are able to prove the existence of a maximal existence time $T_0 >$ such that for all $t \in [0, T_0)$, the curve $\eta(t, \cdot)$ is contained in Ω while $\eta(T_0, \cdot)$ has a nonempty intersection with the boundary. To the best knowledge of the author, this is the first successful application of the machinery introduced in works such as [36, 37] to the case of free endpoints, and may pave the way for studying more complicated configurations, such as networks of curves with freely moving junctions. Furthermore, we choose a geometrically motivated constant speed parametrization. In contrast to this, the authors of previous results such as for example [36] parametrize the curves as graphs over some fixed interval $[a, b]$. In the present setting such a strategy would require to replace $[a, b]$ by a time dependent interval in order to account for the shortening and elongating motion of the endpoints making many of the necessary computations quite lengthy. Finally, we remark that the result presented here is a generalization of the one found in [14], where the author considered the special case $\Omega = \mathbb{R}^2$.

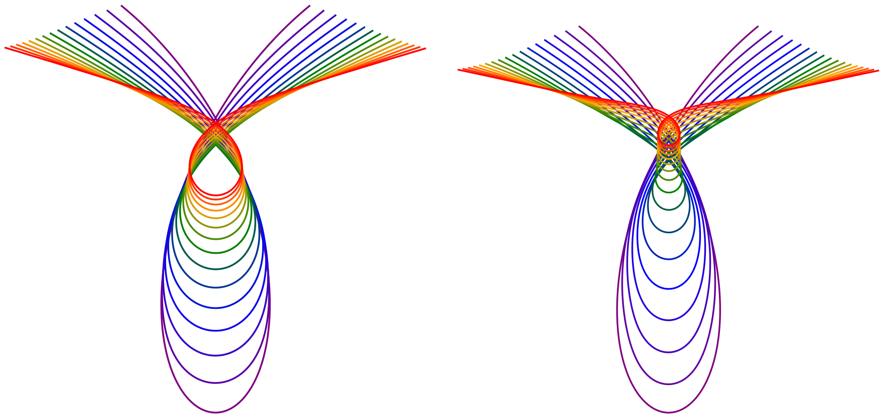


Figure 1.4: Minimizing movements starting from a “”γ-shaped” curve for two different values of ε ($\varepsilon = .025$ one the left and $\varepsilon = .005$ on the right). The color of each curve depicts the temporal ordering, starting with violet and ending with red.

In Figure 1.4, we show a numerical computation of the sequence of step-by-step minimizers (as defined in (1.9)) for $\Omega = \mathbb{R}^2$, starting from an initially “γ-shaped” curve for two different values of ε. In both cases, an interesting long time behavior can be perceived: the curve does not unfold into a straight line of length 1 (the global minimizer), and instead converges towards an optimal “γ-shape.” We can also see the influence of the regularizing term, as the size of the limit “knot” is smaller for smaller ε. This is also in accordance to the fact that the regularizing curvature term has less weight in (1.8) for smaller values of ε. It is worth mentioning at this point that the Willmore flow of curves (or more generally networks) has been studied from a PDE point of view in [30, 51, 29, 38, 39, 50], mainly assuming additional boundary conditions. It is not clear whether these techniques can be used in our free boundary setting. Furthermore, we finish this paragraph by mentioning an interesting open problem, namely the study of the long time behavior of the evolution described above and its asymptotic behavior in the limit $\varepsilon \to 0$.

In the last chapter, we will investigate a generalization of the Ginzburg-Landau model on a compact oriented 2-dimensional Riemannian manifold. The classic Ginzburg-Landau model, formulated in the Euclidean setting, is very well studied model in the mathematical physics community (see e.g. the book [16]). In the Euclidean setting, we consider functions $u \in W^{1,2}(\Omega; \mathbb{R}^2)$, where $\Omega \subset \mathbb{R}^2$ is an open set, and assign the following energy to them:

$$GL_\varepsilon(u) \coloneqq \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \, \mathrm{d}x, \tag{1.13}$$

where $\varepsilon > 0$ is a fixed scalar. There is a strong connection between the XY and the Ginzburg-Landau model, which was investigated in [5]. In fact, under the same logarithmic energy bounds, one can show similar results to the XY model (e.g., see also [8]). The authors of [40] study the analog of the generalized XY model in the context of the Ginzburg-Landau energy. More precisely, given $\mathfrak{m} \in \mathbb{N}$ and an open simply connected set Ω, they consider admissible spin

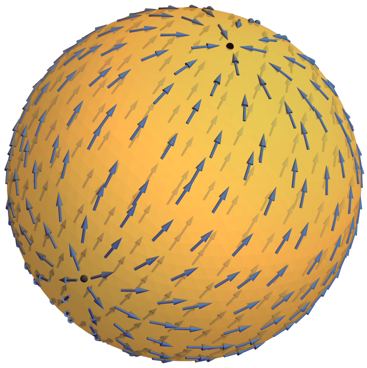


Figure 1.5: A singular unit-length vector field on the sphere. Note that $\chi(S) = 2$ in this case.

configurations $u \in SBV^2(\Omega; \mathbb{R}^2)$, such that $u^+ = -u^-$ at $\mathcal{H}^1$ a.e. point in J_u, and $u^2 = g^2$ $\mathcal{H}^1$-a.e. on $\partial\Omega$ in the sense of traces, where $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$ is a fixed boundary datum. For $\varepsilon > 0$ the energy functional is given by:

$$\tilde{GL}_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \,\mathrm{d}x + \mathcal{H}^1(J_u).$$

Through a Γ-convergence analysis, they show that minimizers of $\tilde{GL}_\varepsilon$ defined above converge towards limit spin configurations that have $2|\deg(g, \partial\Omega)|$ vortices, each with degree $\frac{\operatorname{sgn}(\deg(g,\partial\Omega))}{2}$. Furthermore, the Γ-limit of $\tilde{GL}_\varepsilon(u) - \frac{|\deg(g,\partial\Omega)|}{2}\pi|\log\varepsilon|$ with respect to L^1-convergence is given by:

$$\mathcal{W}(u) + \mathcal{H}^1(J_u) + 2|\deg(g, \partial\Omega)|\tilde{\gamma}, \tag{1.14}$$

where $\mathcal{W}$ is the renormalized energy from (1.4), $\mathcal{H}^1(J_u)$ is the Euclidean length of the jump set of u, and $\tilde{\gamma}$ is a fixed scalar that possibly differs from γ of (1.4). Note that the main difference to the XY case is the interchange of the crystalline length term with the Euclidean one. Both, the XY model and the Ginzburg-Landau model, already have found their extension to the setting of compact oriented 2-dimensional Riemannian manifolds (see also [24, 43]). The mathematical interest for such generalizations lies in the fact that vortices may naturally arise from the nontrivial topology of the manifold. More precisely, by the celebrated hairy ball theorem there exist no continuous tangent vector field on a 2-dimensional Riemannian manifold S with nonzero Euler characteristic $\chi(S)$ (see also Figure 1.5). Consequently, minimizing the analog of (1.13) among tangent vector fields $u \in W^{1,2}(TS)$ results in nontrivial solutions, even without the use of Dirichlet constraints.

The classical XY and Ginzburg-Landau model already found their generalization to the manifold setting (see [24, 43, 44]). In particular, both results show the emergence of $|\chi(S)|$ vortices, each with degree given by $\operatorname{sgn}(\chi(S))$ and compute a similar Γ-limit. In this thesis, we wish to fill in one of the missing pieces by providing a generalization of the Ginzburg-Landau model on a compact oriented 2-dimensional Riemannian manifold. The author believes that the techniques introduced in this regard may also be helpful in a future generalization of the XY model to a manifold setting.

In the research literature, important properties of functions of bounded variation were already successfully generalized to the manifold setting (e.g., see [52, 41]). As in this thesis we will deal with tangent vector fields the need arises for a generalization of the above results to the more general case of sections. To this end we study the fine properties of sections of bounded variation, which are a natural generalization of vector valued functions of bounded variation in the plane. We provide a proper intrinsic definition of blow-up quantities such as the approximate gradient (see also Definition 4.6), and eventually we prove the decomposition theorem for sections of bounded variation (Theorem 4.3).

An important tool in dealing with vortex models is the ball construction independently introduced in [58, 46] and one of the technical difficulties is its generalization to the manifold setting. This was already successfully achieved in [25] (see also Corollary 1 of Section 5.2), where the authors follow the strategy of [46]. Ours is more aligned to results such as [58, 60] and we tried to make out proof as self-contained as possible in order to be accessible also by non-experts. The main idea behind the ball construction can be shortly described as follows: Given a tangent vector field $u \in W^{1,2}(TS)$ whose Ginzburg-Landau energy is of order $\mathrm{O}(|\log \varepsilon|)$ and an admissible $r > 0$, the ball construction allows us to find a finite family $\mathfrak{B}$ of disjoint closed geodesic balls surrounding the zeroes of u, whose radii sum up to r, and:

$$GL_\varepsilon(u, \cup_{B\in\mathfrak{B}} B) = \frac{1}{2}\int_S |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \,\mathrm{vol} \geq \pi D_r \left(\frac{r}{D_r \varepsilon} - C\right), \quad (1.15)$$

where D_r is the sum of the unsigned degrees of u around each ball $B \in \mathfrak{B}$, and C is a universal constant independent of ε and u, and vol denotes the standard volume form of S (see also Theorem 4.6). We remark that this is exactly the same estimate (with a possibly larger C) as in the Euclidean case.

The main result of Chapter 4 is the following Γ-convergence result: Given a manifold S, as above, admissible spin fields are SBV^2 regular tangent vector fields on S (sections of TS), additionally satisfying $u^+ = -u^-$ for $\mathcal{H}^1$-a.e. point on J_u. Furthermore, for fixed $\varepsilon > 0$ the generalized Ginzburg-Landau functional $\tilde{GL}_\varepsilon$ is defined as:

$$\tilde{GL}_\varepsilon(u) := \frac{1}{2}\int_S |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \,\mathrm{vol} + \mathcal{H}^1(J_u).$$

We show by Γ-convergence that minimizers of $\tilde{GL}_\varepsilon$ in $\mathcal{AS}_\varepsilon$ converge, up to extracting a subsequence, towards a limit spin field u, which has $2|\chi(S)|$ isolated vortices each with fractional degree $\frac{\mathrm{sgn}(\chi(S))}{2}$. The Γ-limit of $GL_\varepsilon - \frac{|\chi(S)|}{2}\pi|\log\varepsilon|$ is equal to (see also Theorem 4.7):

$$\tilde{\mathcal{W}}(u) + \mathcal{H}^1(J_u) + 2|\chi(S)|\tilde{\gamma},$$

where $\tilde{\gamma}$ is the scalar from (1.14), and:

$$\tilde{\mathcal{W}}(u) := \lim_{r\to 0}\left(\frac{1}{2}\int_S |\nabla u|^2 \,\mathrm{vol} - \frac{\chi(S)}{2}\pi\log(r)\right), \quad (1.16)$$

with S_r denoting the complement of the union of geodesic balls of radius r around each vortex center of u (see also Lemma 4.17 for well-definedness). Note that the more general case of fractions $\frac{1}{\mathfrak{m}}$ for $\mathfrak{m} \in \mathbb{N}$ is also considered.

From the statement itself, the above result seems to be a very natural generalization of the one found in [40]. Nevertheless, our proof strays away on several occasions from the one in [40]. This is mainly due to the fact that the crucial trick employed in all results concerning fractional vortices does not pass over trivially to the manifold setting. In the flat setting we have a simple procedure at hand that can transform a vector field u with fractional vortices of degree in $\frac{1}{2}\mathbb{Z}$, and jumps satisfying $u^+ = -u^-$ into a vector field v without jumps and with nonfractional vortices instead. More precisely, given such a vector field u, we define $v(x)$ by "doubling" the angle of $u(x)$ with respect to the x_1-axis for every $x \in \Omega$. By the chain rule, the resulting vector field v satisfies the desired properties, and we can bound from above the Dirichlet energy of v with the one of u as follows:

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x \leq 4 \int_\Omega |\nabla u|^2 \,\mathrm{d}x. \tag{1.17}$$

This procedure is constrained in the case of a compact oriented 2-dimensional Riemannian manifold S by the following two reasons: The first problem arises due to the nontrivial topology of S. In order to double angles on S, in each tangent space T_pS, we need to choose a reference unit vector $e(p)$ that represents the zero angle. By the hairy ball theorem, if S has a nonzero Euler characteristic, it is impossible to find a smooth unit length vector field $e\colon S \to TS$.

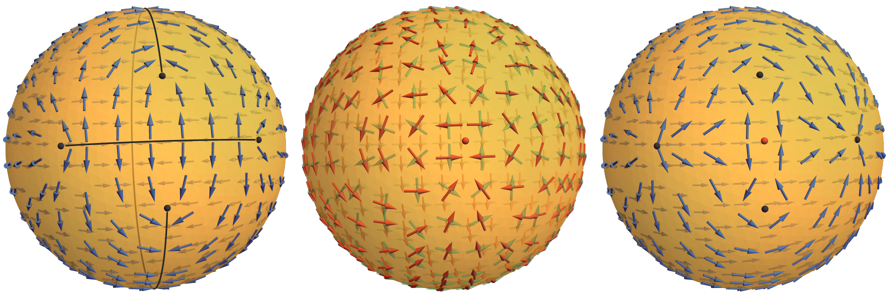


Figure 1.6: Doubling the angles on a sphere. On the left, one can see a tangent vector field u on the sphere with 4 fractional vortices, each of degree $\frac{1}{2}$. In the middle, we depicted the singular frame (vortex of degree 2 at a single point), with respect to which we double the angles of u. The resulting vector field v can be seen on the right. Note that not only all the fractional vortices of u became full ones, but also a new vortex (red dot) was created in this process, located exactly at the singular point of the frame and having degree equal to -2. This is in accordance with the Morse index formula, which enforces the net sum of the vortices to be equal to the Euler characteristic ($\chi(S) = 2$ in this setting). See (4.38) for the general relation between the vorticity of u, v, and e.

As a result, doubling the angles with respect to a singular field e with nonzero vorticity may create additional synthetic vortices (see also Figure 1.6). Hence, a doubling procedure is only available locally. The second problem stems from the nontrivial geometry of S. More precisely, given a tangent vector field $u \in \mathcal{AS}_\varepsilon$ with fractional vortices, a unit vector field $e \in C^\infty(TU)$, and $v\colon U \to TU$ being the resulting vector field after doubling the angles of u with respect to e, the analog of the estimate (1.17) in the present setting turns up to be more involved (see (4.36) for more details), namely

$$\int_U |\nabla v|^2 \operatorname{vol} \leq 4 \int_U |\nabla u|^2 \operatorname{vol} - 4 \int_U \langle \operatorname{jac}(u), \operatorname{jac}(e) \rangle \operatorname{vol} + \mathrm{O}_{\varepsilon \to 0}(1). \tag{1.18}$$

Here the 1-form $\operatorname{jac}(u)$ is called the pre-Jacobian of u defined as $\operatorname{jac}(e)(X) := \langle \nabla_X u, u^\perp \rangle$ for $X \in TM$, $u^\perp$ being the vector u rotated by $\frac{\pi}{2}$ in anticlockwise direction. It is not clear a priori if the second term in (1.18) might – compared to (1.17) – perturb the relation between the Dirichlet energies of v and u in a non-negligible fashion. But knowledge of the precise scaling of the Dirichlet energy of v is a crucial component in the proof in the nonfractional result in [43]. We will overcome this hurdle by making a "good" choice for e, more precisely, we choose a harmonic vector field in U, minimizing the Dirichlet energy:

$$\frac{1}{2} \int_U |\nabla e|^2 \operatorname{vol}.$$

For such a choice of e, we will be able to control the second term in (1.18) from below and above by a constant independent of ε.

Chapter 2

Generalized XY model

2.1 Preliminaries

2.1.1 Gamma-convergence

In this thesis we will investigate several physically motivated problems from the variational point of view. More precisely, we will investigate pairs $(E_{\varepsilon_0}, X_{\varepsilon_0})$ where $\varepsilon_0 > 0$ is a fixed parameter, X_{ε_0} a metric space, and $E_{\varepsilon_0}: X_{\varepsilon_0} \to \mathbb{R}$ an energy functional on $\mathbb{R}$. The correct choice for the parameter $\varepsilon_0 > 0$ will vary from problem to problem. Nevertheless, in all cases which are of interest to us, ε_0 will be a small positive scalar (e.g., the lattice spacing in a crystal). The following prototypical problems are the main interest here:

1) Determine the infimum of E_{ε_0} in X_{ε_0}, classify its minimizers.
2) Starting from a general initial datum $u_0 \in X_{\varepsilon_0}$, describe a motion of u_0 driven by the energy E_ε.

As we shall see later on, the first problem will be challenging to solve in the ε_0-dependent setting, at least. One way to approach this difficulty is to consider a family of models $\{(E_\varepsilon, I_\varepsilon)\}_{\varepsilon\in I}$ – instead of the fixed model $(E_{\varepsilon_0}, X_{\varepsilon_0})$ –, where $I := (0, \alpha)$ for some $\alpha > 0$, and pass in the limit $\varepsilon \to 0$ to an effective model (E, X), that may be easier to handle. Once a correct notion of convergence is chosen, one may aspire that the model described by $(E_{\varepsilon_0}, X_{\varepsilon_0})$ is "close" (in a sense that will be specified later on) for $0 < \varepsilon_0 << 1$ to the effective model (E, X).

It is an ardent wish to shortly comment on a minor technical issue. A definition of convergence for $\{(E_\varepsilon, X_\varepsilon)\}_\varepsilon$ could be greatly simplified if we were to remove the dependence of the metric spaces $\{X_\varepsilon\}_\varepsilon$ on ε from the get-go. In all cases that interest us, we shall thus be able to isometrically embed X_ε into a common metric space $\tilde{X}$. Without further mention, the functional E_ε is then implicitly assumed to be extended to $\tilde{E}_\varepsilon: \tilde{X} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, defined as

$$\tilde{E}_\varepsilon(u) = \begin{cases} E_\varepsilon(u) & \text{if } u \in X_\varepsilon, \\ \infty & \text{else.} \end{cases}$$

Note that this kind of extension is favorable for variational problems, seeing that it does not perturb the minimum of E_ε. From this point on we will identify E_ε with its extension $\tilde{E}_\varepsilon$, as well as X with $\tilde{X}$.

The notion of convergence for variational problems is the one of *Gamma-Convergence*, first introduced by De Giorgi in the 1970s. Its precise statement is as follows:

Definition 2.1 (Γ-convergence)
Given a metric space X, we say that a sequence of energy functionals $(E_n)_{n\in\mathbb{N}}$ with $E_n\colon X \to \bar{\mathbb{R}}$ Γ-converges towards a functional $E\colon X \to \bar{\mathbb{R}}$ as $n \to \infty$ if and only if for all $u \in X$ the following is satisfied:

(i) (Γ-liminf) For every sequence $(u_n) \subset X$ converging to u

$$E(u) \leq \liminf_{n\to\infty} E_n(u_n). \tag{2.1}$$

(ii) (Γ-limsup) There exists a sequence $(u_n) \subset X$ (also called recovery sequence for u) converging to u such that

$$E(u) \geq \limsup_{n\to\infty} E_n(u_n). \tag{2.2}$$

Remark 2.1. The conditions in (2.1) and (2.2) compete with each other, while depending on the choice of metric on X. On the one hand, if we weaken the metric on X, the Gamma-liminf in (2.1) must be tested against more convergent sequences and, therefore, is harder to satisfy. On the other hand, if we strengthen the metric on X, we have less sequences converging towards u, and won't be able to find a recovery sequence satisfying (2.2) anymore. Therefore, the correct choice of metric on X – keeping the balance between the condition in Item (i) and Item (ii) – turns up to be a crucial part of a Γ-convergence result and should be considered first. Finally, it is worth remarking that this definition can be generalized to the setting of topological spaces. For references, see [28].

The main justification for the notion of convergence introduced in Definition 2.1 is that – given a compact set $K \subset X$ – it relates for sufficiently small $\varepsilon > 0$ the infimum of E_ε in K to the minimum of E in K as well as (almost-)minimizers of $E_\varepsilon|_K$ to the ones of $E|_K$. The following condition allows for a generalization of this relation in the case of $K = X$:

Definition 2.2
A sequence (E_n) of energy functionals on metric space X is called equi-mildly coercive if and only if there exists a non-empty compact set $K \subset X$ so that $\inf_X E_n = \inf_K E_n$ for all $n \in \mathbb{N}$.

Then the precise statement of the relation between the relation could be phrased as follows:

Theorem 2.1
Let X be a metric space and (E_n) a sequence of equi-mildly coercive energy functionals on X Γ-converging towards E. Then E attains a minimum (even though each approximating functional might not attain one) and:

$$\min_X E = \lim_{n\to\infty} \inf_X E_n. \tag{2.3}$$

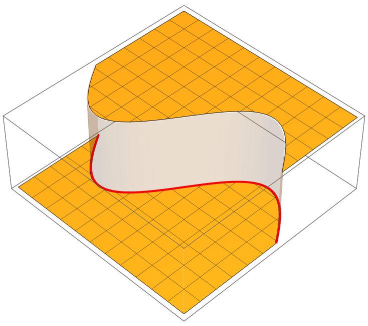
Moreover, if $(u_n) \subset X$ is a precompact sequence such that:

$$\lim_{n\to\infty} (E_n(u_n) - \inf_X E_n) = 0,$$

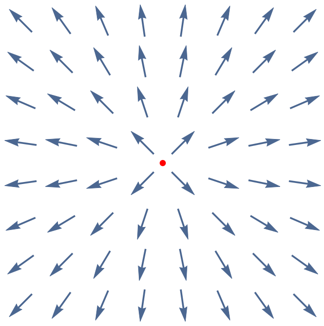
then every limit of a subsequence of (u_n) *is a minimizer of* E*.*

In order to prove that result we need to refer to Theorem 1.21 in [17].

2.1.2 Jump and vortex singularities



(a) Jump discontinuity



(b) Vortex discontinuity

Figure 2.1: Two examples of singular vector-fields. The red curve on the left and the red dot on the right depict the supports of the respective singularities.

Throughout this subsection, m, $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ will denote an open set. We will shortly discuss several important results concerning two different types of singularities of (vector-)valued maps $u\colon \Omega \to \mathbb{R}^m$. In Figure 2.1, an example for each of the two types of singular maps can be seen. Figure 2.1(a) shows the jump discontinuity of the indicator function $u(x) := \mathbb{1}_A(x)$ for some $A \subset \mathbb{R}^2$, while Figure 2.1(b) shows the vortex created by the vector-valued map $v(x) := \frac{x}{|x|}$. Even though, from the geometrical point of view, these two types of singularities are rather simple, their correct analytical description is quite challenging and remains an active area of research up to this day. The problem of main interest to us is the detection of the geometrically or topologically relevant singularities of vector-fields through analytical means (this will be made more precise in a moment). A more thorough presentation containing all relevant proofs can be found in [11] (for the first type of singularities) and in [1] or [21] (for the second).

Jump-type singularities: The starting point for the study of jump-type singularities is the definition of the *total variation*:

Definition 2.3
The total variation of a function $u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ in an open subset $U \subset \Omega$ is defined as

$$\operatorname{var}(u, U) := \sup\left\{ \int_U \langle u, \operatorname{div} \varphi \rangle \,\mathrm{d}x \colon \varphi \in C_c^\infty(\Omega; \mathbb{R}^{m\times d}),\ \|\varphi\|_\infty \le 1 \right\}, \tag{2.4}$$

where $\operatorname{div} \varphi\colon \Omega \to \mathbb{R}^m$ has entries

$$(\operatorname{div} \varphi)^j := \operatorname{div} \varphi^j := \sum_{i=1}^d \frac{\partial}{\partial x_i} \varphi^j, \quad j \in \{1, \dots, m\}.$$

In the case $U = \Omega$ the abbreviation $\mathrm{var}(u)$ – instead of $\mathrm{var}(u, \Omega)$ – will be used, calling the total variation of u.

Note that the total variation extends to a unique positive measure (still denoted by $\mathrm{var}(u, \cdot)$) on Ω. We shall also define:

Definition 2.4 (Functions of bounded variation)
The set $BV(\Omega; \mathbb{R}^m)$ of *functions of bounded variation* on Ω with values in $\mathbb{R}^m$ is defined as:

$$BV(\Omega; \mathbb{R}^m) := \big\{ u \in L^1(\Omega) \colon \ \mathrm{var}(u) < \infty \big\}. \tag{2.5}$$

Given $u \in BV(\Omega; \mathbb{R}^m)$ we defined its BV-norm as:

$$\|u\|_{BV} := \|u\|_{L^1} + \mathrm{var}(u).$$

Equipped with the norm above $BV(\Omega; \mathbb{R}^m)$ turns into a separable Banach space. As we will shortly see, the distributional derivative of a function $u \in BV(\Omega; \mathbb{R}^m)$ can be identified with a vector-valued *Radon measure*. Such measures are defined as follows:

Definition 2.5 (Radon measure)
Let $\mathcal{B}(\Omega)$ bet the set of Borel subsets of Ω. A function $\mu \colon \mathcal{B}(\Omega) \to \mathbb{R}^m$ is called a finite ($\mathbb{R}^m$-valued) Radon measure if and only if:

(i) μ is additive, this means that for all A_1, $A_2 \in \mathcal{B}(\Omega)$:

$$A_1 \cap A_2 = \emptyset \implies \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2);$$

(ii) μ is σ-additive, this means that for all $(A_n)_{h \in \mathbb{N}}$, it holds that:

$$\mu\left(\bigcup_{n=0}^{\infty} A_n \right) = \sum_{n=0}^{\infty} \mu(A_n).$$

The set of such measures will be denoted by $\mathcal{M}(\Omega; \mathbb{R}^m)$. In the case $m = 1$ the abbreviation $\mathcal{M}(\Omega)$ – instead of $\mathcal{M}(\Omega, \mathbb{R})$ – will be used. The set of positive scalar Radon measures will be called $\mathcal{M}_+(\Omega)$. Finally the total variation $|\mu|$ of a Radon measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ is defined for any $A \in \mathcal{B}(\Omega)$ as:

$$|\mu|(A) := \sup \left\{ \sum_{n=0}^{\infty} |\mu(E_n)| \colon (E_n) \text{ pairwise disjoint, } A = \bigcup_{n=0}^{\infty} A_n \right\}$$

Note that the total variation $|\mu|$ turns out to be a positive measure (see also Theorem 1.6 in [11]). The *distributional derivative* of ($\mathbb{R}^m$-valued) function of bounded variation can be identified with a ($\mathbb{R}^m$-valued) Radon measure. Or more precisely:

Theorem 2.2
For any $u \in BV(\Omega; \mathbb{R}^m)$ there exists a $\mathbb{R}^m$-valued Radon measure $Du \in \mathcal{M}(\Omega; \mathbb{R}^{m \times d})$ such that for any $\varphi \in C_c^\infty(\Omega; \mathbb{R}^{m \times d})$, it holds that:

$$\int_\Omega \langle u, -\operatorname{div} \varphi \rangle \,\mathrm{d}x = \int_\Omega \varphi \, dDu = \sum_{i=1}^{m} \sum_{j=1}^{d} \int_\Omega \varphi_i^j \, dD_i u^j,$$

where $D_i u^j \in \mathcal{M}(\Omega) := \mathcal{M}(\Omega;\mathbb{R})$ *the* i*-th component of* Du^j *or equivalently the component of* Du *in the* i*-th row and* j*-th column. Furthermore, the total variation measure* $\mathrm{var}(u,\cdot)$ *of* u *coincides in the sense of measures with the total variation* $|Du|$ *of* Du*, as defined in Definition 2.5.*

The identification in the theorem above allows us to define a notion of weak* convergence for a sequence of functions of bounded variation:

Definition 2.6 (weak* convergence)
A sequence $\{\mu_n\} \subset \mathcal{M}(\Omega;\mathbb{R}^k)$, $k \in \mathbb{N}$, is weakly*-convergent towards $\mu \in \mathcal{M}(\Omega;\mathbb{R}^k)$ ($\mu_n \overset{*}{\rightharpoonup} \mu$) if and only if for any $\varphi \in C_c(\Omega;\mathbb{R}^k)$

$$\lim_{h\to\infty}\int_\Omega \varphi\, dDu_n = \int_\Omega \varphi\, dDu.$$

Consequently a sequence $\{u_n\} \subset BV(\Omega;\mathbb{R}^m)$ is said to be weak*-convergent towards $u \in BV(\Omega;\mathbb{R}^m)$ ($u_n \overset{*}{\rightharpoonup} u$) if and only if $u_n \to u$ in $L^1(\Omega;\mathbb{R}^{m\times d})$ and $Du_n \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega;\mathbb{R}^{m\times d})$.

A special case of functions of bounded variation are those taking values in the set $\{0,1\}$ only.

Definition 2.7 (*Sets of finite (generalized) perimeter*)
For an $\mathcal{L}^d$-measurable set $A \subset \mathbb{R}^d$ we define its (generalized) perimeter in an open set $\Omega \subset \mathbb{R}^d$ as:

$$\mathcal{P}(A,\Omega) := \sup\bigg\{\int_A \operatorname{div}\varphi\,\mathrm{d}x \colon \varphi \in C_c^\infty(\Omega;\mathbb{R}^d),\ \|\varphi\|_\infty \le 1\bigg\}.$$

We say that A has finite (generalized) perimeter in the open set Ω if and only if $\mathcal{P}(A,\Omega) < \infty$.

Equivalently, a set $A \subset \mathbb{R}^d$ has finite perimeter in an open set $\Omega \subset \mathbb{R}^d$ if and only if the indicator function $\mathbb{1}_A \in BV(\Omega)$ equals a scalar function of bounded variation in Ω. Furthermore, we have that:

$$\mathcal{P}(A,\Omega) = \mathrm{var}(\mathbb{1}_A,\Omega).$$

We will now discuss the fine properties of BV-functions, which allow us to describe the distributional derivative of a BV-map in a more constructive fashion. Let us consider two motivating examples: First, let $u \in W^{1,1}(\Omega;\mathbb{R}^m)$. In this case one can show that $u \in BV(\Omega;\mathbb{R}^m)$ and that its distributional derivative Du is equal to $\nabla u\,\mathcal{L}^d$, where ∇u denotes the Sobolev derivative of u. Secondly, we consider $u := \mathbb{1}_A$ for an open set $A \subset \mathbb{R}^d$ with smooth boundary. We can show for any open set $\Omega \subset \mathbb{R}^d$ that $\mathbb{1}_A \in BV(\Omega)$ with a distributional derivative equal to $\nu_A\,\mathcal{H}^{d-1}|_{\partial A\cap\Omega}$, $\nu_A \colon \partial A \to \mathbb{R}^d$ denoting the inward-pointing unit normal-field and $\mathcal{H}^{d-1}$ the $(n-1)$-dimensional Hausdorff-measure in $\mathbb{R}^d$. In other words for any $\varphi \in C_c^\infty(\Omega;\mathbb{R}^d)$, we have:

$$\int_A -\operatorname{div}\varphi\,\mathrm{d}x = \int_{\Omega\cap\partial A}\langle\varphi,\nu_A\rangle\,\mathrm{d}\mathcal{H}^{d-1}.$$

This shows in particular how we can recover the singular set of the function u on the left of Figure 2.1, by investigating the fine properties of its distributional derivative. In the following we will show how to generalize the procedure in the two examples above to the general case. The starting point lies in the definition of *blow-up quantities*:

Definition 2.8 (Approximate continuity point)
Given a function $u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ we say that u has an *approximate limit* $z \in \mathbb{R}^d$ at $x \in \Omega$ if and only if:

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - z| \, dy = 0.$$

We call $x \in \Omega$ an *approximate continuity point* if u has the approximate limit $z = u(x)$ at x. A point in which u does not attain an approximate limit is called an *approximate discontinuity point* of u.

The set of approximate discontinuity points of u will be denoted by S_u. The approximate limit of u at x, if exists, is unique and will be denoted by $\tilde{u}(x)$. One can show (see also Proposition 3.64 in [11]) that S_u is an $\mathcal{L}^d$-negligible ($|\Omega \setminus S_u| = 0$) Borel set and the function $\tilde{u} \colon \Omega \setminus S_u \to \mathbb{R}^m$ is a Borel function coinciding $\mathcal{L}^d$-a.e. with u. With this notation $x \in \Omega$ is an approximate continuity point of u if and only if $x \notin S_u$ and $\tilde{u}(x) = u(x)$.

Definition 2.9 (Approximate jump point)
Given $u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ we say that a point $x \in S_u$ is an *approximate jump point* of u if and only if there exists $a, b \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{d-1}$ such that:

$$\begin{aligned} &\lim_{r \to 0} \frac{1}{|B_r^+(x,\nu)|} \int_{B_r^+(x,\nu)} |u(y) - a| \, dy \\ &\quad = \lim_{r \to 0} \frac{1}{|B_r^-(x,\nu)|} \int_{B_r^-(x,\nu)} |u(y) - b| \, dy = 0, \end{aligned}$$

where

$$\begin{aligned} B_r^+(x,\nu) &\coloneqq \{y \in B_r(x) \colon \langle y - x, \nu \rangle > 0\}, \\ B_r^-(x,\nu) &\coloneqq \{y \in B_r(x) \colon \langle y - x, \nu \rangle < 0\}. \end{aligned}$$

The set of approximate jump points of u will be denoted by J_u. The triple (a, b, ν) in the definition above is unique up to a permutation of (a, b) and a change of sign of ν. Implicitly assuming this equivalence, we will write the triple as $(u^+(x), u^-(x), \nu_u(x))$. One can show (see Proposition 3.69 in [11]) that J_u is a Borel subset of S_u and $u^+ \colon J_u \to \mathbb{R}^m$, $u^- \colon J_u \to \mathbb{R}^m$ and $\nu_u \colon J_u \to \mathbb{S}^{d-1}$ are Borel functions.

Definition 2.10 (Approximate differentiability point)
Given $u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ we say that $x \in \Omega \setminus S_u$ is an *approximate differentiability point* of u if and only if there exists a matrix $L \in \mathbb{R}^{m \times d}$ such that:

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} \frac{1}{r} |u(y) - \tilde{u}(x) - L(y - x)| \, dy = 0,$$

where $\tilde{u}(x)$ is the approximate limit of u at x.

The set of approximate differentiability points of u will be written as D_u. The matrix L from the definition above is unique and is therefore usually written as $\nabla u(x)$. We can show (see Proposition 3.71 in [11]) that D_u is a Borel set and $\nabla u\colon D_u \to \mathbb{R}^{m\times d}$ is a Borel function. We shall continue by discussing the *Lebesgue differentiation* of the distributional derivative Du of a BV-map u which decomposes Du into an absolutely continuous (with respect to $\mathcal{L}^d$) and singular part. In the general setting of Radon measures these two notions are defined as follows:

Definition 2.11 (Absoutely continuous & singular)
Let $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ and $\lambda \in \mathcal{M}_+(\Omega)$. We say μ is *absolutely continuous* with respect to λ ($\mu << \lambda$) if and only if for any Borel set $A \in \mathcal{B}(\Omega)$ with $\lambda(A) = 0$ it holds that $|\mu|(A) = 0$. We say μ is *singular* with respect to λ ($\mu \perp \lambda$) if and only if there exists a Borel set $A \in \mathcal{B}(\Omega)$ such that $|\mu|(A) = \lambda(\Omega \setminus A) = 0$.

The statement of the Lebesgue differentiation is then as follows:

Theorem 2.3 (Lebesgue differentiation)
For any $u \in BV(\Omega;\mathbb{R}^m)$ there exist unique Radon measures $D^a u$ and $D^s u \in \mathcal{M}(\Omega;\mathbb{R}^m)$ such that $D^a u$ is absolutely continuous with respect to $\mathcal{L}^d$, $D^s u$ is singular with respect to $\mathcal{L}^d$ and $Du = D^a u + D^s u$.

Lastly, we will discuss a notion of regularity for $\mathcal{H}^{d-1}$-dimensional subsets of $\mathbb{R}^d$:

Definition 2.12 ($\mathcal{H}^{d-1}$-rectifiabile set)
$\Gamma \subset \mathbb{R}^d$ is called a $(n-1)$-dimensional C^1-graph if and only if there exists a $(n-1)$-dimensional plane π and a C^1-map $f\colon \pi \to \pi^\perp$ (,where $\pi^\perp$ is the line through the origin orthogonal to π) such that:

$$\Gamma = \{x + f(x)\colon x \in \pi\}.$$

We call an $\mathcal{H}^{d-1}$-dimensional set $E \subset \mathbb{R}^d$ is $\mathcal{H}^{d-1}$*-rectifiable* if and only if there exist countably many $(n-1)$-dimensional Lipschitz-graphs $\{\Gamma_n\}_{h\in\mathbb{N}}$ such that:

$$\mathcal{H}^{d-1}\left(E \setminus \bigcup_{n=0}^{\infty} \Gamma_n\right) = 0.$$

We are ready to state the decomposition theorem in BV:

Theorem 2.4 (Decomposition in BV)
A function $u \in BV(\Omega;\mathbb{R}^d)$ is approximately differentiable at $\mathcal{L}^d$-a.e. point ($\mathcal{L}^d(\Omega \setminus D_u) = 0$) and the absolutely continuous part $D^a u$ of Du can be written as:

$$D^a u = \nabla u\, \mathcal{L}^d,$$

where ∇u is the approximate gradient of u. Furthermore, the set of discontinuity points S_u is $\mathcal{H}^{d-1}$-rectifiable, $\mathcal{H}^d(S_u \setminus J_u) = 0$ and the jump part $D^j u := D^s u|_{S_u}$ of u satisfies:

$$D^j u = (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{d-1}|_{J_u},$$

where $\otimes$ denotes the tensor product and therefore $(u^+ - u^-)\otimes\nu_u = (u^+ - u^-)\cdot\nu_u^T$ in matrix notation. Combining both results leads to the following decomposition:

$$Du = \nabla u\, \mathcal{L}^d + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{d-1}|_{J_u} + D^c u, \tag{2.6}$$

where $D^c u := D^s u|_{\Omega\setminus S_u}$ is the so called Cantor-part of u.

The chain rule in the setting of Sobolev functions generalizes in the following way to the BV setting:

Theorem 2.5 (Chain rule in BV)
Let $u \in BV(\Omega;\mathbb{R}^{d_2})$ and $f \in C^1(\mathbb{R}^{d_2};\mathbb{R}^{d_3})$, where $\Omega \subset \mathbb{R}^{d_1}$ is an open and bounded set and d_1, d_2, $d_3 \in \mathbb{N}$. Then $v := f \circ u \in BV(\Omega;\mathbb{R}^{d_3})$ with:

$$\begin{cases} D^a v = \nabla f(u)\nabla u\,\mathcal{L}^d, \\ D^j v = (f(u^+) - f(u^-)) \otimes \nu_u\,\mathcal{H}^{d-1}|_{J_u}, \\ D^c v = \nabla f(\tilde{u})\,D^c u, \end{cases}$$

where $\tilde{u}(x)$ is equal to the approximate limit of u for any $x \in \Omega \setminus S_u \supset \mathrm{spt}(D^c u)$.

For a proof, see Theorem 3.78 and Theorem 3.83 in [11]. Another important tool is the notion of traces:

Theorem 2.6 (Traces in BV)
Let $\Omega \subset \mathbb{R}^d$ be an open set with Lipschitz boundary and $u \in BV(\Omega;\mathbb{R}^m)$. Then for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ there exists $\mathrm{Tr}_{\partial\Omega}(x) \in \mathbb{R}^m$ such that:

$$\lim_{r\to 0} r^{-n} \int_{\Omega\cap B_r(x)} |u(y) - u^{\Omega}(x)|\,dy = 0.$$

Moreover, the trace is continuous with respect to strict convergence in BV, this means that if $u_n \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and $|Du_n|(\Omega) \to |Du|(\Omega)$ we have that:

$$\mathrm{Tr}_{\partial\Omega}(u_n) \to \mathrm{Tr}_{\partial\Omega}(u) \text{ in } L^1(\partial\Omega;\mathbb{R}^m).$$

For a proof we refer to Theorem 3.87 in [11]. Finally, we mention the following lemma:

Lemma 2.1 (Cut and paste)
Let u, $v \in BV(\Omega;\mathbb{R}^m)$ and $A \subset \Omega$ a set of finite perimeter with $\partial^* A$ oriented by the inward-pointing normal ν_A. Let $u^+_{\partial^* A}$ and $u^-_{\partial^* A}$ be the BV-traces on $\partial^* A$ of u and v, respectively. Then $w := u\mathbb{1}_A + v\mathbb{1}_{\Omega\setminus A} \in BV(\Omega;\mathbb{R}^m)$ if and only if:

$$\int_{\partial^* A\cap\Omega} |u^+_{\partial^* A} - v^-_{\partial^* A}|\,\mathrm{d}\mathcal{H}^{d-1} < \infty.$$

Also, if $w \in BV(\Omega;\mathbb{R}^m)$ we can decompose Dw as follows:

$$Dw = Du|_{A^1} + Dv|_{A^0} + (u^+_{\partial^* A} - u^-_{\partial^* A}) \otimes \nu_A\,\mathcal{H}^{d-1}|_{\partial^* A\cap\Omega},$$

where A^1 (A^0) is the measure-theoretic interior (exterior) of A, defined as:

$$A^1 := \left\{x \in \mathbb{R}^d \colon \lim_{r\to 0} \frac{|B_r(x)\cap\Omega|}{|B_r(x)|} = 1\right\},$$
$$A^0 := \left\{x \in \mathbb{R}^d \colon \lim_{r\to 0} \frac{|B_r(x)\cap\Omega|}{|B_r(x)|} = 0\right\}.$$

While the blow-up quantities defined above characterize the absolutely continuous as well as the jump part of a BV function, they provide no information on the Cantor part. This motivates the definition of the set of special functions of bounded variation.

Definition 2.13
We define the space of *special ($\mathbb{R}^m$-valued) functions of bounded variation* $SBV(\Omega;\mathbb{R}^m)$ as:

$$SBV(\Omega;\mathbb{R}^m) \coloneqq \{u \in BV(\Omega;\mathbb{R}^m) \colon D^c u = 0\}.$$

Furthermore, for any $p \in (1,\infty]$ we set:

$$SBV^p(\Omega;\mathbb{R}^m) \coloneqq \{u \in SBV(\Omega;\mathbb{R}^m) \colon \nabla u \in L^p(\Omega;\mathbb{R}^{m\times d}),\ \mathcal{H}^{d-1}(J_u) < \infty\}.$$

Note that the space $SBV(\Omega;\mathbb{R}^m)$ is a closed in $BV(\Omega;\mathbb{R}^m)$ with respect to the strong but not the weak* convergence. In fact, for each $u \in BV(\Omega;\mathbb{R}^m)$ we can find a sequence $\{u_n\} \subset C^\infty(\Omega;\mathbb{R}^m) \subset SBV(\Omega;\mathbb{R}^m)$ weakly converging towards u (see also Theorem 3.9 in [11]). Therefore, any compactness or closure statement in SBV with respect to weak* convergence must assure that no Cantor-part can be created in the limit:

Theorem 2.7 (Compactness in SBV^p)
Let $\{u_n\} \subset SBV^p(\Omega;\mathbb{R}^m)$ for some $p \in (1,\infty]$ such that:

$$\sup_n \left(\int_\Omega |\nabla u_n|^p \, d\mathcal{L}^d + \mathcal{H}^{d-1}(J_{u_n}) + \|u_n\|_\infty \right) < \infty. \tag{2.7}$$

Then there exists $u \in SBV^p(\Omega;\mathbb{R}^m)$ such that, up to a subsequence,

$$\begin{cases} u_n \to u \ \text{in } L^1(\Omega;\mathbb{R}^m), \\ \nabla u_n \rightharpoonup u \ \text{in } L^p(\Omega;\mathbb{R}^{m\times d}), \\ \mathcal{H}^{d-1}|_{J_{u_n}} \overset{*}{\rightharpoonup} \mathcal{H}^{d-1}|_{J_u} \ \text{in } \mathcal{M}(\Omega;\mathbb{R}^{m\times d}). \end{cases} \tag{2.8}$$

Definition 2.14 (Weak convergence in SBV^p)
A sequence $\{u_n\} \subset SBV^p(\Omega;\mathbb{R}^m)$ is said to be weakly convergent towards u in $SBV^p(\Omega;\mathbb{R}^m)$ ($u_n \rightharpoonup u$) if and only if $u_n \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and (2.7) is satisfied.

The next theorem identifies a class of integral functionals in SBV^p that are lower-semicontinuous with respect to weak convergence:

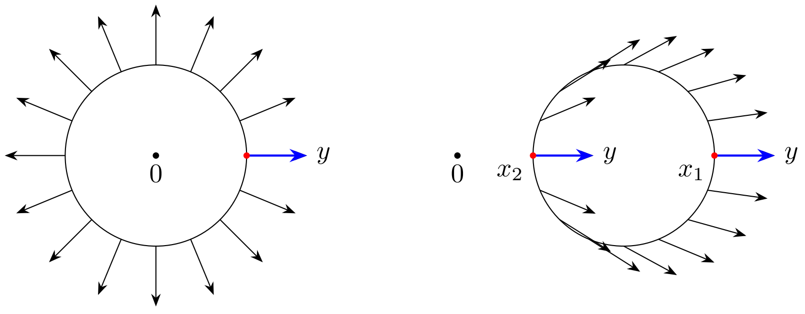
Theorem 2.8 (Lower semicontinuity)
Let $\vartheta \in C(K \times K; [c_0,\infty))$, where $K \subset \mathbb{R}^m$ is a compact set and $c_0 > 0$, be a positive, symmetric function such that for all a, b, $c \in K$:

$$\vartheta(a,c) \le \vartheta(a,b) + \vartheta(b,c).$$

Moreover, let $\varphi \in C(\mathbb{R}^d;\mathbb{R}_+)$ be an even, convex, and positively 1-homogeneous ($\varphi(\lambda x) = \lambda\varphi(x)$) for all $x \in \mathbb{R}^d$ and $\lambda > 0$) such that $\varphi(\nu) \ge c_0$ for all $\nu \in S^{d-1}$. Then for any sequence $\{u_n\} \subset SBV^p(\Omega;\mathbb{R}^m)$ weakly convergent towards $u \in SBV^p(\Omega;\mathbb{R}^m)$ for some $p \in (1,\infty]$ we have

$$\liminf_{h\to\infty} \int_{J_{u_n}} \vartheta(u_n^+, u_n^-)\, \varphi(\nu_{u_n})\, \mathrm{d}\mathcal{H}^{d-1} \ge \int_{J_u} \vartheta(u^+, u^-)\, \varphi(\nu_u)\, \mathrm{d}\mathcal{H}^{d-1}. \tag{2.9}$$

For a proof we refer to Theorem 5.22 and Example 5.23 in [11]. Lastly, we mention an approximation result for SBV-functions taking values in a discrete set:



(a) Case: $0 \in U$. (b) Case: $0 \notin U$.

Figure 2.2: The Brouwer degree of $u(x) \coloneqq \frac{x}{|x|}$, $x \in \mathbb{R}^2$, around two circles.

Theorem 2.9 (Approximation of finite Caccioppoli partitions)
Let $\mathcal{Z} \subset \mathbb{R}^m$ be finite, $\Omega \subset \mathbb{R}^d$ open and with Lipschitz boundary, and let $u \in SBV(\Omega; \mathcal{Z})$. Then there exists a sequence $\{u_n\} \subset SBV(\Omega; \mathcal{Z})$ such that J_{u_n} is polyhedral, $u_n \to u$ in $L^1(\Omega; \mathbb{R}^m)$, and:

$$\lim_{h\to\infty} \int_{J_{u_n}} \psi(u_n^+, u_n^-, \nu_{u_n}) \,\mathrm{d}\mathcal{H}^{d-1} = \int_{J_u} \psi(u^+, u^-, \nu_u) \,\mathrm{d}\mathcal{H}^{d-1} \tag{2.10}$$

for any continuous function $\psi \colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{d-1} \to \mathbb{R}_+$ (,where $\mathbb{R}_+ \coloneqq (0, \infty)$) satisfying $\psi(a, b, \nu) = \psi(b, a, -\nu)$ for all $(a, b, \nu) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{d-1}$. In particular, we have that $u_n \to u$ strictly in BV.

For a proof we refer to Theorem 2.1 and Corollary 2.4 in [19].

Vortex-type singularities Firstly, we introduce the classic notion of (see section 2.10 in [1]):

Definition 2.15 (Brouwer degree)
Let M and M' be m-dimensional, oriented, compact manifolds (without boundary) such that M' is connected, let $u \colon M \to M'$ be a smooth map and let $y \in u(M)$ be an arbitrary regular value of u (i.e., $Du(x) \neq 0$ for all $x \in u^{-1}(y)$.) Then the degree $\deg(u, M, M')$ of u is defined as:

$$\deg(u, M, M') \coloneqq \sum_{x \in u^{-1}(y)} \operatorname{sgn}(\det Du(x)). \tag{2.11}$$

Remark 2.2. The definition of degree in (2.11) is well defined due to the following reasoning: As y above is taken to be a regular value of u, the set $u^{-1}(y)$ turns out to be discrete. By the continuity of f it is also compact and hence finite. Thus the sum in (2.11) is, in fact, finite. Furthermore, by the connectedness of M' one can show that the degree does not depend on the choice of the regular value y.

Intuitively the Brouwer degree counts the "number of times" $u(M)$ covers M' where we also have to take into account the orientation of $u(M)$. We consider an illuminating example: Let $u(x) \coloneqq \frac{x}{|x|}$ for $x \in \mathbb{R}^2 \setminus \{0\}$ and let $U \subset \mathbb{R}^2$ be an open, connected set with smooth boundary such that $0 \notin \partial U$.

We are interested in the Brouwer degree of the map $u\colon \partial U \to \mathbb{S}^1$ (see also figure Figure 2.2). In the case $0 \in U$ we have that for any $y \in \mathbb{S}^{d-1}$ there exists a unique $x \in \partial U$ such that $u(x) = y$. As u is orientation preserving on ∂U, this leads to – combined with (2.11) – $\deg(u, \partial U, \mathbb{S}^{d-1}) = 1$. In the case $0 \notin U$ we can find for any regular value y of $u|_{\partial U}$ exactly 2 points x_1 and x_2 such that $u(x_1) = u(x_2) = y$. Moreover, u is orientation-preserving on one point while being orientation-reversing on the other. This leads to $\deg(u, \partial U, \mathbb{S}^{d-1}) = 0$. And it also coincides with the idea that $u(\partial U)$ covers $\mathbb{S}^1$ exactly one time in the first case and 0 times in the second.

It can further be shown that the Brouwer degree is invariant under uniform convergence. With that being said, we can extend this very notion of degree to continuous maps $u\colon M \to M'$. Furthermore, the following integral formula holds true for any $u \in C^\infty(M, M')$:

$$\deg(u, M, M') = \frac{1}{\mathrm{vol}'(M')} \int_M \det(Du)\,\mathrm{dvol}, \tag{2.12}$$

where vol and vol$'$ are the volume forms on M and M', respectively. (Surprisingly enough, the integral on the right-hand side of (2.12) is invariant under change of Riemannian metric of M or M', respectively.) Hence, by approximation and the formula in (2.12), we can then extend the notion of degree to the case of Sobolev maps $u \in W^{1,m}(M, M')$. Note that the L^m-integrability of Du is optimal: For a map $u \in W^{1,p}(M, M')$ with $1 \le p < m$ the integrand in (2.12) may not be in L^1 – so we could possibly not find an approximating sequence of smooth maps. For further details, see also [21].

Next, we would like to introduce the basics needed in order to study vortex-type singularities of maps $u \in W^{1,1}(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$, where $\Omega \subset \mathbb{R}^2$ is an open set. (Note that we could deal more generally with maps $u \in W^{1,1}(\Omega;\mathbb{R}^k) \cap L^\infty(\Omega, \mathbb{R}^k)$, where $\Omega \subset \mathbb{R}^d$ is an open set and $k \le n$. For readers who are more interested in general theory, see also the discussion found in [1].) Due to the aforementioned remark we cannot define the degree in this setting by approximation alone. Here we shall take the distributional approach similar to the one used in the study of functions of bounded variation instead. The object allowing us to detect singularities is the (signed) Jacobian Jac(u), which is pointwise defined as follows:

$$\mathrm{Jac}(u) := \det(Du) = \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1},$$

where u_1 and u_2 are the components of u. On first glance, this does not seem to be a desirable choice: For $u(x) := \frac{x}{|x|}$ on some bounded domain Ω containing the origin we saw that u has a vortex at 0. But as u maps $\Omega \subset \mathbb{R}^2$ into the lower-dimensional set $\mathbb{S}^1$ we have by the area formula that $\mathrm{Jac}\, u = 0$ for all $x \neq 0$. (The very same thing can be observed by direct computation.) Consequently, the pointwise Jacobian is "blind" to the vortex singularity of u. Thus, we are motivated to take on a distributional point of view, as in the theory of BV-functions. In this regard, we will first define the pre-Jacobian of $u \in C^\infty(\Omega, \mathbb{R}^2)$ as:

$$\begin{aligned}\mathrm{jac}(u) &:= \langle \nabla u, u^\perp \rangle = (\langle \nabla_{x_1} u, u^\perp\rangle, \langle \nabla_{x_2} u, u^\perp\rangle)^T \\ &= \left(u_2\frac{\partial u_1}{\partial x_1} - u_1\frac{\partial u_2}{\partial x_1}, u_2\frac{\partial u_1}{\partial x_2} - u_1\frac{\partial u_2}{\partial x_2}\right)^T,\end{aligned}$$

where $u^\perp := (u_2, -u_1)^T$. It follows from Schwartz's theorem that the pointwise Jacobian can be rewritten in the following way:

$$\begin{aligned}\operatorname{div}(\operatorname{jac}(u)^\perp) &= \frac{\partial}{\partial x_1}\left(-u_2\frac{\partial u_1}{\partial x_2} + u_1\frac{\partial u_2}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(u_2\frac{\partial u_1}{\partial x_1} - u_1\frac{\partial u_2}{\partial x_1}\right)\\ &= -\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\\ &= 2\det(Du) = 2\operatorname{Jac}(u).\end{aligned}$$

Now, if $u \in L^\infty \cap W^{1,1}$, we still have that $\operatorname{jac}(u) \in L^1$. As a result, the divergence above can be taken in the distributional sense. From this point on we implicitly assume that $\operatorname{Jac}(u)$ is defined in this sense. We conclude this section with the following relation between the Brouwer degree and the distributional Jacobian:

Proposition 2.1
Let $\Omega \subset \mathbb{R}^2$ be an open set, $S := \{x_k\}_{k=1}^K$ for some $K \in \mathbb{N}_+$, and $u \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus S; \mathbb{S}^1)$. Then the distributional Jacobian $\operatorname{Jac}(u)$ *is a measure supported in S, and:*

$$\operatorname{Jac}(u) = \pi \sum_{k=1}^{K} \deg(u, \partial B_r(x_k), \mathbb{S}^1)\, \delta_{x_i},$$

where $r > 0$ is chosen sufficiently small so that the balls $\{B_r(x_k)\}_k$ are disjoint.

2.2 Previous work

2.2.1 Binary discrete spin system

Binary spin systems were first investigated from a variational point of view in [3], based on which we want to present the relevant results for this thesis. Such spin systems can be shortly described as follows: Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset with Lipschitz regular boundary. Given $\varepsilon > 0$ we define:

$$\Omega_\varepsilon^{(0)} := \varepsilon\mathbb{Z}^2 \cap \Omega, \tag{2.13}$$

$$\Omega_\varepsilon^{(1)} := \{(i,j) \colon i, j \in \Omega_\varepsilon^{(0)},\ |i-j| = \varepsilon\}. \tag{2.14}$$

(We shall retain this notation throughout the entire chapter.) A pair $(i,j) \in \Omega_\varepsilon^{(0)}$ is called a *nearest neighbor pair*. A binary spin system on a square grid in Ω with grid spacing $\varepsilon > 0$ is a map $u \colon \Omega_\varepsilon^{(0)} \to \{-1, 1\}$ (1 and -1 encoding a binary choice for the spin.) The set of all such spin configurations will be denoted by $\mathcal{AS}_\varepsilon$.

The authors of [3] take into account several different choices of energies defined on $\mathcal{AS}_\varepsilon$ in the vanishing ε-limit through a Γ-convergence analysis. However, here we will only focus on the so called *ferro-magnetic* interaction with the energy $E_\varepsilon \colon \mathcal{AS}_\varepsilon \to \mathbb{R}$ given by:

$$E_\varepsilon(u) := -\frac{1}{2} \sum_{(i,j)\in\Omega_\varepsilon^{(1)}} \varepsilon^2 u(i)u(j), \tag{2.15}$$

where the sum above is over all nearest neighbor pairs $(i,j) \in \Omega_\varepsilon^{(1)}$ without repetition. The energy in (2.15) prefers the spins on a nearest neighbor pair to coincide (ferro-magnetic case). The factor ε^2 assures that E_ε does not blow up as $\varepsilon \to 0$. (Note that the number of grid points lying in Ω, this means $\#\Omega_\varepsilon^{(0)}$, scales like ε^{-2} as $\varepsilon \to 0$.) As already mentioned in the introduction to Γ-convergence, we first need to embed the sets $\mathcal{AS}_\varepsilon$ into a common topological space. In the limit $\varepsilon \to 0$ we expect the discrete spin configurations to accumulate at continuum spin fields, defined on the whole set Ω. Hence, we must identify each $u \in \mathcal{AS}_\varepsilon$ with a map $\tilde{u}\colon \Omega \to \mathbb{R}$, defined on the whole Ω. In the present case this is done through a *piecewise constant interpolation*

$$\tilde{u}(x) := u(i) \text{ for all } x \in i + \varepsilon Q,\, i \in \Omega_\varepsilon^{(0)},$$

where $Q := [0,1)^2$. In [3] it is shown that $L^\infty(\Omega)$ equipped with the weak*-topology is the correct embedding space. We extend E_ε from (2.15) into $L^\infty(\Omega)$ by setting it to be ∞ for any $u \in L^\infty(\Omega)$ that is not a piecewise constant interpolation of a discrete spin field in $\mathcal{AS}_\varepsilon$. The following Γ-convergence result holds true (see also Theorem 3.1 in [3]):

Theorem 2.10 (Zero order Γ-convergence, binary spins)
The functional $E_\varepsilon\colon L^\infty(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ as described above Γ-converges with respect to the weak-topology of $L^\infty(\Omega)$ towards the functional $E\colon L^\infty(\Omega) \to \bar{\mathbb{R}}$, defined as:*

$$E(u) := \begin{cases} -4|\Omega| & \textit{if } u \in L^\infty(\Omega; [-1,1]), \\ \infty & \textit{otherwise.} \end{cases}$$

Intuitively speaking, this result shows that a sequence (u_ε) can arbitrarily mix the uniform states -1 and 1, respectively, at a mesoscopic scale and with a negligible variation of the minimal energy as $\varepsilon \to 0$. In order to gain more insights into the model, we will now attempt a Γ-convergence analysis at a next order where we won't keep track of the energy but its perturbation from the minimal value instead, which is restricted to lie in close range $\delta_\varepsilon = \mathrm{o}(1)$ as $\varepsilon \to 0$. More precisely, we consider sequences (u_ε) satisfying:

$$E(u_\varepsilon) = \min E_\varepsilon + \mathrm{O}(\delta_\varepsilon).$$

The minimum of E_ε is achieved when all spins are either $+1$ or -1, respectively, hence:

$$\min E_\varepsilon = -\varepsilon^2 \#\Omega_\varepsilon^{(1)}.$$

In [3] the authors choose $\delta_\varepsilon = \varepsilon$, and we are thus led to study the energy functional $E_\varepsilon^{(1)}\colon \mathcal{AS}_\varepsilon \to \mathbb{R}$:

$$E_\varepsilon^{(1)}(u) = \frac{E_\varepsilon - \min E_\varepsilon}{\varepsilon} = \sum_{\langle i,j \rangle} \varepsilon(1 - u(i)u(j)),$$

which is assumed to be extended to $L^\infty(\Omega)$ in the same way as E_ε. The following Γ-convergence result holds true (see also Theorem 4.1 in [3]):

Theorem 2.11 (First order Γ-convergence, binary spins)
The functionals $E_\varepsilon^{(1)}\colon L^\infty(\Omega) \to \bar{\mathbb{R}}$ Γ-converge with respect to the strong topology of $L^1(\Omega)$ towards the functional $E^{(1)}\colon L^\infty(\Omega) \to \bar{\mathbb{R}}$, defined as:

$$E^{(1)} := \begin{cases} 4\int_{J_u} |\nu_u| \,\mathrm{d}\mathcal{H}^{d-1} & u \in SBV(\Omega; \{-1,1\}), \\ \infty & \textit{otherwise,} \end{cases}$$

where J_u is the jump set of u, ν_u the approximate normal and $|\cdot|_1$ denotes the l^1-norm in $\mathbb{R}^n$.

In contrast to the zero-order Γ-convergence result, the spins now accumulate in domains $\{u = 1\}$ and $\{u = -1\}$, and the limit energy penalizes the l^1-perimeter (also called *crystalline perimeter*) of the interface between them. As a further illustration, consider $c \in (-1, 1)$ and let $\{c_\varepsilon\} \subset \mathbb{R}_+$ be sequence converging towards c as $\varepsilon \to 0$, and such that $c_\varepsilon \cdot \#\Omega_\varepsilon^{(0)} \in \mathbb{N}$ for all $\varepsilon > 0$. Applying the aforementioned result, any sequence $(u_\varepsilon) \subset L^\infty(\Omega)$, where u_ε is a piecewise constant interpolation of an element of:

$$\operatorname{argmin}\{E_\varepsilon^{(1)}(u) \colon E_\varepsilon(u) = c_\varepsilon\}$$

converges strongly in L^1 towards a nonconstant $u \in SBV(\Omega, \{-1, 1\})$ satisfying $\int_\Omega u \, \mathrm{d}x = c$ while minimizing the crystalline perimeter of the interface J_u between the two phases.

2.2.2 XY spin system

The respective proofs as well as a more thorough discussion of the results presented in this subsection can be found in [4] and [6], respectively. Similar to the previous section we will consider spin configurations living on a rectangular grid (with spacing $\varepsilon > 0$) contained in an open bounded set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary. The crucial difference in the present case of the *XY model* is that the spins take values in the continuum $\mathbb{S}^1 \subset \mathbb{R}^2$ instead of the discrete set $\{-1, 1\}$. More precisely, we consider the following set of admissible spin fields:

$$\mathcal{AS}_\varepsilon := \{u \colon \Omega_\varepsilon^{(0)} \to \mathbb{S}^1\}.$$

The extension of the energy in (2.15) to the XY setting is as follows:

$$E_\varepsilon(u) := -\frac{1}{2} \sum_{(i,j) \in \Omega_\varepsilon^{(1)}} \varepsilon^2 \langle u(i), u(j) \rangle, \quad u \in \mathcal{AS}_\varepsilon, \tag{2.16}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^2$. As before, we identify each $u \in \mathcal{AS}_\varepsilon$ with its piecewise constant interpolation, extending E_ε to $L^\infty(\Omega, \mathbb{R}^2)$ by ∞ through this identification. The zero-order Γ-convergence result coincides with the corresponding one for discrete spin systems:

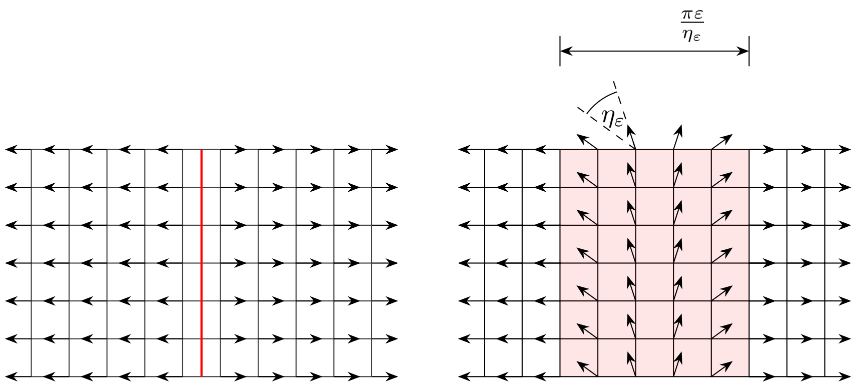
Theorem 2.12 (Zero order Γ-convergence, xy)
The functional $E_\varepsilon \colon L^\infty(\Omega, \mathbb{R}^2) \to \bar{\mathbb{R}}$ as defined above Γ-converges in the weak-topology of $L^\infty(\Omega; \mathbb{R}^2)$ towards $E \colon L^\infty(\Omega; \mathbb{R}^2) \to \bar{\mathbb{R}}$, defined as:*

$$E(u) := \begin{cases} -4|\Omega| & \text{if } u \in L^\infty(\Omega; B), \\ \infty & \text{otherwise}, \end{cases}$$

where B denotes the closed unit-ball in $\mathbb{R}^2$.

Motivated by the similarity to the binary spin setting, we might attempt to prove a higher-order Γ-convergence result by investigating the following energy functional:

$$E_\varepsilon^{(1)} := \frac{E_\varepsilon - \varepsilon^2 \#\Omega_\varepsilon^{(1)}}{\delta_\varepsilon}, \quad \delta_\varepsilon = \varepsilon. \tag{2.17}$$



(a) sharp interface (b) diffuse interface

Figure 2.3: Sharp vs. diffuse interface in the XY model.

Interestingly enough, the above scaling does *not* give rise to interfacial surface energies in the limit. (This fact holds true, even when $\delta_\varepsilon = \omega(\varepsilon^2)$ as $\varepsilon \to 0$ (this means $\lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{\varepsilon^2} = 0$).) In [4] it is shown that instead of developing in the limit $\varepsilon \to 0$ a sharp interface between some vectors $a, b \in \mathbb{S}^1$ it is more energy efficient to "continuously transition" at a certain rate η_ε from a to b (see also Figure 2.3 and Example 1 in [4] for further clarification.) Motivated by this example, the authors of [4] investigate the scaling regime $\delta_\varepsilon = \varepsilon^2|\log\varepsilon|$ and prove the emergence of *vortices* instead of interfaces as $\varepsilon \to 0$. In this case the rescaled energy functional $E_\varepsilon^{(1)}$ is given by:

$$E_\varepsilon^{(1)}(u) = \frac{E_\varepsilon(u) - \varepsilon^2 \#\Omega_\varepsilon^{(1)}}{\varepsilon^2|\log\varepsilon|} = \frac{1}{2|\log\varepsilon|} \sum_{(i,j)\in\Omega_\varepsilon^{(1)}} (1 - \langle u(i), u(j)\rangle). \tag{2.18}$$

One possibility of tracking the vorticity of a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ is to employ the distributional Jacobian on a proper interpolation of u_ε. Since the extension must be at least Sobolev regular, we cannot simply apply the piecewise constant interpolation. Given $u \in \mathcal{AS}_\varepsilon$ we define its piecewise affine interpolation as follows: For any $i \in \varepsilon\mathbb{Z}^2$ we let T_i^- and T_i^+ denote the triangles

$$T_i^- := \mathrm{Conv}\{i,\, i+\varepsilon e_1,\, i+\varepsilon(e_1+e_2)\}, \quad T_i^+ := \mathrm{Conv}\{i,\, i+\varepsilon(e_1+e_2),\, i+\varepsilon e_2\},$$

respectively, where $\mathrm{Conv}\, S$ stands for the convex hull of the set S. Furthermore, let $Q := [0,1)^2$ and $\Omega_\varepsilon^{(2)} := \{i \in \Omega_\varepsilon^{(0)} : i + \varepsilon Q \subset \Omega\}$. Then for $i \in \Omega_\varepsilon^{(2)}$ we consider the discrete differences:

$$\Delta_1 u(i) := u(i+\varepsilon e_1) - u(i), \quad \Delta_2 u(i) := u(i+\varepsilon e_2) - u(i).$$

Given any open set Ω and $\varepsilon > 0$ we set:

$$\Omega_\varepsilon := \bigcup_{i\in\Omega_\varepsilon^{(2)}} i + \varepsilon Q.$$

Thus, the piece-wise affine interpolation $\tilde{u}\colon \Omega_\varepsilon \to \mathbb{R}^2$ of u is defined as:

$$\tilde{u}(x) := \begin{cases} u(i) + \frac{x_1 - i_1}{\varepsilon}\Delta_1 u(i) + \frac{x_2 - i_2}{\varepsilon}\Delta_2 u(i+\varepsilon e_1) & \text{if } x \in T_i^-,\ i \in \Omega_\varepsilon^{(2)}, \\ u(i) + \frac{x_1 - i_1}{\varepsilon}\Delta_1 u(i+\varepsilon e_2) + \frac{x_2 - i_2}{\varepsilon}\Delta_2 u(i) & \text{if } x \in T_i^+,\ i \in \Omega_\varepsilon^{(2)}. \end{cases} \tag{2.19}$$

Moreover, let X denote the following set of Dirac measures:

$$X := \left\{ \mu = \sum_{k=1}^{K} d_k \delta_{x_k} : K \in \mathbb{N}_+,\ d_k \in \mathbb{Z},\ x_k \in \Omega \right\}.$$

The following first-order Γ-convergence result holds true (see also Theorem 3 [4]):

Theorem 2.13 (First-order Γ-convergence, xy)
Let $E_\varepsilon^{(1)}\colon \mathcal{AS}_\varepsilon \to \mathbb{R}$ be defined as in (2.18). *Then the following Γ-convergence result holds true:*

(i) (Compactness) Let $\{u_\varepsilon\}$ be a sequence of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ satisfying $\sup_\varepsilon E_\varepsilon^{(1)}(u_\varepsilon) < \infty$, then there exists a measure $\mu \in X$ such that, up to a subsequence, $\mathrm{Jac}(\tilde{u}_\varepsilon) \overset{\flat}{\rightharpoonup} \pi\mu$, where $\tilde{u}_\varepsilon$ is the piecewise affine interpolation of u_ε as described above.

(ii) (Gamma-liminf) Given a sequence $\{u_\varepsilon\}$ be a sequence of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that $\mathrm{Jac}(\tilde{u}_\varepsilon) \overset{\flat}{\rightharpoonup} \pi\mu$ for some $\mu \in X$, then

$$\liminf_{\varepsilon \to 0} E_\varepsilon^{(1)}(u_\varepsilon) \geq \pi|\mu|, \tag{2.20}$$

where $|\mu|$ denotes the total variation of μ.

(iii) (Gamma-limsup) For any $\mu \in X$ there exists a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that $J(\tilde{u}_\varepsilon) \overset{\flat}{\rightharpoonup} \pi\mu$, and:

$$\limsup_{\varepsilon \to 0} E_\varepsilon^{(1)}(u_\varepsilon) \leq \pi|\mu|.$$

Therefore, in the energy regime, where $E_\varepsilon^{(1)}(u_\varepsilon)$ stays bounded ($E_\varepsilon(u_\varepsilon)$ is $\varepsilon^2|\log\varepsilon|$-close to $\min E_\varepsilon$), a finite amount of vortices emerge whose total number (counting multiplicities) cannot exceed $\frac{\sup_\varepsilon E_\varepsilon(u_\varepsilon)}{\pi}$.

While the above first-order Γ-convergence result can, in fact, be proved in the general setting of spin fields on the n-dimensional grid (see also [4]), all the results from this point on are – to the best knowledge of the author, that is – only available in the two-dimensional setting. Theorem 3.1 in [6] generalizes Theorem 2.13 in two ways. On the one hand, the authors consider more general energy functionals:

Definition 2.16 (Admissible angular potentials, classical setting)
We call a 2π-periodic, bounded function $f\colon \mathbb{R} \to \mathbb{R}$ an admissible angular potential if and only if:

(i) $f(t) \geq 1 - \cos(t)$ for all $t \in [0, 2\pi]$;

(ii) $f(t) - 1 + \cos(t) = \mathrm{O}(|t|^3)$ as $t \to 0$.

Given an admissible angular potential f they investigate the energy functional $XY_\varepsilon\colon \mathcal{AS}_\varepsilon \to \mathbb{R}$, defined as:

$$XY_\varepsilon(u) := \frac{1}{2} \sum_{(i,j)\in\Omega_\varepsilon^{(1)}} f(\varphi(i) - \varphi(j)) \tag{2.21}$$

where φ is an arbitrary angular lift of u. Note that for $f(t) := 1 - \cos(t)$ we recover the energy functional $XY_\varepsilon = |\log\varepsilon| E_\varepsilon^{(1)}$ from Theorem 2.13. Given an arbitrary open set $U \subset \Omega$ we also define the localized version of the XY energy:

$$XY_\varepsilon(u, U) := \frac{1}{2} \sum_{(i,j)\in U_\varepsilon^{(1)}} f(\varphi(i) - \varphi(j)),$$

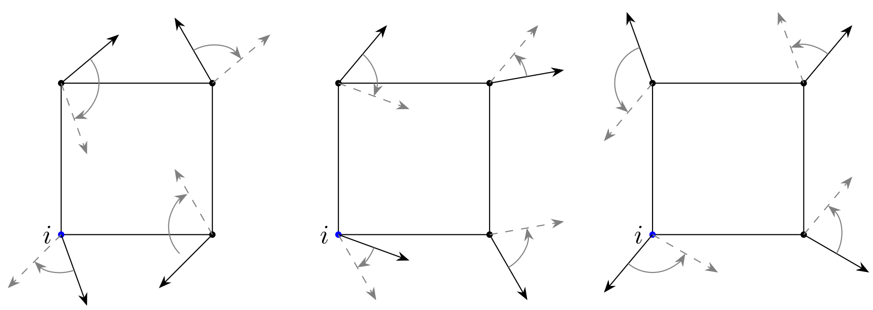
for φ as before, and $U_\varepsilon^{(1)}$ being the set of nearest-neighbor pairs contained in U. On the other hand, the authors of [6] choose a "more discrete" notion of vorticity for a spin field $u \in \mathcal{AS}_\varepsilon$ where one assigns to each cell $i + \varepsilon Q$ $i \in \Omega_\varepsilon^{(2)}$ a vorticity value in $\{-1, 0, 1\}$. More precisely, given $u \in \mathcal{AS}_\varepsilon$ we consider an angular lift $\varphi\colon \Omega_\varepsilon^{(0)} \to \mathbb{R}$ of u satisfying $u = e^{\imath\varphi}$, where $\imath$ is the imaginary unit. (We identify $\mathbb{R}^2$ with $\mathbb{C}$.) For φ and $i \in \Omega_\varepsilon^{(2)}$ we introduce the so called "elastic" discrete differences:

$$\Delta_1^{\mathrm{el}}\varphi(i) := \Delta_1\varphi(i) - 2\pi \left\lfloor \frac{\Delta_1\varphi(i)}{2\pi} \right\rceil, \quad \Delta_2^{\mathrm{el}}\varphi(i) := \Delta_2\varphi(i) - 2\pi \left\lfloor \frac{\Delta_2\varphi(i)}{2\pi} \right\rceil,$$

where $\lfloor\cdot\rceil$ is the rounding function that rounds down in the case of a tie-brake. Furthermore we will use the following handy notation: For any $i \in \Omega_\varepsilon^{(2)}$ let us shortly write for the remaining three vertices of the cell $i + \varepsilon Q$ in anticlockwise order:

$$j = j(i) := i + \varepsilon e_1, \quad k = k(i) := i + \varepsilon(e_1 + e_2), \quad l = l(i) = i + \varepsilon e_2.$$

With φ still denoting an angular lift of u, we then assign the vorticity $\alpha_u(i) \in$



(a) $\alpha_\varphi(i) = -1$ (b) $\alpha_\varphi(i) = 0$ (c) $\alpha_\varphi(i) = 1$

Figure 2.4: Discrete vorticity

$\{-1, 0, 1\}$ to the cell $i + \varepsilon Q$, where $\alpha_\varphi(i)$ is defined

$$\alpha_u(i) := \frac{1}{2} \left(\Delta_1^{\mathrm{el}}\varphi(i) + \Delta_2^{\mathrm{el}}\varphi(j) - \Delta_1^{\mathrm{el}}\varphi(k) - \Delta_2^{\mathrm{el}}\varphi(l) \right)$$

(See also Figure 2.4 for further illustration.) Consequently, we consider the discrete vorticity measure:

$$\mu_u := \sum_{i \in \Omega_\varepsilon^{(2)}} \alpha_u(i)\, \delta_i$$

which puts a Dirac with mass $\alpha_u(i)$ at the center of each cell $i + \varepsilon Q \subset \Omega$. In [6], it is shown that for a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$, satisfying the energy bound $\sup_\varepsilon \frac{XY_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} < \infty$ it holds that $\|\mathrm{Jac}(\tilde{u}_\varepsilon) - \pi\mu_{u_\varepsilon}\|_\flat \to 0$ as $\varepsilon \to 0$. For this reason, the same Γ-convergence result as in Theorem 2.13 holds true with $\mathrm{Jac}(\tilde{u}_\varepsilon)$ replaced by $\pi\mu_{u_\varepsilon}$ and $E_\varepsilon^{(1)}$ replaced by $\frac{XY_\varepsilon}{|\log \varepsilon|}$ (see also Theorem 3.1 in [6]). The authors also show a stronger, localized version of the Γ-liminf inequality (see also eq. (3.1) in theorem 3.1 of [6]):

Theorem 2.14 (Localized liminf-inequality)
Given a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that $\mu_{u_\varepsilon} \xrightarrow{\flat} \mu = \sum_{k=1} d_k \delta_{x_k} \in X$, where $d_k \in \mathbb{Z}$ for all k and $x_k \neq x_l$ for all $k \neq l$. Then for any $k \in \{1, \dots, K\}$ and $r > 0$ small enough such that the balls of $\{B_r(x_k)\}_k$ are disjoint, it holds that:

$$\liminf_{\varepsilon \to 0} \left(XY_\varepsilon(u_\varepsilon, B_r(x_k)) - \pi |d_k| \log\left(\frac{r}{\varepsilon}\right)\right) > -\infty. \tag{2.22}$$

Applying (2.22) for each vortex center thus leads to the improved global liminf inequality

$$\liminf_{\varepsilon \to 0} \left(XY_\varepsilon(u_\varepsilon) - \pi |\mu| |\log \varepsilon| \right) > \infty. \tag{2.23}$$

This version is stronger than the one in (2.20), since it excludes terms that are diverging towards $-\infty$ at a lower order than $|\log \varepsilon|$. So, for example, a scaling such as:

$$XY_\varepsilon(u_\varepsilon) \approx \pi |\mu| |\log \varepsilon| - \log |\log \varepsilon|$$

would still be compatible with (2.20) but not with (2.23). Note that this does not specify the divergent part of $XY_\varepsilon(u_\varepsilon)$ completely, as the liminf in (2.23) may still be equal to $+\infty$. Nevertheless, we can extract a special case wherein we can fully characterize the divergence of XY_ε. In this regard, we consider a sequence $\{u_\varepsilon\}$ of admissible spin field $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that:

$$XY_\varepsilon(u_\varepsilon) \leq K\pi |\log \varepsilon| \tag{2.24}$$

for some fixed $K \in \mathbb{N}_+$. By the compactness statement of Theorem 2.13 we can extract a subsequence without the need of relabeling, such that

$$\mu_{u_\varepsilon} \xrightarrow{\flat} \mu = \sum_{k=1}^{K} d_k \delta_{x_k} \in X,\ x_k \neq x_l \text{ for } k \neq l.$$

Taking the energy bound in (2.24) and the inequality in (2.23) into consideration, it follows that $|\mu| \leq K$. If we further assume that $|\mu| = K$, we become able classify the divergent part of $XY_\varepsilon(u_\varepsilon)$. In fact:

$$XY_\varepsilon(u_\varepsilon) = K\pi |\log \varepsilon| + \mathrm{O}(1),\ \text{as } \varepsilon \to 0.$$

This is the starting point of an "approximate" second-order Γ-convergence result for E_ε where we study the Γ-limit of the energy functional:

$$XY_\varepsilon - K\pi|\log\varepsilon| = \frac{E_\varepsilon^{(1)} - K\pi}{\frac{1}{|\log\varepsilon|}} \tag{2.25}$$

for some $K \in \mathbb{N}_+$.

Before proceding to the Γ-convergence result for the functionals defined through (2.25), we first need to introduce several important things. Let $K \in \mathbb{N}_+$ be fixed; we define $\tilde{X}_K \subset X$ as:

$$\tilde{X}_K := \left\{ \mu = \sum_{k=1}^{K} d_k \delta_{x_k} : d_k \in \{-1, 1\},\, x_k \neq x_l \text{ for } k \neq l \right\}.$$

Accordinly we set:

$$\mathcal{D}_K := \left\{ u \in W^{1,1}(\Omega;\mathbb{S}^1) : \pi^{-1}\operatorname{Jac}(u) \in \tilde{X}_K,\, u \in W^{1,2}_{\text{loc}}(\Omega \setminus \operatorname{spt}\operatorname{Jac}(u);\mathbb{S}^1) \right\}.$$

Given a meassure μ supported in finitely many points $\{x_k\}_{k=1}^K$, $K \in \mathbb{N}_+$, we define

$$\Omega_r(\mu) := \Omega \setminus \bigcup_{k=1}^{K} B_r(x_k). \tag{2.26}$$

The *renormalized energy* is then defined $\mathcal{W} : \mathcal{D}_K \to \mathbb{R}$:

$$\mathcal{W}(u) := \lim_{r\to 0} \frac{1}{2} \int_{\Omega_r(\operatorname{Jac}(u))} |\nabla u|^2 \,\mathrm{d}x - K\pi|\log r|. \tag{2.27}$$

One can show that $\mathcal{W}$ is well defined on the set $\mathcal{D}_K$. Finally, we will introduce the *core energy*. Fix $\varepsilon > 0$ and $r > 0$, and let $\gamma(\varepsilon, r)$ be the scalar given by:

$$\gamma(\varepsilon, r) := \min\left\{ XY_\varepsilon(u_\varepsilon, B_r) : u(x) = \frac{x}{|x|} \text{ on } \partial_\varepsilon B_r \right\}, \tag{2.28}$$

where B_r is the closed ball centered at 0 with radius r, and $\partial_\varepsilon B_r := \partial(B_r)_\varepsilon \cap \varepsilon\mathbb{Z}^2$. One can show that there exists a scalar $\gamma \in \mathbb{R}$ (core energy) *independent* of r such that:

$$\lim_{\varepsilon\to 0} \left(\gamma(\varepsilon, r) - \pi \log\left(\frac{\varepsilon}{r}\right) \right) = \gamma. \tag{2.29}$$

As a result, the following theorem can be stated (see also Theorem 4.5 in [6] for a proof):

Theorem 2.15 (Approximate second order Γ-convergence, xy)
Let $K \in \mathbb{N}_+$ be fixed, then the subsequent Γ-convergence result holds true for XY_ε, as defined in (2.21)*:*

(i) (Compactness) Let $\{u_\varepsilon\}$ be a sequence of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that:

$$XY_\varepsilon(u_\varepsilon) \leq K\pi|\log\varepsilon| + C$$

for some constant $C < \infty$ independent of ε. Then there exists a measure $\mu \in X$ with $|\mu| \leq K$ such that, up to some subsequence, the following applies:

$$\mu_{u_\varepsilon} \overset{\flat}{\rightharpoonup} \mu, \quad \operatorname{Jac}(u_\varepsilon) \overset{\flat}{\rightharpoonup} \pi\mu. \tag{2.30}$$

Furthemore, if $|\mu| = K$*, then we have* $\mu \in \tilde{X}_K$ *and again, up to some subsequence, the following applies:*

$$\tilde{u}_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,2}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\mu; \mathbb{R}^2)$$

for some $u \in \mathcal{D}_K$*, where* $\tilde{u}_\varepsilon$ *is the piecewise affine interpolation of* u_ε*, as defined in* (2.19)*.*

(ii) (Γ*-liminf) Let* $\{u_\varepsilon\}$ *be a sequence of admissible spin field* $u_\varepsilon \in \mathcal{AS}_\varepsilon$ *such that* $\tilde{u}_\varepsilon \to u$ *in* $L^1(\Omega; \mathbb{R}^2)$ *for some* $u \in \mathcal{D}_K$*, then:*

$$\liminf_{\varepsilon \to 0} \left(XY_\varepsilon(u_\varepsilon) - K\pi|\log \varepsilon| \right) \geq \mathcal{W}(u) + K\gamma. \tag{2.31}$$

(iii) (Γ*-limsup) Given* $u \in \mathcal{D}_K$*, there exists a sequence* $\{u_\varepsilon\}$ *of admissible spin fields* $u_\varepsilon \in \mathcal{AS}_\varepsilon$ *such that* $\tilde{u}_\varepsilon \rightharpoonup u$ *weakly in* $W^{1,2}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt} \operatorname{Jac}(u); \mathbb{R}^2)$ *and:*

$$\limsup_{\varepsilon \to 0} \left(XY_\varepsilon(u_\varepsilon) - K\pi|\log \varepsilon| \right) \leq \mathcal{W}(u) + K\gamma.$$

The above result can be summarized as follows: The only sequences $\{u_\varepsilon\}$ whose XY energy scales as

$$XY_\varepsilon(u_\varepsilon) = K\pi|\log\varepsilon| + \mathrm{O}(1) \text{ for } \varepsilon \to 0$$

are exactly those that develop unit vortices in the limit $\varepsilon \to 0$. The Γ-convergence result can be outlined in the following way: Minimizers of XY_ε in the logarithmic energy regime $K\pi|\log\varepsilon| + \mathrm{O}(1)$ (as $\varepsilon \to 0$) develop K distinct unit vortices in the limit $\varepsilon \to 0$ with asymptotic energy given by the sum $\mathcal{W}(u)+K\gamma$. Qualitatively speaking, the renormalized energy $\mathcal{W}$ prefers vortices that have the same sign to be located far apart from each other and vortices of different sign, close to each other (see [16] for further details.) Additionally, $\mathcal{W}(u)$ also measures how "efficiently" u achieves its vortex singularity $\operatorname{Jac}(u)$. In fact, two different limit configurations u, $u' \in \mathcal{D}_K$ with $\operatorname{Jac}(u) = \operatorname{Jac}(u')$ may have different renormalized energies $\mathcal{W}(u)$ and $\mathcal{W}(u')$, repsectively.

2.3 Problem setup

In this section, we will discuss the generalization of the XY model and state the corresponding Γ-convergence result. Without further mention, $\mathfrak{m} \in \mathbb{N}_+$ denotes a fixed natural number and $\Omega \subset \mathbb{R}^2$ a simply connected, open set with smooth boundary. We define $\Omega_\varepsilon^{(0)}$, $\Omega_\varepsilon^{(1)}$, $\Omega_\varepsilon^{(2)}$, Ω_ε, $\partial_\varepsilon\Omega$, $\mathcal{AS}_\varepsilon$ and XY_ε as in Section 2.2.2. Given $u \in \mathcal{AS}_\varepsilon$ and an angular lift φ of u, we rewrite $XY_\varepsilon(u)$ as follows:

$$XY_\varepsilon(u) = \sum_{\langle i,j \rangle} f(\varphi(i) - \varphi(j)), \quad f(t) = 1 - \cos(t).$$

In the case of the *generalized XY model*, wa have to consider a modified 2π-periodic functional $f_\varepsilon^{(\mathfrak{m})}$ (depending on $\varepsilon > 0$) with $\mathfrak{m}$ wells at 0, $\frac{2\pi}{\mathfrak{m}}$, ..., $(\mathfrak{m}-1)\frac{2\pi}{\mathfrak{m}}$ such that $f_\varepsilon^{(\mathfrak{m})}(0) = 0$ and $f_\varepsilon^{(\mathfrak{m})}(k\frac{2\pi}{\mathfrak{m}}) \approx \varepsilon$. More precisely, we take the following class of admissible families of angular potentials:

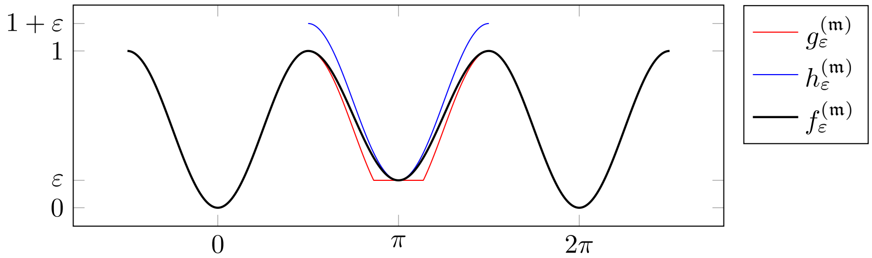


Figure 2.5: An admissible potential with $\mathfrak{m} = 2$ wells.

Definition 2.17 (Admissible angular potentials, fractional setting)
Let $f\colon \mathbb{R} \to \mathbb{R}$ be an angular potential as in Definition 2.16 and given $\varepsilon > 0$ let $g_\varepsilon^{(\mathfrak{m})}$, $h_\varepsilon^{(\mathfrak{m})}$ be defined as follows

$$g_\varepsilon^{(\mathfrak{m})}(t) := f(\mathfrak{m}t) \vee \varepsilon \mathbb{1}_{[\frac{\pi}{\mathfrak{m}}, 2\pi - \frac{\pi}{\mathfrak{m}}]}(t), \quad t \in \left[\frac{\pi}{\mathfrak{m}}, 2\pi - \frac{\pi}{\mathfrak{m}}\right],$$
$$h_\varepsilon^{(\mathfrak{m})}(t) := f(\mathfrak{m}t) + \varepsilon \mathbb{1}_{[\frac{\pi}{\mathfrak{m}}, 2\pi - \frac{\pi}{\mathfrak{m}}]}(t), \quad t \in \left[\frac{\pi}{\mathfrak{m}}, 2\pi - \frac{\pi}{\mathfrak{m}}\right],$$

where $x \vee y := \min\{x, y\}$ for all $x, y \in \mathbb{R}$. We call a sequence $\{f_\varepsilon^{(\mathfrak{m})}\}_\varepsilon$ of 2π-periodic and uniformly bounded functions $f_\varepsilon^{(\mathfrak{m})}\colon \mathbb{R} \to \mathbb{R}$ an admissible sequence of fractional angular potentials, with $\mathfrak{m}$ wells, if and only if

(i) $g_\varepsilon^{(\mathfrak{m})}(t) \le f_\varepsilon^{(\mathfrak{m})}(t) \le f^{(\mathfrak{m})}(t)$ for all $t \in [0, 2\pi]$;

(ii) $|f_\varepsilon^{(\mathfrak{m})}(t) - 1 + \cos(\mathfrak{m}t)| \le C|t|^3$ for all $t \in (-t_0, t_0)$, where $t_0 > 0$ and C are independent of ε;

The conditions imposed in Definition 2.17 are more of technical nature. We single out the very properties of $f_\varepsilon^{(\mathfrak{m})}$ that will be necessary for the derivation of the upcoming Γ-convergence result. The conditions are still flexible enough to allow for angular potentials appearing in the physics literature (see also [26]). An example of $f_\varepsilon^{(\mathfrak{m})}$ is given in Figure 2.5. Note that the upper bound of Item (i) in the vicinity of $t = \frac{2\pi k}{\mathfrak{m}}$ and Item (ii) in Definition 2.16 imply that there exists $\tilde{t}_0 \in (0, \frac{\pi}{\mathfrak{m}})$ and $\tilde{C} > 0$, which are independent of ε such that for all $k \in \{1, \ldots, \mathfrak{m} - 1\}$, the following applies:

$$f_\varepsilon^{(\mathfrak{m})}\left(t - \frac{2\pi k}{\mathfrak{m}}\right) - \varepsilon \le C\left|t - \frac{2\pi k}{\mathfrak{m}}\right|^2 \text{ for all } t \in \left(\frac{2\pi k}{\mathfrak{m}} - \tilde{t}_0, \frac{2\pi k}{\mathfrak{m}} + \tilde{t}_0\right). \quad (2.32)$$

Definition 2.18 (Energy functionals)
Given a sequence of fractional angular potentials $(f_\varepsilon^{(\mathfrak{m})})_\varepsilon$ we define for each $\varepsilon > 0$ the generalized XY energy functional $XY_\varepsilon^{(\mathfrak{m})}\colon \mathcal{AS}_\varepsilon \to \mathbb{R}$ as:

$$XY_\varepsilon^{(\mathfrak{m})}(u) := \frac{1}{2} \sum_{(i,j) \in \Omega_\varepsilon^{(1)}} f_\varepsilon^{(\mathfrak{m})}(\varphi(i) - \varphi(j)), \quad (2.33)$$

where φ is an angular lift of u. As before, the localized version of $XY_\varepsilon^{(\mathfrak{m})}$ is also taken into consideration for an open set $U \subset \Omega$: We also consider for an open set $U \subset \Omega$ the localized version

$$XY_\varepsilon^{(\mathfrak{m})}(u, U) \coloneqq \frac{1}{2} \sum_{(i,j)\in U_\varepsilon^{(1)}} f_\varepsilon^{(\mathfrak{m})}(\varphi(i) - \varphi(j))$$

with φ as before. From this point on we assume XY_ε to be defined as in (2.21), where f is the nonfractional angular potential f from Definition 2.17.

This time, in contrast to the previously discussed models, we will assume a Dirichlet boundary condition on the admissible spin fields. That is, we will fix a boundary datum $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$. We will then consider the following set of admissible spin fields $\mathcal{AS}_\varepsilon^{(g)}$:

$$\mathcal{AS}_\varepsilon^{(g)} \coloneqq \{u \in \mathcal{AS}_\varepsilon \colon u(i) = g(i) \text{ for } i \in \partial_\varepsilon\Omega\},$$

where $\partial_\varepsilon\Omega = \varepsilon\mathbb{Z}^2 \cap \partial\Omega_\varepsilon$, and Ω_ε is the union of squares $i + \varepsilon[0,1]^2$, $i \in \varepsilon\mathbb{Z}^2$ contained in Ω.

Given $u \in \mathcal{AS}_\varepsilon$ a pair $(i,j) \in \Omega_\varepsilon^{(1)}$ of nearest neighbors is called a jump pair (of u) if and only if $|\Delta^{\mathrm{el}}\varphi(i,j)| > \frac{\pi}{\mathfrak{m}}$, where φ is an arbitrary angular lift of u, and:

$$\Delta^{\mathrm{el}}\varphi(i,j) = \varphi(i) - \varphi(j) - 2\pi\left\lfloor \frac{\varphi(i) - \varphi(j)}{2\pi} \right\rceil. \tag{2.34}$$

It is important to add that there is nothing special about the choice of $\frac{\pi}{\mathfrak{m}}$ indeed. In fact, any positive scalar in $(0, \frac{2\pi}{\mathfrak{m}})$ would have also worked here. A cell $i + \varepsilon Q$, where $i \in \varepsilon\mathbb{Z}^2$ lies in Ω, with the other three vertices denoted by j, k and l (in anticlockwise order starting at i) is called a jump cell if and only if one of the pairs (i,j), (j,k), (k,l) or (l,i) is a jump pair at least. The set of all indices $i \in \Omega_\varepsilon^{(2)}$ such that $i + \varepsilon Q$ is a jump cell of u will be denoted by JC_u. Let $A(u)$ denote the piecewise affine interpolation of $u \in \mathcal{AS}_\varepsilon$, as defined in (2.19). Let us then consider the interpolation $AC(u) \colon \Omega_\varepsilon \to \mathbb{R}^2$:

$$AC(u)(x) \coloneqq \begin{cases} u(i) & \text{for } x \in i + \varepsilon Q \text{ with } i \in JC_u, \\ A(u)(x) & \text{for } x \in \Omega_\varepsilon. \end{cases}$$

With $d_g \coloneqq \deg(g, \partial\Omega)$, we shall denote the following set of measures by $X_g^{(\mathfrak{m})}$:

$$X_g^{(\mathfrak{m})} \coloneqq \left\{ \mu = \operatorname{sgn}(d_g) \sum_{k=1}^{\mathfrak{m}|d_g|} \frac{1}{\mathfrak{m}} \delta_{x_k} \colon x_k \neq x_l \text{ for } k \neq l \right\}.$$

Here the set $\mathcal{D}_g^{(\mathfrak{m})}$ contains all $u \in SBV(\Omega; \mathbb{S}^1)$, additionally satisfying:

(i) $(u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$-a.e. point on J_u;

(ii) $\mathcal{H}^1(J_u) < \infty$;

(iii) $u \in SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\operatorname{Jac}(u); \mathbb{S}^1)$ with $\frac{1}{\pi}\operatorname{Jac}(u) \in X_g^{(\mathfrak{m})}$;

(iv) $u^{\mathfrak{m}} = g^{\mathfrak{m}}$ on $\partial\Omega$ in the trace sense.

By applying the chain rule in BV (see also Theorem 2.5) and the definition of $\mathcal{D}_g^{(\mathfrak{m})}$, we can show that $u^{\mathfrak{m}} \in W^{1,1}(\Omega; \mathbb{S}^1)$ and $\operatorname{Jac}(u) = \frac{1}{\mathfrak{m}} \operatorname{Jac}(u^{\mathfrak{m}})$. Let γ denote the core energy as in the classical XY setting (see (2.29)) and the renormalized energy $\mathcal{W}^{(\mathfrak{m})} \colon \mathcal{D}_g^{(\mathfrak{m})} \to \mathbb{R}$ as

$$\mathcal{W}^{(\mathfrak{m})}(u) := \lim_{r \to 0} \left(\mathfrak{m}^2 \int_{\Omega_r(\operatorname{Jac}(u))} |\nabla u|^2 \, \mathrm{d}x - |d_g| \pi \mathfrak{m} |\log r| \right), \tag{2.35}$$

where $\Omega_r(\operatorname{Jac}(u))$ equals (2.26). According to the chain rule in BV, $\mathcal{W}^{(\mathfrak{m})}(u) = \mathcal{W}(u^{\mathfrak{m}})$.

Given a $\mathcal{H}^1$-rectifiable set $S \subset \mathbb{R}^2$ with normal vector field ν we will use the abbreviation

$$\mathcal{H}^1_{\mathrm{cr}}(S) := \int_S |\nu|_1 \, \mathrm{d}\mathcal{H}^1, \tag{2.36}$$

for the crystalline length of S, where $|\cdot|_1$ is the l^1-norm of $\mathbb{R}^2$. Finally, we state the main theorem of this chapter:

Theorem 2.16

With the notation described above, the following Γ-convergence result holds true:

(i) (Compactness) Let $\{u_\varepsilon\}$ be sequence of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon^{(g)}$ such that $XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) \leq \mathfrak{m}|d_g|\pi|\log \varepsilon| + C$ for some constant $C < \infty$ independent of ε. Then, there exists a measure $\mu \in X_g^{(\mathfrak{m})}$ such that, up to some subsequence:

$$\frac{1}{\mathfrak{m}} \mu_{u_\varepsilon^{\mathfrak{m}}} \overset{\flat}{\rightharpoonup} \mu.$$

Furthermore, there also exists $u \in \mathcal{AS}_\varepsilon^{(g)}$ such that, up to taking a subsequence:

$$AC(u_\varepsilon) \rightharpoonup u \text{ weakly in } SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{spt} \operatorname{Jac}(u); \mathbb{R}^2).$$

(ii) (Γ-liminf) Given a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon^{(g)}$ such that $AC(u_\varepsilon) \to u$ in $L^1(\Omega; \mathbb{R}^2)$ for some $u \in \mathcal{D}_g^{(\mathfrak{m})}$, then:

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \left(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) - \mathfrak{m}|d_g|\pi|\log \varepsilon| \right) \\ &\quad \geq \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\} \cap \partial\Omega) + \mathfrak{m}|d_g|\gamma, \end{aligned} \tag{2.37}$$

where we shortly wrote

$$\{u \neq g\} := \{x \in \partial\Omega \colon u(x) \neq g(x)\},$$

where the condition $u(x) \neq g(x)$ should be understood in the trace sense.

(iii) (Γ-limsup) For any $u \in \mathcal{D}_g^{(\mathfrak{m})}$ there exists a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon^{(g)}$ such that $AC(u_\varepsilon^{\mathfrak{m}}) \rightharpoonup u$ in $SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{spt} \operatorname{Jac}(u); \mathbb{R}^2)$, and:

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \left(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) - \mathfrak{m}|d_g|\pi|\log \varepsilon| \right) \\ &\quad = \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\} \cap \partial\Omega) + \mathfrak{m}|d_g|\gamma. \end{aligned}$$

The above theorem shows that correctly chosen interpolations of a sequence of minimizers $\{u_\varepsilon\}$ converge as $\varepsilon \to 0$ towards a limit spin configuration u with fractional vortices. Instead of assuming the divergence of $\{XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon)\}$ (e.g., as we did in Theorem 2.15), it is the nontrivial degree d_g of g that leads to the emergence of vortices with net-vorticity d_g. Similarly to Theorem 2.15, vortices with the lowest absolute value (in the present case $\pm\frac{1}{\mathfrak{m}}$) are the ones most energy-efficient. Therefore, $\frac{1}{\pi}\operatorname{Jac}(u)$ is a sum of $\mathfrak{m}|d_g|$ disjoint Diracs each with weight $\frac{\operatorname{sgn}(d_g)}{\mathfrak{m}}$. In addtition to having fractional vortices the limit spin fields possibly also jump on an $\mathcal{H}^1$-rectifiable set J_u such that $(u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$ a.e. point of J_u. Moreover, the boundary condition is not fully preserved in the limit $\varepsilon \to 0$, which is typical of Dirichlet problems in SBV. Nevertheless, we still have $u^{\mathfrak{m}} = g^{\mathfrak{m}}$ in the trace sense $\partial\Omega$ (by the discussion above $u^{\mathfrak{m}} \in W^{1,1}(\Omega)$). For small ε, the O(1)-terms of $XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon)$ are for small ε approximately equal to the sum of a vortex interaction potential (the same as in the classic setting), which forces the vortices to stay as much as possible away from each other as well as the boundary, the crystalline perimeter of J_u, the crystalline perimeter of the part of the boundary on which u does not attain the boundary condition g, and a fixed scalar term γ (the same as the classic setting) for each of the $\mathfrak{m}|d_g|$ vortices.

Together with M. Cicalese, L. De Luca and M. Ponsiglione, the author has derived a similar result in [15], but without the Dirichlet constraint which can be seen as a more natural generalization of Theorem 2.15 to the fractional setting. In this regard, let $\tilde{X}_K^{(\mathfrak{m})}$, for K, $\mathfrak{m} \in \mathbb{N}_+$, be the set of measures given by:

$$\tilde{X}_K^{(\mathfrak{m})} := \left\{ \mu = \sum_{k=1}^{K} d_k \delta_{x_k} : d_k = \pm 1,\ x_k \neq x_l \text{ for } k \neq l \right\}.$$

Furthemore, $\tilde{\mathcal{D}}_K^{(\mathfrak{m})}$ contains all $u \in SBV(\Omega;\mathbb{S}^1)$ such that:

(i) $(u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$-a.e. point on J_u;

(ii) $\mathcal{H}^1(J_u) < \infty$;

(iii) $u \in SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\operatorname{Jac}(u);\mathbb{S}^1)$ with $\frac{1}{\pi}\operatorname{Jac}(u) \in \tilde{X}_K^{(\mathfrak{m})}$.

Theorem 2.17
With the notation described above, the following Γ-convergence result holds true:

(i) (Compactness) Let $\{u_\varepsilon\}$ be a sequence of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that $XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) \le K\pi|\log\varepsilon| + C$ for some $K \in \mathbb{N}$, and constant $C < \infty$ independent of ε. Then there exists a measure:

$$\mu = \frac{1}{\mathfrak{m}} \sum_{k=1}^{K} d_k \delta_{x_k}$$

with $d_k \in \mathbb{Z}$ and $x_k \neq x_l$ for $k \neq l$, such that, up to some subsequence:

$$\frac{1}{\mathfrak{m}} \mu_{u_\varepsilon^{\mathfrak{m}}} \xrightarrow{\flat} \mu.$$

The measure μ satisfies $|\mu| \leq K$, and if we additionally assume that $|\mu| \geq K$ (and hence $|\mu| = K$) it follows that $\mu \in \tilde{X}_K^{(\mathfrak{m})}$ and, up to taking a further subsequence:

$$AC(u_\varepsilon) \rightharpoonup u \text{ weakly in } SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{spt} \operatorname{Jac}(u); \mathbb{R}^2),$$

where $u \in \mathcal{D}_k^{(\mathfrak{m})}$ with $\operatorname{Jac}(u) = \pi\mu$.

(ii) (Γ-liminf) Given a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that $AC(u_\varepsilon) \to u$ in $L^1(\Omega; \mathbb{R}^2)$ for some $u \in \tilde{\mathcal{D}}_K^{(\mathfrak{m})}$, we have:

$$\liminf_{\varepsilon \to 0} \Big(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) - K\pi|\log \varepsilon| \Big) \geq \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + K\gamma \qquad (2.38)$$

with $\mathcal{H}^1_{\mathrm{cr}}$ as in Theorem 2.16.

(iii) (Γ-limsup) For any $u \in \mathcal{D}_K^{(\mathfrak{m})}$ there exists a sequence $\{u_\varepsilon\}$ of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon$ such that $AC(u_\varepsilon) \rightharpoonup u$ in $SBV^2_{\mathrm{loc}}(\Omega \setminus \operatorname{spt} \operatorname{Jac}(u); \mathbb{R}^2)$, and:

$$\lim_{\varepsilon \to 0} \Big(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) - K\pi|\log \varepsilon| \Big) = \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + K\gamma.$$

Regarding the proof of Theorem 2.16, it will be useful to have a shorthand notation for a tubular neighborhood of $\partial\Omega$. In this regard fix $\delta > 0$ and set:

$$\begin{aligned} T_\delta = T_\delta(\partial\Omega) &:= \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \partial\Omega) < \delta\}, \\ T_\delta^+ = T_\delta^+(\partial\Omega) &:= \{x \in \mathbb{R}^2 \setminus \Omega \colon \operatorname{dist}(x, \partial\Omega) < \delta\}, \\ T_\delta^- = T_\delta^+(\partial\Omega) &:= \{x \in \bar{\Omega} \colon \operatorname{dist}(x, \partial\Omega) < \delta\}. \end{aligned}$$

2.4 Proof of Gamma-convergence

2.4.1 Compactness

In this section, we will prove the compactness statements in Theorem 2.16. We assume the same notation as in Section 2.3. Furthermore, we set $N_g := \mathfrak{m}|d_g|$, and assume without further mention that (u_ε) is a sequence of admissible spin fields $u_\varepsilon \in \mathcal{AS}_\varepsilon^{(g)}$, satisfying the energy bound:

$$XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) \leq N_g \pi |\log \varepsilon| + C, \qquad (2.39)$$

for some constant $C < \infty$ independent of ε. Additionally, we will shortly write $v_\varepsilon := u_\varepsilon^{\mathfrak{m}}$, $\tilde{u}_\varepsilon := AC(u_\varepsilon)$ and $\tilde{v}_\varepsilon := A(v_\varepsilon)$, and

$$J_\varepsilon := \bigcup_{i \in JC_{u_\varepsilon}} i + \varepsilon[0,1]^2,$$

where JC_{u_ε} is the union of all grid points $i \in \Omega_\varepsilon^{(2)}$ such that $i + \varepsilon[0,1]^2$ is a jump cell. Let $\delta_0 > 0$ be chosen sufficiently small such that the projection $\Pi = \Pi_{\partial\Omega}$ onto $\partial\Omega$ is well defined and smooth in $T^+_{2\delta_0}$. Then we implicitly suppose that g is extended into $T^+_{2\delta_0}$ through $g(x) := g(\Pi(x))$ and define $h := g^{\mathfrak{m}}$ in $T^+_{2\delta_0}$. Furthermore, we will shortly write $O := \Omega^{\delta_0}$. Let us start by deriving a compactness result concerning the sequence (v_ε):

Proposition 2.2
There exists a measure $\nu \in X_h^{(1)}$ *such that, up to a subsequence,*

$$\mu_{v_\varepsilon} \stackrel{\flat}{\rightharpoonup} \nu \text{ flat in } \Omega. \tag{2.40}$$

We can also find $v \in \mathcal{D}_h^{(1)}$ *such that* $\mathrm{Jac}(v) = \pi\nu$*, and such that, up to taking a further subsequence:*

$$A(v_\varepsilon) \rightharpoonup v \text{ weakly in } W^{1,2}(\Omega \setminus \operatorname{spt}\nu, \mathbb{R}^2). \tag{2.41}$$

Before getting to the proof of Proposition 2.2 we need to collect several preliminary results:

Lemma 2.2
For any $U \subset\subset \Omega$ *open,* $\varepsilon > 0$ *small enough (*$\varepsilon < \frac{1}{\sqrt{2}} \operatorname{dist}(U, \partial\Omega)$*) and* $v \in \mathcal{AS}_\varepsilon$*, the following holds true:*

$$\frac{1}{2}\int_{U_\varepsilon} |\nabla A(v)|^2 \,\mathrm{d}x \le XY_\varepsilon(v, U) \le \frac{1}{2}\int_{\tilde{U}_\varepsilon} |\nabla A(v)|^2 \,\mathrm{d}x, \tag{2.42}$$

where:

$$XY_\varepsilon(u, U) := \frac{1}{2} \sum_{(i,j)\in U_\varepsilon^{(1)}} f(\varphi(i) - \varphi(j)), \quad e^{\imath\varphi} = u,$$

and:

$$\tilde{U}_\varepsilon := \bigcup_{i \in U_\varepsilon^{(0)}} i + \varepsilon\tilde{Q}, \quad \tilde{Q} := [-1, 1)^2.$$

Proof. The proof can be found in [4] (see also (2.13)). □

Lemma 2.3
For any $\varepsilon > 0$*,* $u \in \mathcal{AS}_\varepsilon$ *and* $v := u^{\mathfrak{m}}$*, it holds that:*

$$XY_\varepsilon(v) \le XY_\varepsilon^{(\mathfrak{m})}(u), \tag{2.43}$$

Proof. Let φ be an arbitrary angular lift of u. Following the definition for v, we know that $\mathfrak{m}\varphi$ is an angular lift of v. Using the lower bound in Item (i) of Definition 2.17 and the definition of XY_ε, we conclude:

$$\begin{aligned} XY_\varepsilon^{(\mathfrak{m})}(u) &= \frac{1}{2} \sum_{(i,j)\in\Omega_\varepsilon^{(2)}} f_\varepsilon^{(\mathfrak{m})}(\varphi(i) - \varphi(j)) \\ &\ge \frac{1}{2} \sum_{(i,j)\in\Omega_\varepsilon^{(2)}} f(\mathfrak{m}(\varphi(i) - \varphi(j))) = XY_\varepsilon(v), \end{aligned}$$

as desired. □

Lemma 2.4
Let $K \in \mathbb{N}$*,* $v \in \mathcal{D}_K$ *such that* $\mathcal{W}(v) < \infty$*,* $\mathrm{Jac}(v)(\Omega) > 0$*,* $x_0 \in \operatorname{spt}\mathrm{Jac}(v)$ *and* $r_0 \in (0, \operatorname{dist}(x_0, \partial\Omega))$ *small enough such that* $\operatorname{spt}\mathrm{Jac}(v) \cap B_{r_0}(x_0) = \{x_0\}$*. Then there exists a sequence* $(\alpha_n) \subset \mathbb{S}^1$ *such that:*

$$\lim_{n\to\infty} \left\| v|_{A_n} - \alpha_n \frac{x - x_0}{|x - x_0|} \right\|_{W^{1,2}(A_n)} = 0, \tag{2.44}$$

where A_n is the annulus:

$$A_n := A_{2^{-(n+1)}r_0, 2^{-n}r_0}(x_0).$$

Proof. It was shown in [6] (see also the proof of Theorem 4.13) that for any $r > 0$

$$\min\left\{\frac{1}{2}\int_{A_{\frac{r}{2},r}(x_0)} |\nabla w|^2 \,\mathrm{d}x \colon w \in W^{1,2}(A_{\frac{r}{2},r}(x_0); \mathbb{S}^1),\ \deg(w, \partial B_{\frac{r}{2}}(x_0)) = 1\right\} \\ = \pi \log 2 \tag{2.45}$$

with the set of minimizers (independent of r) given by:

$$\mathcal{K} := \left\{\alpha \frac{x - x_0}{|x - x_0|} : \alpha \in \mathbb{S}^1\right\}.$$

The authors of [6] have also derived (see also Remark 4.4) the following representation for the renormalized energy:

$$\begin{aligned} \mathcal{W}(v, B_{r_0}(x_0)) &:= \lim_{r \to 0} \left(\frac{1}{2}\int_{A_{r,r_0}(x_0)} |\nabla v|^2 \,\mathrm{d}x - \pi |\log r|\right) \\ &= \sum_{h=0}^{\infty} \left(\frac{1}{2}\int_{A_n} |\nabla v|^2 \,\mathrm{d}x - \pi \log 2\right). \end{aligned}$$

By (2.45), each term in the series above is non-negative. Hence as:

$$\mathcal{W}(v, B_{r_0}(x_0)) \le \mathcal{W}(v) < \infty$$

the series turns out to be convergent, thus:

$$\lim_{n \to \infty} \frac{1}{2}\int_{A_n} |\nabla v|^2 = \pi \log(2). \tag{2.46}$$

Let us suppose by contradiction that there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$

$$\inf_{n \in \mathbb{N}} \inf_{\alpha \in \mathbb{S}^1} \left\| v|_{A_n} - \alpha \frac{x - x_0}{|x - x_0|} \right\|_{W^{1,2}(A_n)} > \delta.$$

This (see also (4.19) in [6]) implies that there exists a scalar $\omega(\delta) > 0$ such that for all $n \in \mathbb{N}$:

$$\frac{1}{2}\int_{A_n} |\nabla v|^2 \,\mathrm{d}x \ge \pi \log 2 + \omega(\delta)$$

which directly contradicts (2.46). □

Lemma 2.5
Let $v \in W^{1,2}(\Omega; \mathbb{R}^2)$ (Ω simply connected) such that $v = h$ on $\partial\Omega$ (in the trace sense) for some $n \in C^\infty(\partial\Omega; \mathbb{S}^1)$, then:

$$\int_\Omega \mathrm{Jac}(v) \,\mathrm{d}x = \pi \deg(h, \partial\Omega). \tag{2.47}$$

Proof. This is a classic result (see for example (0.3) in [21]). □

Lemma 2.6
Let $U \subset \mathbb{R}^2$ be an open bounded set and (v_n), $(w_n) \subset W^{1,2}(U;\mathbb{R}^2)$ such that:

$$\lim_{n\to\infty} \|v_n - w_n\|_{L^2}(\|\nabla v_n\|_{L^2} + \|\nabla w_n\|_{L^2}) = 0,$$

then:

$$\mathrm{Jac}(v_n) - \mathrm{Jac}(w_n) \stackrel{\flat}{\rightharpoonup} 0 \textit{ flat in } \Omega.$$

Proof. See also Lemma 2.1 in [8]. □

Proof of Proposition 2.2. 1. step: By the estimate in (2.43) and the energy bound in (2.39), we have that:

$$XY_\varepsilon(v_\varepsilon) \le N_g \pi |\log \varepsilon| + C \tag{2.48}$$

for a constant $C < \infty$ independent of ε.

1. step: For each $\varepsilon > 0$, we extend v_ε into $\Omega^{(2\delta_0)}$ by setting $v_\varepsilon(i) := h(i)$ outside of $\Omega_\varepsilon^{(0)}$. We will shortly define by $\tilde{v}_\varepsilon$ the piecewise affine interpolation of the extended spin field v_ε. Let us fix $i \in O_\varepsilon^{(0)} \setminus \Omega_\varepsilon^{(0)}$ and let j, k, l denote the remaining vertices of the cell $i + \varepsilon[0,1]^2$ (in anticlockwise order). Furthermore, let $x \in T_i^- := \mathrm{Conv}(\{i,j,k\})$. By the definition of $\tilde{v}_\varepsilon$ (see also (2.19)) and the mean value theorem we can estimate:

$$\begin{aligned}
|\nabla \tilde{v}_\varepsilon(x)|^2 &= |\nabla_{e_1} \tilde{v}_\varepsilon|^2 + |\nabla_{e_2} \tilde{v}_\varepsilon|^2 \\
&= \varepsilon^{-2}(|h(j) - h(i)|^2 + |h(k) - h(j)|^2) \\
&= \varepsilon^{-2}(|\nabla h(\xi)(j-i)|^2 + |\nabla h(\nu)(k-j)|^2) \\
&= 2\|\nabla h\|_{L^\infty(T^+_{2\delta_0})},
\end{aligned}$$

where $\xi \in [i,j]$ and $\nu \in [j,k]$. Hence, by (2.42) and (2.48) the following energy bound holds true for all $\varepsilon > 0$:

$$\begin{aligned}
XY_\varepsilon(v_\varepsilon, O) &\le XY_\varepsilon(v_\varepsilon, \Omega) + |T_{2\delta_0}| \|\nabla h\|_{L^\infty(T^+_{2\delta_0})} \\
&\le N_g \pi |\log \varepsilon| + \tilde{C},
\end{aligned}$$

where $\tilde{C} < \infty$ is a constant independent of ε. By Theorem 2.15 applied in O, we can find a point measure $\nu \in X(O)$ with $|\nu| \le N_g$ such that, up to a subsequence:

$$\mu_{v_\varepsilon} \stackrel{\flat}{\rightharpoonup} \nu = \sum_{k=1}^K d_k \delta_{x_k}, \quad d_k \in \mathbb{Z},\ x_k \in O,\ x_k \neq x_l \text{ for } k \neq l.$$

Note that for the same subsequence we also have that $\mathrm{Jac}(\tilde{v}_\varepsilon) \stackrel{\flat}{\rightharpoonup} \pi\nu$ flat in O.

2. step: Let us fix a $\delta \in (0, \delta_0)$ and consider the continuum spin field:

$$\tilde{w}_\varepsilon(x) := \begin{cases} \tilde{v}_\varepsilon(x) & \text{if } x \in \Omega, \\ (1 - \frac{1}{\delta}\operatorname{dist}(x,\partial\Omega))\tilde{v}_\varepsilon(x) + \frac{1}{\delta}\operatorname{dist}(x,\partial\Omega) h(x), & \text{if } x \in T^+_\delta \\ h(x) & \text{if } x \in T^+_{\delta_0} \setminus T^+_\delta. \end{cases}$$

By the definition of $\tilde{v}_\varepsilon$, the bound on $XY_\varepsilon(v_\varepsilon, O)$ from before and (2.42) we see that:

$$\|\tilde{v}_\varepsilon - \tilde{w}_\varepsilon\|_{L^2(T^+_{\delta_0})}(\|\tilde{v}_\varepsilon\|_{L^2(T^+_{\delta_0})} + \|\tilde{w}_\varepsilon\|_{L^2(T^+_{\delta_0})}) \le C(h)\varepsilon^2\sqrt{|\log\varepsilon|} = \mathrm{o}(1)$$

as $\varepsilon \to 0$, where $C(h)$ is a constant independent of ε. Consequently, due to Lemma 2.6 and the compactness result for $(\mathrm{Jac}(\tilde{v}_\varepsilon))_\varepsilon$ we can see that, up to a subsequence, $\mathrm{Jac}(\tilde{w}_\varepsilon) \overset{\flat}{\rightharpoonup} \nu$ flat in O. Let us now consider an arbitrary test function $\rho \in C^\infty_c(O)$ such that $\rho \equiv 1$ in $\Omega \cup T_\delta$. By definition of weak convergence, we have:

$$\int_O \mathrm{Jac}(\tilde{w}_\varepsilon)\rho\,\mathrm{d}x \to \langle \nu, \rho\rangle = \pi\sum_{k=1}^K d_k\rho(x_k).$$

As $|\tilde{w}_\varepsilon| = |h| = 1$ in $T^+_{\delta_0} \setminus T^+_\delta$, and hence $\mathrm{Jac}(\tilde{w}_\varepsilon) = 0$ in $T^+_{\delta_0} \setminus T^+_\delta$, we can write by (2.47):

$$\int_O \mathrm{Jac}(\tilde{w}_\varepsilon)\rho\,\mathrm{d}x = \int_{\Omega\cup T_\delta} \mathrm{Jac}(\tilde{w}_\varepsilon)\,\mathrm{d}x = \pi\deg(h, \partial(\Omega\cup T_\delta)) = \pi N_g.$$

Since this reasoning works for any $\rho \in C^\infty_c(O)$ as long as $\rho \equiv 1$ in $\Omega^{(\delta)}$, we follow that $\mathrm{spt}\,\nu \subset O^{(\delta)}$ and $\nu(O) = N_g$. By the arbitrariness of δ, we conclude that $\mathrm{spt}(\nu) \subset \bar{\Omega}$.

3. step: We have already shown that $|\nu| = N_g$. Thus, it follows, by the compactness statement of Theorem 2.15 applied to the set O instead of Ω, that there exists $v \in \mathcal{D}^{(\mathrm{m})}_g(O)$ such that, up to a subsequence:

$$\tilde{v}_\varepsilon \rightharpoonup v \text{ weakly in } W^{1,2}_{\mathrm{loc}}(O \setminus \mathrm{spt}\,\mathrm{Jac}(v); \mathbb{R}^2).$$

Let us assume by contradiction that there exists a vortex center $x_0 \in \mathrm{spt}\,\mathrm{Jac}(v) \cap \partial\Omega$. By the smoothness of $\partial\Omega$, there also exists a scalar $r \in (0, \delta_0)$ and cone C with vertex x_0, as well as an opening angle $\alpha \in (0, \pi)$ such that $C \cap B_r(x_0) \subset T^+_{\delta_0}$ and $B_r(x_0) \cap \mathrm{spt}\,\mathrm{Jac}(v) = \{x_0\}$. So, on the one hand, we have by the definition of extended spin fields (v_ε) that $v = g$ in $T^+_{\delta_0}$, and hence:

$$\int_C |\nabla v|^2\,\mathrm{d}x < \infty. \tag{2.49}$$

On the other hand, with Lemma 2.4 we can find a sequence $(\lambda_n) \subset \mathbb{S}^1$ such that:

$$\lim_{n\to\infty}\left\| v|_{A_n} - \lambda_n\frac{x - x_0}{|x - x_0|}\right\|_{W^{1,2}}, \tag{2.50}$$

where $A_n = A_{2^{-(n+1)}r, 2^{-n}r}(x_0)$, and hence:

$$\begin{aligned}\int_{C\cap A_n} |\nabla v|^2\,\mathrm{d}x &\ge \int_{C\cap A_n}\left|\nabla\left(\lambda\frac{x - x_0}{|x - x_0|}\right)\right|^2\mathrm{d}x + \mathrm{o}_{n\to\infty}(1)\\ &\ge \frac{\alpha}{2}\log(2)\end{aligned}$$

for $n \ge N$, where $N \in \mathbb{N}$ is chosen sufficiently large enough. This is a contradiction to (2.50). □

In the next lemma we will show that $\mathcal{H}^1(J_{AC(u_\varepsilon)})$ remains bounded as $\varepsilon \to 0$. This is a nontrivial property since upon first glance the logarithmic energy bound in (2.39) does not seem to exclude $\mathcal{H}^1(J_{AC(u_\varepsilon)}) \approx |\log(\varepsilon)|$ as $\varepsilon \to 0$.

Lemma 2.7 (Uniform bound on the length of the jump set)
Let $(\varepsilon_n) \subset \mathbb{R}_+$ be the subsequence from Proposition 2.2, then:

$$\sup_{\varepsilon} \mathcal{H}^1(J_{AC(u_{\varepsilon_n})}) < \infty. \tag{2.51}$$

Proof. For the sake of simplicity, we will write (u_ε) instead of (u_{ε_n}), and (v_ε) instead of (v_{ε_n}). Furthermore, we will set $\tilde{u}_\varepsilon := AC(u_\varepsilon)$ and $\tilde{v}_\varepsilon := A(v_\varepsilon)$.

1. step: By definition $J_{\tilde{u}_\varepsilon}$ is contained in the edges of jump cells of u_ε. Moreover each jump cell must contain one nearest-neighbor pair (i, j) for which we have, by the lower bound in Item (i) of Definition 2.17, that:

$$f_\varepsilon^{(\mathtt{m})}(\varphi_\varepsilon(i) - \varphi_\varepsilon(j)) \geq \varepsilon, \quad e^{\imath\varphi_\varepsilon} = u_\varepsilon.$$

Consequently, we can estimate:

$$\mathcal{H}^1(J_{\tilde{u}_\varepsilon}) \leq 4\varepsilon \cdot \#JC_{u_\varepsilon} \leq 8XY_\varepsilon^{(\mathtt{m})}(u_\varepsilon, J_\varepsilon). \tag{2.52}$$

$$J_\varepsilon := \bigcup_{i \in JC_{u_\varepsilon}} i + \varepsilon[0,1]^2.$$

By the energy bound, (2.39) this leads to:

$$\mathcal{H}^1(J_{\tilde{u}_\varepsilon}) \leq C|\log \varepsilon| \tag{2.53}$$

for a constant $C < \infty$ independent of ε.

2. step: Now our goal is to improve this estimate. Let ν denote the limit from (2.40), and fix $r > 0$ such that the balls $B_r(x_k)$, $k = 1, \ldots, N_g$, are disjoint, where $\{x_k\} = \operatorname{spt}(\nu)$. By the convergence in (2.41), the smoothness of g, (2.43), the localized lower bound in (2.22), (2.42), and (2.52) we have for ε small enough

$$\begin{aligned}
C &\geq XY_\varepsilon^{(\mathtt{m})}(u_\varepsilon) - N_g\pi|\log \varepsilon| \\
&\geq \sum_{k=1}^{N_g} \left(XY_\varepsilon(v_\varepsilon, B_r(x_k)) - \pi \log\left(\frac{r}{\varepsilon}\right)\right) + XY_\varepsilon^{(\mathtt{m})}(v_\varepsilon, \Omega_r(\nu)) - N_g\pi|\log r| \\
&\geq \tilde{C} + XY_\varepsilon(v_\varepsilon, \Omega_r(\nu) \setminus J_\varepsilon) + XY_\varepsilon^{(\mathtt{m})}(u_\varepsilon, \Omega_r(\nu) \cap J_\varepsilon) - N_g\pi|\log r| \\
&\geq \tilde{C} + \frac{1}{2}\int_{\Omega_{2r}(\nu)\setminus J_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 \,\mathrm{d}x - N_g|\log r| + c\,\mathcal{H}^1(J_{\tilde{u}_\varepsilon} \cap \Omega_r)
\end{aligned} \tag{2.54}$$

for constants $C < \infty$, $\tilde{C} > -\infty$ and $c > 0$ independent of ε. (Note that in the last estimate above we have used that $\Omega_{2r} \subset (\Omega_r)_\varepsilon$ for sufficiently small ε.) Also, by (2.52) and (2.41), we see that:

$$|J_\varepsilon| \leq \#JC_{U_\varepsilon} \cdot \varepsilon^2 \leq 2XY_\varepsilon^{(\mathtt{m})}(u_\varepsilon)\varepsilon \leq C\varepsilon|\log \varepsilon|$$

for a constant $C < \infty$ independent of ε, which implies:

$$|\nabla \tilde{v}_\varepsilon| \mathbb{1}_{\Omega_{2r}(\nu)\setminus J_\varepsilon} \rightharpoonup |\nabla v| \mathbb{1}_{\Omega_{2r}(\nu)} \text{ weakly in } L^2(\Omega_{2r}(\nu)),$$

where v is the limit from (2.40). As a consequence, by the lower semicontinuity of the norm and the definition of the renormalized energy $\mathcal{W}$, this leads to:

$$\begin{aligned}
&\lim_{r\to 0}\liminf_{\varepsilon\to 0}\left(\frac{1}{2}\int_{\Omega_{2r}(\nu)\setminus J_\varepsilon}|\nabla\tilde{v}_\varepsilon|^2\,\mathrm{d}x - N_g\pi|\log r|\right)\\
&\geq \lim_{r\to 0}\left(\frac{1}{2}\int_{\Omega_{2r}}|\nabla v|^2\,\mathrm{d}x - N_g\pi|\log 2r|\right) - N_g\pi\log 2\\
&\geq \mathcal{W}(v) - N_g\pi\log 2. \qquad (2.55)
\end{aligned}$$

Combining (2.54) and (2.55), we derive:

$$\mathcal{H}^1(J_{\tilde{u}_\varepsilon}\cap\Omega_r) \leq C(v) + \alpha(\varepsilon, r), \qquad (2.56)$$

where $C(v) < \infty$ is a constant independent of r and ε, and the remainder $\alpha(\varepsilon, r) \geq 0$ satisfies: $\limsup_{r\to 0}\limsup_{\varepsilon\to 0}\alpha(\varepsilon, r) = 0$. Passing to the limit $\varepsilon \to 0$ and $r \to 0$ (precisely in this order) in (2.56) concludes the proof. □

The following lemma compares the gradients of $AC(u_\varepsilon)$ and $A(v_\varepsilon)$:

Lemma 2.8
There exists a constant $C(\mathfrak{m})$ such that for all $\varepsilon > 0$ and $u \in \mathcal{AS}_\varepsilon^{(g)}$, it holds for all $x \in \Omega_\varepsilon$, that:

$$|\nabla AC(u)(x)|^2 \leq C(\mathfrak{m})|A(v)(x)|^2, \qquad (2.57)$$

where $v := u^{\mathfrak{m}}$.

Proof. By the definition of $AC(u)$, all we have to do is check the inequality for $x \in i + \varepsilon Q$ with $i \in \Omega_\varepsilon^{(2)} \setminus JC_u$, as $AC(u_\varepsilon) = 0$ a.e. in J_ε. Let us assume that $x \in T_i^-$ with vertices denoted by i, j, k (in anticlockwise order) and take an angular lift φ of u. Using half-angle formulas we can express the squared distance between $u(i)$ and $u(j)$ as

$$|u(i) - u(j)|^2 = 2\sin^2\left(\frac{\varphi(i) - \varphi(j)}{2}\right) = 1 - \cos(\varphi(i) - \varphi(j)).$$

Similarly we also have

$$|v(i) - v(j)|^2 = 1 - \cos(\mathfrak{m}(\varphi(i) - \varphi(j))),$$

and therefore we can write

$$\frac{|u(i) - u(j)|^2}{|v(i) - v(j)|^2} = \frac{1 - \cos(\varphi(i) - \varphi(j))}{1 - \cos(\mathfrak{m}(\varphi(i) - \varphi(j)))}.$$

The function $t \mapsto \frac{1-\cos(t)}{1-\cos(\mathfrak{m}t)}$ has its only singularities at $\frac{2\pi}{\mathfrak{m}} + 2\pi\mathbb{Z}$. Since (i, j) is not a jump pair, and hence $\Delta_1^{\mathrm{el}}\varphi(i) \leq \frac{\pi}{\mathfrak{m}}$, the difference angle $\varphi(i) - \varphi(j)$ is uniformly bounded away from $\frac{2\pi}{\mathfrak{m}} + 2\pi\mathbb{Z}$ and we can bound:

$$\frac{|u(i) - u(j)|^2}{|v(i) - v(j)|^2} \leq C(\mathfrak{m})$$

for a constant $C(\mathfrak{m}) < \infty$ only depending on $\mathfrak{m}$. The same holds true if we replace (i,j) with (j,k). By the definition of the $AC(u)(x)$ as well as of $A(v)(x)$, we conclude:

$$\begin{aligned}|\nabla AC(u)(x)|^2 &= \varepsilon^{-2}(|u(j)-u(i)|^2 + |u(k)-v(j)|^2)\\ &\le C(\mathfrak{m})\varepsilon^{-2}(|v(j)-v(i)|^2 + |v(k)-v(j)|^2)\\ &= C(\mathfrak{m})|\nabla A(v)(x)|^2.\end{aligned}$$

The case $x \in T_i^+$ works similarly. □

Starting from a discrete spin field u_ε we either take the $\mathfrak{m}$-th power $u_\varepsilon^{\mathfrak{m}}$ of u_ε and interpolate the resulting discrete spins – an operation which would lead to $A(u_\varepsilon^{\mathfrak{m}})$ –, or we could first interpolate u_ε and then take the $\mathfrak{m}$-th power $(AC(u_\varepsilon))^{\mathfrak{m}}$, respectively. The next lemma shows that $(AC(u_\varepsilon))^{\mathfrak{m}}$ and $A(u_\varepsilon^{\mathfrak{m}})$ are close in L^2-sense for small ε.

Lemma 2.9

$$\lim_{\varepsilon\to 0}\int_\Omega |(AC(u_\varepsilon))^{\mathfrak{m}} - A(u_\varepsilon^{\mathfrak{m}})|^2 \,\mathrm{d}x = 0. \tag{2.58}$$

Proof. Let us shortly write $\tilde{u}_\varepsilon := AC(u_\varepsilon)$, $\tilde{v}_\varepsilon := A(u_\varepsilon^{\mathfrak{m}})$ and $\tilde{w}_\varepsilon := \tilde{u}_\varepsilon^{\mathfrak{m}}$. Consider $i \in \Omega_\varepsilon^{(2)} \setminus JC_{u_\varepsilon}$, and denote the other remaining vortices of $i+\varepsilon[1,1]^2$ by j, k, l in anticlockwise order. Let $x \in T_i^- := \mathrm{Conv}(\{i,j,k\})$ (similar as in the case $x \in T_i^+ := \mathrm{Conv}(\{i,k,l\})$). By (2.57), it follows that:

$$|\nabla \tilde{w}_\varepsilon(x)|^2 = \mathfrak{m}^2|\nabla\tilde{u}_\varepsilon|^2 \le \mathfrak{m}^2 C(\mathfrak{m})|\nabla\tilde{v}_\varepsilon|^2.$$

As $\tilde{w}_\varepsilon$ and $\tilde{v}_\varepsilon$ coincide at grid points, we derive by the intermediate value theorem and the bound above:

$$\begin{aligned}|\tilde{w}_\varepsilon(x)-\tilde{v}_\varepsilon(x)|^2 &\le 2(|\tilde{w}_\varepsilon(x)-w_\varepsilon(i)|^2 + |\tilde{v}_\varepsilon(x)-\tilde{v}_\varepsilon(i)|^2)\\ &\le 2(1+\mathfrak{m}^2C(\mathfrak{m}))|x-i|^2(|\nabla w_\varepsilon(\xi_x)|^2 + |\nabla\tilde{v}_\varepsilon(x)|^2)\\ &\le \tilde{C}(\mathfrak{m})\varepsilon^2|\nabla\tilde{v}_\varepsilon|^2,\end{aligned}$$

for a constant $\tilde{C}(\mathfrak{m}) < \infty$ only depending on $\mathfrak{m}$. In the case $x \in i+\varepsilon[0,1]^2$ for $i \in JC_{u_\varepsilon}$ we use the estimate

$$|\tilde{w}_\varepsilon(x)-\tilde{v}_\varepsilon(x)|^2 \le 4$$

instead. As each jump cell contains at least one jump-pair, which itself contributes at least ε to the energy, we see, by (2.39), that:

$$|J_\varepsilon| = \#JC_{u_\varepsilon}\cdot\varepsilon^2 \le C\varepsilon|\log\varepsilon|$$

for a constant $C < \infty$ independent of ε. Combining all the aforementioned estimates leads to:

$$\begin{aligned}\int_\Omega |w_\varepsilon(x)-\tilde{v}_\varepsilon(x)|^2\,\mathrm{d}x &= \int_{\Omega_\varepsilon\setminus J_\varepsilon}|w_\varepsilon(x)-\tilde{v}_\varepsilon(x)|^2\,\mathrm{d}x + \int_{J_\varepsilon}|w_\varepsilon(x)-\tilde{v}_\varepsilon(x)|^2\,\mathrm{d}x\\ &\le C(\mathfrak{m})\varepsilon^2\int_{\Omega_\varepsilon}|\nabla\tilde{v}_\varepsilon|^2\,\mathrm{d}x + 4|J_\varepsilon| = \mathrm{o}_{\varepsilon\to 0}(1)\end{aligned}$$

as is desired. □

Proof of the compactness statement in Theorem 2.16. Set $\tilde{u}_\varepsilon := AC(u_\varepsilon)$ and $\tilde{v}_\varepsilon := A(v_\varepsilon)$, and assume that we have already selected – without relabeling – the subsequence from Proposition 2.2. We already know that, up to a subsequence, $\frac{1}{\mathfrak{m}}\mu_{v_\varepsilon} \overset{\flat}{\rightharpoonup} \mu := \frac{1}{\mathfrak{m}}\nu \in X_g^{(\mathfrak{m})}$, where ν is the limit from (2.40). Fix $r > 0$ small enough such that the balls in $B_r(x_k)$, $k = 1, \dots, N_g$ are disjoint, where $\{x_k\} = \operatorname{spt}(\mu)$. We wish to prove that:

$$\sup_{\varepsilon>0}\left(\|\tilde{u}_\varepsilon\|_{L^\infty(\Omega_r(\mu);\mathbb{R}^2)} + \int_{\Omega_r(\mu)} |\nabla\tilde{u}_\varepsilon|^2\,\mathrm{d}x + \mathcal{H}^1(J_{\tilde{u}_\varepsilon}\cap\Omega_r(\mu))\right) < \infty. \tag{2.59}$$

By the fact that $\|\tilde{u}_\varepsilon\|_{L^\infty} \leq 1$ and Lemma 2.7, we only need to bound the approximate gradient in (2.59). With (2.57) and Proposition 2.2 we have:

$$\sup_{\varepsilon>0}\int_{\Omega_r(\mu)}|\nabla\tilde{u}_\varepsilon|^2\,\mathrm{d}x \leq C(\mathfrak{m})\sup_{\varepsilon>0}\int_{\Omega_r(\mu)}|\nabla\tilde{v}_\varepsilon|^2\,\mathrm{d}x < \infty,$$

as is desired, where $C(\mathfrak{m})$ is the ε-independent constant from (2.57). Using Theorem 2.7 we can find $u = u^{(r)} \in SBV^2(\Omega_r(\mu);\mathbb{R}^2)$ such that, up to a subsequence, $\tilde{u}_\varepsilon \rightharpoonup u$ weakly in $SBV^2(\Omega_r(\mu);\mathbb{R}^2)$. By a standard diagonal sequence argument, we can find a common subsequence such that $\tilde{u}_\varepsilon \rightharpoonup u$ weakly in $SBV^2_{\mathrm{loc}}(\Omega\setminus\operatorname{spt}\mu;\mathbb{R}^2)$. It remains to prove that $u \in \mathcal{D}_g^{(\mathfrak{m})}$. For this, note that by Proposition 2.2, we have that, up to a subsequence, $\tilde{v}_\varepsilon \rightharpoonup v$, where $v \in \mathcal{D}^1_{g^{\mathfrak{m}}}(\Omega)$. Furthermore, by (2.58) it follows that $u^{\mathfrak{m}} = v$ at a.e. point in Ω. With Theorem 2.5 this shows that $|u| = 1$ a.e., $(u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$-a.e. on J_u, and:

$$\operatorname{Jac}(u) = \frac{1}{\mathfrak{m}}\operatorname{Jac}(u)^{\mathfrak{m}} = \mu.$$

Finally, by the lower semicontinuity of the perimeter with respect to weak convergence in SBV^2 and (2.51) for any $r > 0$ small enough, it holds that:

$$\mathcal{H}^1(J_u\cap\Omega_r(\mu)) \leq \liminf_{\varepsilon\to 0}\mathcal{H}^1(J_{\tilde{u}_\varepsilon}\cap\Omega_r(\mu)) \leq \sup_{\varepsilon>0}\mathcal{H}^1(J_{\tilde{u}_\varepsilon}) < \infty.$$

Taking the limit $r \to 0$ in the estimate above, concludes the proof. □

2.4.2 Gamma-liminf

In this subsection, we will prove the Γ-liminf inequality stated in Theorem 2.16. We assume the same notation as in Section 2.4.1. It is not restrictive to assume that the energy bound in (2.39) is achieved by the sequence (u_ε), since in the other case the liminf inequality is trivially satisfied. Furthermore, we select a subsequence (without relabeling) for which the liminf in (2.37) is, in fact, a limit. As (2.39) is satisfied, we can apply the compactness statement in Item (i) of Theorem 2.16, which shows the existence of a $u \in \mathcal{D}_g^{(\mathfrak{m})}$ such that, up to taking a subsequence:

$$\begin{aligned} AC(u_\varepsilon) &\rightharpoonup u \text{ weakly in } SBV^2_{\mathrm{loc}}(\Omega\setminus\operatorname{spt}\operatorname{Jac}(u);\mathbb{R}^2),\\ A(v_\varepsilon) &\rightharpoonup v \text{ weakly in } W^{1,2}_{\mathrm{loc}}(\Omega\setminus\operatorname{spt}\operatorname{Jac}(u);\mathbb{R}^2, \end{aligned}$$

where $v := u^{\mathfrak{m}}$.

Let $\delta_0 > 0$ be small enough such that the projection $\Pi_{\partial\Omega}$ is well defined and smooth. We extend g into T_{δ_0} by setting $g(x) := g(\Pi_{\partial\Omega}(x))$, and define $h := g^{\mathfrak{m}}$ in T_{δ_0}. From this point on, we implicitly assume that u_ε and v_ε are extended into $(\Omega^{(\delta_0)})_\varepsilon^{(0)}$ by setting $u_\varepsilon(i) := g(i)$ and $v_\varepsilon(i) := h(i)$ outside of $\Omega_\varepsilon^{(0)}$, respectively. Directly by the definition and the previous convergences, we see that:

$$AC(u_\varepsilon) \rightharpoonup u \text{ weakly in } SBV^2_{\mathrm{loc}}(\Omega^{(\delta_0)} \setminus \operatorname{spt}\operatorname{Jac}(u); \mathbb{R}^2), \tag{2.60}$$

$$A(v_\varepsilon) \rightharpoonup v \text{ weakly in } W^{1,2}_{\mathrm{loc}}(\Omega^{(\delta_0)} \setminus \operatorname{spt}\operatorname{Jac}(u); \mathbb{R}^2), \tag{2.61}$$

where $v = u^{\mathfrak{m}}$ a.e. in $\Omega^{(\delta_0)} :=$ and $u = g$ a.e. in $T^+_{\delta_0}$ (and hence also $v = h$ a.e. in $T^+_{\delta_0}$). Our main goal is to show a corresponding liminf inequality for the sequence (u_ε) of extended spin fields:

Proposition 2.3
With the notation above, it holds for any $\delta \in (0, \delta_0)$:

$$\begin{aligned}\lim_{\varepsilon\to 0} &\left(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, \Omega^{(\delta)}) - N_g \pi |\log\varepsilon| \right) \\ &= \mathcal{W}^{(\mathfrak{m})}(u, \Omega^{(\delta)}) + \int_{J_u \cap \Omega^{(\delta)}} |\nu_u|_1 \,\mathrm{d}\mathcal{H}^1 + N_g\gamma,\end{aligned} \tag{2.62}$$

where γ is the core-energy defined in (2.28) *and:*

$$\mathcal{W}^{(\mathfrak{m})}(u, \Omega^{(\delta)}) := \lim_{r\to 0} \left(\mathfrak{m}^2 \int_{(\Omega^{(\delta)})_r(\operatorname{Jac}(u))} |\nabla u|^2 \,\mathrm{d}x - N_g \pi |\log r| \right).$$

Before coming to the proof of Proposition 2.3, let us show that it leads directly to the desired liminf inequality:

Proof of the Γ-liminf of Theorem 2.16. By the definition of $\mathcal{W}^{(\mathfrak{m})}(u, \Omega^{(\delta)})$ and the smoothness of g, we can write

$$\mathcal{W}^{(\mathfrak{m})}(u, \Omega^{(\delta)}) = \mathcal{W}^{(\mathfrak{m})}(u) + \mathfrak{m}^2 \int_{T^+_\delta} |\nabla g|^2 \,\mathrm{d}x = \mathcal{W}^{(\mathfrak{m})}(u) + \mathrm{o}_{\delta\to 0}(1).$$

By Lemma 2.1 and the smoothness of g we have that:

$$\int_{J_u} |\nu_u|_1 \,\mathrm{d}\mathcal{H}^1 = \int_{J_u\cap\Omega} |\nu_u|_1 \,\mathrm{d}x + \int_{\{u\neq g\}\cap\partial\Omega} |\nu_\Omega|_1 \,\mathrm{d}\mathcal{H}^1,$$

where ν_Ω is the outer unit-normal of Ω. Moreover, with (2.42) and the smoothness of g, we see that for $\delta \in (0, \frac{\delta_0}{2})$:

$$\begin{aligned}\limsup_{\varepsilon\to 0} XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, T^+_\delta) &\le \limsup_{\varepsilon\to 0} \int_{(T_{2\delta}(\partial\Omega))_\varepsilon} |\nabla A(g)|^2 \,\mathrm{d}x \\ &\le C \|\nabla g\|_{L^\infty} |T_{2\delta}| = \mathrm{o}_{\delta\to 0}(1).\end{aligned}$$

Combining all the aforementioned results with (2.62) then leads to:

$$\begin{aligned}\liminf_{\varepsilon\to 0} &\left(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, \Omega) - N_g\pi|\log\varepsilon| \right) \\ &\ge \liminf_{\varepsilon\to 0} \left(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, \Omega^{(\delta)}) - N_g\pi|\log\varepsilon| \right) - \limsup_{\varepsilon\to 0} XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, T^+_\delta) \\ &\ge \mathcal{W}(u) + \int_{J_u} |\nu_u|_1 \,\mathrm{d}\mathcal{H}^1 + \int_{\{u\neq g\}\cap\partial\Omega} |\nu_\Omega|_1 \,\mathrm{d}\mathcal{H}^1 + \mathrm{o}_{\delta\to 0}(1).\end{aligned}$$

The desired liminf inequality follows by sending $\delta \to 0$. □

The remainder of this subsection will be concerned with the proof of the convergence in Proposition 2.3. Without further mention, we assume that u_ε and v_ε are properly extended into $\varepsilon\mathbb{Z}^2 \cap \Omega^{(\delta_0)}$ (as described in the beginning of this subsection), and shortly write $\tilde{u}_\varepsilon := AC(u_\varepsilon)$ as well as $\tilde{v}_\varepsilon := A(v_\varepsilon)$. Let us also fix $\delta \in (0, \delta_0)$ and define $O := \Omega^{(\delta)}$. The main idea at this point is to split the domain into two components: a tubular neighborhood around a part of the jump set J_u and its complement (see also figure Figure 2.6 for further clarification). More precisely, as J_u is $\mathcal{H}^1$-rectifiable, we can find a

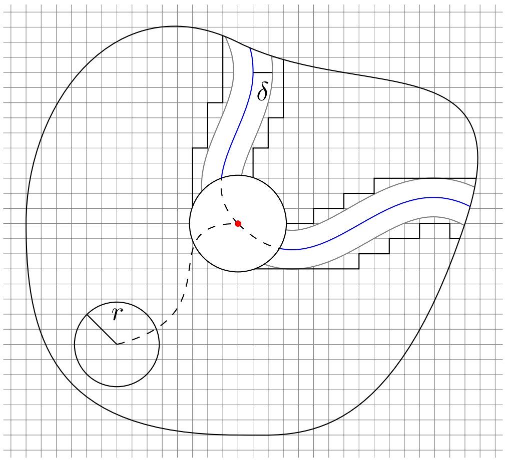


Figure 2.6: Discretized tubular neighborhood around J_N (blue).

family $\{C_i\}_{i\in\mathbb{N}}$ of compact C^1-graphs such that $\mathcal{H}^1(J_u \setminus \bigcup_{i\in\mathbb{N}} C_i) = 0$. Let $\mu := \frac{1}{\pi} \operatorname{Jac}(u)$, then for fixed $d > 0$, $r > 0$, and $N \in \mathbb{N}$ we define

$$T^N_{d,r} := \{x \in O_r(\mu) \colon \operatorname{dist}(x, J^N_u) < \delta\}, \quad J^N_u := J_u \cap \left(\bigcup_{i=1}^N C_i\right).$$

From the piecewise C^1-regularity of the curves $\{C_i\}$ we cannot assure a priori that $\partial T^N_{d,r}$ is Lipschitz regular. We will now replace $T^N_{d,r}$ by a set $\tilde{T}^N_{d,r}$ which has a Lipschitz boundary and satisfies $T^N_{d,r} \subset \tilde{T}^N_{d,r} \subset T^N_{2d,r}$. Note that for all $\varepsilon > 0$ the set $(O \setminus T^N_{d,r})_\varepsilon$ has a Lipschitz regular boundary with:

$$\operatorname{dist}(\partial(O \setminus T^N_{d,r})_\varepsilon, T^N_{d,r}) < \sqrt{2}\varepsilon.$$

Consequently:

$$\tilde{T}^N_{d,r} := O \setminus (O \setminus T^N_{d,r})_{\frac{d}{\sqrt{2}}}$$

has all the desired properties. Outside $\tilde{T}^N_{d,r}$, the following estimate holds true:

Lemma 2.10
Together with the above specified notation, we have:

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\Big(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, O\setminus \tilde{T}_{d,r}^N) - N_g\pi|\log\varepsilon|\Big)\\ &\quad\geq \mathcal{W}^{(\mathfrak{m})}(u) + N_g\gamma - \frac{\mathfrak{m}}{2}\int_{\tilde{T}_{d,r}^N}|\nabla u|^2\,\mathrm{d}x,\end{aligned}\tag{2.63}$$

where $\mathcal{W}^{(\mathfrak{m})}(u)$ is the renormalized energy of u in the extended domain O.

Proof. By construction we assured that $\partial(O\setminus\tilde{T}_{d,r}^N)$ has Lipschitz boundary, hence we can apply the liminf inequality (2.31) for the sequence $\{v_\varepsilon\}$ restricted to the set $(O\setminus\tilde{T}_{d,r}^N)$. With Item (i) of Definition 2.16 and the definition of $\mathcal{W}^{(\mathfrak{m})}$, this lead to:

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\Big(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, O\setminus \tilde{T}_{d,r}^N) - N_g\pi|\log\varepsilon|\Big)\\ &\quad\geq \liminf_{\varepsilon\to 0}\Big(XY_\varepsilon(\tilde{v}_\varepsilon, O\setminus \tilde{T}_{d,r}^N) - N_g\pi|\log\varepsilon|\Big)\\ &\quad\geq \mathcal{W}(\tilde{v}, O\setminus\tilde{T}_{d,r}^N) + N_g\gamma\\ &\quad\geq \mathcal{W}^{(\mathfrak{m})}(u) + N_g\gamma - \frac{\mathfrak{m}}{2}\int_{\tilde{T}_{d,r}^N}|\nabla u|^2\,\mathrm{d}x,\end{aligned}$$

as desired. □

Inside the tubular neighborhood $\tilde{T}_{d,r}^N$ the following liminf inequality holds true:

Proposition 2.4
Together with the above specified notation, the following holds true:

$$\liminf_{\varepsilon\to 0} XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, \tilde{T}_{d,r}^N) \geq \int_{J_u^N\cap O_r(\mu)}|\nu_u|_1\,\mathrm{d}\mathcal{H}^1\,.\tag{2.64}$$

Before getting to the proof of Proposition 2.4, let us first show that it leads together with (2.63) to (2.62):

Proof of Proposition 2.3. Combining (2.63) and (2.64), we derive:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, O) - N_g\pi|\log\varepsilon|\Big)\\ &\quad\geq \liminf_{\varepsilon\to 0}\Big(XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, O\setminus \tilde{T}_{d,r}^N) - N_g\pi|\log\varepsilon|\Big) + \liminf_{\varepsilon\to 0} XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, \tilde{T}_{d,r}^N)\\ &\quad\geq \mathcal{W}^{(\mathfrak{m})}(u) + \int_{J_u}|\nu_u|_1\,\mathrm{d}\mathcal{H}^1 + N_g\gamma - \alpha(d,r,N),\end{aligned}\tag{2.65}$$

where:

$$\alpha(d,r,N) := -\frac{\mathfrak{m}}{2}\int_{\tilde{T}_{d,r}^N}|\nabla u|^2\,\mathrm{d}x - 2\,\mathcal{H}^1(J_u\setminus(J_u^N\cap O_r(\mu))).$$

As $|\nabla u|\in L^2(O_r(\mu))$, we have that:

$$\lim_{d\to 0}\alpha(d,r,N) = -2\,\mathcal{H}^1(J_u\setminus(J_u^N\cap O_r(\mu)))$$

By the definition of J_u^N and the fact that $\mathcal{H}^1(J_u) \le \mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1(\partial\Omega) < \infty$ (see also (2.51)) it follows that:

$$\lim_{r\to 0} \lim_{N\to\infty} \lim_{d\to 0} \alpha(d,r,N) = 0.$$

Thus, letting $d \to 0$, then $N \to \infty$, and finally $r \to 0$ in (2.65) (in exactly this order) leads to the desired result. □

As the proof of Proposition 2.4 is rather technical, we will split it up into several lemmata. We start by defining *exotic nearest-neighbor pairs*:

Definition 2.19 (Exotic pairs)
Given $w \in \mathcal{AS}_\varepsilon(O)$ we call $(i,j) \in O_\varepsilon^{(1)}$ an *exotic nearest-neighbor pair* if and only if:

$$\operatorname{dist}(\varphi(i) - \varphi(j), \frac{2\pi}{\mathfrak{m}}\mathbb{Z}) > \sqrt[3]{\varepsilon}. \tag{2.66}$$

The set of all exotic pairs will be denoted by $N_e(w)$.

Note that there is nothing special about the choice of the power $\frac{1}{3}$ in (2.66). In fact any power in $(0, \frac{1}{2})$ would also work. In the next lemma, we estimate the number of exotic pairs of u_ε as $\varepsilon \to 0$.

Lemma 2.11
There exists a constant $C > 0$ independent of ε such that:

$$\# N_e(u_\varepsilon) \le C\varepsilon^{-\frac{2}{3}} |\log \varepsilon|. \tag{2.67}$$

Proof. Given any exotic pair $(i,j) \in N_e(u_\varepsilon)$, we have by Item (i) of Definition 2.16 and Taylor expansion:

$$\begin{aligned} \frac{1}{2}\mathfrak{m}^2 \varepsilon^{\frac{2}{3}} &\le 1 - \cos(\mathfrak{m}\sqrt[3]{\varepsilon}) \\ &\le 1 - \cos(\mathfrak{m}(\varphi_\varepsilon(i) - \varphi_\varepsilon(j))) \le f_\varepsilon^{(\mathfrak{m})}(\varphi_\varepsilon(i) - \varphi_\varepsilon(j)), \end{aligned}$$

where φ_ε is an arbitrary angular lift of u_ε. Summing the above inequality over all exotic pairs and employing the energy bound (2.39), we see that:

$$\begin{aligned} \# N_e(u_\varepsilon) \cdot \frac{1}{2}\mathfrak{m}^2 \varepsilon^{\frac{2}{3}} &\le \sum_{(i,j)\in N_e(u_\varepsilon)} f_\varepsilon^{(\mathfrak{m})}(\varphi_\varepsilon(i) - \varphi_\varepsilon(j)) \\ &\le 2XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon) \le 2N_g \pi |\log \varepsilon|. \end{aligned}$$

Dividing both sides of the estimate above by $\frac{1}{2}\mathfrak{m}\varepsilon^{\frac{2}{3}}$, leads to (2.67). □

Let us define $E_\varepsilon^{(\mathrm{ng})}$ as the union of $\partial(i + \varepsilon[0,1]^2)$ over all cells $i + \varepsilon[0,1]^2$ containing at least one exotic pair of u_ε. Due to (2.67) we see that the proportion of the jump set $J_{\tilde{u}_\varepsilon}$ contained in $E_\varepsilon^{(\mathrm{ng})}$ has negligible perimeter, as $\varepsilon \to 0$:

$$\mathcal{H}^1(J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{ng})}) \le \mathcal{H}^1(E_\varepsilon^{(\mathrm{ng})}) = \frac{1}{2} \cdot 8 \# N_e(u_\varepsilon) \varepsilon \le C |\log \varepsilon| \sqrt[3]{\varepsilon} \to 0 \text{ as } \varepsilon \to 0, \tag{2.68}$$

where $C < \infty$ is a constant independent of ε. Hence, in order to prove (2.64), we can ignore the part of $J_{\tilde{u}_\varepsilon}$ contained in $E_\varepsilon^{(\mathrm{ng})}$. Let us finally define the map $\mathcal{J}\colon (\mathbb{R}^2)_\varepsilon^{(1)} \to (\mathbb{R}^2)_\varepsilon^{(1)}$ through:

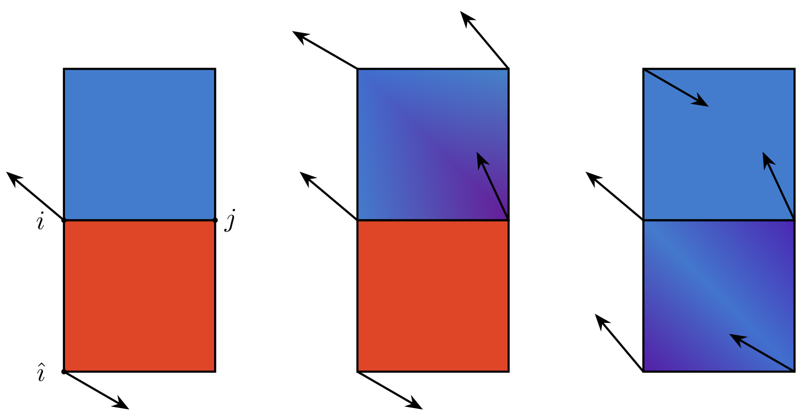
$$\begin{aligned}\mathcal{J}(i, i+\varepsilon e_1) &= (i-\varepsilon e_1, i) \text{ for all } i \in \varepsilon\mathbb{Z}^2,\\ \mathcal{J}(i, i+\varepsilon e_2) &= (i-\varepsilon e_1, i) \text{ for all } i \in \varepsilon\mathbb{Z}^2,\\ \mathcal{J}(i,j) &= \mathcal{J}(j,i) \text{ for all } (i,j) \in (\mathbb{R}^2)_\varepsilon^{(1)}.\end{aligned}$$

Note that we will usually write $(\hat{\imath}, \hat{\jmath})$ instead of $\mathcal{J}(i,j)$. In the next lemma, we will show that given an edge $[i,j]$ such that $[i,j] \cap E_\varepsilon^{(\mathrm{ng})} = \emptyset$, the jump of $\tilde{u}_\varepsilon$ on $[i,j]$ is equal to $|u_\varepsilon(\hat{\imath}) - u_\varepsilon(\hat{\jmath})|$ up to an error that is negligible, as $\varepsilon \to 0$.

Lemma 2.12
Let $(i,j) \in O_\varepsilon^{(1)}$ such that the edge $[i,j] \cap E_\varepsilon^{(\mathrm{ng})} = \emptyset$, then for all $x \in [i,j]$ it holds that:

$$|u_\varepsilon(\hat{\imath}) - u_\varepsilon(\hat{\jmath})| - 2\sqrt[3]{\varepsilon} \le |\tilde{u}_\varepsilon^+(x) - \tilde{u}_\varepsilon^-(x)| \le |u_\varepsilon(\hat{\imath}) - u_\varepsilon(\hat{\jmath})| + 2\sqrt[3]{\varepsilon}. \tag{2.69}$$

Proof. Without loss of generality, we assume that $j = i + \varepsilon e_1$ (the proof is similar in the other case). Furthermore, we choose the normal $\nu_{\tilde{u}_\varepsilon}|_{[i,j]}$ such that it points upwards. The estimate in (2.69) is trivially satisfied if $\mathcal{H}^1(J_{\tilde{u}_\varepsilon} \cap [i,j]) = 0$. Let us therefore assume that $\mathcal{H}^1(J_{\tilde{u}_\varepsilon} \cap [i,j]) > 0$, which by the definition of $\tilde{u}_\varepsilon = AC(u_\varepsilon)$ can only hold true if at least one of the cells $\hat{\imath}+\varepsilon[0,1]^2$ or $\hat{\jmath}+\varepsilon[0,1]^2$ is a jump cell, where in our current setting $\hat{\imath} = i - \varepsilon e_2$ and $\hat{\jmath} = i$.



(a) Both cells jump-cells. (b) Bottom cell jump-cell. (c) Top cell jump-cell.

Figure 2.7: Possible cases for cells attached to the edge $[i,j]$

In the case that both cells are jump cells (see also Figure 2.7(a)), we have for all $x \in [i,j]$:

$$|\tilde{u}_\varepsilon^+(x) - \tilde{u}_\varepsilon^-(x)| = |u_\varepsilon(\hat{\imath}) - u_\varepsilon(\hat{\jmath})|,$$

and (2.69) is satisfied. In the case that $\hat{\imath}+\varepsilon[0,1]^2$ is a jump cell while $\hat{\imath}+\varepsilon[0,1]^2$ is not (see also Figure 2.7(b)), we see that for all $x \in [i,j]$:

$$\tilde{u}_\varepsilon^-(x) = u_\varepsilon(\hat{\imath}).$$

As $\hat{\imath} + \varepsilon[0,1]^2$ is not a jump cell, (i,j) cannot be a jump pair. Furthermore, as $[i,j] \cap E_\varepsilon^{(\mathrm{ng})} = \emptyset$ it follows that:

$$|u_\varepsilon(i) - u_\varepsilon(j)| \le \sqrt[3]{\varepsilon}.$$

Hence:

$$\begin{aligned}
|\tilde{u}_\varepsilon^+(x) - \tilde{u}_\varepsilon^-(x)| &= |\tilde{u}_\varepsilon^+(x) - u_\varepsilon(\hat{\imath})| \\
&= |u_\varepsilon(\hat{\jmath}) + \frac{x_1 - \hat{\jmath}_1}{\varepsilon}(u_\varepsilon(j) - u_\varepsilon(i)) - u_\varepsilon(\hat{\imath})| \\
&\le |u_\varepsilon(\hat{\jmath}) - u_\varepsilon(\hat{\imath})| + \sqrt[3]{\varepsilon}
\end{aligned}$$

which is the desired upper bound in (2.69). The lower bound follows similarly. Lastly, in the remaining case that $\hat{\imath} + \varepsilon[0,1]^2$ is a jump cell while $\hat{\jmath} + \varepsilon[0,1]^2$ is not, we have that:

$$\tilde{u}_\varepsilon^+(x) = u_\varepsilon(\hat{\jmath}).$$

Moreover, as $[i,j] \cap E_\varepsilon^{(\mathrm{ng})} = \emptyset$ we also have $[\hat{\imath},\hat{\jmath}] \cap E_\varepsilon^{(\mathrm{ng})} = \emptyset$. Consequently:

$$\max\{|u_\varepsilon(j) - u_\varepsilon(i)|, |u_\varepsilon(i) - u_\varepsilon(\hat{\jmath})|\} \le \sqrt[3]{\varepsilon}.$$

Combining the aforementioned results, we thus conclude:

$$\begin{aligned}
|\tilde{u}_\varepsilon^+(x) - \tilde{u}_\varepsilon^-(x)| &= |u_\varepsilon(\hat{\jmath}) - u_\varepsilon(i) - \frac{x_1 - i_1}{\varepsilon}(u_\varepsilon(j) - u_\varepsilon(i))| \\
&\le |u_\varepsilon(i) - u_\varepsilon(j)| \\
&\le |u_\varepsilon(\hat{\jmath}) - u_\varepsilon(\hat{\jmath})| + |u_\varepsilon(j) - u_\varepsilon(\hat{\imath})| \\
&\le |u_\varepsilon(\hat{\jmath}) - u_\varepsilon(\hat{\jmath})| + 2\sqrt[3]{\varepsilon}
\end{aligned}$$

which is the desired upper bound in (2.69). The lower bound follows similarly. □

Let $T_\varepsilon^{(\mathrm{ng})}$ be the union of all cells $i + \varepsilon[0,1]^2 \subset O$, each one containing at least one exotic nearest neighbor pair, and:

$$T_\varepsilon := \tilde{T}^N_{\frac{d}{2},r} \setminus T_\varepsilon^{(\mathrm{ng})}.$$

We also define:

$$E_\varepsilon^{(\mathrm{s})} := \bigcup\Big\{[i,j] \colon (i,j) \in (T_\varepsilon)_\varepsilon^{(1)},\, |\tilde{u}_\varepsilon^+(x) - \tilde{u}_\varepsilon^-(x)| \le 3\sqrt[3]{\varepsilon} \text{ for all } x \in [i,j]\Big\},$$

as wells as:

$$E_\varepsilon^{(\mathrm{b})} := \bigcup\Bigg\{[i,j] \colon (i,j) \in (T_\varepsilon)_\varepsilon^{(1)},\, |\tilde{u}_\varepsilon^+(x) - \tilde{u}_\varepsilon^-(x) - 2\sin(\frac{l\pi}{\mathfrak{m}})| \le 3\sqrt[3]{\varepsilon} \\ \text{for all } x \in [i,j] \text{ and some } l = 1, \dots \mathfrak{m} - 1\Bigg\}.$$

By Lemma 2.12, we see that for ε small enough any edge $[i,j] \subset T_\varepsilon$ must be either contained in $E_\varepsilon^{(\mathrm{s})}$ or in $E_\varepsilon^{(\mathrm{b})}$. We are now well equipped in order to prove (2.64).

Proof of Proposition 2.4. 1. step: We wish to employ Theorem 2.8. For this consider for given $t \in (0,1)$ the function $\vartheta_t \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\vartheta_t(a,b) := \begin{cases} t + \frac{1-t}{2\sin(\frac{\pi}{\mathrm{m}})}|a-b| & \text{if } |a-b| < 2\sin(\frac{\pi}{\mathrm{m}}), \\ 1 & \text{otherwise.} \end{cases}$$

By a straightforward computation one can see that ϑ_t is positive, symmetric, and satisfies the triangular inequality. As $|u^+(x) - u^-(x)| \geq 2\sin(\frac{\pi}{\mathrm{m}})$ for $\mathcal{H}^1$-a.e. $x \in J_u$ we see by (2.9) with ϑ_t as above and $\varphi := |\cdot|_1$

$$\begin{aligned} \int_{J_u^N \cap O_r(\mu)} |\nu_1|_1 \,\mathrm{d}\mathcal{H}^1 &= \int_{J_u \cap \tilde{T}^N_{\frac{d}{2},r}} |\nu_u|_1 \,\mathrm{d}\mathcal{H}^1 \\ &= \int_{J_u \cap \tilde{T}^N_{\frac{d}{2},r}} \vartheta_t(u^+, u^-)|\nu|_1 \,\mathrm{d}\mathcal{H}^1 \\ &\leq \liminf_{\varepsilon\to 0} \int_{J_{\tilde{u}_\varepsilon} \cap \tilde{T}^N_{\frac{d}{2},r}} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-) \,\mathrm{d}\mathcal{H}^1 . \end{aligned} \tag{2.70}$$

2. step: We will now show that the last integral in (2.70) can be restricted to $J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{b})}$ without perturbing the liminf "too much." In this regard, notice that by Lemma 2.12, we can write

$$J_{\tilde{u}_\varepsilon} \cap \tilde{T}^N_{\frac{d}{2},r} \subset \left(J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{b})}\right) \dot\cup \left(J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{s})}\right) \dot\cup \left(J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{ng})}\right). \tag{2.71}$$

By (2.68) and the boundedness of ϑ_t, we have:

$$\limsup_{\varepsilon\to 0} \int_{J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{ng})}} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-) \,\mathrm{d}\mathcal{H}^1 \leq C \limsup_{\varepsilon\to 0} |\log\varepsilon| \sqrt[3]{\varepsilon} = 0, \tag{2.72}$$

where $C < \infty$ is a constant independent of ε. Due to (2.51), as well as our choice of ϑ_t and $E_\varepsilon^{(\mathrm{s})}$, we derive that:

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{s})}} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-) \,\mathrm{d}\mathcal{H}^1 &\leq \limsup_{\varepsilon\to 0} \left(t + \frac{1-t}{2\sin(\frac{\pi}{\mathrm{m}})} \sqrt[3]{\varepsilon} \right) \mathcal{H}^1(J_{\tilde{u}_\varepsilon}) \\ &\leq Ct \end{aligned} \tag{2.73}$$

for some constant $C < \infty$ independent of ε and t. Combining (2.6), (2.72), and (2.73) leads to:

$$\begin{aligned} \int_{J_{\tilde{u}_\varepsilon} \cap \tilde{T}^N_{\frac{d}{2},r}} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-) \,\mathrm{d}\mathcal{H}^1 &\leq \liminf_{\varepsilon\to 0} \int_{J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{b})}} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-) \,\mathrm{d}\mathcal{H}^1 \\ &\quad + \limsup_{\varepsilon\to 0} \int_{J_{\tilde{u}_\varepsilon} \cap \left(E_\varepsilon^{(\mathrm{s})} \cup E_\varepsilon^{(\mathrm{ng})}\right)} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-) \,\mathrm{d}\mathcal{H}^1 \\ &\leq \liminf_{\varepsilon\to 0} \int_{J_{\tilde{u}_\varepsilon} \cap E_\varepsilon^{(\mathrm{b})}} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-) \,\mathrm{d}\mathcal{H}^1 + Ct \end{aligned} \tag{2.74}$$

for the same constant C as in (2.73).

3. step: It remains to estimate the liminf in (2.74) from above. Let us consider an arbitrary edge $[i,j] \subset E_\varepsilon^{(\mathrm{b})}$, then by the definition of $E_\varepsilon^{(\mathrm{b})}$ and (2.69), we see that the pair $(\hat{\imath},\hat{\jmath})$ satisfies:

$$|u_\varepsilon(\hat{\imath}) - u_\varepsilon(\hat{\jmath})| \geq \min_{x\in[i,j]} |\tilde{u}_\varepsilon^+(x) - \tilde{u}_\varepsilon^-(x)| - 2\sqrt[3]{\varepsilon} \geq 2\sin\left(\frac{\pi}{\mathfrak{m}}\right) - 5\sqrt[3]{\varepsilon}.$$

Hence, we can find $\varepsilon_0 > 0$ independent of i and j such that for all $\varepsilon < \varepsilon_0$ the pair $(\hat{\imath},\hat{\jmath})$ is forced to be a jump pair. By Item (i) of Definition 2.16, this leads to:

$$\int_{J_{\tilde{u}_\varepsilon}\cap[i,j]} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-)\, d\,\mathcal{H}^1 \leq \varepsilon \leq f_\varepsilon^{(\mathfrak{m})}(\varphi_\varepsilon(\hat{\imath}) - \varphi_\varepsilon(\hat{\jmath})), \tag{2.75}$$

where φ_ε is an angular lift of u_ε. Note that for each edge $[i,j] \subset E_\varepsilon^{(\mathrm{b})}$ we have that $(\hat{\imath},\hat{\jmath}) \in (\tilde{T}_{d,r}^N)_\varepsilon^{(1)}$. Furthermore, the map $\mathcal{J}$ is one-to-one, and thus summing up (2.75) over all edges $[i,j]$ contained in $E_\varepsilon^{(\mathrm{b})}$ and eventually taking the liminf as $\varepsilon \to 0$ leads to:

$$\liminf_{\varepsilon\to 0} \int_{J_{\tilde{u}_\varepsilon}\cap E_\varepsilon^{(\mathrm{b})}} \vartheta_t(\tilde{u}_\varepsilon^+, \tilde{u}_\varepsilon^-)\, \mathrm{d}\mathcal{H}^1 \leq \liminf_{\varepsilon\to 0} XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, \tilde{T}_{d,r}^N). \tag{2.76}$$

We combine (2.70), (2.74), and (2.76):

$$\liminf_{\varepsilon\to 0} XY_\varepsilon^{(\mathfrak{m})}(u_\varepsilon, \tilde{T}_{d,r}^N) \geq \int_{J_u^N \cap O_r(\mu)} |\nu_1|_1\, \mathrm{d}\mathcal{H}^1 - Ct$$

for some constant C independent of t. This concludes the proof – after letting $t \to 0$. □

2.4.3 Gamma-limsup

In this subsection, we will write $u \in \mathcal{D}_g^{(\mathfrak{m})}$ for a general limit spin field and set $\mu := \frac{1}{\mathfrak{m}}\,\mathrm{Jac}(u) = \frac{1}{\mathfrak{m}}\sum_{k=1}^{N_g} \delta_{x_k}$, where $N_g := \mathfrak{m} d_g$ and define $\mathcal{Z} := \{\frac{2\pi k}{\mathfrak{m}} : k = 1,\dots,\mathfrak{m}-1\}$. Without loss of generality, we will assume that $\mathrm{sgn}(\deg(g,\partial\Omega)) > 0$ (the other case works in the same manner). Fixing $r > 0$, we will first construct the recovery sequence "close" to the vortices of μ. In the case of the non-fractional theory, the authors of [6] have investigated for fixed $\varepsilon > 0$, $r > 0$, $x_0 \in \Omega$ and $\lambda \in \mathbb{S}^1$ the following minimum problem:

$$\begin{aligned}&\gamma_\varepsilon(r, x_0, \lambda)\\ &\quad := \min\left\{ XY_\varepsilon(v) \colon v \in \mathcal{AS}_\varepsilon(B_r(x_0)),\, v(x) = \lambda\frac{x - x_0}{|x - x_0|} \text{ on } \partial_\varepsilon B_r(x_0)\right\}.\end{aligned} \tag{2.77}$$

They derive the subsequent limit behavior for $\gamma_\varepsilon(r, x_0, \lambda)$ as defined above: There exists a scalar $\gamma \in \mathbb{R}$ (also called the *core energy*) independent of r, x_0 and λ such that:

$$\lim_{\varepsilon\to 0} \left(\gamma_\varepsilon(r, x_0, \lambda) - \pi \log\left(\frac{r}{\varepsilon}\right)\right) = \gamma. \tag{2.78}$$

In our fractional setting, correspondingly, the following minimum problem is of interest to us:

$$\gamma_\varepsilon^{(\mathtt{m})}(r, x_0, \lambda) \\ := \min\left\{ XY_\varepsilon^{(\mathtt{m})}(u) \colon u \in \mathcal{AS}_\varepsilon^{(\mathtt{m})}(B_r(x_0)),\, u^{\mathtt{m}}(x) = \lambda \frac{x - x_0}{|x - x_0|} \text{ on } \partial_\varepsilon B_r(x_0) \right\} \tag{2.79}$$

We wish to transfer the convergence result in (2.78) to our modified minimum problem. Let us start with an elementary estimate:

Lemma 2.13
For all $\varepsilon > 0$, $r > 0$ and $x_0 \in \mathbb{R}^2$:

$$\gamma_\varepsilon^{(\mathtt{m})}(r, x, \lambda) \geq \gamma_\varepsilon(r, x, \lambda). \tag{2.80}$$

Proof. Let $u_{\varepsilon,r,\lambda} \in \mathcal{AS}_\varepsilon(B_r(x_0))$ be a minimizer for $\gamma_\varepsilon^{(\mathtt{m})}(r, x_0, \lambda)$ and set $v_{\varepsilon,r,\lambda} := u^{\mathtt{m}}$. The boundary condition in (2.79) implies that $v_{\varepsilon,r,\lambda} = \lambda \frac{x-x_0}{|x-x_0|}$ on $\partial_\varepsilon B_r(x_0)$. Consequently, $v_{\varepsilon,r,\lambda}$ is a competitor for the minimum problem in (2.77). By the lower bound in Item (i) in Definition 2.17:

$$XY_\varepsilon^{(\mathtt{m})}(u_{\varepsilon,r,\lambda}) \geq XY_\varepsilon(v_{\varepsilon,r,\lambda})$$

which leads to (2.80). □

In contrast, the reverse inequality to the one stated above is in general *false*. We would like to briefly highlight the main difficulty in attempting to prove the reverse estimate: Given a minimizer $v_{\varepsilon,r,\lambda}$ of the problem in (2.77) with $\nu_{\varepsilon,r,\lambda}$ denoting one of its angular lifts, we might consider $u_{\varepsilon,r} := e^{\frac{\nu_{\varepsilon,r}}{\mathtt{m}}}$ which by construction satisfies $u_{\varepsilon,r,\lambda}^{\mathtt{m}} = v_{\varepsilon,r,\lambda}$. In particular, this is a competitor for the minimum problem in (2.79). The crux of the problem is that we generally have: $XY_\varepsilon^{(\mathtt{m})}(u_{\varepsilon,r}) > XY_\varepsilon(v_{\varepsilon,r})$, as $u_{\varepsilon,r}$ might have "many jump pairs". More precisely, any nearest-neighbor pair (i, j) satisfying $\nu_{\varepsilon,r}(i) - \nu_{\varepsilon,r}(j) \in 2\pi\mathbb{Z} \setminus 2\pi\mathtt{m}\mathbb{Z}$ is a jump pair of $u_{\varepsilon,r,\lambda}$ and contributes ε to the difference $XY_\varepsilon^{(\mathtt{m})}(u_{\varepsilon,r}) - XY_\varepsilon(v_{\varepsilon,r})$. We will not be able to show that the number of such nearest-neighbor pairs is negligible, meaning $o(\frac{1}{\varepsilon})$, as $\varepsilon \to 0$. Nevertheless, we will prove that it is of order $O_{\varepsilon\to 0}(\frac{r}{\varepsilon})$, and hence small if $r > 0$ is small. (We will later see that this estimate will still suffice for the construction of the recovery sequence.)

The proof of the above estimate partly employs basics from the theory of 1-currents, which we would like to present now (see also [35]): Let $\Lambda^1(\mathbb{R}^n)$ denote the set of linear functionals on $\mathbb{R}^n$ and $\{\mathrm{d}x_k\}_{k=1}^n$ its canonical basis dual to the standard basis $\{e_i\}_{k=1}^n$ ($\mathrm{d}x_k(e_j) = \delta_{ij}$). Given an open set $\Omega \subset \mathbb{R}^n$, we define $\mathcal{D}^1(\Omega)$ the set of all smooth, compactly supported *differential* 1*-forms* $\omega \colon \Omega \to \Lambda^1(\mathbb{R}^n)$. An example of such an ω is the differential of a function $\varphi \in C_c^\infty(\Omega)$, defined as:

$$\mathrm{d}\varphi := \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}\, \mathrm{d}x_k.$$

A 1*-current* T is a bounded linear functional on $\mathcal{D}^1(\Omega)$ (,where $\mathcal{D}^1(\Omega)$ is equipped with the $\|\cdot\|_\infty$-norm). We denote the set of all 1-currents on Ω as

$\mathcal{D}_1(\Omega)$. For any current $T \in \mathcal{D}_1(\Omega)$, we can define a generalized notion of boundary: The boundary ∂T of T is a distribution such that for any $\varphi \in C_c^\infty(\Omega)$, we have:

$$\partial T(\varphi) = T(\mathrm{d}\varphi).$$

The mass $\mathbb{M}(T)$ of a 1-current T is simply its dual norm:

$$\mathbb{M}(T) := \sup\{T(\omega) \colon \omega \in \mathcal{D}^1(\Omega),\ \|\omega\|_\infty \leq \infty\}.$$

By an application of Riesz representation theorem one can show that there exists a unique scalar Radon-measure μ_T on Ω and a μ_T-measurable function $\vec{T} \colon \Omega \to \Lambda^1(\mathbb{R}^n)$ such that for any $\omega \in \mathcal{D}^1(\Omega)$ it holds that

$$T(\omega) := \int_\Omega \omega(\vec{T})\, \mathrm{d}\mathcal{H}^1 .$$

Let us consider a simple example: Any regular curve $\gamma \colon [0,1] \to \Omega$ can be identified with a 1-current $[[\gamma]]$ acting on any $\omega \in \mathcal{D}^1(\Omega)$ as follows:

$$[[\gamma]](\omega) := \int_0^1 \omega\left(\frac{\gamma_s}{|\gamma_s|}\right) \mathrm{d}s$$

where s is the curve parameter. We remark that under this identification and the fundamental theorem of calculus it holds that $\partial[[\gamma]] = \delta_{\gamma(1)} - \delta_{\gamma(0)}$. Furthermore, the mass $\mathbb{M}(T)$ of T is equal to the length of γ.

Lastly, we discuss a relation between 1-currents and the flat norm of a signed measure μ on Ω, defined as:

$$\|\mu\|_\flat := \sup\left\{\int_\Omega \varphi\, \mathrm{d}\mu \colon \varphi \in C_c^\infty(\Omega),\ \|\varphi\|_\infty + \mathrm{Lip}(\varphi) \leq 1\right\}, \tag{2.81}$$

where $\mathrm{Lip}(\varphi)$ is the Lipschitz constant of φ. Moreover, we have that

$$\begin{aligned} \min_{\partial T \cap \Omega = \mu} \mathbb{M}(T) &:= \min\left\{\mathbb{M}(T) \colon T \in \mathcal{D}^1(\Omega),\ \partial T = \mu\right\} \\ &= \sup\left\{\int_\Omega \varphi\, \mathrm{d}\mu \colon \varphi \in C_c^\infty(\Omega),\ \mathrm{Lip}(\varphi) \leq 1\right\}. \end{aligned} \tag{2.82}$$

Note that – in contrast to (2.81) – the sup in (2.82) is taken over a larger set since we only constrain the Lipschitz norm instead of the sum $\|\varphi\|_\infty + \mathrm{Lip}(\Phi)$. Nevertheless, for a bounded open Ω, we have:

$$\min_{\partial T \cap \Omega = \mu} \mathbb{M}(T) \geq \|\mu\|_\flat \geq \frac{1}{1 + \mathrm{diam}(\Omega)} \min_{\partial T \cap \Omega = \mu} \mathbb{M}(T). \tag{2.83}$$

This can be seen as follows: Taking a function $\varphi \in C_c^\infty$ with $\mathrm{Lip}(\varphi)$, we can bound:

$$\|\varphi\|_\infty \leq \sup_{x \in \Omega} |\varphi(x) - \varphi(x_0)| \leq \mathrm{Lip}(\varphi)\, \mathrm{diam}(\Omega)$$

where $x_0 \in \partial\Omega$.

Lemma 2.14
Let γ be the limit in (2.78) *and $\gamma_\varepsilon^{(\mathfrak{m})}(r, x_0, \lambda)$ as defined in* (2.79)*, then:*

$$\begin{aligned} \gamma &= \limsup_{r \to 0} \limsup_{\varepsilon \to 0} \left(\gamma_\varepsilon^{(\mathfrak{m})}(r, x_0, \lambda) - \pi \log \frac{r}{\varepsilon}\right) \\ &= \liminf_{r \to 0} \liminf_{\varepsilon \to 0} \left(\gamma_\varepsilon^{(\mathfrak{m})}(r, x_0, \lambda) - \pi \log \frac{r}{\varepsilon}\right). \end{aligned} \tag{2.84}$$

Proof. 1. step: By (2.80) and (2.78), we see that:

$$\begin{aligned}\liminf_{r\to 0}\liminf_{\varepsilon\to 0}&\left(\gamma_\varepsilon^{(\mathfrak{m})}(r,x_0,\lambda)-\pi\log\frac{r}{\varepsilon}\right)\\ &\geq \liminf_{r\to 0}\liminf_{\varepsilon\to 0}\left(\gamma_\varepsilon(r,x_0,\lambda)-\pi\log\frac{r}{\varepsilon}\right)=\gamma.\end{aligned}$$

Hence, it is enough to show that there exists a remainder term $\alpha(\varepsilon,r)$ such that

$$\limsup_{r\to 0}\limsup_{\varepsilon\to 0}\alpha(\varepsilon,r)=0, \tag{2.85}$$

$$\gamma_\varepsilon^{(\mathfrak{m})}(r,x,\lambda)\leq\gamma_\varepsilon(r,x,\lambda)+\alpha(\varepsilon,r). \tag{2.86}$$

2. step: For any given $r>0$ and $\varepsilon>0$ let $v_{\varepsilon,r}$ denote a solution of the minimization problem in (2.77). By the minimality of $v_{\varepsilon,r}$, Item (ii) of Definition 2.17, as well as the fact that $h(x):=\lambda\frac{x-x_0}{|x-x_0|}$ (restricted to $(B_r)_\varepsilon^{(0)}$) is a competitor for (2.77), we have for all $\varepsilon>0$:

$$XY_\varepsilon(v_{\varepsilon,r})\leq XY_\varepsilon(h)\leq\pi|\log\varepsilon|+C$$

for some constant $C<\infty$ independent of ε. We can therefore apply the compactness statement in Theorem 2.16 for the sequence $(v_{\varepsilon,r})_\varepsilon$, where $\Omega=B_r(x_0)$, $g=h$ and $\mathfrak{m}=1$. Consequently, we can find $x^*\in B_r(x_0)$ such that, up to a subsequence:

$$\mu_{v_{\varepsilon,r}}:=\mu_{v_{\varepsilon,r}}\overset{\flat}{\rightharpoonup}\delta_{x^*}\text{ flat in }B_r(x_0),$$

where $\mu_{v_{\varepsilon,r}}$ is the discrete vorticity measure of $v_{\varepsilon,r}$. Let us consider a minimal (with regard to mass) current $T_{\varepsilon,r}\in\mathcal{D}_1(\Omega)$ such that $\partial T_{\varepsilon,r}\cap B_r(x_0)=\mu_{v_{\varepsilon,r}}-\delta_{x^*}$. It is a classic result that such a current exists and is a finite union of oriented line segments. By (2.83), we see that:

$$\lim_{\varepsilon\to 0}\mathbb{M}(T_{\varepsilon,r})\leq 2r\limsup_{\varepsilon\to 0}\|\mu_{v_{\varepsilon,r}}-\delta_{x^*}\|_\flat=0. \tag{2.87}$$

Moreover, let S be the oriented segment passing from x^* to a point on the boundary $\partial B_r(x_0)$. Then the current $\tilde{T}_{\varepsilon,r}:=T_{\varepsilon,r}+S$ satisfies:

$$\limsup_{\varepsilon\to 0}\mathbb{M}(\tilde{T}_{\varepsilon,r})\leq\limsup_{\varepsilon\to 0}\mathbb{M}(T_{\varepsilon,r})+\mathbb{M}(S)\leq r. \tag{2.88}$$

3. step: Let us define the set $A_{\varepsilon,r}$:

$$A_{\varepsilon,r}:=\bigcup\left\{i+\varepsilon[0,1]^2\colon i\in(B_r)^2_\varepsilon,\ (i+\varepsilon[0,1]^2)\cap\tilde{T}_{\varepsilon,r}\neq\emptyset\right\}$$

as the union of all grid cells that have a nonempty intersection with $\tilde{T}_{\varepsilon,r}$. We will write $U_{\varepsilon,r}:=B_r(x_0)\setminus A_{\varepsilon,r}$ for its complement in $B_r(x_0)$. Given a connected component $V_{\varepsilon,r}$ of $U_{\varepsilon,r}$, a seed point $i^*\in\varepsilon\mathbb{Z}^2\cap V_{\varepsilon,r}$, and an angular lift $\nu_{\varepsilon,r}$ of $v_{\varepsilon,r}$, we wish to define a $u_{\varepsilon,r}\colon\varepsilon\mathbb{Z}^2\cap V_{\varepsilon,r}\to\mathbb{S}^1$ without jump pairs such that $u^{\mathfrak{m}}_{\varepsilon,r}=v_{\varepsilon,r}$. As it is easier to work with angles instead of vectors, we will find $\varphi_{\varepsilon,r}\colon\varepsilon\mathbb{Z}^2\cap V_{\varepsilon,r}\to\mathbb{R}$ such that $u_{\varepsilon,r}:=e^{\imath\varphi_{\varepsilon,r}}$ is the desired spin field. For the construction, it will be useful to make the following convention: We call a finite sequence $i_0,\dots,i_K$ of grid points in $V_{\varepsilon,r}$ a discrete path from i to j if and only

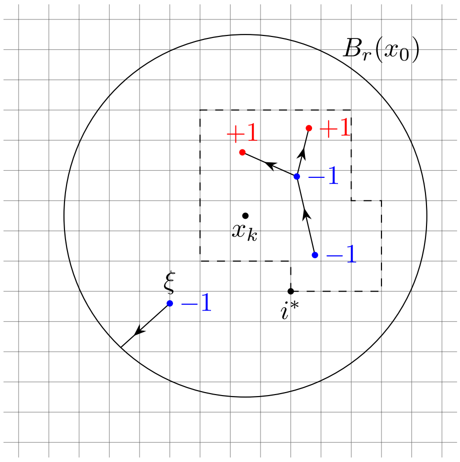


Figure 2.8: A closed path around $S_{\varepsilon,r}$.

if $i_0 = i$, $i_K = j$ and $|i_{k+1} - i_k| = \varepsilon$ for every $k = 0, \dots, K-1$. Now back to the definition of $\varphi_{\varepsilon,r}$. We will simply set $\varphi_{\varepsilon,r}(i^*) := \frac{1}{\mathfrak{m}}\nu_{\varepsilon,r}(i^*)$ at the seed point. For any other $i \in \varepsilon\mathbb{Z}^2 \cap V_{\varepsilon,r}$, we first select a discrete path $i_0, \dots, i_K$ from i^* to i, and then set:

$$\varphi_{\varepsilon,r}(i) := \varphi_{\varepsilon,r}(i_1) + \frac{1}{\mathfrak{m}} \sum_{k=0}^{K-1} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i_k, i_{k+1}). \tag{2.89}$$

By the connectedness of $V_{\varepsilon,r}$ there exists at least one such path if we take ε small enough. We will now check if the definition in (2.89) is path-independent. Given another discrete path $\tilde{i}_0, \dots, \tilde{i}_K$ from i^* to i, we need to show that:

$$\varphi_{\varepsilon,r}(i_0) + \frac{1}{\mathfrak{m}} \sum_{k=0}^{K-1} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i_k, i_{k+1}) = \varphi_{\varepsilon,r}(\tilde{i}_0) + \frac{1}{\mathfrak{m}} \sum_{k=0}^{\tilde{K}-1} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(\tilde{i}_k, \tilde{i}_{k+1}).$$

As $i_0 = \tilde{i}_0 = i^*$, we can equivalently show:

$$\sum_{k=0}^{K-1} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i_k, i_{k+1}) + \sum_{k=\tilde{K}-1}^{0} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(\tilde{i}_{k+1}, \tilde{i}_k) = 0.$$

(Note that the ordering is reversed in the second sum.) It remains to show for any discrete path $i_0, \dots, i_K$ with $i_0 = i_K = i^*$, that:

$$\sum_{k=0}^{K-1} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i_k, i_{k+1}) = 0. \tag{2.90}$$

In this regard, let $W_{\varepsilon,r}$ be the interior of the set encircled by the discrete path $i_0, \dots, i_K$ and let $S_{\varepsilon,r} := \tilde{T}_{\varepsilon,r}|_{W_{\varepsilon,r}}$ (see also Figure 2.8 for further clarification). By construction, $S_{\varepsilon,r}$ does not contain S, and hence $\partial S_{\varepsilon,r} = \mu_{v_{\varepsilon,r}}|_{W_{\varepsilon,r}}$. Moreover, as $S_{\varepsilon,r}$ is compactly supported in $W_{\varepsilon,r}$, we have that $\partial S_{\varepsilon,r}(W_{\varepsilon,r}) = 0$. (This can be seen by testing with $\rho \in C_c^\infty(W_{\varepsilon,r})$ such that

$\rho \equiv 1$ in $\operatorname{spt}(\partial S_{\varepsilon,r}) \cap W_{\varepsilon,r}$, and using the definition of $\partial S_{\varepsilon,r}$.) Consequently, $\mu_{v_{\varepsilon,r}}(W_{\varepsilon,r}) = 0$ – and by applying a discrete version of Stoke's theorem (see also [6]) –, we have:

$$\sum_{k=0}^{K-1} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i_k, i_{k+1}) = \mu_{v_{\varepsilon,r}}(W_{\varepsilon,r}) = 0.$$

4. step: Let us now show that our choice of $u_{\varepsilon,r}$ satisfies all the desired properties. Fix $i \in \varepsilon\mathbb{Z}^2 \cap V_{\varepsilon,r}$ (with $V_{\varepsilon,r}$ as before) and consider a discrete path $i_0, \dots, i_K$ from i^* to i. By definition of $\varphi_{\varepsilon,r}$ and the definition of Δ^{el}, it holds that:

$$\begin{aligned}
\mathfrak{m}\varphi_{\varepsilon,r}(i) &= \mathfrak{m}\varphi_{\varepsilon,r}(i_0) + \sum_{k=0}^{K-1} \Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i_k, i_{k+1}) \\
&= \nu_{\varepsilon,r}(i_0) + \sum_{k=0}^{K-1} (\nu_{\varepsilon,r}(i_{k+1}) - \nu_{\varepsilon,r}(i_k)) \mod 2\pi \\
&= \nu_{\varepsilon,r}(i) \mod 2\pi,
\end{aligned}$$

and therefore $u^{\mathfrak{m}}_{\varepsilon,r}(i) = e^{\imath \cdot \mathfrak{m}\varphi_{\varepsilon,r}(i)} = v_{\varepsilon,r}(i)$. We will now prove that $u_{\varepsilon,r}$ has no jump pairs in $V_{\varepsilon,r}$. Let (i,j) be a nearest-neighbor pair in $V_{\varepsilon,r}$, and $i_0, \dots, i_K$ a discrete path from i^* to i. Then, $i_0, \dots, i_K, j$ is a discrete path from i^* to j, and by (2.89) we follow:

$$|\varphi_{\varepsilon,r}(i) - \varphi_{\varepsilon,r}(j)| = |\Delta^{\mathrm{el}} \varphi_{\varepsilon,r}(i,j)| = \frac{1}{\mathfrak{m}} |\Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i,j)| \le \frac{\pi}{\mathfrak{m}}, \tag{2.91}$$

as desired. Note that in the last estimate we used $\Delta^{\mathrm{el}} \nu_{\varepsilon,r}(i,j) \in [-\pi, \pi]$.

Repeating the above construction in each of the connected components of $U_{\varepsilon,r}$ and setting for all $i \in \varepsilon\mathbb{Z}^2 \cap A_{\varepsilon,r}$:

$$u_{\varepsilon,r}(i) := e^{\imath \varphi_{\varepsilon,r}(i)}, \quad \varphi_{\varepsilon,r}(i) := \frac{1}{\mathfrak{m}} \nu_{\varepsilon,r}(i)$$

we end up with a globally defined spin field in $(B_r(x_0))^{(1)}_\varepsilon$ satisfying $u^{\mathfrak{m}}_{\varepsilon,r} = v_{\varepsilon,r}$. Furthermore, for any jump pair (i,j) of $u_{\varepsilon,r}$, we must have $[i,j] \subset A_{\varepsilon,r}$ as well as $\Delta^{\mathrm{el}} \varphi_{\varepsilon,r}(i,j) \in \mathcal{Z}$. As each such edge $[i,j]$ must have at least one point of intersection with $\tilde{T}_{\varepsilon,r}$, we derive (using Item (i) of Definition 2.17):

$$\begin{aligned}
XY^{(\mathfrak{m})}_\varepsilon(u_{\varepsilon,r}) &= \sum_{\langle i,j \rangle} f^{(\mathfrak{m})}_\varepsilon(\varphi_{\varepsilon,r}(i) - \varphi_{\varepsilon,r}(j)) \\
&\le XY_\varepsilon(v_{\varepsilon,r}) + \#\Big\{(i,j) \in (B_r)^{(1)}_\varepsilon : [i,j] \cap \tilde{T}_{\varepsilon,r}\Big\} \cdot \varepsilon \\
&\le XY_\varepsilon(v_{\varepsilon,r}) + 2\mathbb{M}(\tilde{T}_{\varepsilon,r}).
\end{aligned}$$

By passing to the limit $\varepsilon \to 0$ as well as using (2.78) and (2.88), we follow that:

$$\limsup_{\varepsilon \to 0} XY^{(\mathfrak{m})}_\varepsilon(u_{\varepsilon,r}) \le \gamma + 4r.$$

We conclude the proof by consequently passing to the limit $r \to 0$. □

We will now introduce necessary tools for the construction of the recovery sequence "away" from the vortices. The main idea will be to first find an appropriate approximation w of u (this will be made more precise in a moment), and consider $(w|_{(\Omega_r(\mu))_\varepsilon})_\varepsilon$ ($r > 0$ fixed) as a competitor for the recovery sequence away from the vortices of u. Let us fix some notation first. For each $x_k \in \operatorname{spt}\mu$ ($\operatorname{vort}(u) = \pi\mu$), we select a closed line segment L_k connecting x_k with the boundary $\partial\Omega$ such that all line segments $L_1, \ldots, L_{N_g}$ are pairwise disjoint and:

$$\mathcal{H}^1(J_u \cap L) = 0, \text{ where } L \coloneqq \bigcup_{k=1}^{N_g} L_k. \tag{2.92}$$

For fixed $r > 0$, we define:

$$\Omega_r^{(L)}(\mu) \coloneqq \Omega_r(\mu) \setminus L \tag{2.93}$$

with $\Omega_r(\mu)$ as in (2.26). Note that as Ω was assumed to be simply connected and $\Omega_r(\mu)$ has exactly N_g holes centered in $\{x_1, \ldots, x_{N_g}\}$, we see that $\Omega_r^{(L)}(\mu)$ must also be simply connected. The following version of Poincaré's theorem will be used later on:

Theorem 2.18 (Poincaré's theorem)
Let $\Omega \subset \mathbb{R}^2$ be a simply connected set, then for any $w \in L^2(\Omega;\mathbb{R}^2)$ satisfying $\operatorname{curl}(w) = 0$ *(in the distributional sense) we can find $\varphi \in W^{1,2}(\Omega)$ such that $\nabla\varphi = w$ at a.e. point in Ω.*

Moreover, we assume implicitly that L is oriented by ν such that for each $k \in \{1, \ldots, \mathfrak{m} - 1\}$, the vector field $\nu_k \coloneqq \nu|_{L_k}$ points in anticlockwise direction with respect to the orientation of L_k. We can decompose u as follows:

Lemma 2.15 (Decomposition of u)
For each $u \in \mathcal{D}_g^{(\mathfrak{m})}$, there exist $\alpha \in SBV^2(\Omega_r(\mu))$ and $\psi \in SBV(\Omega_r(\mu); \mathcal{Z})$ such that:

$$D^j\alpha = -\frac{2\pi}{\mathfrak{m}} \otimes \nu\mathcal{H}^1|_L, \tag{2.94}$$

$$e^{\imath(\alpha+\psi)} = u \text{ in } \Omega_r(\mu). \tag{2.95}$$

Proof. 1. step: Let us first find an admissible α. By the SBV-lifting result found in [31], we can find a scalar function $\varphi \in SBV^2(\Omega_r(\mu))$ such that $(\sin(\varphi), \cos(\varphi))^T = e^{\imath\varphi} = u$ a.e. in $\Omega_r(\mu)$. Furthermore, with the chain rule:

$$\begin{aligned}\langle \nabla_{x_1} u, u^\perp\rangle &= u_2\frac{\partial u_1}{\partial x_1} - u_1\frac{\partial u_2}{\partial x_1}\\ &= \cos(\varphi)\cos(\varphi)\frac{\partial\varphi}{\partial x_1} - \sin(\varphi)(-\sin(\varphi))\frac{\partial\varphi}{\partial x_1} = \frac{\partial\varphi}{\partial x_1}\end{aligned}$$

and similarly:

$$\langle \nabla_{x_2} u, u^\perp\rangle = \frac{\partial\varphi}{\partial x_2}.$$

Hence, $\operatorname{jac}(u) = \nabla\varphi$ at a.e. point in $\Omega_r(\mu)$ and $\operatorname{curl}(w) = 2\operatorname{Jac}(u) = 0$ a.e. in $\Omega_r(\mu)$ for $w \coloneqq \nabla\varphi$. As $\Omega_r^{(L)}(\mu)$ is simply connected, there exists by Theorem 2.18 a scalar function $\alpha \in W^{1,2}(\Omega_r^{(L)}(\mu))$ such that $\nabla\alpha = w = \nabla\varphi$ in $\Omega_r^{(L)}(\mu)$.

Let us fix L_k. We wish to show that for almost all $x \in L_k$, we have that $\alpha^+(x) - \alpha^-(x) = -\frac{2\pi}{\mathfrak{m}}$. By slicing theory, we can find for almost every $x \in L_k$ a simple closed curve $\gamma\colon [0,1] \to \mathbb{R}^2$ encircling x_k (and not the remaining vortex centers) in anticlockwise direction, and crossing L_k exactly at x such that $\alpha \circ \gamma$ is absolutely continuous on $[0,1]$. By possibly removing a set of negligible $\mathcal{H}^1$-measure we can further assume that $\alpha^+(x) = \alpha(\gamma(0))$ and $\alpha^-(x) = \alpha(\gamma(1))$. By the Fundamental Theorem of Calculus and the definition of degree, we then have:

$$
\begin{aligned}
\alpha^-(x) - \alpha^+(x) &= \int_0^1 (\alpha \circ \gamma)'(s)\,\mathrm{d}s = \int_\gamma \langle \nabla \alpha, \tau_\gamma \rangle \,\mathrm{d}\mathcal{H}^1 \\
&= \int_\gamma \langle \operatorname{jac} u, \tau_\gamma \rangle \,\mathrm{d}\mathcal{H}^1 = \frac{2\pi}{\mathfrak{m}}.
\end{aligned}
$$

Hence, $\alpha \in SBV^2(\Omega_r(\mu))$ and (2.94) holds true.

2. step: It remains to find the desired ψ. Notice that by construction, $\varphi - \alpha \in SBV^2(\Omega_r(\mu))$ has a vanishing approximate gradient. Consequently, there exists a Caccioppoli partition $\{U_l\}_{l\in\mathbb{N}}$ of $\Omega_r(\mu)$ subordinate to $J_\varphi \cup L$ and constants $\{c_l\}_{l\in\mathbb{N}} \subset \mathbb{R}$ such that:

$$
\varphi - \alpha = \sum_{l\in\mathbb{N}} c_l \mathbb{1}_{U_l}.
$$

By (2.94) and the fact that $u^{\mathfrak{m}} = e^{\imath \mathfrak{m}\varphi} \in W^{1,2}(\Omega_r(\mu))$, we can find $c \in \mathbb{R}$ such that $c_l - c \in \frac{2\pi}{\mathfrak{m}}\mathbb{Z}$ for all $l \in \mathbb{N}$. Furthermore, replacing α with $\alpha + c$ we can without loss of generality assume that $c_l \in \frac{2\pi}{\mathfrak{m}}\mathbb{Z}$ for all $l \in \mathbb{N}$. Let us now define for any $k \in \{1, \ldots, \mathfrak{m}-1\}$ the set:

$$
E_k \coloneqq \bigcup \left\{ U_l \colon c_l \in \frac{2\pi k}{\mathfrak{m}} + 2\pi\mathbb{Z} \right\}
$$

and set $\psi \coloneqq \sum_{k=1}^{\mathfrak{m}-1} \frac{2\pi k}{\mathfrak{m}} \mathbb{1}_{E_k}$. By construction, $\psi \in SBV(\Omega_r(\mu), \mathcal{Z})$ and $\alpha + \psi - \varphi \in 2\pi\mathbb{Z}$ a.e. in $\Omega_r(\mu)$. As φ is an angular lift for u, this implies that $\alpha + \psi$ is one also. □

We will approximate α and ψ from the above lemma separately. The former can be approximated as follows:

Lemma 2.16 (Approximation of α)
Given α as in Lemma 2.15 we can find a sequence:

$$
(\alpha_n) \subset SBV^2(\Omega_r(\mu)) \cap C^\infty(\Omega_r^{(L)}(\mu))
$$

such that $\alpha_n \to \alpha$ in $SBV^2(\Omega_r(\mu))$ and:

$$
D^j \alpha_n = -\frac{2\pi}{\mathfrak{m}} \otimes \nu \mathcal{H}^1|_L, \tag{2.96}
$$

$$
\mathcal{H}^1(\{\alpha_n - \alpha \notin 2\pi\mathbb{Z}\} \cap \partial\Omega) \to 0. \tag{2.97}
$$

Proof. 1. step: Fix $z \coloneqq e^{\imath \mathfrak{m}\alpha}$, then by the chain rule and (2.94), it follows that $z \in W^{1,2}(\Omega_r(\mu))$. Being able to find a sequence $(z_n) \subset C^\infty(\Omega_r(\mu); \mathbb{S}^1)$ such that $z_n = z$ on $\partial\Omega$ and $z_n \to z$ strongly in $W^{1,2}(\Omega_r(\mu); \mathbb{R}^2)$ is a rather classic result (see also, for instance, [61]). By Poincaré's theorem, for each $n \in \mathbb{N}$ there exists a function $\beta_n \in C^\infty(\Omega_r^{(L)}(\mu))$ such that $e^{\imath\beta_n} = z_n$ in $\Omega_r^{(L)}(\mu)$ and $\beta_n = \mathfrak{m}\alpha \mod 2\pi$ on $\partial\Omega$. By the continuity of the degree with respect to $W^{1,2}$-convergence and an argument similar to the one in the proof of Lemma 2.15, we also see that $\beta_n^+ - \beta_n^- = -2\pi$ on L for n big enough. By possibly removing finitely many elements from the sequence, we can assume – without loss of generality – that this is true for the whole sequence.

2. step: We will now show that $\nabla(\beta_n - \beta) \to 0$ in $L^2(\Omega_r^{(L)}(\mu);\mathbb{R}^2)$, where $\beta := \mathfrak{m}\alpha$. Consider:

$$a_n := \sin(\beta_n)\frac{\partial\beta_n}{\partial x_1}, \qquad a := \sin(\beta)\frac{\partial\beta}{\partial x_1},$$
$$b_n := \cos(\beta_n)\frac{\partial\beta_n}{\partial x_1}, \qquad b := \cos(\beta)\frac{\partial\beta}{\partial x_1}.$$

From the convergence of $\nabla(z_n - z)$, we follow that

$$a_n \to a, \quad b_n \to b, \quad \text{both strongly in } L^2(\Omega_r^{(L)}(\mu)).$$

By the L^2-convergence of z_n, we see that, up to a subsequence, $\sin(\beta_n) \to \sin(\beta)$ a.e. in $\Omega_r(\mu)$. Lastly, as the sinus function is bounded and the sequence (a_n) convergent in L^2, the sequence $(\sin(\beta_n)a_n)$ cannot concentrate mass in L^2. With Vitali's convergence theorem this leads to:

$$\sin^2(\beta_n)\frac{\partial\beta_n}{\partial x_1} = \sin(\beta_n)a_n \to \sin(\beta)a = \sin^2(\beta)\frac{\partial\beta}{\partial x_1} \text{ in } L^2(\Omega_r^{(L)}(\mu)).$$

In the same manner, we can show $\cos^2(\beta_n)\frac{\partial\beta_n}{\partial x_1} \to \cos^2(\beta)\frac{\partial\beta}{\partial x_1}$, and hence $\frac{\partial\beta_n}{\partial x_1} \to \frac{\partial\beta}{\partial x_1}$ in $L^2(\Omega_r^{(L)}(\mu))$. The proof of $\frac{\partial\beta_n}{\partial x_2} \to \frac{\partial\beta}{\partial x_2}$ in $L^2(\Omega_r(\mu))$ works the same way.

2. step: From the convergence of $\nabla(\beta_n - \beta)$ and the Poincaré-Wirtinger inequality we see for $c_n := -\int_{\Omega_r^{(L)}(\mu)} \beta_n - \beta \,\mathrm{d}x$ that

$$\beta_n + c_n \to \beta \text{ in } L^2(\Omega_r^{(L)}(\mu)).$$

As the angular lifts provided in [31] are uniformly bounded we can therefore find a constant $c \in \mathbb{R}$ such that, up to a subsequence:

$$\frac{\beta_n}{\mathfrak{m}} + c \to \alpha \text{ in } L^2(\Omega_r^{(L)}(\mu)). \tag{2.98}$$

Note that by construction, $e^{\imath\beta_n} = z_n = z = e^{\imath\mathfrak{m}\alpha}$ on $\partial\Omega$, and therefore $\frac{\beta_n}{\mathfrak{m}} - \alpha \in \frac{2\pi}{\mathfrak{m}}\mathbb{Z}$ $\mathcal{H}^1$-a.e. on $\partial\Omega$. Combined with (2.98), this leads to $c = \frac{2\pi k}{\mathfrak{m}}$ for some $k \in \mathbb{Z}$. Let us set $\alpha_n := \frac{\beta_n}{\mathfrak{m}} + \frac{2\pi k}{\mathfrak{m}}$. By the previous reasoning, $\alpha_n \to \alpha$ in $SBV^2(\Omega_r(\mu))$ and $\alpha_n(x) - \alpha(x) \in \frac{2\pi}{\mathfrak{m}}\mathbb{Z}$ for $\mathcal{H}^1$-a.e. point of $\partial\Omega$. Therefore, for $\mathcal{H}^1$-a.e. $x \in \{\alpha_n - \alpha \notin 2\pi\mathbb{Z}\} \cap \partial\Omega$ we must have $|\alpha_n(x) - \alpha(x)| \geq \frac{2\pi}{\mathfrak{m}}$, and consequently:

$$\mathcal{H}^1(\{\alpha_n \neq \alpha\} \cap \partial\Omega) \leq \frac{\mathfrak{m}}{2\pi}\int_{\partial\Omega} |\alpha_n - \alpha| \,\mathrm{d}\mathcal{H}^1 \to 0 \text{ as } n \to \infty$$

by the continuity of the trace operator, which shows (2.97). □

The following approximation result holds true for Ψ:

Lemma 2.17 (Approximation of ψ)
Let $\psi \in SBV(\Omega_r(\mu); \mathcal{Z})$ as in the statement of Lemma 2.15, then we can find a sequence $(\psi_n) \subset SBV(\Omega_r(\mu); \mathcal{Z})$ such that J_{ψ_n} is polyhedral set (being a union of finitely many line segments) for all $n \in \mathbb{N}$, and:

$$\psi_n \to \psi \text{ in } L^1(\Omega_r(\mu)), \tag{2.99}$$

$$\mathcal{H}^1(\{\psi_n^\pm \neq \psi^\pm\} \cap L) \to 0, \quad \mathcal{H}^1(\{\psi_n \neq \psi\} \cap \partial\Omega) \to 0. \tag{2.100}$$

Moreover, for any continuous bounded function $\Phi\colon \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}_+$ satisfying $\Phi(a, b, \nu) = \Phi(b, a, -\nu)$ for all a, $b \in \mathbb{R}^2$ and $\nu \in \mathbb{S}^1$, it holds that:

$$\lim_{n\to\infty} \int_{J_{\psi_n}\setminus L} \Phi(\psi_n^+, \psi_n^-, \nu_{\psi_n})\, \mathrm{d}\mathcal{H}^1 = \int_{J_\psi \setminus L} \Phi(\psi^+, \psi^-, \nu_\psi)\, \mathrm{d}\mathcal{H}^1 . \tag{2.101}$$

Before coming to the proof, we wish to point out one of the main difficulties in proving the lemma above: It is not possible here to directly apply Theorem 2.9, since we do not know a priori that $\mathcal{H}^1(J_{\psi_n} \cap L) \to 0$, as $n \to \infty$, for the sequence provided by Theorem 2.9. One possible solution to this problem is to repeat the proof of [19], and to make sure that the extra condition is satisfied. As this is rather time-consuming, we take another – more modular – approach by employing Theorem 2.9 "away" from the segments L_k. In a second step, we will extend the constructed approximation to the whole domain $\Omega_r(\mu)$, while also making sure that $\mathcal{H}^1(J_{\psi_n} \cap L) \to 0$, as $n \to \infty$.

Proof of Lemma 2.17. 1. step: Let us start with some notation: The intersection point of L_k and $\partial\Omega$ will be denoted by p_k; the length of the segment will be shortly written as $|L_k|$. Given $t > 0$, we define:

$$\mathrm{Rect}_i(t, \eta) := \left\{ x_k + a\nu_i + b\frac{p_k - x_k}{|p_k - x_k|} : a \in [-t, t],\, b \in [r + \eta, |L_k| - \eta] \right\}$$

as well as

$$\mathrm{Rect}_i^\pm(t, \eta) := \left\{ x_k \pm a\nu_i + b\frac{p_k - x_k}{|p_k - x_k|} : a \in [0, t],\, b \in [r + \eta, |L_k| - \eta] \right\},$$

$$L_k^\pm(t, \eta) := \left\{ x_k \pm t\nu_i + b\frac{p_k - x_k}{|p_k - x_k|} : b \in [r + \eta, |L_k| - \eta] \right\}.$$

Furthermore, we fix $\eta > 0$ and $\delta > 0$ small enough such that $\mathrm{Rect}_i(2\delta, \eta) \subset \Omega_r(\mu)$. We apply Theorem 2.9 for ψ restricted to the Lipschitz domain $\Omega^{(R)}_{r,\delta,\eta}$ given by

$$\Omega^{(R)}_{r,\delta,\eta} := \Omega_r(\mu) \setminus \bigcup_{k=1}^{N} \mathrm{Rect}_i(\delta, \eta).$$

Hence, there exists a sequence $\{\tilde{\psi}_{n,\delta,\eta}\}_n \subset SBV(\Omega^{(R)}_{r,\delta,\eta}; \mathcal{Z})$ such that $J_{\tilde{\psi}_{n,\delta,\eta}}$ is polyhedral, $\tilde{\psi}_{n,\delta,\eta} \to \psi$ strict in $BV(\Omega^{(R)}_{r,\delta,\eta})$, and:

$$\lim_{n\to\infty} \int_{J_{\tilde{\psi}_{n,\delta,\eta}}} \Phi(\tilde{\psi}^+_{n,\delta,\eta}, \tilde{\psi}^-_{n,\delta,\eta}, \nu_{\tilde{\psi}_{n,\delta,\eta}})\, \mathrm{d}\mathcal{H}^1 = \int_{J_\psi \cap \Omega^{(R)}_{r,\delta,\eta}} \Phi(\psi^+, \psi^-, \nu_\psi)\, \mathrm{d}\mathcal{H}^1, \tag{2.102}$$

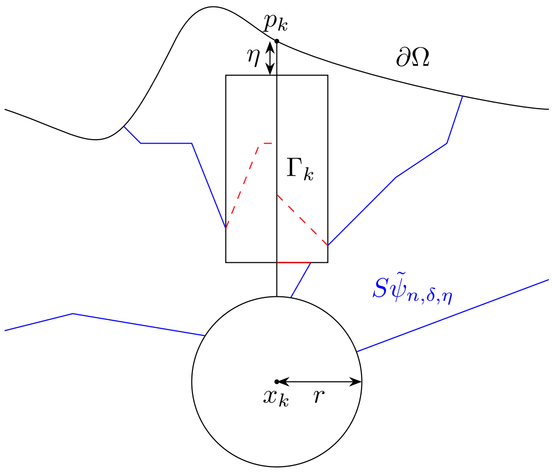


Figure 2.9: Extension of $\tilde{\Psi}_{n,\delta,\eta}$ into $\mathrm{Rect}_i(\delta,\eta)$ by reflection.

for all Φ as in the statement of the lemma.

2. step: Let us now extend $\tilde{\psi}_n$ into the rectangle $\mathrm{Rect}_i(\delta,\eta)$. In this regard, $R^{\pm}_{i,\delta}\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflections:

$$R^{\pm}_{i,\delta}(x) := x + 2\,\mathrm{dist}(x, L^{+}_i(\delta,\eta))\nu_i.$$

The extension $\psi_{n,\delta,\eta} \in SBV(\Omega_r(\mu);\mathcal{Z})$ is then defined as:

$$\psi_{n,\delta,\eta}(x) := \begin{cases} \tilde{\psi}_{n,\delta,\eta}(x) & \text{if } x \in \Omega^{(R)}_{r,\delta,\eta}, \\ \tilde{\psi}_{n,\delta,\eta}(R^{\pm}_{i,\delta}(x)) & \text{if } x \in \mathrm{Rect}^{\pm}_i(\delta,\eta). \end{cases}$$

The jump set $J_{\psi_{n,\delta,\eta}}$ remains polyhedral after this extension procedure (see also figure Figure 2.9). Furthermore, by construction and the L^1-convergence of $\tilde{\psi}_{n,\delta,\eta}$, we see that:

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\|\psi_{n,\delta,\eta} - \psi\|_{L^1(\Omega_r(\mu))} = 0. \tag{2.103}$$

By the definition of $\psi_{n,\delta,\eta}$ and the strict convergence of $(\tilde{\psi}_{n,\delta,\eta})_n$, we also have that:

$$\lim_{n\to\infty}\|\psi^{\pm}_{n,\delta,\eta} - \psi^{\pm}\|_{L^1(\tilde{L}_k)} = \|\psi(\cdot \pm 2\delta\nu_i) - \psi^{\pm}\|_{L^1(\tilde{L}_k)},$$

where $\tilde{L}_k := L_k \cap \mathrm{Rect}_i(\delta,\eta)$. Furthermore, as $\mathcal{H}^1(J_\psi \cap \tilde{L}_k) = 0$, we follow:

$$\lim_{\delta\to 0}\|\psi(\cdot \pm 2\delta\nu_i) - \psi^{\pm}\|_{L^1(\tilde{L}_k)} = 0$$

and consequently:

$$\limsup_{\delta\to 0}\lim_{n\to\infty}\|\psi^{\pm}_{n,\delta,\eta} - \psi^{\pm}\|_{L^1(\tilde{L})} = 0, \quad \tilde{L} := \bigcup_{k=1}^{N_g}\tilde{L}_k. \tag{2.104}$$

By construction, we have for $\mathcal{H}^1$-a.e. $x \in L$ that both $\psi^\pm(x)$ and $\psi^\pm_{n,\delta,\eta}(x)$ lie in $\frac{2\pi}{\mathfrak{m}}\mathbb{Z}$. Hence $|\psi^\pm_{n,\delta,\eta}(x) - \psi^\pm(x)| \geq \frac{2\pi}{\mathfrak{m}}$ for $\mathcal{H}^1$-a.e. $x \in \{\psi^\pm_{n,\delta,\eta} \neq \psi^\pm\} \cap L$. With (2.104), this eventually leads to

$$\begin{aligned}&\limsup_{\delta\to0}\lim_{n\to\infty}\mathcal{H}^1(\{\psi^\pm_{n,\delta,\eta} - \psi^\pm\} \cap L)\\&\quad\leq \limsup_{\delta\to0}\lim_{n\to\infty}\frac{\mathfrak{m}}{2\pi}\|\psi^\pm_{n,\delta,\eta} - \psi^\pm\|_{L^1(\tilde{L})} + 2\eta \leq 2\eta.\end{aligned} \tag{2.105}$$

3. step: We wish to show (2.101). By (2.102) and the definition of $\psi_{n,\delta,\eta}$, we see that:

$$\begin{aligned}&\lim_{n\to\infty}\int_{J_{\psi_{n,\delta,\eta}}\cap\Omega^{(R)}_{r,\delta,\eta}}\Phi(\psi^+_{n,\delta,\eta},\psi^-_{n,\delta,\eta},\nu_{\psi_{n,\delta,\eta}})\,\mathrm{d}\mathcal{H}^1\\&\quad=\int_{J_\psi\cap\Omega^{(R)}_{r,\delta,\eta}}\Phi(\psi^+,\psi^-,\nu_\psi)\,\mathrm{d}\mathcal{H}^1\,.\end{aligned}$$

Thus, it remains to investigate the situation inside a rectangle $\mathrm{Rect}_i(\delta,\eta)$. Moreover, as we extended $\tilde{\psi}_{n,\delta,\eta}$ by reflecting on the lines $L^\pm_k(\delta,\eta)$, it follows that:

$$\begin{gathered}\mathcal{H}^1(J_{\psi_{n,\delta,\eta}}\cap L^\pm_k(\delta,\eta)) = 0,\\ \mathcal{H}^1(J_{\psi_{n,\delta,\eta}}\cap(\mathrm{Rect}_i(\delta,\eta))^0) \leq \mathcal{H}^1(J_{\tilde{\psi}_{n,2\delta,\eta}}\cap\mathrm{Rect}_i(2\delta,\eta)).\end{gathered}$$

Additionally, $\psi_{n,\delta,\eta}$ may jump on the shorter sides of $\mathrm{Rect}_i(\delta,\eta)$ (see also Figure 2.9.) Hence:

$$\mathcal{H}^1(J_{\psi_{n,\delta,\eta}}\cap\mathrm{Rect}_i(\delta,\eta)) \leq \mathcal{H}^1(J_{\tilde{\psi}_{n,\delta,\eta}}\cap\mathrm{Rect}_i(2\delta,\eta)) + 4\delta,$$

and by the strict convergence of $(\tilde{\psi}_{n,\delta,\eta})_n$ and the fact that $\mathcal{H}^1(J_\psi \cap L) = 0$:

$$\begin{aligned}&\limsup_{\delta\to0}\limsup_{n\to\infty}\mathcal{H}^1(J_{\psi_{n,\delta,\eta}}\cap\mathrm{Rect}_i(\delta,\eta))\\&\quad\leq\sum_{k=1}^N\limsup_{\delta\to0}\mathcal{H}^1(J_\psi\cap\mathrm{Rect}_i(2\delta,\eta)) = 0.\end{aligned}$$

Combining both estimates results in:

$$\begin{aligned}\limsup_{\delta\to0}\limsup_{n\to\infty}\Bigg|&\int_{J_{\psi_{n,\delta,\eta}}}\Phi(\psi^+_{n,\delta,\eta},\psi^-_{n,\delta,\eta},\nu_{\psi_{n,\delta,\eta}})\,\mathrm{d}\mathcal{H}^1\\&-\int_{J_\psi}\Phi(\psi^+,\psi^-,\nu_\psi)\,\mathrm{d}\mathcal{H}^1\Bigg| = 0\end{aligned} \tag{2.106}$$

With (2.103), (2.105), (2.106), and a standard diagonal sequence argument, we can select $\delta_n \to 0$ and $\eta_n \to 0$ such that $\psi_n := \psi_{n,\delta_n,\eta_n}$ satisfies (2.99), (2.101), and $\mathcal{H}^1(\{\psi^\pm_n \neq \psi^\pm\}\cap L) \to 0$ as $n \to \infty$. It remains to show $\mathcal{H}^1(\{\psi_n \neq \psi\} \cap \partial\Omega) \to 0$. In this regard, we first use (2.101) with $\Phi(a,b,\nu) := |a-b|$, which shows – together with the L^1-convergence of (ψ_n) – that $\psi_n \to \psi$ strictly in $BV(\Omega_r(\mu))$. Due to the continuity of the trace in BV with respect to strict convergence, this implies $\psi_n \to \psi$ in $L^1(\partial\Omega)$. Both ψ_n and ψ take values in

$\mathcal{Z}$, and hence for $\mathcal{H}^1$-a.e. $x \in \{\psi_n \neq \psi\} \cap \partial\Omega$ we have $|\psi_n(x) - \psi(x)| \geq \frac{2\pi}{\mathfrak{m}}$. In conclusion, we see:

$$\mathcal{H}^1(\{\psi_n \neq \psi\} \cap \partial\Omega) \leq \frac{\mathfrak{m}}{2\pi} \int_{\partial\Omega} |\psi_n - \psi| \, \mathrm{d}\mathcal{H}^1 \to 0.$$

□

We are ready to state and prove the desired approximation result for u outside its vortices:

Lemma 2.18 (Approximation of u away from its vortices)
Given $r > 0$ small enough, there exists a sequence $(u_n) \subset SBV^2(\Omega_r(\mu); \mathbb{S}^1)$ such that each u_n is smooth outside its polyhedral jump set J_{u_n}, and:

$$\lim_{n\to\infty} \|u - u_n\|_{L^2(\Omega_r(\mu);\mathbb{R}^2)} + \|\nabla(u_n - u)\|_{L^2(\Omega_r(\mu);\mathbb{R}^{2\times 2})} = 0, \tag{2.107}$$

$$\lim_{n\to 0} \mathcal{H}^1_{\mathrm{cr}}(J_{u_n}) + \mathcal{H}^1_{\mathrm{cr}}(\{u_n \neq g\} \cap \partial\Omega) = \mathcal{H}^1_{\mathrm{cr}}(J_u) + \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\}). \tag{2.108}$$

Proof. 1. step: We start by proving (2.107). Let α and ψ be as in Lemma 2.15, $\varphi := \alpha + \psi$, (α_n) the sequence from Lemma 2.16, and (ψ_n) the one from Lemma 2.17. For each $n \in \mathbb{N}$, define $u_n := e^{\imath\varphi_n}$ for $\varphi_n := \alpha_n + \psi_n$. By construction, each u_n is $\mathbb{S}^1$-valued and smooth outside its polyhedral jump set $J_{u_n} \subset J_{\psi_n} \cup L$. As $|u_n - u| \leq 2\pi|\alpha_n - \alpha|$ and $\alpha_n \to \alpha$ in $L^2(\Omega_r(\mu))$, we have:

$$\lim_{n\to\infty} \|u - u_n\|_{L^2(\Omega_r(\mu);\mathbb{R}^2)} = 0.$$

Furthermore, with the $L^2(\Omega_r(\mu))$-convergence of $(\frac{\partial\alpha_n}{\partial x_1})$ and the boundedness of $(\sin(\alpha_n))$, we can show similarly to the 2. step of the proof of Lemma 2.16:

$$\sin(\alpha_n)\frac{\partial\alpha_n}{\partial x_1} \to \sin(\alpha)\frac{\partial\alpha}{\partial x_1} \text{ in } L^2(\Omega_r(\mu)).$$

With the chain rule, this leads to:

$$\lim_{n\to\infty} \|\nabla(u - u_n)\|_{L^2(\Omega_r(\mu);\mathbb{R}^{2\times 2})} = 0$$

and (2.107) is proved.

2. step: Let us show (2.108). By (2.97), $\mathcal{H}^1\,\{\psi_n \neq \psi\} \cap \partial\Omega \to 0$, the second convergence in (2.100), and the fact that

$$\{u_n \neq g\} \cap \partial\Omega \subset (\{\alpha_n \neq \alpha\} \cap \partial\Omega) \cup (\{\psi_n \neq \alpha\} \cap \partial\Omega)$$

we follow that:

$$\begin{aligned}&\limsup_{n\to\infty} |\mathcal{H}^1_{\mathrm{cr}}(\{u_n \neq g\} \cap \partial\Omega)| \\ &\quad\leq 2 \limsup_{n\to\infty} \big(\mathcal{H}^1(\{\alpha_n \neq \alpha\} \cap \partial\Omega) + \mathcal{H}^1(\{\psi_n \neq \alpha\} \cap \partial\Omega)\big) = 0.\end{aligned} \tag{2.109}$$

As α_n is smooth outside L, it follows that $J_{u_n} \setminus L = J_{\psi_n} \setminus L$. Consequently, with (2.10) for $\Phi(a, b, \nu) := |\nu|_1$, we have:

$$\lim_{n\to\infty} \int_{J_{u_n}\setminus L} |\nu_{u_n}|_1 \, \mathrm{d}\mathcal{H}^1 = \int_{J_u \cap \Omega_r^{(L)}(\mu)} |\nu_u|_1 \, \mathrm{d}\mathcal{H}^1 = \int_{J_u \cap \Omega_r(\mu)} |\nu_u|_1 \, \mathrm{d}\mathcal{H}^1. \tag{2.110}$$

Note that we have used $\mathcal{H}^1(J_u \cap L) = 0$ in the last equality above. Furthermore, as $\alpha_n^+ - \alpha_n^- = \alpha^+ - \alpha^-$ and $\psi^+ = \psi^-$ at $\mathcal{H}^1$-a.e. point in L, it holds for $\mathcal{H}^1$-a.e. $x \in (\{\psi_n^+ = \psi^+\} \cap L) \cap (\{\psi_n^- = \psi^-\} \cap L)$ that:

$$\begin{aligned}\varphi_n^+(x) - \varphi_n^-(x) &= \alpha_n^+(x) + \psi_n^+(x) - \alpha_n^-(x) - \psi_n^-(x)\\ &= \alpha^+(x) - \alpha^-(x) = 0 \mod 2\pi\end{aligned}$$

and therefore $u_n^+(x) = u_n^-(x)$. Consequently,

$$\begin{aligned}&\limsup_{n\to\infty} \int_L |\nu_{u_n}|_1 \,\mathrm{d}\mathcal{H}^1\\ &\quad\le 2\limsup_{n\to\infty} \big(\mathcal{H}^1(\{\psi_n^+ \neq \psi^+\} \cap L) + \mathcal{H}^1(\{\psi_n^- \neq \psi^-\} \cap L)\big) = 0\end{aligned}$$

which – together with (2.110) – leads to

$$\lim_{n\to\infty} \int_{J_{u_n}} |\nu_{u_n}|_1 \,\mathrm{d}\mathcal{H}^1 = \int_{J_u \cap \Omega_r^{(L)}(\mu)} |\nu_u|_1 \,\mathrm{d}\mathcal{H}^1 = \int_{J_u\cap\Omega_r(\mu)} |\nu_u|_1 \,\mathrm{d}\mathcal{H}^1 . \quad (2.111)$$

By combining (2.109) and (2.51), we see that (2.108) holds true. □

Let δ_0 be chosen sufficiently small such that the projection $\Pi_{\partial\Omega}$ is well defined and smooth in $T_{\delta_0}^- := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \delta_0\}$. In the following we will further modify the approximating sequence (u_n) from Lemma 2.18 into another approximating sequence $(\tilde{u}_n)$ with the additional property that each $\tilde{u}_n$ is equal to g in T_δ^- for some $\delta = \delta(u_n) \in (0, \delta_0)$ possibly depending on u_n.

Lemma 2.19
Let u_n be an arbitrary element of the sequence constructed in lemma 2.18. Given $r > 0$ small enough and $\sigma > 0$, we can find $\tilde{u}_n \in SBV^2(\Omega_r(\mu))$ and $\delta = \delta(\sigma, n)$ such that $\tilde{u}_n = g$ in T_δ^-, and:

$$\begin{aligned}\left|\int_{\Omega_r(\mu)} |\nabla u_n|^2 \,\mathrm{d}x - \int_{\Omega_r(\mu)} |\nabla \tilde{u}_n|^2 \,\mathrm{d}x\right| &\le \sigma,\\ \left|\mathcal{H}^1_{\mathrm{cr}}(J_{u_n} \cap \Omega_r(\mu)) + \mathcal{H}^1_{\mathrm{cr}}(\{u_n \neq g\} \cap \partial\Omega) - \mathcal{H}^1_{\mathrm{cr}}(J_{\tilde{u}_n})\right| &\le \sigma.\end{aligned}$$

Proof. For the sake of clearer notation we will shortly write u for u_n and $\tilde{u}$ for $\tilde{u}_n$. *1. step:* As the proof employs standard techniques for dealing with Dirichlet conditions in SBV, we will only provide the crucial ingredients leaving out some of the technical points (see [40] and the references therein). The idea of the proof is to dilate the spin field u "away from the boundary" into $\Omega \setminus T_\rho^-(\Omega)$ and to set it equal to g in T_δ^- for $\delta \in (0, \delta_0)$ to be fixed later on. Let us describe more precisely how $\tilde{u}$ is constructed. We first define the signed distance $d_\Omega\colon \mathbb{R}^2 \to \mathbb{R}$ in the tubular neighborhood $T_{\delta_0} := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega < \delta_0)\}$ as:

$$d_\Omega(x) := \begin{cases} -\operatorname{dist}(x, \Omega) & \text{if } x \in \Omega,\\ \operatorname{dist}(x, \Omega) & \text{if } x \in \mathbb{R}^2 \setminus \Omega.\end{cases}$$

In contrast to the standard distance function, $d_\Omega\colon T_{\delta_0} \to \mathbb{R}$ is smooth in T_{δ_0}. (As this is a classic result in differential geometry we will not further comment on this.) By possibly decreasing δ_0 we can assure that $\operatorname{spt}\mu \cap T_{\delta_0}^- = \emptyset$. Let

$O \coloneqq \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x,\partial\Omega) < \delta_0\}$ and $X \in C_c^\infty(T_{\delta_0};\mathbb{R}^2)$ such that $X = \nabla d_\Omega$ in $T_{\frac{\delta_0}{2}}$. The existence of such a vector field X follows from the smoothness of d_Ω in T_{δ_0} and a standard cut-off procedure. We consider the integral flow $\{\Phi_t\}_{t\geq 0}$ generated by the field X. By construction, each $\Phi_t \colon O \to O$ is a smooth diffeomorphism for each $t \geq 0$. A competitor for $\tilde{u}$ is constructed in three steps: First, extend u by setting $u \coloneqq g$ in $T_{\delta_0}^+$. Second, pre-compose u with the diffeomorphism Φ_t (for some fixed $t > 0$) resulting in $u_t \coloneqq u \circ \Phi_t$. Third, restrict u_t to Ω. As $\operatorname{spt} X \subset\subset T_{\delta_0}$, we have $\Phi_t \equiv \mathrm{Id}$ in $\Omega \setminus T_{\delta_0}^-$, and consequently $u_t = u$ in $\Omega \setminus T_{\delta_0}^-$.

2. step: In order to conclude the proof, it will suffice to show:

$$\lim_{t\to 0} \int_{T_{\delta_0}^-} |\nabla u_t|^2 \,\mathrm{d}x = \int_{T_{\delta_0}^-} |\nabla u|^2 \,\mathrm{d}x, \tag{2.112}$$

$$\lim_{t\to 0} \mathcal{H}^1_{\mathrm{cr}}(J_{u_t} \cap T_{\delta_0}^-) = \mathcal{H}^1_{\mathrm{cr}}(J_u \cap T_{\delta_0}^-) + \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\} \cap \partial\Omega). \tag{2.113}$$

Note that as $X = \nabla d_\Omega$, and by the definition of the signed distance, we have for any $t \in (0, \frac{\delta_0}{2})$ that $u_t \equiv g$ in T_t^-. Furthermore, by the chain rule, it follows in $T_{\delta_0}^- \setminus T_t^-$ that:

$$\nabla u_t = \nabla u(\Phi_t)\nabla\Phi_t, \tag{2.114}$$

$$\begin{aligned}(u_t^+ - u_t^-) \otimes \mathcal{H}^1|_{J_{u_t}} &= (u^+(\Phi_t) - u^-(\Phi_t)) \otimes \mathcal{H}^1|_{\Phi_t^{-1}(J_u)} \\ &\quad + (g - u_{\partial\Omega}^-) \otimes \mathcal{H}^1|_{\Phi_t^{-1}},\end{aligned} \tag{2.115}$$

where in the second equality we implicitly take the approximately continuous representative of u and $u_{\partial\Omega}^-$ denotes the inner trace of u onto $\partial\Omega$. On the one hand, this shows that $u_t \in \mathcal{D}_g^{(\mathfrak{m})}$. On the other, we see that – by the smoothness of $t \mapsto \Phi_t$ and by $\Phi_0 = \mathrm{Id}$ in O – (2.112) and (2.113) are satisfied. □

Let $r > 0$ be small enough such that the balls $B_r(x_k)$, $k = 1,\dots,N_g$, are disjoint. For $n \in \mathbb{N}$ and $k \in \{1,\dots,N_g\}$, we define the sets:

$$\Omega_n \coloneqq \Omega \setminus \bigcup_{k=1}^{N_g} B_{2^{-n}r}(x_k), \tag{2.116}$$

$$A_{k,n} \coloneqq B_{2^{-n}r}(x_k) \setminus B_{2^{-n-1}r}(x_k), \quad \tilde{A}_{k,n} \coloneqq A_{k,n} \cup A_{k,n-1}.$$

Proposition 2.5

Let $r > 0$ be small enough and $u \in \mathcal{D}_g^{(\mathfrak{m})}$ with $\mathcal{W}^{(\mathfrak{m})}(u) < \infty$, then we are able to find for every n a spin field $u_n \in SBV^2(\Omega_n;\mathbb{S}^1)$, which is smooth outside its polyhedral jumpset J_{u_n}, and $\lambda_{k,n} \in \mathbb{S}^1$, $k = 1,\dots,N_g$ such that:

- *(i) $u_n = g$ in a tubular neighborhood around $\partial\Omega$ for every $n \in \mathbb{N}$;*
- *(ii) $u_n^{\mathfrak{m}}(x) = \lambda_{k,n} \frac{x-x_k}{|x-x_k|}$ in $A_{k,n}$ for every $n \in \mathbb{N}$ and $k \in \{1,\dots,N_g\}$;*
- *(iii) the following convergences hold true:*

$$\lim_{n\to\infty} \left(\|u - u_n\|_{L^2(\Omega_n;\mathbb{R}^2)} + \left| \int_{\Omega_n} |\nabla u_n|^2 \,\mathrm{d}x - \int_{\Omega_n} |\nabla u|^2 \,\mathrm{d}x \right| \right) = 0, \tag{2.117}$$

$$\lim_{n\to\infty} \left(|\mathcal{H}^1_{\mathrm{cr}}(J_{u_n}) - \mathcal{H}^1_{\mathrm{cr}}(J_u \cap \Omega_n) - \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\} \cap \partial\Omega)| \right) = 0. \tag{2.118}$$

Proof. 1. step: Combining Lemma 2.18 and Lemma 2.19, we can find for each $n \in \mathbb{N}$ a spin field $u_n \in SBV^2(\Omega_n;\mathbb{R}^2)$ that is smooth outside its jump set J_{u_n} – the latter being a union of finitely many smooth curves such that Item (i) of the statement and (2.118) are satisfied. Moreover:

$$\lim_{n\to 0}\left(\|u-u_n\|_{L^2(\Omega_n;\mathbb{R}^2)}+\|\nabla(u-u_n)\|_{L^2(\Omega_n;\mathbb{R}^{2\times 2})}\right)=0. \tag{2.119}$$

Note that in the construction of Lemma 2.19 we may lose the straightness of the jump set close to $\partial\Omega$. Nevertheless, by a further standard approximation, we can replace u_n by a spin field that has all the aforementioned properties and a polyhedral jump set. We will still denote the resulting spin field by u_n.

2. step: Let us further modify u_n in the annuli $\tilde{A}_{k,n}$, $k=1,\dots,N_g$ in order to assure that Item (ii) in the statement is satisfied: By Lemma 2.4, we can find for each $n \in \mathbb{N}$ vectors $\lambda_{k,n} \in \mathbb{S}^1$, $k=1,\dots,N_g$, such that:

$$\left\| u_n^{\mathfrak{m}} - \lambda_{k,n}\frac{x-x_k}{|x-x_k|}\right\|_{W^{1,2}(A_{k,n};\mathbb{R}^2)} \to 0 \text{ as } n\to\infty.$$

Let $\varphi_{k,n} \in SBV^2(A_{k,n})$ denote the angular lift of u_n provided by [31], then the above convergence – together with the chain rule – imply:

$$\|\nabla(\mathfrak{m}\varphi_{k,n}-\vartheta_{k,n})\|_{L^2(A_{k,n})} \to 0 \text{ as } n\to\infty, \tag{2.120}$$

where $\vartheta_{k,n}$ is the angle of $x \mapsto \lambda_{k,n}\frac{x-x_k}{|x-x_k|}$ with respect to standard polar coordinates. As $\nabla(\mathfrak{m}\varphi_{k,n}-\vartheta_{k,n})$ is a conservative vector field, we can find $\delta_{k,n} \in W^{1,2}(\tilde{A}_{k,n}\setminus L_k)$ with zero average such that $\nabla\delta_{k,n}=\nabla(\mathfrak{m}\varphi_{k,n}-\vartheta_{k,n})$, where L_k denotes an arbitrary line segment emanating from x_k. By the same argument as in the proof of Lemma 2.15, we can show that $\delta^+_{k,n}-\delta^-_{k,n}=0$ on $L_k\cap\tilde{A}_{k,n}$. (Note that in contrast to Lemma 2.15, we have in the present setting $\mathrm{Jac}(u)^{\mathfrak{m}} - \mathrm{Jac}\,\frac{x-x_k}{|x-x_k|}=0$ in $B_r(x_k)$.) Therefore, we can see $\delta_{k,n}$ also as a function in $W^{1,2}(\tilde{A}_{k,n})$. By (2.120) and Poincaré's inequality, it holds that:

$$\|\delta_{k,n}\|_{W^{1,2}(\tilde{A}_{k,n})} \to 0 \text{ as } n\to\infty. \tag{2.121}$$

Let $\eta \in C^\infty(\mathbb{R}_+;[0,1])$ be a smooth cutoff function with $\eta=1$ in $[0,\frac{5}{4}]$ and $\eta=0$ in $[\frac{7}{4},2]$. For each k, we redefine each u_n in $\tilde{A}_{k,n}$ as follows:

$$u_n := e^{\imath\psi_{k,n}}, \quad \psi_{k,n}(x) := \varphi_{k,n}(x) - \eta(2^h\rho^{-1}|x-x_k|)\delta_{k,n}(x).$$

By (2.121), the modified sequence (u_n) still has all the aforementioned properties and additionally satisfies Item (ii) in the statement with $\lambda_{k,n}$, $k=1,\dots,N_g$, as chosen above. □

We are ready to prove the Γ-limsup:

Proof of the Γ-limsup of Theorem 2.16. 1. step: Let (u_n) be the sequence from Proposition 2.5. For fixed $n\in\mathbb{N}$, we define the following sequence of admissible discrete spin fields $\{u_{\varepsilon,n}\}_\varepsilon$:

$$u_{\varepsilon,n}(i) = \begin{cases} u_n(i) & \text{if } i \in \Omega_n, \\ w_{\varepsilon,k,n}(i) & \text{if } i \in B_{2^{-n}r}(x_k) \end{cases}$$

where

$$w_{\varepsilon,k,n} \in \operatorname{argmin}\left\{ XY_\varepsilon^{(\mathfrak{m})}(w) \colon w = \lambda_{k,n} \frac{x - x_k}{|x - x_k|} \text{ on } \partial_\varepsilon B_{2^{-n}r}(x_k) \right\}.$$

By Item (ii) of Proposition 2.5, the sequence $(u_{\varepsilon,n})_\varepsilon \subset \mathcal{AS}_\varepsilon^{(g)}$ is admissible for $\varepsilon > 0$ small enough.

2. step: Let us now investigate the convergence of $XY_\varepsilon^{(\mathfrak{m})}(u_{\varepsilon,n}) - N_g\pi|\log\varepsilon|$ as $\varepsilon \to 0$: By the choice of $w_{\varepsilon,k,n}$ and (2.84), we have that:

$$\limsup_{n\to\infty} \limsup_{\varepsilon\to 0} \left(XY_\varepsilon^{(\mathfrak{m})}(u_{\varepsilon,n}, B_{2^{-n}r}(\mu)) - N_g\pi \log\left(\frac{2^{-n}r}{\varepsilon}\right)\right) = N_g\gamma, \quad (2.122)$$

where

$$B_r(\mu) \coloneqq \bigcup_{k=1}^{N_g} B_r(x_k).$$

By the regularity of u_n, we have the following characterization of the jump pairs of u_n: A nearest-neighbor pair (i,j) in $(\Omega_n)_\varepsilon^{(1)}$ is a jump pair of u_n and therefore, by construction, also of $u_{\varepsilon,n}$ if and only if $[i,j] \cap J_{u_n} \neq \emptyset$. Furthermore, we have – again by the smoothness of u_n outside its jump-set – that:

$$\operatorname{dist}(\Delta^{\text{el}}\varphi_n(i,j), \mathcal{Z}) \leq C(n)\varepsilon,$$

where φ_n is an angular lift of u_n and $C(n) < \infty$ a constant not depending on ε. By (2.32), this leads to

$$\sum_{(i,j)\in JP(u_n)} f_\varepsilon^{(\mathfrak{m})}(\varphi_n(i) - \varphi_n(j)) = \mathcal{H}^1_{\text{cr}}(J_{u_n}) + C(n)\varepsilon.$$

For the remaining nearest-neighbor pairs, we see by standard interpolation estimates and Item (ii) in Definition 2.16 that:

$$\sum_{(i,j)\notin JP(u_{\varepsilon,n})} f_\varepsilon^{(\mathfrak{m})}(\varphi_n(i) - \varphi_n(j)) = \int_{\Omega_{2^{-n-1}r}(\mu)} |\nabla u_n|^2 \,\mathrm{d}x + C(n)\varepsilon,$$

again, for a constant $C(n) < \infty$ independent of ε. Consequently, by also employing the definition of $\mathcal{W}^{(\mathfrak{m})}$, (2.117), and (2.118), we see that:

$$\begin{aligned}&\limsup_{n\to 0} \limsup_{\varepsilon\to 0} \left(XY_\varepsilon^{(\mathfrak{m})}(u_{\varepsilon,n}, B_{2^{-n}r}(\mu)) - N_g\pi|\log(2^{-n}r)|\right)\\ &\quad\leq \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1_{\text{cr}}(J_u) + \mathcal{H}^1_{\text{cr}}(\{u \neq g\} \cap \partial\Omega). \end{aligned} \quad (2.123)$$

Finally, with (2.122), (2.123), and a standard diagonal sequence argument, the desired result follows. □

Chapter 3

Dynamics of the generalized XY model

3.1 Preliminaries

3.1.1 Minimizing movements for the heat equation

In this section, we wish to introduce the main ideas behind the notion of *minimizing movements*, introduced by *De Giorgi* in [32], through an illuminating example. Consider the energy functional $E\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ given by

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \text{ for all } u \in W_0^{1,2}(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is a open bounded set with smooth boundary. Furthermore, we define $D\colon W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ as:

$$D(u, \tilde{u}) := \frac{1}{2}\int_\Omega |u - \tilde{u}|^2 \,\mathrm{d}x.$$

For fixed $u_0 \in W_0^{1,2}(\Omega)$ and $\lambda > 0$, we then consider the iteratively defined sequence $(u_n^\lambda)_n$ as:

$$\begin{cases} u_n^\lambda \in \operatorname{argmin}\big\{E(u) + \lambda D(u, u_{n-1}^\lambda)\big\} \text{ for all } n \geq 1, \\ u_0^\lambda = 0. \end{cases} \tag{3.1}$$

The well-definedness of (u_n^λ) follows from a standard application of the direct method and induction on $n \in \mathbb{N}$. The minimization described above is nontrivial due to the competition between E and D. The former aspires u_n^λ to be as close as possible to the constant 0-function on Ω (the global minimum of E in $W_0^{1,2}(\Omega)$), while the latter prefers u_n^λ to be as close as possible to u_{n-1}^λ, respectively. Furthermore, the bigger we chose λ to be, the more dominant λD becomes in the minimization, and hence the closer u_n^λ is to u_{n-1}^λ. We can therefore think of λ^{-1} as a discrete time step. The piecewise constant interpolation $u^\lambda\colon [0,\infty) \to W_0^{1,2}(\Omega)$ is defined as:

$$u^\lambda(t) := u_{\lceil \lambda t \rceil}^\lambda \text{ for all } t \in [0,\infty),$$

and piecewise affine interpolation $\hat{u}^\lambda$ by:

$$\hat{u}^\lambda(t) := (\lceil \lambda t \rceil - \lambda t)\, u^\lambda_{\lfloor \lambda t \rfloor} + (\lambda t - \lfloor \lambda t \rfloor)\, u^\lambda_{\lceil \lambda t \rceil}.$$

Note that $\lceil x \rceil$ is the smallest number in $\mathbb{Z}$ above x, and $\lfloor x \rfloor$ is the largest smaller than x.

In the following, we will study the limit behavior of (u^λ) and $(\hat{u}^\lambda)$ as $\lambda \to \infty$. The following basic a priori estimates come into play: Using u^λ_{n-1} as a competitor for the minimizer of (3.1), we can estimate:

$$E(u^\lambda_n) + \lambda D(u^\lambda_n, u^\lambda_{n-1}) \le E(u^\lambda_{n-1}) + \lambda D(u^\lambda_{n-1}, u^\lambda_{n-1}) = E(u^\lambda_{n-1}), \tag{3.2}$$

and hence by the nonnegativity of D and an inductive argument

$$\int_\Omega |\nabla u^\lambda_n|^2 \,\mathrm{d}x = E(u^\lambda_n) \le E(u^\lambda_0) = E(u_0) < \infty \text{ for all } n \in \mathbb{N}.$$

By standard compactness in Sobolev spaces there exists a $u \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_0)$ such that, up to taking a subsequence:

$$u^\lambda \rightharpoonup u \text{ weakly in } L^2_{\mathrm{loc}}([0,\infty); W^{1,2}_0).$$

Let us now substract $E(u^\lambda_n)$ from both sides of (3.2) :

$$\lambda D(u^\lambda_n, u^\lambda_{n-1}) \le E(u^\lambda_{n-1}) - E(u^\lambda_n).$$

We then sum the above inequality over all $n \in \mathbb{N}$ and unfold the telescopic sum on the right-hand side. With the non-negativity of E, this leads to:

$$\sum_{n=1}^\infty \lambda D(u^\lambda_n, u^\lambda_{n-1}) \le \sum_{n=1}^\infty (E(u^\lambda_{n-1}) - E(u^\lambda_n)) \le E(u_0) < \infty.$$

Note that with the definition of $\hat{u}^\lambda$, the sum on the left-hand side above can be written as:

$$\begin{aligned}\sum_{n=1}^\infty \lambda D(u^\lambda_n, u^\lambda_{n-1}) &= \frac{1}{2}\sum_{n=1}^\infty \lambda^{-1} \int_\Omega \left|\lambda(u^\lambda_n - u^\lambda_{n-1})\right|^2 \mathrm{d}x \\ &= \frac{1}{2}\int_0^\infty \int_\Omega \left|\partial_t \hat{u}^\lambda\right|^2 \mathrm{d}x dt,\end{aligned}$$

and hence:

$$\sup_{n\in\mathbb{N}} \|\partial_t \hat{u}^\lambda_n\|_{L^2([0,\infty);L^2(\Omega))} \le C(u_0) < \infty. \tag{3.3}$$

Consequently, there exists $\tilde{u} \in W^{1,2}_{\mathrm{loc}}([0,\infty); L^2(\Omega)])$ such that, up to a further subsequence:

$$\hat{u}^\lambda \rightharpoonup \tilde{u} \text{ weakly in } W^{1,2}_{\mathrm{loc}}([0,\infty); L^2(\Omega)).$$

We wish to show that $u = \tilde{u}$. For this, note that by the fundamental theorem of calculus, the Cauchy-Schwarz inequality, and (3.3), we have for any t_1, $t_2 \in \mathbb{R}$ such that $0 \le t_1 < t_2 < \infty$:

$$\begin{aligned}\|\hat{u}^\lambda(t_2,\cdot) - \hat{u}^\lambda(t_1,\cdot)\|^2_{L^2(\Omega)} &= \int_\Omega \left|\int_{t_1}^{t_2} \partial_t \hat{u}^\lambda(t,x)\, dt\right|^2 \mathrm{d}x \\ &\le \int_\Omega |t_2 - t_1| \int_{t_1}^{t_2} \left|\partial_t \hat{u}^\lambda\right|^2 dt dx \\ &\le C(u_0)|t_2 - t_1|\end{aligned}$$

with a constant $C(u_0) < \infty$ independent from λ, t_1, and t_2. By the definition of $\hat{u}^\lambda$ and u^λ, we can estimate for any $t \in [0, \infty)$

$$\|\hat{u}^\lambda(t, \cdot) - u^\lambda(t, \cdot)\|_{L^2(\Omega)} \leq C(u_0)\lambda^{\frac{1}{2}}$$

for the same constant as before. As right-hand side is independent of t it follows that

$$\|\hat{u}^\lambda - u^\lambda\|_{L^\infty([0,\infty);L^2)} \to 0 \text{ as } \lambda \to 0.$$

Consequently,

$$u = \tilde{u} \in L^2_{\text{loc}}([0, \infty); W_0^{1,2}(\Omega)) \cap W^{1,2}_{\text{loc}}([0, \infty); L^2(\Omega)).$$

It is often possible to derive more information on u beyond its mere existence. As u_n^λ is minimal, it must satisfy the Euler-Lagrange equation corresponding to the minimum problem in (3.1):

$$\int_\Omega \langle \nabla u_n^\lambda, \nabla \varphi \rangle + \lambda(u_n^\lambda - u_{n-1}^\lambda)\varphi \, \mathrm{d}x = 0,$$

for any test function $\varphi \in C_c^\infty(\Omega)$. With the definition of u^λ and $\hat{u}^\lambda$, this relation can be rewritten as

$$\int_0^\infty \int_\Omega \langle \nabla u^\lambda, \nabla \varphi \rangle + \langle \partial_t \hat{u}^\lambda, \varphi \rangle \, \mathrm{d}x dt = 0 \tag{3.4}$$

for every test function $\varphi \in C_c^\infty([0, \infty); C_c^\infty(\Omega))$. Employing the aforementioned convergence of $(\hat{u}^\lambda)$ and (u^λ) (up to taking subsequences), we pass to the limit $\lambda \to \infty$ in (3.4), which results in:

$$\int_0^\infty \int_\Omega \langle \nabla u, \nabla \varphi \rangle + \langle \partial_t u, \varphi \rangle \, \mathrm{d}x dt = 0$$

for all test functions $\varphi \in C_c^\infty([0, \infty); C_c^\infty(\Omega))$. This is the classic weak formulation of (3.5) and by elliptic theory, we see that u is also a strong solution of the heat equation:

$$\begin{cases} \partial_t u(t, x) = \Delta u & \text{for all } t \in (0, T),\ x \in \Omega, \\ u(0, x) = u_0(x) & \text{for all } x \in \Omega. \end{cases} \tag{3.5}$$

Lastly, we mention that by the uniqueness of the heat equation, any convergent subsubsequence of (u^λ) must converge towards the u found above. Hence, the full sequence also converges towards u in all the previously mentioned topologies.

3.2 Problem setup

3.2.1 Reduction to a simpler model

The main task of this chapter is to study – from a *variational* point of view – the dynamics of a model related to the one derived in the previous chapter through a Γ-convergence analysis. The Γ-limit of Theorem 2.16 was given by the energy $E\colon \mathcal{D}_g^{(\mathfrak{m})} \to \mathbb{R}$, defined as:

$$E(u) := \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\} \cap \partial\Omega) + N_g\gamma, \tag{3.6}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected set with smooth boundary, $\mathfrak{m} \in \mathbb{N}_+$, $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$, $N_g := \mathfrak{m}|\deg(g), \partial\Omega|$, $\gamma > 0$ the core energy, $\mathcal{H}^1_{\mathrm{cr}}$ the crystalline length (as defined in (2.36)), and $\mathcal{W}^{(\mathfrak{m})}$ the renormalized energy (as defined in (2.35)). Admissible spin fields $u \in \mathcal{D}_K$ satisfy the Dirichlet condition $u^{\mathfrak{m}} = g^{\mathfrak{m}}$ in the sense of Sobolev traces, and have exactly N_g vortices in Ω, each with fractional degree $\frac{\mathrm{sgn}(\deg(g))}{\mathfrak{m}}$. (For a more precise definition, see also Section 2.3.)

We wish to study of the evolution of an initial spin field $u_0 \in \mathcal{D}_g^{(\mathfrak{m})}$ driven by the energy functional E. To the best knowledge of the author, this is a very challenging problem as it would entail to work in the class of SBV-functions for which – in other well-known cases, such as the *Mumford-Shah* functional – satisfactory results concerning dynamics are still missing. In the following, we will succinctly modify the model. As a first step, let us simplify things by working with the singularities of u (described by $\mathrm{Jac}\, u$ and J_u) only. Instead of the whole spin field. The variationally rigorous way to achieve this is to first fix $u \in \mathcal{D}_K$, and then consider the following minimization problem:

$$\min\Big\{E(\tilde{u})\colon \tilde{u} \in \mathcal{D}_g^{(\mathfrak{m})},\ \mathrm{Jac}\,\tilde{u} = \mathrm{Jac}\, u,\ J_{\tilde{u}} = J_u\Big\}. \tag{3.7}$$

In this regard, let us assume – without loss of generality – that $\deg(g, \partial\Omega) > 0$ (the other case works similarly), denote by $\mathrm{spt}(\mathrm{Jac}\, u) = \{x_k\}_{k=1}^{N_g}$ the set of vortex centers of u, and consider for each $k \in \{1, \dots, N_g\}$ a straight line segment Γ_k connecting x_k with $\partial\Omega$ such that $\Gamma := \bigcup_{k=1}^N \Gamma_k$ satisfies $\mathcal{H}^1(\Gamma \cap J_u) = \emptyset$. Similar to the proof of the Γ-limsup in the previous chapter, we take an angular lift $\varphi \in SBV^2_{\mathrm{loc}}(\Omega \setminus \mathrm{spt}\,\mathrm{Jac}(u))$ such that $u = e^{\imath\varphi}$ a.e. in Ω (see also [31]). Furthermore, let $\tilde{\varphi} \in SBV^2_{\mathrm{loc}}(\Omega \setminus \mathrm{spt}\,\mathrm{Jac}(u))$ with:

$$\nabla\tilde{\varphi} = \nabla\varphi \text{ a.e. in } \Omega, \quad D^j\tilde{\varphi} = -\frac{2\pi}{\mathfrak{m}} \otimes \nu\, \mathcal{H}^1|_\Gamma,$$

where for each $k \in \{1, \dots, N\}$, the restriction $\nu_k := \nu|_{\Gamma_k}$ is the normal field on Γ_k pointing in anticlockwise direction with respect to the orientation of Γ_k, and $D^j\tilde{\varphi}$ is the jump part. As in the last chapter, we can find a Caccioppoli partition $\psi \in SBV(\Omega; \mathcal{Z})$ with $\mathcal{Z} := \{0, \frac{2\pi}{\mathfrak{m}}, \dots, (\mathfrak{m}-1)\frac{2\pi}{\mathfrak{m}}\}$ such that:

$$\psi = \varphi - \tilde{\varphi} \mod 2\pi \text{ a.e. in } \Omega.$$

Let $h := g^{\mathfrak{m}}$. We are ready to show the following reformulation of the minimum problem in (3.7):

$$\begin{aligned}\min&\Big\{E(\tilde{u})\colon \tilde{u} \in \mathcal{D}_g^{(\mathfrak{m})},\ \mathrm{Jac}\,\tilde{u} = \mathrm{Jac}\, u,\ J_{\tilde{u}} = J_u\Big\}\\ &= \min\{\mathcal{W}(v)\colon v \in \mathcal{D}_h^{(1)}(\Omega),\ \mathrm{Jac}\, v = \mathfrak{m}\,\mathrm{Jac}\, u\}\\ &\quad + \mathcal{H}^1_{\mathrm{cr}}(J_u) + \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\} \cap \Omega) + N_g\gamma.\end{aligned} \tag{3.8}$$

Given $\tilde{u} \in \mathcal{D}_h^{(1)}(\Omega)$ with $\mathrm{Jac}\,\tilde{u} = \mathrm{Jac}\, u$ and $J_{\tilde{u}} = J_u$, we see by the chain rule and the definition of $\mathcal{W}$ as well as $\mathcal{W}^{(\mathfrak{m})}$ that $v := \tilde{u}^{\mathfrak{m}} \in \mathcal{D}_h^{(1)}(\Omega)$ with

$\operatorname{Jac} v = \mathfrak{m} \operatorname{Jac} \tilde{u} = \mathfrak{m} \operatorname{Jac} u$, and $\mathcal{W}(v) = \mathcal{W}^{(\mathfrak{m})}(\tilde{u})$. This shows the "$\leq$" direction in (3.8). It remains to show the "$\geq$" direction, respectively. In this regard, let $v \in \mathcal{D}_h^{(1)}(\Omega)$ with $\operatorname{Jac} v = \mathfrak{m} \operatorname{Jac} u$. We will construct $\tilde{u} \in \mathcal{D}_g^{(\mathfrak{m})}$ with $J_{\tilde{u}} = J_u$ and $v = \tilde{u}^{\mathfrak{m}}$. With the definition of $\mathcal{W}$ and $\mathcal{W}^{(\mathfrak{m})}$, this would directly lead to (3.8). By a similar reasoning as before, we can find ϑ, $\tilde{\vartheta} \in SBV^2_{\mathrm{loc}}(\Omega \backslash \operatorname{spt} \operatorname{Jac}(u))$ such that:

$$e^{\imath\vartheta} = v, \quad \nabla\tilde{\vartheta} = \nabla\vartheta \text{ a.e. in } \Omega, \quad D^j\tilde{\vartheta} = -2\pi \otimes \nu\, \mathcal{H}^1|_\Gamma$$

with ν and Γ as before. In particular we see that $D^j \frac{\tilde{\vartheta}}{\mathfrak{m}} = D^j\tilde{\varphi}$, with $\tilde{\varphi}$ as defined before, and by the chain rule:

$$\tilde{u} := e^{\imath\left(\frac{\tilde{\vartheta}}{\mathfrak{m}} + \psi\right)}$$

satisfies all the desired properties. It remains to investigate the minimum problem:

$$\min\{\mathcal{W}(v)\colon v \in \mathcal{D}_h^{(1)}(\Omega),\ \operatorname{Jac} v = \mathfrak{m} \operatorname{Jac} u\}. \tag{3.9}$$

As this problem was already thoroughly studied by the authors of [16] (see also, in particular, Chapter 1) we will keep our presentation as short as possible. Let us write $\mu := \mathfrak{m} \operatorname{Jac} u$, then the minimum in (3.9) can be determined by first finding the solution Φ of the following auxiliary linear Neumann problem:

$$\begin{cases} \dfrac{1}{2}\Delta\Phi = \mu & \text{in } \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} = h \times \dfrac{\partial h}{\partial\tau} & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where $\times\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\begin{pmatrix} a \\ b \end{pmatrix} \times \begin{pmatrix} c \\ d \end{pmatrix} := ad - bc.$$

In [16], it is shown that under the additional condition:

$$\int_{\partial\Omega} \Phi \, \mathrm{d}\mathcal{H}^1 = 0$$

there exists a unique solution Φ of (3.10). Given Φ, we also define:

$$R(x) := \Phi(x) - \sum_{k=1}^{N} \log|x - x_k|. \tag{3.11}$$

Note that as:

$$\frac{1}{2}\Delta \sum_{k=1}^{N} \log|x - x_k| = \mu$$

R turns out to be a harmonic function in Ω. The minimum in (3.9) can be represented as follows (see also (47) in [16]):

$$\begin{aligned} \mathbb{W}(x_1, \dots, x_{N_g}) &:= \min\Big\{\mathcal{W}(v)\colon v \in \mathcal{D}_h^{(1)}(\Omega),\ \operatorname{Jac} v = \mu\Big\} \\ &= -\pi \sum_{k \neq l} \log|x_k - x_l| - \pi \sum_{k=1}^{N} R(x_k) + \frac{1}{2}\int_{\partial\Omega} \Phi \left(h \times \frac{\partial h}{\partial\tau}\right) \mathrm{d}\mathcal{H}^1. \end{aligned} \tag{3.12}$$

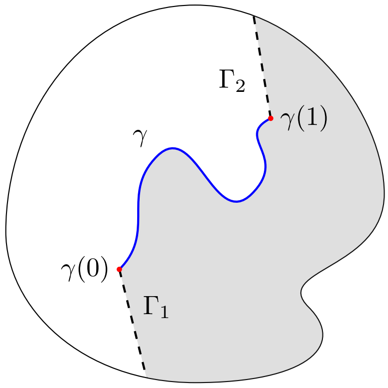


Figure 3.1: Caccioppoli partition used in the construction of u.

Generally $\mathbb{W}$ cannot be written in a more explicit fashion as this would entail having explicit expressions for the Green's functions in Ω. Nevertheless, the authors of [16] derive the following properties of $\mathbb{W}$:

(i) It is smooth on the set:

$$\left\{(x_1, \dots, x_{N_g}) \in \Omega^{N_g} : x_k \neq x_l \text{ for } k \neq l\right\};$$

(ii) It has the following divergence behavior:

$$\mathbb{W}(x_1, \dots, x_N) \to \infty \text{ as } \min\left\{\min_{k\neq l}|x_k - x_l|, \min_k \operatorname{dist}(x_k, \partial\Omega)\right\}. \tag{3.13}$$

(For a proof, we refer to Theorem I.10 in [16].) Intuitively, this means that the vortices repel each other, while also being repelled by the boundary of Ω.

A minimizer of (3.9) is given by a solution v_μ of the following Dirichlet problem:

$$\begin{cases} \operatorname{jac}(v_\mu) = (\nabla\Phi)^\perp & \text{in } \Omega, \\ v_\mu = h & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Again, the authors of [16] show that such a v_μ exists and is, in fact, unique. We are thus left with studying the dynamics of the reduced energy functional $E_1 \colon \mathcal{D}_g^{(\mathfrak{m})} \to \mathbb{R}$, defined as:

$$E_1(u) \coloneqq \mathbb{W}(x_1, \dots, x_N) + \mathcal{H}^1_{\mathrm{cr}}(J_u) + \mathcal{H}^1_{\mathrm{cr}}(\{u \neq g\} \cap \partial\Omega), \tag{3.15}$$

where $\operatorname{Jac} u = \frac{2\pi}{\mathfrak{m}} \sum_{k=1}^N \delta_{x_k}$. Note that we have removed the core energy term as it is the same for all admissible u. This can be justified by (3.13), which disallows the collision of two vortices or one vortex with $\partial\Omega$ during the evolution.

In the next step, we restrict ourselves to the special singularity configuration where the jump set J_u is parameterized by a curve $\gamma \in W^{1,2}([0,1];\Omega)$ connecting two half-vortices at its endpoints $\gamma(0)$ and $\gamma(1)$. In order to show that this singularity configuration is admissible, we need to assume that $\deg(g, \partial\Omega) = 1$. Then, a spin field u with jumpset γ (note that we identify γ with its image im(γ)) and two half-vortices at $\gamma(0)$ and $\gamma(1)$ can be constructed as follows: For $\mu := \pi(\delta_{\gamma(0)} + \delta_{\gamma(1)})$, let v_μ be the solution of (3.14). Moreover, let $\vartheta_\mu \in SBV^2_{\mathrm{loc}}(\Omega \setminus \{\gamma(0), \gamma(1)\})$ such that:

$$\nabla v_\mu = \nabla e^{\imath\vartheta_\mu} \text{ a.e. in } \Omega, \quad D^j\vartheta_\mu = -\pi \otimes \nu\, \mathcal{H}^1|_\Gamma$$

with $\Gamma = \Gamma_1 \cup \Gamma_2$ and ν as before. Then:

$$u := e^{\imath\frac{\vartheta_\mu}{\mathrm{m}}} e^{\imath\psi}$$

for the Caccioppoli partition Ψ as depicted in figure Figure 3.1 is the desired spin configuration with jump set γ and vortices located at $\gamma(0)$ and $\gamma(1)$. Lastly, we replace the crystalline perimeter of γ in (3.15) by the Euclidean one and add a regularizing second-order term. More precisely, given a fixed scalar $\varepsilon > 0$, we consider the energy functional $E_2 \colon W^{1,2} \to \mathbb{R}$ given by:

$$E_2(\gamma) := W(\gamma(0), \gamma(1)) + L(\gamma) + \frac{\varepsilon^2}{2} \int_\gamma \kappa^2 \, \mathrm{d}s, \tag{3.16}$$

where $L(\gamma)$ is the length of γ, κ its curvature, s denotes the arc-length parameter, and:

$$W(p, q) := -\pi \log|p - q| - \pi(R(p) + R(q)) + \frac{1}{2} \int_{\partial\Omega} \Phi\left(h \times \frac{\partial h}{\partial\tau}\right) \mathrm{d}\mathcal{H}^1 . \tag{3.17}$$

Since from this point on we will only deal with the energy functional E_2, we redefine $E := E_2$ for the sake of shorter notation. The additional curvature term in (3.16) is a standard regularizing term in the study of the evolution of the perimeter function (see also, e.g. [27] and [36]). Speaking more generally, it is also an interesting problem to take networks of curves with multiple junctions (possibly containing vortices) into consideration. For the sake of clarity, general networks are not studied in this thesis. Nevertheless, the author believes that the proof strategy found in this chapter can be generalized to the case of more complex singularity configurations.

3.2.2 Description of the minimizing movements scheme

Let us first fix some notation: The interval $[0, 1]$ will be shortly written as I. We will be dealing with curves $\gamma \colon I \to \mathbb{R}^2$ as well as families of curves of the type $\gamma \colon I \times [0, \infty) \to \mathbb{R}^2$ or $\gamma \colon I \times [0, T] \to \mathbb{R}$ for some $T > 0$. It will be clear from the context whether γ denotes a single curve or a family of curves. If not explicitly stated, γ can have values outside of Ω. The following shorthand notation for function spaces of such (family of) curves is assumed: The space $F(I; \mathbb{R}^2)$ will be shortly written as F. So, for example $W^{2,2}$ is the same as $W^{2,2}(I; \mathbb{R}^2)$. Given a space of curves $G = G(I; \mathbb{R}^2)$ and $T > 0$, we will shortly write the following spaces of families of curves as:

$$FG := F([0, \infty); G), \quad F_T G := F([0, T]; G).$$

For example:

$$W^{1,2}_{\mathrm{loc}} L^2 := W^{1,2}_{\mathrm{loc}}([0, \infty); L^2(I; \mathbb{R}^2)), \quad W^{1,2}_T L^2 := W^{1,2}([0, T]; L^2(I; \mathbb{R}^2)).$$

For a curve γ, we will write x for its curve parameter, s for the arc-length parameter, κ for its curvature, $L = L(\gamma)$ for its length, τ for the unit norm tangent vector, and $\nu := \tau^\perp$ for the corresponding normal vector field. For a family of curves γ, we will denote the first parameter (which is the curve parameter of $\gamma(\cdot, t)$ for any t) by x, and the second parameter (which corresponds to time) by t. The same notation will be used for all geometric quantities of γ, such as κ, L, τ, and ν where we implicitly assume that they all may depend on time. Derivatives will be written with lower index notation such as for example: κ_{xx}, κ_x, γ_t, etc. If γ is weakly differentiable in time, we will shortly write $V := \gamma_t$, $V^\top :=:= \langle \gamma_t, \tau\rangle$, and $V^\perp := \langle \gamma_t, \nu\rangle$. Furthermore, we will write $\gamma^p := \gamma(0, \cdot)$ and $\gamma^q := \gamma(1, \cdot)$. The same notation will be used for geometric quantities and their derivatives, respectively. So, for example $\kappa^p := \kappa(0, \cdot)$ and $\kappa^p_x = \kappa_x(0, \cdot)$. Correspondingly, when dealing with two (families of) curves γ and $\tilde\gamma$, we write the geometric quantities of $\tilde\gamma$ as $\tilde\kappa$, $\tilde L$, $\tilde\tau$, and $\tilde\nu$.

We are ready to describe the *minimizing movements scheme* and state the main result of this chapter. The set of admissible curves is given by:

$$\mathcal{AC} := \big\{\gamma \in W^{2,2} \colon |\gamma_x| \equiv \text{const} = L,\, \gamma(0), \gamma(1) \in \Omega,\, \gamma(0) \neq \gamma(1)\big\}. \tag{3.18}$$

Furthermore, for any $\tilde\gamma \in \mathcal{AC}$ we define:

$$\mathcal{AC}(\tilde\gamma) := \{\gamma \in \mathcal{AC} \colon \langle \gamma_x, \tilde\gamma_x\rangle \geq 0\}. \tag{3.19}$$

For fixed $\varepsilon > 0$, the energy $E \colon \mathcal{AC} \to \mathbb{R}$ is given by:

$$E(\gamma) := W(\gamma(0), \gamma(1)) + L + \frac{\varepsilon}{2}\int_\gamma \kappa^2 \,\mathrm{d}s. \tag{3.20}$$

The dissipation functional $D \colon \mathcal{AC}^2 \to \mathbb{R}$ is defined as:

$$\begin{aligned} D(\gamma, \tilde\gamma) := {}& \frac{1}{4}\int_0^1 \langle \gamma - \tilde\gamma, \tilde\nu\rangle^2 \,\mathrm{d}x + \frac{1}{4}\int_0^1 \langle \gamma - \tilde\gamma, \nu\rangle^2 \,\mathrm{d}x \\ &+ \frac{1}{2}|\gamma(0) - \tilde\gamma(0)|^2 + \frac{1}{2}|\gamma(1) - \tilde\gamma(1)|^2. \end{aligned} \tag{3.21}$$

For a given initial curve $\gamma_0 \in \mathcal{AC}$ such that $\gamma_0 \subset \Omega$ (again, we identify γ and $\mathrm{im}(\gamma)$) and a given $\lambda \in (0, 1)$ we define the following sequence of step-by-step minimizers $\{\gamma^\lambda_n\}_n \subset \mathcal{AC}$ as:

$$\begin{cases} \gamma^\lambda_n \in \operatorname{argmin}\big\{E(\gamma) + \lambda D(\gamma, \gamma^\lambda_{n-1}) \colon \gamma \in \mathcal{AC}(\gamma^\lambda_{n-1})\big\} \text{ for all } n \in \mathbb{N}_+, \\ \gamma^\lambda_0 = \gamma_0. \end{cases} \tag{3.22}$$

(Note that in the minimization scheme above we do *not* enforce the curves $\{\gamma_n\}_{n\in\mathbb{N}_+}$ to be contained in Ω.)

For the proof of the well-definedness of (3.22), the following lemma will be used:

Lemma 3.1
The function W, as defined in (3.17)*, is bounded from below:*

$$\inf_{p,q\in\Omega} W(p, q) > -\infty. \tag{3.23}$$

Proof. By the boundedness of Ω and the estimate $\log t \leq t$, we see that:

$$\inf_{p,q\in\Omega} -\pi \log|p-q| \geq -\pi \sup_{p,q\in\Omega} |p-q| \geq -\pi \operatorname{diam}(\Omega) > -\infty.$$

Moreover, in Lemma I.6 of [16] it was shown that there exists a constant $C = C(\Omega, g) < \infty$, independent of the vortex centers p and q, such that Φ, as defined in (3.10) with $\mu = \pi(\delta_{\gamma(0)} + \delta_{\gamma(1)})$, satisfies:

$$\int_{\partial\Omega} |\Phi| \leq C. \tag{3.24}$$

Consequently, with the smoothness of g we derive

$$\inf_{p,q\in\Omega} \int_{\partial\Omega} \Phi \left(g^{\mathrm{m}} \times \frac{\partial g^{\mathrm{m}}}{\partial \tau} \right) \mathrm{d}\mathcal{H}^1 > -\infty.$$

Hence, in order to show (3.23), it is enough to prove:

$$\sup_{p,q\in\Omega} R(p) < \infty, \tag{3.25}$$

where R is as in (3.11). Note that by symmetry, it would also follow that:

$$\sup_{p,q\in\Omega} R(q) < \infty.$$

By Lemma I.7 in [16]:

$$R(p) \to \infty \text{ as } \min\{\operatorname{dist}(p, \partial\Omega), \operatorname{dist}(q, \partial\Omega)\} \to 0.$$

Thus, we can find $\delta \in (0,1)$ small enough such that $R(p) \geq 0$, whenever at least one of the points p and q has distance smaller than δ from the boundary. It remains to investigate the case $\min\{\operatorname{dist}(p, \partial\Omega), \operatorname{dist}(q, \partial\Omega)\} > \delta$. We can estimate:

$$\sup_{x\in\partial\Omega} (-\log|x-p| - \log|x-q|) \leq -\log\delta < \infty.$$

Hence, with (3.24) and the definition of R (see (3.11)), we see that:

$$\int_{\partial\Omega} |R| \,\mathrm{d}\mathcal{H}^1 \leq C = C(\delta, \Omega, g) < \infty$$

for a constant C independent of p and q. Let G denote the Green's function of Ω. By the smoothness of G restricted to $\{x \in \Omega \colon \operatorname{dist}(x, \Omega) > \delta\} \times \partial\Omega$ and the estimate above, we derive:

$$R(p) = \int_{\partial\Omega} R(x) \frac{\partial G}{\partial \nu}(x, p) \,\mathrm{d}\mathcal{H}^1(x) \leq C = C(\delta, \Omega, g) < \infty$$

for a constant C independent of p and q, which concludes the proof. □

Proposition 3.1 (Existence of step-by-step minimizers)
For any $\lambda > 0$ and $\tilde{\gamma} \in \mathcal{AC}$, there exists

$$\gamma \in \operatorname{argmin}\{E(\gamma) + \lambda D(\gamma, \tilde{\gamma}) \colon \gamma \in \mathcal{AC}(\tilde{\gamma})\}. \tag{3.26}$$

Proof. 1. step: Let us fix $F\colon \mathcal{AC}\times\mathcal{AC}\to\mathbb{R}$ as:

$$F(\mu,\tilde{\mu}) \coloneqq E(\mu)+\lambda D(\mu,\tilde{\mu}).$$

Using $\tilde{\gamma}$ as a competitor in (3.26), we see that

$$\inf_{\gamma\in\mathcal{AC}(\tilde{\gamma})} F(\gamma,\tilde{\gamma}) \le F(\tilde{\gamma},\tilde{\gamma}) = E(\tilde{\gamma}). \tag{3.27}$$

Let m denote the infimum on the left side of (3.23). By (3.23) and the definition of F:

$$-\infty < m \le \inf_{\gamma\in\mathcal{AC}(\tilde{\gamma})} F(\gamma,\tilde{\gamma}) \le E(\tilde{\gamma}) < \infty. \tag{3.28}$$

Consequently, we can find a minimizing sequence $\{\gamma_n\}\subset\mathcal{AC}(\tilde{\gamma})$ such that:

$$\lim_{n\to\infty} F(\gamma_n,\tilde{\gamma}) = \inf_{\gamma\in\mathcal{AC}(\tilde{\gamma})} F(\gamma,\tilde{\gamma}), \tag{3.29}$$

$$F(\gamma_n,\tilde{\gamma}) \le E(\tilde{\gamma})+1<\infty \text{ for all } n\in\mathbb{N}. \tag{3.30}$$

2. step: We wish to show that $\sup_n\|\gamma_n\|_{W^{2,2}}<\infty$. In this regard, by (3.23) and (3.30), we derive:

$$E(\tilde{\gamma})+1 \ge F(\gamma_n,\tilde{\gamma}) \ge E(\gamma_n) \ge m + L_n + \frac{\varepsilon L_n}{2}\int_0^1 \kappa_n^2\,\mathrm{d}x, \tag{3.31}$$

where L_n is the length of γ_n and κ_n is its curvature. Moreover, by the definition of D and (3.30):

$$\lambda|\gamma_n(0)-\tilde{\gamma}(0)|^2 \le D(\gamma_n,\tilde{\gamma}) \le F(\gamma_n,\tilde{\gamma}) - m \le E(\tilde{\gamma})+1-m<\infty. \tag{3.32}$$

Thus, by the fundamental theorem of calculus, the fact that $|(\gamma_n)_x| = L_n$, (3.31), and (3.32), we derive:

$$\begin{aligned}\int_0^1 |\gamma_n|^2\,\mathrm{d}x + \int_0^1 |(\gamma_n)_x|^2\,\mathrm{d}x &\le (|\gamma_n(0)|+L_n)^2 + L_n^2 \\ &\le (|\tilde{\gamma}(0)| + |\gamma_n(0)-\tilde{\gamma}(0)| + L_n)^2 + L_n^2 \\ &\le 2|\tilde{\gamma}(0)|^2 + \frac{4}{\lambda}\cdot\lambda|\gamma_n(0)-\tilde{\gamma}(0)|^2 + 3L_n^2 \\ &\le 2|\tilde{\gamma}(0)|^2 + \frac{4}{\lambda}(E(\tilde{\gamma})+1-m) \\ &\quad + 3(E(\tilde{\gamma})+1-m)^2 < \infty. \end{aligned} \tag{3.33}$$

With (3.31) and the constant speed of γ_n ($|(\gamma_n)_x| = L_n$), we follow that:

$$\begin{aligned}\int_0^1 |(\gamma_n)_{xx}|^2\,\mathrm{d}x = \int_0^1 L_n^4\kappa_n^2\,\mathrm{d}x &= \frac{2}{\varepsilon}L_n^3\left(\frac{\varepsilon}{2}L_n\int_0^1\kappa_n^2\,\mathrm{d}x\right) \\ &\le \frac{2}{\varepsilon}(E(\tilde{\gamma})+1-m)^4<\infty. \end{aligned} \tag{3.34}$$

Combining (3.33) and (3.34) eventually leads to $\sup_n\|\gamma_n\|_{W^{2,2}}<\infty$, as is desired. By the weak compactness in $W^{2,2}$ and by the Sobolev embedding theorem, we can find $\gamma\in W^{2,2}$ such that, up to taking a subsequence:

$$\gamma_n \rightharpoonup \gamma \text{ weakly in } W^{2,2}, \tag{3.35}$$

$$\gamma_n \to \gamma \text{ in } C^1. \tag{3.36}$$

3. step: We continue by showing $\gamma \in \mathcal{AC}(\tilde{\gamma})$. By (3.36) and $(\gamma_n) \subset \mathcal{AC}(\tilde{\gamma})$, we already have that:

$$|\gamma_x| \equiv L_\gamma, \quad \langle \gamma_x, \tilde{\gamma}_x \rangle \geq 0.$$

By (3.13) and (3.30), it also follows that there exists a $\delta > 0$ satisfying:

$$\min\{|\gamma_n(0) - \gamma_n(1)|, \operatorname{dist}(\gamma_n(0), \partial\Omega), \operatorname{dist}(\gamma_n(1), \partial\Omega)\} \geq \delta \text{ for all } n \in \mathbb{N}.$$

With the smoothness of W outside the diagonal and (3.36), we can pass to the limit $n \to \infty$ which leads to

$$\min\{|\gamma(0) - \gamma(1)|, \operatorname{dist}(\gamma(0), \partial\Omega), \operatorname{dist}(\gamma(1), \partial\Omega)\} \geq \delta > 0.$$

With this, we conclude that $\gamma \in \mathcal{AC}(\tilde{\gamma})$.

4. step: It remains to show that γ is also minimizing. To this end, note that:

$$\begin{aligned}\int_{\gamma_n} \kappa_n^2 \,\mathrm{d}s &= \int_0^1 \left(\frac{\langle (\gamma_n)_{xx}, \nu_n \rangle}{|(\gamma_n)_x|^2} \right)^2 |(\gamma_n)_x| \,\mathrm{d}x \\ &= \int_0^1 \langle (\gamma_n)_{xx}, L_n^{-\frac{5}{2}} \nu_n \rangle^2 \,\mathrm{d}x,\end{aligned}$$

where ν_n is the unit normal field on γ_n. By (3.35) and (3.36), the following convergences hold true:

$$\begin{aligned}(\gamma_n)_{xx} &\rightharpoonup \gamma \text{ weakly in } L^2, \\ L_n^{-\frac{5}{2}} \nu_n &\to L^{-\frac{5}{2}} \nu \text{ in } L^2.\end{aligned}$$

Hence by weak-strong convergence

$$\langle (\gamma_n)_{xx}, L_n^{-\frac{5}{2}} \nu_n \rangle \rightharpoonup \langle \gamma_{xx}, L^{-\frac{5}{2}} \nu \rangle \text{ weakly in } L^2,$$

and therefore:

$$\liminf_{n\to\infty} \frac{\varepsilon}{2} \int_{\gamma_n} \kappa_n^2 \,\mathrm{d}s \geq \frac{\varepsilon}{2} \int_\gamma \kappa^2 \,\mathrm{d}s. \tag{3.37}$$

By (3.36), we have that:

$$\lim_{n\to\infty} F(\gamma_n, \tilde{\gamma}) - \frac{\varepsilon}{2} \int_{\gamma_n} \kappa_n^2 d\,\mathcal{H}^1 = F(\gamma, \tilde{\gamma}) - \frac{\varepsilon}{2} \int_\gamma \kappa_\gamma^2 d\,\mathcal{H}^1 . \tag{3.38}$$

Combining (3.37) and (3.38) as well as using (3.29) eventually leads to:

$$\inf_{\gamma \in \mathcal{AC}(\tilde{\gamma})} F(\gamma, \tilde{\gamma}) = \lim_{n\to\infty} F(\gamma_n, \tilde{\gamma}) \geq F(\gamma, \tilde{\gamma}). \tag{3.39}$$

□

With the scheme described above, we will be able to derive the following maximal existence result of an L^2-type gradient flow of E:

Theorem 3.1 (Maximal existence)
For any $\gamma_0 \in \mathcal{AC}$ with $\gamma_0 \subset \Omega$, there exists $T_0 \in (0,\infty]$ and a family of curves γ with the following regularity:

$$\begin{gathered}\gamma \in L^2_{T_0}H^4 \cap C^0_{T_0}C^1 \cap W^{1,2}_{T_0}L^2, \quad V^\top \in L^2([0,T_0); W^{1,2}(I)), \\ \gamma^p, \gamma^q \in W^{1,2}([0,T_0); \mathbb{R}^2)\end{gathered} \tag{3.40}$$

such that $\gamma(\cdot,0) = \gamma_0$, for every $t \in [0,T_0)$, it holds that $\gamma(t,\cdot) \subset \Omega$, and for a.e. $t \in [0,T_0]$ and a.e. $s \in [0,L]$ (s denotes the arc-length parameter):

$$V^\perp = \kappa - \varepsilon(\kappa_{ss} + \tfrac{1}{2}\kappa^3), \tag{3.41}$$
$$V^p = -\nabla_p W(\gamma^p,\gamma^q) + \tau^p - \varepsilon\kappa_s^p \nu^p, \tag{3.42}$$
$$V^q = -\nabla_q W(\gamma^p,\gamma^q) - \tau^q + \varepsilon\kappa_s^q \nu^q, \tag{3.43}$$
$$\kappa^p = \kappa^q = 0, \tag{3.44}$$

as well as:

$$V_s^\top = \frac{L'}{L} + \kappa\gamma_t^\perp. \tag{3.45}$$

T_0 is optimal in the sense that if $T_0 < \infty$, we have that $\gamma(\cdot,T_0) \cap \partial\Omega \neq \emptyset$.

We remark that a similar result was proved by the author in [14] (see also Theorem 9). In contrast to the setting described above, the curves in [14] were not restrained to lie in a bounded domain Ω. The endpoints still repelled each other, but through the simpler interaction potential:

$$\tilde{W}(p,q) := -\pi \log|p-q|.$$

Note that the evolution in [14] satisfied the same system as in (3.41) to (3.45) up to replacing W with $\tilde{W}$. Lastly, the existence result in [14] stands in contrast to Theorem 3.1, which is global in time.

3.3 Proof of the minimizing movements result

3.3.1 Compactness

In this subsection, the sequence $(\gamma_n^\lambda)_n$ will always denote the one from (3.22). We also assume that $\lambda \geq 1$. (As we are interested in the limit $\lambda \to \infty$, this is not restrictive.)

Lemma 3.2 (A priori bounds)
There exist constants $C = C(\gamma_0)$, $c = c(\gamma_0)$ independent of λ and n such that:

$$c < |\gamma_n^\lambda(1) - \gamma_n^\lambda(0)| \leq L_n^\lambda \leq C, \tag{3.46}$$
$$\int_0^1 |(\gamma_n^\lambda)_{xx}|^2 \,\mathrm{d}x \leq C. \tag{3.47}$$

Proof. Let us fix $n \in \mathbb{N}$ and use γ_{n-1}^λ as a competitor for the minimization in (3.22) at step n:

$$E(\gamma_n^\lambda) + \lambda D(\gamma_n^\lambda, \gamma_{n-1}^\lambda) \leq E(\gamma_{n-1}^\lambda) + \lambda D(\gamma_{n-1}^\lambda, \gamma_{n-1}^\lambda) = E(\gamma_{n-1}^\lambda).$$

By the nonnegativity of D and an induction argument, this eventually leads to:

$$E(\gamma_n^\lambda) \le E(\gamma_{n-1}^\lambda) \le \cdots \le E(\gamma_0^\lambda) = E(\gamma_0) \tag{3.48}$$

for all $n \in \mathbb{N}$. Let m denote the infimum in (3.23). By the definition of E:

$$L_n^\lambda + (L_n^\lambda)^5 \int_0^1 |(\gamma_n^\lambda)_{xx}|^2 \, \mathrm{d}x \le E(\gamma_0) - m \tag{3.49}$$

for all $n \in \mathbb{N}$. Furthermore, by the nonnegativity of the second term above, this leads directly to:

$$L_n^\lambda \le E(\gamma_0) - m < \infty,$$

again, for all $n \in \mathbb{N}$. Notice that as the straight line is the shortest connection between two points:

$$L_n^\lambda \ge |\gamma_n^\lambda(1) - \gamma_n^\lambda(0)|.$$

Now we want to show that there exists $c > 0$ independent of λ and n such that for all $n \in \mathbb{N}$ and $\lambda \ge 1$:

$$|\gamma_n^\lambda(1) - \gamma_n^\lambda(0)| \ge c.$$

This fact follows from (3.13). More precisely, we can find $\delta > 0$ small enough such that for all $p, q \in \Omega$ with $|p - q| < \delta$, it holds that $\mathcal{W}(p, q) > E(\gamma_0)$. So let us suppose by contradiction that $|\gamma_n^\lambda(1) - \gamma_n^\lambda(0)| < \delta$. Then, by the choice δ:

$$W(\gamma_n^\lambda(0), \gamma_n^\lambda(1)) > E(\gamma_0).$$

This contradicts (3.48), which implies by the definition of E

$$W(\gamma_n^\lambda(0), \gamma_n^\lambda(1)) \le E(\gamma_n^\lambda) \le E(\gamma_0).$$

It remains to show (3.47). By the lower bound in (3.46) and (3.49), we see that:

$$\int_0^1 |(\gamma_n^\lambda)_{xx}|^2 \, \mathrm{d}x = \frac{1}{(L_n^\lambda)^5} (L_n^\lambda)^5 \int_0^1 |(\gamma_n^\lambda)_{xx}|^2 \, \mathrm{d}x \le \frac{1}{c^5}(E(\gamma_0) - m) < \infty,$$

as is desired. □

Definition 3.1 (Interpolations)
The piecewise constant interpolation $\gamma^\lambda \colon I \times [0, \infty) \to \mathbb{R}^2$ of $(\gamma_n^\lambda)_n$ is defined as:

$$\gamma^\lambda(t, x) := \gamma_{\lceil \lambda t \rceil}^\lambda(x).$$

Correspondingly, L^λ will denote its length, κ^λ its curvature, τ^λ its unit tangent vector field, and ν^λ its unit normal vector field. Later, it will be useful to have a notation for the translation in time :

$$\tilde{\gamma}^\lambda(t, x) := \gamma^\lambda(x, t - \lambda^{-1}).$$

Similarly, $\tilde{L}^\lambda$ will denote its length, $\tilde{\kappa}^\lambda$ its curvature, $\tilde{\tau}^\lambda$ its unit tangent vector field, and $\tilde{\nu}^\lambda$ its unit normal vector field. Furthermore, we denote by $\hat{\gamma}^\lambda \colon I \times [0, \infty) \to \mathbb{R}^2$ the piecewise affine interpolation of $(\gamma_n^\lambda)_n$, defined as:

$$\hat{\gamma}^\lambda(t, x) := (\lceil \lambda t \rceil - \lambda t)\gamma_{\lfloor \lambda t \rfloor}^\lambda(x) + (\lambda t - \lfloor \lambda t \rfloor)\gamma_{\lceil \lambda t \rceil}^\lambda(x).$$

Finally, we also fix the piecewise affine interpolation $\hat{L}^\lambda\colon [0,\infty) \to \mathbb{R}$ of $(L_{\gamma_n^\lambda})_n$ as:

$$\hat{L}^\lambda(t) := (\lceil \lambda t \rceil - \lambda t)L^\lambda(\lfloor \lambda t \rfloor) + (\lambda t - \lfloor \lambda t \rfloor)L^\lambda(\lceil \lambda t \rceil),$$

and $V^\lambda\colon I \times [0,\infty) \to \mathbb{R}^2$ to be

$$V^\lambda(t,x) := \hat{\gamma}_t^\lambda(t,x).$$

In the following lemma we will derive an important coupling relation between the tangential and the orthogonal projection of the velocity. It will eventually be used in the proof of Proposition 3.2 in order to derive a bound on the time-discrete velocity.

Lemma 3.3 (Coupling relation)
For every $x \in I$ and $t \in [0,\infty)$, it holds that:

$$\langle V^\lambda, \tilde{L}^\lambda \tilde{\tau}^\lambda + L^\lambda \tau^\lambda \rangle_x = (\tilde{L}^\lambda + L^\lambda)\hat{L}_t^\lambda + \langle V^\lambda, (\tilde{L}^\lambda)^2 \tilde{\kappa}^\lambda \tilde{\nu}^\lambda + (L^\lambda)^2 \kappa^\lambda \nu^\lambda \rangle. \tag{3.50}$$

Proof. The derivation of the coupling relation (3.50) is the result of the following computation: Since γ_n belongs to $\mathcal{AC}$, we have $\gamma_x^\lambda(t,x) = L^\lambda(t)$ for all $x \in I$ and $t \in [0,\infty)$. Defining $\mu^\lambda := \gamma_x^\lambda + \tilde{\gamma}_x^\lambda = \tilde{L}^\lambda \tilde{\tau}^\lambda + L\tau^\lambda$, we derive for all $x \in I$ and $t \in [0,\infty)$:

$$\begin{aligned}(\tilde{L}^\lambda + L^\lambda)\hat{L}_t^\lambda &= \lambda(\tilde{L}^\lambda + L^\lambda)(L^\lambda - \tilde{L}^\lambda)\\ &= \lambda\Big((L^\lambda)^2 - (\tilde{L}^\lambda)^2\Big)\\ &= \lambda(\langle \gamma_x^\lambda, \gamma_x^\lambda\rangle - \langle \tilde{\gamma}_x^\lambda, \tilde{\gamma}_x^\lambda\rangle) = \langle V_x^\lambda, \mu^\lambda\rangle.\end{aligned}$$

Furthermore, by the product rule we have:

$$\begin{aligned}\langle V^\lambda, \mu^\lambda\rangle_x &= \langle V_x^\lambda, \mu^\lambda\rangle + \langle V^\lambda, \tilde{\gamma}_{xx}^\lambda + \gamma_{xx}^\lambda\rangle\\ &= \langle V_x^\lambda, \mu^\lambda\rangle + \langle V^\lambda, (\tilde{L}^\lambda)^2 \tilde{\kappa}^\lambda \tilde{\nu}^\lambda + (L^\lambda)^2 \kappa^\lambda \nu^\lambda\rangle.\end{aligned}$$

By combining both computations, we conclude the proof. □

In the next proposition, we will employ the coupling relation:

Proposition 3.2 (L^2-bound on the velocity)
There exists a constant $C = C(\varepsilon) < \infty$ independent of λ such that:

$$\int_0^\infty \int_0^1 |V^\lambda|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \le C. \tag{3.51}$$

Proof. 1. Step: In order to shorten notations, we will set $\mu^\lambda := \tilde{L}^\lambda \tilde{\tau}^\lambda + L^\lambda \tau^\lambda$. Note that by (3.46) and the fact that $\gamma_n^\lambda \in \mathcal{AC}(\gamma_{n-1}^\lambda)$ for all $n \in \mathbb{N}_+$, we have:

$$\begin{aligned}|\mu^\lambda|^2 &= (\tilde{L}^\lambda)^2 + 2\langle \tilde{\gamma}_x^\lambda, \gamma_x^\lambda\rangle + (L^\lambda)^2\\ &\ge (\tilde{L}^\lambda)^2 + (L^\lambda)^2 \ge c > 0\end{aligned} \tag{3.52}$$

for some constant $c > 0$ independent of λ. Moreover, by comparison (see also (3.22)):

$$\lambda D(\gamma_n^\lambda, \gamma_{n-1}^\lambda) \le E(\gamma_{n-1}^\lambda) - E(\gamma_n^\lambda)$$

for all $n \in \mathbb{N}_+$. Summing the above over $n \in \mathbb{N}_+$ and using $E \geq m$, where m is the infimum in (3.23), we derive that:

$$\begin{aligned} \lambda \sum_{n=1}^{\infty} D(\gamma_n^\lambda, \gamma_{n-1}^\lambda) &\leq \sum_{n=1}^{\infty} (E(\gamma_{n-1}^\lambda) - E(\gamma_n^\lambda)) \\ &\leq E(\gamma_0) - \liminf_{n\to\infty} E(\gamma_n^\lambda) \leq E(\gamma_0) - m. \end{aligned} \tag{3.53}$$

The sum on the left-hand side of (3.53) can be written as:

$$\begin{aligned} &\lambda \sum_{n=1}^{\infty} D(\gamma_n^\lambda, \gamma_{n-1}^\lambda) \\ &= \sum_{n=1}^{\infty} \frac{1}{4\lambda} \int_0^1 \langle \lambda(\gamma_n^\lambda - \gamma_{n-1}^\lambda), \nu_n^\lambda \rangle^2 \,\mathrm{d}x + \sum_{n=1}^{\infty} \frac{1}{4\lambda} \int_0^1 \langle \lambda(\gamma_n^\lambda - \gamma_{n-1}^\lambda), \tilde{\nu}_n^\lambda \rangle^2 \,\mathrm{d}x \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{2\lambda} |\lambda(\gamma_n^\lambda(0) - \gamma_{n-1}^\lambda(0))|^2 + \sum_{n=1}^{\infty} \frac{1}{2\lambda} |\lambda(\gamma_n^\lambda(1) - \gamma_{n-1}^\lambda(1))|^2 \\ &= \frac{1}{4} \int_0^\infty \int_0^1 \langle V^\lambda, \tilde{\nu}^\lambda \rangle^2 + \langle V^\lambda, \nu^\lambda \rangle^2 \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{2} \int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t. \end{aligned} \tag{3.54}$$

Combining (3.53) and (3.54), then leads to:

$$\int_0^\infty \int_0^1 \langle V^\lambda, \tilde{\nu}^\lambda \rangle^2 + \langle V^\lambda, \nu^\lambda \rangle^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \leq C \tag{3.55}$$

for some constant C independent of λ. With (3.52), this shows:

$$\int_0^\infty \int_0^1 \langle V^\lambda, \frac{(\mu^\lambda)^\perp}{|\mu^\lambda|} \rangle^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \leq C,$$

again, for a constant C independent of λ.

1. Step: In order to obtain (3.51), we are left to control the following quantity from above:

$$\int_0^\infty \int_0^1 \langle V^\lambda, \frac{\mu^\lambda}{|\mu^\lambda|} \rangle^2 \,\mathrm{d}x \,\mathrm{d}t.$$

To this end, we integrate (3.50) in the curve parameter over I and solve for $\hat{L}_t^\lambda$:

$$\hat{L}_t^\lambda = \frac{1}{\tilde{L}^\lambda + L^\lambda} \left(\langle V^\lambda, \mu^\lambda \rangle \big|_{t=0}^1 - \int_0^1 \langle V^\lambda, (\tilde{L}^\lambda)^2 \tilde{\kappa}^\lambda \tilde{\nu}^\lambda + (L^\lambda)^2 \kappa^\lambda \nu^\lambda \rangle \,\mathrm{d}x \right)$$

Squaring both sides of the above equality, integrating them over $t \in [0, \infty)$ as well as using (3.46), (3.47), (3.55), and Hölder's inequality, we get:

$$\begin{aligned}\int_0^\infty (\hat{L}_t^\lambda)^2 \,\mathrm{d}t &\le C\int_0^\infty |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2 \,\mathrm{d}t \\ &\quad + C\int_0^\infty \left(\int_0^1 \langle V^\lambda, (\tilde{L}^\lambda)^2\tilde{\kappa}^\lambda\tilde{\nu}^\lambda + (L^\lambda\kappa^\lambda)^2\nu^\lambda\rangle \,\mathrm{d}x\right)^2 \mathrm{d}t \\ &\le C + C\int_0^\infty \left(\int_0^1 (\tilde{\kappa}^\lambda)^2 \,\mathrm{d}x\right)\left(\int_0^1 \langle V^\lambda, \tilde{\nu}^\lambda\rangle^2 \,\mathrm{d}x\right)\mathrm{d}t \\ &\quad + C\int_0^\infty \left(\int_0^1 (\kappa^\lambda)^2 \,\mathrm{d}x\right)\left(\int_0^1 \langle V^\lambda, \nu^\lambda\rangle^2 \,\mathrm{d}x\right)\mathrm{d}t \\ &\le C(\varepsilon)\left(1 + \int_0^\infty\int_0^1 \langle V^\lambda, \tilde{\nu}^\lambda\rangle^2 + \langle V^\lambda, \nu^\lambda\rangle^2 \,\mathrm{d}x\,\mathrm{d}t\right) \le C(\varepsilon),\end{aligned} \tag{3.56}$$

where C and $C(\varepsilon)$ are constants independent of λ. (Note that $\lim_{\varepsilon\to 0} C(\varepsilon) = \infty$.) Next, we fix $t \in [0,\infty)$ and again integrate (3.50) in the curve parameter, but this time over $[0,x]$ for some fixed $x \in I$:

$$\begin{aligned}\langle V^\lambda, \mu^\lambda\rangle(x,t) &= \langle V^\lambda, \mu^\lambda\rangle(t,0) + (\tilde{L}^\lambda(t) + L^\lambda(t))\hat{L}_t^\lambda(t)x \\ &\quad + \int_0^x \langle V^\lambda(\tilde{x},t), (\tilde{L}^\lambda(t))^2\tilde{\kappa}^\lambda(\tilde{x},t)\tilde{\nu}^\lambda(\tilde{x},t)\rangle \, d\tilde{x} \\ &\quad + \int_0^x \langle V^\lambda(\tilde{x},t), (L^\lambda(t))^2\kappa^\lambda(\tilde{x},t)\nu^\lambda(\tilde{x},t)\rangle \, d\tilde{x}.\end{aligned}$$

Taking the square of both sides of the equality above, integrating over (t,x) in $I \times [0,\infty)$ as well as employing (3.46), (3.47), (3.55), (3.56), and Hölder's Inequality, then results in:

$$\begin{aligned}&\int_0^\infty\int_0^1 \langle V^\lambda, \mu^\lambda\rangle^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le C\int_0^\infty \left(|V^\lambda(t,0)|^2 + \hat{(}L_t^\lambda)^2(t)\right)\mathrm{d}t \\ &\qquad + C\int_0^\infty\int_0^1 \left(\int_0^x (\tilde{\kappa}^\lambda)^2(\tilde{x},t)\, d\tilde{x}\right)\left(\int_0^x \langle V^\lambda, \tilde{\nu}^\lambda\rangle^2(\tilde{x},t)\, d\tilde{x}\right)\mathrm{d}x\,\mathrm{d}t \\ &\qquad + C\int_0^\infty\int_0^1 \left(\int_0^x (\kappa^\lambda)^2(\tilde{x},t)\, d\tilde{x}\right)\left(\int_0^x \langle V^\lambda, \nu^\lambda\rangle^2(\tilde{x},t)\, d\tilde{x}\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad\le C(\varepsilon)\left(1 + \int_0^\infty\int_0^1 \langle V^\lambda, \tilde{\nu}^\lambda\rangle^2 + \langle V^\lambda, \nu^\lambda\rangle^2 \,\mathrm{d}x\,\mathrm{d}t\right) \le C(\varepsilon)\end{aligned}$$

for some constant $C(\varepsilon) < \infty$ independent of λ. By (3.52), we conclude:

$$\int_0^\infty\int_0^1 \langle V^\lambda, \frac{\mu^\lambda}{|\mu^\lambda|}\rangle^2 \,\mathrm{d}x\,\mathrm{d}t \le C\int_0^\infty\int_0^1 \langle V^\lambda, \mu^\lambda\rangle^2 \,\mathrm{d}x\,\mathrm{d}t \le C(\varepsilon) \tag{3.57}$$

for constant $C(\varepsilon)$ independent of λ. □

With the bounds on the velocity, we can derive the following uniform Hölder estimates:

Lemma 3.4 (Uniform Hölder continuity in time)
There exist constants $C(\varepsilon)$, C independent of λ such that for all $t_1, t_2 \in [0, \infty)$ with $0 \le t_1 < t_2 < \infty$, it holds that:

$$\|\hat{\gamma}^\lambda(\cdot, t_2) - \hat{\gamma}^\lambda(\cdot, t_1)\|_{L^2} \le C(\varepsilon)(t_2 - t_1)^{\frac{1}{2}}, \tag{3.58}$$

and:

$$\begin{aligned} |\hat{\gamma}^\lambda(0, t_2) - \hat{\gamma}^\lambda(0, t_1)| &\le C(t_2 - t_1)^{\frac{1}{2}}, \\ |\hat{\gamma}^\lambda(1, t_2) - \hat{\gamma}^\lambda(1, t_1)| &\le C(t_2 - t_1)^{\frac{1}{2}}. \end{aligned} \tag{3.59}$$

Furthermore, for any $T > 0$:

$$\|\gamma^\lambda\|_{L^\infty_T H^2} \le C(\varepsilon, T), \tag{3.60}$$

where $C(\varepsilon, T)$ is a constant independent of λ.

Proof. By the absolute continuity of $\hat{\gamma}^\lambda(x, \cdot)$ for every $x \in I$, (3.51), Hölder's inequality, and Fubini's theorem, we derive:

$$\begin{aligned} \|\hat{\gamma}^\lambda(\cdot, t_2) - \hat{\gamma}^\lambda(\cdot, t_1)\|^2_{L^2} &= \int_0^1 \left| \int_{t_1}^{t_2} V^\lambda \, \mathrm{d}t \right|^2 \mathrm{d}x \\ &\le \int_0^1 (t_2 - t_1) \left(\int_{t_1}^{t_2} |V^\lambda|^2 \, \mathrm{d}t \right) \mathrm{d}x \\ &\le (t_2 - t_1) \int_0^\infty \|V^\lambda\|^2_{L^2} \, \mathrm{d}t \le C(\varepsilon)(t_2 - t_1) \end{aligned}$$

for some constant $C(\varepsilon)$ independent of λ, and (3.58) follows. The proof of (3.59) is similar.

Let us now fix $T > 0$, then with the definition of $\hat{\gamma}^\lambda$ and (3.58), we can derive for any $t \in [0, T]$:

$$\|\hat{\gamma}^\lambda(\cdot, t)\|_{L^2} \le \|\hat{\gamma}^\lambda(\cdot, t) - \hat{\gamma}^\lambda(\cdot, 0)\|_{L^2} + \|\gamma_0\|_{L^2} \le C(\varepsilon) T^{\frac{1}{2}} + C \le C(\varepsilon, T) \tag{3.61}$$

for some constant $C(\varepsilon, T)$ independent of λ. Applying (3.61) for all $t \in [0, T]$ of the form $t = \lambda^{-1} n$ with $n \in \mathbb{N}$, and using the definition of $\hat{\gamma}^\lambda$, we see that:

$$\|\gamma^\lambda(\cdot, t)\|_{L^2_T L^2} \le C(\varepsilon, T)$$

for the same constant $C(\varepsilon, T)$ as in (3.61). Furthermore, by (3.46) and (3.47), we have:

$$\|\gamma^\lambda_x\|_{L^\infty H^1} \le C$$

for some constant C independent of λ. Using the last two estimates, we conclude that (3.60) is satisfied. □

Throughout the chapter, we will be employing the interpolation inequality in Sobolev spaces. (For a proof, we refer to Theorem 6.4 in [36].) Its precise formulation is as follows:

Theorem 3.2 (Interpolation in Sobolev spaces)
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, and let i, j, and m be integers such that $0 \le i \le j \le m$. Let $p, q \in [1, \infty]$ satisfying $1 \le p \le q < \infty$ if

$(m-j)p \geq n$, or satisfying $1 \leq p \leq q \leq \infty$ if $(m-j)p > n$. Then, there exists $C = C(\Omega) > 0$ such that for all $u \in W^{m,p}(\Omega)$, it holds:

$$\|D^j u\|_{L^q(\Omega)} \leq C \left(\|D^m u\|^{\alpha}_{L^p(\Omega)} \|D^i u\|^{1-\alpha}_{L^p(\Omega)} + \|D^i u\|_{L^p(\Omega)} \right), \tag{3.62}$$

where:

$$\alpha := \frac{1}{m-i} \left(\frac{n}{p} - \frac{n}{q} + j - i \right).$$

We will employ the interpolation inequality (3.62) for u being a curve parameterization $\gamma \colon I \to \mathbb{R}^2$ ($\Omega = I$). For the sake of clarity, in the next table we list all possible combinations of i, j, m, p, q that we are going to encounter along the way.

	i	j	m	p	q	α
a)	0	1	2	2	∞	$\frac{3}{4}$
b)	2	3	4	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{11}{18}$
c)	2	2	4	$\frac{3}{2}$	$\frac{9}{2}$	$\frac{2}{9}$
d)	2	2	4	$\frac{3}{2}$	6	$\frac{1}{4}$
e)	2	3	4	$\frac{3}{2}$	$\frac{39}{23}$	$\frac{7}{13}$
f)	2	3	4	2	3	$\frac{7}{12}$
g)	2	2	4	2	6	$\frac{1}{6}$

Table 3.1: Admissible choices for the parameters of the interpolation inequality.

Thanks to the uniform L^2 bound on the curvature in (3.47), we will improve the Hölder continuity results from the previous lemma by interpolation.

Lemma 3.5
For any, $\alpha \in (0, \frac{1}{2})$ and $T > 0$, there exists a constant $C(\varepsilon, T)$ independent of λ such that for all $t_1, t_2 \in [0, T)$ with $t_1 < t_2$, it holds that:

$$\|\hat{\gamma}^{\lambda}(\cdot, t_2) - \hat{\gamma}^{\lambda}(\cdot, t_1)\|_{C^{1,\alpha}} \leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1-2\alpha}{8}}. \tag{3.63}$$

Remark 3.1. Take any $\alpha \in (0, \frac{1}{2})$, and $T > 0$. Using (3.46) and (3.63), we derive:

$$\begin{aligned} |\langle \gamma^{\lambda}_x, \tilde{\gamma}^{\lambda}_x \rangle - L^{\lambda} \tilde{L}^{\lambda}| &= |\langle \gamma^{\lambda}_x - \tilde{\gamma}^{\lambda}_x, \tilde{\gamma}^{\lambda}_x \rangle + (\tilde{L}^{\lambda})^2 - L^{\lambda} \tilde{L}^{\lambda}| \\ &\leq \tilde{L}^{\lambda} \left(\|\gamma^{\lambda}_x - \tilde{\gamma}^{\lambda}_x\|_{L^\infty} + \int_0^1 |\gamma^{\lambda}_x - \tilde{\gamma}^{\lambda}_x| \,\mathrm{d}x \right) \\ &\leq C \|\gamma^{\lambda}_x - \tilde{\gamma}^{\lambda}_x\|_{L^\infty} \leq C(\varepsilon, T) \lambda^{\frac{2\alpha-1}{8}} \to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

Hence, by (3.46), there exists $\lambda_0 := \lambda_0(\varepsilon, T) > 0$ independent of ε such that for all $\lambda > \lambda_0$, we have:

$$\langle \gamma^{\lambda}_x, \tilde{\gamma}^{\lambda}_x \rangle > 0 \text{ in I.}$$

In particular, we derive the following crucial result: For $\lambda > \lambda_0$ and $n \in \{1, \dots, \lfloor T/\lambda \rfloor\}$, the step-by-step minimizer γ^{λ}_n satisfies:

$$\gamma^{\lambda}_n \in \operatorname{argmin}\big\{E(\gamma) + \lambda D(\gamma, \gamma^{\lambda}_{n-1}) \colon \gamma \in \mathcal{AC}\big\}. \tag{3.64}$$

The difference is that γ_n^λ is only restrained to lie in $\mathcal{AC}$ instead of $\mathcal{AC}(\gamma_{n-1}^\lambda)$. This will become relevant once we compute the Euler-Lagrange equation corresponding to the step-by-step minimization in (3.22), as (3.64) tells us that the additional angle constraint coming from the definition of $\mathcal{AC}(\gamma_{n-1}^\tau)$ is not influencing the minimization, at least not for $\lambda > \lambda_0$ and $n \leq \lfloor T/\lambda \rfloor$.

Proof. In order to shorten notation, we define:

$$\Delta\hat{\gamma}^\lambda := \hat{\gamma}^\lambda(\cdot, t_2) - \hat{\gamma}^\lambda(\cdot, t_1).$$

Using the interpolation inequality for the curve $\Delta\hat{\gamma}^\lambda$ with parameters listed in row a) of Table 3.1, it follows:

$$\|(\Delta\hat{\gamma}^\lambda)_x\|_{L^\infty} \leq C\left(\|(\Delta\hat{\gamma}^\lambda)_{xx}\|_{L^2}^{\frac{3}{4}}\|\Delta\hat{\gamma}^\lambda\|_{L^2}^{\frac{1}{4}} + \|\Delta\hat{\gamma}^\lambda\|_{L^2}\right).$$

By the very definition of $\Delta\hat{\gamma}^\lambda$, (3.58), and (3.60), we can control the right-hand side of the equation above as follows:

$$\begin{aligned}\|(\Delta\hat{\gamma}^\lambda)_x\|_{L^\infty} &\leq C(\varepsilon, T)\left((t_2 - t_1)^{\frac{1}{8}} + (t_2 - t_1)^{\frac{1}{2}}\right)\\ &= C(\varepsilon, T)\left(1 + (t_2 - t_1)^{\frac{1}{2}-\frac{1}{8}}\right)(t_2 - t_1)^{\frac{1}{8}}\\ &\leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1}{8}} \end{aligned} \tag{3.65}$$

for a constant $C(\varepsilon, T)$ independent of λ. Note that in the last inequality we have used the fact that t_1, t_2 are in the bounded interval $[0, T]$. By the fundamental theorem of calculus, (3.59), and (3.65), we also derive:

$$\begin{aligned}\|\Delta\hat{\gamma}^\lambda\|_{L^\infty} &\leq |\Delta\hat{\gamma}^\lambda(0)| + \int_0^1 |\Delta\hat{\gamma}_x^\lambda| \, \mathrm{d}x\\ &\leq C(t_2 - t_1)^{\frac{1}{2}} + \|\Delta\hat{\gamma}_x^\lambda\|_{L^\infty}\\ &\leq C(t_2 - t_1)^{\frac{1}{2}} + C(\varepsilon, T)(t_2 - t_1)^{\frac{1}{8}}\\ &\leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1}{8}}. \end{aligned} \tag{3.66}$$

Thus, in order to conclude on these statements, it remains to control the Hölder semi-norm $|\Delta\hat{\gamma}_x^\lambda|_\alpha$. By Morrey's inequality, (3.60), and (3.65), we have for any $x_1, x_2 \in I$:

$$\begin{aligned}&\frac{|\Delta\hat{\gamma}_x^\lambda(x_2) - \Delta\hat{\gamma}_x^\lambda(x_1)|}{|x_2 - x_1|^\alpha}\\ &= \left(\frac{|\Delta\hat{\gamma}_x^\lambda(x_2) - \Delta\hat{\gamma}_x^\lambda(x_1)|}{|x_2 - x_1|^{\frac{1}{2}}}\right)^{2\alpha} |\Delta\hat{\gamma}_x^\lambda(x_2) - \Delta\hat{\gamma}_x^\lambda(x_1)|^{1-2\alpha}\\ &\leq C|\Delta\hat{\gamma}_x^\lambda|_{\frac{1}{2}}^{2\alpha}\|\Delta\hat{\gamma}_x^\lambda\|_{L^\infty}^{1-2\alpha}\\ &\leq C\|\Delta\hat{\gamma}^\lambda\|_{H^2}^{2\alpha}\|\Delta\hat{\gamma}_x^\lambda\|_{L^\infty}^{1-2\alpha} \leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1-2\alpha}{8}}\end{aligned}$$

for a constant $C(\varepsilon, T)$ independent of x_1, x_2, and λ. Taking the supremum over all $x_1, x_2 \in I$ then leads to:

$$|\Delta\hat{\gamma}_x^\lambda|_\alpha \leq C(\varepsilon, T)(t_2 - t_1)^{\frac{1-2\alpha}{8}} \tag{3.67}$$

for the same $C(\varepsilon, T)$ as before. Combining (3.66), (3.65), and (3.67), results in (3.63). □

Theorem 3.3 (Initial compactness)
There exists $\gamma\colon I \times [0,\infty) \to \mathbb{R}^2$ *such that for any* $\alpha \in (0, \frac{1}{2})$ *and* $\beta \in (0, \frac{1-2\alpha}{8})$*, up to some subsequences, it holds:*

$$\hat{\gamma}^\lambda \to \gamma \text{ in } C^{0,\beta}_{\mathrm{loc}} C^{1,\alpha}, \tag{3.68}$$

$$\hat{\gamma}^\lambda \rightharpoonup \gamma \text{ weakly in } H^1_{\mathrm{loc}} L^2, \tag{3.69}$$

$$\gamma^\lambda \to \gamma \text{ in } L^\infty_{\mathrm{loc}} C^{1,\alpha}, \tag{3.70}$$

as well as

$$\begin{aligned} \hat{\gamma}^\lambda(0,\cdot) \rightharpoonup \gamma(0,\cdot) \text{ weakly in } H^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^2). \\ \hat{\gamma}^\lambda(1,\cdot) \rightharpoonup \gamma(1,\cdot) \text{ weakly in } H^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^2). \end{aligned} \tag{3.71}$$

Proof. The proof of (3.68) follows by a standard diagonal sequence argument. For this, let $(T_k) \subset [0,\infty)$ be an auxiliary sequence with $T_k \nearrow \infty$ as $k \to \infty$. By (3.63) and the Arzelá-Ascoli theorem, there exists $(\lambda_i^{(0)})$ converging to 0 and $\gamma^{(0)}\colon I \times [0,T_0] \to \mathbb{R}^2$ such that for any $\alpha \in (0, \frac{1}{2})$, $\beta \in (0, \frac{1-2\alpha}{8})$:

$$\hat{\gamma}^{\lambda_i^{(0)}} \to \gamma^{(0)} \text{ in } C^{0,\beta}_{T_0} C^{1,\alpha} \text{ as } i \to \infty.$$

We continue by applying for every $k \in \mathbb{N}_+$ the Arzelá-Ascoli theorem to the sequence $(\hat{\gamma}^{\lambda_i^{(k-1)}})_i$ in order to construct $(\lambda_i^{(k)})_i$ as a subsequence of $(\lambda_i^{(k-1)})_i$ and $\gamma^{(k)}\colon I \times [0,T_{k+1}] \to \mathbb{R}^2$ such that for any $\alpha \in (0, \frac{1}{2})$, $\beta \in (0, \frac{1-2\alpha}{8})$:

$$\hat{\gamma}^{\lambda_i^{(k)}} \to \gamma^{(k)} \text{ in } C^{0,\beta}_{T_k} C^{1,\alpha} \text{ as } i \to \infty.$$

Note that as $C^{0,\beta}_{T_k} C^{1,\alpha}$ convergence implies $C^{0,\beta}_{T_{k-1}} C^{1,\alpha}$ convergence for any $k \geq 1$, we see:

$$\gamma^{(k)}|_{[0,T_k]} = \gamma^{(k-1)} \text{ for all } k \geq 1.$$

Therefore, we can define $\gamma\colon I \times [0,\infty) \to \mathbb{R}^2$ by setting:

$$\gamma|_{[0,T_k]} \coloneqq \gamma^{(k)} \text{ for } k \in \mathbb{N}.$$

Consequently, along the diagonal sequence $\lambda_i \coloneqq \lambda_i^{(i)}$, it holds for any $\alpha \in (0, \frac{1}{2})$, $\beta \in (0, \frac{1-2\alpha}{8})$ that:

$$\hat{\gamma}^{\lambda_i} \to \gamma \text{ in } C^{0,\beta}_{\mathrm{loc}} C^{1,\alpha}.$$

From this point on, we assume that we have already extracted the subsequence $(\hat{\gamma}^{\lambda_i})$ and will be denoting it, for the sake of abbreviation, just by $(\hat{\gamma}^\lambda)$. By the definition of $\hat{\gamma}^\lambda$ and γ^λ, and thanks to (3.63), we see that for any $\alpha \in (0, \frac{1}{2})$, $T > 0$, and $0 \leq t \leq T$, it holds that:

$$\|\gamma^\lambda(\cdot,t) - \hat{\gamma}^\lambda(\cdot,t)\|_{C^{1,\alpha}} \leq C(\varepsilon,T)\lambda^{\frac{2\alpha-1}{8}} \overset{\lambda\to\infty}{\to} 0. \tag{3.72}$$

As a consequence of (3.68) and (3.72), we deduce (3.70). Finally, thanks to (3.51), and the already proven convergence in (3.68), we have – up to a further subsequence – that (3.69) and (3.71) hold true. □

Next, we wish to compute the first variation of the minimization problem:

$$\min\{E + \lambda D(\gamma,\tilde{\gamma})\colon \gamma \in \mathcal{AC}\}, \tag{3.73}$$

for some fixed $\tilde{\gamma} \in \mathcal{AC}$. Due to the nonlinearity of the velocity constraint of $\mathcal{AC}$, the additive variation $\gamma + \delta\eta$, with $\gamma \in \mathcal{AC}$, $\delta > 0$ a small scalar, and $\eta \in H^2$, is in general not admissible. Consequently, we need to reparametrize $\gamma + \delta\eta$ via a map $\Psi(\delta, \cdot)\colon I \to I$ (depending on δ) such that $(\gamma + \delta\eta) \circ \Psi(\delta, \cdot) \in \mathcal{AC}$. More precisely:

Lemma 3.6 (Admissible variations in $\mathcal{AC}$)
For any $\gamma \in \mathcal{AC}$, $\eta \in H^2(I; \mathbb{R}^2)$ and δ such that:

$$0 < \delta < \|\eta_x\|_{L^\infty}^{-1} \min\{L, \operatorname{dist}(\{\gamma(0), \gamma(1)\}, \partial\Omega)\}$$

(L length of γ) there exists a unique bijective map $\Psi(\delta, \cdot)\colon I \to I$ such that $\mu(\delta, \cdot)\colon I \to \mathbb{R}^2$, defined as

$$\mu(\delta, x) \coloneqq (\gamma + \delta\eta)(\Psi(\delta, x)), \tag{3.74}$$

satisfies:

$$\mu(\delta, \cdot) \in \mathcal{AC}, \quad \mu(\delta, 0) = \gamma(0) + \delta\eta(0). \tag{3.75}$$

Furthermore, we have for every $x \in I$:

$$\partial_\delta \Psi(0, x) = \frac{1}{L}\left(x \int_0^1 \langle \eta_x, \tau \rangle \, d\tilde{x} - \int_0^x \langle \eta_x, \tau \rangle \, d\tilde{x}\right), \tag{3.76}$$

$$\partial_x \Psi(0, x) = 1, \tag{3.77}$$

$$\partial_{x\delta} \Psi(0, x) = \frac{1}{L}\left(\int_0^1 \langle \eta_x, \tau \rangle \, d\tilde{x} - \langle \eta_x(x), \tau(x) \rangle\right), \tag{3.78}$$

where τ is the unit tangent vector field of γ.

Proof. Let $\delta > 0$ be as described in the statement, and $\mu(\delta, \cdot)$ as in (3.74), then with $\delta < \|\eta\|_\infty^{-1} \operatorname{dist}(\gamma(0), \partial\Omega)$, we have that:

$$\operatorname{dist}(\mu(\delta, 0), \partial\Omega) \geq \operatorname{dist}(\gamma(0), \partial\Omega) - \delta\|\eta\|_\infty > 0,$$

and hence $\mu(\delta, 0) \in \Omega$. The same holds true for the other endpoint. Let us consider the differentiable function $F(\delta, \cdot)\colon I \to \mathbb{R}$ given by:

$$F(\delta, y) \coloneqq \frac{1}{L_\delta} \int_0^y |\gamma_x + \delta\eta_x| \, d\tilde{x}, \tag{3.79}$$

where L_δ denotes the length of $\gamma + \delta\eta$. Then, as long $\delta < \|\eta_x\|_{L^\infty}^{-1} L$, we have:

$$|\gamma_x + \delta\eta_x| \geq |\gamma_x| - \delta\|\eta_x\|_{L^\infty} = L - \delta\|\eta_x\|_{L^\infty} > 0$$

and consequently $F_y(\delta, y) > 0$ for all $y \in I$. Together with $F(\delta, 0) = 0$ and $F(\delta, 1) = 1$, this implies that $F(\delta, \cdot)$ is a diffeomorphism from I to I. Given δ as described in the statement, let us consider $\Psi(\delta, \cdot)\colon I \to I$, defined as:

$$\Psi(\delta, x) \coloneqq F(\delta, \cdot)^{-1}(x).$$

We will show that Ψ is the desired diffeomorphism. As $F(\delta, 0) = 0$, we also have $\Psi(\delta, 0) = 0$, and therefore:

$$\mu(\delta, 0) = \gamma(0) + \delta\eta(0)$$

From $F(\delta, \Psi(\delta, x)) = x$, we see by the chain rule that:

$$\partial_y F(\delta, \Psi(\delta, x))\partial_x \Psi(\delta, x) = 1 \tag{3.80}$$

$$\partial_\delta F(\delta, \Psi(\delta, x)) + \partial_y F(\delta, \Psi(\delta, x))\partial_\delta \Psi(\delta, x) = 0. \tag{3.81}$$

Moreover, for any $y \in I$:

$$\partial_y F(\delta, y) = \frac{|\gamma_x(y) + \delta\eta_x(y)|}{L_\delta}, \tag{3.82}$$

$$\begin{aligned} \partial_\delta F(\delta, y) = {}& \frac{1}{L_\delta}\int_0^y \langle \frac{\gamma_x + \delta\eta_x}{|\gamma_x + \delta\eta_x|}, \eta_x\rangle \, d\tilde{x} \\ & - \frac{1}{L_\delta^2}\left(\int_0^y |\gamma_x + \delta\eta_x| \, d\tilde{x}\right)\left(\int_0^1 \langle \frac{\gamma_x + \delta\eta_x}{|\gamma_x + \delta\eta_x|}, \eta_x\rangle \, d\tilde{x}\right). \end{aligned} \tag{3.83}$$

With (3.80) and (3.82), we derive that:

$$\partial_x \Psi(\delta, x) = \frac{L_\delta}{|(\gamma_x + \delta\eta_x)(\Psi(\delta, x))|}, \tag{3.84}$$

by which (3.77) follows. Furthermore, from the same equation, we also conclude that:

$$|(\mu(\delta, \cdot))_x(x)| = |(\gamma + \delta\eta)(\Psi(\delta, x))||\partial_x \Psi(\delta, x)| = L_\delta$$

and therefore $\mu \in \mathcal{AC}$. It remains to check (3.76) and (3.78). We use (3.81), (3.82), (3.83), and $\Psi(0, x) = x$ in order to compute

$$\begin{aligned} \partial_\delta \Psi(0, x) = -\frac{\partial_\delta F(0, \Psi(0, x))}{\partial_y F(0, \Psi(0, x))} &= -\partial_\delta F(0, x) \\ &= \frac{1}{L^2}\left(x\int_0^1 \langle \eta_x, \gamma_x\rangle \, d\tilde{x} - \int_0^x \langle \eta_x, \gamma_x\rangle \, d\tilde{x}\right), \end{aligned}$$

which is (3.76). In order to finish the proof, we differentiate (3.84) with respect to δ:

$$\begin{aligned} \partial_{x\delta}\Psi(\delta, x) = {}& \frac{1}{|(\gamma_x + \delta\eta_x)(\Psi(\delta, x))|}\int_0^1 \left\langle \frac{\gamma_x + \delta\eta_x}{|\gamma_x + \delta\eta_x|}, \eta_x \right\rangle d\tilde{x} \\ & - \frac{L_\delta}{|(\gamma_x + \delta\eta_x)(\Psi(\delta, x))|^3}\big\langle (\gamma_x + \delta\eta_x)(\Psi(\delta, x)), \eta_x(\Psi(\delta, x)) \\ & + \gamma_{xx}(\Psi(\delta, x))\partial_\delta \Psi(\delta, x) + \delta\eta_{xx}(\Psi(\delta, x))\partial_\delta \Psi(\delta, x)\big\rangle. \end{aligned}$$

Plugging in $\delta = 0$ above and using $\langle \gamma_x, \gamma_{xx}\rangle = 0$, eventually leads to (3.78). ☐

Remark 3.2. We wish to provide intuition behind formula (3.79). Suppose that there exists a $\Psi(\delta, \cdot)\colon I \to I$ such that $\mu(\delta, \cdot)$, as defined in (3.74), satisfies (3.75). Hence, we can follow that:

$$\int_0^{\Psi(\delta, x)} |\gamma_x + \delta\eta_x| \, d\tilde{x} = \int_0^x |\mu_x(\delta, \tilde{x})| \, d\tilde{x} = xL_\delta$$

for all $x \in I$. After dividing by L_δ above, we see that $\Psi(\delta, \cdot)$ is the inverse of:

$$F(\delta, y) := \frac{1}{L_\delta}\int_0^y |\gamma_x + \delta\eta_x| \, d\tilde{x},$$

as long as one such inverse exists.

The above lemma motivates the following definition:

Definition 3.2
Given γ, $\eta \in H^1$, we define the functions $P_1(\eta,\gamma)\colon I \to \mathbb{R}$ and $P_2(\eta,\gamma)\colon I \to \mathbb{R}$ as:

$$P_1(\eta,\gamma)(x) \coloneqq x\int_0^1 \langle \eta_x, \tau\rangle \, d\tilde{x} - \int_0^x \langle \eta_x, \tau\rangle \, d\tilde{x}, \tag{3.85}$$

$$P_2(\eta,\gamma)(x) \coloneqq \int_0^1 \langle \eta_x, \tau\rangle \, d\tilde{x} - \langle \eta_x(x), \tau(x)\rangle. \tag{3.86}$$

We are finally ready to compute the first variation of the minimization problem (3.73), eventually leading to the weak formulation of the time-discrete evolution in Proposition 3.3.

Lemma 3.7 (First variation)
Fix $\tilde{\gamma} \in \mathcal{AC}$ and let $\gamma \in \mathcal{AC}$ be a minimizer of (3.73). Then for all $\eta \in C^\infty$, it holds that:

$$E(\eta,\gamma) + D(\eta,\gamma) + Err(\eta,\gamma) = 0, \tag{3.87}$$

where:

$$\begin{aligned}
E(\eta,\gamma) &\coloneqq \int_0^1 \left(\frac{\varepsilon}{L^2}\kappa\langle \eta_{xx}, \nu\rangle + \frac{1}{L}(1-\frac{3\varepsilon}{2}\kappa^2)\langle \eta_x, \tau\rangle\right) Ldx \\
&\quad + \langle \eta(0), \nabla_p W(\gamma(0),\gamma(1))\rangle + \langle \eta(1), \nabla_q W(\gamma(0),\gamma(1))\rangle, \\
D(\eta,\gamma) &\coloneqq \frac{1}{2}\int_0^1 \langle \lambda(\gamma-\tilde{\gamma}), \tilde{\nu}\rangle\langle \eta, \tilde{\nu}\rangle \, \tilde{L}dx + \frac{1}{2}\int_0^1 \langle \lambda(\gamma-\tilde{\gamma}), \nu\rangle\langle \eta, \nu\rangle \, Ldx \\
&\quad + \langle \lambda(\gamma(0)-\tilde{\gamma}(0)), \eta(0)\rangle + \langle \lambda(\gamma(1)-\tilde{\gamma}(1)), \eta(1)\rangle, \\
Err(\eta,\gamma) &\coloneqq \frac{1}{2}\int_0^1 \tilde{L}\langle \lambda(\gamma-\tilde{\gamma}), \tilde{\nu}\rangle\langle \tilde{\nu}, \tau\rangle P_1(\eta,\gamma)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_0^1 \langle \lambda(\gamma-\tilde{\gamma}), \tilde{\nu}\rangle\langle \gamma-\tilde{\gamma}, \tau\rangle (P_2(\eta,\gamma) - L\kappa P_1(\eta,\gamma))\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_0^1 \langle \lambda(\gamma-\tilde{\gamma}), \tilde{\nu}\rangle\langle (\gamma-\tilde{\gamma})^\perp, \eta_x\rangle \,\mathrm{d}x \\
&\quad - \frac{1}{4}\left(\int_0^1 \langle \eta_x, \tau\rangle \,\mathrm{d}x\right)\left(\int_0^1 \langle \lambda(\gamma-\tilde{\gamma}), \nu\rangle\langle \gamma-\tilde{\gamma}, \nu\rangle \,\mathrm{d}x\right),
\end{aligned}$$

where $P_1(\eta,\gamma)$ and $P_2(\eta,\gamma)$ are as in (3.85) and (3.86), respectively.

Remark 3.3. It will prove useful to equivalently write $E(\eta,\gamma)$ from the statement of Lemma 3.7 as:

$$\begin{aligned}
E(\eta,\gamma) &\coloneqq \int_0^1 \left(\frac{\varepsilon}{L^4}\langle \eta_{xx}, \gamma_{xx}\rangle + \frac{1}{L^2}(1-\frac{3\varepsilon}{2}\kappa^2)\langle \eta_x, \gamma_x\rangle\right) Ldx \\
&\quad + \langle \nabla_p W(\gamma(0),\gamma(1)), \eta(0)\rangle + \langle \nabla_q W(\gamma(0),\gamma(1)), \eta(1)\rangle.
\end{aligned}$$

Furthermore, one can think of $Err(\eta,\gamma)$ as an error term, which should vanish in the limit $\lambda \to \infty$, as long as we can derive "good enough" compactness for the sequence $(\hat{\gamma}_\lambda)$. (The red terms in the definition of $Err(\eta,\gamma)$ should be small for $\lambda >> 1$.)

Proof. For $\gamma, \tilde{\gamma} \in \mathcal{AC}$, let us shortly write $F(\gamma, \tilde{\gamma}) \coloneqq E(\gamma) + \lambda D(\gamma, \tilde{\gamma})$ with E and D as in (3.73). From this point on, γ will denote a minimizer of (3.73). By the minimality of γ, it follows that $\frac{\partial}{\partial \delta}|_{\delta=0} F(\mu(\delta,\cdot)) = 0$ with $\mu(\delta,\cdot)$ as defined in (3.74). It remains to show that:

$$\frac{\partial}{\partial \delta}|_{\delta=0} F(\mu(\delta,\cdot)) = E(\eta,\gamma) + D(\eta,\gamma) + Err(\eta,\gamma).$$

Given $\mu \in \mathcal{AC}$, for the reader's convenience, we split the dissipation functional D into the following three terms:

$$\begin{aligned}
D_1(\mu,\tilde{\gamma}) &\coloneqq \frac{\lambda}{2}|\mu(0) - \tilde{\gamma}(0)|^2 + \frac{\lambda}{2}|\mu(1) - \tilde{\gamma}(1)|^2, \\
D_2(\mu,\tilde{\gamma}) &\coloneqq \frac{\lambda}{4\tilde{L}} \int_0^1 \langle \mu - \tilde{\gamma}, \gamma_x^\perp \rangle^2 \,\mathrm{d}x, \\
D_3(\mu,\tilde{\gamma}) &\coloneqq \frac{\lambda}{4L(\mu)} \int_0^1 \langle \mu - \tilde{\gamma}, \mu_x^\perp \rangle^2 \,\mathrm{d}x.
\end{aligned}$$

Hence, we can split F as follows:

$$F(\mu,\tilde{\gamma}) = E(\mu) + D_1(\mu,\tilde{\gamma}) + D_2(\mu,\tilde{\gamma}) + D_3(\mu,\tilde{\gamma}).$$

In the following, we will separately compute the first variation of E, D_1, D_2, and D_3. We will repeatedly make use of the fact that all intrinsic quantities, as well as all quantities depending only on the endpoints, remain unchanged under the reparametrization of $\gamma + \delta\eta$ into $\mu(\delta,\cdot)$.

First variation of E: As:

$$\begin{aligned}
E(\mu(\delta,\cdot)) = {} & W(\gamma(0) + \delta\eta(0), \gamma(1) + \delta\eta(1)) \\
& + \int_0^1 \frac{\varepsilon}{2} \frac{\langle \gamma_{xx} + \delta\eta_{xx}, \gamma_x^\perp + \delta\eta_x^\perp \rangle^2}{|\gamma_x + \delta\eta_x|^5} + |\gamma_x + \delta\eta_x| \,\mathrm{d}x
\end{aligned}$$

we derive using the dominated convergence theorem, and $\gamma_{xx} = L\kappa\gamma_x^\perp$

$$\begin{aligned}
\frac{\partial}{\partial \delta}|_{\delta=0} E(\mu(\delta,\cdot)) = {} & \int_0^1 \varepsilon \frac{\langle \gamma_{xx}, \gamma_x^\perp \rangle}{|\gamma_x|^5} (\langle \gamma_x^\perp, \eta_{xx} \rangle + \langle \gamma_{xx}, \eta_x^\perp \rangle) \,\mathrm{d}x \\
& - \int_0^1 \frac{5\varepsilon}{2} \frac{\langle \gamma_{xx}, \gamma_x^\perp \rangle^2}{|\gamma_x|^7} \langle \gamma_x, \eta_x \rangle + \frac{\langle \gamma_x, \eta_x \rangle}{|\gamma_x|} \,\mathrm{d}x \\
& + \langle \nabla_p W(\gamma(0),\gamma(1)), \eta(0) \rangle + \langle \nabla_q W(\gamma(0),\gamma(1)), \eta(1) \rangle \\
= {} & \int_0^1 \left(\frac{\varepsilon}{L^2} \kappa \langle \eta_{xx}, \nu \rangle + \frac{1}{L} \left(1 - \frac{3\varepsilon}{2} \kappa^2 \right) \langle \eta_x, \tau \rangle \right) L dx. \\
& + \langle \nabla_p W(\gamma(0),\gamma(1)), \eta(0) \rangle + \langle \nabla_q W(\gamma(0),\gamma(1)), \eta(1) \rangle.
\end{aligned}$$

First variation of D_1:

$$\begin{aligned}
\frac{\partial}{\partial \delta}|_{\delta=0} D_1(\mu(\delta,\cdot),\tilde{\gamma}) &= \frac{\partial}{\partial \delta}|_{\delta=0} D_1(\gamma + \delta\eta, \tilde{\gamma}) \\
&= \langle \lambda(\gamma(0) - \tilde{\gamma}(0)), \eta(0) \rangle + \langle \lambda(\gamma(1) - \tilde{\gamma}(1)), \eta(1) \rangle.
\end{aligned}$$

First variation of D_2: Let $\Psi(\delta,\cdot)$ (for $\delta>0$ small enough) be the diffeomorphism from Lemma 3.6. Comparing (3.76) and (3.78) with (3.85) and (3.86), respectively, we see that:

$$\Psi_\delta(0,\cdot)=\frac{1}{L}P_1(\eta,\gamma),\quad \Psi_{x\delta}(0,\cdot)=\frac{1}{L}P_2(\eta,\gamma). \tag{3.88}$$

Furthermore, we preliminary compute:

$$\mu_\delta(\delta,x)=\Psi_\delta(\delta,x)\gamma_x(\Psi(\delta,x))+\eta(\Psi(\delta,x))+\delta\Psi_\delta(\delta,x)\eta_x(\Psi(\delta,x)).$$

Hence, by the dominated convergence theorem:

$$\begin{aligned}\frac{\partial}{\partial\delta}|_{\delta=0}D_2(\mu(\delta,\cdot))&=\frac{1}{2}\int_0^1\langle\lambda(\gamma-\tilde\gamma),\tilde\nu\rangle\langle\eta,\tilde\nu\rangle\,\tilde L dx\\&+\frac{1}{2}\int_0^1\tilde L\langle\lambda(\gamma-\tilde\gamma),\tilde\nu\rangle\langle\tilde\nu,\tau\rangle P_1(\eta,\gamma)\,\mathrm{d}x\end{aligned}$$

First variation of D_3: We preliminary compute:

$$\begin{aligned}\mu_x(\delta,x)&=\Psi_x(\delta,x)\gamma_x(\Psi(\delta,x))+\delta\Psi_x(\delta,x)\eta_x(\Psi(\delta,x)),\\\mu_{x\delta}(\delta,x)&=\Psi_x(\delta,x)\Psi_\delta(\delta,x)\gamma_{xx}(\Psi(\delta,x))+\Psi_{x\delta}(\delta,x)\gamma_x(\Psi(\delta,x))\\&\quad+\Psi_x(\delta,x)\eta_x(\Psi(\delta,x))+\delta\Psi_x(\delta,x)\Psi_\delta(\delta,x)\eta_{xx}(\Psi(\delta,x))\\&\quad+\delta\Psi_{x\delta}(\delta,x)\eta_x(\Psi(\delta,x)),\end{aligned}$$

and:

$$\frac{\partial}{\partial\delta}|_{\delta=0}\frac{1}{L(\mu(\delta,\cdot))}=-\frac{1}{L^2}\int_0^1\langle\eta_x,\tau\rangle\,\mathrm{d}x.$$

Therefore, by the dominated convergence theorem and (3.88), we see that:

$$\begin{aligned}&\frac{\partial}{\partial\delta}|_{\delta=0}D_3(\mu(\delta,\cdot))\\&=\frac{\lambda}{4L}\int_0^1 2\langle\gamma-\tilde\gamma,\gamma_x^\perp\rangle\langle\frac{1}{L}P_1(\eta,\gamma)\gamma_x+\eta,\gamma_x^\perp\rangle\,\mathrm{d}x\\&+\frac{\lambda}{4L}\int_0^1 2\langle\gamma-\tilde\gamma,\gamma_x^\perp\rangle\langle\gamma-\tilde\gamma,\frac{1}{L}P_1(\eta,\gamma)\gamma_{xx}^\perp+\frac{1}{L}P_2(\eta,\gamma)\gamma_x^\perp+\eta_x^\perp\rangle\,\mathrm{d}x\\&-\frac{1}{L^2}\int_0^1\langle\frac{\gamma_x}{L},\eta_x\rangle\,\mathrm{d}x\int_0^1\frac{1}{4\lambda}\langle\gamma-\tilde\gamma,\gamma_x^\perp\rangle^2\,\mathrm{d}x.\end{aligned}$$

The above equality can be further simplified to:

$$\begin{aligned}\frac{\partial}{\partial\delta}|_{\delta=0}D_3(\mu(\delta,\cdot))&=\frac{1}{2}\int_0^1\langle\lambda(\gamma-\tilde\gamma),\tau\rangle\langle\eta,\nu\rangle\,Ldx\\&+\frac{1}{2}\int_0^1\langle\lambda(\gamma-\tilde\gamma),\tilde\nu\rangle\langle\gamma-\tilde\gamma,\tau\rangle(P_2(\eta,\gamma)-L\kappa P_1(\eta,\gamma))\,\mathrm{d}x\\&+\frac{1}{2}\int_0^1\langle\lambda(\gamma-\tilde\gamma),\tilde\nu\rangle\langle(\gamma-\tilde\gamma)^\perp,\eta_x\rangle\,\mathrm{d}x\\&-\frac{1}{4}\left(\int_0^1\langle\eta_x,\tau\rangle\,\mathrm{d}x\right)\left(\int_0^1\langle\lambda(\gamma-\tilde\gamma),\nu\rangle\langle\gamma-\tilde\gamma,\nu\rangle\,\mathrm{d}x\right)\end{aligned}$$

Combining all the aforementioned computations, we conclude the proof. □

Proposition 3.3 (Time-discrete evolution)
For any $T > 0$, there exists a $\lambda_0 = \lambda_0(\varepsilon, T) > 0$ such that for every $\eta \in C^\infty_T C^\infty$ and every $\lambda > \lambda_0$, it holds that:

$$\int_0^T E(\gamma^\lambda(t,\cdot), \eta(t,\cdot)) + D(\gamma^\lambda(t,\cdot), \eta(t,\cdot)) + Err(\gamma^\lambda(t,\cdot), \eta(t,\cdot))\, \mathrm{d}t = 0, \tag{3.89}$$

where E, D, and Err are as in the statement of Lemma 3.7.

Proof. The proof follows by using Remark 3.1, (3.87), and a simple induction argument. □

The weak formulation (3.89) of the time-discrete evolution will now be used to derive further compactness results. We start with

Proposition 3.4
Up to a subsequence, it holds that:

$$\gamma^\lambda \to \gamma \text{ strongly in } L^2_{\mathrm{loc}} H^2. \tag{3.90}$$

Proof. Fix $T > 0$ and let λ_0 be as in Remark 3.1. We wish to show that (γ^λ) is a Cauchy sequence in $L^2_T H^2$. A standard diagonal sequence argument would then conclude the proof. Let us fix $\delta > 0$. Due to (3.69), there exists $\lambda_1 = \lambda_1(\delta) > 0$ such that for all $0 < \lambda < \Lambda < \lambda_1$, we have for $\Delta\gamma := \gamma^\Lambda - \gamma^\lambda$

$$\|\Delta\gamma\|_{L^\infty_T C^1} < \delta. \tag{3.91}$$

Given $0 < \lambda < \Lambda < \min\{\lambda_0, \lambda_1\}$, we write:

$$\begin{aligned}
&\frac{\varepsilon}{(L^\Lambda)^3} \int_0^T \int_0^1 |\Delta\gamma_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_0^1 \frac{\varepsilon}{(L^\Lambda)^3} \langle \Delta\gamma_{xx}, \gamma^\Lambda_{xx}\rangle - \frac{\varepsilon}{(L^\lambda)^3} \langle \Delta\gamma_{xx}, \gamma^\lambda_{xx}\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad + \int_0^T \int_0^1 \varepsilon \left(\frac{1}{(L^\Lambda)^3} - \frac{1}{(L^\lambda)^3} \right) \langle \Delta\gamma_{xx}, \gamma^\lambda_{xx}\rangle \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.92}$$

Subtracting (3.89) with time step Λ^{-1} and $\eta = \Delta\gamma$ from (3.89) with time step λ^{-1}, and again $\eta = \Delta\gamma$, we can reformulate (3.92) as the sum:

$$\frac{\varepsilon}{(L^\Lambda)^3} \int_0^T \int_0^1 |\Delta\gamma_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t = A + B^\lambda_1 - B^\Lambda_1 + B^\lambda_2 - B^\Lambda_2 + B^\lambda_3 - B^\Lambda_3, \tag{3.93}$$

where

$$\begin{aligned}
A &= \int_0^T \int_0^1 \varepsilon \left(\frac{1}{(L^\lambda)^3} - \frac{1}{(L^\Lambda)^3} \right) \langle \Delta\gamma_{xx}, \gamma^\lambda_{xx}\rangle \,\mathrm{d}x\,\mathrm{d}t, \\
B^\lambda_1 &= \int_0^T \int_0^1 (1 - \frac{3\varepsilon}{2}(\kappa^\lambda)^2) \langle \Delta\gamma_x, \tau^\lambda\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \langle \nabla_p W(\gamma^\lambda(t,0), \gamma^\lambda(t,1)), \Delta\gamma(t,0)\rangle \,\mathrm{d}t \\
&\quad + \int_0^T \langle \nabla_q W(\gamma^\lambda(t,0), \gamma^\lambda(t,1)), \Delta\gamma(t,1)\rangle \,\mathrm{d}t,
\end{aligned}$$

$$
\begin{aligned}
B_2^\lambda = {} & \frac{\tilde{L}^\lambda}{2}\int_0^T\int_0^1 \langle V^\lambda, \tilde{\nu}^\lambda\rangle\langle\Delta\gamma, \tilde{\nu}^\lambda\rangle \,\mathrm{d}x\,\mathrm{d}t + \frac{L^\lambda}{2}\int_0^T\int_0^1 \langle V^\lambda, \nu^\lambda\rangle\langle\Delta\gamma, \nu^\lambda\rangle \,\mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T \langle V^\lambda(t,0), \Delta\gamma(t,0)\rangle + \langle V^\lambda(t,1), \Delta\gamma(t,1)\rangle \,\mathrm{d}t, \\
B_3^\lambda = {} & \frac{L^\lambda}{2}\int_0^T\int_0^1 \langle V^\lambda, \tilde{\nu}\rangle\langle\tilde{\nu}, \tau\rangle P_1(\Delta\gamma, \gamma^\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
& + \frac{1}{2}\int_0^T\int_0^1 \langle V^\lambda, \tilde{\nu}^\lambda\rangle\langle\gamma^\lambda - \tilde{\gamma}^\lambda, P_2(\Delta\gamma, \gamma^\lambda) - L^\lambda\kappa^\lambda P_1(\Delta\gamma, \gamma^\lambda)\rangle \,\mathrm{d}x\,\mathrm{d}t \\
& + \frac{1}{2}\int_0^1 \langle V^\lambda, \nu\rangle(\gamma^\lambda - \tilde{\gamma}^\lambda)^\perp \Delta\gamma_x \,\mathrm{d}x\,\mathrm{d}t \\
& - \frac{1}{4}\int_0^T \left(\int_0^1 \langle\Delta\gamma_x, \tau\rangle \,\mathrm{d}x\right)\left(\int_0^1 \langle V^\lambda, \nu^\lambda\rangle\langle\gamma^\lambda - \tilde{\gamma}^\lambda, \nu^\lambda\rangle \,\mathrm{d}x\right)\mathrm{d}t,
\end{aligned}
$$

and B_i^Λ, $i \in \{1,2,3\}$, defined by the same formula as B_i^λ, but with each λ exchanged with Λ. We wish to bound the right-hand side of (3.93). This will be achieved by taking advantage of (3.91), thanks to which we can estimate every $\Delta\gamma$ -and $\Delta\gamma_x$-term appearing on the right-hand side of (3.93) by δ from above. For all the remaining terms, it will be enough to find an upper bound $C(\varepsilon, T) < \infty$ independent of Λ and λ. Without further mention, all constants encountered in the following will be independent from the choice of λ and Λ:

A-term: Since L^λ, $L^\Lambda \geq c > 0$, due to the Lipschitz continuity of $x \mapsto x^{-3}$ away from 0, we have:

$$
\begin{aligned}
\left|\frac{1}{(L^\lambda)^3} - \frac{1}{(L^\Lambda)^3}\right| \leq C|L^\lambda - L^\Lambda| = C\left|\int_0^1 |\gamma_x^\lambda| - |\gamma_x^\Lambda| \,\mathrm{d}x\right| \\
\leq C(\varepsilon)\int_0^1 |\gamma_x^\lambda - \gamma_x^\Lambda| \,\mathrm{d}x \leq C(\varepsilon)\delta.
\end{aligned}
$$

Combining this with the bound on the curvature in (3.47), we have:

$$
|A| \leq C(\varepsilon)\int_0^T\int_0^1 |\gamma_{xx}^\lambda|^2 + |\gamma_{xx}^\Lambda|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C(\varepsilon)\delta.
$$

B_1^λ-term: Thanks to (3.46) and (3.47), we have:

$$
\begin{aligned}
|B_1^\lambda| &\leq C(\varepsilon)\int_0^T\int_0^1 \left(1 + (\kappa^\lambda)^2\right)|\Delta\gamma_x| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_0^T |\Delta\gamma(t,1)| + |\Delta\gamma(t,0)| \,\mathrm{d}t \\
&\leq C(\varepsilon)\delta + C\delta = C(\varepsilon)\delta.
\end{aligned}
$$

B_2^λ-term: Due to (3.46), (3.47), and (3.51), we have that:

$$\begin{aligned}
|B_2^\lambda| &\leq C\int_0^T\int_0^1 (|V^\lambda|+|V^\Lambda|)|\Delta\gamma|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T |V^\lambda(t,0)||\Delta\gamma(t,0)| + |V^\lambda(t,1)||\Delta\gamma(t,1)|\,\mathrm{d}t \\
&\leq \sqrt{T}\delta\left(\int_0^T\int_0^1 |V^\lambda|^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \sqrt{T}\delta\left(\int_0^T |V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq C(\varepsilon,T)\delta.
\end{aligned}$$

B_3^λ-term: The bound on the B_3^λ-term can be obtained, by arguing as in the estimate for B_2^λ and noticing that $P_i(\gamma^\lambda,\Delta\gamma)$, $i\in\{1,2\}$, can be bounded as follows:

$$\max_i |P_i(\Delta\gamma,\gamma^\lambda)| \leq C\int_0^1 |\langle\gamma_x^\lambda,\Delta\gamma_x\rangle|\,\mathrm{d}x \leq C\delta.$$

Similarly, one can bound the B_i^Λ-terms by $C\delta$.

Exploiting once more (3.46), and taking into account all the previous estimates, we can find a constant $c(\varepsilon)>0$ such that

$$c(\varepsilon)\int_0^T\int_0^1 |\Delta\gamma_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{\varepsilon}{(L^\Lambda)^3}\int_0^T\int_0^1 |\Delta\gamma_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \leq C(\varepsilon,T)\delta.$$

With all the aforementioned bounds, we see that $(\gamma^\lambda)_\lambda$ is a Cauchy-sequence in $L_T^2H^2$, whose limit must be γ due to (3.70). This concludes the proof. □

Corollary 3.1
As $\gamma^\lambda(\cdot,t)\in\mathcal{AC}$ for all t and by the convergence in (3.70) *and in* (3.90)*, we see that the limit evolution $\gamma(t,\cdot)\in\mathcal{AC}$ for almost all t.*

In the following, we will continue by employing the ellipticity of (3.89), as well as a boot-strapping argument, in order to show the uniform boundedness of higher order x-derivatives of (γ^λ).

Proposition 3.5 (Boot-strapping)
Let $T>0$ be fixed and $\lambda>\lambda_0=\lambda_0(\varepsilon,T)$ with λ_0 as in Remark 3.1. Then $\gamma_{xxxx}^\lambda(\cdot,t)$ exists for all $t\in[0,T]$ and

$$\|\gamma_{xxxx}^\lambda\|_{L_T^{\frac{3}{2}}L^{\frac{3}{2}}} \leq C(\varepsilon,T)<\infty \tag{3.94}$$

for a constant $C(\varepsilon,T)$ independent of λ.

Proof. Let us fix $t\in[0,T]$. In order to keep the notation compact, we will not explicitly write out the dependence of quantities such as γ^λ on t. So, for example, $\gamma^\lambda(\cdot):=\gamma^\lambda(t,\cdot)$ for the fixed $t\in[0,T]$ from above. We define for any $f\colon[0,1]\to\mathbb{R}^2$:

$$D_x^{-1}f(x) := \int_0^s f(\tilde{x})\,d\tilde{x}$$

and D_x^{-n}, for $n \in \mathbb{N}_+$, recursively as $D_x^{-1} D_x^{-(n-1)}$. Integrating by parts in the terms $E(\eta, \gamma^\lambda)$ and $D(\eta, \gamma^\lambda)$ from Lemma 3.7 for a fixed $\eta \in C_c^\infty$, leads to:

$$E(\eta, \gamma^\lambda) = \int_0^1 \left\langle \frac{\varepsilon}{(L^\lambda)^3} \gamma_{xx}^\lambda + D_x^{-1} A_1, \eta_{xx} \right\rangle \mathrm{d}x \tag{3.95}$$

$$D(\eta, \gamma^\lambda) = \int_0^1 \langle D_x^{-2} A_2, \eta_{xx} \rangle \,\mathrm{d}x, \tag{3.96}$$

where

$$A_1 := -(1 - \frac{3\varepsilon}{2}(\kappa^\lambda)^2)\tau^\lambda, \qquad A_2 := \frac{\tilde{L}^\lambda}{2} \langle V^\lambda, \tilde{\nu}^\lambda \rangle \tilde{\nu}^\lambda + \frac{L^\lambda}{2} \langle V^\lambda, \nu^\lambda \rangle \nu^\lambda.$$

Using the definition of P_1 and P_2 (see also (3.85) and (3.86)), Fubini's theorem, and integrating by parts in the term $Err(\eta, \gamma^\lambda)$ from Lemma 3.7 for the same η as before, we derive:

$$Err(\gamma^\lambda, \eta) = \int_0^1 \left\langle D_x^{-1} \tilde{A}, \eta_{xx} \right\rangle \mathrm{d}x \tag{3.97}$$

where:

$$\begin{aligned}
\tilde{A} &:= \left(\int_x^1 A_3 \, d\tilde{x} - \int_0^1 \tilde{x} A_3 \, d\tilde{x} + A_4 - \int_0^1 A_4 \, d\tilde{x} + A_5 \right) \tau^\lambda + A_6, \\
A_3 &:= \frac{\tilde{L}^\lambda}{2} \langle V^\lambda, \tilde{\nu}^\lambda \rangle \langle \tilde{\nu}^\lambda, \tau^\lambda \rangle + \frac{1}{2} \kappa \langle V^\lambda, \tilde{\nu}^\lambda \rangle \langle \gamma^\lambda - \tilde{\gamma}^\lambda, \nu \rangle, \\
A_4 &:= \frac{1}{2} \langle V^\lambda, \tilde{\nu}^\lambda \rangle \langle \gamma^\lambda - \tilde{\gamma}^\lambda, \nu^\lambda \rangle, \\
A_5 &:= \frac{1}{4} \int_0^1 \langle V^\lambda, \nu^\lambda \rangle \langle \gamma^\lambda - \tilde{\gamma}^\lambda, \nu^\lambda \rangle \,\mathrm{d}x, \\
A_6 &:= -\frac{L^\lambda}{2} \langle V^\lambda, \tilde{\nu}^\lambda \rangle (\gamma^\lambda - \tilde{\gamma}^\lambda)^\perp.
\end{aligned}$$

By (3.95), (3.96), (3.97), and (3.89), there exists $v^\lambda = v^\lambda(t)$, $w^\lambda = w^\lambda(t) \in \mathbb{R}^2$ such that:

$$-\frac{\varepsilon}{(L^\lambda)^3} \gamma_{xx}^\lambda(x) = v^\lambda + w^\lambda x + D_x^{-1} A_7(x) + D_x^{-2} A_2(x), \tag{3.98}$$

where:

$$A_7 := A_1 + \left(\int_x^1 A_3 \, d\tilde{x} - \int_0^1 \tilde{x} A_3 \, d\tilde{x} + A_4 - \int_0^1 A_4 \, d\tilde{x} + A_5 \right) \gamma_x^\lambda + A_6.$$

As the right-hand side of (3.98) is weakly differentiable (with respect to x), we can further differentiate γ_{xx}^λ to obtain:

$$-\frac{\varepsilon}{(L^\lambda)^3} \gamma_{xxx}^\lambda = w^\lambda + A_7 + D_x^{-1} A_2.$$

By the very definition of A_7 and the regularity of γ^λ, this shows that γ^λ is four times weakly differentiable and:

$$-\frac{\varepsilon}{(L^\lambda)^3} \gamma_{xxxx}^\lambda = (A_7)_x + A_2.$$

For convenience, we will now split up the right-hand side of the above equation as follows:

$$-\frac{\varepsilon}{(L^\lambda)^3}\gamma^\lambda_{xxxx} = \sum_{i=1}^{5} B_i, \tag{3.99}$$

where

$$\begin{aligned}
B_1 &:= (A_1)_x = \frac{1}{L^\lambda}\left(3\varepsilon\kappa^\lambda\kappa^\lambda_x\gamma^\lambda_x - (1-\frac{3\varepsilon}{2}(\kappa^\lambda)^2)\gamma^\lambda_{xx}\right),\\
B_2 &:= A_2,\\
B_3 &:= A_3\gamma^\lambda_x + \left(\int_x^1 A_3\,d\tilde{x} - \int_0^1 \tilde{x}A_3\,d\tilde{x}\right)\gamma^\lambda_{xx},\\
B_4 &:= (A_4)_x\gamma^\lambda_x + \left(A_4 - \int_0^1 A_4\,d\tilde{x}\right)\gamma^\lambda_{xx},\\
B_5 &:= (A_5)_x\gamma^\lambda_x + A_5\gamma^\lambda_{xx} + (A_6)_x.
\end{aligned}$$

Let us separately estimate each term on the right-hand side of (3.99). We note that we will repeatedly make use of (3.46), (3.47), (3.63), as well as the boundedness implied by the convergence in (3.70). Furthermore, all constants appearing in the following estimates will be independent of λ.

B_1-term:

$$\begin{aligned}
\int_0^1 |B_1|^{\frac{3}{2}}\,\mathrm{d}x &\leq C(\varepsilon)\int_0^1 |\kappa^\lambda|^{\frac{3}{2}}|\kappa^\lambda_x|^{\frac{3}{2}} + |\gamma^\lambda_{xx}|^{\frac{3}{2}} + |\kappa^\lambda|^3|\gamma^\lambda_{xx}|^{\frac{3}{2}}\,\mathrm{d}x\\
&\leq C(\varepsilon)\int_0^1 |\gamma^\lambda_{xx}|^{\frac{3}{2}}(|\gamma^\lambda_{xxx}|^{\frac{3}{2}}+1) + |\gamma^\lambda_{xx}|^{\frac{9}{2}}\,\mathrm{d}x\\
&\leq C(\varepsilon)\int_0^1 1 + |\gamma^\lambda_{xxx}|^{\frac{9}{4}} + |\gamma^\lambda_{xx}|^{\frac{9}{2}}\,\mathrm{d}x,
\end{aligned}$$

where in the third line we employed Young's inequality. Using the interpolation inequality (3.62) with parameters given by row b) and c) in Table 3.1, and eventually Young's inequality with arbitrary $\delta > 0$, leads to:

$$\begin{aligned}
\|\gamma^\lambda_{xxx}\|^{\frac{9}{4}}_{L^{\frac{9}{4}}} &\leq C\left(\|\gamma^\lambda_{xxxx}\|^{\frac{11}{8}}_{L^{\frac{3}{2}}}\|\gamma^\lambda_{xx}\|^{\frac{7}{8}}_{L^{\frac{3}{2}}} + \|\gamma^\lambda_{xx}\|^{\frac{9}{4}}_{L^{\frac{3}{2}}}\right).\\
&\leq C\left(\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} + C(\delta)\|\gamma^\lambda_{xx}\|^{\frac{21}{2}}_{L^{\frac{3}{2}}} + \|\gamma^\lambda_{xx}\|^{\frac{9}{4}}_{L^{\frac{3}{2}}}\right)\\
&\leq C\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} + C(\delta,\varepsilon)
\end{aligned}$$

as well as:

$$\begin{aligned}
\|\gamma^\lambda_{xx}\|^{\frac{9}{2}}_{L^{\frac{9}{2}}} &\leq C\left(\|\gamma^\lambda_{xxxx}\|_{L^{\frac{3}{2}}}\|\gamma^\lambda_{xx}\|^{\frac{7}{2}}_{L^{\frac{3}{2}}} + \|\gamma^\lambda_{xx}\|^{\frac{9}{2}}_{L^{\frac{3}{2}}}\right)\\
&\leq C\left(\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} + C(\delta)\|\gamma^\lambda_{xx}\|^{\frac{21}{2}}_{L^{\frac{3}{2}}} + \|\gamma^\lambda_{xx}\|^{\frac{9}{2}}_{L^{\frac{3}{2}}}\right)\\
&\leq C\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} + C(\delta,\varepsilon).
\end{aligned}$$

Consequently, we derive:

$$\int_0^1 |B_1|^{\frac{3}{2}}\,\mathrm{d}x \leq C(\varepsilon,\delta) + C(\varepsilon)\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}.$$

B_2-term:

$$\int_0^1 |B_2|^{\frac{3}{2}}\,\mathrm{d}x \le C\|B_2\|_{L^2}^{\frac{3}{2}} \le C\|V^\lambda\|_{L^2}^{\frac{3}{2}} \le C(1+\|V^\lambda\|_{L^2}^2).$$

B_3-term:

$$\begin{aligned}
\int_0^1 |B_3|^{\frac{3}{2}}\,\mathrm{d}x &\le C\int_0^1 \left(|A_3|^{\frac{3}{2}} + \left(\int_0^1 |A_3|\,d\tilde{x}\right)^{\frac{3}{2}} |\gamma^\lambda_{xx}|^{\frac{3}{2}}\right)\mathrm{d}x \\
&\le C(T)\int_0^1 |V^\lambda|^{\frac{3}{2}}(1+|\gamma^\lambda_{xx}|^{\frac{3}{2}})\,\mathrm{d}x \\
&\quad + C(T)\int_0^1 (\|V^\lambda\|_{L^1}^{\frac{3}{2}} + \|V^\lambda\|_{L^2}^{\frac{3}{2}}\|\gamma^\lambda_{xx}\|_{L^2}^{\frac{3}{2}})|\gamma^\lambda_{xx}|^{\frac{3}{2}}\,\mathrm{d}x \\
&\le C(T)\int_0^1 \left(1 + C(\delta)|V^\lambda|^2 + \delta|\gamma^\lambda_{xx}|^6\right)\mathrm{d}x \\
&\quad + C(T)(\|V^\lambda\|_{L^1}^{\frac{3}{2}} + \|V^\lambda\|_{L^2}^{\frac{3}{2}}\|\gamma^\lambda_{xx}\|_{L^2}^{\frac{3}{2}})\|\gamma^\lambda_{xx}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \\
&\le C(T,\delta)\left(1+\|V^\lambda\|_{L^2}^2\right) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xx}\|_{L^6}^6 + C(\varepsilon,T)\|V^\lambda\|_{L^2}^{\frac{3}{2}} \\
&\le C(\varepsilon,T,\delta)\left(1+\|V^\lambda\|_{L^2}^2\right) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xx}\|_{L^6}^6,
\end{aligned}$$

where in the third line we used Young's inequality with arbitrary $\delta > 0$ in order to estimate $|V^\lambda|^{\frac{3}{2}}(1+|\gamma^\lambda_{xx}|^{\frac{3}{2}})$. Making use of the interpolation inequality (3.62) with parameters given by row d) of Table 3.1, it follows that:

$$\begin{aligned}
\|\gamma^\lambda_{xx}\|_{L^6}^6 &\le C\left(\|\gamma^\lambda_{xxxx}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}\|\gamma^\lambda_{xx}\|_{L^{\frac{3}{2}}}^{\frac{9}{2}} + \|\gamma^\lambda_{xx}\|_{L^{\frac{3}{2}}}^6\right) \\
&\le C\left(\|\gamma^\lambda_{xxxx}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}\|\gamma^\lambda_{xx}\|_{L^2}^{\frac{9}{2}} + \|\gamma^\lambda_{xx}\|_{L^2}^6\right).
\end{aligned} \tag{3.100}$$

Combining the last two estimates then results in:

$$\int_0^1 |B_3|^{\frac{3}{2}}\,\mathrm{d}x \le C(\varepsilon,T,\delta)(1+\|V^\lambda\|_{L^2}^2) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xxxx}\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}.$$

B_4-term:

$$\begin{aligned}
&\int_0^1 |B_4|^{\frac{3}{2}}\,\mathrm{d}x \\
&\quad\le C\int_0^1 \left(|(A_4)_x|^{\frac{3}{2}} + \left(\int_0^1 |A_4|\,d\tilde{x}\right)^{\frac{3}{2}}|\gamma^\lambda_{xx}|^{\frac{3}{2}} + |A_4|^{\frac{3}{2}}|\gamma^\lambda_{xx}|^{\frac{3}{2}}\right)\mathrm{d}x.
\end{aligned} \tag{3.101}$$

With:

$$\begin{aligned}
2L^\lambda(A_4)_x = {} & \langle\gamma^\lambda_x - \tilde{\gamma}^\lambda_x, \tilde{\nu}^\lambda\rangle\langle V^\lambda, \nu^\lambda\rangle + \tilde{L}^\lambda\tilde{\kappa}^\lambda\langle V^\lambda, \tilde{\nu}^\lambda\rangle\langle\gamma^\lambda - \tilde{\gamma}^\lambda, \nu^\lambda\rangle \\
& + \langle V^\lambda, \tilde{\nu}^\lambda\rangle\langle\gamma^\lambda_x - \tilde{\gamma}^\lambda_x, \nu^\lambda\rangle + L^\lambda\kappa^\lambda\langle V^\lambda, \tilde{\nu}^\lambda\rangle\langle\gamma^\lambda - \tilde{\gamma}^\lambda, \nu^\lambda\rangle,
\end{aligned}$$

and (3.100), we follow that:

$$\begin{aligned}\int_0^1 |(A_4)_x|^{\frac{3}{2}}\,\mathrm{d}x &\leq C(T)\int_0^1 |V^\lambda|^{\frac{3}{2}}(1+|\gamma^\lambda_{xx}|^{\frac{3}{2}})\,\mathrm{d}x\\ &\leq C(T)\int_0^1 1 + C(\delta)|V^\lambda|^2 + \delta|\gamma^\lambda_{xx}|^6\,\mathrm{d}x\\ &= C(T,\delta)(1+\|V^\lambda\|^2_{L^2}) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}.\end{aligned}$$

Furthermore, by (3.100) and again Young's inequality:

$$\begin{aligned}&\int_0^1 \left(\int_0^1 |A_4|\,d\tilde{x}\right)^{\frac{3}{2}} |\gamma^\lambda_{xx}|^{\frac{3}{2}} + |A_4|^{\frac{3}{2}}|\gamma^\lambda_{xx}|^{\frac{3}{2}}\,\mathrm{d}x\\ &\leq C(T)\left(\|V^\lambda\|^{\frac{3}{2}}_{L^1}\|\gamma^\lambda_{xx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} + \int_0^1 |V^\lambda|^{\frac{3}{2}}|\gamma^\lambda_{xx}|^{\frac{3}{2}}\,\mathrm{d}x\right)\\ &\leq C(\varepsilon,T)(1+\|V^\lambda\|^2_{L^2}) + C(T,\delta)\|V^\lambda\|^2_{L^2} + \delta\|\gamma^\lambda_{xx}\|^6_{L^6}\\ &\leq C(\varepsilon,T,\delta)(1+\|V^\lambda\|^2_{L^2}) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}.\end{aligned}$$

Combining the last three estimates, results in:

$$\int_0^1 |B_4|^{\frac{3}{2}}\,\mathrm{d}x \leq C(\varepsilon,T,\delta)(1+\|V^\lambda\|^2_{L^2}) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}.$$

B_5*-term*: Repeating the same argument as in the previous steps, we derive that:

$$\int_0^1 |B_5|^{\frac{3}{2}}\,\mathrm{d}x \leq C(\varepsilon,T,\delta)(1+\|V^\lambda\|^2_{L^2}) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}.$$

Hence, with (3.99) and the bounds we found for the B_i-terms, we have:

$$c(\varepsilon)\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} \leq C(\varepsilon,T,\delta)\left(1+\|V^\lambda\|^2_{L^2}\right) + C(\varepsilon,T)\delta\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}.$$

Taking $\delta > 0$ small enough, we then see:

$$\|\gamma^\lambda_{xxxx}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} \leq C(\varepsilon,T)\left(1+\|V^\lambda\|^2_{L^2}\right).$$

As $t \in [0,T]$ was arbitrary:

$$\|\gamma^\lambda_{xxxx}(t,\cdot)\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} \leq C(\varepsilon,T)\left(1+\|V^\lambda(t,\cdot)\|^2_{L^2}\right) \tag{3.102}$$

for all $t \in [0,T]$. Integrating (3.102) over $t \in [0,T]$ and employing the velocity bound in (3.51), finally leads to (3.94). □

The previously derived bound results in the following compactness result:

Theorem 3.4
Up to taking subsequences, it holds that:

$$\gamma^\lambda \rightharpoonup \gamma \text{ weakly in } L^{\frac{3}{2}}_{\text{loc}}W^{4,\frac{3}{2}} \tag{3.103}$$

$$\gamma^\lambda \to \gamma \text{ strongly in } L^{\frac{39}{23}}_{\text{loc}}W^{3,\frac{39}{23}}, \tag{3.104}$$

where γ is the limit from Theorem 3.3. In particular, we have that for almost all t:

$$\gamma^\lambda(t,\cdot) \to \gamma(t,\cdot) \text{ in } C^2. \tag{3.105}$$

Proof. Let us fix $T > 0$, and suppose that we have selected (without renumbering) the subsequence from Theorem 3.3. Then, directly from the bound in (3.94), we see that:

$$\gamma^\lambda \rightharpoonup \gamma \text{ weakly in } L_T^{\frac{3}{2}} W^{4,\frac{3}{2}}.$$

In the following: we will show that (γ^λ) is Cauchy in $L_T^{\frac{39}{23}} W^{2,\frac{39}{23}}$. For this, fix $\tilde{\delta}$, $\delta \in \mathbb{R}$ such that $\delta \geq \tilde{\delta} > 0$. From (3.90), we know that there exists $\lambda_0 = \lambda_0(\tilde{\delta}) > 0$ big enough such that for any $\lambda, \Lambda \in \mathbb{R}$ with $\Lambda > \lambda > \lambda_0$ and for $\Delta\gamma \coloneqq \gamma^\Lambda - \gamma^\lambda$, it holds that:

$$\|\Delta\gamma\|_{L_T^{\frac{39}{23}} W^{2,\frac{39}{23}}} \leq C\|\Delta\gamma\|_{L_T^2 H^2} < \tilde{\delta}, \tag{3.106}$$

where we have employed Hölder's inequality in the first step above. By the interpolation inequality (3.62) with parameters given by row e) in Table 3.1, we have:

$$\|\Delta\gamma_{xxx}\|_{L^{\frac{39}{23}}} \leq C\left(\|\Delta\gamma_{xxxx}\|_{L^{\frac{3}{2}}}^{\frac{7}{13}} \|\Delta\gamma_{xx}\|_{L^{\frac{3}{2}}}^{\frac{6}{13}} + \|\Delta\gamma_{xx}\|_{L^{\frac{3}{2}}}\right).$$

Hence by Hölder's inequality, (3.94), and (3.106), we derive for all $\Lambda > \lambda > \lambda_0$:

$$\begin{aligned}
&\int_0^T \|\Delta\gamma_{xxx}\|_{L^{\frac{39}{23}}}^{\frac{39}{23}} \,\mathrm{d}t \\
&\leq C\int_0^T \left(\|\Delta\gamma_{xxxx}\|_{L^{\frac{3}{2}}}^{\frac{21}{23}} \|\Delta\gamma_{xx}\|_{L^{\frac{3}{2}}}^{\frac{18}{23}} + \|\Delta\gamma_{xx}\|_{L^{\frac{3}{2}}}^{\frac{39}{23}}\right) \mathrm{d}t \\
&\leq C\|\Delta\gamma_{xxxx}\|_{L_T^{\frac{3}{2}} L^{\frac{3}{2}}} \|\Delta\gamma_{xx}\|_{L_T^2 L^2} + C(T)\|\Delta\gamma_{xx}\|_{L_T^2 L^2}^{\frac{39}{46}} \\
&\leq C(\varepsilon, T)(\tilde{\delta} + \tilde{\delta}^{\frac{39}{46}}).
\end{aligned}$$

Therefore, for $\tilde{\delta}$ small enough, we have for $\Lambda > \lambda > \lambda_0$:

$$\|\Delta\gamma_{xxx}\|_{L_T^{\frac{39}{23}} L^{\frac{39}{23}}} < \delta. \tag{3.107}$$

Thanks to (3.106) and (3.107), we conclude (3.104) through a diagonal sequence argument. Note that (3.105) follows from the Sobolev's embedding Theorem. □

Our last compactness result is derived from the coupling relation (3.50), which will also lead to the equation (compare with (3.110)) satisfied by the tangential component of the velocity of γ.

Theorem 3.5
Let $V \coloneqq \gamma_t$, where γ is the limit from Theorem 3.3, and $\hat{L}^\lambda$ be defined as in Definition 3.1. Then, up to a subsequence, it holds:

$$\hat{L}^\lambda \rightharpoonup L \text{ weakly in } H^1_{\mathrm{loc}}([0,\infty)), \tag{3.108}$$

$$\langle V^\lambda, \tilde{\gamma}_x^\lambda + \gamma_x^\lambda \rangle \rightharpoonup 2LV^\top \text{ weakly in } L^{\frac{3}{2}}_{\mathrm{loc}}([0,\infty); W^{1,\frac{3}{2}}). \tag{3.109}$$

Furthermore, for almost all $t \geq 0$ and $x \in I$, the limit γ satisfies:

$$V_x^\top(t,x) = L'(t) + L(t)\kappa(t,x)V^\perp(t,x). \tag{3.110}$$

Proof. Let us fix $T > 0$. We start by integrating (3.50) for fixed $t \in [0,T]$ over $x \in I$, and consequently solving for $\hat{L}_t^\lambda$:

$$(\tilde{L}^\lambda + L^\lambda)\hat{L}_t^\lambda = \langle V^\lambda, \tilde{\gamma}_x^\lambda + \gamma_x^\lambda\rangle|_{t=0}^1 - \int_0^1 \langle V^\lambda, \tilde{L}^\lambda\tilde{\kappa}^\lambda R(\tilde{\gamma}_x^\lambda) + L^\lambda\kappa^\lambda R(\gamma_x^\lambda)\rangle \,\mathrm{d}x.$$

Integrating the square of the above equation over $t \in [0,T]$ and using (3.46), (3.47), and (3.51), we derive:

$$\begin{aligned}
&\int_0^T (\hat{L}_t^\lambda)^2 \,\mathrm{d}t \\
&\quad\leq C\int_0^T \left(|V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2\right)\mathrm{d}t \\
&\qquad + C\int_0^T \left(\left(\int_0^1 |\tilde{\kappa}^\lambda V^\lambda|\,\mathrm{d}x\right)^2 + \left(\int_0^1 |\kappa^\lambda V^\lambda|\,\mathrm{d}x\right)^2\right)\mathrm{d}t \\
&\quad\leq C\int_0^T \left(|V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2\right)\mathrm{d}t + C\int_0^T \left(\int_0^1 (\tilde{\kappa}^\lambda)^2\,\mathrm{d}x \int_0^1 |V^\lambda|^2\,\mathrm{d}x\right)\mathrm{d}t \\
&\qquad + C\int_0^T \left(\int_0^1 (\kappa^\lambda)^2\,\mathrm{d}x \int_0^1 |V^\lambda|^2\,\mathrm{d}x\right)\mathrm{d}t \\
&\quad\leq C(\varepsilon)\int_0^T \left(|V^\lambda(t,0)|^2 + |V^\lambda(t,1)|^2\right)\mathrm{d}t + C(\varepsilon)\int_0^T\int_0^1 |V^\lambda|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq C(\varepsilon, T).
\end{aligned}$$

Hence, $(\hat{L}^\lambda)$ is uniformly bounded in $H^1(0,T)$ and, up to choosing a subsequence:

$$\hat{L}^\lambda \rightharpoonup L \text{ weakly in } H^1(0,T).$$

Next, we take the absolute value of both sides of (3.50) to the power $\frac{3}{2}$ and integrate over $x \in I$ and $t \in [0,T]$. By the $L^2([0,T])$ bound on $(L_t^\lambda)_\lambda$, (3.51), (3.100), and (3.94), we have:

$$\begin{aligned}
&\int_0^T\int_0^1 |\langle V^\lambda, \tilde{\gamma}_x^\lambda + \gamma_x^\lambda\rangle_x|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq C\int_0^T (\hat{L}_t^\lambda)^{\frac{3}{2}}\,\mathrm{d}t + C\int_0^T\int_0^1 |V^\lambda|^{\frac{3}{2}}|\gamma_{xx}^\lambda|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq C(T)\|\hat{L}_t^\lambda\|_{L^2}^{\frac{3}{2}} + \int_0^T \|V^\lambda\|_{L^2}^2 + \|\gamma_{xx}^\lambda\|_{L^6}^6\,\mathrm{d}t \leq C(\varepsilon, T),
\end{aligned}$$

where in the third line we have employed Young's inequality. Hence, $\langle V^\lambda, \tilde{\gamma}_x^\lambda + \gamma_x^\lambda\rangle_x$ is bounded in $L_T^{\frac{3}{2}}L^{\frac{3}{2}}$ and, up to taking a further subsequence:

$$\langle V^\lambda, \tilde{\gamma}_x^\lambda + \gamma_x^\lambda\rangle \rightharpoonup \langle V, 2\gamma_x\rangle = 2LV^\top \text{ weakly in } L^{\frac{3}{2}}(0,T;W^{1,\frac{3}{2}}).$$

By a diagonal sequence argument, we conclude (3.108) and (3.109). Finally, equation (3.110) follows by combining these two convergences with (3.50). □

3.3.2 Convergence

In this subsection, we proof the main result of this chapter stated in Theorem 3.1. Let us start by employing the compactness results of the previous subsection in order to pass to the limit $\lambda \to \infty$ in the weak formulation (3.89) of the time-discrete evolution.

Theorem 3.6 (Weak form of the geometric evolution)
Let γ be the limit of Theorem 3.3, then for all test functions $\eta \in C_c^\infty C^\infty$, it holds that:

$$\begin{aligned}\int_0^\infty \int_0^1 \left(\frac{\varepsilon}{L^2}\kappa\langle \eta_{xx}, \nu\rangle + \frac{1}{L}(1 - \frac{3\varepsilon}{2}\kappa^2)\langle \eta_x, \tau\rangle \right) L \,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\infty \langle \nabla_p W(\gamma(t,0), \gamma(t,1)), \eta(t,0)\rangle + \langle \nabla_q W(\gamma(t,0),\gamma(t,1)), \eta(t,1)\rangle \,\mathrm{d}t \\ + \int_0^T \int_0^1 V^\perp \langle \eta, \nu\rangle L \,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\infty \langle V(t,0), \eta(t,0)\rangle + \langle V(t,1), \eta(t,1)\rangle \,\mathrm{d}t = 0.\end{aligned} \tag{3.111}$$

Proof. By (3.89), in order to show (3.111) it is enough to prove the following convergences as $\lambda \to \infty$ (up to taking subsequences):

$$\int_0^\infty E(\gamma^\lambda, \eta)\,\mathrm{d}t \to \int_0^\infty E(\gamma, \eta), \tag{3.112}$$

$$\begin{aligned}\int_0^\infty D(\gamma^\lambda, \eta)\,\mathrm{d}t \to \int_0^\infty \int_0^1 \langle V, \nu\rangle\langle \eta, \nu\rangle L\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\infty \langle V(t,0), \eta(t,0)\rangle + \langle V(t,1), \eta(t,1)\rangle\,\mathrm{d}t\end{aligned} \tag{3.113}$$

$$\int_0^\infty Err(\gamma^\lambda, \eta)\,\mathrm{d}t \to 0. \tag{3.114}$$

In the following, let $T > 0$ be big enough such that $\mathrm{spt}(\eta) \subset [0,T] \times I$. Without renumbering, we suppose that we have already selected a subsequence such that all the compactness statements from the previous subsection hold true for the whole sequence (γ^λ) with the limit denoted by γ.

Proof of (3.112): By (3.46), there exists a constant $d_0 > 0$ such that for all λ:

$$|\gamma^\lambda(t,0) - \gamma^\lambda(t,1)| \geq c.$$

With (3.68), this is also satisfied for the limit γ. Consequently by the smoothness of W in the set $\{(p,q) \in \Omega^2 \colon |p-q| \geq d_0\}$, we see that:

$$\begin{aligned}&|\langle \nabla_p W(\gamma^\lambda(t,0), \gamma^\lambda(t,1)), \eta(t,0)\rangle - \langle \nabla_p W(\gamma(t,0),\gamma(t,1)), \eta(t,0)\rangle| \\ &\quad\leq C(d_0)(|\gamma^\lambda(t,0) - \gamma(t,0)| + |\gamma^\lambda(t,1) + \gamma(t,1)|)\|\eta\|_{L_T^\infty L^\infty} \\ &\quad\leq C(d_0,\eta)\|\hat{\gamma}^\lambda - \gamma\|_{C_T^0 C^0} \to 0 \text{ as } \lambda \to 0.\end{aligned}$$

In the same way, one can also show that:

$$\langle \nabla_q W(\gamma^\lambda(t,0), \gamma^\lambda(t,1)), \eta(t,0)\rangle \to \langle \nabla_q W(\gamma^\lambda(t,0), \gamma^\lambda(t,1)), \eta(t,0)\rangle.$$

Employing (3.46) and (3.47), we can find $C(\varepsilon,\eta)<\infty$ independent of λ such that:

$$\int_0^1 \left(\frac{\varepsilon}{(L^\lambda)^2}\kappa^\lambda\langle\eta_{xx},\nu^\lambda\rangle + \frac{1}{L^\lambda}(1-\frac{3\varepsilon}{2}(\kappa^\lambda)^2)\langle\eta_x,\tau\rangle\right) L^\lambda dx \le C(\varepsilon,\eta).$$

Thus, (3.105), the dominated convergence theorem, and the previous convergence of the renormalized energy terms lead to (3.112).

Proof of (3.113): By (3.70), we have that:

$$\tilde{L}^\lambda\langle\eta,\tilde{\nu}^\lambda\rangle\tilde{\nu}^\lambda \to L\langle\eta,\nu\rangle\nu, \quad L^\lambda\langle\eta,\nu^\lambda\rangle\nu^\lambda \to \langle\eta,\nu\rangle\nu$$

strongly in $L^\infty_T L^\infty$, and therefore also strongly in $L^2_T L^2$. Hence, by weak-strong convergence, we derive:

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_0^1 \tilde{L}^\lambda\langle V^\lambda,\langle\eta,\tilde{\nu}^\lambda\rangle\tilde{\nu}^\lambda\rangle + L^\lambda\langle V^\lambda,\langle\eta,\nu^\lambda\rangle\nu^\lambda\rangle\,\mathrm{d}x\,\mathrm{d}t\\ &\to \int_0^T\int_0^1\langle V,\nu\rangle\langle\eta,\nu\rangle L\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore, by additionally using (3.71), we follow (3.113).

Proof of (3.114): From (3.70) and the definition of P_1 and P_2 (see also (3.85) and (3.86)), we derive that:

$$P_1(\eta,\gamma^\lambda)\to P_1(\eta,\gamma), \quad P_2(\eta,\gamma^\lambda)\to P_2(\eta,\gamma)$$

strongly in $L^\infty([0,T];L^\infty(I))$. Thus, (3.70) and (3.90) imply that:

$$\begin{aligned}\tilde{L}^\lambda\langle\tilde{\nu}^\lambda,\tau^\lambda\rangle P_1(\eta,\gamma^\lambda)\tilde{\nu}^\lambda &\to 0,\\ \langle\gamma^\lambda-\tilde{\gamma}^\lambda,\tau^\lambda\rangle(P_2(\eta,\gamma^\lambda)-L\kappa P_1(\eta,\gamma^\lambda))\tilde{\nu}^\lambda &\to 0,\\ \left(\int_0^1\langle\eta_x,\tau^\lambda\rangle\,\mathrm{d}x\right)\langle\gamma^\lambda-\tilde{\gamma}^\lambda,\nu^\lambda\rangle\nu^\lambda &\to 0,\end{aligned}$$

strongly in $L^2_T L^2$. Therefore, as in the previous step, the result follows by weak-strong convergence. □

In the next corollary, we will show that the regularity of the limit evolution can be improved:

Corollary 3.2
For the time-continuous evolution γ from Theorem 3.6, it holds that $\gamma\in L^2_{\mathrm{loc}}H^4$.

From (3.94), only $\gamma\in L^{\frac{3}{2}}_{\mathrm{loc}}W^{4,\frac{3}{2}}$ can be deduced a priori. In order to improve the integrability from $\frac{3}{2}$ to 2, we have to repeat the strategy of the proof of Proposition 3.5 in the time-continuous setting. Instead of (3.89), we will employ (3.111) which – in contrast to (3.89) – is simpler as it misses the Err-term. More precisely:

Proof. Testing (3.111) with $\eta(t,x):=\psi(t)\,\varphi(x)$, where $\psi\in C_c^\infty([0,\infty)$ and $\varphi\in C_c^\infty((0,1);\mathbb{R}^2)$, as well as using the arbitrariness of ψ, we derive that for a.e. $t\in[0,\infty)$, it follows that:

$$\int_0^1 \frac{\varepsilon}{L^3}\langle\gamma_{xx},\varphi_{xx}\rangle + (1-\frac{3\varepsilon}{2}\kappa^2)\langle\varphi_x,\tau\rangle + L\langle V,\nu\rangle\langle\varphi,\nu\rangle\,\mathrm{d}x = 0.$$

Integrating by parts in the above equation and employing the notation from the proof of Proposition 3.5, then leads to:

$$\int_0^1 \left\langle \frac{\varepsilon}{L^3}\gamma_{xx} - D_x^{-1}((1-\frac{3\varepsilon}{2}\kappa^2)\tau) + D_x^{-2}(L\langle V,\nu\rangle\nu), \varphi_{xx} \right\rangle \mathrm{d}x = 0$$

for a.e. $t \in [0,\infty)$. Hence, for all such t, there exist $v(t)$, $w(t) \in \mathbb{R}^2$ satisfying for a.e. $x \in I$:

$$-\frac{\varepsilon}{L^3}\gamma_{xx} = v + wx - D_x^{-1}((1-\frac{3\varepsilon}{2}\kappa^2)\tau) + D_x^{-2}(L\langle V,\nu\rangle\nu).$$

We then twice differentiate the equation above which leads to:

$$-\frac{\varepsilon}{L^3}\gamma_{xxxx} = 3\varepsilon\kappa\kappa_x\gamma_x - L(1-\frac{3\varepsilon}{2}\kappa^2)\kappa\nu + L\langle V,\nu\rangle\nu,$$

again, for a.e. $t \geq 0$ and a.e $x \in I$. Consequently, by Young's Inequality, (3.46) and (3.47) we have for a.e. $t \in [0,\infty)$:

$$\begin{aligned}
\|\gamma_{xxxx}\|_{L^2}^2 &\leq C(\varepsilon)\int_0^1 |\kappa|^2|\kappa_x|^2 + |\gamma_{xx}|^2 + |\kappa|^4|\gamma_{xx}|^2 + |V|^2\,\mathrm{d}x \\
&\leq C(\varepsilon)\int_0^1 |\gamma_{xx}|^2(|\gamma_{xxx}|^2+1) + |\gamma_{xx}|^6 + |V|^2\,\mathrm{d}x \\
&\leq C(\varepsilon)\int_0^1 1 + |\gamma_{xx}|^6 + |\gamma_{xxx}|^3 + |V|^2\,\mathrm{d}x.
\end{aligned}$$

Employing the interpolation inequality (3.62) with parameters given by rows f) and g) in Table 3.1, Young's inequality with $\delta > 0$, and (3.47), we have:

$$\begin{aligned}
\|\gamma_{xxx}\|_{L^3}^3 &\leq C\left(\|\gamma_{xxxx}\|_{L^2}^{\frac{21}{12}}\|\gamma_{xx}\|_{L^2}^{\frac{5}{4}} + \|\gamma_{xx}\|_{L^2}^3\right) \\
&\leq C\left(\delta\|\gamma_{xxxx}\|_{L^2}^2 + C(\delta)\|\gamma_{xx}\|_{L^2}^{\frac{35}{12}} + \|\gamma_{xx}\|_{L^2}^3\right) \\
&\leq C\delta\|\gamma_{xxxx}\|_{L^2}^2 + C(\delta,\varepsilon),
\end{aligned}$$

as well as:

$$\begin{aligned}
\|\gamma_{xx}\|_{L^6}^6 &\leq C\left(\|\gamma_{xxxx}\|_{L^2}^1\|\gamma_{xx}\|_{L^2}^5 + \|\gamma_{xx}\|_{L^2}^6\right) \\
&\leq C\left(\delta\|\gamma_{xxxx}\|_{L^2}^2 + C(\delta)\|\gamma_{xx}\|_{L^2}^{10} + \|\gamma_{xx}\|_{L^2}^6\right) \\
&\leq C\delta\|\gamma_{xxxx}\|_{L^2}^2 + C(\delta,\varepsilon).
\end{aligned}$$

We combine the last two estimates:

$$\|\gamma_{xxxx}\|_{L^2}^2 \leq C(\varepsilon)\delta\|\gamma_{xxxx}\|_{L^2}^2 + C(\delta,\varepsilon)\int_0^1 1 + |V|^2\,\mathrm{d}x.$$

Hence, choosing δ small enough, we have for a.e. $t \in [0,\infty)$:

$$\|\gamma_{xxxx}\|_{L^2}^2 \leq C(\varepsilon)\int_0^1 1 + |V|^2\,\mathrm{d}x.$$

Integrating the above inequality over $t \in [0,T]$ with arbitrary $T > 0$ and using the velocity bound in (3.51), we conclude the proof. □

We are finally ready to prove the main result of this chapter (see also Theorem 3.1):

Proof of Theorem 3.1. The regularity statements in (3.40) directly follow from (3.68), (3.69), (3.71), and Corollary 3.2. We wish to show that γ satisfies all five equations stated in Theorem 3.1 at a.e. $t \in [0,\infty)$ and a.e. $x \in I$. Due to Corollary 3.2, we can find a sequence $(\mu^i) \subset C_c^\infty C^\infty \cap L^2_{\text{loc}} H^4$ such that:

$$\mu^i \to \gamma \text{ strongly in } L^2_{\text{loc}} H^4 \text{ as } i \to \infty.$$

So, in particular:

$$\mu^i \to \gamma \text{ strongly in } L^2_{\text{loc}} C^3. \tag{3.115}$$

Furthermore, by (3.68), we can also assume that:

$$\mu^i \to \gamma \text{ strongly in } C_c^0 C^1. \tag{3.116}$$

1. step: We will first show the geometric evolution of the interior of the curve. Fix $\varphi \in C^\infty([0,T] \times I)$ for some arbitrary $T > 0$. By (3.115), the definition of E from Lemma 3.7, Hölder's inequality, (3.46), and the interpolation estimate in (3.100), we derive:

$$\begin{aligned}
&\left| \int_0^\infty E(\gamma, \varphi(\mu_x^i)^\perp)\,\mathrm{d}t - \int_0^\infty E(\gamma, \varphi\gamma_x^\perp)\,\mathrm{d}t \right| \\
&\quad \le \int_0^T \int_0^1 C(\varphi)(1 + |\gamma_{xx}| + |\gamma_{xx}|^2)\|\mu^i(t,\cdot) - \gamma(t,\cdot)\|_{C^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C(\varphi) \left(\int_0^T 1 + \|\gamma_{xx}(t,\cdot)\|_{L^6}^6\,\mathrm{d}t \right)^{\frac{1}{3}} \left(\int_0^T \|\mu^i(t,\cdot) - \gamma(t,\cdot)\|_{C^3}^{\frac{3}{2}}\,\mathrm{d}t \right)^{\frac{2}{3}} \\
&\quad \le C(\varphi, \varepsilon, T)\|\mu^i - \gamma\|_{L^2_T C^3} \overset{h\to\infty}{\to} 0.
\end{aligned}$$

Furthermore, by (3.46) and (3.116), we also see that:

$$\begin{aligned}
&\left| \int_0^T \int_0^1 LV^\perp \langle \varphi(\mu_x^i)^\perp, \nu \rangle\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad \le C(\varphi)\|V^\perp\|_{L^2_T L^2}\|\mu^i - \gamma\|_{L^\infty_T C^1} \overset{h\to\infty}{\to} 0.
\end{aligned}$$

In a similar fashion, we can show;

$$\begin{aligned}
&\left| \int_0^\infty \langle V(t,0), \varphi(t,0)(\mu_x^i)^\perp \rangle + \langle V(t,0), \varphi(t,0)(\mu_x^i)^\perp \rangle\,\mathrm{d}t \right. \\
&\quad \left. - \int_0^\infty \langle V(t,0), \varphi(t,0)\gamma_x^\perp \rangle - \langle V(t,0), \varphi(t,0)\gamma_x^\perp \rangle\,\mathrm{d}t \right| \to 0 \text{ as } i \to \infty.
\end{aligned}$$

Let us now test (3.111) with $\eta = \varphi(\mu_x^i)^\perp$ and pass to the limit $i \to \infty$:

$$\begin{aligned}\int_0^\infty \int_0^1 & \varepsilon\kappa\varphi_{xx} - \varepsilon L^2\kappa^3\varphi - L^2(\kappa - \frac{3\varepsilon}{2}\kappa^3)\varphi + L^2 V^\perp \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T L\varphi(t,0)\langle \nabla_p W(\gamma(t,0),\gamma(t,1)), \nu(t,0)\rangle \,\mathrm{d}t \\ &+ \int_0^T L\varphi(t,1)\langle \nabla_q W(\gamma(t,0),\gamma(t,1)), \nu(t,1)\rangle \,\mathrm{d}t \\ &+ \int_0^T \big(\varphi(t,0)V^\perp(t,0) + \varphi(t,1)V^\perp(t,1)\big)\, L\,\mathrm{d}t = 0\end{aligned} \tag{3.117}$$

Assuming additionally $\varphi(t,0) = \varphi(t,1) = 0$ for all $t \in [0,\infty)$ and integrating by parts in (3.117), then leads to:

$$\int_0^\infty \int_0^1 \Big(\frac{\varepsilon}{L^2}\kappa_{xx} + \frac{\varepsilon}{2}\kappa^3 - \kappa + V^\perp\Big) L^2\varphi \,\mathrm{d}x\,\mathrm{d}t = 0.$$

By the arbitrariness of φ, we see that (3.41) holds true.

2. step: We wish to investigate the equations governing the evolution of the endpoints. Let us now take $\varphi \in C_c^\infty([0,\infty); C^\infty(I))$, possibly nonzero at ∂I. Integrating by parts in (3.117) and using (3.41), we derive that:

$$\begin{aligned}\int_0^T & \varepsilon\kappa\varphi_x \,\mathrm{d}t - \varepsilon\kappa_x\varphi|_{t=0}^1 \\ &+ \int_0^T L\varphi(t,0)\langle \nabla_p W(\gamma(t,0),\gamma(t,1)), \nu(t,0)\rangle \,\mathrm{d}t \\ &+ \int_0^T L\varphi(t,1)\langle \nabla_q W(\gamma(t,0),\gamma(t,1)), \nu(t,1)\rangle \,\mathrm{d}t \\ &+ \int_0^T \big(\varphi(t,0)V^\perp(t,0) + \varphi(t,1)V^\perp(t,1)\big)\, L\,\mathrm{d}t = 0\end{aligned} \tag{3.118}$$

Choosing φ such that $\varphi(\cdot,0) = \varphi(\cdot,1) = \varphi_x(\cdot,1) = 0$ in (3.118) leads to:

$$\int_0^T \varepsilon\kappa(t,0)\varphi_x(t,0)\,\mathrm{d}t = 0,$$

and due to the arbitrariness of $\varphi_x(\cdot,0)$ and T to:

$$\kappa(t,0) = 0$$

for a.e. $t \in [0,\infty)$. In a similar fashion, we can derive the same natural boundary condition at $x = 1$ which leads to (3.44). Using the natural boundary conditions of (3.44) in (3.118) for φ additionally satisfying $\varphi(\cdot,1) = 0$, we see that:

$$\begin{aligned}\int_0^T & \varepsilon\kappa_x(t,0)\varphi(t,0)\,\mathrm{d}t \\ &+ \int_0^T \varphi(t,0)\langle \nabla_p W(\gamma(t,0),\gamma(t,1)), \nu(t,0)\rangle \,\mathrm{d}t \\ &+ \int_0^\infty V^\perp(t,0)\varphi(t,0)L\,\mathrm{d}t = 0.\end{aligned}$$

Hence, by the arbitrariness of $\varphi(\cdot,0)$ and T, we have for a.e. $t\in[0,\infty)$:

$$V^{\perp}(t,0) = -\langle \nabla_p W(\gamma(t,0),\gamma(t,1)), \nu(t,0)\rangle - \varepsilon \frac{1}{L}\kappa_x(t,0). \tag{3.119}$$

We continue by testing (3.111) with $\eta = \varphi\,\mu^i_x$, where (μ^i) is the approximating sequence from before and $\varphi \in C_c^\infty([0,\infty);C^\infty)$ with $\varphi(\cdot,1)\equiv 0$. Passing to the limit $i\to\infty$, as done previously, then leads to:

$$\begin{aligned} 0 &= \int_0^\infty \int_0^1 \left(\left(1+\frac{\varepsilon}{2}\kappa^2\right)\varphi_x + \varepsilon\kappa\kappa_x\varphi\right) L\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\infty \big(\langle \nabla_p W(\gamma(t,0),\gamma(t,0)),\tau(t,0)\rangle + V^{\top}(t,0)\big)\,\varphi(t,0)L\,\mathrm{d}t \\ &= \int_0^\infty \big(-1 + \langle \nabla_p W(\gamma(t,0),\gamma(t,0)),\tau(t,0)\rangle + V^{\top}(t,0)\big)\,\varphi(t,0)L\,\mathrm{d}t. \end{aligned}$$

By the arbitrariness of $\varphi(\cdot,0)$, we have:

$$V^{\top}(t,0) = 1 - \langle \nabla_p W(\gamma(t,0),\gamma(t,1)),\tau\rangle \tag{3.120}$$

for a.e. $t\in[0,\infty)$. We can show (3.42) by combining (3.119) and (3.120). The proof of (3.43) works similarly. Moreover, (3.45) follows directly from (3.110).

3. step: It is important to note at this point that even though γ exists for all times $t\in[0,\infty)$ and satisfies all the equations stated in Theorem 3.1, it is not guaranteed that $\gamma\subset\Omega$ for all $t\in[0,\infty)$ (the definition of $\mathcal{AC}$ does not enforce this). Let us define the function $d\colon[0,\infty)\to\mathbb{R}$ as:

$$d(t) \coloneqq \operatorname{dist}(\gamma(t,\cdot),\partial\Omega),$$

which is well defined due to the compactness of γ and $\partial\Omega$. Furthermore, as $\gamma(0,\cdot)=\gamma_0\subset\Omega$, we have that $d(0)>0$. Define:

$$T_0 \coloneqq \sup\{t\geq 0\colon d(t)>0\}.$$

By the $C_c^0C^1$ regularity of γ and the smoothness of $\partial\Omega$, we see that d is continuous, and therefore $T_0\in(0,\infty]$. Moreover, in the case $T_0<\infty$, we must have $d(T_0)=0$, and thus $\gamma(T_0,\cdot)\cap\partial\Omega\neq\emptyset$ which concludes the proof. □

Chapter 4

Generalized Ginzburg-Landau on a manifold

4.1 Preliminaries

In this chapter, it is assumed that the reader is familiar with basics of differential geometry (see also, e.g. [63] and [64]). Let us fix some notation: We denote by S a two-dimensional oriented compact (without boundary) and connected Riemannian manifold. The letter M will be reserved to a more general d-dimensional oriented Riemannian manifold (possibly with boundary). In both cases, g will be the metric tensor, and vol the corresponding volume form. Provided with local coordinates, we will denote by $\sqrt{|g|}$ the corresponding area factor. Furthermore, E will denote an r-dimensional vector bundle over M with its own metric tensor written as h. TM will be the tangent vector bundle on M, and T^*M the cotangent vector bundle both with their natural metrics induced by g. We will write the set of smooth sections of TM as $C^\infty(TM)$ and, more generally, the set of all smooth sections of E as $C^\infty(E)$. The interior product with $X \in C^\infty(TM)$ will be written as $X\llcorner$. Where no danger of confusion is present, we will denote the Levi-Civita connection on M as well as E by ∇, and we will denote by Γ the Christoffel tensor (in both cases, respectively). Furthermore, $\otimes$ will be the *tensor product*, $\wedge$ the *wedge product*, and $\star$ the *Hodge star*. A subset $U \subset M$ is called a domain if and only if it is open, connected, and $E|_U$ is trivial. In particular, any coordinate neighborhood U is a domain. In this case, let us write (φ, Ω), where $\Omega \subset \mathbb{R}^d$ and $\varphi\colon \Omega \to U$, for the chart on U. For the sake of simpler notation, we identify a function u on U with its coordinate representation $u \circ \varphi$ living on Ω. We call $\nabla^*\colon C^\infty(T^*M \otimes E) \to C^\infty(E)$ a formal adjoint to the covariant derivative ∇ if and only if the following integration by parts formula holds true for all $u \in C_c^\infty(E)$ and $v \in C_c^\infty(T^*M \otimes E)$:

$$\int_M \langle u, \nabla^* v\rangle \,\mathrm{vol} = \int_M \langle \nabla u, v\rangle \,\mathrm{vol}\,.$$

It is a classic result in differential geometry (see also, e.g. Proposition 10.1.30 and (10.1.8) in [55]) that such an adjoint exists, is unique, and has the following local representation inside a domain U:

$$\nabla^* = -\left[\sqrt{|g|}^{-1} \partial_{x^k}\left(\sqrt{|g|} g^{ki}\right) + g^{ki}\nabla_{\partial_{x^k}}\right] \partial_{x^i}\llcorner. \tag{4.1}$$

Note that (g^{ki}) denotes the inverse of the metric tensor. As was already done above, we use the Einstein summation convention where any index appearing twice is implicitly summed over. With regard to indices, we will use Latin letters such as k and i for indices running in $\{1, \ldots, d\}$ and Greek letters such as α and β for indices running in $\{1, \ldots, r\}$. In general, we denote by $F(M; \mathbb{S}^1)$ the set of all sections $u\colon M \to TM$ with $|u| = 1$ and the regularity given by F. So, for example, $W^{1,1}(M; \mathbb{S}^1)$ is the set of all $W^{1,1}$ sections $u\colon M \to TM$ with $|u| = 1$ at a.e. point. Norms will be usually written in short-hand notation; so, for instance $\|u\|_{W^{1,1}} := \|u\|_{W^{1,1}(TM)}$. We define $\bar{B}(x, r)$ to be the closed Euclidean ball centered at $x \in \mathbb{R}^2$ with radius r, and by $B(p, r)$ the geodesic ball centered at $p \in M$ with radius r. Finally, for any $0 < r_1 < r_2 < r^*$ (r^* will denote the injectivity radius of M) and $p \in M$ we denote by $A_{r_1,r_2}(p)$ the geodesic annulus given by:

$$A_{r_1,r_2}(p) := B_{r_2}(p) \setminus B_{r_1}(p).$$

4.1.1 Sections of bounded variation

In this subsection, we wish to introduce all the necessary results with regard to *sections of bounded variation* of a general vector bundle E over M; this means Borel regular sections u of E that have bounded variation (this will be defined in a moment). They can be seen as a natural generalization of vector-valued functions of bounded variation. Our main goal will be the correct intrinsic definition of the blow-up quantities of u as well as the statement and proof of the decomposition theorem for sections of bounded variation (see also Theorem 4.3).

Let us start by defining the total variation of an L^1-section:

Definition 4.1 (Total variation)
The total variation of a section $u \in L^1_{\text{loc}}(E)$ is defined as

$$\operatorname{var}(u) := \sup\left\{ \int_M \langle u, \nabla^* v\rangle \operatorname{vol}\colon v \in C^\infty(TM \otimes E),\ \|v\|_\infty \leq 1 \right\}. \tag{4.2}$$

For any open subset $U \subset M$, we define the total variation $\operatorname{var}(u, U)$ of u in U as

$$\operatorname{var}(u, U) := \sup\left\{ \int_U \langle u, \nabla^* v\rangle \operatorname{vol}\colon v \in C_c^\infty(T^*U \otimes E|_U),\ \|v\|_\infty \leq 1 \right\}. \tag{4.3}$$

The definition of sections of bounded variation then naturally follows:

Definition 4.2 (Sections of bounded variation)
A section $u \in L^1_{\text{loc}}(E)$ is of bounded variation if and only if $\operatorname{var}(u) < \infty$. The set of all sections of E with bounded variation will be denoted by $BV(E)$. This space is equipped with the following norm:

$$\|u\|_{BV} := \|u\|_{L^1} + \operatorname{var}(u), \text{ for all } u \in BV(E).$$

Using (4.1), a simple computation leads to the following local representation for the integral term in (4.2):

Lemma 4.1
Given a coordinate neighborhood U with coordinates $\{x^1, \ldots, x^d\}$ and orthonormal frame $\{e_1, \ldots, e_r\}$ (of $E|_U$), $u \in L^1_{\mathrm{loc}}(E|_U)$, and $v \in C^\infty(TU \otimes E|_U)$, we have:

$$\int_U \langle u, \nabla^* v\rangle \ \mathrm{vol} = -\int_\Omega u^\alpha \partial_{x^k}\left(\sqrt{|g|}g^{ki}v_i^\alpha\right) dx - \int_\Omega \Gamma^\beta_{k\alpha} u^\beta g^{ki} v_i^\alpha \ \mathrm{d}x. \tag{4.4}$$

Furthermore, there exist constants C_1, $C_2 \in (0, \infty)$ only depending on M such that:

$$C_1(\|u\|_{L^1(\Omega)} + \bar{\mathrm{var}}(u)) \le \mathrm{var}(u) \le C_2(\|u\|_{L^1(\Omega)} + \bar{\mathrm{var}}(u)), \tag{4.5}$$

where $\bar{\mathrm{var}}(u)$ denotes the Euclidean total variation of the coordinate representation of u. In particular, this shows that $u \in BV(E|_U)$ if and only if its coordinate representation is in $BV(\Omega; \mathbb{R}^r)$.

By (4.2) and an approximation argument, any $u \in BV(E)$ can be identified with a bounded linear functional T on $C(TM \otimes E)$ with its operator norm satisfying: $\|T\| = \mathrm{var}(u)$. In the next theorem, we will identify such a functional T with a generalized vector measure in $\mathcal{M}(E)$, which is defined as follows:

Definition 4.3 (Generalized vector measures)
Let $\mathcal{M}_+(M)$ be the set of finite positive Radon measures on M, and $\mathcal{B}(TM\otimes E)$ the set of Borel regular sections on $TM \otimes E$. We define the set $\mathcal{M}(E)$ of *generalized vector-valued measures* as:

$$\mathcal{M}(E) := \{(\sigma, \mu)\colon \mu \in \mathcal{M}_+(M),\ \sigma \in \mathcal{B}(TM \otimes E) \text{ with } |\sigma| = 1\ \mu\text{-a.e.}\}.$$

Given a generalized vector-valued measure $\nu = (\sigma, \mu) \in \mathcal{M}(E)$ we call μ the total variation of ν, and σ the polar density of ν. (This will be motivated in Theorem 4.1.)

Theorem 4.1 (Riesz's representation)
For any $u \in BV(E)$, there exist $\mu \in \mathcal{M}_+(M)$ and $\sigma_u \in \mathcal{B}(TM \otimes E)$ with $|\sigma_u| = 1$ at μ-a.e. point such that for any $v \in C^\infty(TM \otimes E)$:

$$\int_M \langle u, \nabla^* v\rangle \ \mathrm{vol} = \int_M \langle \sigma_u, v\rangle \, d\mu. \tag{4.6}$$

In particular, for any open subset $U \subset M$, we have that $\mu(U) = \mathrm{var}(u, U)$ with $\mathrm{var}(u, U)$, as defined in (4.3).

The measure μ and the section σ are unique in the following sense: Suppose that there exist another $\mu' \in \mathcal{M}_+(M)$ and $\sigma' \in \mathcal{B}(TM \otimes E)$ with $|\sigma'| = 1$ at μ-a.e. such that (4.6) is satisfied for any $v \in C^\infty(TM \otimes E)$. Then $\mu' = \mu$ in the sense of measures and $\sigma' = \sigma_u$ at μ-a.e. point in M.

Proof. In [41], the authors have proved (see also Theorem 2.5) the statement in the special case $E = M \times \mathbb{R}$. We will extend their result to the setting of a general vector bundle by closely following their reasoning.

1. step: We will first prove existence of the positive Radon measure μ in (4.6). Let us consider the linear functional $T\colon C^\infty(TM \otimes E) \to \mathbb{R}$, defined as:

$$T(v) := \int_M \langle u, \nabla^* v\rangle \ \mathrm{vol}\,.$$

From (4.2), we follow that T is a bounded functional on $C^\infty(TM \otimes E)$ equipped with the L^∞-norm, and that its operator norm satisfies $\|T\| = \operatorname{var}(u)$. Consequently, we can extend T by approximation to a bounded linear functional on $C(TM \otimes E)$, while leaving its norm unchanged. Let $\tilde{T} \colon C_+(M) \to \mathbb{R}$ be defined as:

$$\tilde{T}(f) \coloneqq \sup_{\|v\|_\infty \le 1} T(fv),$$

where the supremum above is over $v \in C(TM \otimes E)$. Furthermore, extend $\tilde{T}$ to the functional $\hat{T} \colon C(M) \to \mathbb{R}$ defined as

$$\hat{T}(f) = \tilde{T}(f^+) - \tilde{T}(f^-),$$

where f^+ and f^- are the positive and negative part of f, respectively. The following intuition lies behind our choice of $\hat{T}$: Suppose that there exist $\mu \in \mathcal{M}_+(M)$ and $\sigma_u \in \mathcal{B}(TM \otimes E)$ with $|\sigma_u| = 1$ at μ-a.e. point such that (4.6) is satisfied for any $v \in C^\infty(TM \otimes E)$. Then for any $f \in C^\infty_+(M)$, it holds that:

$$\tilde{T}(f) = \sup_{\|v\|_\infty \le 1} \int_M \langle \nabla^*(fv), u \rangle \operatorname{vol} = \sup_{\|v\|_\infty \le 1} \int_M f \langle v, \sigma_u \rangle \, d\mu = \int_M f \, d\mu.$$

By approximation, the above statement also holds true for any $f \in C_+(M)$. With the definition of $\hat{T}$, we therefore derive for any $f \in C(M)$:

$$\hat{T}(f) = \tilde{T}(f^+) - \tilde{T}(f^-) = \int_M f^+ \, d\mu - \int_M f^- \, d\mu = \int_M f \, d\mu.$$

Applying the classic Riesz representation theorem on the functional $\hat{T}$ would result in the existence of a desired measure μ. For its application, we need to check that $\hat{T}$ is a positive linear functional. Let $\lambda \ge 0$ and $f \in C_+(M)$, then:

$$\tilde{T}(\lambda f) = \sup_{\|v\|_\infty \le 1} T(\lambda f v) = \sup_{\|v\|_\infty \le 1} \lambda T(fv) = \lambda \tilde{T}(f).$$

Furthermore, for any $f, g \in C_+(M)$, we have that:

$$\tilde{T}(f+g) = \sup_{\|v\|_\infty \le 1} T((f+g)v) = \sup_{\|v\|_\infty \le 1} T(fv) + T(gv) \le \tilde{T}(f) + \tilde{T}(g).$$

It remains to prove the reverse inequality: Let v, $w \in C(TM \otimes E)$ with $\|v\|_\infty$, $\|w\|_\infty \le 1$. Let $\rho \in C(M)$ be a cutoff function with $\rho \equiv 1$ in $\{f + g > \varepsilon\}$ for some fixed $\varepsilon > 0$, and $\operatorname{spt}(\rho) \subset\subset \{f + g > 0\}$. Then by linearity of T:

$$\begin{aligned} T(fv) + T(gw) &= T(f(1-\rho)v) + T(f\rho v) + T(g(1-\rho)w) + T(g\rho w) \\ &\le 2\|T\|\varepsilon + T\left((f+g)\left(\frac{\rho f}{f+g} v + \frac{\rho g}{f+g} w\right)\right) \\ &\le 2\|T\|\varepsilon + \tilde{T}(f+g), \end{aligned}$$

where in the last estimate we employed:

$$\left\| \frac{\rho f}{f+g} v + \frac{\rho g}{f+g} w \right\|_\infty \le \frac{\rho f}{f+g} \|v\|_\infty + \frac{\rho g}{f+g} \|w\|_\infty \le 1.$$

Also note that the cutoff function ρ assures the continuity of $\frac{\rho f}{f+g}v + \frac{\rho g}{f+g}w$. By the arbitrariness of v, w, and $\varepsilon > 0$ this leads to the desired inequality. Hence, $\hat{T}$ is a positive linear functional on the space $C(TM \otimes E)$ and by the scalar version of Riesz representation Theorem there exists a positive bounded Radon measure μ on M such that for all $f \in C(M)$, it holds that:

$$\hat{T}(f) = \int_M f \, \mathrm{d}\mu.$$

2. step: We continue by proving the existence of σ_u in (4.6). Let $\{U_h\}_{h=1}^N$ be a finite cover of M with domains, and let $\{\rho_h\}_{h=1}^N$ be a subordinate partition of unity. For each $h \in \{1, \dots, N\}$ let $\{\varphi_i^h\}_{i=1}^{nr}$ be an orthonormal frame on $T^*U_h \otimes E|_{U_h}$. By Riesz representation for $\hat{T}$, we have for any $f \in C_c(U_h)$ that:

$$\begin{aligned} |T(f\rho_h\varphi_i^h)| &= |T(f^+\rho_h\varphi_i^h)| + |T(f^-\rho_h\varphi_i^h)| \\ &\leq \tilde{T}(f^+) + \tilde{T}(f^-) \\ &= \hat{T}(|f|) = \int_M |f| d\mu. \end{aligned}$$

Therefore, by approximation, the map $f \mapsto T(f\rho_h\varphi_i^h)$ is a linear bounded functional on $L^1(U_h, \mu)$. Consequently, we can find $\sigma_i^h \in L^\infty(U_h, \mu)$ with $|\sigma_i^h| \leq 1$ μ-a.e. and:

$$T(f\rho_h\varphi_i^h) = \int_{U_h} f\sigma_i^h d\mu.$$

Setting:

$$\sigma_u := \sum_{h=1}^N \sum_{i=1}^{nr} \rho_h \sigma_i^h \varphi_i^h.$$

we then have for any $v \in C^\infty(TM \otimes E)$ with coefficients given by $\{v_i^h\}_i$ in the local frame $\{\varphi_i^h\}_i$ of $TM \otimes E$ from before that:

$$\begin{aligned} T(v) &= \sum_{h=1}^N \sum_{i=1}^{nr} T(\rho_h v_i^h \varphi_i^h) \\ &= \int_M \sum_{h=1}^N \rho_h \sum_{i=1}^{nr} v_i^h \sigma_i^h \, d\mu \\ &= \int_M \sum_{h=1}^N \rho_h \langle v, \sigma_u \rangle \, d\mu = \int_M \langle v, \sigma_u \rangle \, d\mu, \end{aligned}$$

which shows (4.6). As $|\sigma_i^h| \leq 1$ μ-a.e., we see that:

$$|\sigma_u| \leq \sum_{h=1}^N \rho_h |\sigma_i^h| \leq \sum_{h=1}^N \rho_h = 1.$$

The reverse inequality can be proved as follows:

$$\mu(M) \geq \int_M |\sigma_u| \, d\mu = \sup_{\|v\|_\infty \leq 1} T(v) \geq \sup_{\|f\|_\infty \leq 1} \hat{T}(f) = \|\hat{T}\| = \mu(M).$$

By the definition of $\operatorname{var}(u, U)$, (4.6), and an approximation procedure, we see that $\mu(U) = \operatorname{var}(u, U)$ for any open subset $U \subset M$.

3. step: It remains to prove uniqueness. Let μ' and σ' be as in the statement of this theorem. Then, for any $v \in C(TM \otimes E)$:

$$\int_M \langle v, \sigma_u \rangle \, d\mu = \int_M \langle v, \sigma' \rangle \, d\mu'.$$

By the arbitrariness of v and an approximation argument:

$$\mu(B) = \int_B \langle \sigma_u, \sigma_u \rangle \, d\mu = \int_B \langle \sigma_u, \sigma' \rangle \, d\mu' \leq \mu'(B).$$

Note that we have used $|\sigma_u|, |\sigma'| \leq 1$ above. By symmetry, this leads to the uniqueness of μ. Finally, testing the above equation with $B = M$, implies that $\langle \sigma_u, \sigma' \rangle = 1$ at μ-a.e. point. □

Definition 4.4
We say that $\nu \in \mathcal{M}(E)$ is absolutely continuous with respect to $\mu \in \mathcal{M}_+(M)$ (shortly written as $\nu << \mu$) if and only if for every Borel set $A \subset M$ with $\mu(A) = 0$, it holds that $|\nu|(A)$, where $|\nu|$ is the total variation of ν.

Furthermore, ν is said to be singular with respect to μ (shortly written as $\nu \perp \mu$) if and only if there exists a Borel set $A \subset M$ satisfying $|\nu|(A) = \mu(M \setminus A) = 0$.

Theorem 4.2 (Lebesgue decomposition)
For any $\nu \in \mathcal{M}(TM \otimes E)$ and $\mu \in \mathcal{M}_+(M)$ there exists $\nu^a \in \mathcal{M}(T^*M \otimes E)$ with $\nu^a << \mu$, as well as $\nu^s \in \mathcal{M}(T^*M \otimes E)$ with $\nu^s \perp \mu$ such that $\nu = \nu^a + \nu^s$. Both ν^a and ν^s, are unique in the sense of measures. Furthermore, we can find a unique $\sigma^a \in L^1(TM \otimes E, \mu)$ such that $\nu^a = \sigma^a \mu$.

Proof. Let $|\nu|$ be the total variation of ν, and σ the polar density of ν. As $|\nu|$ is a scalar Radon measure, we can apply the classical Lebesgue decomposition theorem to the pair $|\nu|$ and μ. Consequently, there exist unique scalar measures $|\nu|^a$ and $|\nu|^s$ such that $|\nu|^a << \mu$, $|\nu|^s \perp \mu$ and $|\nu| = |\nu|^a + |\nu|^s$. We can also find $\tilde{\sigma}^a \in L^1(\mu)$ satisfying $|\nu|^a = \tilde{\sigma}^a \mu$. Let us define:

$$\nu^a := \sigma |\nu|^a, \quad \nu^s := \sigma |\nu|^s, \quad \sigma^a := \sigma \tilde{\sigma}^a.$$

As $|\nu|^a << \mu$ and $|\sigma| \leq 1$ μ-a.e., we see that $\nu^a << \mu$. Similarly, we can show that $\nu^s \perp \mu$. As $\tilde{\sigma}^a \in L^1(\mu)$ and $|\sigma| = 1$ at $|\nu|^a$-a.e. point, we follow that $\sigma^a \in L^1(TM \otimes E, \mu)$. Hence, ν^a, ν^s, and σ^a satisfy all the desired properties. The uniqueness of ν^a, ν^s, σ^a follows from the uniqueness of $|\nu|^a$, $|\nu|^s$, $\tilde{\sigma}^a$, and σ, respectively. □

Definition 4.5
For any $u \in BV(E)$, we will denote the unique generalized vector-valued measure provided by Theorem 4.1 as Du. Furthermore $|Du|$, will stand for the total variation of Du, and $\frac{Du}{|Du|}$ will be the polar density of Du. We write $D^a u$ and $D^s u$ for the absolutely continuous and the singular port of Du with respect to $|Du|$ (see also Theorem 4.2). Lastly, $\frac{D^a u}{|Du|}$ will stand for the density of $D^a u$ with respect to $|Du|$.

We wish to prove the decomposition theorem for sections of bounded variation. Given $u \in BV(E)$, it provides an explicit representation of $D^a u$, as well as the $(n-1)$-dimensional part of $D^s u$ through quantities defined by blow-ups at certain points $p \in M$. For the intrinsic definition of these *blow-up quantities*, we will need to compare – in an intrinsic fashion – a vector in E_p with another vector in E_q, where p, $q \in M$. This can be achieved – at least for points p, q close enough to each other – through *parallel transport* on M, which can be defined as follows: Given a smooth curve $\gamma\colon [0,1] \to M$, and vector $v_0 \in E_{\gamma(0)}$, there exists a unique family $\{P_t\}_{t\in[0,1]}$ of linear isomorphisms:

$$P_t = P_t^{(\gamma)}\colon E_{\gamma(0)} \to E_{\gamma(t)}$$

such that $v(t) := P_t(v_0)$ satisfies:

$$\begin{cases} \nabla_{\frac{d}{dt}} v(t) = 0, & t \in (0,1), \\ \quad v(0) = v_0. \end{cases}$$

Given a local orthonormal frame in a neighborhood of $\gamma(0)$, we can identify P_t (for $t > 0$ small enough) with a matrix (still denoted by P_t) whose Taylor expansion at $t = 0$ is given by:

$$P_t = Id - t\Gamma_0 + O(t^2), \quad \Gamma_0 := (\dot\gamma(0) \llcorner \Gamma)|_{\gamma(0)}. \tag{4.7}$$

For any $p_0 \in M$ and $r < r^*$, we can then define the smooth map $T\colon E|_{B_r(p_0)} \to E_{p_0}$ as:

$$T(v) := P_1^{(\gamma_p)}(v), \quad v \in E_p,$$

where $\gamma_p\colon [0,1] \to M$ is the geodesic starting at p and ending at p_0 with constant speed equal to the geodesic distance $\mathrm{dist}(p_0, p)$ between p_0 and p.

Definition 4.6 (Blow-up quantities)
Given $u \in L^1(E)$, a point $p \in M$ is called an *approximate continuity point* of u if and only if:

$$\lim_{r\to 0} ⨍_{B_r(p)} |T(u(q)) - u(p)| \ \mathrm{vol}(q) = 0.$$

The set of approximate continuity points of u will be denoted by S_u.

A point $p \in M$ is called an *approximate jump point* of u if and only if there exist $u^+ = u^+(p)$, $u^- = u^-(p) \in E_p$ with $u^+ \neq u^-$, and a unit vector $\nu = \nu(p) \in T_pM$ such that:

$$\lim_{r\to 0} ⨍_{B_r^+(p,\nu)} |T(u(q)) - u^+| \ \mathrm{vol}(q) = 0,$$
$$\lim_{r\to 0} ⨍_{B_r^-(p,\nu)} |T(u(q)) - u^-| \ \mathrm{vol}(q) = 0,$$

where:

$$B_r^+(p,\nu) := \exp_p(\{X \in T_pM\colon |X| \le 1,\ \langle X, \nu\rangle \ge 0)\},$$
$$B_r^-(p,\nu) := \exp_p(\{X \in T_pM\colon |X| \le 1,\ \langle X, \nu\rangle \le 0)\}$$

– $\exp_p$ being the exponential map at p – are geodesic half-balls. The set of approximate jump points of u will be denoted by J_u.

Lastly, $p \in M$ is called an *approximate differentiability point* of u if there exists $\nabla u = \nabla u(p) \in T_p^* M \otimes E_p$ such that:

$$\lim_{r \to 0} \fint_{B_r(p)} r^{-1} |T(u(q)) - u(p) - \nabla u(\exp_p^{-1}(q))| \operatorname{vol}(q) = 0.$$

Proposition 4.1 (Blow-up and coordinates)
Let $u \in L^1_{\mathrm{loc}}(E|_U)$, where U is a domain with coordinate chart (φ, Ω) and a fixed orthonormal frame. Then a point $p_0 \in U$ is an approximate continuity (jump, differentiability) point of u if and only if $x_0 := \varphi^{-1}(p_0)$ is an approximate continuity (jump, differentiability) in the Euclidean sense of the coordinate representation of u.

Additionally, if p_0 is an approximate jump point of u, we have:

$$(u^\pm)^k = (u^k)^\pm, \quad \nu^k = |\mu|_g^{-1} \mu^k \text{ with } \mu^k := g^{ki} \bar{\nu}^i, \tag{4.8}$$

where ν is the approximate normal of u at p_0 and $\bar{\nu}$ the approximate normal of the coordinate representation $\bar{u} := u \colon \varphi$ of u. If p_0 is an approximate differentiability point of u, it holds that:

$$(\nabla u)_k^\alpha = \partial_{x^k} u^\alpha + \Gamma_{k\beta}^\alpha u^\beta. \tag{4.9}$$

Proof. Let $\lambda, \Lambda \in \mathbb{R}$ such that:

$$\bar{B}_{\lambda r} \subset \varphi^{-1}(B_r(p)) \subset \bar{B}_{\Lambda r}, \tag{4.10}$$

for all $r > 0$ small enough. Furthermore, let $v \colon \Omega \to \mathbb{R}^r$ be defined as $v(x) := T(u(\varphi(x)))$ and $w \colon \Omega \to \mathbb{R}^n$ denote the map $w(x) := \exp_p^{-1}(\varphi(x))$. From (4.7), we have the following Taylor expansion of v at 0:

$$v(x) = u(x) + \Gamma_{i\alpha}^\beta(0)\, w^i(x) + O(|x|^2) = u(x) + O(|x|). \tag{4.11}$$

1. step: Let us assume that p is an approximate continuity point for u. By the smoothness of g and (4.11), we can find a universal constant C such that:

$$\begin{aligned}
&\fint_{B_r(p)} |T(u(q)) - u(p)| \operatorname{vol}(q) \\
&\quad = \left(\int_{\varphi^{-1}(B_r(p))} \sqrt{|g|}\, \mathrm{d}x \right)^{-1} \int_{\varphi^{-1}(B_r(p))} |v(x) - u(0)| \sqrt{|g|}\, \mathrm{d}x \\
&\quad \geq \frac{\sqrt{|g(0)|} - Cr}{\sqrt{|g(0)|} + Cr} \fint_{\varphi^{-1}(B_r(p))} |u(x) - u(0)|\, \mathrm{d}x - Cr.
\end{aligned}$$

With (4.10) and the definition of approximate continuity on M, we follow:

$$\begin{aligned}
\fint_{B_{\lambda r}(0)} |u(x) - u(0)|\, \mathrm{d}x &= \frac{\Lambda^n}{\lambda^n} \frac{1}{|\bar{B}_{\Lambda r}|} \int_{\bar{B}_{\lambda r}} |u(x) - u(0)|\, \mathrm{d}x \\
&\leq C \fint_{\varphi^{-1}(B_r(p))} |u(x) - u(0)|\, \mathrm{d}x \to 0 \text{ as } r \to 0.
\end{aligned}$$

As for any sequence $(r_h)_h$ converging to 0, we also have that $(\frac{1}{\lambda} r_h)_h$ is convergent to 0 and (vice versa) it follows that 0 is an approximate continuity point of u. The reverse direction can be proved in a similar fashion.

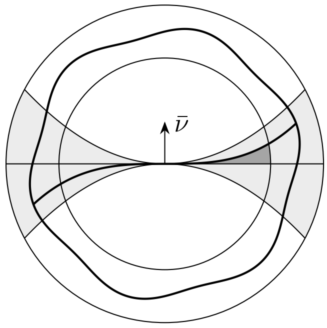


Figure 4.1: Geodesic half-ball in coordinates. The light gray area represents $C^{\alpha}_{\Lambda r}$ while the dark gray area corresponds to $\bar{H}^{+}_{\lambda r}\Delta H^{+}_{r}$.

2. step: Let p be an approximate jump point of u. As before we can prove that:

$$\lim_{r\to 0} \fint_{\varphi^{-1}(B_r^{\pm}(p,\nu_u(p)))} |u(x) - \varphi^* u^{\pm}(p)| \,\mathrm{d}x = 0. \tag{4.12}$$

To show that 0 is a jump point of u we, first and foremost, need to choose the appropriate Euclidean normal $\bar{\nu}$. In this regard, consider the geodesic disk:

$$\begin{aligned} D_r(p, \nu_u(p)) &:= \exp_p(\{X \in T_pM : |X| \le 1,\ \langle X, \nu\rangle = 0)\} \\ &= B_r^{+}(p, \nu(p)) \cap B_r^{-}(p, \nu(p)). \end{aligned}$$

Its preimage $\varphi^{-1}(D_r(p, \nu_u(p)))$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ and we choose $\bar{\nu}$ as its Euclidean normal at the origin. Using the fact that $\varphi^*\nu_u(p)$ is g-normal to $\varphi^{-1}(D_r(p, \nu_u(p)))$, we can derive the following relation between $\varphi^*\nu_u(p)$ and $\bar{\nu}$:

$$\varphi^*\nu_u(p) = \|\mu\|_g^{-1}\mu, \text{ with } \mu^k := g^{ki}(0)\bar{\nu}^i. \tag{4.13}$$

For the sake of shorter notation, we will write for any admissible r, $\alpha > 0$ (with r small enough):

$$\bar{H}_r^{\pm} := \bar{B}_r^{\pm}(0, \bar{\nu}), \qquad H_r^{\pm} := \varphi^{-1}(B_r^{\pm}(p, \nu_u(p))),$$
$$C_r^{\alpha} := \left\{ x \in B_r : \langle X, \bar{\nu}\rangle^2 \le \frac{\alpha}{1+\alpha}|x|^2 \right\}.$$

As $\varphi^{-1}(D_r(p, \nu_u(p)))$ and the hyperplane orthogonal to $\bar{\nu}$ have first-order contact, we can find $\alpha > 0$ such that for $r > 0$ small enough (see also Figure 4.1) such that $\bar{H}^{+}_{\lambda r}\Delta H^{+}_{r} \subset C^{\alpha}_{\Lambda r}$ and therefore:

$$|\bar{H}^{+}_{\lambda r}\Delta H^{+}_{r}| \le |C^{\alpha}_{\Lambda r}| \le C(\alpha, \Lambda) r^{n+1}.$$

Consequently, by (4.10) and (4.12), we can estimate:

$$\begin{aligned}
\fint_{\bar H^+_{\lambda r}} |u(x) - \varphi^* u^+(p)| \,\mathrm{d}x &= \frac{2\Lambda^n}{\lambda^n} \frac{1}{|\bar B^+_{\Lambda r}(0)|} \int_{\bar H^+_{\lambda r}} |u(x) - \varphi^* u^+(p)| \,\mathrm{d}x \\
&\leq \frac{2\Lambda^n}{\lambda^n} \fint_{H^+_r} |u(x) - \varphi^* u^+(p)| \,\mathrm{d}x \\
&\quad + \frac{2\Lambda^n}{\lambda^n} \frac{1}{|\bar B^+_{\Lambda r}(0)|} \int_{H^-_r \cap \bar H^+_{\lambda r}} |u(x) - \varphi^* u^+(p)| \,\mathrm{d}x \\
&\leq C \fint_{H^+_r} |u(x) - \varphi^* u^+(p)| \,\mathrm{d}x \\
&\quad + \fint_{H^-_r} |u(x) - \varphi^* u^-(p)| \,\mathrm{d}x \\
&\quad + C|\varphi^* u^+(p) - \varphi^* u^-(p)| \frac{|C^\alpha_{\Lambda r}|}{|B^+_{\Lambda r}|} \overset{r\to 0}{\to} 0.
\end{aligned}$$

In a similar fashion, we can also show:

$$\lim_{r\to 0} \fint_{\bar H^-_{\lambda r}} |u(x) - \varphi^* u^-(p)| \,\mathrm{d}x = 0.$$

It follows that 0 is an approximate jump point of u. Furthermore, (4.13) leads directly to (4.8). The reverse direction follows similarly.

3. step: Let us assume that p is an approximate differentiability point. We will first show that p must also be an approximate continuity point: By the definition of approximate differentiability, we have:

$$\begin{aligned}
&\fint_{B_r(p)} |u(q) - T(q, u(p))| \operatorname{vol}(q) \\
&\quad \leq r \fint_{B_r(p)} r^{-1} |u(q) - T(q, u(p)) - \nabla u(p)(\exp_p^{-1}(q))| \operatorname{vol}(q) \\
&\qquad + r \fint_{B_r(p)} |\nabla u(p)| \operatorname{vol}(q) \overset{r\to 0}{\to} 0
\end{aligned}$$

which implies that 0 is an approximate continuity point of u. In particular, it makes sense to evaluate u at 0 (where we may need to redefine $u(0)$ to be equal to the approximate limit). Passing to coordinates, we can derive in the same fashion as was done before:

$$\lim_{r\to 0} \fint_{\varphi^{-1}(B_r(p))} r^{-1} |u(x) - T(x, u(0)) - \operatorname{dist}_g(x,0) L(\dot\gamma^x(0))| \,\mathrm{d}x = 0, \tag{4.14}$$

where $\operatorname{dist}_g(x,0)$ is the geodesic distance between x and the origin, L is the coordinate representation of $\nabla u(p)$, and γ^x denotes the coordinate representation of the unit speed geodesic starting at p and ending in $\varphi(x)$ ($\gamma^x(0) = 0$, $\gamma^x(\operatorname{dist}_g(x,0)) = x$). By the definition of γ^x and the smoothness of the map $(s,x) \mapsto \gamma^x(s)$, we have:

$$x = \gamma^x(\operatorname{dist}_g(x,0)) = \operatorname{dist}_g(x,0)\dot\gamma^x(0) + O(|x|^2).$$

Therefore, using (4.11) we can write:

$$\begin{aligned}
&u(x) - T(x, u(0)) - \operatorname{dist}_g(x, 0)L(\dot\gamma^x(0)) \\
&\quad = u(x) - u(0) + \operatorname{dist}_g(x, 0)\Gamma^\alpha_{i\beta}(0)u^\beta(0)\dot\gamma^x_i(0)e_\alpha + O(|x|^2) \\
&\qquad - \operatorname{dist}_g(x, 0)L(\dot\gamma^x(0)) \\
&\quad = u(x) - u(0) - \operatorname{dist}_g(x, 0)\bar L(\dot\gamma^x(0)) + O(|x|^2) \\
&\quad = u(x) - u(0) - \bar L(x) + O(|x|^2),
\end{aligned}$$

where:

$$\bar L = L - (\Gamma^\alpha_{i\beta}(0)u^\beta(0))_{i\alpha}. \tag{4.15}$$

By (4.10), it follows:

$$\begin{aligned}
&\fint_{\bar B_{\lambda r}} r^{-1}|u(x) - u(0) - \bar L(x)| \,\mathrm{d}x \\
&\quad \le \frac{\Lambda^n}{\lambda^{n+1}} \frac{1}{|\bar B_{\Lambda r}|} \int_{\bar B_{\lambda r}} r^{-1}|u(x) - T(x, u(0)) - \operatorname{dist}_g(x, 0)\bar L(\dot\gamma^x(0))| \,\mathrm{d}x + Cr \\
&\quad \le C \fint_{\varphi^{-1}(B_r(p))} r^{-1}|u(x) - T(u(x), u(0)) - \operatorname{dist}_g(x, 0)L(\dot\gamma^x(0))| \,\mathrm{d}x + Cr.
\end{aligned}$$

With (4.14), we see that u is approximately differentiable at 0 with approximate gradient given by $\bar L$. Moreover, (4.9) is satisfied by (4.15). The reverse direction can be shown in a similar fashion. □

Lemma 4.2
For differential forms $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$ as well as an $(n-1)$-dimensional oriented embedded C^1 submanifold N of M with unit normal denoted by ν, it holds that:

$$(\alpha \wedge \star\beta)|_N = \alpha|_N \wedge \star_N(\nu \llcorner \beta)|_N, \tag{4.16}$$

where $\star$ is the Hodge star operator on M, while $\star_N$ denotes the corresponding Hodge star on N, induced by the restriction of g onto N.

Proof. For a proof, we refer, for example, to Proposition 4.1.54 in [55]. □

Lemma 4.3 (Integration on submanifolds in coordinates)
Let N be an $(n-1)$-dimensional oriented embedded C^1-submanifold of M and U a coordinate neighborhood with orientation preserving chart φ. Then for any integrable $f\colon N \to \mathbb{R}$, it holds that:

$$\int_{N\cap U} f \operatorname{vol}_N = \int_{\varphi^{-1}(N\cap U)} f\|\mu\|_g\sqrt{|g|}\,\mathrm{d}x,$$

where $\mu^k := g^{ki}\bar\nu^i$ and $\bar\nu$ is the upper normal onto $\varphi^{-1}(N \cap U)$.

Proof. We will shortly write $\bar{N} := \varphi^{-1}(N \cap U)$. Using (4.16) with $\alpha = f \in \Omega^0(M)$ and $\beta = \alpha^\nu := \langle \nu, \cdot \rangle \in \Omega^1(M)$ and a change of coordinates, leads to:

$$\begin{aligned}\int_{N\cap U} f \,\mathrm{vol}_N &= \int_{N\cap U} f \wedge \star_N(\alpha^\nu(\nu)) \\ &= \int_{N\cap U} f \wedge \star_N(\nu \llcorner \alpha^\nu) \\ &= \int_{N\cap U} \star(f\alpha^\nu) = \int_{\bar{N}} \varphi^*(\star f\alpha^\nu).\end{aligned}$$

Note that by the definition of $\star$ for any differential form α, it holds that:

$$(\star\alpha) \wedge \alpha = \mathrm{vol}\,.$$

Now we take the pullback with respect to φ on both sides of the above equation:

$$\varphi^*(\star\alpha) \wedge \varphi^*\alpha = \sqrt{|g|}\mathcal{L}^n.$$

Hence:

$$\varphi^*(\star\alpha) = \sqrt{|g|}\,\bar{\star}(\varphi^*\alpha),$$

where $\bar{\star}$ is the Euclidean Hodge star operator. Substituting $\alpha = f\alpha^\nu$, and employing (4.16) in the Euclidean setting leads to:

$$\int_{N\cap U} f \,\mathrm{vol}_N = \int_{\bar{N}} f\sqrt{|g|}\bar{\star}\alpha^\nu_i \,\mathrm{d}x^i = \int_{\bar{N}} f\sqrt{|g|}(\alpha^\nu_i \bar{\nu}^i)^{-1}\,\mathrm{d}x^i,$$

where $\bar{\nu}$ is the Euclidean normal onto $\bar{N}$ having the same orientation ν. Furthermore, by the definition of α^ν it holds that $\alpha^\nu_i\bar{\nu}^i = \langle \varphi^*\nu, \bar{\nu}\rangle_g$. The following relation holds true between $\varphi^*\nu$ and $\bar{\nu}$:

$$\varphi^*\nu = \|\mu\|_g^{-1}\mu, \quad \mu^k := g^{ki}\bar{\nu}^i.$$

In conclusion:

$$\langle \nu, \bar{\nu}\rangle_g = \|\mu\|_g^{-1} g_{kl} g^{ki}\bar{\nu}^i\bar{\nu}^l = \|\mu\|_g^{-1},$$

which leads to the desired result. □

Definition 4.7 ($\mathcal{H}^1$-rectifiable sets)
A set $J \subset M$ is called $\mathcal{H}^1$-rectifiable if and only if there exists a countable family $\{N_i\}$ of compact embedded C^1 manifolds with or without boundary such that:

$$\mathcal{H}^1(J \setminus \bigcup_{i=1}^N) = 0.$$

Let $u \in BV(E)$; applying the Lebesgue decomposition theorem (see also Theorem 4.2) leads to:

$$Du = D^a u + D^s u,$$

where Du^a denotes the absolutely continuous part of Du with respect to $|Du|$, and Du^s the singular part of Du with respect to $|Du|$. The following theorem provides a more explicit version of the above decomposition, employing the blow-up quantities introduced in Definition 4.6.

Theorem 4.3 (Decomposition in $BV(E)$)
Let $u \in BV(E)$, then u is approximately differentiable at almost every point, and the approximate gradient ∇u is the density of $D^a u$ with respect to vol*:*

$$D^a u = \nabla u \,\mathrm{vol}\,. \tag{4.17}$$

The set J_u of approximate jump points is $(n-1)$-rectifiable. Furthermore, it is $\mathcal{H}^1$-a.e. contained in S_u; this means $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, and $D^j u$, which is the singular part $D^s u$ restricted to S_u, can be written as:

$$D^j u := D^s u|_{S_u} = \nu_u \otimes (u^+ - u^-)\mathcal{H}^{n-1}|_{J_u}, \tag{4.18}$$

where we implicitly identify ν_u with $\alpha^{\nu_u} := \langle \nu_u, \cdot \rangle$. Note that we will usually write this decomposition as:

$$Du = \nabla u \,\mathrm{vol} + \nu_u \otimes (u^+ - u^-)\mathcal{H}^{n-1}|_{J_u} + D^c u, \tag{4.19}$$

$D^c u = D^s u|_{M \setminus S_u}$ denoting the Cantor part of u.

Proof. It is enough to prove a localized version of the theorem in a coordinate neighborhood U with an orientation preserving chart (φ, Ω). The general result then follows by a partition of unity argument.

As the coordinate representation of u (which will be identified with u) is a BV regular vector field in the Euclidean sense, we can employ the classic decomposition theorem (see also (3.89) in [11]). This implies that the coordinate representation of u is approximately differentiable at a.e. point in Ω, the Euclidean approximate jump set $\bar{J}_u$ is $(n-1)$-rectifiable with $\mathcal{H}^{n-1}(\bar{S}_u \setminus \bar{J}_u) = 0$ ($\bar{S}_u$ denotes the Euclidean singular set), and:

$$\bar{D}u = \bar{\nabla} u + (u^+ - u^-) \otimes \bar{\nu}_u + \bar{D}^c u. \tag{4.20}$$

With Proposition 4.1, it follows that u is approximately differentiable at a.e. in U, $J_u \cap U$ is $\mathcal{H}^1$-rectifiable, and $\mathcal{H}^1((S_u \setminus J_u) \cap U) = 0$.

Fix a general $v \in C_c^\infty(T^*U \otimes E|_U)$. On the one hand, (4.4) and the Euclidean Lebesgue decomposition, we derive that:

$$\begin{aligned}\int_U \langle u, \nabla^* v \rangle \,\mathrm{vol} = &\int_{\varphi^{-1}(U)} g^{ki} v_i^\alpha \sqrt{|g|}\,\mathrm{d}\bar{D}_k^a u^\alpha - \int_{\varphi^{-1}(U)} \Gamma_{k\alpha}^\beta u^\beta g^{ki} v_i^\alpha \sqrt{|g|}\,\mathrm{d}x \\ &+ \int_{\varphi^{-1}(U)} g^{ki} v_i^\alpha \sqrt{|g|}\,\mathrm{d}\bar{D}_k^s u^\alpha,\end{aligned}$$

where

$$\begin{gathered} g^{ki}\sqrt{|g|}\bar{D}_k^a u^\alpha - \Gamma_{k\alpha}^\beta u^\beta g^{ki}\sqrt{|g|}\mathcal{L}^n << \mathcal{L}^n, \\ g^{ki}\sqrt{|g|}\bar{D}_k^s u^\alpha \perp \mathcal{L}^n. \end{gathered}$$

On the other hand, by the Lebesgue decomposition on M, we can also write:

$$\int_U \langle u, \nabla^* v \rangle \,\mathrm{vol} = \int_{\varphi^{-1}(U)} v_i^\alpha \,\mathrm{d}(\varphi^{-1})_\# D_i^a u^\alpha + \int_{\varphi^{-1}(U)} v_i^\alpha \,\mathrm{d}(\varphi^{-1})_\# D_i^s u^\alpha,$$

where $(\varphi^{-1})_\#$ is the pushforward with respect to φ^{-1}. With the uniqueness of the Lebesgue decomposition, we then derive that:

$$\int_U v \, \mathrm{d}D^a u = \int_{\varphi^{-1}(U)} g^{ki} v_i^\alpha \sqrt{|g|} \, \mathrm{d}\bar{D}_k^a u^\alpha - \int_{\varphi^{-1}(U)} \Gamma_{k\alpha}^\beta u^\beta g^{ki} v_i^\alpha \sqrt{|g|} \, \mathrm{d}x, \quad (4.21)$$

$$\int_U v \, \mathrm{d}D^j u = \int_{\varphi^{-1}(U)} g^{ki} v_i^\alpha \sqrt{|g|} \, \mathrm{d}\bar{D}_k^j u^\alpha. \quad (4.22)$$

By (4.21), (4.20), and (4.9) as well as the relation $\Gamma_{k\alpha}^\beta = -\Gamma_{k\beta}^\alpha$, we conclude that:

$$\int_U v \, \mathrm{d}D^a u = \int_{\varphi^{-1}(U)} g^{ki} \left(\partial_{x^k} u^\alpha - \Gamma_{k\alpha}^\beta u^\beta \right) v_i^\alpha \sqrt{|g|} \, \mathrm{d}x = \int_U \langle \nabla u, v \rangle \, \mathrm{vol}$$

which shows by the arbitrariness of v that (4.17) is satisfied in U. Note that the following identity was employed:

$$\begin{aligned} \langle \nabla u, v \rangle &= \langle \nabla_{\partial_{x^k}} u^\alpha dx^k \otimes e_\alpha, v_i^\beta dx^i \otimes e_\beta \rangle \\ &= \nabla_{\partial_{x^k}} u^\alpha v_i^\beta \langle dx^k, dx^i \rangle \langle e_\alpha, e_\beta \rangle \\ &= g^{ki} \nabla_{\partial_{x^k}} u^\alpha v_i^\alpha. \end{aligned}$$

As $\bar{J}_u \cap U$ is $(n-1)$-rectifiable, we can assume without loss of generality that it is,in fact, a C^1 submanifold of $\varphi^{-1}(U)$ whose normal we will denoted by $\bar{\nu}_u$. (The precise argument here follows by approximation of $\bar{J}u$ with finite unions of compact C^1 manifolds.) The following relation holds true between $\varphi^*\nu_u$ and $\bar{\nu}_u$:

$$\varphi^*\nu_u = \|\mu\|_g^{-1} \mu, \quad \mu^i = g^{ki} \bar{\nu}_u^k.$$

We see that:

$$\begin{aligned} \langle \alpha^\nu \otimes (u^+ - u^-), v \rangle &= \langle (u^+ - u^-)^\alpha e_\alpha, v_i^\beta \mu^i e_\beta \rangle \\ &= g^{ki} (u^+ - u^-)^\alpha v_i^\alpha \bar{\nu}_u^k \|\mu\|_g^{-1}, \end{aligned}$$

where $\alpha^\nu := \langle \nu, \cdot \rangle$. Consequently, by Lemma 4.3, (4.20), and (4.8), we have:

$$\begin{aligned} \int_U v \, \mathrm{d}D^j u &= \int_{\bar{J}_u} g^{ki} (u^+ - u^-)^\alpha \bar{\nu}_u^k v_i^\alpha \|\mu\|_g^{-1} \|\mu\|_g \sqrt{|g|} \, \mathrm{d}\bar{\mathcal{H}}^{n-1} \\ &= \int_{J_u \cap U} \langle \alpha^\nu \otimes (u^+ - u^-), v \rangle, \end{aligned}$$

where $\bar{\mathcal{H}}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. By the arbitrariness of v, this shows (4.18) locally in U. □

We will now discuss two last important theorems: the *chain rule* as well as a *compactness theorem* concerning the space of *special sections of bounded variation* (shortly written as *SBV*).

Theorem 4.4 (Chain rule in *BV*)
Let $G \colon E \to E$ be a smooth bundle map ($\pi_E \circ G = \pi_E$, where π_E is the projection onto M) and $u \in BV(E)$. Then $v := G \circ u \in BV(E)$ and in any local coordinates $(x^1, \ldots, x^d, a^1, \ldots, a^r)$ of $E|_U$ for a domain $U \subset M$:

$$\nabla_{\partial_{x^i}} v^\alpha = \frac{\partial}{\partial a^\beta} G^\alpha(u) \partial_{x^k} u^\beta + \Gamma_{i\beta}^\alpha v^\beta + \partial_{x^i} G^\alpha(u). \quad (4.23)$$

Here $\{a^\beta\}_{\beta=1}^r$ Furthermore, $J_v \subset J_u$ with:

$$D^s v|_{J_v} = \nu_u \otimes (G(u^+) - G(u^-))\, \mathcal{H}^{n-1}|_{J_u}. \tag{4.24}$$

Definition 4.8 (Special sections of bounded variation)
We define the set $SBV(E)$ of special sections of bounded variation as:

$$SBV(E) := \{u \in BV(E) \colon D^c u = 0\},$$

as well as for any $p \in (1, \infty)$ the space:

$$SBV^p(E) := \{u \in SBV(E) \colon \nabla u \in L^p,\ \mathcal{H}^{n-1}(J_u) < \infty\}.$$

We say that a sequence $\{u_h\} \subset SBV^p(E)$ if and only if

$$u_h \to u \text{ in } L^p(E),$$
$$\nabla u_h \rightharpoonup \nabla u \text{ in } L^p(T^*M \otimes E), \quad D^j u_h \overset{*}{\rightharpoonup} D^j u \text{ in } \mathcal{M}(E).$$

Theorem 4.5 (Compactness in SBV)
Let $p \in (1, \infty)$ and $\{u_h\} \subset SBV^p(E)$ satisfying the following bound:

$$\sup_h \left(\|u_h\|_\infty + \|\nabla u_h\|_p + \mathcal{H}^{n-1}(J_{u_h}) \right) < \infty. \tag{4.25}$$

Then, up to a subsequence, we have that $u_h \rightharpoonup u$ weakly in $SBV^p(E)$.

Both theorems follow from their Euclidean counterparts and a standard partition of unity argument.

4.1.2 Vortices on surfaces

From this point on, $\mathfrak{m} \in \mathbb{N}_+$ will denote a positive natural number. Given a unit vector $e \in T_pM$ for some $p \in M$, we identify any vector $X \in T_pM$ with the complex number:

$$z = z(X) := \langle X, e\rangle + \langle X, e^\perp\rangle i. \tag{4.26}$$

Under this identification, we have $|X| = |z|$, where $|X|$ is the length of X measured with $\langle \cdot, \cdot \rangle_g(p)$, and $|z|$ is the absolute value of $z \in \mathbb{C}$. Similarly, we will identify any complex number z with the vector:

$$X = X(z) := \Re(z)e + \Im(z)e^\perp.$$

Note that both maps *depend* on the choice of unit vector e. Let us now consider a domain $U \subset M$ such that there exists $e \in C^\infty(U; \mathbb{S}^1)$. Then we can define the map $P_e \colon TU \to TU$ as:

$$P_e(v) := \Re(z(v)^{\mathfrak{m}})e + \Im(z(v)^{\mathfrak{m}})e^\perp, \tag{4.27}$$

with $z(v)$ as in (4.26).

In the following, we wish to introduce a notion of degree on two-dimensional compact oriented Riemannian manifolds (see also [21] and [22]). First note that the volume form induces a unique map $(\cdot)^\perp \colon TS \to TS$ that is characterized by the following properties:

$$(v^\perp)^\perp = -v, \quad \langle v^\perp, w\rangle = -\langle v, w^\perp\rangle \text{ for all } v, w \in TS.$$

Intuitively speaking, it is the map that rotates each vector $v \in T_pM$, for $p \in M$, by $\frac{\pi}{2}$ in positive direction (which is induced by the orientation of M). As in the flat setting, we first define the *pre-Jacobian*: Given $u \in C^\infty(TS)$, the pre-Jacobian $\operatorname{jac}(u)$ is a 1-form on M defined as

$$\operatorname{jac}(u)(X) \coloneqq \langle \nabla_X u, u^\perp\rangle \qquad \text{for any } X \in C^\infty(TS).$$

In the flat setting, we saw that for any simply connected set $\Omega \subset \mathbb{R}^2$ such that $|u| = 1$ on $\partial\Omega$, it holds that:

$$\int_{\partial\Omega} \operatorname{jac}(u)(\tau)\, \mathrm{d}\mathcal{H}^1 \in 2\pi\mathbb{Z},$$

where τ is the unit tangent vector-field pointing in anticlockwise direction. In the case of a manifold, this is not true in general. In fact, given a simply connected domain $U \subset M$, we can parallelly transport a unit vector $u_0 \in T_{p_0}M$ with $p_0 \in \partial U$ along ∂U such that the integral above satisfies:

$$\int_{\partial U} \operatorname{jac}(u) = -\int_U \kappa \operatorname{vol}.$$

This motivates the following definition of *degree* of a map $u \in C^\infty(TS)$ around ∂U, where U is a simply connected domain, and $|u| = 1$ on ∂U:

$$\deg(u, \partial U) \coloneqq \frac{1}{2\pi}\left(\int_{\partial U} \operatorname{jac}(u) + \int_U \kappa \operatorname{vol}\right). \tag{4.28}$$

In fact, it is a classic result (see also [34]) that the degree defined above is valued in $\mathbb{Z}$. As boundary integrals can be problematic once generalized to less regular settings such as Sobolev spaces, we wish to express the degree in (4.28) through a single integral of a 2-form. This can be achieved by an application of Stokes' theorem, which allows us to rewrite $\deg(u, \partial B)$ as follows:

$$\deg(u, \partial U) = \frac{1}{2\pi}\int_U \operatorname{djac}(u) + \kappa \operatorname{vol}. \tag{4.29}$$

This motivates the definition of the vorticity 2-form:

$$\operatorname{vort}(u) \coloneqq \frac{1}{2\pi}(\operatorname{djac}(u) + \kappa \operatorname{vol}). \tag{4.30}$$

In the following, we will generalize vorticity defined in (4.30) to the less regular setting of vector fields $u \in W^{1,1}(TS) \cap L^\infty(TS)$. In this case, by the Cauchy-Schwarz inequality, we can bound $|\operatorname{jac}(u)|$ at a.e. point as follows:

$$|\operatorname{jac}(u)| = |\langle \nabla u, u^\perp\rangle| \leq |\nabla u||u| \in L^1(M).$$

Hence, $\operatorname{jac}(u) \in L^1(T^*S)$ and we can define $\operatorname{djac}(u)$ as well as $\operatorname{vort}(u)$ in the distributional sense. More precisely, assume for the moment that u is smooth and take $\varphi \in C^\infty(S)$. Then by integration by parts on S (see also Proposition 4.1.54 in [55]) and the fact that the adjoint differential satisfies $\mathrm{d}^* := \star d \star$, we can write:

$$\begin{aligned}\int_S \varphi \,\mathrm{djac}(u) &= \int_S \langle \varphi \,\mathrm{vol}, \mathrm{djac}(u)\rangle \,\mathrm{vol} \\ &= \int_S \langle \star \,\mathrm{d}(\varphi \star \mathrm{vol}), \mathrm{jac}(u)\rangle \,\mathrm{vol} \\ &= \int_S \langle \star \,\mathrm{d}\varphi, \mathrm{jac}(u)\rangle \,\mathrm{vol} \\ &= \int_S \mathrm{d}\varphi \wedge \mathrm{jac}(u), \end{aligned} \tag{4.31}$$

where in the last line we have used that $\langle \star\alpha, \beta\rangle \,\mathrm{vol} = \alpha \wedge \beta$ for any pair of 1-forms α and β. As the integral in (4.31) is still well defined for $u \in W^{1,1}(TS) \cap L^\infty(TS)$, we can use it to define $\mathrm{vort}(u)$ in the distributional sense as follows: On a given test function $\varphi \in C^\infty(S)$ the distributional vorticity $\mathrm{vort}(u)$ acts as:

$$\frac{1}{2\pi}\langle \mathrm{vort}(u), \varphi\rangle := \int_S \mathrm{d}\varphi \wedge \mathrm{jac}(u) + \kappa \,\mathrm{vol}\,. \tag{4.32}$$

In the above construction, we have only employed that $\nabla u \in L^1$ and $u \in L^\infty$. Hence, in the same manner, we can define the distributional vorticity for $u \in SBV(TS) \cap L^\infty(TS)$ by replacing the Sobolev gradient with the approximate gradient of u. It is possible to define distributional vorticity locally in a domain U by restricting the test functions to lie in $C_c^\infty(U)$.

Note that in the Euclidean setting it was seen that for $v \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $\Omega \subset \mathbb{R}^2$ open the distributional Jacobian was defined in the pointwise sense. The same holds true in our setting:

Lemma 4.4
Let $v \in W^{1,2}(TU)$ for some domain $U \subset S$, then $\mathrm{vort}(v) \in L^1(\Lambda^2(U))$ and:

$$|\mathrm{vort}(v)| \le C\|v\|^2_{W^{1,2}}, \tag{4.33}$$

where $C = C(S)$ is a universal constant only depending on S.

Proof. Let us first consider $v \in C^\infty(TU) \cap W^{1,2}(TU)$. In a coordinate neighborhood, we derive by employing Proposition 2 of Chapter 5.3 in [34] that at any point in $\{v \neq 0\}$:

$$\begin{aligned}\mathrm{djac}(v) &= \mathrm{d}(|v|^2 \,\mathrm{jac}(|v|^{-1}v)) \\ &= \mathrm{d}(|v|^2) \wedge \mathrm{jac}(|v|^{-1}v) + |v|^2 \,\mathrm{djac}(|v|^{-1}v) \\ &= \langle \nabla v, |v|^{-1}v\rangle \wedge \langle \nabla v, |v|^{-1}v^\perp\rangle - |v|^2\kappa \,\mathrm{vol}\,.\end{aligned}$$

For almost all points in $\{v = 0\}$, the equality above still remains true. By the equivalence of norms, we can find $\lambda > 0$ such that for any $\varphi = \varphi_i dx^i$ it holds that:

$$g^{ij}\varphi_i\varphi_j \ge \lambda\varphi_i\varphi_j \ge \lambda\delta^{ij}\varphi_i\varphi_j, \tag{4.34}$$

where δ^{ij} denotes the Kronecker delta. Setting $\alpha := \langle \nabla v, |v|^{-1}v\rangle$ and $\beta := \langle \nabla v, |v|^{-1}v^\perp\rangle$, we derive by using the definition of $\wedge$, Young's inequality, and

(4.34):

$$\begin{aligned}|\langle\nabla v,|v|^{-1}v\rangle\wedge\langle\nabla v,|v|^{-1}v^{\perp}\rangle| &\le \frac{1}{\sqrt{|g|}}|\alpha_1\beta_2-\alpha_2\beta_1|\\ &\le \frac{1}{2\sqrt{|g|}}(\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2)\\ &\le \frac{1}{2\lambda\sqrt{|g|}}(|\alpha|^2+|\beta|^2)=\frac{1}{2\lambda c}|\nabla v|^2.\end{aligned}$$

Note that in the last line above, we have used $\sqrt{|g|}\ge c>0$ for some constant depending on the coordinate neighborhood. As S is compact, we can cover it by finitely many coordinate neighborhoods. Consequently, all constants above can be chosen uniformly over S. Thus, we derive:

$$|\mathrm{djac}(v)|\le C(|\nabla v|^2+|v|^2\|\kappa\|_\infty)\le C\|v\|^2_{W^{1,2}},$$

for a constant $C=C(S)$. This shows (4.33) which leads directly to $\mathrm{vort}(v)\in L^1(\Lambda^2(U))$ by integrating (4.33) over U. The general case of nonsmooth v can be shown by a standard approximation procedure. □

In the next lemma, we study how geometric quantities such as, for example, the covariant derivative ∇ change under composition with the map P_e.

Lemma 4.5
Given a domain U, let $u\in SBV(TU)\cap L^\infty(TU)$ satisfying $(u^+)^{\mathfrak{m}}=(u^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$-a.e. point in J_u, and let $e\in C^\infty(U;\mathbb{S}^1)$ for some domain U. Then, setting $v:=P_e(u)$, the following relation holds true between ∇v and ∇u at a.e. point in U:

$$\nabla v=|u|^{-1}\,\mathrm{d}|u|\otimes v+(\mathfrak{m}|u|^{-2}\,\mathrm{jac}(u)-(\mathfrak{m}-1)\,\mathrm{jac}(e))\otimes v^{\perp}, \tag{4.35}$$

where the right-hand side above is defined to be 0 in $\{u=0\}$. Furthermore, we also have a.e. point in U:

$$\begin{aligned}|\nabla v|^2 &= \mathfrak{m}^2|\nabla u|^2+(1-\mathfrak{m}^2)|\mathrm{d}|u||^2+(\mathfrak{m}-1)^2|u|^2|\nabla e|^2\\ &\quad -2\mathfrak{m}(\mathfrak{m}-1)\langle\mathrm{jac}(u),\mathrm{jac}(e)\rangle, \qquad (4.36)\\ \mathrm{jac}(v) &= \mathfrak{m}\,\mathrm{jac}(u)+(1-\mathfrak{m})|u|^2\,\mathrm{jac}(e)\\ &= \mathfrak{m}\,\mathrm{jac}(u)-(\mathfrak{m}-1)\,\mathrm{jac}(|u|e), \qquad (4.37)\\ \mathrm{vort}(v) &= \mathfrak{m}\,\mathrm{vort}(u)-(\mathfrak{m}-1)\,\mathrm{vort}(|u|e). \qquad (4.38)\end{aligned}$$

We shortly remark that the condition $(u^+)^{\mathfrak{m}}=(u^-)^{\mathfrak{m}}$ in the statement of the lemma above is independent of the choice of base vector e.

Proof. 1. Step: By the smoothness of P_e and the chain rule in BV (see also Theorem 4.4) $v\in SBV(TU)$ with $v^+=P_e(u^+)=P_e(u^-)=v^-$ at $\mathcal{H}^1$-a.e. point in $J_v\subset J_u$. (We employed the fact that the equality $(u^+(p))^{\mathfrak{m}}=(u^-(p))^{\mathfrak{m}}$ locally at a point p does not depend on the choice of $e(p)$.) Consequently, $\mathcal{H}^1(J_u)=0$, and hence $v\in W^{1,1}(TU)$. As $|v|=|u|$, we have $\nabla v=0$ at a.e. point in $\{|u|=0\}$. Thus, it is enough to show (4.35) at every approximate differentiability point of u contained in $\{|u|\neq 0\}$. The coordinate representation

$\bar{P}_e$ of P_e in a neighborhood of such a point can be expressed in polar coordinates as follows:

$$\bar{P}_e(r,\varphi) = \begin{pmatrix} r\cos(m\varphi) \\ r\sin(m\varphi) \end{pmatrix}.$$

By the chain rule in the Euclidean setting, we compute:

$$\begin{aligned} \bar{\nabla}_{x_1}\bar{P}_e &= \frac{x_1}{x_1^2+x_2^2}\bar{P}_e - \mathfrak{m}\frac{x_2}{x_1^2+x_2^2}P_e^{\perp}, \\ \bar{\nabla}_{x_2}\bar{P}_e &= \frac{x_1}{x_1^2+x_2^2}\bar{P}_e + \mathfrak{m}\frac{x_1}{x_1^2+x_2^2}\bar{P}_e^{\perp}. \end{aligned}$$

With (4.23), this leads for $i \in \{1,2\}$ to:

$$\begin{aligned} \nabla_{\partial_{x^i}} v &= |u|^{-2}\left[(u^1\partial_{x^i}u^1 + u^2\partial_{x^i}u^2)v + \mathfrak{m}(-u^2\partial_{x^i}u^1 + u^1\partial_{x^i}u^2 v^{\perp})\right] \\ &\quad + v^1\nabla_{\partial_{x^i}}e + v^2\nabla_{\partial_{x^i}}e^{\perp} \\ &= |u|^{-1}d|u|(\partial_{x^i})v + \mathrm{jac}(e)(\partial_{x^i})v^{\perp} + \mathfrak{m}|u|^{-2}(-u^2\partial_{x^i}u^1 + u^1\partial_{x^i}u^2)v^{\perp}. \end{aligned}$$

With:

$$\begin{aligned} \mathrm{jac}(u)(\partial_{x^i}) &= \langle \nabla_{\partial_{x^i}}u + u^1\nabla_{\partial_{x^i}}e + u^2\nabla_{\partial_{x^i}}e^{\perp}, u^{\perp}\rangle \\ &= -u^2\partial_{x^i}u^1 + u^1\partial_{x^i}u^2 + |u|^2\,\mathrm{jac}(e)(\partial_{x^i}), \end{aligned}$$

we complete the proof of (4.35).

2. Step: We now take the norm squared of both sides of (4.35) and employ $|u| = |v|$:

$$\begin{aligned} |\nabla v|^2 &= |\mathrm{d}|u||^2 + \mathfrak{m}^2|u|^{-2}|\mathrm{jac}(u)| \\ &\quad + (\mathfrak{m}-1)^2|u|^2|\mathrm{jac}(e)|^2 - 2\mathfrak{m}(\mathfrak{m}-1)\langle \mathrm{jac}(u), \mathrm{jac}(e)\rangle. \end{aligned}$$

By:

$$|\nabla u|^2 = \left||u|^{-2}\,\mathrm{jac}(u)\otimes u^{\perp} + |u|^{-2}\langle \nabla u, u\rangle \otimes u\right|^2 = |u|^{-2}|\mathrm{jac}(u)|^2 + |\mathrm{d}\,|u||^2,$$

(4.36) is satisfied. Using (4.35), the pre-jacobian of v can be written as

$$\mathrm{jac}(v) = \mathfrak{m}\,\mathrm{jac}(u) - (\mathfrak{m}-1)|u|^2\,\mathrm{jac}(e)$$

which combined with:

$$\mathrm{jac}(|u|e) = \langle |u|\,\mathrm{d}|u| \otimes e + |u|\nabla e, |u|e^{\perp}\rangle = |u|^2\,\mathrm{jac}(e)$$

leads to (4.37). Finally, by the linearity of the distributional exterior derivative and (4.37), we have:

$$\begin{aligned} \mathrm{vort}(v) &= \frac{1}{2\pi}(\mathfrak{m}\,\mathrm{djac}(u) - (\mathfrak{m}-1)\,\mathrm{djac}(|u|e) + \kappa\,\mathrm{vol}) \\ &= \mathfrak{m}\,\mathrm{vort}(u) - (\mathfrak{m}-1)\,\mathrm{vort}(|u|e) \end{aligned}$$

which is (4.38). □

The following lemma relates the vorticity of a spin field and the topology of S:

Lemma 4.6 (Morse index formula)
For any or $u \in W^{1,1}(TS) \cap L^\infty(TS)$ or $u \in SBV(TS) \cap L^\infty(TS)$ with $(u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}$ on $\mathcal{H}^1$-a.e. point in J_u, it holds that:

$$\operatorname{vort}(u)(S) := \langle \operatorname{vort}(u), \mathbb{1}_S \rangle = \chi(S), \tag{4.39}$$

where $\chi(S)$ is the Euler characteristic of S.

Proof. With the definition of $\operatorname{vort}(u)$ and the Gauss-Bonnet theorem we have

$$\begin{aligned} \langle \operatorname{vort}(u), \mathbb{1}_S \rangle &= \frac{1}{2\pi} \int_S \mathrm{d}\mathbb{1}_S \wedge \operatorname{jac}(u) + \kappa \operatorname{vol} \\ &= \frac{1}{2\pi} \int_S 0 \wedge \operatorname{jac}(u) + \kappa \operatorname{vol} = \chi(S). \end{aligned}$$

□

Note that the distributional vorticity satisfies the index formula, even in cases where one cannot find a sensible notion for the vorticity (such as $u \equiv 0$ on S).

Lastly, we directly follow from (4.32):

Lemma 4.7
Let $(u_\varepsilon) \subset W^{1,1}(TS)$ (or $(u_\varepsilon) \subset SBV(TS)$) such that $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $W^{1,1}(TS)$ (or $SBV(TS)$), then $\operatorname{vort}(u_\varepsilon)$ converges flat towards $\operatorname{vort}(u)$.

4.1.3 Ball construction on a compact surface

In this subsection, we will generalize the celebrated *ball construction* – independently introduced in [58] and [46] – from the Euclidean to the manifold setting. The presentation closely follows the one in Chapter 4 of [60]. All balls we encounter in this subsection are assumed to have a radius smaller than the injectivity radius of S. If not otherwise stated, all families $\mathfrak{B}$ of geodesic balls are assumed to contain only finitely many disjoint closed balls. The radius of a ball B will be written as r_B; the sum of the radii of all balls in a family $\mathfrak{B}$ will be denoted by $r(\mathfrak{B})$, and for any set $A \subset S$ with a slight abuse of notation, we will write:

$$\mathfrak{B} \cap A := \{B \in \mathfrak{B} \colon B \subset A\}.$$

Fixing an open connected set U and a map $v \in C^\infty(U \setminus \omega^o; \mathbb{S}^1)$, where $\omega := \bigcup_{B \in \mathfrak{B}} B$ and ω^o is the interior of ω. We define the degree $d_B(U)$ (also depending on v) of a ball B in U as:

$$d_B(U) := \begin{cases} \deg(v, \partial B) & \text{if } \partial B \subset U \setminus \omega^o, \\ 0 & \text{else,} \end{cases} \tag{4.40}$$

and the degree $D_{\mathfrak{B}}(U)$ of the family $\mathfrak{B}$ in U as:

$$D_{\mathfrak{B}}(U) := \sum_{B \in \mathfrak{B}} d_B(U). \tag{4.41}$$

Note that the degree of v around ∂B in (4.40) is defined as described in the previous subsection (see also (4.28)):

$$\deg(v, \partial B) \coloneqq \frac{1}{2\pi}\left(\int_{\partial B} \operatorname{jac}(v) + \int_B \kappa \operatorname{vol}\right).$$

Our main goal in this subsection is to prove the following theorem:

Theorem 4.6
Given an open connected set $U \subset S$, $\alpha \in (0,1)$ and $\beta \in (0, \frac{1-\alpha}{2})$, there exist $\varepsilon_0 = \varepsilon_0(S,\alpha,\beta) \in (0,1)$ and a universal constant $C = C(S)$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and $u \in C^\infty(TU)$, whose energy is bounded by:

$$\frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \operatorname{vol} \leq C_0 |\log \varepsilon| \tag{4.42}$$

for some universal constant $C_0 = C_0(S)$ and $r \in [\varepsilon^\alpha, r']$ with:

$$r' \coloneqq \min\{r^*, C^{-1}, 1\}$$

(r^* being the injectivity radius of S), we can find a finite family $\mathfrak{B}$ of disjoint closed balls satisfying:

(i) $r(\mathfrak{B}) = r$.

(ii) For $r_1, r_2 \in [\varepsilon^\alpha, r']$ with $r_1 < r_2$ and the corresponding families $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively, there exists for any $B \in \mathfrak{B}_1$ a ball $\tilde{B} \in \mathfrak{B}_2$ such that $B \subset \tilde{B}$.

(iii) Setting $V \coloneqq U_\varepsilon \cap \bigcup_{B\in\mathfrak{B}} B$, where $U_\varepsilon \coloneqq \{p \in U\colon \operatorname{dist}(p,\partial U) > \varepsilon\}$, we have:

$$\{|1-|u|| \geq \varepsilon^\beta\} \cap U_\varepsilon \subset V.$$

(iv) The following energy lower bound holds true:

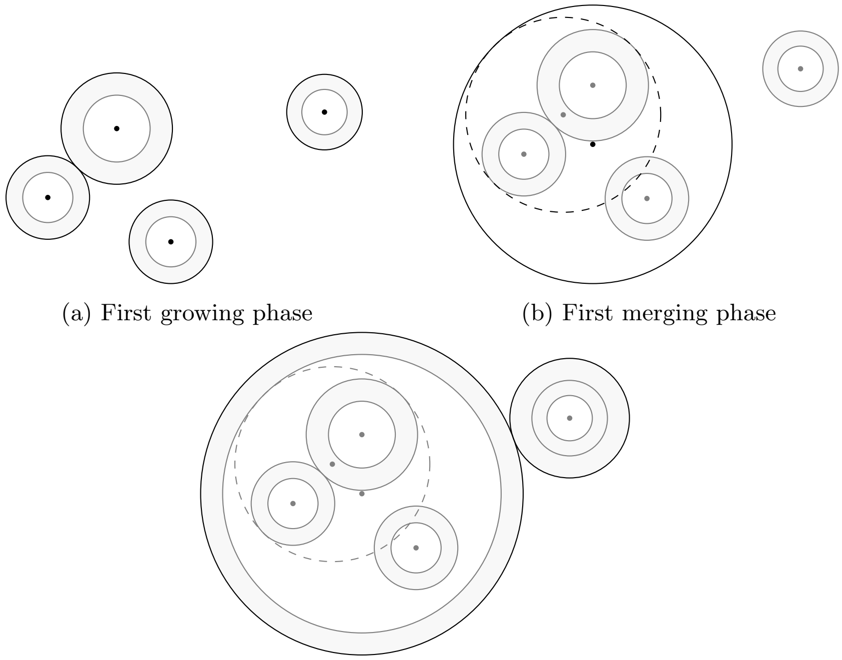
$$\frac{1}{2}\int_V |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \operatorname{vol} \geq \pi D_r \left(\log \frac{r}{D_r \varepsilon} - C\right), \tag{4.43}$$

where $D_r \coloneqq D_{\mathfrak{B}}(U_\varepsilon)$ (with $D_{\mathfrak{B}}(U_\varepsilon)$ defined in (4.41)) satisfies:

$$\sup\{D_r \colon r \in [\varepsilon^\alpha, r'],\ \varepsilon \in (0,\varepsilon_0)\} < \infty. \tag{4.44}$$

In the following, we will describe the central construction of this section. Simply speaking, given an initial family $\mathfrak{B}_0$, we wish to grow the balls contained in $\mathfrak{B}_0$ into larger and larger balls (see also Figure 4.2) More precisely, setting $T \coloneqq \log \frac{r^*}{r_0}$ ($r_0 \coloneqq r(\mathfrak{B}_0)$), we will construct for each $t \in [0,T)$ a family $\mathfrak{B}(t)$ of balls as follows: For $t = 0$, we set $\mathfrak{B}(0) \coloneqq \mathfrak{B}_0$. Given $t > 0$ small enough, we let $\mathfrak{B}(t) \coloneqq \{e^t B \colon B \in \mathfrak{B}_0\}$. We continue this either until $t = T$ or there exists (a smallest) $t_1 \in (0,T)$ such that at least two balls in $\mathfrak{B}(t)$ have a nonempty intersection. In the latter case, we stop the *growing phase* and initiate the *merging phase*. In the merging phase, we select an arbitrary pair of balls $B_1 \coloneqq B_{r_1}(p_1)$, $B_2 \coloneqq B_{r_2}(p_2) \in \mathfrak{B}(t_1)$ such that $B_1 \cap B_2 \neq \emptyset$. Then we define the geodesic ball $\tilde{B} = \tilde{B}(\tilde{p},\tilde{r})$, where $\tilde{r} = r_1 + r_2$, and $\tilde{p}$ is defined as

$$\tilde{p} \coloneqq \gamma_{p_1,p_2}(\min\{r_2, \operatorname{dist}(p_1,p_2)\}),$$



(a) First growing phase

(b) First merging phase

(c) Second growing phase

Figure 4.2: The first three phases of the ball growing procedure.

γ_{p_1,p_2} denotes the unique, unit-speed geodesic from p_1 to p_2. (Uniqueness follows from $r_1 + r_2 \leq r(\mathfrak{B}(t)) < r*$.) Note that for any $p \in B_1$, we have by the triangular inequality and the definition of $\tilde{p}$ that:

$$\operatorname{dist}(p, \tilde{p}) \leq \operatorname{dist}(p, p_1) + \operatorname{dist}(p_1, \tilde{p}) < r_1 + r_2 = \tilde{r},$$

which implies $B_1 \subset \tilde{B}$. One can also show $B_2 \subset \tilde{B}$, and hence $B_1 \cup B_2 \subset \tilde{B}$. We remove B_1, B_2, as well as any other ball $B \in \mathfrak{B}(t_1)$, which is contained in $\tilde{B}$ from $\mathfrak{B}(t_1)$, and add $\tilde{B}$ to the collection $\mathfrak{B}(t_1)$ instead. As the number of balls in $\mathfrak{B}(t_1)$ decreases in each such merging steps by at least one, we will arrive after finitely many steps at a collection $\mathfrak{B}(t_1)$, which is again disjoint and has the same total radius as the family $\mathfrak{B}(t_1^-)$ we started with. We have completed the first merging phase and initiate the second growing phase by again setting for $t > t_1$, close enough to t_1, $\mathfrak{B}(t) := \{e^{t-t_1}B \colon C \in \mathfrak{B}(t_1)\}$. As before, we continue doing this either until $t = T$ or there exists a (smallest) $t_2 \in (t_1, T)$ such that we can find two balls $B_1, B_2 \in \mathfrak{B}(t_2)$ that have a nonempty intersection. In the latter case, we stop the second growing phase and enter the second merging phase. As in each merging phase the number of balls decreases at least by one, we finish the construction after finitely many collision times $t_1, \ldots, t_N$, $N \in \mathbb{N}$. In conclusion, we end up with families of balls $\{\mathfrak{B}(t)\}_{t\in[0,T)}$ (T as before) satisfying the following properties:

(i) $\mathfrak{B}(0) = \mathfrak{B}_0$.

(ii) $\mathfrak{B}(t)$ is a finite union of disjoint balls for any $t \in [0, T)$.

(iii) For any t, $\tilde{t} \in [0,T)$ such that $t < \tilde{t}$, it holds that for any ball $\tilde{B} \in \mathfrak{B}(\tilde{t})$, we can find at least one ball B in $\mathfrak{B}(t)$ such that $B \subset \tilde{B}$.

(iv) There exists a finite set $\{t_1, \dots, t_N\} \subset [0,T)$ (exactly the collision times from before), where $N \in \mathbb{N}$ and $0 < t_1 < \cdots < t_N$ such that for any $t \in [t_k, t_{k+1})$, $k \in \{1, \dots, N-1\}$, it holds that:

$$\mathfrak{B}(t) = e^{t-t_k}\mathfrak{B}(t_k) := \{e^{t-t_k}B \colon B \in \mathfrak{B}(t_k)\}.$$

(v) For all $t \in [0,T)$:

$$r(\mathfrak{B}(t)) = e^t r(\mathfrak{B}(0)). \tag{4.45}$$

In the next lemma, we relate the degree between two different families of balls where one covers the other:

Lemma 4.8
Given an open set U, two families $\mathfrak{B}$ and $\tilde{\mathfrak{B}}$ such that $\tilde{\mathfrak{B}}$ covers $\bigcup_{B\in\mathfrak{B}} B$, and a map $v \in C^\infty(U \setminus \omega^{\mathrm{o}}; \mathbb{S}^1)$, where $\omega := \bigcup_{B\in\mathfrak{B}_0} B$, it holds that:

$$|d_{\tilde{B}}(U)| \le \sum_{B\in\mathfrak{B}(t)\cap\tilde{B}} |d_B(U)|, \tag{4.46}$$

$$D_{\tilde{\mathfrak{B}}}(U) \le D_{\mathfrak{B}}(U). \tag{4.47}$$

Proof. First, we wish to prove (4.46). Therefore, let us first consider the case $\tilde{B} \subset U$ and $\mathfrak{B} \cap \tilde{B} \neq \emptyset$. As v is smooth in $\tilde{B} \setminus \bigcup_{B\in\mathfrak{B}\cap\tilde{B}} B$ and has unit length, we see by the definition of $d_B(U)$ and $d_{\tilde{B}}(U)$:

$$d_{\tilde{B}}(U) = \deg(v, \partial\tilde{B}) = \sum_{B\in\mathfrak{B}(t)\cap\tilde{B}} \deg(v, \partial B) = \sum_{B\in\mathfrak{B}(t)\cap\tilde{B}} d_B(U).$$

Hence, by the triangular inequality (4.46) follows. In the remaining cases ($\tilde{B} \setminus U \neq \emptyset$ or $\tilde{B} \subset U$ but $\mathfrak{B} \cap \tilde{B} = \emptyset$), the inequality in (4.46) is trivially satisfied as $d_{\tilde{B}}(U) = 0$. Summing up (4.46) over all $\tilde{B} \in \mathfrak{B}(\tilde{t})$, we derive (4.47). □

Later, it will be useful to have a generalized notion of radius for closed subsets of S at hand. We will implicitly assume that all sets ω, for which we will define a generalized radius, are closed and can be covered by finite disjoint union of closed balls with their sum of radii strictly smaller than r^*. The precise definition of the generalized radius is as follows: For an admissible set ω, we define:

$$r(\omega) := \inf\left\{\sum_{i=1}^{N} r_{B_i} : , n \in \mathbb{N}, \omega \subset \cup_{i=1}^{N} B_i\right\}, \tag{4.48}$$

where the infimum is taken over all finite families (of possibly nondisjoint) balls. Note that by our implicit assumptions on ω, we always have that $r(\omega) < r^*$. Applying the merging procedure from before, we can equivalently take the infimum in (4.48) over *disjoint* families of closed balls. Furthermore, for ω as before and an open set $U \subset S$, we define the localized radius $r_U(\omega)$ as:

$$r_U(\omega) := \sup\{r(K \cap \omega) \colon K \subset\subset U \text{ compact }, \partial K \cap \omega = \emptyset\}. \tag{4.49}$$

We can show that in the case of $\omega = \bigcup_{B\in\mathfrak{B}} B$ for a finite family of disjoint closed balls, we have:

$$r_U(\omega) = r(\mathfrak{B}\cap U) = \sum_{B\in\mathfrak{B}\cap U} r_B.$$

Lemma 4.9 (Properties of the generalized radius)
The generalized radius enjoys the following properties:

(i) *It is monotone; that means given admissible sets ω and $\tilde{\omega}$ with $\omega \subset \tilde{\omega}$, we have $r(\omega) \le r(\tilde{\omega})$.*

(ii) *It is subadditive; that means given admissible sets ω and $\tilde{\omega}$, it holds that $r(\omega\cup\tilde{\omega}) \le r(\omega) + r(\tilde{\omega})$.*

(iii) *Given an admissible set ω such that:*

$$|\omega| + \frac{\Lambda\pi}{4}(\mathcal{H}^1(\partial\omega))^2 < |S|, \quad \Lambda := \sup\left\{\frac{|B_r(p)|}{\pi r^2} : p\in S, r\in[0,r*]\right\}, \tag{4.50}$$

we have:

$$r(\omega) \le \frac{1}{2}\,\mathcal{H}^1(\partial\omega), \tag{4.51}$$

as well as for any open subset U:

$$r_U(\omega) \le \frac{1}{2}\,\mathcal{H}^1(\partial\omega\cap U). \tag{4.52}$$

Proof. Monotonicity and subadditivity follow directly from the definition of the generalized radius. It remains to prove (iii). In this regard, fix $\delta > 0$ and take a cover of $\partial\omega$ by a (possibly infinite) family $\mathfrak{B}$ of open balls such that:

$$2r(\mathfrak{B}) \le \mathcal{H}^1(\partial\omega) + \delta. \tag{4.53}$$

That such a cover exists, simply follows from the definition of the Hausdorff measure. By the compactness of ω, we can assume – without loss of generality – that $\mathfrak{B}$ is finite. Taking the closure of each ball in $\mathfrak{B}$ and employing a merging procedure results in a disjoint collections $\tilde{\mathfrak{B}}$, whose radius $r(\tilde{\mathfrak{B}}) = r(\mathfrak{B})$ such that each ball of $\mathfrak{B}$ is contained in a ball from $\tilde{\mathfrak{B}}$. As the balls in $\tilde{\mathfrak{B}}$ are disjoint and S was assumed to be connected, we follow that $A := S\setminus\bigcup_{B\in\tilde{\mathfrak{B}}} B$ is an open connected subset of S with $\partial\omega\cap A = \emptyset$. Furthermore, A must be either contained in $S\setminus\omega$ or in ω^{o}, as else $A\cap(S\setminus\omega^{\mathrm{o}})$, $A\cap\omega^{\mathrm{o}}$ would be a nontrivial open partition of A contradicting the connectedness of A. We wish to exclude the case $A\subset\omega^{\mathrm{o}}$. Note that in the Euclidean setting this simply followed from the compactness of ω and the unboundedness of A. In the manifold, this is not generally true. In fact, if ω is too "large", it may happen that $A\subset\omega$. (On this occasion we will employ the condition in (4.50).) Assume by contradiction that $A\subset\omega^{\mathrm{o}}$, then using (4.53), we derive:

$$|\omega| \ge |A| = |S| - \sum_{B\in\tilde{\mathfrak{B}}} |B| \ge |S| - \Lambda\pi r(\mathfrak{B})^2 \ge |S| - \frac{\Lambda\pi}{4}(\mathcal{H}^1(\partial\omega)+\delta)^2.$$

For $\delta > 0$ small enough, this is a contradiction to (4.50). Consequently, the balls in $\tilde{\mathfrak{B}}$ must cover ω. Hence, by the definition of $r(\omega)$ (see also (4.48)):

$$r(\omega) \leq r(\tilde{\mathfrak{B}}) = r(\mathfrak{B}) \leq \mathcal{H}^1(\partial\omega) + \delta.$$

(4.51) follows by passing to the limit $\delta \to 0$. Now consider a compact set $K \subset\subset U$ such that $\partial K \cap \omega = \emptyset$, then $\omega \cap K$ is closed, $\partial(\omega \cap K) = \partial\omega \cap K$, and by (4.51):

$$2r(K \cap \omega) \leq \mathcal{H}^1(\partial(\omega \cap K)) = \mathcal{H}^1(\partial\omega \cap K) \leq \mathcal{H}^1(\partial\omega \cap U).$$

The inequality (4.52) follows by taking the supremum over all such sets K. □

In the next proposition, we provide an estimate on the Dirichlet energy of unit length spin fields:

Proposition 4.2 (Lower bounds for unit length spin fields)
Let $U \subset S$ be an open connected subset of S, $\mathfrak{B}_0$ be a finite family of closed disjoint balls with $r_0 := r(\mathfrak{B}_0) < r^*$, and $\{\mathfrak{B}(t)\}_{t\in[0,T)}$, where $T := \log \frac{r^*}{r_0}$ be the corresponding collection of families of balls arising from the aforementioned ball growing procedure starting at $\mathfrak{B}_0$. Furthermore, set $\omega := \bigcup_{B\in\mathfrak{B}_0} B$ and fix $v \in C^\infty(U \setminus \omega^{\mathrm{o}}; \mathbb{S}^1)$. We can then find a universal constant $C = C(S)$ such that for any $t \in [0,T)$ and $B \in \mathfrak{B}(t)$ contained in U, it holds that:

$$\frac{1}{2}\int_{B\setminus\omega} |\nabla v|^2 \,\mathrm{vol} \geq \pi |d_B(U)| \left(\log\left(\frac{r_1}{r_0}\right) - C(r_1 - r_0)\right), \tag{4.54}$$

where $r_1 := r(\mathfrak{B}(t))$ and d_B is defined in (4.40).

Proof. For any closed ball $B = B_r(p)$ such that $\partial B \subset U \setminus \omega^{\mathrm{o}}$ and $r < r^*$, we set:

$$\mathcal{F}(p,r) := \frac{1}{2}\int_{B\setminus\omega^{\mathrm{o}}} |\nabla v|^2 \,\mathrm{vol}.$$

We can show that $\mathcal{F}$ is nondecreasing in r (wherever it is defined), and:

$$\frac{\partial}{\partial r}\mathcal{F}(p,r) = \frac{1}{2}\int_{\partial B} |\nabla v|^2.$$

In [24] the authors proved the following lower bound for the derivative above (see Lemma 21):

$$\frac{1}{2}\int_{\partial B} |\nabla v|^2 \geq \frac{1}{4\pi r + C_1 r^2}\left|2\pi d_B - \int_B \kappa \,\mathrm{vol}\right|^2,$$

where $C_1 = C_1(S) > 0$ is a universal constant. By the compactness of S, we have that $\|\kappa\|_\infty < \infty$, and hence we can bound:

$$\int_B \kappa \,\mathrm{vol} \leq \|\kappa\|_\infty |B| \leq \|\kappa\|_\infty \Lambda \pi r^2 = C_2 r^2,$$

with Λ as in (4.50) and a universal constant $C_2 = C_2(S)$. In the case $d_B \neq 0$ (and therefore $|d_B| \geq 1$), we can estimate:

$$\begin{aligned}\frac{1}{4\pi r + C_1 r^2}\left|2\pi d_B - \int_B \kappa\,\mathrm{vol}\right|^2 &\geq \frac{1}{4\pi r + C_1 r^2}(4\pi^2 d_B^2 - 4\pi|d_B|C_2 r^2)\\ &\geq \pi d_B^2 \cdot \frac{1 - \frac{C_2}{\pi|d_B|}r^2}{r + \frac{C_1}{4\pi}r^2}\\ &\geq \pi d_B^2 \cdot \frac{1 - C_3 r^2}{r + C_3 r^2},\end{aligned}$$

with $C_3 := \max\{\frac{C_2}{\pi}, \frac{C_1}{4\pi}\}$. Hence:

$$\begin{aligned}\frac{\partial}{\partial r}\mathcal{F}(p,r) &\geq \pi d_B^2 \cdot \frac{1 - C_3 r^2}{r + C_3 r^2}\\ &= \pi d_B^2 \left(\frac{1}{r} - \frac{C_3(1+r)}{1 + C_3 r}\right)\\ &\geq \pi d_B^2 \left(\frac{1}{r} - C\right),\end{aligned} \tag{4.55}$$

where $C_4 := C_3(1 + r^*)$. Note that the above estimate is trivially satisfied in the case $d_B = 0$. Let $s = \log\left(\frac{r_1}{r_0}\right)$ and fix a balls $B \in \mathfrak{B}(s)$ such that $B \subset U$. By the choice of s, we have that $r(\mathfrak{B}(s)) = e^s r_0 = r_1$. Using the estimate (4.6) in Lemma 4.1 of [60] for the functional $\mathcal{F}$ above, (4.55), (4.46), and (4.45), we derive that:

$$\begin{aligned}\int_{B\setminus\omega} |\nabla v|^2\,\mathrm{vol} &\geq \int_0^t \sum_{B_r(p)\in\mathfrak{B}(s)\cap B} r\frac{\partial}{\partial r}\mathcal{F}(p,r)\,ds\\ &\geq \int_0^t \sum_{B'\in\mathfrak{B}(s)\cap B} \pi d_B^2(1 - C_4 r_{B'})\,ds\\ &\geq \pi\int_0^t (1 - C_4 r(\mathfrak{B}(s))) \sum_{B'\in\mathfrak{B}(s)\cap B} |d_{B'}|\,ds,\\ &\geq \pi|d_B|(\log(t) - C_4(e^t - 1)r_0)\\ &= \pi|d_B|\left(\log\left(\frac{r_1}{r_0}\right) - C_4(r_1 - r_0)\right),\end{aligned}$$

which leads to the desired estimate in (4.54) with $C := C_4$. □

In the next lemma, we will estimate the generalized radius of the set of points where $|u|$ (for $u \in C^\infty(TU)$) strays away from 1.

Lemma 4.10
There exists a universal constant $C = C(S)$ such that for any $M > 0$, any ε, $\delta \in (0,1)$ with:

$$\frac{\varepsilon^2 M^2}{\delta^4} \leq \frac{|S|}{C}, \tag{4.56}$$

and any $u \in C^\infty(TU)$ (,where $U \subset S$ is an open connected set) satisfying the energy bound:

$$\frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2\,\mathrm{vol} \leq M, \tag{4.57}$$

we have:

$$r(\{|1-|u||\geq\delta\}\cap U_\varepsilon)\leq C\frac{\varepsilon M}{\delta^2}, \tag{4.58}$$

with $U_\varepsilon := \{p\in U\colon \operatorname{dist}(p,\partial U)>\varepsilon\}$.

Proof. By the Cauchy-Schwarz inequality, we derive:

$$2|u|\,\mathrm{d}|u| = \mathrm{d}\langle u,u\rangle = 2\langle\nabla u,u\rangle \leq 2|\nabla u||u|.$$

In the case $|u|>0$, we can divide the above inequality by $2|u|$, which leads to:

$$\mathrm{d}|u|\leq|\nabla u|. \tag{4.59}$$

As $\mathrm{d}|u|=0$ at a.e. point in $\{u=0\}$, equation (4.59) holds true a.e. in U. Moreover, due to the energy bound in (4.57):

$$\frac{1}{2}\int_U |\mathrm{d}|u||^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2\,\mathrm{vol}\leq M.$$

Using the Cauchy-Schwarz inequality, it follows that:

$$\int_U |\mathrm{d}|u||\frac{|1-|u|^2|}{\sqrt{2}\varepsilon}\,\mathrm{vol}\leq M.$$

With the coarea formula on manifolds (see also, e.g. example [23]), this estimate turns into:

$$\int_0^\infty \frac{|1-t^2|}{\varepsilon}\,\mathcal{H}^1(\{|u|=t\})\,\mathrm{d}t\leq\sqrt{2}M. \tag{4.60}$$

Note that the integrand in (4.60) cannot be strictly bigger than $\frac{2\sqrt{2}M}{\delta}$ at a.e. point in $(1-\delta,1-\frac{\delta}{2})$, as otherwise:

$$\int_{1-\delta}^{1-\frac{\delta}{2}} \frac{|1-t^2|}{\varepsilon}\,\mathcal{H}^1(\{|u|=t\})\,\mathrm{d}t>\sqrt{2}M,$$

contradicts (4.60). Consequently, we can find a regular value $t^*\in(1-\delta,1-\frac{\delta}{2})$ of $|u|$; this means $\{|u|=t^*\}=\partial\omega$ with $\omega:=\{|u|\geq t^*\})$ such that:

$$\mathcal{H}^1(\partial\omega\cap U)\leq\frac{2\sqrt{2}\varepsilon M}{\delta|1-(t^*)^2|}.$$

Note that as $t^*\in(1-\delta,1-\frac{\delta}{2})$, we can estimate $|1-(t^*)^2|=|1+t^*||1-t^*|\geq\frac{\delta}{2}$, and therefore derive that:

$$\mathcal{H}^1(\partial\omega\cap U)\leq 4\sqrt{2}\frac{\varepsilon M}{\delta^2}. \tag{4.61}$$

From (4.59), the definition of ω, and $|1-(t^*)^2|^2\geq\frac{\delta^2}{4}$, we then follow:

$$\frac{\delta^2}{16\varepsilon^2}|\omega|\leq\frac{1}{4\varepsilon^2}\int_U(1-|u|^2)^2\,\mathrm{vol}\leq M,$$

and by solving for $|\omega|$:

$$|\omega|\leq\frac{16\varepsilon M^2}{\delta^2}. \tag{4.62}$$

We wish to employ Lemma 4.9 in order to show (4.58). But in order to do so, we need an estimate on $\mathcal{H}^1(\partial\omega \cap U_\varepsilon)$ instead of $\mathcal{H}^1(\partial\omega \cap U)$. In this regard, by (4.62) and Fubini's theorem, we can find an $s \in (0, \varepsilon)$ such that:

$$\mathcal{H}^1(\omega \cap U_s) \leq \frac{16\varepsilon M}{\delta^2} \tag{4.63}$$

with $U_s := \{p \in U \colon \operatorname{dist}(p, \partial U) > s\}$, and therefore:

$$\mathcal{H}^1(\partial(\omega \cap U_s)) \leq \mathcal{H}^1(\partial\omega \cap U) + \mathcal{H}^1(\omega \cap U_s) = C_1 \frac{\varepsilon M}{\delta^2}, \tag{4.64}$$

where $C_1 = 16 + 4\sqrt{2}$. By (4.64) and (4.62), we can estimate:

$$|\omega \cap U_s| + \frac{\Lambda\pi}{4}\left(\mathcal{H}^1(\partial(\omega \cap U_s))\right)^2 \leq C_2 \frac{\varepsilon M^2}{\delta^4},$$

where Λ is the universal constant defined in (4.50) and $C_2 := 16(1 + \frac{C_1^2 \Lambda\pi}{4})$. Hence, setting $C := \max\{\frac{1}{2}C_1, C_2\}$, we see by (4.56) that (4.50) is satisfied for the set $\omega \cap U_s \subset\subset U$. Consequently, by (4.51), (4.64), the definition of C, and the monotonicity of the generalized radius, we finally derive:

$$r(\omega \cap U_\varepsilon) \leq r(\omega \cap U_s) \leq \frac{1}{2}\mathcal{H}^1(\partial(\omega \cap U_s)) \leq C\frac{\varepsilon M}{\delta^2},$$

as is desired. □

In the following proposition, we will select the initial family of balls. More precisely:

Proposition 4.3 (Initial ball selection)
Let $\alpha \in (0,1)$ and $\beta \in (0, \frac{1-\alpha}{2})$, then there exists an universal scalar $\varepsilon_0 = \varepsilon_0(S, \alpha, \beta) \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, and any $u \in C^\infty(TU)$ ($U \subset S$ open and connected) satisfying the energy bound:

$$\frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \operatorname{vol} \leq C_0 |\log \varepsilon| \tag{4.65}$$

for some universal constant $C_0 = C_0(S)$, we can find a finite family $\mathfrak{B}_0$ of disjoint closed balls with the following properties:

(i) $r(\mathfrak{B}_0) = \varepsilon^\alpha$.

(ii) $\{|1 - |u|| \geq \varepsilon^\beta\} \subset V_0 := U_\varepsilon \cap \bigcup_{B \in \mathfrak{B}_0} B$.

(iii) Setting $v := \frac{u}{|u|}$ wherever this makes sense, we have for any $t \in (0, 1 - \varepsilon^\beta)$:

$$\frac{1}{2}\int_{V_0 \setminus \omega_t} |\nabla v|^2 \operatorname{vol} \geq \pi D_{\mathfrak{B}_0}(U_\varepsilon)\left(\log\left(\frac{\varepsilon^\alpha}{r_{U_\varepsilon}(\omega_t)}\right) - C\right), \tag{4.66}$$

where $C = C(S)$ is a universal constant, $U_\varepsilon := \{p \in U \colon \operatorname{dist}(p, \partial U) > \varepsilon\}$, $\omega_t := \{|u| \leq t\}$, and $D_{\mathfrak{B}_0}(U_\varepsilon)$ is defined in (4.41).

Proof. In the following, we will fix $\varepsilon \in (0, \varepsilon_0)$, where ε_0 will be appropriately defined along the way.

1. Step: We wish to apply Lemma 4.10 with $M := C_0|\log\varepsilon|$ (where C_0 is the constant from (4.65)) and $\delta := \varepsilon^\beta$. As $2 - 4\beta > 2 - 2(1-\alpha) = 2\alpha > 0$, it follows that:

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon^2 M^2}{\delta^4} \le \limsup_{\varepsilon\to 0} C_0\varepsilon^{2\alpha}|\log\varepsilon|^2 = 0.$$

Hence, we can find a scalar $\varepsilon_0 = \varepsilon_0(C_0, C_1, \alpha) \in (0,1)$, where C_1 is the universal constant from Lemma 4.10 such that for any $\varepsilon < \varepsilon_0$ the condition Lemma 4.10 is satisfied for δ, as defined above. Let us set $\tilde\alpha := 1 - 2\beta$, then $\tilde\alpha > 1 - 2\cdot\frac{1-\alpha}{2} = \alpha$ and $\frac{1-\tilde\alpha}{2} = \frac{1-1+2\beta}{2} = \beta$. Consequently, by (4.58) and the definition of the generalized radius, we can find a finite family $\tilde{\mathfrak{B}}$ of disjoint closed balls covering the set $\{|1-|u|| \ge \delta\}$ and satisfying:

$$\tilde r := r(\tilde{\mathfrak{B}}) \le 2C_1 \cdot \frac{\varepsilon M}{\delta^2} = 2C_1C_0\varepsilon^{1-2\frac{1-\tilde\alpha}{2}}|\log\varepsilon| = 2C_1C_0\varepsilon^{\tilde\alpha}|\log\varepsilon|. \tag{4.67}$$

As:

$$\lim_{\varepsilon\to 0} C_1C_0\varepsilon^{\tilde\alpha-\alpha}|\log\varepsilon| = 0,$$

by possibly decreasing ε_0 (dependent on C_0, C_1, α, and β), we see that $\tilde r \le \frac{1}{6}\varepsilon^\alpha$. Using the ball growing procedure introduced in this subsection, we can assure that $\tilde r = \frac{1}{6}\varepsilon^\alpha$. Note that the choice of the prefactor $\frac{1}{6}$ will be made clear at a later point in the proof.

2. Step: Let $\tilde V := \bigcup_{B\in\tilde{\mathfrak{B}}} B$, $\tilde V' := \bigcup_{B\in\tilde{\mathfrak{B}}\cap U_\varepsilon} B$, $\tilde V''$ the union of the remaining balls in $\tilde{\mathfrak{B}}$, and $\tilde U_\varepsilon := U_\varepsilon \setminus V_0'$. By the compactness of V_0'' and the definition of $r_{U_\varepsilon}(\omega_t)$, we see that for any $t \in (0, 1-\delta)$:s

$$r_{U_\varepsilon}(\omega_t) \ge r(\omega_t \cap \tilde V') = r(\omega_t \cap \tilde U_\varepsilon). \tag{4.68}$$

Consequently, we can cover for any given $t \in (0, 1-\delta)$ the set $\omega_t \cap \tilde U_\varepsilon$ with a family $\tilde{\mathfrak{B}}_t$ of balls with a radius no greater than $2r(\omega_t \cap \tilde U_\varepsilon)$. As $r(\omega_t \cap \tilde U_\varepsilon) \le r(\omega_t) \le r(\tilde{\mathfrak{B}}) = \tilde r$, we can grow these balls into the family $\mathfrak{B}_t$ with $r(\mathfrak{B}_t) = 2\tilde r$. Employing (4.67), Proposition 4.2 with the domain $\tilde U_\varepsilon$, $v := \frac{u}{|u|}$ (defined in $\{u \ne 0\}$), as well as the two families $\tilde{\mathfrak{B}}_t$ and $\mathfrak{B}_t$, we derive by the monotonicity of the logarithm:

$$\begin{aligned} \frac{1}{2}\int_{W_t\setminus\omega_t} |\nabla v|^2 \,\mathrm{vol} &\ge \pi D_{\mathfrak{B}_t}(\tilde U_\varepsilon)\left(\log\left(\frac{2\tilde r}{2r(\omega_t\cap\tilde U_\varepsilon)}\right) - C_2(2\tilde r - 2r(\omega_t \cap \tilde U_\varepsilon))\right) \\ &\ge \pi D_{\mathfrak{B}_t}(\tilde U_\varepsilon)\left(\log\left(\frac{\tilde r}{r_{U_\varepsilon}(\omega_t)}\right) - 2C_2 r^*\right) \end{aligned} \tag{4.69}$$

where $C_2 = C_2(S)$ is the universal constant from Proposition 4.2 and:

$$W_t := \bigcup_{B\in\mathfrak{B}_t} B.$$

As $D_{\mathfrak{B}_t}(\tilde U_\varepsilon)$ is a natural number for any $t \in (0, 1-\delta)$ we can find $\bar t \in (0, 1-\delta)$ such that $D_{\mathfrak{B}_{\bar t}}(\tilde U_\varepsilon)$ is minimal among all $t \in (0, 1-\delta)$. We then define $\mathfrak{B} := \mathfrak{B}_{\bar t}$.

3. Step: Let $K \subset\subset U$ such that $r(K) \leq 2\tilde{r}$ and $\mathcal{F}(K) \geq m-1$, where:

$$\mathcal{F}(K) \coloneqq \frac{1}{2}\int_{(K\cap U_\varepsilon)\setminus\tilde{V}} |\nabla u|^2 \operatorname{vol},$$
$$m \coloneqq \sup\Big\{\mathcal{F}(\tilde{K})\colon \tilde{K} \subset\subset U \text{ compact}, r(\tilde{K}) \leq 2\tilde{r}\Big\}.$$

By the previous reasoning and the subadditivity of the generalized radius, we then have:

$$r\left(K \cup \tilde{V} \cup \bigcup_{B\in\mathfrak{B}} B\right) \leq r(K) + r(\tilde{V}) + r(\mathfrak{B}) = 5\tilde{r}.$$

Consequently, we can cover $K \cup \tilde{V} \cup \bigcup_{B\in\mathfrak{B}} B$ by a family $\mathfrak{B}_0$ with $r(\mathfrak{B}_0) \leq 6\tilde{r}$. By possible growing the balls in $\mathfrak{B}_0$, we can assume – without loss of generality – that $r(\mathfrak{B}_0) = \varepsilon^\alpha$ and Item (i) of Proposition 4.3 is satisfied. Furthermore, as the balls of $\mathfrak{B}_0$ cover $\tilde{V}$, Item (ii) of Proposition 4.3 is also satisfied.

4.step: It remains to show (4.66). Let $V_0 \coloneqq \bigcup_{B\in\mathfrak{B}_0} \cap U_\varepsilon$, then as $\mathfrak{B}_0$ covers K:

$$\frac{1}{2}\int_{V_0\setminus\omega_t} |\nabla v|^2 \operatorname{vol} \geq \mathcal{F}(K) + \frac{1}{2}\int_{\tilde{V}\setminus\omega_t} |\nabla v|^2 \operatorname{vol}.$$

Hence, with (4.69), the definition of K, and the fact that W_t is a competitor for the supremum m:

$$\begin{aligned}\frac{1}{2}\int_{V_0\setminus\omega_t} |\nabla v|^2 \operatorname{vol} &\geq \mathcal{F}(W_t) - 1 + \frac{1}{2}\int_{\tilde{V}\setminus\omega_t} |\nabla v|^2 \operatorname{vol}\\ &= \frac{1}{2}\int_{W_t\setminus\omega_t} |\nabla v|^2 \operatorname{vol} - 1\\ &\geq \pi D_{\mathfrak{B}}(\tilde{U}_\varepsilon)\left(\log\left(\frac{\frac{1}{6}\varepsilon^\alpha}{r_{U_\varepsilon}(\omega_t)}\right) - 2C_2 r^*\right) - 1\\ &\geq \pi D_{\mathfrak{B}}(\tilde{U}_\varepsilon)\left(\log\left(\frac{\varepsilon^\alpha}{r_{U_\varepsilon}(\omega_t)}\right) - \log(6) - 2C_2 r^*\right) - 1.\end{aligned}$$

As $\mathfrak{B}_0$ covers $\bigcup_{B\in\mathfrak{B}} B$, we have by (4.47) that $D_{\mathfrak{B}}(\tilde{U}_\varepsilon) \geq D_{\mathfrak{B}_0}(\tilde{U}_\varepsilon)$. Moreover, as $\mathfrak{B}_0$ also covers $\tilde{V}$, any ball $B \in \mathfrak{B}_0$ with $B \setminus \tilde{U}_\varepsilon \neq \emptyset$ must intersect ∂U_ε, and therefore has zero degree $d_B(U_\varepsilon) = 0$ in $\tilde{U}_\varepsilon$. Consequently, $D_{\mathfrak{B}}(\tilde{U}_\varepsilon) \geq D_{\mathfrak{B}_0}(\tilde{U}_\varepsilon) = D_{\mathfrak{B}_0}(U_\varepsilon)$. Note that if $D_{\mathfrak{B}_0}(\tilde{U}_\varepsilon)$, the estimate in (4.66) is trivially satisfied for $C = 0$. In the case $D_{\mathfrak{B}_0}(U_\varepsilon) > 0$, and hence also $D_{\mathfrak{B}_0}(U_\varepsilon) \geq 1$, we see that (4.66) holds true for $C = \log(6) + 2C_2 r^* + \frac{1}{\pi}$. □

We are ready to proof the main result of this subsection:

Proof of Theorem 4.6. The correct values of ε_0 will be fixed along the way. The proof follows closely the one of Theorem 4.1 in [60]:

1. Step: Let $\mathfrak{B}_0$ denote the initial family provided by Proposition 4.3, with α and β as stated and $C_1 = C_1(S)$ denoting the constant in (4.66). We let the balls in $\mathfrak{B}_0$ grow and merge as described in the beginning of this subsection, resulting in $\{\mathfrak{B}(s)\}_{s\in[0,T)}$, where $T \coloneqq \log\left(\frac{r^*}{r_0}\right)$ with $r_0 \coloneqq r(\mathfrak{B}_0)$. Let us now set:

$$\mathfrak{B}_r \coloneqq \mathfrak{B}(s), \quad s \coloneqq \log\left(\frac{r}{r_0}\right)$$

By construction, $r(\mathfrak{B}(s)) = e^s r(\mathfrak{B}_0)$, and hence $\mathfrak{B}_r$ – as chosen above – satisfies Item (i). From the definition of the growing procedure, Item (ii) of Theorem 4.6 directly follows. By Item (iii) and Item (ii) in Proposition 4.3, we also derive Item (iii) of Theorem 4.6.

2. Step: It remains to prove the energy lower bound (4.43). First note that as $u \neq 0$ in $U_\varepsilon \setminus W_0$ with $W_0 := \bigcup_{B\in\mathfrak{B}_0} B$, we can apply Proposition 4.2 in U_ε for $v = \frac{u}{|u|}$, which implies for every $B \in \mathfrak{B}_r \cap U_\varepsilon$:

$$\frac{1}{2}\int_{B\setminus W_0} |\nabla v|^2 \,\mathrm{vol} \geq \pi |d_B(U_\varepsilon)| \left(\log\left(\frac{r}{r_0}\right) - C_2 r^*\right), \tag{4.70}$$

where r^* is the injectivity radius, $C_2 = C_2(S)$ is the constant from Proposition 4.2, and $d_B(U_\varepsilon)$ is defined in (4.40). Let us assume for the moment that ε_0 is as in Proposition 4.3; then summing up (4.70) over all balls $B \in \mathfrak{B}_r \cap U_\varepsilon$ and using (4.66), leads – for every $t \in (0, 1-\delta)$ with $\delta := \varepsilon^\beta$ – to:

$$\begin{aligned}\frac{1}{2}\int_{V_t} |\nabla v|^2 \,\mathrm{vol} &= \frac{1}{2}\int_{V_t\setminus W_0} |\nabla v|^2 \,\mathrm{vol} + \frac{1}{2}\int_{W_0} |\nabla v|^2 \,\mathrm{vol} \\ &\geq \pi D_r \left(\log\left(\frac{r}{r(\mathfrak{B}_0)}\right) + \log\left(\frac{r(\mathfrak{B}_0)}{r_{U_\varepsilon}(\omega_t)}\right) - C_2 r^* - C_3\right), \\ &= \pi D_r \left(\log\left(\frac{r}{r_{U_\varepsilon}(\omega_t)}\right) - C_4\right),\end{aligned} \tag{4.71}$$

where $D_r := D_{\mathfrak{B}_r}(U_\varepsilon)$, $\omega_t := \{|u| \leq t\}$, $C_4 = C_2 + C_3$ with $C_3 = C_3(S)$ being the universal constant from Proposition 4.3, and

$$V_t := V \setminus \omega_t, \quad V := U_\varepsilon \cap \bigcup_{B\in\mathfrak{B}} B.$$

3. Step: The desired estimate will follow by integrating (4.71) over $t \in (0, 1-\delta)$. In this regard, we will apply the coarea formula on Riemannian manifolds (see also [23]). Let us first fix some notation: For any $t > 0$, we define:

$$\gamma_t := \{|u| = t\} \cap U_\varepsilon, \quad \Theta(t) := \frac{1}{2}\int_{V_t} |\nabla v|^2 \,\mathrm{vol}.$$

By the Cauchy-Schwarz inequality, we have for any $t > 0$:

$$|\mathrm{d}|u||^2 + \frac{(1-t^2)^2}{2\varepsilon^2} \geq |\mathrm{d}|u|| \frac{\sqrt{2}|1-t^2|}{\varepsilon},$$

and hence with the coarea formula, this leads to

$$\frac{1}{2}\int_U |\mathrm{d}|u||^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \,\mathrm{vol} \geq \frac{1}{2}\int_0^\infty \frac{\sqrt{2}|1-t^2|}{\varepsilon} \mathcal{H}^1(\gamma_t) \,\mathrm{d}t. \tag{4.72}$$

Furthermore, by Fubini's theorem:

$$\frac{1}{2}\int_U |u|^2 |\nabla u|^2 \,\mathrm{vol} = -\int_0^\infty t^2 \Theta'(t) \,\mathrm{d}t \geq 2\int_0^\infty t\Theta(t). \tag{4.73}$$

(Note that the last inequality can be strict if $|u|$ is constant on sets of positive area.) Let us shortly write:

$$I := \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \,\mathrm{vol}. \tag{4.74}$$

By the product rule, we derive in $\{u \neq 0\}$ that:

$$\nabla u = \nabla(|u|v) = d|u| \otimes v + |u|\nabla v,$$

and by taking the norm on both sides:

$$\begin{aligned}|\nabla u|^2 &= \langle \mathrm{d}|u| \otimes v + |u|\nabla v, \mathrm{d}|u| \otimes v + |u|\nabla v\rangle \\ &= |\mathrm{d}|u||^2 + |u|^2|\nabla v|^2 + 2|u|\langle \mathrm{d}|u| \otimes v, \nabla v\rangle.\end{aligned}$$

As for any section $X \in C^\infty(TU)$, we have that $\langle \nabla_X v, v\rangle = \frac{1}{2}\,\mathrm{d}_X\langle v, v\rangle = 0$, the above equation simplifies to:

$$|\nabla u|^2 = |\mathrm{d}|u||^2 + |u|^2|\nabla v|^2. \tag{4.75}$$

Hence, with (4.72) and (4.73), the following estimate follows:

$$\begin{aligned}I\frac{1}{2}\int_U |u|^2|\nabla v|^2 \,\mathrm{vol} + \frac{1}{2}\int_U |\mathrm{d}|u||^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \,\mathrm{vol} \\ \geq \int_0^{1-\delta} 2t\Theta(t) + \frac{1-t^2}{\sqrt{2}\varepsilon}\,\mathcal{H}^1(\gamma_t)\,\mathrm{d}t.\end{aligned} \tag{4.76}$$

Let us take ε_0 as in Proposition 4.3. In this case, we have already seen in the proof of Proposition 4.3 that $2r_{U_\varepsilon}(\omega_t) \leq \mathcal{H}^1(\partial\omega_t \cap U_\varepsilon)$ for any $t \in (0, 1-\delta)$. By the continuity of u and the definition of γ_t, we follow that $\partial\omega_t \cap U_\varepsilon \subset \gamma_t$, and therefore:

$$2r_{U_\varepsilon}(\omega_t) \leq \mathcal{H}^1(\gamma_t). \tag{4.77}$$

Due to (4.71) and the definition of $\Theta(t)$, it holds that:

$$\Theta(t) \geq \pi D_r \left(\log\left(\frac{r}{r_{U_\varepsilon}(\omega_t)}\right) - C_4\right),$$

for each $t \in (0, 1-\delta)$. Consequently, with (4.77), we have that:

$$I \geq \int_0^{1-\delta} 2t\pi D_r \left(\log\left(\frac{r}{r_{U_\varepsilon}(\omega_t)}\right) - C_4\right) + \frac{\sqrt{2}(1-t^2)}{\varepsilon} r_{U_\varepsilon}(\omega_t)\,\mathrm{d}t.$$

Minimizing the above integrand with respect to the value of $r_{U_\varepsilon}(\omega_t)$, we see that the global minimum is achieved for:

$$r_{U_\varepsilon}(\omega_t) = \frac{2t\pi D_r \varepsilon}{\sqrt{2}(1-t^2)}.$$

Hence:

$$I \geq \int_0^{1-\delta} 2t\pi D_r \left(\log\left(\frac{r}{D_r\varepsilon}\right) + \log\left(\frac{1-t^2}{\sqrt{2}\pi t}\right) - C_4\right)\mathrm{d}t.$$

By computing the integral in the estimate above, we see that:

$$I \geq \pi D_r \left((1-\delta)^2 \log\left(\frac{r}{D_r\varepsilon}\right) - C_5\right),$$

where:

$$C_5 := -\int_0^1 2t \log\left(\frac{1-t^2}{\sqrt{2}\pi t}\right) \mathrm{d}t > 0.$$

In the case $\log \frac{r}{D\varepsilon} - C_5 \le 0$, the inequality in (4.43) is trivially satisfied with C_5, as chosen above. We can therefore reduce ourselves to the case $\log \frac{r}{D\varepsilon} \ge C_5 > 0$. Under this additional assumption, we have:

$$I \ge \pi D_r \left(\log\left(\frac{r}{D_r\varepsilon}\right) - 2\delta \log\left(\frac{r}{D_r}\right) - 2\delta |\log\varepsilon| - C_5 \right).$$

Without loss of generality, we can assume that $D_r \ge 1$ (in the case $D_r = 0$, the right-hand side of (4.43) is 0). Then, as additionally $r \le r' \le 1$, the term $\delta \log\left(\frac{r}{D}\right)$ is nonpositive which – together with $\delta = \varepsilon^\beta$ – leads to

$$I \ge \pi D_r \left(\log\left(\frac{r}{D_r\varepsilon}\right) - 2\varepsilon^\beta |\log\varepsilon| - C_5 \right).$$

By possibly decreasing ε_0, we can assure that $2\varepsilon_0^\beta |\log\varepsilon_0| \le 1$, showing:

$$I \ge \pi D_r \left(\log\left(\frac{r}{D_r\varepsilon}\right) - 1 - C_5 \right),$$

which is (4.43) for $C := 1 + C_5$.

4. Step: It remains to check (4.44). Suppose that for a.a. $t \in (\frac{1}{2}, \frac{3}{4})$, it holds that $\mathcal{H}^1 \gamma_t \ge l$ for some $l > 0$. Then by (4.72) and (4.42), we have:

$$\frac{7}{64\sqrt{2}} \cdot \frac{l}{\varepsilon} \le \frac{1}{2} \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{\sqrt{2}|1-t^2|}{\varepsilon} \mathcal{H}^1(\gamma_t) \le C_0 |\log\varepsilon|.$$

For $l > \frac{\sqrt{2}\cdot 64}{7} C_0 |\log\varepsilon|\varepsilon$, this leads to a contradiction. Hence we can find a regular value $t^* \in (\frac{1}{2}, \frac{3}{4})$ of $|u|$ such that $\mathcal{H}^1(\gamma_t) \le C_6 \varepsilon |\log\varepsilon|$, where $C_6 = \frac{\sqrt{2}\cdot 64}{7} C_0$. For ε_0 small enough, we have that $1 - \delta = 1 - \varepsilon^\beta > \frac{3}{4}$, and by (4.77), we see that $r_{U_\varepsilon}(\omega_{t^*}) \le \mathcal{H}^1(\gamma_{t^*}) \le C_6 \varepsilon |\log\varepsilon|$. With (4.66), and the fact that $v := \frac{u}{|u|}$ satisfies $|\nabla v|^2 \le \frac{|\nabla u|^2}{|u|^2} \le 4|\nabla u|^2$ (see (4.75)), this leads to:

$$4C_0 |\log\varepsilon| \ge \frac{1}{2} \int_{V_{t^*}} |\nabla v|^2 \,\mathrm{vol} \ge \pi D_r \left(\log\left(\frac{r}{C_6 \varepsilon |\log\varepsilon|} \right) - C \right),$$

where C is the constant from (4.66). By possibly further decreasing ε_0 (depending on C_6 and α), we can assure that $C_6 \varepsilon |\log\varepsilon| \le \varepsilon^{-\frac{1-\alpha}{2}}$, and therefore:

$$\frac{8}{(1-\alpha)\pi} C_0 |\log\varepsilon| \ge D_r(|\log\varepsilon| - \log r - C) \ge D_r(|\log\varepsilon| - C), \tag{4.78}$$

where we have used $r \le r' \le 1$. Note that the assumption $D_r > \frac{8}{(1-\alpha)\pi} C_0$ contradicts (4.78) for sufficiently small ε_0, and (4.44) follows. □

4.2 Problem setup

The Γ-convergence result stated in this section can be seen as a natural generalization of Theorem 3.1 in [40] to the setting of a compact oriented two-dimensional Riemannian manifold.

In this regard, we will investigate the following admissible spin fields:

$$\mathcal{AS}^{(\mathfrak{m})} = \mathcal{AS}^{(\mathfrak{m})}(S) := \{u \in SBV^2(TS) \colon (u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}\ \mathcal{H}^1\text{-a.e. on } J_u\}.$$

We also consider the restriction to a general domain $U \subset S$ (open, simply connected):

$$\mathcal{AS}^{(\mathfrak{m})}(U) := \{u \in SBV^2(TU) \colon (u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}\ \mathcal{H}^1\text{-a.e. on } J_u\}.$$

Given $\varepsilon > 0$, we define the generalized Ginzburg-Landau energy functional GGL_ε on the set $\mathcal{AS}^{(\mathfrak{m})}$ as follows:

$$GGL_\varepsilon(u) := \frac{1}{2}\int_S |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \operatorname{vol} + \mathcal{H}^1(J_u).$$

Correspondingly, we define the localized version in a domain U as:

$$GGL_\varepsilon(u, U) := \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \operatorname{vol} + \mathcal{H}^1(J_u \cap U).$$

We will now describe the Γ-limit: The set of limit vortex measures $X^{(\mathfrak{m})}$ is given by:

$$X^{(\mathfrak{m})} := \left\{ \operatorname{sgn}(\chi(S)) \sum_{k=1}^{\mathfrak{m}|\chi(S)|} \frac{1}{\mathfrak{m}} \delta_{p_k} : p_k \in S,\, p_k \neq p_l \text{ for } k \neq l \right\}.$$

It will be useful to also have a notation for the following superset of $X^{(\mathfrak{m})}$:

$$\tilde{X}^{(\mathfrak{m})} := \left\{ \mu = \sum_{k=1}^{K} \frac{d_k}{\mathfrak{m}} \delta_{p_k} : d_k \in \mathbb{Z},\, p_k \neq p_l \text{ for } k \neq l,\, \sum_{k=1}^{K} d_k = \mathfrak{m}\chi(S) \right\}.$$

Furthermore, $\mathcal{LS}^{(\mathfrak{m})}$ will denote the set of limit spin fields $u \in SBV(S; \mathbb{S}^1)$, additionally satisfying:

(i) $(u^+)^{\mathfrak{m}} = (u^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$-a.e. point in J_u, where $\mathcal{H}^1(J_u) < \infty$.

(ii) $\frac{1}{\pi} \operatorname{vort}(u) \in X^{(\mathfrak{m})}$, and $u \in SBV^2_{\mathrm{loc}}(S \setminus \operatorname{spt} \operatorname{vort}(u); \mathbb{S}^1)$.

On $\mathcal{LS}^{(\mathfrak{m})}$, we define the (fractional) renormalized energy as:

$$\mathcal{W}^{(\mathfrak{m})}(u) := \lim_{r \to 0} \left(\frac{1}{2} \int_{S_r} |\nabla u|^2 \operatorname{vol} - \frac{|\chi(S)|}{\mathfrak{m}} \pi |\log r| \right), \tag{4.79}$$

where S_r is defined as:

$$S_r := S \setminus \bigcup_{k=1}^{\mathfrak{m}|\chi(S)|} B_r(p_k), \text{ where } \operatorname{vort}(u) = \operatorname{sgn}(\chi(S)) \sum_{k=1}^{\mathfrak{m}|\chi(S)|} \frac{1}{\mathfrak{m}} \delta_{p_k}. \tag{4.80}$$

Note that in Lemma 4.17, we will prove the well-definedness of the renormalized energy $\mathcal{W}^{(\mathfrak{m})}$ on $\mathcal{LS}^{(\mathfrak{m})}$. Given $r > 0$, $\varepsilon > 0$, and $\lambda \in \mathbb{S}^1$ we consider:

$$\begin{aligned}&\bar{\gamma}_\varepsilon^{(\mathfrak{m})}(r,\lambda)\\ &\quad := \min\left\{E_\varepsilon(v,\bar{B}_r(0))\colon v \in W^{1,2}(\bar{B}_r(0);\mathbb{R}^2), v = \lambda\frac{x}{|x|} \text{ on } \partial\bar{B}_r(0)\right\},\end{aligned} \tag{4.81}$$

where $\bar{B}_r(0)$ is the ball in $\mathbb{R}^2$ of radius r, and E_ε is given by:

$$E_\varepsilon(v,\bar{B}_r(0)) := \frac{1}{2\mathfrak{m}^2}\int_{\bar{B}_r(0)} |\nabla v|^2 + (\mathfrak{m}^2-1)|\nabla|v||^2 + \frac{\mathfrak{m}^2}{2\varepsilon^2}(1-|v|^2)^2\,\mathrm{d}x.$$

Note that the minimum problem above is formulated in the Euclidean setting, so, for example, $B_r(0)$ denotes the Euclidean ball of radius r centered at 0. By a change of coordinates, we can rewrite for $v \in W^{1,2}(\bar{B}_r(0);\mathbb{R}^2)$ with $v = \lambda\frac{x}{|x|}$ on $\partial\bar{B}_r(0)$:

$$\begin{aligned}&\frac{1}{2\mathfrak{m}^2}\int_{B_r(0)} |\nabla v|^2 + (\mathfrak{m}-1)^2|\nabla|v||^2 + \frac{\mathfrak{m}^2}{2\varepsilon^2}(1-|v|^2)^2\,\mathrm{d}x\\ &\quad = \frac{1}{2\mathfrak{m}^2}\int_{\bar{B}_{\frac{r}{\varepsilon}}(0)} |\nabla \tilde{v}|^2 + (\mathfrak{m}-1)^2|\nabla|\tilde{v}||^2 + \frac{\mathfrak{m}^2}{2}(1-|\tilde{v}|^2)^2\,\mathrm{d}x,\end{aligned}$$

where $\tilde{v}(x) := \lambda^{-1}v(\varepsilon x)$ is admissible for the minimization problem in the definition of $\bar{\gamma}^{\mathfrak{m}}(\frac{r}{\varepsilon}) := \bar{\gamma}_1^{\mathfrak{m}}(\frac{r}{\varepsilon},1,1)$. By symmetry, we therefore see that:

$$\bar{\gamma}^{(\mathfrak{m})}\left(\frac{r}{\varepsilon}\right) = \bar{\gamma}_\varepsilon^{(\mathfrak{m})}(r,\lambda).$$

It was proved in [40] (see also Lemma 3.9) that there exists a scalar $\gamma_{\mathfrak{m}}$ (possibly depending on $\mathfrak{m}$) such that the following convergence holds true:

$$\gamma_{\mathfrak{m}} = \lim_{R\to\infty}\left(\bar{\gamma}^{(\mathfrak{m})}(R) - \frac{\pi}{\mathfrak{m}^2}\log(R)\right). \tag{4.82}$$

We are well equipped to state the main result of this chapter:

Theorem 4.7
With the notation defined above the following Γ-convergence result holds true:

(i) (Compactness) Given a sequence $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}$ bounded in $L^\infty(TS)$ and satisfying the energy bound:

$$GGL_\varepsilon(u_\varepsilon) \le \frac{|\chi(S)|}{\mathfrak{m}}\pi|\log\varepsilon| + C, \tag{4.83}$$

for some constant $C > 0$ independent of ε, we can find a measure $\mu \in X^{\mathfrak{m}}$ such that, up to a subsequence:

$$\operatorname{vort}(u_\varepsilon) \overset{\flat}{\rightharpoonup} \mu \text{ flat in } S. \tag{4.84}$$

Furthermore, there exists $u \in \mathcal{LS}^{(\mathfrak{m})}$ with $\operatorname{vort}(u) = \mu$ such that, up to possibly taking a further subsequence:

$$u_\varepsilon \rightharpoonup u \text{ in } SBV^2_{\mathrm{loc}}(T(S\setminus \operatorname{spt}\mu)) \cap SBV(TS). \tag{4.85}$$

(ii) *(Γ-liminf) Let $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}$ such that $u_\varepsilon \to u \in L^1(S)$, then:*

$$\liminf_{\varepsilon\to 0} GGL_\varepsilon(u_\varepsilon) - \frac{|\chi(S)|}{\mathfrak{m}}\pi|\log\varepsilon| \geq \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1(J_u) + \mathfrak{m}|\chi(S)|\gamma_{\mathfrak{m}}. \tag{4.86}$$

(iii) *(Γ-limsup) For any $u \in \mathcal{LS}^{(\mathfrak{m})}$, we can find a sequence $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}$ such that:*

$$\limsup_{\varepsilon\to 0} GGL_\varepsilon(u_\varepsilon) - \frac{|\chi(S)|}{\mathfrak{m}}\pi|\log\varepsilon| \leq \mathcal{W}^{(\mathfrak{m})}(u) + \mathcal{H}^1(J_u) + \mathfrak{m}|\chi(S)|\gamma_{\mathfrak{m}}. \tag{4.87}$$

4.3 Proof of Gamma-convergence

4.3.1 Compactness

The following estimate will turn up to be useful:

Lemma 4.11
Given a simply connected, open subset $U \subset S$ and sequences $(v_n), (w_n) \subset \mathcal{AS}^{(\mathfrak{m})}(U)$ such that:

$$\lim_{n\to\infty} \|v_n - w_n\|_{L^2}(\|\nabla v_n\|_{L^2} + \|\nabla w_n\|_{L^2}) = 0,$$

it holds that:

$$\mathrm{djac}(v_n) - \mathrm{djac}(w_n) \overset{\flat}{\rightharpoonup} 0 \text{ flat in } U. \tag{4.88}$$

Proof. The proof is the same as in the flat setting. (See also Lemma 2.1 in [8].) □

On several occasions, we will employ:

Lemma 4.12
Given an open subset $U \subset S$ and $v \in W^{1,1}_{\mathrm{loc}}(TU)$, the following holds true at a.e. point in U:

$$\nabla v = \begin{cases} \mathrm{d}|v| \otimes \frac{v}{|v|} + \frac{1}{|v|}\,\mathrm{jac}(v) \otimes \frac{v^\perp}{|v|}, & \text{if } v \neq 0, \\ 0 & \text{else.} \end{cases} \tag{4.89}$$

Proof. With the product rule, we compute at a.e. point in $\{v \neq 0\}$:

$$\begin{aligned} \nabla v = \nabla\left(|v|\frac{v}{|v|}\right) &= \mathrm{d}|v| \otimes \frac{v}{|v|} + |v|\nabla\frac{v}{|v|} \\ &= \mathrm{d}|v| \otimes \frac{v}{|v|} + |v|\,\mathrm{jac}\left(\frac{v}{|v|}\right) \otimes \frac{v^\perp}{|v|} \\ &= \mathrm{d}|v| \otimes \frac{v}{|v|} + \frac{1}{|v|}\,\mathrm{jac}(v) \otimes \frac{v^\perp}{|v|}. \end{aligned}$$

As for almost all points in $\{v = 0\}$ we have that $\nabla v = 0$, the equality in (4.89) remains true at a.e. point in $\{v = 0\}$. □

Given an open set $U \subset S$, we will shortly write:

$$\mathcal{AS}(U) \coloneqq \mathcal{AS}^{(1)}(U).$$

On $\mathcal{AS}(U)$ as well as on $\mathcal{AS}^{(\mathfrak{m})}(U)$, we define the classic Ginzburg-Landau energy functional as:

$$GL_\varepsilon(u) \coloneqq \frac{1}{2}\int_U |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \operatorname{vol}.$$

Furthermore, for $\tau > 0$, we define $\mathrm{T}_\tau \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ by:

$$\mathrm{T}_\tau(x) \coloneqq \min\{\tau^{-1}|x|, 1\}\frac{x}{|x|}.$$

Lemma 4.13
Let $U \subset S$ be a simply connected open set, $e \in C^\infty(\bar{U};\mathbb{S}^1)$, and $(v_\varepsilon) \subset \mathcal{AS}(U)$ be a bounded sequence in $L^\infty(TU)$ satisfying the energy bound:

$$GL_\varepsilon(v_\varepsilon) \le C|\log\varepsilon|$$

for some constant C independent of ε, then:

$$\operatorname{vort}(|v_\varepsilon|e) \overset{\flat}{\rightharpoonup} 0 \textit{ flat in } U. \tag{4.90}$$

Proof. By approximation and Lemma 4.11, we can assume without loss of generality that each v_ε is smooth in U. Due to the energy bound and the definition of GGL_ε, we can apply Theorem 4.6 for sufficiently small $\varepsilon > 0$ with $\alpha = \frac{1}{2}$, $\beta = \frac{1}{5}$, and $r = \varepsilon^{\frac{1}{3}}$. Let $\mathfrak{B}_\varepsilon$ denote the family of balls corresponding to the choices made above, and $V_\varepsilon \coloneqq U_\varepsilon \cap \bigcup_{B\in\mathfrak{B}_\varepsilon} B$, where $U_\varepsilon \coloneqq \{p \in U \colon \operatorname{dist}(p,\partial U) > \varepsilon\}$. Furthermore, we will shortly write $w_\varepsilon \coloneqq |v_\varepsilon|e$ and $\tilde{w}_\varepsilon = \mathrm{T}_{1-\delta_\varepsilon} w_\varepsilon$ with $\delta_\varepsilon \coloneqq \varepsilon^{\frac{1}{5}}$. We then derive with the boundedness of (v_ε) in L^∞:

$$\begin{aligned}\|w_\varepsilon - \tilde{w}_\varepsilon\|_{L^2(TU)} &= \|w_\varepsilon - \tilde{w}_\varepsilon\|_{L^2(TV_\varepsilon)} + \|w_\varepsilon - \tilde{w}_\varepsilon\|_{L^2(T(U\setminus V_\varepsilon))}\\ &\le (\sup_\varepsilon \|v_\varepsilon\|_{L^\infty(TU)} + 1)\operatorname{vol}(V_\varepsilon) + \operatorname{vol}(U\setminus V_\varepsilon)\delta_\varepsilon\\ &\le C\varepsilon^{\frac{1}{6}} + \operatorname{vol}(U)\varepsilon^{\frac{1}{5}} \le C(U)\varepsilon^{\frac{1}{6}}.\end{aligned}$$

Moreover, by the product rule:

$$\nabla w_\varepsilon = \nabla(|v_\varepsilon|e) = \mathrm{d}|v_\varepsilon| \otimes e + |v_\varepsilon|\nabla e = \mathrm{d}|v_\varepsilon| \otimes e + |v_\varepsilon| \operatorname{jac}(e) \otimes e^\perp,$$

and therefore:

$$|\nabla w_\varepsilon|^2 = |\mathrm{d}|v_\varepsilon||^2 + |v_\varepsilon|^2|\operatorname{jac}(e)|^2. \tag{4.91}$$

Due to (4.89), we also have:

$$|\mathrm{d}|v_\varepsilon||^2 = |\nabla v_\varepsilon|^2 - \frac{1}{|v_\varepsilon|^2}|\operatorname{jac}(v_\varepsilon)|^2 \le |\nabla v_\varepsilon|^2.$$

Hence with (4.91), the energy-bound on (v_ε), and the boundedness of (v_ε) in L^∞:

$$\|\nabla w_\varepsilon\|_{L^2} + \|\nabla\tilde{w}_\varepsilon\|_{L^2} \le 3\|\nabla v_\varepsilon\|_{L^2} + C(e) \le C(e)|\log\varepsilon|, \tag{4.92}$$

where in the first estimate we have used the fact that $|\nabla\tilde{w}_\varepsilon| \leq 2|\nabla w_\varepsilon|$ for ε small enough. By Lemma 4.11, it is therefore sufficient to show $\operatorname{vort}(\tilde{w}_\varepsilon) \stackrel{\flat}{\rightharpoonup} 0$ in U. For this purpose, let $\varphi \in C_0^{0,1}(U)$ be an admissible test function with Lipschitz constant $L_\varphi \leq 1$. We first split up $\int_U \varphi \operatorname{vort}(\tilde{w}_\varepsilon)$ as follows:

$$\int_U \varphi \operatorname{vort}(\tilde{w}_\varepsilon) = \int_{U_\varepsilon \setminus V_\varepsilon} \varphi \operatorname{vort}(\tilde{w}_\varepsilon) + \int_{V_\varepsilon} \varphi \operatorname{vort}(\tilde{w}_\varepsilon) + \int_{U \setminus U_\varepsilon} \varphi \operatorname{vort}(\tilde{w}_\varepsilon).$$

As $|\tilde{w}_\varepsilon| = 1$ in $U_\varepsilon \setminus V_\varepsilon$, we have that $\operatorname{vort}(w_\varepsilon) = 0$ in $U_\varepsilon \setminus V_\varepsilon$, and therefore:

$$\int_{U \setminus V_\varepsilon} \varphi \operatorname{vort}(\tilde{w}_\varepsilon) = 0.$$

By the definition of $\tilde{w}_\varepsilon$ and the smoothness of e, it holds that $\deg(\tilde{w}_\varepsilon, \partial B) = \deg(e, \partial B) = 0$ for every $B \in \tilde{\mathfrak{B}}_\varepsilon := \{B \in \mathfrak{B}_\varepsilon \colon B \cap U_\varepsilon \neq \emptyset\}$, and hence with (4.33) we can estimate:

$$\begin{aligned}\int_{V_\varepsilon} \varphi \operatorname{vort}(\tilde{w}_\varepsilon) &= \sum_{B_r(p) \in \tilde{\mathfrak{B}}_\varepsilon} \int_{B_r(p) \cap U} (\varphi - \varphi(p)) \operatorname{vort}(\tilde{w}_\varepsilon) \\ &\leq \sum_{B_r(p) \in \tilde{\mathfrak{B}}_\varepsilon} L_\varphi r \int_{B_r(p) \cap U} |\operatorname{vort}(\tilde{w}_\varepsilon)| \operatorname{vol} \\ &\leq C(S)\varepsilon^{\frac{1}{3}} \left(1 + \int_U |\nabla \tilde{w}_\varepsilon|^2 \operatorname{vol}\right) \leq C(S)\varepsilon^{\frac{1}{3}} |\log\varepsilon|.\end{aligned}$$

As each point $p \in U \setminus U_\varepsilon$ has distance at most ε from the boundary we derive by using $\varphi = 0$, and again (4.33):

$$\begin{aligned} int_{U \setminus U_\varepsilon} \varphi \operatorname{vort}(\tilde{w}_\varepsilon) &\leq L_\varphi \varepsilon int_{U \setminus U_\varepsilon} |\operatorname{vort}(\tilde{w}_\varepsilon)| \operatorname{vol} \\ &\leq C(S)\varepsilon \left(1 + \int_U |\nabla \tilde{w}_\varepsilon|^2 \operatorname{vol}\right) \leq C(S)\varepsilon |\log\varepsilon|.\end{aligned}$$

By the arbitrariness of φ, we then derive for ε sufficiently small that:

$$\|\operatorname{vort}(\tilde{w}_\varepsilon)\|_\flat \leq C(S)\varepsilon^{\frac{1}{3}} |\log\varepsilon| \to 0,$$

which concludes the proof. □

Proposition 4.4 (Initial Compactness)
Given a sequence $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}$ bounded in $L^\infty(TS)$ and satisfying the energy bound

$$GL_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$$

for a constant C independent of ε, there exists a measure $\mu \in \tilde{X}^{(\mathfrak{m})}$ such that, up to a subsequence:

$$\operatorname{vort}(u_\varepsilon) \stackrel{\flat}{\rightharpoonup} \mu \text{ flat in } S. \tag{4.93}$$

Proof. 1. Step: We start by localizing the problem. Let $U^{(1)}, \ldots, U^{(N)}$ be a cover of S with coordinate neighborhoods. By possibly shrinking each $U^{(i)}$, we can assume that for each i, there exists another coordinate neighborhood $\tilde{U}^{(i)}$ with $U^{(i)} \subset\subset \tilde{U}^{(i)}$. In the following, we will fix i and shortly write $\tilde{U} = \tilde{U}^{(i)}$, as well as $U := U^{(i)}$. Furthermore, we choose an arbitrary spin field $e \in C^\infty(\tilde{U}; \mathbb{S}^1)$ and set $v_\varepsilon := P_e(u_\varepsilon)$ in $\tilde{U}$. Note first that by (4.36) we have:

$$\begin{aligned}\int_U |\nabla v_\varepsilon|^2 \operatorname{vol} = \mathfrak{m}^2 \int_U |\nabla u_\varepsilon|^2 \operatorname{vol} + (1-\mathfrak{m}^2) \int_U |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} \\ + (\mathfrak{m}-1)^2 \int_U |u_\varepsilon|^2 |\nabla e|^2 \operatorname{vol} - 2\mathfrak{m}(\mathfrak{m}-1) \int_U \langle \operatorname{jac}(u_\varepsilon), \operatorname{jac}(e)\rangle \operatorname{vol}.\end{aligned}$$

With Young's inequality and (4.89) we can estimate:

$$\int_U \langle \operatorname{jac}(u_\varepsilon), \operatorname{jac}(e)\rangle \operatorname{vol} \le \int_U |u_\varepsilon||\nabla u_\varepsilon||\nabla e| \operatorname{vol} \le \frac{1}{2}\int_U |u_\varepsilon|^2 |\nabla u_\varepsilon|^2 \operatorname{vol} + \frac{1}{2}\int_U |\nabla e|^2 \operatorname{vol}.$$

Consequently, by the energy bound on (u_ε) and the boundedness of (u_ε) in L^∞ we follow:

$$\begin{aligned}\int_U |\nabla v_\varepsilon|^2 \operatorname{vol} &\le C + \mathfrak{m}^2 \int_U |\nabla u_\varepsilon|^2 \operatorname{vol} + \int_U |u_\varepsilon|^2 |\nabla u_\varepsilon|^2 \operatorname{vol} \\ &\le C(1 + |\log \varepsilon|) \le C|\log \varepsilon|\end{aligned}$$

for a constant C independent of ε. Using standard approximation results in Sobolev spaces, we can find a sequence $(w_\varepsilon) \subset C^\infty(T\tilde{U})$ such that:

$$\lim_{\varepsilon\to 0} \|w_\varepsilon - v_\varepsilon\|_{W^{1,2}(\tilde{U})} = 0. \tag{4.94}$$

Hence, with the previous reasoning, (w_ε) satisfies the logarithmic energy bound in (4.42). Consequently, for each $\varepsilon > 0$ small enough, we can apply Theorem 4.6 for sufficiently small $\varepsilon > 0$ with $\alpha = \frac{1}{2}$, $\beta = \frac{1}{5}$, and $r = \varepsilon^{\frac{1}{3}}$. Let $\tilde{\mathfrak{B}}_\varepsilon$ denote the finite family of geodesic balls provided by Theorem 4.6 and $\tilde{D}_{r_\varepsilon} := D_{\tilde{\mathfrak{B}}_\varepsilon}(\tilde{U}_\varepsilon)$. Moreover:

$$\mathfrak{B}_\varepsilon := \{B \in \tilde{\mathfrak{B}}_\varepsilon : B \cap U \neq \emptyset\}, \quad D_{r_\varepsilon} := D_{\mathfrak{B}_\varepsilon}(U).$$

We end up with a collection $\mathfrak{B}_\varepsilon$, satisfying:

$$\sup_\varepsilon D_{r_\varepsilon} \le \sup_\varepsilon \tilde{D}_{r_\varepsilon} < \infty, \tag{4.95}$$

$$V_\varepsilon = U \cap \bigcup_{B \in \mathfrak{B}_\varepsilon} B \supset \{|1 - |w_\varepsilon|| \cap U \ge \varepsilon^{\frac{1}{5}}\}, \tag{4.96}$$

$$r(\mathfrak{B}_\varepsilon) \le r(\tilde{\mathfrak{B}}_\varepsilon) = \varepsilon^{\frac{1}{3}}. \tag{4.97}$$

Note that (4.95) follows from (4.44) and the fact that $U \subset \tilde{U}_\varepsilon$ for $\varepsilon > 0$ small enough. The set relation in (4.96) can be proved using Item (iii) in Theorem 4.6, and again the fact that $U \subset \tilde{U}_\varepsilon$ for $\varepsilon > 0$ small enough. Lastly, (4.97) is implied by Item (i) of Theorem 4.6. Let us set:

$$\nu_\varepsilon := \sum_{B_r(p) \in \mathfrak{B}_\varepsilon} \deg(\tilde{w}_\varepsilon, \partial B_r(p))\delta_p, \text{ where } \tilde{w}_\varepsilon := \mathrm{T}_{1-\varepsilon^{\frac{1}{5}}} v_\varepsilon.$$

2. Step: We wish to derive a compactness result for (ν_ε). Note that by the smoothness of $\tilde{w}_\varepsilon$, the degree $\deg(\tilde{w}_\varepsilon, \partial B) \in \mathbb{Z}$ for any $B \in \mathfrak{B}_\varepsilon \cap U$, hence the measure ν_ε has only weights in $\mathbb{Z}$. Furthermore, by (4.44) and the definition of ν_ε:

$$\sup_{\varepsilon\in(0,\varepsilon_0)} |\nu_\varepsilon|(U) = \sup_{\varepsilon\in(0,\varepsilon_0)} D_{\varepsilon^{\frac{1}{3}}} < \infty.$$

As a consequence, we can find a point measure ν with weights in $\mathbb{Z}$ such that, up to a subsequence:

$$\nu_\varepsilon \overset{*}{\rightharpoonup} \nu \text{ weak* in } U, \text{ and therefore also } \nu_\varepsilon \overset{\flat}{\rightharpoonup} \nu \text{ flat in } U. \tag{4.98}$$

Our aim now is to estimate the flat distance between $\mathrm{vort}(v_\varepsilon)|_U$ and ν_ε. With (4.94) and Lemma 4.11, we derive that $\mathrm{vort}(v_\varepsilon)|_U - \mathrm{vort}(w_\varepsilon)|_U \overset{\flat}{\rightharpoonup} 0$ flat in U. By the definition of $\tilde{w}_\varepsilon$, (4.96), and again Lemma 4.11, we also have that $\mathrm{vort}(w_\varepsilon)|_U - \mathrm{vort}(\tilde{w}_\varepsilon)|_U \overset{\flat}{\rightharpoonup} 0$ flat in U. It remains to estimate the flat distance between $\mathrm{vort}(\tilde{w}_\varepsilon)|_U$ and ν_ε. We follow the same strategy as in the proof of Lemma 4.13: Consider an arbitrary test function $\varphi \in C_0^{0,1}(U)$ with Lipschitz constant $L_\varphi \le 1$. As $|\tilde{w}_\varepsilon| = 1$ in $U \setminus V_\varepsilon$, where V_ε was defined as $U \cap \bigcup_{B\in\mathfrak{B}_\varepsilon} B$, we have $\mathrm{vort}(\tilde{w}_\varepsilon) = 0$ in $U \setminus V_\varepsilon$, and therefore:

$$\int_{U\setminus V_\varepsilon} \varphi \,\mathrm{vort}(\tilde{w}_\varepsilon) = 0.$$

For each ball $B \in \mathfrak{B}_\varepsilon \cap U$, we have by the definition of ν_ε:

$$\int_B \mathrm{vort}(\tilde{w}_\varepsilon) = \deg(\tilde{w}_\varepsilon, \partial B) = \nu_\varepsilon(B).$$

Consequently, we derive by (4.97) and (4.33), respectively:

$$\begin{aligned}
&\left(\int_{V_\varepsilon} \varphi \,\mathrm{vort}(\tilde{w}_\varepsilon)\right) - \langle \nu_\varepsilon, \varphi\rangle \\
&= \sum_{B_r(p)\in\mathfrak{B}_\varepsilon} \left(\int_{B_r(p)\cap U} \varphi \,\mathrm{vort}(\tilde{w}_\varepsilon)\right) - \deg(\tilde{w}_\varepsilon, \partial B_r(p))\varphi(p) \\
&= \sum_{B_r(p)\in\mathfrak{B}_\varepsilon} \int_{B_r(p)\cap U} (\varphi - \varphi(p)) \,\mathrm{vort}(\tilde{w}_\varepsilon) \\
&\le \sum_{B_r(p)\in\mathfrak{B}_\varepsilon\cap u} L_\varphi r \int_{B_r(p)} |\mathrm{vort}(\tilde{w}_\varepsilon)| \le C\varepsilon^{\frac{1}{3}} |\log\varepsilon|.
\end{aligned}$$

By the arbitrariness of φ, this then leads to:

$$\|\mathrm{vort}(\tilde{w}_\varepsilon)|_U - \nu_\varepsilon\|_\flat \le C\varepsilon^{\frac{1}{3}} |\log\varepsilon| \to 0,$$

showing that, up to a subsequence:

$$\mathrm{vort}(v_\varepsilon) \overset{\flat}{\rightharpoonup} \nu \text{ in } U,$$

and by (4.38) as well as Lemma 4.13, we then conclude for the same subsequence as before:

$$\mathrm{vort}(u_\varepsilon)|_U = \frac{1}{\mathfrak{m}} \mathrm{vort}(v_\varepsilon)|_U + \frac{\mathfrak{m}-1}{\mathfrak{m}} \mathrm{vort}(|v_\varepsilon| e) \overset{\flat}{\rightharpoonup} \frac{1}{\mathfrak{m}} \nu \text{ flat in } U.$$

3. Step: The global result follows by a partition of unity argument: In this regard, let $\{\rho_1, \dots, \rho_N\}$ denote a partition of unity subordinate to the cover $\{U^{(i)}\}_i$. In the previous step, we have seen that we can find for each $i \in \{1, \dots, N\}$ a point measure $\mu^{(i)}$ with weights in $\frac{1}{\mathtt{m}}\mathbb{Z}$ such that, up to taking subsequences:

$$\operatorname{vort}(u_\varepsilon)|_U \overset{\flat}{\rightharpoonup} \mu^{(i)} \text{ flat in } U^{(i)}.$$

Let μ denote the point measure on S such that $\mu = \mu^{(i)}$ in $U^{(i)}$ for every i. Note that the measure μ is well defined as $\mu^{(i)} = \mu^{(j)}$ in $U^{(i)} \cap U^{(j)}$, which can be seen by testing with functions compactly supported in $U^{(i)} \cap U^{(j)}$. We have found a point measure μ with weights in $\frac{1}{\mathtt{m}}\mathbb{Z}$ such that for any test function $\varphi \in C^{0,1}(S)$, up to a subsequence:

$$\int_S \varphi \operatorname{vort}(u_\varepsilon) = \sum_{i=1}^N \int_{U^{(i)}} \rho_i \varphi \operatorname{vort}(u_\varepsilon) \to \sum_{i=1}^N \langle \rho_i \varphi, \mu \rangle = \langle \varphi, \mu \rangle,$$

and hence $\operatorname{vort}(u_\varepsilon) \overset{\flat}{\rightharpoonup} \mu$ flat in S. In order to conclude the proof, it remains to show $\mu(S) = \chi(S)$ (see also the definition of $\tilde{X}^{(\mathtt{m})}$), which follows by (4.39) and the flat convergence of $\operatorname{vort}(u_\varepsilon)$:

$$\chi(S) = \langle \operatorname{vort}(u_\varepsilon), \mathbb{1}_S \rangle \to \langle \mu, \mathbb{1}_S \rangle = \mu(S).$$

□

Definition 4.9
Given a simply connected open subset $U \subset S$, we call a vector field $e \in C^\infty(U, \mathbb{S}^1)$ *harmonic* if and only if $\operatorname{jac}(e) = \mathrm{d}^*\Phi$ (d^* denotes the adjoint of the exterior derivative), where $\Phi \in \Omega^2(U)$ is the 2-form satisfying:

$$\begin{cases} \Delta\Phi = -\kappa \operatorname{vol} & \text{in } U, \\ \Phi = 0 & \text{on } \partial U, \end{cases} \tag{4.99}$$

Δ being the Laplace-Beltrami operator on S (in the current setting $\Delta = \mathrm{d}\mathrm{d}^*$, see also, e.g., [42] for further clarification).

Let us show that the notion above is not vacuous:

Lemma 4.14 (Existence of harmonic frames)
On any simply connected, open subset $U \subset S$, there exists a harmonic vector field e, as described in Definition 4.9.

Proof. In [42], it is shown that there exists a (unique) two-form Φ, solving (4.99). Fix an arbitrary seed point $p_0 \in U$ and a unit vector $u_0 \in T_{p_0}U$. We define e at a point $p \in U$ as follows: Let $\gamma \colon [0,1] \to U$ be a smooth curve with $\gamma(0) = p_0$ and $\gamma(1) = p$. Then by classic ODE theory there exists a unique smooth vector-field $X \colon [0,1] \to TU$ on γ, solving

$$\begin{cases} \nabla_{\gamma'(s)} X(s) = \mathrm{d}^*\Phi(\gamma'(s)) X^\perp(s) & \text{in } (0,1), \\ X(0) = u_0. \end{cases}$$

We need to check that setting $u(p) := X(1)$ is path-independent. In this regard, consider another curve $\mu\colon [0,1] \to U$ with $\mu(0) = p_0$, $\mu(1) = p$, and $Y\colon [0,1] \to TU$ being the solution of:

$$\begin{cases} \nabla_{\mu'(s)} Y(s) = \mathrm{d}^*\Phi(\mu'(s)) Y^\perp(s) & \text{in } (0,1), \\ \qquad Y(0) = u_0, \end{cases}$$

We need to show that $Y(1) = X(1)$: As U is simply connected, the curves γ and μ enclose a (possibly disconnected) region ω. Then by Stokes' theorem, the difference angle δ between $X(1)$ and $Y(1)$ satisfies:

$$\begin{aligned} &\int_0^1 \mathrm{d}^*\,\Phi(\gamma'(s))\,\mathrm{d}s - \int_0^1 \mathrm{d}^*\,\Phi(\mu(s))\,\mathrm{d}s + \int_\omega \kappa\,\mathrm{vol} \\ &\quad = \int_\omega \mathrm{d}\mathrm{d}^*\Phi + \int_\omega \kappa\,\mathrm{vol} = \int_\omega \Delta\Phi + \int_\omega \kappa\,\mathrm{vol} = 0 \mod 2\pi. \end{aligned}$$

This completes the proof of well-definedness. Finally, note that by construction, the condition $\mathrm{jac}(e) = \mathrm{d}^*\Phi$ is automatically satisfied. □

The main reason why we have investigated harmonic vector fields is that they allow us to control a defect term appearing on the right-hand side of (4.36):

Lemma 4.15 (Convergence of the defect term)
Let $U \subset m$ be a simply connected open set and $(v_\varepsilon) \subset \mathcal{AS}(U)$ such that:

$$\mathrm{vort}(v_\varepsilon) \overset{\flat}{\rightharpoonup} k\delta_p, \quad k \in \mathbb{Z},\, p \in U,$$

and let e be a harmonic vector field on U (see also Lemma 4.14). Then, as $\varepsilon \to 0$, we have:

$$\int_U \langle \mathrm{jac}(v_\varepsilon), \mathrm{jac}(e)\rangle\,\mathrm{vol} \to k \cdot \star\Phi(p) - \int_U \kappa\,\Phi, \tag{4.100}$$

where $\mathrm{jac}(e) = \mathrm{d}^*\Phi$ *for* $\Phi \in \Omega^2(U)$ *satisfying* (4.99).

Proof. Employing (4.99) and integrating by parts leads together with the definition of $\mathrm{vort}(v_\varepsilon)$ to

$$\begin{aligned} \int_U \langle \mathrm{jac}(v_\varepsilon), \mathrm{jac}(e)\rangle\,\mathrm{vol} &= \int_U \langle \mathrm{jac}(v_\varepsilon), \mathrm{d}^*\Phi\rangle\,\mathrm{vol} \\ &= \int_U \langle \mathrm{d}\mathrm{jac}(v_\varepsilon), \Phi\rangle\,\mathrm{vol} + \int_{\partial U} \star\Phi\,\mathrm{jac}(v_\varepsilon) \\ &= \int_U \langle \mathrm{d}\mathrm{jac}(v_\varepsilon), \Phi\rangle\,\mathrm{vol} \overset{\varepsilon\to 0}{\to} k \cdot \star\Phi(p) - \int_U \kappa\,\Phi, \end{aligned}$$

as is desired. □

We continue by showing a localized 0-order Γ-liminf inequality:

Lemma 4.16 (Localized Γ-liminf inequality)
Let $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}(S)$ be a bounded sequence in $L^\infty(TS)$ such that:

$$\mathrm{vort}(u_\varepsilon) \overset{\flat}{\rightharpoonup} \mu := \sum_{k=1}^{K} \frac{d_k}{\mathfrak{m}} \delta_{p_k} \in \tilde{X}^{(\mathfrak{m})}.$$

Then there exists a constant C independent of ε such that for every $k \in \{1, \dots, K\}$ and $r > 0$ small enough, it holds that:

$$\liminf_{\varepsilon\to 0}\left(GL_\varepsilon(u_\varepsilon, B_r(p_k)) - \frac{\pi|d_k|}{\mathfrak{m}^2}\log\left(\frac{r}{\varepsilon}\right)\right) \geq C. \tag{4.101}$$

Proof. Without loss of generality, we can assume that $\operatorname{sgn}(\chi(S)) > 0$ (the other case works in the same fashion). Let $r_0 > 0$ be small enough such that the balls in $\{B_{r_0}(p_k)\}_k$ are disjoint. Furthermore, fix $B_0 := B_{r_0}(p_k)$ for some k, and let e be a harmonic vector field in B_0. Setting $v_\varepsilon := P_e(u_\varepsilon)$ in B_0, we see by Lemma 4.13 and (4.38) that for any test function $\varphi \in C_0^{0,1}(B_0)$:

$$\int_{B_0}\varphi\operatorname{vort}(v_\varepsilon) = \mathfrak{m}\int_{B_0}\varphi\operatorname{vort}(u_\varepsilon) - (\mathfrak{m}-1)\int_{B_0}\varphi\operatorname{vort}(|v_\varepsilon|e) \to \delta_{p_k}.$$

By the arbitrariness of φ, we follow:

$$\operatorname{vort}(v_\varepsilon) \xrightarrow{\flat} \delta_{p_k} \text{ flat in } B_0.$$

Moreover, with (4.36) and the boundedness of (u_ε) in L^∞, we derive that:

$$\begin{aligned}
GL_\varepsilon(u_\varepsilon, B_0) \geq{}& \frac{1}{2\mathfrak{m}^2}\int_{B_0}|\nabla v_\varepsilon|^2\operatorname{vol} + \frac{1}{4\varepsilon^2}\int_{B_0}(1-|v_\varepsilon|^2)^2\operatorname{vol}\\
&- \frac{(\mathfrak{m}-1)^2}{2\mathfrak{m}^2}\int_{B_0}|u_\varepsilon|^2|\nabla e|^2\operatorname{vol} + \frac{\mathfrak{m}^2-1}{2\mathfrak{m}^2}\int_{B_0}|\mathrm{d}|u_\varepsilon||^2\operatorname{vol}\\
&+ \frac{2m(\mathfrak{m}-1)}{2\mathfrak{m}^2}\int_{B_0}\langle\operatorname{jac}(u_\varepsilon), \operatorname{jac}(e)\rangle\operatorname{vol}\\
\geq{}& C + \frac{1}{\mathfrak{m}^2}GL_\varepsilon(v_\varepsilon) + \frac{1}{2}\frac{\mathfrak{m}-1}{\mathfrak{m}}\int_{B_0}\langle\operatorname{jac}(u_\varepsilon), \operatorname{jac}(e)\rangle\operatorname{vol}
\end{aligned}$$

for a constant C independent of ε. By (4.37), we have:

$$\operatorname{jac}(u_\varepsilon) = \frac{1}{\mathfrak{m}}\operatorname{jac}(v_\varepsilon) + \frac{\mathfrak{m}-1}{\mathfrak{m}}|u_\varepsilon|^2\operatorname{jac}(e),$$

and therefore, we can rewrite the previous estimate as follows:

$$GL_\varepsilon(u_\varepsilon, B_0) \geq C + \frac{1}{\mathfrak{m}^2}GL_\varepsilon(v_\varepsilon) + c(\mathfrak{m})\int_B\langle\operatorname{jac}(v_\varepsilon), \operatorname{jac}(e)\rangle\operatorname{vol},$$

where C is a constant independent of ε and r, and $c(\mathfrak{m}) := \frac{1}{2}\frac{\mathfrak{m}-1}{\mathfrak{m}^2}$. Consequently, with Lemma 4.15 and the localized liminf inequality in the nonfractional setting (see also [43]), there exists a constant $C \in \mathbb{R}$ independent of ε, r and k such that for r small enough:

$$\liminf_{\varepsilon\to 0}\left(GL_\varepsilon(u_\varepsilon, B_0) - \pi\frac{|d_k|}{\mathfrak{m}^2}\log\frac{r}{\varepsilon}\right) \geq \frac{C}{\mathfrak{m}^2} + c(\mathfrak{m})\left(d_k(\star\Phi)(p_k) - \int_B\kappa\,\Phi\right).$$

It remains to estimate the second term above, independently from r. Let G denote the Green's function in B, we derive for any $p \in B$

$$\begin{aligned}
|\Phi(p)| = \left|-\int_B G(q,p)\kappa(q)\operatorname{vol}(q)\right| &\leq \int_B G(q,p)|\kappa(q)|\operatorname{vol}(q)\\
&\leq \|\kappa\|_\infty\int_B G(q,p)\operatorname{vol}(q) = \|\kappa\|_\infty,
\end{aligned}$$

which concludes the proof. □

We are ready to prove the first half of the compactness statement in Theorem 4.7:

Theorem 4.8 (Vortex compactness)
Let $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}(S)$ be a bounded sequence in $L^\infty(TS)$ satisfying the energy bound:

$$GL_\varepsilon(u_\varepsilon) \leq \frac{|\chi(S)|}{\mathfrak{m}}\pi|\log\varepsilon| + C, \tag{4.102}$$

for some constant C independent of ε. Then there exists a point measure $\mu \in X^{(\mathfrak{m})}$ such that, up to a subsequence, $\operatorname{vort}(u_\varepsilon) \stackrel{\flat}{\rightharpoonup} \mu$ flat in S.

Proof. By Proposition 4.4, up to a subsequence, it holds that:

$$\operatorname{vort}(u_\varepsilon) \stackrel{\flat}{\rightharpoonup} \mu = \sum_{k=1}^{K} \frac{d_k}{\mathfrak{m}}\delta_{p_k} \quad d_k \in \mathbb{Z},\, p_k \neq p_l \text{ for } k \neq l.$$

As $\mu(S) = \chi(S)$, in order to conclude, it remains to show that $d_k = \operatorname{sgn}(\chi(S))$ for all k, which would directly lead to $\mu \in X^{(\mathfrak{m})}$. By Lemma 4.16 and (4.102) there exists a constant C independent of ε, r, and k, such that for r and ε small enough:

$$\begin{aligned}\frac{|\chi(S)|}{\mathfrak{m}}|\log\varepsilon| &\geq \sum_{k=1}^{K} GL(u_\varepsilon, B_r(p_k)) \\ &\geq \sum_{k=1}^{K} \frac{|d_i|}{\mathfrak{m}^2}\log\frac{r}{\varepsilon} \geq \frac{|\mu|}{\mathfrak{m}}|\log\varepsilon| - C.\end{aligned}$$

By making ε sufficiently small, the above inequality can only be true if $|\mu| \leq |\chi(S)|$. As $\mu \in \tilde{X}^{(\mathfrak{m})}$, the reverse inequality $|\mu| \geq |\mu(S)| = |\chi(S)|$ also holds true, and therefore $|\mu| = |\chi(S)|$. This means that for all k, we have $\operatorname{sgn}(d_k) = \operatorname{sgn}(\chi(S))$. It remains to show that there exists no k with $|d_k| \geq 2$. Fix $k \in \{1, \ldots, K\}$, and consider the geodesic ball $B := B_r(p_k)$ for $r > 0$ small enough, so that no other vortex center p_l, where $l \neq k$ is contained in B. Furthermore, consider an arbitrary harmonic vector field e in B and set $v_\varepsilon := P_e(u_\varepsilon)$ in B. With (4.36) and (4.37), we then derive

$$\begin{aligned}GL_\varepsilon(v_\varepsilon) = &\frac{\mathfrak{m}^2}{2}\int_B |\nabla u_\varepsilon|^2 \operatorname{vol} + \frac{1}{4\varepsilon^2}\int_B (1-|u_\varepsilon|^2)^2 \operatorname{vol} + \frac{1-\mathfrak{m}^2}{2}\int_B |d|u||^2 \operatorname{vol} \\ &+ \frac{(\mathfrak{m}-1)^2}{2}\int_B |u_\varepsilon|^2|\nabla e|^2 \operatorname{vol} - \mathfrak{m}(\mathfrak{m}-1)\int_B \langle \operatorname{jac}(u_\varepsilon), \operatorname{jac}(e)\rangle \operatorname{vol} \\ \leq\ &\mathfrak{m}^2 GL_\varepsilon(u_\varepsilon, B) - \frac{(\mathfrak{m}-1)^2}{2}\int_B |u_\varepsilon|^2|\nabla e|^2 \operatorname{vol} \\ &+ (\mathfrak{m}-1)\int_B \langle \operatorname{jac}(v_\varepsilon), \operatorname{jac}(e)\rangle \operatorname{vol} \\ \leq\ &\mathfrak{m}^2 GL_\varepsilon(u_\varepsilon, B) + (\mathfrak{m}-1)\int_B \langle \operatorname{jac}(v_\varepsilon), \operatorname{jac}(e)\rangle \operatorname{vol}.\end{aligned}$$

By (4.101), (4.102), $|d_l| = \operatorname{sgn}(\chi(S))d_l$ for all $l \in \{1, \ldots, K\}$, and $|\mu| = |\chi(S)|$, the first term in the last line above can be estimated for $r > 0$ small enough as follows:

$$\begin{aligned}
GL_\varepsilon(u_\varepsilon, B) &\le GL_\varepsilon(u_\varepsilon) - \sum_{l\neq k} GL_\varepsilon(u_\varepsilon, B_r(p_l)) \\
&\le \pi \frac{|\chi(S)|}{\mathfrak{m}} |\log\varepsilon| - \pi \operatorname{sgn}(\chi(S)) \frac{1}{\mathfrak{m}} \sum_{l\neq k} \frac{d_l}{\mathfrak{m}} \log\frac{r}{\varepsilon} \\
&\le \pi \frac{|\chi(S)|}{\mathfrak{m}} |\log\varepsilon| - \pi \operatorname{sgn}(\chi(S)) \frac{1}{\mathfrak{m}} (\mu - \frac{d_k}{\mathfrak{m}}) |\log\varepsilon| + C(r) \\
&\le \pi \frac{|\chi(S)|}{\mathfrak{m}} |\log\varepsilon| - \pi \left(\frac{|\mu|}{\mathfrak{m}} - \frac{|d_i|}{\mathfrak{m}^2} \right) |\log\varepsilon| + C(r) \\
&= \pi \frac{|d_i|}{\mathfrak{m}^2} |\log\varepsilon| + C(r)
\end{aligned}$$

for some constant $C(r) < \infty$ independent of ε. By the convergence in (4.100), it follows that $\sup_\varepsilon \int_U \langle \mathrm{jac}(v_\varepsilon), \mathrm{jac}(e)\rangle \,\mathrm{vol} < \infty$. Hence, $GL(v_\varepsilon) \le \pi |d_i| |\log\varepsilon| + C$ for some constant independent of ε, and by a corresponding result in the nonfractional Ginzburg-Landau setting (see also [43]), we see that $|d_k| = 1$. By the arbitrariness of k, this concludes the proof. □

Given $u \in \mathcal{D}_g^{(\mathfrak{m})}$, we will shortly write:

$$S_r = S_r(\mathrm{vort}(u)) := S \setminus \bigcup_{k=1}^{\mathfrak{m}|\chi(S)|} B_r(p_k),$$

where $\{p_k\}$ is the set of vortex centers of u. Before coming to the proof of the compactness statement of Theorem 4.7, we will need to show the well-definedness of $\mathcal{W}^{(\mathfrak{m})}$:

Lemma 4.17 (Well-definedness of the renormalized energy)
For any $u \in \mathcal{LS}^{(\mathfrak{m})}$, the limit in (4.79) exists and lies in $(-\infty, \infty]$. More precisely, for any r_0 small enough such that the balls $\{B_{r_0}(p_k)\}_k$ around the vortex centers $\{p_k\}$ of u are disjoint, we have:

$$\begin{aligned}
\mathcal{W}^{(\mathfrak{m})}(u) = \frac{1}{2} \int_{S_{r_0}} |\nabla u|^2 \,\mathrm{vol} + &\sum_{k=1}^{\mathfrak{m}|\chi(S)|} \frac{1}{\mathfrak{m}^2} \mathcal{W}(v^{(k)}) \\
&- \frac{(\mathfrak{m}-1)^2}{2\mathfrak{m}^2} \int_{B_{r_0}(p_k)} |\nabla e^{(k)}|^2 \,\mathrm{vol} \\
&+ \frac{\mathfrak{m}-1}{\mathfrak{m}} \int_{B_{r_0}(p_k)} \langle \mathrm{jac}(v^{(k)}), \mathrm{jac}(e^{(k)}) \rangle \,\mathrm{vol},
\end{aligned} \tag{4.103}$$

where $v^{(k)} := P_{e^{(k)}}(u)$ in $B_{r_0}(p_k)$ for an arbitrary smooth frame $e^{(k)}$, and:

$$\mathcal{W}(v^{(k)}) = \lim_{r\to 0} \left(\frac{1}{2} \int_{A_{r,r_0}(p_k)} |\nabla v^{(k)}|^2 \,\mathrm{vol} - \pi |\log r| \right) \tag{4.104}$$

is the nonfractional renormalized energy.

Proof. Let $0 < r_0 < 1$ be small enough such that the balls in $\{B_{r_0}(p_k)\}_k$ are disjoint, and where $\{p_k\}$ is the set of vortex centers of u. For any $0 < r < r_0$, we can split the difference appearing in (4.79) as follows:

$$\begin{aligned}&\frac{1}{2}\int_{S_r} |\nabla u|^2 \operatorname{vol} - \frac{|\chi(S)|}{\mathfrak{m}}\pi|\log r| \\ &\quad = \frac{1}{2}\int_{S_{r_0}} |\nabla u|^2 \operatorname{vol} + \sum_{k=1}^{\mathfrak{m}|\chi(S)|}\left(\frac{1}{2}\int_{A_{r,r_0}(p_k)} |\nabla u|^2 \operatorname{vol} - \frac{\pi}{\mathfrak{m}^2}|\log r|\right).\end{aligned}$$

Therefore, it is enough to show for any $k \in \{1, \dots \mathfrak{m}|\chi(S)|\}$ that:

$$\lim_{r\to 0} \frac{1}{2}\int_{A_{r,r_0}(p_k)} |\nabla u|^2 \operatorname{vol} - \frac{\pi}{\mathfrak{m}^2}|\log r| \in (-\infty, \infty].$$

Let us fix $k \in \{1, \dots, \mathfrak{m}|\chi(S)|\}$, and shortly write $B := B_{r_0}(p_k)$ as well as $A_r := A_{r,r_0}(p_k)$. Furthermore, let $e \in C^\infty(B; \mathbb{S}^1)$ be a harmonic vector field and $v := P_e(u)$ in B, then with (4.36), we follow that:

$$\begin{aligned}&\frac{1}{2}\int_{A_r} |\nabla u|^2 \operatorname{vol} - \frac{\pi}{\mathfrak{m}^2}|\log r| \\ &\quad = \frac{1}{\mathfrak{m}^2}\left(\frac{1}{2}\int_{A_r} |\nabla v|^2 \operatorname{vol} - \pi|\log r|\right) - \frac{(\mathfrak{m}-1)^2}{2\mathfrak{m}^2}\int_{A_r} |\nabla e|^2 \operatorname{vol} \\ &\qquad + \frac{\mathfrak{m}-1}{\mathfrak{m}}\int_{A_r} \langle \operatorname{jac}(v), \operatorname{jac}(e)\rangle \operatorname{vol}.\end{aligned}$$

In Section 6 of [24], it was shown that the limit:

$$\mathcal{W}(v) := \lim_{r\to 0}\left(\frac{1}{2}\int_{A_r} |\nabla v|^2 \operatorname{vol} - \pi|\log r|\right)$$

exists and is contained $(-\infty, \infty]$, which – due to the previous reasoning – leads directly to (4.103). □

Lemma 4.18
Given a geodesic ball $B_0 := B_{r_0}(p_0) \subset S$, a sequence $(v_\varepsilon) \subset C^\infty(TB_0)$ with:

$$\operatorname{vort}(v_\varepsilon) \xrightarrow{\flat} \pm\delta_{p_0} \text{ flat in } B_0,$$

and satisfying the following energy bound:

$$GL_\varepsilon(v_\varepsilon) \le \pi|\log\varepsilon| + C$$

for a constant C independent of ε. Then, there exists $r_1 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $r \in [\varepsilon^{\frac{1}{2}}, r_1]$, Theorem 4.6 is applicable (,where $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{5}$), with the corresponding collection of disjoint balls denoted by $\mathfrak{B}_r^\varepsilon$, $D_r^\varepsilon := D_{\mathfrak{B}_r^\varepsilon}((B_0)_\varepsilon)$, and $V_r^\varepsilon := (B_0)_\varepsilon \cap \bigcup_{B\in\mathfrak{B}_r^\varepsilon} B$. Furthermore, for any $r \in [\varepsilon^{\frac{1}{3}}, r_1]$ and $\varepsilon \in (0, \varepsilon_0)$, it holds that:

$$D_r^\varepsilon = 1, \tag{4.105}$$

$$\tag{4.106}$$

and for any $\varepsilon \in (0, \varepsilon_0)$*:*

$$V_{r_1}^{\varepsilon} \supset B_{\frac{r_1}{2}}(p_0) \tag{4.107}$$

Finally $p_\varepsilon \to p_0$ *as* $\varepsilon \to 0$*, where* p_ε *is the center of the unique ball in* $\mathfrak{B}_r^\varepsilon$ *with nonzero degree (*p_ε *is well defined due to* (4.105)*.*

Proof. Without loss of generality, we can assume that $\mathrm{vort}(v_\varepsilon) \overset{\flat}{\rightharpoonup} \delta_{p_0}$. (The case $\mathrm{vort}(v_\varepsilon) \overset{\flat}{\rightharpoonup} -\delta_{p_0}$ can be proved similarly.) Let ε_0 and r_1 be as in the statement of Theorem 4.6. Then employing Theorem 4.6 (with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{5}$), we can find for any $r \in [\varepsilon^{\frac{1}{2}}, r_1]$ a collection $\mathfrak{B}_r^\varepsilon$ such that all the properties stated in Theorem 4.6 hold true.

1. Step: We wish to show, by possibly decreasing ε_0 and r_1, that (4.105) and (4.107) are satisfied. In this regard, let us define the point measures μ_ε and ν_ε as:

$$\mu_\varepsilon := \sum_{B_r(p) \in \mathfrak{B}_{r_\varepsilon}^\varepsilon \cap (B_0)_\varepsilon} \deg(\tilde{v}_\varepsilon, \partial B_r(p))\delta_p, \quad r_\varepsilon := \varepsilon^{\frac{1}{3}},$$

$$\nu_\varepsilon := \sum_{B_r(p) \in \mathfrak{B}_{r_1}^\varepsilon \cap (B_0)_\varepsilon} \deg(\tilde{v}_\varepsilon, \partial B_r(p))\delta_p,$$

where $\tilde{v}_\varepsilon := \mathrm{T}_{1-\varepsilon^\beta}(v_\varepsilon)$. By (4.44), we have:

$$\sup_{\varepsilon \in (0,\varepsilon_0)} |\mu_\varepsilon| = \sup_{\varepsilon \in (0,\varepsilon_0)} D_{\varepsilon^{\frac{1}{3}}}^{\varepsilon} < \infty.$$

Hence, up to taking a subsequence, $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ weak* in B_0 for a point measure μ in B_0. In the same fashion as in the proof of Proposition 4.4, we can show that $\lim_{\varepsilon\to 0} \|\mathrm{vort}(v_\varepsilon) - \mu_\varepsilon\|_\flat = 0$, and therefore follow $\mu_\varepsilon \overset{*}{\rightharpoonup} \delta_{p_0}$. In the same manner as for (μ_ε), we can prove that $\nu_\varepsilon \overset{*}{\rightharpoonup} \delta_{p_0}$ weak* in B_0. Consequently, by the lower semicontinuity of the total variation with respect to weak* convergence, we see that $\liminf_{\varepsilon\to 0} |\nu_\varepsilon| \geq |\delta_{p_0}| = 1$, and by possibly decreasing ε_0, we can assure that for all $\varepsilon \in (0, \varepsilon_0)$ and $r \in [\varepsilon^{\frac{1}{3}}, r_1]$:

$$D_r^\varepsilon \geq D_{r_1}^\varepsilon = |\nu_\varepsilon| \geq 1,$$

where we have used the monotonicity of $r \mapsto D_r^\varepsilon$. In order to prove (4.105), it remains to show $D_r^\varepsilon \leq 1$. With (4.108), (4.43), (4.44), and by making sure that $r_1 \leq \frac{7}{8C_1}$, where C_1 is the constant from (4.43), we can estimate for any $r \in [\varepsilon^{\frac{1}{3}}, r_1]$:

$$\begin{aligned}
\pi|\log\varepsilon| &\geq \pi D_r^\varepsilon \left(\log\left(\frac{r}{D_r^\varepsilon \varepsilon}\right) - C_1\right) \\
&\geq \pi D_r^\varepsilon \log\frac{\varepsilon^{\frac{1}{3}}}{\varepsilon} - \frac{1}{2}\pi D_r^\varepsilon \log D_r^\varepsilon - C_1 \\
&\geq \frac{2}{3}\pi D_r^\varepsilon |\log\varepsilon| - \pi C_2 \log C_2 - C_1,
\end{aligned}$$

where C_2 is the supremum in (4.44). This leads – for ε_0 small enough – to a contradiction, except that:

$$D_r^\varepsilon \leq \frac{3}{2} < 2,$$

and as $D_r^\varepsilon \in \mathbb{N}$, $D_r^\varepsilon \leq 1$ follows, as is desired.

2. Step: It remains to show (4.107). By (4.105), the measure μ_ε must be of the form $\mu_\varepsilon = \delta_{p_\varepsilon}$ for some $p_\varepsilon \in B_0$. As $\mu_\varepsilon \overset{*}{\rightharpoonup} \delta_{p_0}$, it follows that $p_\varepsilon \to p_0$. Hence, for ε_0 small enough, we have $p_\varepsilon \in B_{\frac{r_1}{3}}(p_0)$, and by the definition of the ball-growing procedure $B_{r_1}(p_\varepsilon) \subset V_{r_1}^\varepsilon$. Finally, due to $B_{\frac{r_1}{2}}(p_0) \subset B_{r_1}(p_\varepsilon)$, the desired result follows. □

In the following, we generalize Lemma 2.12 found in [40] to the manifold setting.

Lemma 4.19
Let $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}(S)$ be a bounded sequence in $L^\infty(TS)$ satisfying the energy bound

$$GL_\varepsilon(u_\varepsilon) \leq \frac{|\chi(S)|}{\mathfrak{m}}\pi|\log\varepsilon| + C \tag{4.108}$$

for a constant C independent of ε, and such that $\mathrm{vort}(u_\varepsilon) \overset{\flat}{\rightharpoonup} \mu \in X^{(\mathfrak{m})}$ *flat in S, then for any $r > 0$ small enough and $q \in [1, 2)$, it holds that:*

$$\sup_\varepsilon \|u_\varepsilon\|_{SBV^2(TS_r)} < \infty, \tag{4.109}$$

$$\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^q(T^*S\otimes TS)} < \infty, \tag{4.110}$$

Proof. In parts, we follow the proof found in [40]. Let us – without loss of generality – assume that $\mathrm{sgn}(\chi(S)) > 0$ (the other case works in the same way), and let $\{p_k\}$ denote the set of vortex centers of u and shortly write $\mu \coloneqq \mathrm{vort}(u)$. Furthermore, we fix $r_0 > 0$ small enough such that the balls in $\{B_{r_0}(p_k)\}_k$ are disjoint. All constants we will encounter in this proof are implicitly assumed to be independent of ε.

1. Step: We start by deriving the SBV^2-bound away from the vortices. By the localized liminf inequality in (4.101) and the energy bound in (4.83), we derive for any $0 < r < r_0$:

$$\begin{aligned} GGL_\varepsilon(u_\varepsilon, S_r) &\leq GGL_\varepsilon(u_\varepsilon) - \sum_{k=1}^{\mathfrak{m}|\chi(S)|} GL_\varepsilon(u_\varepsilon, B_r(p_k)) \\ &\leq \frac{|\chi(S)|}{\mathfrak{m}}\pi|\log\varepsilon| - \sum_{k=1}^{m|\chi(S)|} \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r}{\varepsilon}\right) + C \\ &\leq \frac{\pi|\chi(S)|}{\mathfrak{m}}|\log r| + C = \mathrm{O}(|\log r|), \end{aligned} \tag{4.111}$$

where C is the constant from (4.101). By the definition of GGL_ε and the boundedness of (u_ε) in L^∞, this leads to:

$$\sup_\varepsilon \|u_\varepsilon\|_{SBV^2(TS_r)} \leq \mathrm{O}(|\log r|),$$

as is desired.

2. Step: Due to (4.109), it remains to show that:

$$\sup_\varepsilon \int_{B_{r_0}(p_k)} |\nabla u_\varepsilon|^q \,\mathrm{vol} < \infty \tag{4.112}$$

for all vortex centers p_k, and where r_0 is chosen sufficiently small so that all balls in $\{B_{r_0}(p_k)\}_k$ are disjoint. Fix $k \in \{1, \dots, \mathfrak{m}|\chi(S)|\}$ and shortly write $B_0 := B_{r_0}(p_k)$. Furthermore, let $e \in C^\infty(B_0; \mathbb{S}^1)$ be a harmonic vector-field, and set $v_\varepsilon := P_e(u_\varepsilon)$ in B_0. By the chain rule and the definition of $\mathcal{AS}^{(\mathfrak{m})}(S)$, we have that $v_\varepsilon \in W^{1,2}(TB_0)$, and instead of deriving (4.112), we can equivalently show:

$$\sup_\varepsilon \int_{B_0} |\nabla v_\varepsilon|^q \operatorname{vol} < \infty. \tag{4.113}$$

We wish to construct an appropriate decomposition of B_0. Afterwards, we will estimate the L^q-norms of (w_ε) on each component separately. In this regard, we see by (4.36), Lemma 4.15, and the boundedness of (u_ε) in $L^\infty(TB_0)$, that:

$$\begin{aligned} GL(v_\varepsilon) &\leq \frac{\mathfrak{m}^2}{2} \int_{B_0} |\nabla(u_\varepsilon)|^2 \operatorname{vol} + \frac{1}{4\varepsilon^2} \int_{B_0} (1 - |u_\varepsilon|^2)^2 + C_1 \\ &\leq \mathfrak{m}^2 GL(u_\varepsilon, B_0) + C_1. \end{aligned}$$

By the local liminf inequality in (4.101), the definition of $X^{(\mathfrak{m})}$, and the energy upper bound in (4.108), it holds that:

$$\begin{aligned} GL(u_\varepsilon, B_0) &\leq GL(u_\varepsilon) - \sum_{l \neq k} GL(u_\varepsilon, B_{r_0}(p_l)) \\ &\leq \frac{\pi}{\mathfrak{m}^2} |\log \varepsilon| + C|\log(r_0)|. \end{aligned} \tag{4.114}$$

With the previous estimate, this results in:

$$GL(v_\varepsilon) \leq \pi |\log \varepsilon| + C|\log(r_0)|.$$

By approximation in Sobolev spaces, we can find a sequence $(w_\varepsilon) \in C^\infty(TB_0)$ such that $\|v_\varepsilon - w_\varepsilon\|_{W^{1,2}(TB_0)} \to 0$ as $\varepsilon \to 0$ and:

$$\lim_{\varepsilon \to 0} (GL_\varepsilon(v_\varepsilon) - GL_\varepsilon(w_\varepsilon)) = 0.$$

Therefore:

$$GL_\varepsilon(w_\varepsilon) \leq \pi |\log \varepsilon| + C|\log(r_0)|. \tag{4.115}$$

Also, by the flat convergence of $(\operatorname{vort}(u_\varepsilon))$ and (4.38), (4.88) and (4.100), we have:

$$\operatorname{vort}(w_\varepsilon) \overset{\flat}{\rightharpoonup} \delta_{p_k} \text{ flat in } B_0.$$

We are in a position to apply Lemma 4.18 for the sequence (w_ε) and the ball B_0. Let $\mathfrak{B}_r^\varepsilon$, V_r^ε (for $\varepsilon \in (0, \varepsilon_0)$ and $r \in [\varepsilon^{\frac{1}{3}}, r_1]$), ε_0 and r_1 be as in the statement of Lemma 4.18. We define $J = J(\varepsilon) \in \mathbb{N}$ as the largest natural number satisfying $2^{-J} r_1 > \varepsilon^{\frac{1}{3}}$. As $2^{-J} r_1 \geq \varepsilon^{\frac{1}{3}}$, we have:

$$J(\varepsilon) \leq \frac{1}{\log(2)} \left(\frac{1}{3} |\log \varepsilon| - \log r_1 \right) = C|\log \varepsilon|.$$

Let us decompose B_0 into the following sets

$$B_0 = V_{2^{-J} r_1} \cup \bigcup_{j=0}^{J-1} A_j^\varepsilon \cup (B_0 \setminus V_{r_1},{}^\varepsilon)$$

where:

$$A_j^\varepsilon := V^\varepsilon_{2^{-j}r_1} \setminus V^\varepsilon_{2^{-(j+1)}r_1}.$$

3. Step: We continue by estimating the L^q-norm of (∇w_ε) outside the annuli $\{A_j^\varepsilon\}_{j=0}^{J-1}$. From the definition of J, we see that $2^{-(J+1)}r_1 \le \varepsilon^{\frac{1}{3}}$, and therefore $2^{-J}r_1 \le 2\varepsilon^{\frac{1}{3}}$. This leads to:

$$|V^\varepsilon_{2^{-J}r_1}| \le Cr(\mathfrak{B}^\varepsilon_{2^{-J}r_1})^2 \le C\varepsilon^{\frac{1}{9}}.$$

Consequently, with Hölder's inequality and the energy bound (4.115):

$$\begin{aligned}\int_{V^\varepsilon_{2^{-J}r_1}} |\nabla w_\varepsilon|^q \operatorname{vol} &\le |V^\varepsilon_{2^{-J}r_1}|^{\frac{2}{2-1}} \int_{B_0} |\nabla w_\varepsilon|^2 \operatorname{vol} \\ &\le C\varepsilon^{\frac{2}{9(2-q)}} |\log\varepsilon| = \mathrm{O}_{\varepsilon\to 0}(1).\end{aligned}$$

Hence:

$$\sup_\varepsilon \int_{V^\varepsilon_{2^{-J}r_1}} |\nabla w_\varepsilon|^q \operatorname{vol} < \infty.$$

By (4.107), Hölder's inequality and (4.109), we also derive:

$$\begin{aligned}\int_{B_0\setminus V^\varepsilon_{r_1}} |\nabla w_\varepsilon|^q \operatorname{vol} &\le \int_{B_0\setminus B_{\frac{r_1}{2}}(p_k)} |\nabla w_\varepsilon|^q \operatorname{vol} \\ &\le |B_0|^{\frac{2}{2-q}} \int_{B_0\setminus B_{\frac{r_1}{2}}(p_k)} |\nabla w_\varepsilon|^2 \operatorname{vol} = \mathrm{O}_{\varepsilon\to 0}(1).\end{aligned}$$

4. Step: Lastly, we consider the situation inside the annuli $\{A_j^\varepsilon\}_{j=1}^{J-1}$. Given any $r \in [\varepsilon^{\frac{1}{3}}, r_1]$ (with r_1 as in Lemma 4.19), we have by (4.70) the following lower bound:

$$GL_\varepsilon(w_\varepsilon, V_r^\varepsilon) \ge \pi \log\left(\frac{r}{\varepsilon}\right) - C.$$

With (4.115), this results in:

$$\begin{aligned}GL_\varepsilon(w_\varepsilon, B_0 \setminus V_r^\varepsilon) &= GL_\varepsilon(w_\varepsilon) - GL_\varepsilon(w_\varepsilon, V_r^\varepsilon) \\ &\le \pi|\log\varepsilon| - \pi\log\left(\frac{r}{\varepsilon}\right) + C \le \pi|\log r| + C. \qquad (4.116)\end{aligned}$$

Furthermore, setting $\tilde{w}_\varepsilon := \frac{w_\varepsilon}{|w_\varepsilon|}$, we see with (4.54) for all $r \in [\varepsilon^{\frac{1}{3}}, \frac{r_1}{2}]$ that:

$$\begin{aligned}\frac{1}{2}\int_{V^\varepsilon_{2r}\setminus V^\varepsilon_r} |\nabla \tilde{w}_\varepsilon|^2 \operatorname{vol} &\ge \frac{1}{2}\int_{B^*_\varepsilon\setminus V^\varepsilon_r} |\nabla \tilde{w}_\varepsilon|^2 \operatorname{vol} \\ &\ge \pi\left(\log\left(\frac{2r}{r}\right) - Cr\right) \ge \pi\log(2) - Cr\end{aligned}$$

where B^*_ε is the unique ball in $\mathfrak{B}^\varepsilon_{2r}$ with $d_{B^*_\varepsilon} = 1$. As by the product rule

$$|\nabla w_\varepsilon|^2 = |\mathrm{d}|w_\varepsilon||^2 + |w_\varepsilon|^2|\nabla\tilde{w}_\varepsilon|^2,$$

and $|w_\varepsilon| \ge 1 - \varepsilon^\beta = 1 - \varepsilon^{\frac{1}{4}}$ in $B_0 \setminus V_r^\varepsilon$, this implies:

$$\begin{aligned}\frac{1}{2}\int_{V^\varepsilon_{2r}\setminus V^\varepsilon_r} |\nabla w_\varepsilon|^2 \operatorname{vol} &\ge \frac{1}{2}\int_{V^\varepsilon_{2r}\setminus V^\varepsilon_r} |w_\varepsilon|^2|\nabla\tilde{w}_\varepsilon|^2 \operatorname{vol} \\ &\ge (1 - C\varepsilon^{\frac{1}{4}})(\pi\log(2) - Cr). \qquad (4.117)\end{aligned}$$

Fix an arbitrary $\lambda > 0$ and suppose that, up to taking a subsequence, we can find for each ε a $j^* = j^*(\varepsilon) \in \{0, \dots, J-1\}$ such that:

$$\frac{1}{2} \int_{A^\varepsilon_{j^*}} |\nabla w_\varepsilon|^2 \,\mathrm{vol} > \lambda.$$

Combining (4.116) and (4.117) then results in the following condition on λ:

$$\begin{aligned}
&(J-1)(1 - C\varepsilon^{\frac{1}{4}})\pi \log(2) - C \sum_{j \neq j^*} 2^{-j} r_1 \\
&\leq \sum_{j \neq j^*} \frac{1}{2} \int_{A^\varepsilon_j} |\nabla w_\varepsilon|^2 \,\mathrm{vol} \\
&\leq GL_\varepsilon(w_\varepsilon, B_0 \setminus V^\varepsilon_{2^{-J} r_1}) - \frac{1}{2} \int_{A^\varepsilon_{j^*}} |\nabla w_\varepsilon|^2 \,\mathrm{vol} \\
&< J\pi \log(2) + O(1) - \lambda.
\end{aligned}$$

Solving for λ above and using $J = O(|\log \varepsilon|)$, then leads to:

$$\begin{aligned}
\lambda &\leq \pi \log(2) + C \left(1 + (J-1)\varepsilon^{\frac{1}{4}} \pi \log(2) + \sum_{j \neq j^*} 2^{-j} r_1 \right) \\
&= C \left(1 + \varepsilon^{\frac{1}{4}} |\log \varepsilon| + \sum_{j=0}^{\infty} 2^{-j} \right) = \mathrm{O}_{\varepsilon \to 0}(1),
\end{aligned}$$

which leads to a contradiction for λ sufficiently large. Consequently:

$$\int_{A^\varepsilon_j} |\nabla w_\varepsilon|^2 \,\mathrm{vol} = \mathrm{O}_{\varepsilon \to 0}(1), \tag{4.118}$$

uniformly in j. By Hölder's inequality, and the definition of A^ε_j, we can therefore derive for $q \in [0, 2)$:

$$\begin{aligned}
\int_{V^\varepsilon_{r_1} \setminus V^\varepsilon_{2^{-J} r_1}} |\nabla w_\varepsilon|^q \,\mathrm{vol} &= \sum_{j=0}^{J-1} \int_{A^\varepsilon_j} |\nabla w_\varepsilon|^q \,\mathrm{vol} \\
&\leq \sum_{j=0}^{J-1} |A^\varepsilon_j|^{\frac{2}{2-q}} \int_{A^\varepsilon_j} |\nabla w_\varepsilon|^2 \,\mathrm{vol} \\
&\leq C \sum_{j=1}^{\infty} |V^\varepsilon_{2^{-j} r_1}|^2 \\
&\leq C \sum_{j=1}^{\infty} (2^{-\frac{4}{2-q}})^j = C \frac{1}{1 - 2^{-\frac{4}{2-q}}} = \mathrm{O}_{\varepsilon \to 0}(1).
\end{aligned}$$

5. Step: It remains to estimate the L^q-norms of (v_ε). With the previous reasoning and the fact that $\|v_\varepsilon - w_\varepsilon\|_{W^{1,2}(TB_0)} = 0$ as $\varepsilon \to 0$, we derive that:

$$\sup_\varepsilon \int_{B_0} |\nabla v_\varepsilon|^q \,\mathrm{vol} < \infty. \tag{4.119}$$

It remains to transfer this estimate to ∇u_ε which will be achieved employing (4.36). Before coming to this, we will need to show the intermediate estimate:

$$\sup_\varepsilon \int_{B_0} |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} < \infty. \tag{4.120}$$

This can be proved as follows: By the localized zero-order liminf inequality for (v_ε), (4.36), (4.114), (4.100), and the boundedness of (u_ε) in L^∞, we see that:

$$\begin{aligned}
\pi|\log\varepsilon| - C &\leq \int_{B_0} |\nabla v_\varepsilon|^2 \operatorname{vol} \\
&= \mathfrak{m}^2 \int_{B_0} |\nabla u_\varepsilon|^2 \operatorname{vol} + (1-\mathfrak{m}^2) \int_{B_0} |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol} \\
&\quad + (\mathfrak{m}-1)^2 \int_{B_0} |u_\varepsilon|^2 |\nabla e|^2 \operatorname{vol} - 2\mathfrak{m}(\mathfrak{m}-1) \int_{B_0} \langle \operatorname{jac}(u_\varepsilon), \operatorname{jac}(e)\rangle \operatorname{vol} \\
&\leq \pi|\log\varepsilon| + \tilde{C} - (\mathfrak{m}^2 - 1) \int_{B_0} |\mathrm{d}|u_\varepsilon||^2 \operatorname{vol}
\end{aligned}$$

which leads to (4.120). With (4.35), we can write:

$$|u_\varepsilon|^{-2} \operatorname{jac}(u_\varepsilon) \otimes v_\varepsilon^\perp = \nabla v_\varepsilon - \frac{1}{\mathfrak{m}} |u|^{-1} \,\mathrm{d}|u_\varepsilon| \otimes v_\varepsilon + \frac{\mathfrak{m}-1}{\mathfrak{m}} \operatorname{jac}(e) \otimes v_\varepsilon^\perp,$$

hence, by the triangular inequality and (4.119) and (4.120), this shows:

$$\begin{aligned}
\| |u_\varepsilon|^{-1} \operatorname{jac}(u_\varepsilon) \|_{L^q(B_0)} &\leq \|\nabla v_\varepsilon\|_{L^q(B_0)} + \frac{1}{\mathfrak{m}} \|\mathrm{d}\, u_\varepsilon\|_{L^q(B_0)} + C \\
&\leq C(1 + \|\mathrm{d}\, u_\varepsilon\|_{L^2(B_0)}) = \mathrm{O}_{\varepsilon\to 0}(1),
\end{aligned} \tag{4.121}$$

where C is a constant independent of ε. Finally, with (4.121), (4.120), and

$$\begin{aligned}
\nabla u_\varepsilon &= |u_\varepsilon|^{-2} \langle \nabla u_\varepsilon, u_\varepsilon^\perp\rangle \otimes u_\varepsilon^\perp + |u_\varepsilon|^{-2} \langle \nabla u_\varepsilon, u_\varepsilon\rangle \otimes u_\varepsilon \\
&= |u_\varepsilon|^{-2} \langle \nabla u_\varepsilon, u_\varepsilon^\perp\rangle \otimes u_\varepsilon^\perp + |u_\varepsilon|^{-1} \,\mathrm{d}|u_\varepsilon| \otimes u_\varepsilon,
\end{aligned}$$

we follow for any $q \in [1,2)$:

$$\|\nabla u_\varepsilon\|_{L^q(B_0)} \leq \| |u_\varepsilon|^{-1} \operatorname{jac}(u_\varepsilon)\|_{L^q(B_0)} + \|\mathrm{d}\, u_\varepsilon\|_{L^q(B_0)} = \mathrm{O}_{\varepsilon\to 0}(1),$$

as is desired. □

We are ready to proof the compactness statement of Theorem 4.7:

Proof of the compactness statement in Theorem 4.7. Note that we have shown (4.84) in Theorem 4.8. Let us assume that (4.84) holds true, without relabeling, for the whole sequence (u_ε).

1. Step: In the first step, we wish to derive the local SBV^2-compactness of (u_ε). Given $r > 0$ small enough, we can select by (4.109) and the compactness theorem for SBV-sections (see also Theorem 4.5) a further subsequence, without relabeling, such that:

$$u_\varepsilon \rightharpoonup u^{(r)} \text{ weakly in } SBV^2(TS_r).$$

For r_1, r_2 small enough such that $0 < r_1 < r_2$, we see by the L^1-convergence of $(u_\varepsilon|_{S_{r_1}})$ and $(u_\varepsilon|_{S_{r_2}})$ that $u^{(r_1)} = u^{(r_2)}$ a.e. in S_{r_2}. Hence, $u(p) \coloneqq u^{(r)}(p)$ for some $r > 0$ such that $r < \operatorname{dist}(p, \operatorname{spt}(\mu))$ is a well-defined section in $SBV^2_{\mathrm{loc}}(T(S\setminus \operatorname{spt}(\mu)))$ and by a standard diagonal sequence argument, (4.85) follows. Let us assume that we have already selected a subsequence such that (4.84) and (4.85) are satisfied. It remains to show that $u \in \mathcal{LS}^{(\mathfrak{m})}(S)$.

2. Step: We continue by showing the unit length of u. By the energy-bound in (4.83) and the definition of GGL_ε, we derive that:

$$\int_S (1 - |u_\varepsilon|^2)^2 \operatorname{vol} \leq C\varepsilon^2 |\log \varepsilon| = \mathrm{o}_{\varepsilon\to 0}(1). \tag{4.122}$$

For fixed $r > 0$ small enough, we derive by the L^2 convergence of $(u_\varepsilon|_{S_r})$ that, up to a subsequence, $u_\varepsilon \to u$ pointwise a.e. in S_r. By the boundedness of (u_ε) in $L^\infty(TS)$ and the dominated convergence theorem, we see that:

$$\int_{S_r} (1 - |u|^2)^2 \operatorname{vol} = \lim_{\varepsilon\to 0} \int_{S_r} (1 - |u_\varepsilon|^2)^2 \operatorname{vol} = 0,$$

and therefore $|u| = 1$ a.e. in S_r. By the arbitrariness of $r > 0$, we follow that $|u| = 1$ a.e. in S.

3. Step: We wish to derive a uniform SBV-bound on (u_ε). By (4.110) for $q = 1$, we can select a subsequence such that $\nabla u_\varepsilon \rightharpoonup \Psi$ weakly in $L^1(T^*S \otimes TS)$. Due to (4.85), we have already seen that for any $r > 0$ small enough $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(TS_r \otimes TS_r)$. Hence, (and by also using the arbitrariness of $r > 0$,) we derive that $\nabla u = \Psi = L^1(T^*S \otimes TS)$ (in particular, $\operatorname{vort}(u)$ is well defined in the distributional sense). We will now show $\mathcal{H}^1(J_u) < \infty$. As in the proof of Lemma 4.19, we can prove that for $r > 0$ small enough:

$$GGL_\varepsilon(u_\varepsilon, S_r) \leq \pi \frac{|\chi(S)|}{\mathfrak{m}} |\log r| + C$$

for some constant C independent of r and ε. Solving for $\mathcal{H}^1(J_\varepsilon)$, we derive:

$$\mathcal{H}^1(J_{u_\varepsilon} \cap S_r) \leq \frac{|\chi(S)|}{\mathfrak{m}} |\log r| + C - \frac{1}{2} \int_{S_r} |\nabla u_\varepsilon|^2 \operatorname{vol}$$

for the same constant C as before. By (4.85):

$$\begin{aligned}\mathcal{H}^1(J_u \cap S_r) &\leq \liminf_{\varepsilon\to 0} \mathcal{H}^1(J_{u_\varepsilon} \cap S_r) \\ &\leq \limsup_{\varepsilon\to 0} \mathcal{H}^1(J_{u_\varepsilon} \cap S_r) \\ &\leq C - \left(\frac{1}{2} \int_{S_r} |\nabla u|^2 \operatorname{vol} - \pi \frac{|\chi(S)|}{\mathfrak{m}} |\log r| \right).\end{aligned}$$

With $\mathcal{W}^{(\mathfrak{m})}(u) > -\infty$ (see Lemma 4.17), it then follows:

$$\mathcal{H}^1(J_u) = \limsup_{r\to 0} \mathcal{H}^1(J_u \cap S_r) \leq C - \mathcal{W}^{(\mathfrak{m})}(u) < \infty,$$

as desired.

4. Step: It remains to show that $\operatorname{vort}(u) = \mu$, where μ is the limit from (4.141). (Note that in the previous step we have seen that $\operatorname{vort}(u)$ is well defined in the distributional sense.) As $|\mathrm{jac}(u_\varepsilon)| \le |\nabla u_\varepsilon||u_\varepsilon|$, we conclude from the boundedness of (u_ε) in $L^\infty(TS)$ and (4.110) for $q = 1$ that $(\mathrm{jac}(u_\varepsilon))$ is bounded in $L^1(T^*S)$ and hence, up to a subsequence, weakly convergent towards $\Psi \in L^1(T^*S)$. By (4.85), we have for $r > 0$ small enough:

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^2(TS_r \otimes TS_r), \quad u_\varepsilon \to u \text{ strongly in } L^2(TS_r).$$

Hence by weak-strong convergence, we follow for any $\varphi \in L^\infty(T^*S_r)$ and coordinate neighborhood U on S:

$$\begin{aligned}\int_{U\cap S_r} \langle \varphi, \Psi\rangle \,\mathrm{vol} &= \lim_{\varepsilon\to 0} \int_{U\cap S_r} \langle \varphi, \mathrm{jac}(u_\varepsilon)\rangle \,\mathrm{vol} \\ &= \lim_{\varepsilon\to 0} \int_{U\cap S_r} g^{ij}\varphi^i \langle \nabla_{\partial_{x^j}} u_\varepsilon, u_\varepsilon^\perp\rangle \,\mathrm{vol} \\ &= \lim_{\varepsilon\to 0} \int_{U\cap S_r} \langle \nabla_{\partial_{x^j}} u_\varepsilon, g^{ij}\varphi^i u_\varepsilon^\perp\rangle \,\mathrm{vol} \\ &= \int_{U\cap S_r} \langle \varphi, \mathrm{jac}(u)\rangle \,\mathrm{vol}\,.\end{aligned}$$

With a partition of unity argument and the arbitrariness of φ we follow that $\Psi = \mathrm{jac}(u)$ a.e. in S. Furthermore, by the weak convergence of $(\mathrm{jac}(u_\varepsilon))$ towards $\mathrm{jac}(u)$ in $L^1(T^*S)$, the definition of vort, and (4.84), we see that for any $\varphi \in C^{0,1}(S)$:

$$\begin{aligned}\langle \mathrm{vort}(u), \varphi\rangle &= \int_S \mathrm{d}\,\varphi \wedge \mathrm{jac}(u) \\ &= \lim_{\varepsilon\to 0} \int_S \mathrm{d}\,\varphi \wedge \mathrm{jac}(u_\varepsilon) \\ &= \lim_{\varepsilon\to 0} \langle \mathrm{vort}(u_\varepsilon), \varphi\rangle = \langle \mu, \varphi\rangle.\end{aligned}$$

The arbitrariness of φ concludes the proof. □

4.3.2 Gamma-liminf

In this subsection, if not explicitly stated otherwise, all instances of asymptotic notation in this chapter are meant as $\varepsilon \to 0$. Given an open set $U \subset S$, we define the modified Ginzburg-Landau energy $GL^{(\mathfrak{m})} \colon \mathcal{AS}(U) \to \mathbb{R}$ as:

$$GL^{(\mathfrak{m})} = GL_\varepsilon^{(\mathfrak{m})}(v, U) \coloneqq \frac{1}{2\mathfrak{m}^2} \int_B |\nabla v|^2 + (\mathfrak{m}^2 - 1)|\mathrm{d}|v||^2 + \frac{\mathfrak{m}^2}{2\varepsilon^2}(1 - |v|^2)^2 \,\mathrm{vol}\,. \tag{4.123}$$

The following intuition lies behind this definition: Let $U = B$ be a geodesic ball on S, $u \in \mathcal{AS}^{(\mathfrak{m})}(B)$, $e \in C^\infty(B; \mathbb{S}^1)$ a harmonic vector field, and set $v \coloneqq P_e(u) \in \mathcal{AS}(B)$ in B. Then, by (4.36):

$$\begin{aligned}GL(u) \ge GL_\varepsilon^{(\mathfrak{m})}(v) &- \frac{\mathfrak{m}^2 - 1}{\mathfrak{m}^2} \int_B |u_\varepsilon|^2 |\nabla e|^2 \,\mathrm{vol} \\ &+ \frac{2\mathfrak{m}(\mathfrak{m} - 1)}{\mathfrak{m}^2} \int_B \langle \mathrm{jac}(u_\varepsilon), \mathrm{jac}(e)\rangle \,\mathrm{vol},\end{aligned} \tag{4.124}$$

where the latter two terms will turn up to be negligible for small balls. (This will be made more precise later on.)

Remark 4.1. Given a (possibly nonsmooth) section v of $TB_r(p_0)$ ($r \in (0, r^*)$), we implicitly assume that $\bar{v}\colon B_r(0) \to \mathbb{R}^2$ is the coordinate representation of v induced by centered (at p_0) normal coordinates and an auxiliary frame $\{e, e^\perp\}$ on $TB_r(p_0)$. Objects such as g_{ij}, $\sqrt{|g|}$, $\Gamma^\beta_{i\alpha}$, etc. will always correspond to the above choice of coordinates.

In the following, we will study two minimum problems and their relation to each other. The first such minimum problem is formulated in the Euclidean setting: Given $r > 0$ and $\lambda \in \mathbb{S}^1$, we consider:

$$\begin{aligned}\bar{\gamma}_\varepsilon^{(\mathfrak{m})}&(r, \lambda)\\ &:= \min\left\{ E_\varepsilon(v, \bar{B}_r(0))\colon v \in W^{1,2}(\bar{B}_r(0); \mathbb{R}^2), v = \lambda \frac{x}{|x|} \text{ on } \partial \bar{B}_r(0) \right\},\end{aligned} \tag{4.125}$$

where $\bar{B}_r(0)$ is the ball in $\mathbb{R}^2$ of radius r, and E_ε is the "flat version" of $GL_\varepsilon^{(\mathfrak{m})}$:

$$E_\varepsilon(v, B_r(0)) := \frac{1}{2\mathfrak{m}^2} \int_{B_r(0)} |\nabla v|^2 + (\mathfrak{m}^2 - 1)|\nabla |v||^2 + \frac{\mathfrak{m}^2}{2\varepsilon^2}(1 - |v|^2)^2 \,\mathrm{d}x.$$

Note that by direct methods, we can show that the minimum in (4.125) exists. As $E_\varepsilon(v, B_r(0)) = E_\varepsilon(\tilde{v}, B_r(0))$ for $\tilde{v}(x) := \lambda^{-1} v(\varepsilon x)$, and $\tilde{v}$ is admissible for the minimum problem in the definition of $\bar{\gamma}_1^{(\mathfrak{m})}\left(\frac{r}{\varepsilon}, 1\right)$, we see that for any $r > 0$, $\varepsilon > 0$, and $\lambda \in \mathbb{S}^1$:

$$\bar{\gamma}^{(\mathfrak{m})}\left(\frac{r}{\varepsilon}\right) := \bar{\gamma}_1^{(\mathfrak{m})}\left(\frac{r}{\varepsilon}, 1, 1\right) = \bar{\gamma}_\varepsilon^{(\mathfrak{m})}(r, \varepsilon, \lambda).$$

The following convergence result was proved in [40]:

Lemma 4.20
There exists $\gamma_\mathfrak{m} \in \mathbb{R}$ such that:

$$\lim_{R\to\infty} \left(\bar{\gamma}^{(\mathfrak{m})}(R) - \frac{\pi}{\mathfrak{m}^2} |\log(R)| \right) = \gamma_\mathfrak{m}. \tag{4.126}$$

Consequently, for any $\lambda \in \mathbb{S}^1$ and sequence $(r_\varepsilon) \subset \mathbb{R}_+$ satisfying $\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = \infty$, we have that:

$$\lim_{\varepsilon\to 0} \left(\bar{\gamma}_\varepsilon^{(\mathfrak{m})}(r_\varepsilon, \lambda) - \frac{\pi}{\mathfrak{m}^2} \log\left(\frac{r_\varepsilon}{\varepsilon}\right) \right) = \gamma_\mathfrak{m}. \tag{4.127}$$

We will now introduce the second minimization problem that is, in contrast to (4.125), defined on S. In this regard, given $r \in (0, r^*)$, $\varepsilon > 0$, $p_0 \in S$, and $\lambda \in \mathbb{S}^1$, we define:

$$\gamma_\varepsilon^{(\mathfrak{m})}(p_0, r, \lambda) := \min\left\{ GL_\varepsilon^{(\mathfrak{m})}(v, B_r(p_0))\colon v \in \mathcal{AS}(B_r(p)),\ \bar{v} = \lambda \frac{x}{|x|} \text{ on } \partial \bar{B}_r(0) \right\}, \tag{4.128}$$

where $\bar{v}$ is the coordinate representation of v as described in Remark 4.1, $B_r(p_0)$ denotes the geodesic ball on S centered at p_0 with radius r, and $\bar{B}_r(0)$ is the Euclidean ball of radius r centered at the origin. Again, by direct methods, we can show that a minimizer for the problem above exists. We are only able to specify the convergence behavior of γ_ε in "small" balls:

Lemma 4.21
Let $(r_\varepsilon) \subset \mathbb{R}_+$ *such that* $r_\varepsilon|\log\varepsilon| \to 0$ *and* $\frac{r_\varepsilon}{\varepsilon} \to \infty$, $(p_\varepsilon) \subset S$, *as well as* $(\lambda_\varepsilon) \subset \mathbb{S}^1$, *then the following convergence holds true:*

$$\lim_{\varepsilon\to 0}\left(\gamma_\varepsilon^{(\mathfrak{m})}(p_\varepsilon, r_\varepsilon, \lambda_\varepsilon) - \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r_\varepsilon}{\varepsilon}\right)\right) = \gamma_{\mathfrak{m}} \tag{4.129}$$

with the same $\gamma_{\mathfrak{m}}$ *as in Lemma 4.20.*

Proof. As $r_\varepsilon \leq r_\varepsilon|\log\varepsilon| \to 0$, we can define centered (at p_ε) normal coordinates for ε small enough. Fix for the moment $\varepsilon > 0$ (sufficiently small) and let $v_\varepsilon \in \mathcal{AS}(B_{r_\varepsilon}(p_\varepsilon))$ be a minimizer for the minimum problem in the definition of $\gamma_\varepsilon^{(\mathfrak{m})}(p_\varepsilon, r_\varepsilon, \lambda_\varepsilon)$. By definition, its coordinate representation $\bar{v}_\varepsilon$ is a competitor for the minimum problem in the definition of $\bar{\gamma}_\varepsilon^{(\mathfrak{m})}(r_\varepsilon, \lambda_\varepsilon)$, thus we derive:

$$\begin{aligned}\bar{\gamma}_\varepsilon^{(\mathfrak{m})}(r_\varepsilon, \lambda_\varepsilon) &\leq E_\varepsilon(\bar{v}_\varepsilon, \bar{B}_{r_\varepsilon}(0)) \\ &= E_\varepsilon(\bar{v}_\varepsilon, \bar{B}_{r_\varepsilon}(0)) - GL_\varepsilon^{(\mathfrak{m})}(v, B_{r_\varepsilon}(p_\varepsilon)) + \gamma_\varepsilon(r_\varepsilon, p_\varepsilon, \lambda_\varepsilon).\end{aligned} \tag{4.130}$$

Note that $w_\varepsilon := \min\{1, \frac{1}{|v_\varepsilon|}\}\, v_\varepsilon$ satisfies $GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon) \leq GL_\varepsilon^{(\mathfrak{m})}(v_\varepsilon)$, hence we can assume without loss of generality that $\|v_\varepsilon\|_{L^\infty} \leq 1$ for all ε. Due to equivalence of norms, this implies that $\sup_\varepsilon \|\bar{v}_\varepsilon\|_\infty < \infty$. By a further comparison argument, we can also suppose that:

$$\int_{\bar{B}_{r_\varepsilon}(0)} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x = \mathrm{O}(|\log\varepsilon|), \tag{4.131}$$

where $\bar{\nabla}$ stands for the Euclidean gradient. As we used centered normal coordinates on $B_{r_\varepsilon}(p_\varepsilon)$:

$$g^{ij}\sqrt{|g|} = \delta^{ij} + \mathrm{O}(r_\varepsilon) = \delta^{ij} + \mathrm{o}(|\log\varepsilon|^{-1}), \quad \Gamma_{i\alpha}^{\beta} = \mathrm{O}(1).$$

With (4.131), the boundedness of $(barv_\varepsilon)$ in L^∞, Hölder's and Young's inequality, we then follow:

$$\begin{aligned}&\int_{B_{r_\varepsilon}(p_\varepsilon)} |\nabla v_\varepsilon|^2 \,\mathrm{vol} \\ &= \int_{\bar{B}_{r_\varepsilon}(0)} \sum_{\alpha=1}^{2} \left(\frac{\partial(\bar{v}_\varepsilon)^\alpha}{\partial x^i} + \Gamma_{i\beta}^{\alpha}(\bar{v}_\varepsilon)^\beta\right)\left(\frac{\partial(\bar{v}_\varepsilon)^\alpha}{\partial x^j} + \Gamma_{j\gamma}^{\alpha}(\bar{v}_\varepsilon)^\gamma\right) g^{ij}\sqrt{|g|}\,\mathrm{d}x \\ &\geq \int_{\bar{B}_{r_\varepsilon}(0)} |\bar{\nabla}\bar{v}_\varepsilon|^2 - \mathrm{O}(1)(1 + |\bar{\nabla}\bar{v}_\varepsilon|)\,\mathrm{d}x - \mathrm{o}(|\log\varepsilon|^{-1}) \int_{\bar{B}_{r_\varepsilon}(0)} 1 + |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x \\ &\geq \int_{\bar{B}_{r_\varepsilon}(0)} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x - \mathrm{O}(r_\varepsilon)\int_{\bar{B}_{r_\varepsilon}(0)} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x - \mathrm{o}(1) \\ &= \int_{\bar{B}_{r_\varepsilon}(0)} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x - \mathrm{o}(1).\end{aligned}$$

A similar argument for the remaining terms of $GL_\varepsilon^{(\mathfrak{m})}(v_\varepsilon, B_{r_\varepsilon}(p_\varepsilon))$ then leads to:

$$GL_\varepsilon^{(\mathfrak{m})}(v_\varepsilon, B_{r_\varepsilon}(p_\varepsilon)) \geq E_\varepsilon(\bar{v}_\varepsilon, \bar{B}_{r_\varepsilon}(0)) - \mathrm{o}(1).$$

Consequently with (4.130) and (4.126), we see that:

$$\begin{aligned}&\liminf_{\varepsilon\to0}\left(\gamma_\varepsilon^{(\mathfrak{m})}(p_\varepsilon,r_\varepsilon,\lambda_\varepsilon)-\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r_\varepsilon}{\varepsilon}\right)\right)\\&\quad\geq\liminf_{\varepsilon\to0}\left(\bar\gamma_\varepsilon^{(\mathfrak{m})}(r_\varepsilon,\lambda_\varepsilon)-\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r_\varepsilon}{\varepsilon}\right)\right)=\gamma_{\mathfrak{m}},\end{aligned}$$

where we have used that $\frac{r_\varepsilon}{\varepsilon}\to\infty$. In a similar fashion, one can also derive:

$$\limsup_{\varepsilon\to0}\left(\gamma_\varepsilon^{(\mathfrak{m})}(p_\varepsilon,r_\varepsilon,\lambda_\varepsilon)-\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r_\varepsilon}{\varepsilon}\right)\right)\leq\gamma_{\mathfrak{m}}$$

and (4.129) follows. □

Given $r\in(0,r^*)$ and $p_0\in S$, we define the set:

$$\mathcal{H}(p_0,r):=\left\{v\in C^\infty(A_{\frac{r}{2},r}(p_0);\mathbb{S}^1)\colon\bar v=\lambda\frac{x}{|x|}\text{ for some }\lambda\in\mathbb{S}^1\right\},$$

where $A_{\frac{r}{2},r}(p_0)$ denotes the annulus:

$$A_{\frac{r}{2},r}(p_0):=B_r(p_0)\setminus B_{\frac{r}{2}}(p_0)$$

and $\bar v$ is the coordinate representation of v via centered normal coordinates at p_0. We also define for $r>0$ the set $\bar{\mathcal{H}}(r,x_0)$ as:

$$\bar{\mathcal{H}}:=\left\{x\mapsto\lambda\frac{x}{|x|}:\lambda\in\mathbb{S}^1\right\}.$$

In the next lemma, we show that a sequence of smooth vector fields (v_ε) defined on dyadic annuli (A_ε) and degree equal 1 around the larger cirlce of A_ε and with length approximately equal to 1 is either close to an element of $\mathcal{H}(p_\varepsilon,r_\varepsilon)$ (,where p_ε is the center and r_ε the outer radius of A_ε), or has in the limit $\varepsilon\to0$ Dirichlet energy strictly larger that $\pi\log(2)$. More precisely:

Lemma 4.22
Let $(r_\varepsilon)\subset(0,r^*)$ with $r_\varepsilon\to0$, $(p_\varepsilon)\subset S$, and for each $\varepsilon>0$ let $v_\varepsilon\in W^{1,2}(TA_\varepsilon)$, where $A_\varepsilon:=A_{\frac{r_\varepsilon}{2},r_\varepsilon}(p_\varepsilon)$ such that $\sup_\varepsilon\|v_\varepsilon\|_{L^\infty(A_\varepsilon)}<\infty$, $|\deg(v_\varepsilon,\partial B_{r_\varepsilon}(p_\varepsilon))|=1$,

$$r_\varepsilon^{-2}\int_{A_\varepsilon}||v_\varepsilon|-1|\,\mathrm{vol}\to0\ \textit{as}\ \varepsilon\to0,\tag{4.132}$$

and

$$\inf_{z\in\mathcal{H}(p_\varepsilon,r_\varepsilon)}\|v_\varepsilon-z\|_{W^{1,2}(A_\varepsilon)}\geq\delta\tag{4.133}$$

for some fixed $\delta>0$. Then, there exists $\omega(\delta)>0$ only depending on δ such that:

$$\frac12\int_{A_\varepsilon}|\nabla v_\varepsilon|^2\,\mathrm{vol}\geq\pi\log(2)+\omega(\delta)-\mathrm{o}(1).\tag{4.134}$$

Proof. Suppose, by contradiction, that up to taking a subsequence:

$$\limsup_{\varepsilon\to0}\frac12\int_{A_\varepsilon}|\nabla v_\varepsilon|^2\,\mathrm{vol}\leq\pi\log(2).\tag{4.135}$$

1. step: We consider the coordinate representation $\bar{v}_\varepsilon$ of v_ε via centered (at p_ε) normal coordinates, and set $\bar{A}_\varepsilon := \bar{A}_{\frac{r_\varepsilon}{2},r_\varepsilon}(0)$ to be the corresponding Euclidean annulus. By the choice above, we see:

$$g^{ij}\sqrt{|g|} = \delta^{ij} + \mathrm{O}(r_\varepsilon) = \delta^{ij} + \mathrm{o}(1), \quad \Gamma^{\beta}_{i\alpha} = \mathrm{O}(1).$$

As $\sup_\varepsilon \|v_\varepsilon\|_{L^\infty(A_\varepsilon)} < \infty$, we also have $\sup_\varepsilon \|\bar{v}_\varepsilon\|_{L^\infty(\bar{A}_\varepsilon)} < \infty$, due to the equivalence of norms. Similarly:

$$\sup_\varepsilon \int_{\bar{A}_\varepsilon} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x < \infty.$$

Therefore:

$$\begin{aligned}
&\int_{A_\varepsilon} |\nabla v_\varepsilon|^2 \,\mathrm{vol} \\
&= \int_{\bar{A}_\varepsilon} \sum_{\alpha=1}^{2} \left(\frac{\partial(\bar{v}_\varepsilon)^\alpha}{\partial x^i} + \Gamma^{\alpha}_{i\beta}(\bar{v}_\varepsilon)^\beta \right) \left(\frac{\partial(\bar{v}_\varepsilon)^\alpha}{\partial x^j} + \Gamma^{\alpha}_{j\gamma}(\bar{v}_\varepsilon)^\gamma \right) g^{ij}\sqrt{|g|}\,\mathrm{d}x \\
&\geq \int_{\bar{A}_\varepsilon} |\bar{\nabla}\bar{v}_\varepsilon|^2 - \mathrm{O}(1)(1 + |\bar{\nabla}\bar{v}_\varepsilon|)\,\mathrm{d}x - \mathrm{O}(r_\varepsilon)\int_{\bar{A}_\varepsilon} 1 + |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x \\
&\geq \int_{\bar{A}_\varepsilon} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x - \mathrm{o}(1).
\end{aligned}$$

By (4.135), this estimate implies:

$$\limsup_{\varepsilon\to 0} \int_{\bar{A}_\varepsilon} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x \leq \pi\log(2). \tag{4.136}$$

2. step: Given $\bar{w}_\varepsilon(x) := \bar{v}_\varepsilon(r_\varepsilon x)$ and $\bar{A} := A_{\frac{1}{2},1}(0)$, we see by (4.136) that:

$$\limsup_{\varepsilon\to 0} \int_{\bar{A}} |\bar{\nabla}\bar{w}_\varepsilon|^2 \,\mathrm{d}x = \limsup_{\varepsilon\to 0} \int_{\bar{A}_\varepsilon} |\bar{\nabla}\bar{v}_\varepsilon|^2 \,\mathrm{d}x \leq \pi\log(2).$$

Together with the boundedness of (w_ε) in L^∞, this – in particular – implies that up to a subsequence $\bar{w}_\varepsilon \rightharpoonup \bar{w}$ weakly in $W^{1,2}(\bar{A};\mathbb{R}^2)$ with $\bar{w}$ satisfying:

$$\frac{1}{2}\int_{\bar{A}} |\bar{\nabla}\bar{w}|^2 \,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_{\bar{A}} |\bar{\nabla}\bar{w}_\varepsilon|^2 \,\mathrm{d}x \leq \pi\log(2). \tag{4.137}$$

Furthermore, by (4.132) and the definition of $\bar{w}_\varepsilon$, we see that:

$$\int_{\bar{A}} ||\bar{w}_\varepsilon| - 1| \,\mathrm{d}x = r_\varepsilon^{-2} \int_{\bar{A}_\varepsilon} ||\bar{v}_\varepsilon| - 1| \,\mathrm{d}x \leq r_\varepsilon^{-2}(1 + \mathrm{o}(1)) \int_{A_\varepsilon} ||v_\varepsilon| - 1| \,\mathrm{vol} \to 0,$$

as $\varepsilon \to 0$, and therefore $|\bar{w}| = 1$ a.e. in $\bar{A}$. Finally, by the continuity of the degree with respect to weak convergence in $W^{1,2}$, we follow that $|\deg(\bar{w}, \partial B_1(0))| = 1$. It is a classic result in the Euclidean setting (see also, e.g. [8]) that this implies, combined with (4.137):

$$\int_{\bar{A}} |\bar{\nabla}\bar{w}|^2 \,\mathrm{d}x = \pi\log(2),$$

and thus, $\bar{w} = \lambda \frac{x}{|x|}$ for some $\lambda \in \mathbb{S}^1$. Furthermore, with (4.137) we also see that:

$$\lim_{\varepsilon \to 0} \int_{\bar{A}} |\bar{\nabla} \bar{w}_\varepsilon|^2 \,\mathrm{d}x = \int_{\bar{A}} |\bar{\nabla} \bar{w}|^2 \,\mathrm{d}x,$$

and hence $\bar{w}_\varepsilon \to \bar{w}$ strongly in $W^{1,2}(\bar{A};\mathbb{R}^2)$, where we have used that weak convergence together with convergence of the norm leads to strong convergence. By change of coordinates ($\tilde{x} = \frac{x}{r_\varepsilon}$), we have that:

$$\begin{aligned}\int_{\bar{A}_\varepsilon} &|\bar{v}_\varepsilon(x) - \bar{w}(x)|^2 \,\mathrm{d}x + \int_{\bar{A}_\varepsilon} |\bar{\nabla}\bar{v}_\varepsilon(x) - \bar{\nabla}\bar{w}(x)|^2 \,\mathrm{d}x \\ &= r_\varepsilon^2 \int_{\bar{A}_\varepsilon} |\bar{w}_\varepsilon(\tilde{x}) - \bar{w}(\tilde{x})|^2 \,\mathrm{d}\tilde{x} + \int_{\bar{A}_\varepsilon} |\bar{\nabla}\bar{w}_\varepsilon(\tilde{x}) - \bar{\nabla}\bar{w}(\tilde{x})|^2 \,\mathrm{d}\tilde{x} \to 0, \text{ as } \varepsilon \to 0,\end{aligned}$$

and hence $\lim_{\varepsilon\to 0}\|\bar{v}_\varepsilon - \bar{v}\|_{W^{1,2}(\bar{A}_\varepsilon,\mathbb{R}^2)} = 0$. Let z_ε be the section on TA_ε, whose coordinate representation (in centered normal coordinates at p_ε) is $\bar{w}$. As $\bar{w} := \lambda \frac{x}{|x|}$, we have that $z_\varepsilon \in \mathcal{H}(p_\varepsilon, r_\varepsilon)$. Furthermore, as we chose normal coordinates, we derive:

$$\|\cdot\|_{W^{1,2}(TA_\varepsilon)} = \|\cdot\|_{W^{1,2}(\bar{A}_\varepsilon;\mathbb{R}^2)} + \mathrm{o}(1),$$

and consequently:

$$\lim_{\varepsilon\to 0}\|v_\varepsilon - z_\varepsilon\|_{W^{1,2}(TA_\varepsilon)} = 0,$$

which is a contradiction to (4.133) for ε sufficiently small. □

Late, we will need a slight improvement of Lemma 4.18:

Lemma 4.23
Given a sequence $(v_\varepsilon) \subset C^\infty(TB_0)$ (,where $B_0 = B_{r_0}(p_0)$ is a geodesic ball) satisfying the energy bound

$$GL_\varepsilon(v_\varepsilon) \le \pi|\log\varepsilon| + C_0 \tag{4.138}$$

for a constant C_0 independent of ε, and such that:

$$\mathrm{vort}(v_\varepsilon) \xrightarrow{\flat} \pm\delta_{p_0} \textit{ flat in } B_0,$$

then, we can find $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there exists a finite family $\mathfrak{B}_\varepsilon$ of disjoint geodesic balls with the following properties:

(i) $r(\mathfrak{B}_\varepsilon) = \varepsilon^{\frac{1}{3}}$.

(ii) $\left\{||v_\varepsilon| - 1| \ge \varepsilon^{\frac{1}{5}}\right\} \cap (B_0)_\varepsilon \subset V_\varepsilon$, *where* $V_\varepsilon := \bigcup_{B\in\mathfrak{B}_\varepsilon} B \cap (B_0)_\varepsilon$.

(iii) $D_\varepsilon := D_{\mathfrak{B}_\varepsilon}((B_0)_\varepsilon) = \pm 1$, *and the center p_ε of the unique ball $B_\varepsilon \in \mathfrak{B}_\varepsilon$ with nonzero degree converges towards p_0, as $\varepsilon \to 0$.*

(iv) For any r_1, $r_2 > 0$ such that $\varepsilon^{\frac{1}{3}} \le r_1 < r_2 \le r_0 - \varepsilon - \mathrm{dist}(p_\varepsilon, p_0)$ the following energy lower bound holds true:

$$\frac{1}{2}\int_{A_{r_1,r_2}(p_\varepsilon)} |\nabla\tilde{v}_\varepsilon|^2 \,\mathrm{vol} \ge \left(1 - 2\varepsilon^{\frac{1}{4}}\right)\pi\log\left(\frac{r_2}{r_1 + 2\varepsilon^{\frac{1}{3}}}\right) - C(r_2 - r_1), \tag{4.139}$$

where $C = C(S)$ is a universal constant independent of ε.

Proof. Let $\mathfrak{B}_\varepsilon := \mathfrak{B}^\varepsilon_{r_\varepsilon}$ be the family of balls from Lemma 4.18 for $r_\varepsilon = \varepsilon^{\frac{1}{3}}$. Then, directly from Lemma 4.18, we see that the first three items of the statement above are satisfied. It remains to show the energy lower bound in (4.139). We define I_ε as the set of all radii $r \in J_\varepsilon := [\varepsilon^{\frac{1}{3}}, r - \varepsilon - \operatorname{dist}(p_\varepsilon, p_0)]$ such that $\partial B_r(p_\varepsilon) \cap V_\varepsilon \neq \emptyset$. As $r(\mathfrak{B}_\varepsilon) = \varepsilon^{\frac{1}{3}}$, we can estimate $|I_\varepsilon| \leq 2\varepsilon^{\frac{1}{3}}$. Let us now fix $r \in J_\varepsilon \setminus I_\varepsilon$, then by construction $B_\varepsilon \subset B_r(p_\varepsilon)$, and therefore $|\deg(\tilde{v}_\varepsilon, \partial B_r(p_\varepsilon))| = 1$, where $\tilde{v}_\varepsilon := \mathrm{T}_{1-\varepsilon^{\frac{1}{4}}} v_\varepsilon$ is as in Lemma 4.18. Hence, as was done in the proof of Proposition 4.2 (see also (4.55)) we can find a universal constant $\tilde{C} = \tilde{C}(S)$ independent of r and ε such that:

$$\frac{1}{2}\int_{\partial B_r(p_\varepsilon)} |\nabla \tilde{v}_\varepsilon|^2 \geq \pi \left(\frac{1}{r} - \tilde{C}\right),$$

By Fubini's theorem and the monotonicity of $r \mapsto \frac{1}{r}$, we have:

$$\begin{aligned}\frac{1}{2}\int_{A_{r_1,r_2}(p_\varepsilon)} |\nabla \tilde{v}_\varepsilon|^2 \operatorname{vol} &\geq \int_{[r_1,r_2]\setminus I_\varepsilon} \frac{1}{2}\left(\int_{\partial B_r(p_\varepsilon)} |\nabla \tilde{v}_\varepsilon|^2\right) \mathrm{d}r \\ &\geq \pi \int_{r_1+2\varepsilon^{\frac{1}{2}}}^{r_2} \frac{1}{r} - C \,\mathrm{d}r \\ &= \pi \left(\log\left(\frac{r_2}{r_1 + 2\varepsilon^{\frac{1}{3}}}\right) - \tilde{C}(r_1 - r_2)\right).\end{aligned}$$

As:

$$|\nabla v_\varepsilon|^2 \geq |v_\varepsilon|^2 |\nabla \tilde{v}_\varepsilon|^2 \geq (1-\varepsilon^{\frac{1}{4}})^2 |\nabla \tilde{v}_\varepsilon|^2 \geq (1 - 2\varepsilon^{\frac{1}{4}})|\nabla \tilde{v}_\varepsilon|^2 \geq,$$

the desired estimate follows. □

We are ready to proof the Γ-liminf:

Proof of the Γ*-liminf of Theorem 4.7.* Let us first select a subsequence (without relabeling) such that:

$$\liminf_{\varepsilon\to 0}\left(GGL(v_\varepsilon) - \pi\frac{|\chi(S)|}{\mathfrak{m}}|\log\varepsilon|\right) = \lim_{\varepsilon\to 0}\left(GGL(v_\varepsilon) - \pi\frac{|\chi(S)|}{\mathfrak{m}}|\log\varepsilon|\right).$$

Given any $w \in \mathcal{AS}^{(\mathfrak{m})}$, the truncation $\mathrm{T}_1 w$ has lower energy:

$$GGL(w) \leq GGL(v).$$

Consequently, by replacing the sequence (v_ε) with $(\mathrm{T}_1 v_\varepsilon)$, we can assume, without loss of generality, that $\sup_\varepsilon \|v_\varepsilon\|_{L^\infty} \leq 1 < \infty$. Furthermore, it is not restrictive to suppose that:

$$GGL(v_\varepsilon) \leq \pi\frac{|\chi(S)|}{\mathfrak{m}}|\log\varepsilon| + C \tag{4.140}$$

for some constant C independent of ε as in the other case (4.86) trivially follows. By the compactness statement of Theorem 4.7, we can select a subsequence, again without relabeling, such that:

$$\operatorname{vort}(v_\varepsilon) \overset{\flat}{\rightharpoonup} \mu = \operatorname{sgn}(\chi(S)) \sum_{k=1}^{\mathfrak{m}|\chi(S)|} \frac{1}{\mathfrak{m}}\delta_{q_k} \in X^{(\mathfrak{m})} \text{ flat in } S \tag{4.141}$$

$$u_\varepsilon \rightharpoonup u \text{ weakly in } SBV^2_{\mathrm{loc}}(T(S\setminus \operatorname{spt}(\mu))), \tag{4.142}$$

where $u \in \mathcal{LS}^{(\mathfrak{m})}$ with $\operatorname{vort}(v) = \mu$. Let $r_0 \in (0,1)$ be small enough such that the balls in $\{B_{r_0}(q_k)\}_k$ are pairwise disjoint, and for $r \in (0, r_0)$ let:

$$S_r := S \setminus \bigcup_k B_{r_0}(q_k).$$

1. step: We first wish to derive the Γ-liminf inequality in S_r. By the weak convergence of (u_ε) (see also (4.142)), standard lower semicontinuity arguments, the definition of $\mathcal{W}^{(\mathfrak{m})}(u) < \infty$, and the fact that $\mathcal{H}^1(J_u) < \infty$, we derive:

$$\begin{aligned}\liminf_{\varepsilon\to 0} GGL(u_\varepsilon, S_r) &\geq \liminf_{\varepsilon\to 0}\left(\frac{1}{2}\int_{S_r}|\nabla u_\varepsilon|^2 \operatorname{vol} + \mathcal{H}^1(J_{u_\varepsilon}\cap S_r)\right)\\ &\geq \frac{1}{2}\int_{S_r}|\nabla u|^2 \operatorname{vol} + \mathcal{H}^1(J_u \cap S_r)\\ &= \mathcal{W}^{(\mathfrak{m})}(u) + \pi\frac{|\chi(S)|}{\mathfrak{m}}|\log r| + \mathcal{H}^1(J_u) - \mathrm{o}_{r\to 0}(1). \quad (4.143)\end{aligned}$$

2. step: It remains to show:

$$\liminf_{\varepsilon\to 0}\left(GGL(u_\varepsilon, B_r(q_k)) - \frac{\pi}{\mathfrak{m}^2}\log(\tfrac{r}{\varepsilon})\right) \geq \gamma_{\mathfrak{m}} - \mathrm{o}_{r\to 0}(1) \qquad (4.144)$$

for any vortex center q_k of u. In this regard, let $B_{r_0} := B_{r_0}(q_k)$ and $B_r := B_r(q_k)$ for fixed k, $e \in C^\infty(B_{r_0}; \mathbb{S}^1)$ an arbitrary vector field, $v_\varepsilon := P_e(u_\varepsilon)$ (P_e as in (4.27)) for each $\varepsilon > 0$, and $GL_\varepsilon^{(\mathfrak{m})}$ be the energy functional from (4.123). Furthermore, we take a sequence $(w_\varepsilon) \subset C^\infty(TB_{r_0})$ such that:

$$\lim_{\varepsilon\to 0}\|v_\varepsilon - w_\varepsilon\|_{W^{1,2}(TB_{r_0})} = 0. \qquad (4.145)$$

With the energy bound (4.140), Lemma 4.16, and (4.145) we see that:

$$GL(w_\varepsilon, B_{r_0}) \leq \pi|\log\varepsilon| + C, \qquad (4.146)$$

for a constant C independent of ε. Using (4.110), the boundedness of (u_ε) in L^∞, the smoothness of e, and Hölder's inequality, we derive that:

$$\begin{aligned}&\int_{B_r}|u_\varepsilon|^2|\nabla e|^2 + |\langle \operatorname{jac}(u_\varepsilon), \operatorname{jac}(e)\rangle| \operatorname{vol}\\ &\quad\leq \||\nabla e|\|_{L^\infty}(|B_r| + |B_r|^{\frac{1}{3}}\||\nabla u_\varepsilon|\|_{L^{\frac{3}{2}}(S)}) = \mathrm{o}_{r\to 0}(1).\end{aligned}$$

Hence, by (4.124) and (4.145) instead of (4.144), we can equivalently show:

$$\liminf_{\varepsilon\to 0}\left(GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon, B_r) - \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r}{\varepsilon}\right)\right) \geq \gamma_{\mathfrak{m}} - \mathrm{o}_{r\to 0}(1). \qquad (4.147)$$

3. step: By (4.146), we can employ Lemma 4.23. Let $\mathfrak{B}_\varepsilon$, V_ε, and p_ε, be as in the statement of Lemma 4.23. We also set $r_\varepsilon := \varepsilon^{\frac{1}{3}}$ and $R_\varepsilon := \varepsilon^{\frac{1}{4}}$. As $p_\varepsilon \to q_k$ for $\varepsilon \to 0$, we see that for any $s \in (0, r)$ and $\varepsilon > 0$ small enough, we have by (4.139):

$$
\begin{aligned}
&GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon, A_{R_\varepsilon,s}(p_\varepsilon)\setminus V_\varepsilon)\\
&\geq \frac{1}{2\mathfrak{m}^2}\int_{A_{R_\varepsilon,s}(p_\varepsilon)\setminus V_\varepsilon} |\nabla w_\varepsilon|^2 \,\mathrm{vol}\\
&\geq \left(1-2\varepsilon^{\frac14}\right)\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{s}{R_\varepsilon+2r_\varepsilon}\right) - C_1 r\\
&\geq \frac{\pi}{\mathfrak{m}^2}\left(\log\left(\frac{r}{R_\varepsilon}\right) - \log\left(\frac{r}{s}\right) - \log(1+2\varepsilon^{\frac{1}{12}}) - 2\varepsilon^{\frac14}\log\left(\frac{r^*}{3\varepsilon^{\frac13}}\right)\right) - C_1 r\\
&= \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r}{R_\varepsilon}\right) - \mathrm{o}(1) - \mathrm{o}_{s\to r}(1) - \mathrm{o}_{r\to 0}(1),
\end{aligned}
\tag{4.148}
$$

where C_1 is the constant from (4.139), and $\mathrm{o}_{s\to r}(1)+\mathrm{o}_{r\to 0}(1)$ is independent of ε. Also by (4.139), we have for any $K\in\mathbb{N}$ and $\varepsilon>0$ small enough:

$$
\begin{aligned}
&GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon, A_{r_\varepsilon,2^{-K}R_\varepsilon}(p_\varepsilon)\setminus V_\varepsilon)\\
&\geq \frac{1}{2\mathfrak{m}^2}\int_{A_{r_\varepsilon,2^{-K}R_\varepsilon}(p_\varepsilon)\setminus V_\varepsilon} |\nabla w_\varepsilon|^2 \,\mathrm{vol}\\
&\geq \left(1-2\varepsilon^{\frac14}\right)\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{2^{-K}R_\varepsilon}{3r_\varepsilon}\right) - C_1 R_\varepsilon\\
&\geq \frac{\pi}{\mathfrak{m}^2}\left(\log\left(\frac{R_\varepsilon}{r_\varepsilon}\right) - K\log(2) - \log(3) - 2\varepsilon^{\frac14}\log(\varepsilon^{\frac{1}{12}})\right) - C_1 R_\varepsilon\\
&= \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{R_\varepsilon}{r_\varepsilon}\right) - K\frac{\pi}{\mathfrak{m}^2}\log(2) - C_2 - \mathrm{o}(1)
\end{aligned}
\tag{4.149}
$$

with $C_2 \coloneqq \frac{\pi}{\mathfrak{m}^2}\log(2)$. Furthermore, by (4.43) (with a slightly modified double well potential), we have for $\varepsilon>0$ small enough:

$$
\begin{aligned}
GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon, V_\varepsilon) &\geq \frac{1}{\mathfrak{m}^2}\left(\frac12\int_{V_\varepsilon}|\nabla w_\varepsilon|^2 + \frac{\mathfrak{m}^2}{2\varepsilon^2}(1-|w_\varepsilon|^2)^2\,\mathrm{vol}\right)\\
&\geq \frac{\pi}{\mathfrak{m}^2}\left(\log\left(\frac{r_\varepsilon}{\varepsilon}\right) - \tilde{C}_3\right)\\
&= \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r_\varepsilon}{\varepsilon}\right) - C_3,
\end{aligned}
\tag{4.150}
$$

where $\tilde{C}_3$ is the universal constant from (4.139) and $C_3 = \frac{\pi}{\mathfrak{m}^2}\tilde{C}_3$. Then by (4.148), (4.149), and (4.150), it follows that for sufficiently small $\varepsilon>0$:

$$
\begin{aligned}
&GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon, B_r\setminus A_{2^{-K}R_\varepsilon,R_\varepsilon}(p_\varepsilon))\\
&\geq \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r}{\varepsilon}\right) - K\frac{\pi}{\mathfrak{m}^2}\log(2) - C_4 - \mathrm{o}(1) - \mathrm{o}_{s\to r}(1) - \mathrm{o}_{r\to 0}(1),
\end{aligned}
\tag{4.151}
$$

where $C_4 \coloneqq C_2 + C_3$, and the term $\mathrm{o}_{s\to r}(1)+\mathrm{o}_{r\to 0}(1)$ is independent of ε. Fix $\delta>0$ and given C_4 as above, let $K=K(\delta)\in\mathbb{N}$ be chosen (independently of ε, r, and s) big enough so that:

$$
K\omega(\delta) \geq C_4 + \gamma_{\mathfrak{m}},
\tag{4.152}
$$

where $\omega(\delta)>0$ is as in Lemma 4.22. We need to discern between two cases.

4. step: In the first case we assume that, up to a subsequence:

$$\inf_{z\in\mathcal{H}(p_\varepsilon,2^{-k}R_\varepsilon)}\|w_\varepsilon - z\|_{W^{1,2}(TA_{\varepsilon,k})}\geq\delta$$

for all $k\in\{0,\dots,K-1\}$, where $A_{\varepsilon,k}:=A_{2^{-(k+1)}R_\varepsilon,2^{-k}R_\varepsilon}(p_\varepsilon)$. By Lemma 4.22, we therefore have for every $k\in\{0,\dots,K-1\}$ and every $\varepsilon>0$ small enough:

$$\int_{A_{\varepsilon,k}}|\nabla w_\varepsilon|^2\,\mathrm{vol}\geq\frac{\pi}{\mathfrak{m}^2}\log(2)+\omega(\delta)-\mathrm{o}(1). \tag{4.153}$$

Note that (4.132) is satisfied due to the following argument: As $\{|1-|w_\varepsilon||\geq\varepsilon^{\frac14}\}\subset V_\varepsilon$ and $\sup_\varepsilon\|w_\varepsilon\|_{L^\infty}\leq 1$, we have for any $k\in\{0,\dots,K-1\}$:

$$\begin{aligned}(2^{-k}R_\varepsilon)^{-2}\int_{A_{\varepsilon,k}}|1-|w_\varepsilon||\,\mathrm{vol}&\leq(2^{-k}R_\varepsilon)^{-2}\left(|A_{\varepsilon,k}|\varepsilon^{\frac14}+2|V_\varepsilon|\right)\\&\leq 2^{K-1}\varepsilon^{-\frac12}C_5(\varepsilon^{\frac34}+\varepsilon^{\frac23})=\mathrm{o}(1),\end{aligned}$$

where $C_5=C_5(S)$ is a universal constant. Combining (4.153), (4.151), and (4.152) we have for $\varepsilon>0$ small enough

$$\begin{aligned}&GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon,B_r)\\&\quad=GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon,B_r\setminus A_{2^{-K}R_\varepsilon,R_\varepsilon}(p_\varepsilon))+GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon,A_{2^{-K}R_\varepsilon,R_\varepsilon}(p_\varepsilon))\\&\quad\geq\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r}{\varepsilon}\right)+\gamma_{\mathfrak{m}}-\mathrm{o}(1)-\mathrm{o}_{s\to r}(1)-\mathrm{o}_{r\to 0}(1),\end{aligned}$$

where, again, $\mathrm{o}_{s\to r}(1)+\mathrm{o}_{r\to 0}(1)$ is independent of ε. Consequently, (4.147) follows after letting $\varepsilon\to 0$ and $s\to r$ (in exactly this order).

5. step: We will now deal with the second case, in which, up to taking a subsequence, we can find $k_0\in\{0,\dots,K-1\}$, and for each ε a $\lambda_\varepsilon\in\mathbb{S}^1$ such that:

$$\|w_\varepsilon-z_\varepsilon\|_{W^{1,2}(TA_{\varepsilon,k_0})}<\delta,$$

where z_ε is the unique element in $\mathcal{H}(2^{-k_0}R_\varepsilon,p_\varepsilon)$ whose coordinate representation satisfies $\bar z_\varepsilon=\lambda_\varepsilon\frac{x}{|x|}$. By standard cutoff arguments, we can modify w_ε into $\hat w_\varepsilon$ such that $\hat w_\varepsilon=w_\varepsilon$ outside A_{ε,k_0}, $w_\varepsilon=z_\varepsilon$ in $A_{\frac32\cdot 2^{-(k_0+1)}R_\varepsilon,2^{-k_0}R_\varepsilon}(p_\varepsilon)$, and:

$$\|w_\varepsilon-z_\varepsilon\|_{W^{1,2}(TA_{\varepsilon,k_0})}<2\delta. \tag{4.154}$$

Then, by (4.128) and (4.129), we derive that:

$$\begin{aligned}GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon,B_{2^{-k_0}R_\varepsilon}(p_\varepsilon))&=GL_\varepsilon^{(\mathfrak{m})}(\hat w_\varepsilon,B_{2^{-k_0}R_\varepsilon}(p_\varepsilon))+\mathrm{o}_{\delta\to 0}\\&\geq\gamma_\varepsilon(2^{-k_0}R_\varepsilon,p_\varepsilon,\lambda_\varepsilon)+\mathrm{o}_{\delta\to 0}\\&=\gamma_{\mathfrak{m}}+\mathrm{o}(1)+\mathrm{o}_{\delta\to 0}(1),\end{aligned}\tag{4.155}$$

where the term $\mathrm{o}_{\delta\to 0}(1)$ is independent of ε. By (4.139), we have for $\varepsilon>0$ small enough:

$$\begin{aligned}GL_\varepsilon^{(\mathfrak{m})}(w_\varepsilon,A_{2^{-k_0}R_\varepsilon,R_\varepsilon}(p_\varepsilon))&\geq\frac{1}{2\mathfrak{m}^2}\int_{A_{2^{-k_0}R_\varepsilon,R_\varepsilon}}|\nabla w_\varepsilon|^2\,\mathrm{vol}\\&\geq(1-2\varepsilon^{\frac14})\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{R_\varepsilon}{2^{-k_0}R_\varepsilon+2r_\varepsilon}\right)-C_1R_\varepsilon\\&=\frac{\pi}{\mathfrak{m}^2}\log\left(\frac{R_\varepsilon}{2^{-k_0}R_\varepsilon}\right)-\mathrm{o}(1),\end{aligned}$$

where, again, C_1 is the universal constant from (4.139). Combined with (4.148) and (4.155), this shows (4.127), after letting $\varepsilon \to 0$, $s \to r$, and $\delta \to 0$ (in exactly this order). $\square$

4.3.3 Gamma-limsup

In the following lemma, we will relate the non-fractional renormalized energy on a surface to the Euclidean one:

Lemma 4.24
Given a geodesic ball $B_0 = B_{r_0}(p_0)$ and $v \in W^{1,2}_{\text{loc}}(B_0 \setminus \{p_0\}; \mathbb{S}^1) \cap W^{1,1}(TB_0)$ such that $\operatorname{vort}(v) = \pm\delta_{p_0}$ and $\mathcal{W}(v, B_0) < \infty$ (with $\mathcal{W}(v)$ as in (4.104)), we have:

$$\lim_{k\to\infty} \frac{1}{2} \int_{A_k} |\nabla v|^2 \operatorname{vol} = \pi \log(2), \tag{4.156}$$

where $A_k := A_{2^{-(k+1)}r_0, 2^{-k}r_0}(p_0)$. Furthermore, a coordinate representation $\bar{\mathcal{W}}(\bar{v}, \bar{B}_0)$ ($\bar{B}_0 := \bar{B}_{r_0}(0)$ is the Euclidean ball of radius r_0 and centered at the origin) with respect to centered normal coordinates at p_0 and an arbitrary local frame in B_0 satisfies $\bar{\mathcal{W}}(\bar{v}, \bar{B}_0) < \infty$, where:

$$\bar{\mathcal{W}}(\bar{v}, \bar{B}_0) := \lim_{r\to 0} \left(\frac{1}{2} \int_{\bar{A}_r} |\bar{\nabla} \bar{v}|^2 \,\mathrm{d}x - \pi |\log(r)| \right),$$

is the Euclidean analogue of $\mathcal{W}(v, B_0)$ with $\bar{A}_r := \bar{A}_{r,r_0}(0)$ being the Euclidean annulus centered at the origin with inner radius r and outer radius r_0.

Proof. 1. Step: Let us first prove (4.156). We rewrite the renormalized energy $\mathcal{W}(v, B_0)$ as follows:

$$\begin{aligned}
\mathcal{W}(v, B_0) &= \sum_{k=0}^{\infty} \left(\frac{1}{2} \int_{A_k} |\nabla v|^2 \operatorname{vol} - \pi \log\left(\frac{2^{-k} r_0}{2^{-(k+1)} r_0} \right) \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2} \int_{A_k} |\nabla v|^2 \operatorname{vol} - \pi \log(2) \right)
\end{aligned}$$

with A_k as before. As $|v| = 1$ a.e. and $\operatorname{vort}(v) = \pm\delta_{p_0}$, we see by (4.54) that for each $k \in \mathbb{N}$:

$$\begin{aligned}
\frac{1}{2} \int_{A_k} |\nabla v|^2 \operatorname{vol} &\geq \pi \log\left(\frac{2^{-k} r_0}{2^{-(k+1)} r_0} \right) - C_0(2^{-k} r_0 - 2^{-(k+1)} r_0) \\
&= \pi \log(2) - C_0(2^{-k} r_0 - 2^{-(k+1)} r_0)
\end{aligned}$$

for some universal constant $C_0 = C_0(S)$ independent of k and v. Consequently, for the same constant C, we follow:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left(\frac{1}{2} |\nabla v|^2 \operatorname{vol} - \pi \log(2) + C_0(2^{-k} r_0 - 2^{-(k+1)} r_0) \right) \\
&\qquad \leq \mathcal{W}(v, B_0) + C_0 r_0 < \infty,
\end{aligned}$$

where each term in the series on the left side above is nonnegative. Consequently, we must have:

$$\lim_{k\to\infty} \frac{1}{2} |\nabla v|^2 \operatorname{vol} - \pi \log(2) + C_0(2^{-k} r_0 - 2^{-(k+1)} r_0) = 0,$$

and (4.156) follows.

2. Step: We will now shot the finiteness of $\bar{\mathcal{W}}(\bar{v}, \bar{B}_0)$. By (4.156), we can find $K_0 \in \mathbb{N}$ big enough such that for any $k \geq K_0$;

$$\frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} \leq 3\pi \log(2).$$

Hence by our choice of coordinates we can find a universal constant $C_1 = C_1(S)$ such that:

$$\begin{aligned}
\mathcal{W}(\bar{v}, \bar{B}_{2^{-K_0} r_0}(0)) &= \sum_{k=K_0}^{\infty} \left(\frac{1}{2}\int_{\bar{A}_k} |\bar{\nabla}\bar{v}|^2 \,\mathrm{d}x - \pi \log(2) \right) \\
&\leq \sum_{k=K_0}^{\infty} \left(\frac{1}{2}(1 + C_1 2^{-k} r_0) \int_{A_k} |\nabla v|^2 \,\mathrm{vol} - \pi \log(2) \right) \\
&= \sum_{k=K_0}^{\infty} \left(\frac{1}{2}\int_{A_k} |\nabla v|^2 \,\mathrm{vol} - \pi \log(2) \right) + \sum_{k=K_0}^{\infty} C_1 2^{-(k+1)} r_0 \int_{A_k} |\nabla v|^2 \,\mathrm{vol} \\
&\leq \mathcal{W}(v, B_0) + 6C_1 r_0 \pi \log(3) < \infty.
\end{aligned}$$

As $v \in W^{1,2}(TA_{2^{-K_0} r_0})$ and therefore $\bar{v} \in W^{1,2}(\bar{A}_{2^{-K_0} r_0}; \mathbb{R}^2)$, this concludes the proof. □

Lemma 4.25
Given $u \in \mathcal{LS}^{(\mathfrak{m})}(S)$ with $\mathrm{vort}(u) = \mathrm{sgn}(\chi(S)) \sum_{k=1}^{\mathfrak{m}|\chi(S)|} \frac{1}{\mathfrak{m}} \delta_{p_k}$ and for each k let $e^{(k)}$ be a harmonic vector field in a geodesic ball $B_{r'}(p_k)$ (r' is chosen small enough so that $\{B_{r'}(p_k)\}_k$ are disjoint). Then, for any $\delta > 0$ and $r_0 \in (0, r')$, there exists $r \in (0, r_0)$, $(\lambda_k) \in \mathbb{S}^1$, and $w \in \mathcal{AS}^{(\mathfrak{m})}(S_{\frac{r}{2}}; \mathbb{S}^1)$ such that:

- *(i) $w = u$ in S_r.*
- *(ii) For any k, the coordinate representation $\bar{w}$ of w in $B_{r'}(p_k)$, as described in Remark 4.1 satisfies $\bar{w}^{\mathfrak{m}} = \lambda_k \frac{x}{|x|}$ on $\partial \bar{B}_{\frac{r}{2}}(0)$.*
- *(iii) $\|w - u\|_{SBV^2(T(S_{\frac{r}{2}} \setminus S_r))} \leq \delta$.*

Proof. Let us fix k and shortly write $e := e^{(k)}$, $B_{r'} := B_{r'}(p_k)$, $v := P_e(u)$ in $B_{r'}$. By (4.103), we see that $\mathcal{W}(v, B_{r'}) < \infty$ and with Lemma 4.24, it follows that $\mathcal{W}(\bar{v}, \bar{B}_{r'}(0)) < \infty$, where $\bar{v}$ is the coordinate representation of v, as described in Remark 4.1. Let us also denote by $\bar{u}$ the coordinate representation of u in $B_{r'}$ through the same coordinates as before. By the choice of coordinates and the definition of P_e, we have that $\bar{v} = \bar{u}^{\mathfrak{m}}$ (where we identify $\mathbb{R}^2$ with $\mathbb{C}$). It was already shown in the Euclidean setting that for any $r_0 \in (0, r')$ and $\delta' > 0$, there exists $r \in (0, r_0)$, $\lambda \in \mathbb{S}^1$, and $\bar{w} \in SBV^2(\bar{A}_{\frac{r}{2}, r_0}(0); \mathbb{S}^1)$ such that:

- (i) $(\bar{w}^+)^{\mathfrak{m}} = (\bar{w}^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$-a.e. point on $J_{\bar{w}}$;
- (ii) $\bar{w} = \bar{u}$ in $\bar{A}_{r,r'}(0)$;
- (iii) $\|\bar{w} - \bar{u}\|_{SBV^2(\bar{A}_{\frac{r}{2},r}(0))} \leq \delta'$;
- (iv) $(\bar{w})^{\mathfrak{m}} = \lambda \frac{x}{|x|}$ on $\partial \bar{B}_{\frac{r}{2}}(0)$.

Let w be the vector field on $B_{r'}$ whose coordinate representation is $\bar{w}$. By the choice of coordinates, we have that $|w| = 1$ a.e. and:

$$P_e(w^+) = (\bar{w}^+)^{\mathfrak{m}} = (\bar{w}^-)^{\mathfrak{m}} = P_e(w^-)$$

at $\mathcal{H}^1$-a.e. point on J_w. By the equivalence of norms and the choice of $\bar{w}$, it then follows that $w \in \mathcal{AS}^{(\mathfrak{m})}(A_{\frac{r}{2},r'}(p_k))$, $w = u$ in $A_{r,r'}(p_k)$, and:

$$\|w - u\|_{SBV^2(TA_{\frac{r}{2},r}(p_k))} \leq C\delta'$$

for a universal constant $C = C(S)$. For δ' small enough, we therefore have:

$$\|w - u\|_{SBV^2(TA_{\frac{r}{2},r}(p_k))} < \frac{\delta}{\mathfrak{m}|\chi(S)|}.$$

This concludes the proof by the arbitrariness of k. □

The next lemma will construct the recovery sequence in the vicinity of the limit vortices:

Lemma 4.26
Given a geodesic ball $B := B_r(p)$ (for some $p \in S$ and $r \in (0, r^*)$), a smooth vector field $e \in C^\infty(B; \mathbb{S}^1)$, $\lambda \in \mathbb{S}^1$, and $\delta > 0$, there exists a sequence $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}(B)$ such that:

(i) The coordinate representation $\bar{u}_\varepsilon$ (as described in Remark 4.1) satisfies $\bar{u}_\varepsilon^{\mathfrak{m}} = \lambda \frac{x}{|x|}$ on $\partial \bar{B}_r$.

(ii) $\limsup_{\varepsilon \to 0} \left(GL(u_\varepsilon, B) - \frac{\pi}{\mathfrak{m}^2} |\log \varepsilon| \right) \leq \gamma_{\mathfrak{m}} + \delta + \mathrm{O}_{r \to 0}(1)$.

(iii) $\limsup_{\varepsilon \to 0} \mathcal{H}^1(J_{u_\varepsilon}) = \mathrm{O}_{r \to 0}(1)$.

Proof. By Corollary 3.11 in [40], we can find $\varepsilon_0 \in (0, 1)$ small enough and $\bar{z} \in SBV^2(\bar{B}_1(0); \mathbb{S}^1)$ such that $(\bar{z}^+)^{\mathfrak{m}} = (\bar{z}^-)^{\mathfrak{m}}$ at $\mathcal{H}^1$-a.e. point in $J_{\bar{z}}$, $\bar{z}^{\mathfrak{m}} = \lambda x$ on $\partial \bar{B}_1(0)$, and:

$$\bar{E}_{\varepsilon_0}(\bar{z}^{\mathfrak{m}}, \bar{B}_1(0)) - \frac{\pi}{\mathfrak{m}^2} |\log \varepsilon| \leq \gamma_{\mathfrak{m}} + \delta,$$

where:

$$\bar{E}_{\varepsilon_0}(\bar{v}, \Omega) := \frac{1}{2\mathfrak{m}^2} \int_\Omega |\bar{\nabla} \bar{v}|^2 + (\mathfrak{m}^2 - 1)|\bar{\nabla} |\bar{v}||^2 + \frac{\mathfrak{m}^2}{2\varepsilon_0^2}(1 - |\bar{v}|^2)^2 \, \mathrm{d}x$$

for any $\Omega \subset \mathbb{R}^2$ open and $\bar{v} \in W^{1,2}(\Omega; \mathbb{S}^1)$. Note that by the chain rule, we can show:

$$\bar{E}_{\varepsilon_0}(\bar{z}^{\mathfrak{m}}, \bar{B}_1(0)) = \frac{1}{2} \int_{\bar{B}_1(0)} |\nabla z|^2 + \frac{1}{2\varepsilon_0^2}(1 - |z|^2)^2 \, \mathrm{d}x. \tag{4.157}$$

For $\varepsilon \in (0, r\varepsilon_0)$, we will construct the coordinate representation $\bar{u}_\varepsilon$ of the desired u_ε by properly rescaling $\bar{z}$ and filling up the remaining space with a uniform rotation. More precisely, let:

$$\bar{u}_\varepsilon(x) := \begin{cases} \bar{z}(\frac{\varepsilon_0}{\varepsilon} x) & \text{if } |x| \leq \frac{\varepsilon}{\varepsilon_0}, \\ e^{\imath \frac{1}{\mathfrak{m}} \arg(x)} & \text{if } \frac{\varepsilon}{\varepsilon_0} < |x| \leq r, \end{cases}$$

where $\arg(x) \in [-\pi, \pi)$ is the argument of x seen as a complex number. By changing coordinates ($\tilde{x} = \frac{\varepsilon_0}{\varepsilon} x$), we then see that:

$$\bar{E}_{\varepsilon_0}(\bar{u}_\varepsilon, \bar{B}_{\frac{\varepsilon}{\varepsilon_0}}(0)) = \bar{E}_{\varepsilon_0}(\bar{z}, \bar{B}_1(0)).$$

Let u_ε be the vector field on $B_r(p)$ whose coordinate representation, as described in Remark 4.1, is $\bar{u}_\varepsilon$. By construction, $u_\varepsilon \in \mathcal{AS}^{(\mathfrak{m})}(B_r(p))$. Furthermore, by (4.157), we can show with an argument similar to the one in the proof of Lemma 4.21:

$$GL(u_\varepsilon, B_{\frac{\varepsilon}{\varepsilon_0}}) - \frac{\pi}{\mathfrak{m}^2}|\log \varepsilon_0| \leq \gamma_{\mathfrak{m}} + \delta + o_{\varepsilon \to 0}(1). \tag{4.158}$$

Let K be the largest natural number such that $2^{-K} r \geq \frac{\varepsilon}{\varepsilon_0}$, and define for any $k \in \{1, \ldots, K\}$:

$$A_k := \begin{cases} A_{2^{-(k+1)}r, 2^{-k}r}(p) & \text{if } k < K, \\ A_{\frac{\varepsilon}{\varepsilon_0}, 2^{-k}r}(p) & \text{if } k = K, \end{cases} \qquad \bar{A}_k := \begin{cases} \bar{A}_{2^{-(k+1)}r, 2^{-k}r}(0) & \text{if } k < K, \\ \bar{A}_{\frac{\varepsilon}{\varepsilon_0}, 2^{-k}r}(0) & \text{if } k = K. \end{cases}$$

Using a dyadic decomposition and passing to coordinates, we have the following estimate outside of $B_{\frac{\varepsilon}{\varepsilon_0}}(p)$:

$$\begin{aligned} GL(u_\varepsilon, A_{\frac{\varepsilon}{\varepsilon_0}, r}(p)) &= \sum_{k=0}^{K} \frac{1}{2} \int_{A_k} |\nabla u_\varepsilon|^2 \operatorname{vol} \\ &\leq \sum_{k=0}^{K} \frac{1}{2}(1 + C 2^{-k} r) \int_{\bar{A}_k} |\bar{\nabla} \bar{u}_\varepsilon|^2 \, dx \\ &\leq \frac{\pi}{\mathfrak{m}^2} \log\left(\frac{r\varepsilon_0}{\varepsilon}\right) + 2C \frac{\pi}{\mathfrak{m}^2} \log(2) r \end{aligned}$$

for some universal constant $C = C(S)$. Combined with (4.158), this shows Item (ii) of the statement. Finally, by construction and the equivalence of norms, we can find a universal constant $\tilde{C} = \tilde{C}(S)$ such that

$$\mathcal{H}^1(J_{u_\varepsilon} \cap B_r(p)) \leq \tilde{C} \left(\frac{\varepsilon}{\varepsilon_0} \mathcal{H}^1(J_z) + 2\pi r + 2\pi \frac{\varepsilon}{\varepsilon_0} \right),$$

which directly leads to Item (iii) in the statement. □

Proof of the Γ*-limsup of Theorem 4.7.* Let us fix $\delta > 0$ and $r_0 \in (0, \min\{r^*, 1\})$ small enough such that the balls $\{B_{r_0}(p_k)\}_k$ are disjoint, where $\{p_k\}$ is the set of vortex centers of u. Let $r \in (0, r_0)$, $(\lambda_k) \subset \mathbb{S}^1$, and w be as in Lemma 4.25, and for each k, let $(u_\varepsilon^{(k)})_\varepsilon$ be the sequence from Lemma 4.26 for $\lambda = \lambda_k$ and radius $\frac{r}{2}$. For each $\varepsilon > 0$, we then define u_ε to be:

$$u_\varepsilon(p) := \begin{cases} w(p) & \text{if } p \in S_{\frac{r}{2}}, \\ u_\varepsilon^{(k)}(p) & \text{if } p \in B_{\frac{r}{2}}(p_k). \end{cases}$$

By construction, $(u_\varepsilon) \subset \mathcal{AS}^{(\mathfrak{m})}(S)$ with:

$$\begin{aligned}
&GL(u_\varepsilon) - \frac{\pi|\chi(S)|}{\mathfrak{m}}|\log\varepsilon| \\
&\quad = GL(w, S_{\frac{r}{2}}) - \frac{\pi|\chi(S)|}{\mathfrak{m}}|\log r| \\
&\qquad + \sum_{k=1}^{\mathfrak{m}|\chi(S)|} GL(u_\varepsilon^{(k)}, B_{\frac{r}{2}}(p_k)) - \frac{\pi}{\mathfrak{m}^2}\log\left(\frac{r}{2\varepsilon}\right) \\
&\quad \leq \mathcal{W}^{\mathfrak{m}}(u) + |\chi(S)|\mathfrak{m}\gamma_{\mathfrak{m}} + \mathcal{H}^1(J_u) + \delta + \mathrm{o}_{r\to 0}(1) + \mathrm{o}_{\varepsilon\to 0}(1),
\end{aligned}$$

where the term $\mathrm{o}_{r\to 0}(1)$ is independent of ε. Consequently:

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} GL(u_\varepsilon) - \frac{\pi|\chi(S)|}{\mathfrak{m}}|\log\varepsilon| \\
&\quad \leq \mathcal{W}^{\mathfrak{m}}(u) + \mathfrak{m}|\chi(S)|\gamma_{\mathfrak{m}} + \mathcal{H}^1(J_u) + \delta + \mathrm{o}_{r\to 0}(1).
\end{aligned}$$

Finally, a standard diagonal sequence argument in which we send $r \to 0$ and $\delta \to 0$ (in exactly this order), concludes the proof. $\square$

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