# Quadratic Invariance for Distributed Control Systems with Intermittent Observations

P. Ugo Abara, V. Causevic and S. Hirche

*Abstract*— **In this paper we consider a finite-horizon optimization problem with a distributed control policy. The local outputs are sent to a local controller in an intermittent fashion. As a consequence the controller has access to sensor information only if it is sent by the associated local scheduler or by neighboring controllers. We consider generalized event-triggered schedulers (which includes time-triggered schedulers as a special case, where time-instants define the events). This leads to an event-dependent information structure available at each local controller. As a result, the information structure changes, which potentially leads to a non-convex control design problem. For any event-triggered sensing topology, we give a necessary and sufficient condition for convexity of the optimal control problem, by using the quadratic invariance (QI) property. Furthermore, we provide an online algorithm that adapts the communication topology among the local controllers and guarantees a *step-by-step* QI, which translates to a global QI.**

## I. INTRODUCTION

In recent times, networked control applications such as smart grids, transportation networks and other large-scale infrastructure systems have imposed the need for distributed and tractable control design. In distributed control, at each time instant, each controller has a set of measurements, referred to as available information, used for the computation of the control signals to be applied to the corresponding subsystems. Here we focus on intermittent sensing, i.e., the information available to each controller is different and event-triggered, i.e., depends on the event-triggering sequence. An interesting example of a technology that allows to change the packet routes (i.e. have time-varying communication topology) is SDN (Software-Defined Networking) [2].

The sensing topology describes which output each controller has direct access to at each time instant. It is easily seen that an event-triggered sensing topology therefore induces an event-dependent information structure, defined as the set of all available information to the controllers. In a well-known counterexample [10], it is shown that the practical solvability of a distributed optimal control problem depends on the information available to each controller. Finding tractable solutions to this kind of problem and the role of the information structure has been a very prolific research area [1], [7], [8]. A celebrated result can be found in [9], where the set of all information-constrained problems which can be cast a convex problem is given, in terms of quadratically invariant (QI) information constraints. In [5] the condition is proven to be necessary and sufficient. While [5] and [9] consider infinite-dimensional maps, in [4] this result is adapted to finite-dimensional maps. From [4], [5] and [9], however, it cannot be deduced in a straightforward manner how QI extends to the case of non-linear observation processes due to intermittent transmissions. Indeed, the design of QI event-triggered information structures is an open problem.

The main contribution of this paper is a necessary and sufficient condition such that the information structure is quadratically invariant, given an event-triggered sensing topology. This implies an information structure which is a function of the intermittent transmissions, and will translate into an event-dependent information constraint set on the controllers. The directed graph describing the interconnection between the local controllers, namely the communication topology, must satisfy some condition in order to achieve convexity of the optimization problem. Moreover, the problem herein deals also with the design of an event-dependent communication topology that guarantees convexity of the optimization problem given event-triggered sensing topologies. Additionally, we provide an adaptation mechanism for each time instant in order to guarantee QI with suitable communication topology.

The remainder of the paper is outlined as follows. We describe the problem setup in II. In section III we present our approach to the information-constrained problem. Finally, in section IV we provide our main result on necessary and sufficient condition for QI and present an online convexification algorithm. Finally, conclusions are given in V.

*Notation:* In this paper, the operator $(\cdot)^\top$ denotes the transpose. The Expectation operator is denoted by $\mathbb{E}[\cdot]$. The Euclidean norm is denoted by $\|\cdot\|_2$. The vector $\mathbf{1}^\top = (1,\ldots,1)$ where the length will be clear from the context. The sets $\mathbb{Z},\mathbb{R}$ and $\mathbb{B}$ represents the set of integers, real and Boolean matrices, respectively. The Boolean matrix $A \in \mathbb{B}^{n\times m}$ such that $A = \mathscr{B}(B)$ indicates the structure of a matrix $B$, i.e., $A_{ij} = 1$ if and only if $B_{ij} \neq 0\ \forall i,j$. For $X,Y \in \mathbb{B}^{m\times n}$, we say that $X \leq Y$ holds if and only if $X_{ij} \leq Y_{ij}$ for all $i,j$ satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. For appropriate matrices $C_i$, the matrix $D = \mathrm{blkdiag}(C_1,C_2,\ \ldots\ ,C_n)$ is the block-diagonal matrix such that $D_{ii} = C_i$ and $D_{ij} = 0$ for $i \neq j$.

## II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a networked control system composed of $N$ linear time-invariant subsystems which are physically inter-

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connected. The interconnections are represented through a directed graph $\mathscr{G}_p = (\mathscr{V}_p, \mathscr{E}_p)$ such that node $i \in \mathscr{V}_p$ for each subsystem $i \in \{1,\ldots,N\}$ and edge $(j,i) \in \mathscr{E}_p$ if the dynamics of node $i$ is influenced directly by the dynamics of node $j$. The dynamics of the $i$-th subsystem is given by the stochastic difference equation

$$\begin{cases} \bar{x}^i_{k+1} &= \bar{A}_i \bar{x}^i_k + \bar{B}_i \bar{u}^i_k + \sum_{j\in\mathscr{N}^i_p} \bar{A}_{ij}\bar{x}^j_k + \bar{w}^i_k \\ \bar{y}^i_k &= \bar{C}_i \bar{x}^i_k + \bar{v}^i_k \end{cases} \tag{1}$$

where $\bar{x}^i_k \in \mathbb{R}^{n_i}$ is the state of the $i$-th subsystem, $\bar{u}^i_k \in \mathbb{R}^{m_i}$ is the control signal and $A_i \in \mathbb{R}^{n_i\times n_i}$, $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $B_i \in \mathbb{R}^{n_i\times m_i}$. The initial state $\bar{x}^i_0$ is a random variable with finite mean and covariance. Additionally, $\bar{w}^i_k \in \mathbb{R}^{n_i}$ and $\bar{v}^i_k \in \mathbb{R}^{q_i}$ are zero-mean uncorrelated i.i.d. Gaussian noises with finite covariances, which are statistically independent of $\bar{x}^i_0$ for each $k$. The set of direct neighbors $\mathscr{N}^i_p \subset \mathscr{V}_p$ of subsystem $\mathscr{P}^i$ is defined as

$$\mathscr{N}^i_p = \{j \in \mathscr{V}_p | (j,i) \in \mathscr{E}_p\}.$$

For a compact notation, equation (1) can be rewritten as

$$\begin{cases} \bar{x}_{k+1} &= \bar{A}\bar{x}_k + \bar{B}\bar{u}_k + \bar{E}\bar{\xi}_k \\ \bar{y}_k &= \bar{C}\bar{x}_k + \bar{F}\bar{\xi}_k \end{cases} \tag{2}$$

where the stacked vectors are

$$\bar{x}_k = \left(\bar{x}^{1\top}_k,\ldots,\bar{x}^{N\top}_k\right)^\top \in \mathbb{R}^n, \bar{y}_k = \left(\bar{y}^{1\top}_k,\ldots,\bar{y}^{N\top}_k\right)^\top \in \mathbb{R}^q,$$

$$\bar{u}_k = \left(\bar{u}^{1\top}_k,\ldots,\bar{u}^{N\top}_k\right)^\top \in \mathbb{R}^m, \bar{\xi}_k = \left(\bar{\xi}^{1\top}_k,,\ldots,\bar{\xi}^{N\top}_k\right)^\top \in \mathbb{R}^n.$$

and $\bar{\xi}^i_k = \begin{bmatrix} \bar{w}^i_k \\ \bar{v}^i_k \end{bmatrix}$. The matrices $\bar{A}, \bar{B}$ and $\bar{C}$ are of appropriate dimension, meanwhile $\bar{E} = \mathrm{blkdiag}(I \;\; 0, I \;\; 0, \;\ldots\;, I \;\; 0)$ and $\bar{F} = \mathrm{blkdiag}(0 \;\; I, 0 \;\; I, \;\ldots\;, 0 \;\; I)$.

In addition to the process $\mathscr{P}^i$, the network also consists of control units $\mathscr{K}^i$ and schedulers $\mathscr{S}^i$.

### A. Schedulers

We assume that the controllers $\mathscr{K}^i$, $i = 1,\ldots,N$, have no direct access to the local sub-system output but instead the measurement $y^i_k$ is transmitted to $\mathscr{K}^i$ in an intermittent fashion. The output transmission depends on the scheduling variable $\lambda^i_k$ of the local scheduler $\mathscr{S}^i$, $i \in \{1,\ldots,N\}$, which is assumed to be a binary variable i.e.

$$\lambda^i_k = \begin{cases} 1 & \text{transmit } y^i_k, \\ 0 & \text{no transmission.} \end{cases} \tag{3}$$

Since we are not aiming for the design of the triggering laws, we assume equation (3) to encapsulate both time-triggered and event-triggered transmission (in fact we consider time-triggered as special case). We will thus refer to the information structure as event-dependent, or intermittent.

### B. Control

The interconnection of the local controllers is modelled by a time-varying graph $\mathscr{G}_c(k) = (\mathscr{V}_p, \mathscr{E}_c(k))$. The information set $\mathscr{I}^i_k$, $i \in \{1,\ldots,N\}$, available to each controller incorporates the communication history from the local scheduler $\mathscr{S}^i$ and the neighboring control units i.e.

$$\mathscr{I}^i_k = \mathscr{I}^i_{k-1} \cup \{\lambda^i_k y^i_k\} \cup_{j\in\mathscr{N}_c{}^i(k-1)} \{\mathscr{I}^j_{k-1}\}, \tag{4}$$

where $\mathscr{I}^i_{-1} = \emptyset$ and $\mathscr{N}_c{}^i(k)$ is the set of neighbors of $\mathscr{K}^i$ in the control network at time $k$. Formally,

$$\mathscr{N}_c{}^i(k) = \{j | (j,i) \in \mathscr{E}_c(k)\}. \tag{5}$$

Note that $\bar{y}^j_l$ is known to controller $i$ either if it is received directly from the scheduler $\mathscr{S}^j$ or by a one-step communication with a controller $\mathscr{K}^s$, which knows $y^j_l$. We are interested in finding control signals $\bar{u}^i_k$ which are constrained to be of the form

$$\bar{u}^i_k = \gamma^i_k(\mathscr{I}^i_k), \qquad i = 1,\ldots,N. \tag{6}$$

where $\gamma^i_k$ is a causal and measurable function of the information $\mathscr{I}^i_k$ available at $\mathscr{K}^i$ at the beginning of each period indexed by $k$. Finally, we define the tuple $\mathscr{I}_k = \left(\mathscr{I}^1_k,\ldots,\mathscr{I}^N_k\right)$ as the information structure of system (1).

### C. Objective

We consider a finite-horizon quadratic cost

$$J_T = \mathbb{E}\left[\bar{x}^\top_T \Lambda \bar{x}_T + \sum_{k=0}^{T-1} \bar{x}^\top_k Q \bar{x}_k + \bar{u}^\top_k R \bar{u}_k\right], \tag{7}$$

where $T$ indicates the length of the finite-horizon and $Q = Q^{\frac{1}{2}}\left(Q^{\frac{1}{2}}\right)^\top \succeq 0$, $\Lambda = \Lambda^{\frac{1}{2}}\left(\Lambda^{\frac{1}{2}}\right)^\top \succeq 0$, $R = R^{\frac{1}{2}}\left(R^{\frac{1}{2}}\right)^\top \succ 0$. We are interested in characterizing the information structure defined in (4) such that the problem

$$\begin{aligned} \underset{\bar{u}_0,\ldots,\bar{u}_{T-1}}{\text{minimize}} \quad & J_T \\ \text{subject to} \quad & (2),(4),(5) \text{ and } (6) \end{aligned} \tag{8}$$

can be cast as a convex optimization problem.

## III. Approach

In order to define the control policies in (6) as a function of the available output history, it is convenient to introduce the *history* vectors as follows

$$\begin{aligned} \tilde{x} &= [\bar{x}^\top_0,\ldots,\bar{x}^\top_T]^\top \in \mathbb{R}^{n(T+1)}, & \tilde{y} &= [\bar{y}^\top_0,\ldots,\bar{y}^\top_T]^\top \in \mathbb{R}^{q(T+1)}, \\ \tilde{u} &= [\bar{u}^\top_0,\ldots,\bar{u}^\top_{T-1}]^\top \in \mathbb{R}^{mT}, & \tilde{\xi} &= [\bar{\xi}^\top_0,\ldots,\bar{\xi}^\top_{T-1}]^\top \in \mathbb{R}^{nT}. \end{aligned}$$

The corresponding system equations read as

$$\begin{cases} \tilde{x} = A\tilde{x} + B\tilde{u} + E\tilde{\xi} \\ \tilde{y} = C\tilde{x} + F\tilde{\xi} \end{cases} \tag{9}$$

where the matrices $A,B$ are

$$A=\left[\begin{array}{c|cccc} I & 0 & 0 & \dots & 0 \\ \hline \bar{A} & 0 & 0 & \dots & 0 \\ 0 & \bar{A} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & \bar{A} & 0 \end{array}\right],\ B=\left[\begin{array}{ccc} & 0 & \\ \hline \bar{B} & & \\ & \ddots & \\ & & \bar{B} \end{array}\right], \quad (10)$$

$C=diag(\bar{C},\dots,\bar{C})$ and $E,F$ have the same structure as $B$. The equation in (9) can be seen as special case of the dynamical system defined as

$$\begin{cases} x_{t+1} &= Ax_t + Bu_t + E\xi_t \\ y_t &= Cx_t + F\xi_t \end{cases} \quad \text{s.t.} \quad \begin{array}{ll} u_t = \tilde{u}, & \forall t \geq 0 \\ \xi_t = \tilde{\xi}, & \forall t \geq 0. \end{array} \quad (11)$$

where the initial condition is $x_0 = \tilde{x}$. It follows immediately that $x_{t+1} = x_t = \tilde{x}$ for all $t \geq 0$.

### A. Control Structure

*1) Output history feedback:* We restrict our analysis to control laws which are linear in the history. From (4) and (6), the control policy $u_t$ can therefore be expressed as

$$u_t = Ky_t \quad (12)$$

where $K \in \mathbb{R}^{mT\times q(T+1)}$. Since $u_t$ represents the stacked vector defined in (6) and the fact that $y_t$ is the complete history of the system, causality constraints must be imposed on $K$ i.e.

$$K_{ij} = 0, \qquad i > j \text{ or } j = T,$$

where $K$ is partitioned according to $\tilde{u}$ and $\tilde{y}$, and $K_{ij} \in \mathbb{R}^{m\times q}$ $\forall i \in [0,T-1], j \in [0,T]$. In this paper we focus on single-input single-output sub-systems, i.e., $m_h = q_h = 1$, for all $h \in \{1,\dots,N\}$.

*2) Information constraints:* As discussed in the previous section, we assume the sensing mechanism to be event-triggered. Thus it can be represented by a Boolean matrix that describes which outputs can be sensed by the different local controllers $\mathscr{K}^i, i \in \{1,\dots,m\}$, i.e.

$$S_k \in \mathbb{B}^{m\times q} \text{ such that } S_k(i,j) = 1 \text{ iff } \bar{y}_k^j \in \mathscr{I}_k^i. \quad (13)$$

That is, $S_k(i,j) = 1$ if the controller $\mathscr{K}^i$ has access to $\bar{y}_k^j$, and $S_k(i,j) = 0$ otherwise. For example, if each controller is limited to have access to its own local plant output then $S_k = \text{diag}(\lambda_k^1,\dots,\lambda_k^N)$.

The communication topology describes which controllers are able to share information. Analogously to the sensing topology, it can be encoded in a Boolean matrix

$$Z_k \in \mathbb{B}^{m\times m} \text{ s.t. } Z_k(i,j) = 1 \text{ iff } j \in \mathscr{N}_c^i(k), \quad (14)$$

i.e, if controller $\mathscr{K}^i$ receives information from controller $\mathscr{K}^j$ at time instant $k$ then $Z_k(i,j) = 1$, otherwise it is zero.

*Remark 1 (Event-Dependent Information Structure):* Let $Z_k$ be the communication topology as in (14) and indicate with $\mu_k(i,j)$ the element in position $(i,j)$, i.e., $Z_k(i,j) = \mu_k(i,j)$. Under the assumption that each controller has access only to its own local plant output, whenever $\lambda_{k-1}^j = 0$ the controllers do not have access to $y_{k-1}^j$ and therefore do not need to communicate. On the other hand, if $\lambda_{k-1}^j = 1$ the controller $\mathscr{K}^j$ needs to communicate $y_{k-1}^j$ to the neighbors in order to preserve convexity. Therefore, the variable $\mu_k(i,j)$ is clearly a function of $\lambda_{k-1}^j$ and can be seen as an event-triggered variable which indicates whether controller $\mathscr{K}^j$ can communicate to $\mathscr{K}^i$, i.e., from (5), $j \in \mathscr{N}_c^i(k)$ if and only if $\mu_k(i,j) = 1$.

From Remark 1, the information structure induced by event-triggered sensing topologies and the corresponding communication topologies is referred to as event-triggered information structure.

The problem herein deals with the design of an event-dependent information structure that preserves the convexity of the optimization problem in (8) given an event-triggered sensing topology. The distributed nature of the problem which is encoded by the communication and sensing topologies, makes it is necessary to impose sparsity constraints on the gain matrix $K$ introduced in (12) . We adopt the following notation to streamline our use of sparsity constraints. Let us define the variable $\Pi_b^a$, for $a,b \in [0,T-1]$,

$$\Pi_b^a = \begin{cases} I_{m\times m}, & b \leq a, \\ Z_{b-1}\Pi_{b-1}^a, & b > a. \end{cases} \quad (15)$$

*Remark 2:* As we will show in the following lemma, $\Pi_b^a = Z_{b-1}Z_{b-2}\cdot\ldots\cdot Z_a$ is the adjacency matrix indicating whether $(\mathscr{K}^i,\mathscr{K}^j)$ can communicate in $b-a$ time steps for a given initial time $a$, for all pair of controllers $i,j$. Furthermore, $\Pi_b^{b-1}$ represents the communication topology at time $b-1$, i.e., $Z_{b-1}$.

The following lemma is adapted from [4].

*Lemma 1:* A sequence of sensing topologies $S_{0:T-1}$ and a sequence of communication topologies $Z_{0:T-2}$ induce the sparsity constraints on the gain matrix $K$ defined in (12) which can be written as

$$K \in \mathbf{S}_Z \subset \mathbb{R}^{mT\times q(T+1)} \quad (16)$$

where

$$S_Z = \begin{bmatrix} S_0 & 0 & 0 & \dots & 0 & 0 \\ \Pi_1^0 S_0 & S_1 & 0 & \dots & 0 & 0 \\ \Pi_2^0 S_0 & \Pi_2^1 S_1 & S_2 & \dots & 0 & 0 \\ \vdots & & & & & \\ \Pi_{T-1}^0 S_0 & \Pi_{T-2}^1 S_1 & \Pi_{T-3}^2 S_2 & \dots & S_{T-1} & 0 \end{bmatrix} \quad (17)$$

and $\mathbf{S}_Z = \text{Span}(S_Z) = \{J \in \mathbb{R}^{mT \times q(T+1)} \mid J(i,j) = 0 \ \forall i,j \text{ s.t. } S_Z(i,j) = 0\}$ is a subspace generated by the $S_Z$.

*Proof:* From definition (14) the controller $\mathscr{K}_k^i$ knows $y_{k-d}^j$ if and only if there is a controller $\mathscr{K}_{k-d}^l$ such that $S_{k-d}(l,j) = 1$ and $\mathscr{K}_k^i$ receives information from $\mathscr{K}_{k-d}^l$ in $d$ time steps. Let $\Xi = \Pi_k^{k-d} S_{k-d}$, where $\Pi_k^{k-d} = Z_{k-1} Z_{k-2} \cdot \ldots \cdot Z_{k-d}$. Thus $\Xi(i,j) = 1$ indicates if $\mathscr{K}_k^i$ knows $y_{k-d}^l$. The 0-blocks in (17) are due to causality constraints on the control law. Furthermore, each block of (17) with dimension $m \times q$ in position $(k, k-\tau)$ defines the subset of outputs at time $\tau$ which are known to the different controllers at time $k$. Therefore, such block sub-matrix must be equal to $\Pi_k^{k-\tau} S_{k-\tau}$ for all $k \in [0, T-1]$ and all $\tau \in [0,k]$. Hence, the structure of $S_Z$ in (17) holds. Moreover, since $S_Z$ represents the sparsity constraints on $K$, i.e. $\mathscr{B}(K) \leq S_Z$, we must have $K \in \mathbf{S}_Z = \text{Span}(S_Z)$ as in (16). ■

### B. Equivalent Formulation

In this subsection we analyze the convexity of (8) subject to (12) and information constraints (16) through quadratic invariance [9]. To this end, we cast the convexity problem in (8) in a more convenient way. Let us define the performance output $z_t = C_z x_t + D_z u_t$. We extend system (11) as

$$\begin{cases} x_{t+1} &= A x_t + B u_t + E \xi_t \\ y_t &= C x_t + F \xi_t \\ z_t &= C_z x_t + D_z u_t . \end{cases} \tag{18}$$

Moreover, with the choice

$$C_z = \left[\begin{array}{cccc} Q^{\frac{1}{2}} & & & \\ & \ddots & & \\ & & Q^{\frac{1}{2}} & \\ & & & \Lambda^{\frac{1}{2}} \\ \hline & & 0 & \end{array}\right], D_z = \left[\begin{array}{ccc} & 0 & \\ \hline R^{\frac{1}{2}} & & \\ & \ddots & \\ & & R^{\frac{1}{2}} \end{array}\right]$$

and from $C_z^\top D_z = 0$, we have immediately that

$$\mathbb{E}\left[z_t^\top z_t\right] = J_T .$$

Furthermore, let us consider the generalized plant of system (18) given in Fig. 1. Given the importance of the block $P_{22}$, in the remainder of the paper, we lighten our notation and define $G(z) = P_{22}$. Note that for the system under consideration $G(z) = C(zI - A)^{-1} B$ is the impulse response in $z$-domain.

Hence, studying the convexity of problem (8) is equivalent to studying the convexity of the following optimization problem (see [4], [9])

$$\begin{array}{ll} \underset{K}{\text{minimize}} & \|P_{11} + P_{12} K (I - GK)^{-1} P_{21}\| \\ \text{subject to} & K \in \mathbf{S}_Z \\ & K \text{ stabilizing} \end{array} \tag{19}$$

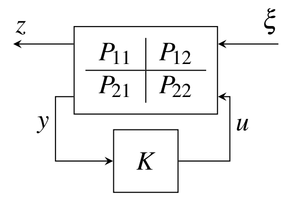


Fig. 1. Generalized plant

Note that the constraints $u_t = u_0$, $t > 0$, from (11), need not to be considered since $y_t = y_0$, $\forall t > 0$.

## IV. MAIN RESULTS

We now give some definitions and results on quadratic invariance.

### A. Quadratic Invariance

Given the form of (19), it is convenient to define the following closed-loop maps.

*Definition 1:* Let $\mathscr{U}$, $\mathscr{Y}$ be Banach spaces. The closed-loop maps $h_0 : \mathscr{L}(\mathscr{U}, \mathscr{Y}) \times \mathscr{L}(\mathscr{Y}, \mathscr{U}) \to \mathscr{L}(\mathscr{Y}, \mathscr{U})$ and $h_1 : \mathbb{R}^{q(T+1) \times mT} \times \mathbb{R}^{mT \times q(T+1)} \to \mathbb{R}^{mT \times q(T+1)}$ are such that

$$h_0(G,K) = -K(I - GK)^{-1}$$
$$h_1(P,L) = -L(I - PL)^{-1}$$

for all $G, K$ such that $I - GK$ is invertible and for all $P, L$ such that $I - PL$ is invertible. We also introduce the sets $h_0(G, \mathbf{S}_Z) = \{h_0(G,K), \ \forall K \in \mathbf{S}_Z\}$ and $h_1(P, \mathbf{S}_Z) = \{h_1(P,L), \ \forall L \in \mathbf{S}_Z\}$.

*Definition 2 (Quadratic Invariance):* Let $K, J : \mathbb{R}^a \to \mathbb{R}^b$, $X : \mathbb{R}^b \to \mathbb{R}^a$ be finite-dimensional linear maps. Let $K, J \in \tilde{S} \subset \mathbb{R}^{b \times a}$, where $\tilde{S} = \text{Span}(K^{bin})$ for some $K^{bin} \in \mathbb{B}^{b \times a}$. Then, $\tilde{S}$ is quadratically invariant (QI) with respect to $X$ if

$$KXJ \in \tilde{S} \quad \forall K, J \in \tilde{S}.$$

*Lemma 2 (From [5], [9]):* The set $h_0(G, \mathbf{S}_Z)$ is convex if and only if it is equal to $\mathbf{S}_Z$.

*Theorem 1 (Theorem 14 in [9]):* The set $h_0(G, \mathbf{S}_Z) = \mathbf{S}_Z$ if and only if $\mathbf{S}_Z$ is quadratically invariant with respect to $G$. We now give a preliminary result of the section.

*Lemma 3 (Quadratic Invariance in Time-Domain):* Let $K, J \in \mathbf{S}_Z$ be two time-independent control gains. The subspace $\mathbf{S}_Z$ is quadratically invariant with respect to $G$ if and only if $\mathbf{S}_Z$ it is quadratically invariant with respect to $P_k$, where

$$P_k = CA^k B, \quad \forall k \in [0, T-1].$$

*Proof:* Define $\Psi(z) = KG(z)J$ and let $\psi(k)$ be its inverse $z$-transform. From the assumption that $\mathbf{S}_Z$ is quadratically invariant with respect to $G$, we have from Definition 2 that $\Psi(z) \in \mathbf{S}_Z$ for almost all $z \in \mathbb{C}$. Furthermore we have that

$$\Psi_{a,b}(z) = 0 \text{ for almost all } z \in \mathbb{C} \Leftrightarrow \psi_{a,b}(k) = 0, \forall k \in \mathbb{Z}.$$

Therefore

$$\Psi(z) \in \mathbf{S}_Z \Leftrightarrow \psi(k) \in \mathbf{S}_Z, \forall k \in \mathbb{Z}. \tag{20}$$

Given the expression $G(z) = C(zI - A)^{-1}B$, and the power series expansion $(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \ldots$, accounting for the assumption that $K$ and $J$ are time-independent, we find

$$\psi(k) = KC\left(\sum_{n=0}^{\infty} A^n \delta_{k-n-1}\right) BJ = K(CA^k BJ),$$

for $k = 0, \ldots, T-1$. Where the impulse function $\delta_k$ is such that $\delta_k = 1$ for $k = 0$ and $\delta_k = 0$ otherwise. Note that the summation is truncated at $T-1$ due to the structure of matrix $A$ defined in (10). Indeed, the sub-matrix $A_{2,2}$, i.e. in position $(2,2)$ of the matrix $A$, is nilpotent since it is lower-triangular and with zero elements on the diagonal. This implies $(A_{2,2})^k = 0$, for $k \geq T$. Furthermore, the first row of $B$ is zero, which therefore yields $A^k B = 0$, for $k \geq T$. The proof is concluded by applying (20) for $k \in [0, T-1]$. ■

*Corollary 1:* The subspace $\mathbf{S}_Z$ in quadratically invariant with respect to $G$ if and only if it is quadratically invariant with respect to $P$, where

$$P = \sum_{k=0}^{T-1} CA^k B. \tag{21}$$

*Proof:* The proof is straightforward from the application of Lemma 3. ■

*Remark 3:* As argued at the beginning of this section, the gain matrix $K$ in (12) is constant. Therefore we can apply Lemma 3 to analyze the convexity of problem (19).

*Remark 4:* Given the structure of $A, B$ and $C$, it is easy to see that $P$, as defined in Corollary 1, is such that $P_{i,i-j} = \bar{C}\bar{A}^{j-1}\bar{B}$, for $i \in [2, T-1], j \in [1, i-1]$ and $P_{i,i-j} = 0$ otherwise.

The following proposition will be important in the derivation of our main result.

*Proposition 1:* Let $K, J \in \mathbf{S}_Z$ and $P = \sum_{k=0}^{T-1} CA^k B$. The subspace $\mathbf{S}_Z$ is QI with respect to $P$ if and only if

$$\mathscr{B}\left(\sum_{h=i-j+1}^{i} \sum_{g=0}^{h-(i-j)-1} K_{i,h}\bar{C}\bar{A}^g\bar{B}J_{h-g-1,i-j}\right) \leq \Pi_i^{i-j} S_{i-j}, \tag{22}$$

for every $i \in [0, T-1]$ and for every $j \in [0, i]$.

*Proof:* Similarly to [4], define $\Phi = KPJ$ and $\phi = PJ$. Denote with $\Phi_{a,b}$ the block $a, b$. Also, let $\phi_{a,b}$ denote the $q \times q$ block sub-matrices of $\phi$. Since $J$ is lower triangular and, from Remark (4), $P$ is strictly lower triangular, we have that

$$\phi_{h,b} = \sum_{g=0}^{h-b-1} \bar{C}\bar{A}^g\bar{B}J_{h-g-1,b}$$

in fact $J_{h-g-1,b} = 0$ for $h - g - 1 < b$. Therefore we can calculate $\Phi$ as

$$\Phi_{a,b} = \sum_{h=b+1}^{a} K_{a,h}\phi_{h,b} = \sum_{h=b+1}^{a} K_{a,b} \sum_{g=0}^{h-b-1} \bar{C}\bar{A}^g\bar{B}J_{h-g-1,b} \tag{23}$$

By substitution of $a = i$, $b = i - j$, where $j \geq 0$ we finally get

$$\Phi_{i,i-j} = \sum_{h=i-j+1}^{i} K_{i,h} \sum_{g=0}^{h-(i-j)-1} \bar{C}\bar{A}^g\bar{B}J_{h-g-1,i-j}. \tag{24}$$

From Lemma 1, a generic element in position $(i, i-j)$ of $S_Z$ is $\Pi_i^{i-j} S_{i-j}$. Therefore $\mathbf{S}_Z$ is QI with respect to $P$ if and only if

$$\mathscr{B}(\Phi_{i,i-j}) \leq \Pi_i^{i-j} S_{i-j}, \quad \forall K, J \in \mathbf{S}_Z \tag{25}$$

■

### B. Event-Dependent Quadratic Invariance

We state the following intermediate result

*Lemma 4:* The sparsity subspace $\mathbf{S}_Z$ given in Lemma 1 is QI with respect to $P = \sum_{k=0}^{T-1} CA^k B$ if and only if

$$\Pi_i^h S_h \Delta_g \Pi_{h-g-1}^{i-j} S_{i-j} \leq \Pi_i^{i-j} S_{i-j}, \tag{26}$$

and

$$\Delta_g = \mathscr{B}(\bar{C}\bar{A}^g\bar{B}) \tag{27}$$

for every $i \in [0, T-1]$ and for every $j \in [0, i]$, $h \in [i-j+1, i]$ and $g \in [0, h-(i-j)-1]$.

*Proof:* Sufficiency: From the inequality $\mathscr{B}(MN) \leq \mathscr{B}(M)\mathscr{B}(N)$, for all $M, N$ of appropriate dimensions, and due to the sparsity conditions $\mathscr{B}(K_{i,h}) \leq \Pi_i^h S_h$ and $\mathscr{B}(J_{h-g-1,i-j}) \leq \Pi_{h-g-1}^{i-j} S_{i-j}$ equation (22) translates into

$$\mathscr{B}(\Phi_{i,i-j}) \leq \sum_{h=i-j+1}^{i} \sum_{g=0}^{h-(i-j)-1} \Pi_i^h S_h \Delta_g \Pi_{h-g-1}^{i-j} S_{i-j} \tag{28}$$

From (27), and from Boolean matrix properties, if (26) is satisfied for every $i \in [0, T-1]$ and for every $j \in [0, i]$, $h \in [i-j+1, i]$ and $g \in [0, h-(i-j)-1]$ then, according to (25), $\mathbf{S}_Z$ is QI with respect to $P$.

Necessity: Suppose

$$\Pi_i^h S_h \Delta_g \Pi_{h-g-1}^{i-j} S_{i-j} \nleq \Pi_i^{i-j} S_{i-j}, \tag{29}$$

i.e, there is an element for which the left side is strictly greater that the right side. Furthermore suppose that $\bar{C} =$

$\bar{A}=\bar{B}=I$ and also $\Pi_i^0=I$ and $S_i=I$, for all $i$. Now suppose $K_{a,b}=0$ for all $(a,b)\neq(r^\star,0)$ and $J_{a,b}=0$ for all $(c,d)\neq(r^\star-k^\star-1,0)$, for some given $k^\star$ and $r^\star$. From equation (23) and by substituting $(a,b)=(r^\star,0)$ we obtain

$$\begin{aligned}\Phi_{r^\star,0}&=\sum_{h=1}^{r^\star}K_{r^\star,h}\sum_{g=0}^{h-1}\bar{C}\bar{A}^g\bar{B}J_{h-g-1,b}\\&=K_{r^\star,r^\star}\bar{C}\bar{A}^g\bar{B}J_{r^\star-k^\star-1,0}\end{aligned}$$

Now, from $K,J\in\mathbf{S}_Z$ clearly we have

$$\begin{aligned}\mathscr{B}(K_{r^\star,r^\star})&\leq\Pi_{r^\star}^{r^\star}S_{r^\star}=I\\ \mathscr{B}(J_{r^\star-k^\star-1,0})&\leq\Pi_{r^\star-k^\star-1}^{0}S_0=I\end{aligned}$$

Note that it is possible to choose $K,J$ such that

$$\mathscr{B}(\Phi_{r^\star,0})=\Delta_g=I.$$

In the new indices $k^\star$ and $r^\star$, condition (26) translates to

$$\Pi_{r^\star}^{r^\star}S_{r^\star}\Delta_g\Pi_{r^\star-k^\star-1}^{0}S_0\leq\Pi_{r^\star}^{r^\star}S_{r^\star},$$

or rather $\Delta_g\leq I$. This is in contrast with (29) which requires $\Delta_g\nleq I$. From our choice of $\Delta_g=I$, the proof is concluded. ■

Since the sums in (28) represent a logical *or* when dealing with Boolean matrices, we observe that if (26) for all $h,g$ then $\mathbf{S}_Z$ is QI with respect to $P$ since $\mathscr{B}(\Phi_{i,i-j})\leq\Pi_i^{i-j}S_{i-j}$.

What follows is the key result of the paper.

*Theorem 2 (Event-Dependent Quadratic Invariance):* Consider the sparsity subspace $\mathbf{S}_Z$ as defined in Lemma 1, and $P$ given by equation (21). Then $\mathbf{S}_Z$ is QI with respect to $P$ if and only if

$$S_i\Delta_r\Pi_{i-r-1}^{i-j}S_{i-j}\leq\Pi_i^{i-j}S_{i-j}\tag{30}$$

for every $i\in[0,T-1]$, $j\in[0,i]$, and $r\in[0,j-1]$. The structural matrix $\Delta_r$ is defined as in Lemma 4.

*Proof:* Starting from equation (24), we can rewrite $\Phi_{i,i-j}$ as

$$\begin{aligned}\Phi_{i,i-j}=K_{i,i}&\sum_{r=0}^{j-1}\bar{C}\bar{A}^r\bar{B}J_{i-r-1,i-j}+\\&+\sum_{h=i-j+1}^{i-1}K_{i,h}\sum_{g=0}^{h-(i-j)-1}\bar{C}\bar{A}^g\bar{B}J_{h-g-1,i-j}\end{aligned}$$

where we extracted $h=i$ from the overall summation in (24). Using the same procedure as in Lemma 4 we find that

$$\begin{aligned}\mathscr{B}(\Phi_{i,i-j})\leq&\sum_{r=0}^{j-1}\Pi_i^iS_i\Delta_r\Pi_{i-r-1}^{i-j}S_{i-j}+\\&+\sum_{h=i-j+1}^{i-1}\sum_{g=0}^{h-(i-j)-1}\Pi_i^hS_h\Delta_g\Pi_{h-g-1}^{i-j}.\end{aligned}\tag{31}$$

Furthermore, with the quadratic invariance inequality (22) of Proposition 1, and applying the same reasoning as in Lemma 4, it must therefore hold for each summand of (31)

$$S_i\Delta_r\Pi_{i-r-1}^{i-j}S_{i-j}+\Pi_i^hS_h\Delta_g\Pi_{h-g-1}^{i-j}\leq\Pi_i^{i-j}S_{i-j}\tag{32}$$

for every $i\in[0,T-1]$, $j\in[0,i]$, $h\in[i-j+1,i-1]$, $g\in[0,h-(i-j)-1]$ and $r\in[0,j-1]$. Recalling that $\Pi_i^i=I$ and $\Pi_i^h=\Pi_i^{i-1}\Pi_{i-1}^h$ we can rewrite (32) as

$$S_i\Delta_r\Pi_{i-r-1}^{i-j}S_{i-j}+\Pi_i^{i-1}\Pi_{i-1}^hS_h\Delta_g\Pi_{h-g-1}^{i-j}\leq\Pi_i^{i-1}\Pi_{i-1}^{i-j}S_{i-j}.$$

Due to the fact that all the matrices involved are Boolean, the previous equation can be separated into

$$S_i\Delta_r\Pi_{i-r-1}^{i-j}S_{i-j}\leq\Pi_i^{i-1}\Pi_{i-1}^{i-j}S_{i-j},\tag{33}$$

$$\Pi_i^{i-1}\Pi_{i-1}^hS_h\Delta_g\Pi_{h-g-1}^{i-j}\leq\Pi_i^{i-1}\Pi_{i-1}^{i-j}S_{i-j},\tag{34}$$

We can observe that equation (34) refers to the QI condition of Lemma 4 at the previous time instant $i-1$ pre-multiplied by $\Pi_i^{i-1}$. Therefore it is always satisfied since the pre-multiplication does not change the inequality. Therefore $\mathbf{S}_Z$ is quadratically invariant with respect to $P$ if and only if (33) holds. The proof is concluded since proof of necessity follows Lemma 4. ■

*Remark 5:* From Theorem 2, we notice that if no event was triggered at time $i$, i.e., no measurement was observed by the controllers, then $S_i=0$. Hence, it can be observed from equation (30) of the previous theorem that if $S_i=0$ then no additional constraints on $\Pi_i^{i-1}$ is needed, i.e., the communication topology at time instant $i$ will be $\Pi_i^{i-1}=Z_{i-1}=I$.

### C. Online Convexification

In this subsection, we exploit Theorem 2 to ensure the convexity of problem (19) at every time instant $i\in[0,T-1]$. In fact, Theorem 2 allows for a more efficient computation if compared to Lemma 4. As a matter of fact, it can be observed that inequality (30) is a function of $r\in[0,j-1]$ and $j\in[0,i]$ for every a given $i$, as opposed to (26) which is function of one additional variable. Therefore the complexity is reduced by an order when implementing a convexification algorithm according to Theorem 2.

We first recall that for two Boolean matrices $X,W\in\mathbb{B}^{a\times b}$

$$X\leq W\Leftrightarrow X+W=W.$$

According to the definition of the communication topology evolution in (15), the intermittent QI condition in (30) of Theorem 2 can equivalently be written as

$$S_i\Delta_r\Pi_{i-r-1}^{i-j}S_{i-j}\leq\Pi_i^{i-1}\Pi_{i-1}^{i-j}S_{i-j}.\tag{35}$$

where we highlighted $\Pi_i^{i-1}$ from the inequality, for every $i \in [0, T-1]$ and for every $j \in [0, i]$ and $r \in [0, j-1]$. Moreover, notice that $\Pi_i^{i-1} = Z_{i-1}$. We propose Algorithm 1 to guarantee convexity in an *online* manner. In fact, it is possible to execute the algorithm without prior knowledge of the sensing topologies since the only unknown variable in (35) is $\Pi_i^{i-1}$. This is easily seen from the fact that $r \geq 0$ and $j \geq 0$ implies that $\Pi_{i-r-1}^{i-j}$ and $\Pi_{i-1}^{i-j}$ are known, as they refer to matrices computed at previous time instants. Whenever inequality (35) is not satisfied, in order to achieve QI it is necessary to add new link the control network to enable communication between to controllers. This is implemented in Algorithm 1.

**Algorithm 1** Convexification

```
procedure PRESERVE CONVEXITY AT TIME i ≥ 1
    suppose Z_k for k = 0,...,i−2 is known
    compute Π_i^{i−1} = Z_{i−1} as follows
    Z_{i−1} = I_{m×m}
    for j = 0 to i do
        for r = 0 to j−1 do
            Y = S_i Δ_r Π_{i−r−1}^{i−j} S_{i−j}
            V = Π_{i−1}^{i−j} S_{i−j}
            if Z_{i−1}(Y + V) != Z_{i−1} V  then
                add ones to Z_{i−1} so that (35) is satisfied
```

### D. Illustrative Example

Consider system (1) with $B = C = I_{4\times4}$ and $A$ such that

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \\ x_{k+1}^3 \\ x_{k+1}^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \\ x_k^4 \end{bmatrix} + \begin{bmatrix} u_k^1 \\ u_k^2 \\ u_k^3 \\ u_k^4 \end{bmatrix} + w_k,$$

Application of Algorithm 1 led to Fig. 2. A green node indicates that the related state was observed (transmitted to the controller by the corresponding event-trigger). Each row indicates a different time instant. The top-down dotted edges in each column represent local measurement history. We can see that at every time instant the obtained information structure is partially nested (see e.g. [3], [6]) just with very few communication links. In fact, consider for example the node $u_3^2$. We have that $u_3^2 = f(u_0^1, u_0^2, u_1^1, u_0^3, u_1^3)$, where $f$ is a linear function. With the links given in Fig. 2 we have that the information structure is partially nested.

## V. CONCLUSIONS

In this paper we proposed a solution to the design problem of a quadratically invariant event-triggered information structure. We showed that an event-triggered sensing topology induces a corresponding event-dependent communication topology, which must be carefully designed in order to achieve a convex optimal control problem. Moreover, necessary and sufficient condition for the sparsity constraints, due to the distributed nature of the problem, to be quadratically invariant were given. We provided an algorithm that renders the optimization problem convex at each time-instant, and does it in an online fashion, i.e., no information about the future is needed.

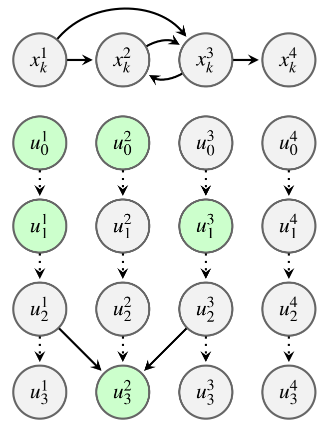

Fig. 2. Evolution of the communication topology $\Pi_k^{k-1}$.

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