

# Polarizability, Consensusability and Neutralizability of Opinion Dynamics on Coopetitive Networks

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**Abstract**—Opinion dynamics on social networks with coopetitive (cooperative-competitive) interactions may result in polarity, consensus or neutrality under different opinion protocols. The antecedent of protocol design is to study the accessibility problem: whether or not there exist admissible control rules to polarize, consensus, or neutralize individual opinions in a large population. From an operational perspective, this technical note is aimed at the investigation of polarizability, consensusability, and neutralizability of opinion dynamics in question. Particular emphasis is on the joint impact of the dynamical properties of individuals and the interaction topology among them on polarizability, consensusability, and neutralizability, respectively. Sufficient and/or necessary conditions for those accessibility problems are provided by using powerful tools from spectral analysis and algebraic graph theory. To characterize the individual diversity in real life, we further investigate the solvability of opinion formation problems in heterogeneous systems with non-identical dynamics. Accordingly, sufficient and/or necessary criteria for heterogeneous network polarizability, consensusability, and neutralizability are derived.

**Index Terms**—Polarizability, bipartite consensusability, neutralizability; opinion dynamics, coopetitive networks.

## I. INTRODUCTION

Public opinions do not always exhibit unanimous behavior (consensus) but lead to persistent disagreement or clustering. Among others, the phenomenon “polarity” or “polarization” under a specific protocol that the opinions of the agents reach two opposite values is of considerable interest. The phenomenon of polar opinions appears broadly in multitudes of fields, e.g., political voting, segregation of residential communities, and cultural conflicts [1]–[3]. Apart from polarity and consensus, people may also keep neutral on topics in which they have no interest. Accordingly, the concept of “neutralization” is another central issue in opinion-dynamics engineering [1].

Although a notable size of the dedicated literature works on opinion clustering problems of DeGroot-type dynamics on signed graphs [4], [5], the primary focus is on the control protocol design and the associated convergence analysis. However, the manipulability of opinion dynamics, which concerns the existence of opinion protocols such that systems in question empower polarity, neutrality, and consensus, is a fundamental problem and of great significance in both theoretic synthesis and engineering implementation. Only very recently, researchers from the control theory field have started to investigate the consensusability [6] and synchronizability [7] of multi-agent systems on cooperative networks. These articles, nevertheless,

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focus only on addressing the existence question for linear time-invariant (LTI) systems with identical continuous dynamics and trustful interactions. Thus, studying polarizability, consensusability, and neutralizability of opinion dynamics in a more general setting becomes a prime desideratum in social network science.

The main contribution of this technical note is to address the fundamental question: Under what conditions, there exists certain kind of distributed protocols such that the opinion dynamics over coopetitive (cooperative-competitive) networks are polarized, consensus and neutralized, respectively. In specific, the formal definitions of these novel concepts, under the umbrella of “modulus consensusability”, are introduced as an appetizer. In view of the bipartite consensus at the heart of modulus consensus, we set out to study the bipartite consensusability that examines whether or not there exist admissible protocols such that the individual opinions asymptotically reach the same value but may differ in signs. Specifically, sufficient and/or necessary conditions for bipartite consensusability of opinion systems with identical dynamics are provided. The developed criterion emphasizes the functional role of interaction topological properties in conjunction with the dynamic structure of the subsystems. Along with the examination of bipartite consensusability, neutralizability is taken into account as well and is characterized by sufficient and necessary conditions. With the emphasis on individual diversity, another significant contribution of this technical note is to extend the procured results to heterogeneous opinion dynamics. Criteria to examine the polarizability, consensusability, and neutralizability of non-identical opinion dynamics are explored. In particular, some common algebraic properties shared among individuals play an essential role in establishing polarization, consensus, and neutralization.

The remainder of the technical note is organized as follows. In Section II, we formulate the problems based on the preliminary of signed graphs. Then the sufficient and necessary conditions for bipartite consensusability, along with neutralizability of opinion dynamics on coopetitive networks are established in Section III. The procured results are then extended to heterogeneous systems in Section IV. Finally, a brief example is given in Section V to show polarization in heterogeneous opinion dynamics.

**Notations:** Let  $\mathbb{R}$ ,  $\mathbb{N}_{>0}$ , and  $\mathbb{C}_{<0}$  ( $\mathbb{C}_{>0}$ ,  $\mathbb{C}_{\leq 0}$ ) be the set of real numbers, positive integers, and negative (positive, non-positive) complex numbers, respectively. For a series of numbers (matrices)  $M_1, M_2, \dots, M_n$ ,  $\text{diag}(M_1, M_2, \dots, M_n)$  is a (block) diagonal matrix in which the  $i$ -th ( $i = 1, 2, \dots, n$ ) element on the diagonal is  $M_i$ .  $\text{sp}\{M\}$  stands for the spectrum of a square matrix  $M$ .  $\ker M$  represents the kernel of matrix  $M$  and  $\dim \ker M$  is the dimension of the  $\ker M$ . Vector  $\mathbf{1}_N$  ( $\mathbf{0}_N$ ) represents the column vector of all ones (zeros) with appropriate dimensions and matrix  $\mathbf{I}_{N \times N}$  ( $\mathbf{0}_{N \times N}$ ) means the identity (zero) matrix. All subscripts of those vectors and matrices will be dropped when their dimensions are clear from the context.

## II. PRELIMINARIES AND PROBLEM FORMULATIONS

Before we formally address the problems of polarizability, consensusability, and neutralizability with respect to (w.r.t.) an admissible control set, a recap of basic notions in graph theory is presented.

### A. Signed Graph Theory

A weighted directed graph  $\mathcal{G}$  is described as a triple  $(\mathbb{V}, \mathbb{E}, W)$ , where  $\mathbb{V} = \{1, 2, \dots, N\}$ ,  $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$  and  $W = [w_{ij}] \in \mathbb{R}^{N \times N}$  ( $w_{ij} \neq 0$  for  $(j, i) \in \mathbb{E}$ ) are the set of vertices, the set of edges and the weighted adjacency matrix, respectively. Furthermore,  $w_{ij} > 0$  indicates that  $j$  is cooperative with  $i$ , while  $w_{ij} < 0$  means that  $j$  is competitive towards  $i$ . We call  $\mathcal{G}$  a signed graph if at least one of the entries of  $W$  is negative; otherwise,  $\mathcal{G}$  is unsigned. Throughout the technical note, we assume graphs under consideration have no self-loops, i.e.,  $w_{ii} = 0$ ,  $\forall i \in \mathbb{V}$ , and are digon sign-symmetric, i.e.,  $w_{ij}w_{ji} \geq 0$ ,  $\forall i, j \in \mathbb{V}$ . More significantly, a signed digraph  $\mathcal{G}$  is *structurally balanced* (SB) if the node set can be split into two disjoint subsets (i.e.,  $\mathbb{V}^+ \cup \mathbb{V}^- = \mathbb{V}$ ,  $\mathbb{V}^+ \cap \mathbb{V}^- = \emptyset$ ) such that the weights of  $(j, i) \in \mathbb{E}$  are positive for all  $i \in \mathbb{V}^+$ ,  $j \in \mathbb{V}^+$  and  $i \in \mathbb{V}^-$ ,  $j \in \mathbb{V}^-$ , and negative for all  $i \in \mathbb{V}^+$ ,  $j \in \mathbb{V}^-$  and  $\forall j \in \mathbb{V}^+, i \in \mathbb{V}^-$ . Without loss of generality, a structurally balanced graph allows one to associate each node  $i = \{1, \dots, N\}$  with a scalar  $d_i \in \{\pm 1\}$  such that  $d_i = 1$  if  $i \in \mathbb{V}^+$  and  $d_i = -1$  if  $i \in \mathbb{V}^-$ . In relation to gauge transformation [4], a graph  $\mathcal{G}$  is SB, if and only if there exists a diagonal matrix  $D = \text{diag}(d_1, \dots, d_N)$ ,  $d_i \in \{\pm 1\}$ ,  $i = 1, 2, \dots, N$  such that all entries of  $DWD$  are nonnegative [8].

A digraph is said to be *quasi-strongly connected* (QSC) if it has at least one node (called *root*) which possesses paths to all other nodes in this graph. A subgraph of  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$  is a graph formed from subsets of  $\mathbb{V}$ ,  $\mathbb{E}$  and the associated weights in  $W$ . A subgraph is *in-isolated* if no edge comes from outside to itself.

For the subsequent analysis, the *Laplacian matrix* of  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$  is introduced by

$$L = [l_{ij}]_{N \times N}, \quad l_{ij} = \begin{cases} \sum_{k=1}^N |w_{ik}|, & i = j, \\ -w_{ij}, & i \neq j. \end{cases}$$

which has at least one eigenvalue 0 associated with the eigenvector  $d = [d_1, \dots, d_N]^T$ , if  $\mathcal{G}$  is SB. Laplacian  $L$  has no eigenvalue at 0 when  $\mathcal{G}$  is neither SB nor contains in-isolated SB subgraphs. Moreover, a SB graph is QSC if and only if  $\dim \ker L = 1$ . See [4], [5] for more details on signed graphs and their properties.

Let  $\bar{\mathcal{G}} = (\mathbb{V}, \mathbb{E}, \bar{W})$  be the associated unsigned graph of the signed graph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$ . The weighted adjacency matrix  $\bar{W} = [\bar{w}_{ij}] \in \mathbb{R}^{N \times N}$  is formed by  $\bar{w}_{ij} = |w_{ij}|$ ,  $\forall i, j \in \mathbb{V}$ . Analogously,  $\bar{L}$  is denoted as the Laplacian of the associated unsigned graph  $\bar{\mathcal{G}}$ .

### B. Problem Formulation

Consider  $N \geq 2$  agents indexed 1 through  $N$  and the interaction among individuals characterized by a digraph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$  with  $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ . In contrast to the conventional opinion models which usually consider scalar-valued opinions, we deal with simultaneous opinion discussion on multiple topics, thus necessitating the consideration of vector-valued opinions [9]–[11].

Each agent is associated with a vector  $x_i \in \mathbb{R}^n$  that represents her attitudes on  $n$  issues (subtopics) and updates continuously her opinion in the following fashion

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \mathbb{V}, \quad (1)$$

where  $u_i \in \mathbb{R}^m$  is the control input of individual  $i$ . The state matrix  $A \in \mathbb{R}^{n \times n}$  characterizes the level of ‘‘anchorage’’ on their topic-specific opinions, which has been formed by some exogenous conditions, e.g., the past social experience, or endogenous factors,

e.g., personal intelligence and character. The input matrix  $B \in \mathbb{R}^{n \times m}$  stands for the susceptibility of agents to the interpersonal influence. We consider a distributed feedback control under cooperative interaction as follows,

$$u_i(t) = -K \sum_{(j,i) \in \mathbb{E}} |w_{ij}|(x_i - \text{sgn}(w_{ij})x_j), \quad i \in \mathbb{V}, \quad (2)$$

where  $K \in \mathbb{R}^{m \times n}$  is the feedback gain matrix. In the recent literature of opinion evolution on signed graphs [4], [5], the control protocol (2) can often be found.

*Remark 1.* This LTI system (1) has been widely adopted in analyzing the collective behavior of multi-agent systems on cooperative networks, including agreement- [7], [12] and disagreement-problems [13]. Only very recently in the context of cooperative networks have researchers started to employ such a state-feedback LTI model to study bipartite consensus [8], [9]. In the special case  $A = \mathbf{0}$ ,  $B = \mathbf{I}$ , the model (1) of opinion evolving on single topic degenerates to Altafini-type model [4]. Furthermore, a nonlinear counterpart of the model (1) with the controller (2) is proposed in [14].

After denoting  $u(t) := [u_1^T(t), \dots, u_n^T(t)]^T$ , we consider the following admissible set,

$$\mathbb{U} = \{u(t) \in \mathbb{R}^{mN} | u_i(t) = -K \sum_{j=1}^N |w_{ij}|(x_i - \text{sgn}(w_{ij})x_j), \quad (3) \\ \forall t > 0, K \in \mathbb{R}^{m \times n}, i = 1, \dots, N\}.$$

The admissible control set (3) covers a relatively large number of distributed protocols with antagonistic interactions. The overarching question before entering into the stage of protocol design is to determine under what conditions, the opinion dynamics of interest is polarizable, consensusable, and neutralizable w.r.t. such an admissible control set  $\mathbb{U}$ . To investigate such question, we first provide the formal definitions of the polarizability, consensusability, and neutralizability in the context of the so-called modulus consensusability of an opinion system w.r.t.  $\mathbb{U}$ .

**Definition 1** (Modulus consensusability). The system (1) is *modulus consensusable* w.r.t.  $\mathbb{U}$ , if one can find a  $u \in \mathbb{U}$  and scalars  $\rho_i, \rho_j \in \{\pm 1\}$ ,  $\forall i, j \in \mathbb{V}$  such that for any initial value  $x_i(0)$ , the solution of (1) on a graph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$  satisfies

$$\lim_{t \rightarrow \infty} \rho_i x_i(t) - \rho_j x_j(t) = \mathbf{0}, \quad \lim_{t \rightarrow \infty} \|x_i(t)\| < \infty, \quad \forall i, j \in \mathbb{V}.$$

Specifically, modulus consensusability can be classified into the following cases:

- if  $\lim_{t \rightarrow \infty} x_i(t) = \mathbf{0}, \forall i \in \mathbb{V}$ , we say the system (1) is *neutralizable* w.r.t.  $\mathbb{U}$ ;
- if  $\rho_i = \rho_j, \forall i, j \in \mathbb{V}$  and a) is not satisfied, we say the system (1) is *consensusable* w.r.t.  $\mathbb{U}$ ;
- if  $\rho_i = -\rho_j$  holds for at least a pair of nodes  $i, j \in \mathbb{V}$  and a) is not satisfied, we say the system (1) is *polarizable* w.r.t.  $\mathbb{U}$ .
- the system (1) is *bipartite consensusable* if it is either consensusable or polarizable. Specially, the system (1) is *stationary bipartite consensusable*, if  $\lim_{t \rightarrow \infty} \rho_i x_i(t) = v$  is further satisfied for all  $i \in \mathbb{V}$ , where  $v \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  is a constant vector.

A Venn diagram is presented in Figure 1 to illustrate the relations of the concepts proposed in Definition 1. Note that modulus consensusability proposed in this technical note is equivalent to consensusability in [6] on unsigned graphs, where the trivial case of neutralization is not specialized. In the literature on opinion dynamics, neutralization is commonly referred to stabilization as an inheritance of the seminal work [4]. Since social actors preferably

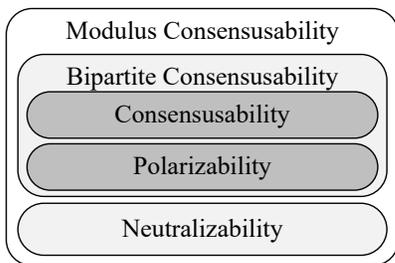


Fig. 1: Venn diagram on the relations between the concepts in Definition 1.

take a neutral stance on sensitive issues and uninteresting topics, we tend to use the sociological terminology, neutralization, instead in this technical note. Furthermore, agents constantly keep neutralized in the neutral configuration of initial opinions. In the following of this technical note, we are more interested in studying opinion dynamics when the initial conditions are non-neutral. The non-trivial case of modulus consensus, i.e., the opinions reach consensus or oppositely separate, is named as bipartite consensus. Polarizability is based on the phenomenon of polarity [4], [5], but the detailed content here is slightly different. First, the opinion variable is a vector rather than a scalar, thus the case when some (not all) of the entries of the opinion vector are 0 is also allowed for bipartite consensusability. Second, Definition 1 enables us to investigate the opinion formation process in a more general setting in which the opinion states may converge to trajectories but not to a set of fixed points. Convergence to trajectories mirrors the fact that the amplitudes of the steady-state opinions may fluctuate in some degree caused by exogenous influence or endogenous vibration, but an about-turn of individual attitudes seldom happens. We also point out that the opinion states should be bounded no matter how extreme they could be since infinite values of opinions make no sense from the perspective of sociology and psychology.

Before embarking on the main results, we fix some notations and terminology. Throughout this technical note, symbols  $\rho_i \in \{\pm 1\}$  characterize the signatures of individual opinions at steady-state, while the structural balance of a graph, if it has, is specified by  $d_i \in \{\pm 1\}$ . As we shall see, there is no explicit dependence between the two sequences of scalars, with few exceptions.

### III. HOMOGENEOUS OPINION DYNAMICS

In this section, we elaborate on whether or not there exist a matrix  $K \in \mathbb{R}^{m \times n}$  and a signed graph  $\mathcal{G}$  such that the system (1) with a controller (2) establishes polarization, consensus, and neutralization.

#### A. Bipartite Consensusability

In comparison to the trivial case in modulus consensus, the bipartite consensus is of great interest for sociological studies. Such bipartition of opinions is known to have a tight connection to the structural balance of the underlying interaction topology. Following this line of thought, the sufficient condition for bipartite consensusability of the system (1) can be derived.

For the convenience of presentation, we introduce the set of marginally stable matrices

$$\mathbb{M} = \left\{ M : \text{sp}\{M\} \subseteq \mathbb{C}_{\leq 0}, \text{sp}\{M\} \cap i\mathbb{R} \neq \emptyset, \right. \\ \left. \lambda \text{ is semi-simple}, \forall \lambda \in \text{sp}\{M\} \cap i\mathbb{R} \right\},$$

where  $i$  is the imaginary unit, i.e.,  $i^2 = -1$ , and a semi-simple eigenvalue possesses equal algebraic and geometric multiplicities. In

the next theorem, we first provides a sufficient condition for bipartite consensusability of opinion dynamics.

**Theorem 1.** *Given a graph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$ , the system (1) is bipartite consensusable for any non-zero initial conditions w.r.t.  $\mathbb{U}$ , if the following conditions are satisfied:*

- i)  $\mathcal{G}$  is structurally balanced and quasi-strongly connected.
- ii).  $A \in \mathbb{M}$  and  $(A, B)$  is stabilizable.

Moreover, the system (1) is consensusable if  $\mathcal{G}$  is an unsigned graph, otherwise polarizable if  $\mathcal{G}$  is a signed graph.

*Proof.* The SB property of  $\mathcal{G}$  means that there exist  $N$  scalars  $d_i \in \{\pm 1\}$  such that  $d_i d_j w_{ij} = \bar{w}_{ij}$ . By denoting  $\bar{x}_i(t) := d_i x_i(t)$ , one can obtain its time-derivative from (1) as follows

$$\dot{\bar{x}}_i(t) = A\bar{x}_i(t) - BK \sum_{j=1}^N \bar{w}_{ij}(\bar{x}_i(t) - \bar{x}_j(t)), i \in \mathbb{V}. \quad (4)$$

By adopting Kronecker product, the system (4) can be rewritten in a compact form as below.

$$\dot{\bar{x}}(t) = (\mathbf{I}_N \otimes A - \bar{L} \otimes BK) \bar{x}(t), \quad (5)$$

where  $\bar{x}(t) = [\bar{x}_1^T(t), \dots, \bar{x}_N^T(t)]^T$  and  $\bar{L}$  is the Laplacian matrix of the associated unsigned graph  $\bar{\mathcal{G}}$ .

The bipartite consensusability problem of (1) can be transformed into the consensusability problem of (5). According to [6, Theorem 2], the system (5) is modulus consensusable if the unsigned graph  $\bar{\mathcal{G}}$  associated to  $\mathcal{G}$  is QSC and  $(A, B)$  is stabilizable. After setting  $\rho_i = d_i, \forall i \in \mathbb{V}$ , the system (5) achieving consensus, i.e.,  $\lim_{t \rightarrow \infty} \bar{x}_i - \bar{x}_j = 0$ , implies the establishment of bipartite consensus in the opinion dynamics (1), specifically,  $\lim_{t \rightarrow \infty} \rho_i x_i - \rho_j x_j = 0$ . Accordingly, it is illuminating to view that the original system (1) is bipartite consensusable if condition i) and ii) occur. It is worthy to note that for non-zero initial opinions, the constraint  $A \in \mathbb{M}$  guarantees that  $x_i(t)$  does not converge to  $\mathbf{0}$ , and is bounded when  $t \rightarrow \infty, \forall i \in \mathbb{V}$ .

Moreover, a signed graph  $\mathcal{G}$  associated with at least one negative weighted edge results in opinion polarization otherwise opinion consensus.  $\square$

*Remark 2.* The conditions in Theorem 1 involve two aspects, i.e., the requirements for the network topology and for the subsystem dynamics. In condition i), quasi-strongly connectivity ensures that there exists at least one agent who can deliver directly or indirectly his/her willing to the remaining members, while the structural balance of graphs paves the way for the bipartite consensus of opinions. Condition ii), on the other hand, emphasizes the importance of the dynamical properties of individuals. Specifically, to excludes the situation of neutralization and the meaningless case in which opinions are unbounded,  $A \in \mathbb{M}$  is of significance. Besides, the stabilizability of  $(A, B)$  implies that the individual is open to interpersonal influence.

In Theorem 1, we only focus on the sufficient conditions for bipartite consensusability, wherein graph being SB is indispensable in the sufficient criterion. Through the next analysis, however, we will address that this graph property is not necessary for an opinion dynamics to achieve stationary bipartite consensus, let alone for the generic case.

**Theorem 2.** *Given a graph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$ , the system (1) is stationary bipartite consensusable w.r.t.  $\mathbb{U}$ , only if the following conditions are satisfied:*

- i).  $\mathcal{G}$  is quasi-strongly connected and there exists a non-negative scalar  $\alpha \geq 0$  such that there holds

$$\alpha_i \equiv \alpha, \quad \forall i \in \mathbb{V} \quad (6)$$

where  $\alpha_i := \sum_{j=1}^N (|\tilde{w}_{ij}| - \tilde{w}_{ij})$  and  $\tilde{w}_{ij} := \rho_i \rho_j w_{ij}$ .

- ii).  $(A, B)$  is stabilizable,  $A - \alpha BK \in \mathbb{M}$  and the spectral subset  $\text{sp}\{A - \alpha BK\} \cap i\mathbb{R}$  contains only 0.

*Proof.* We first prove  $\mathcal{G}$  is QSC by contradiction. If  $\mathcal{G}$  is not QSC, then it has either at least two nodes without inward edges or two separate subgraphs [15]. The first situation means that there exist at least two ‘‘isolated’’ actors (say agent  $p$  and agent  $q$ ) whose opinions remain independent of the others’ thoughts. Accordingly, the dynamics of these two agents reduces to  $\dot{x}_k = Ax_k$  with  $k = p, q$ . It is evident that such subsystems cannot reach bipartite consensus for arbitrary initial states. In the second case, even if a stationary bipartite consensus is achieved in each subgraph, it is unlikely to be established across the entire graph  $\mathcal{G}$  for all initial configuration. As a result, the contradictions arising in both two cases allows us to state that the QSC property of a graph is necessary for bipartite consensusability.

Next, we begin to demonstrate the relation (6). According to Definition 1, the system (1) being stationary bipartite consensusable means there exist a sequence of scalars  $\rho_i, \rho_j \in \{\pm 1\}$  and a constant vector  $v \in \mathbb{R}^n \setminus \{0\}$  such that  $\lim_{t \rightarrow \infty} \rho_i x_i(t) = v$  is valid for all  $i \in \mathbb{V}$ . After denoting  $\tilde{x}_i := \rho_i x_i$  for  $i \in \mathbb{V}$ , in analogy with the system dynamics (4), one can obtain

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) - BK \sum_{j=1}^N (|\tilde{w}_{ij}| \tilde{x}_i(t) - \tilde{w}_{ij} \tilde{x}_j(t)), \quad i \in \mathbb{V}, \quad (7)$$

where  $\tilde{w}_{ij} := \rho_i \rho_j w_{ij}$ . With the notation  $\tilde{x} = [\tilde{x}_1^\top, \dots, \tilde{x}_N^\top]^\top$ , the compact form of (7) can be given by

$$\dot{\tilde{x}}(t) = (\mathbf{I}_N \otimes A - \tilde{L} \otimes BK) \tilde{x}(t), \quad (8)$$

where

$$\tilde{L} := \text{diag}(\rho) L \text{diag}(\rho) = \begin{bmatrix} \sum_{j=1}^N |\tilde{w}_{1j}| & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}.$$

Arisen from the fact that the system (1) is stationary bipartite consensusable, the stationary consensusability of the dynamics (7) is unambiguous.

To promote the analysis, we introduce an auxiliary variable  $\xi_i = \tilde{x}_1 - \tilde{x}_i$  ( $i = 2, 3, \dots, N$ ) associated with the time evolution as follows

$$\begin{aligned} \dot{\xi}_i &= A\xi_i - BK \sum_{j=1}^N [(\tilde{w}_{1j} - \tilde{w}_{ij})\xi_j + |\tilde{w}_{ij}|\xi_i] \\ &\quad - BK\tilde{x}_1 \sum_{j=1}^N [(|\tilde{w}_{1j}| - \tilde{w}_{1j}) - (|\tilde{w}_{ij}| - \tilde{w}_{ij})]. \end{aligned} \quad (9)$$

We can immediately infer that the auxiliary system (9) is asymptotically stable, i.e.,  $\lim_{t \rightarrow \infty} \xi_i(t) = 0$  as the system (7) is stationary consensusable. As a consequence, the third item in the right-hand side of (9) needs to approach 0 as  $t \rightarrow \infty$  for all  $i = 2, \dots, N$ , which is equivalent to the condition

$$\sum_{j=1}^N [(|\tilde{w}_{1j}| - \tilde{w}_{1j}) - (|\tilde{w}_{ij}| - \tilde{w}_{ij})] = 0, \quad \text{for } i = 2, \dots, N. \quad (10)$$

and/or  $\lim_{t \rightarrow \infty} BK\tilde{x}_1 = 0$  succeeds.

In order to rule out the second situation, we first postulate, without loss of generality, that  $\lim_{t \rightarrow \infty} BK\tilde{x}_i = BKv = 0$  holds for all  $i \in \mathbb{V}$  due to the fact  $\lim_{t \rightarrow \infty} \tilde{x}_i = v$  for  $i \in \mathbb{V}$ . In other words, we have  $v \in \ker BK$ . The limiting behavior of the dynamics (7) and its stationary consensusability manifest that  $Av = 0$ , i.e.,  $v \in \ker A$ .

Meanwhile, let  $e(t) := \tilde{x}(t) - \mathbf{1}_N \otimes v$  be an error vector. By noticing the fact that

$$\begin{aligned} &(\mathbf{I}_N \otimes A - \tilde{L} \otimes BK)(\mathbf{1}_N \otimes v) \\ &= \mathbf{1}_N \otimes (Av) - (\tilde{L}\mathbf{1}_N) \otimes (BKv) = \mathbf{0}_{nN}, \end{aligned}$$

the dynamics of the error vector obeys the evolution rule

$$\dot{e}(t) = (\mathbf{I}_N \otimes A - \tilde{L} \otimes BK)e(t).$$

Thanks to  $\lim_{t \rightarrow \infty} e = 0$ , one can obtain  $\text{sp}\{\mathbf{I}_N \otimes A - \tilde{L} \otimes BK\} \subseteq \mathbb{C}_{<0}$ . That is to say, the system (8) is asymptotically stable, i.e.,  $\lim_{t \rightarrow \infty} \tilde{x} = 0$ , which is contradictory to the definition of stationary bipartite consensus. To this end, we illustrate that the equality (10) is necessary for bipartite consensusability of the opinion system (1), which is equivalent to

$$\sum_{j=1}^N (|\tilde{w}_{ij}| - \tilde{w}_{ij}) = \alpha, \quad \forall i \in \mathbb{V}, \quad (11)$$

where  $\alpha \geq 0$  is a non-negative scalar. As a result, the dynamics (9) can be compactly written as

$$\dot{\xi}(t) = (\mathbf{I}_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes BK)\xi(t), \quad (12)$$

where  $\xi(t) = [\xi_2^\top(t), \dots, \xi_N^\top(t)]^\top$  and  $\tilde{\mathcal{L}} = \tilde{L}_{22} - \mathbf{1}_{N-1} \tilde{L}_{12}$ .

Next, we illustrate  $(A, B)$  stabilizable in condition ii). The equality (11) can be rephrased to  $\tilde{L}\mathbf{1} = \alpha\mathbf{1}$  which facilitates the matrix decomposition as follows

$$\begin{bmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & \mathbf{I}_{N-1} \end{bmatrix}^{-1} \tilde{L} \begin{bmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & \mathbf{I}_{N-1} \end{bmatrix} = \begin{bmatrix} \alpha & \tilde{L}_{12} \\ \mathbf{0} & \tilde{\mathcal{L}} \end{bmatrix}. \quad (13)$$

Then, the fact that the transformation  $L \rightarrow \text{diag}(\rho)L\text{diag}(\rho)$  is a similar transformation leads to  $\text{sp}\{L\} = \text{sp}\{\tilde{L}\}$ , which immediately implies  $\text{sp}\{L\} = \text{sp}\{\tilde{\mathcal{L}}\} \cup \{\alpha\}$ . As a result, one can compute

$$\text{sp}\{\mathbf{I}_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes BK\} = \cup_{\lambda \in \text{sp}\{L\} \setminus \{\alpha\}} \text{sp}\{A - \lambda BK\}. \quad (14)$$

Thereby, the asymptotic stability of the auxiliary system (12) accords to

$$\text{sp}\{A - \lambda BK\} \subseteq \mathbb{C}_{<0}, \quad \forall \lambda \in \text{sp}\{\tilde{\mathcal{L}}\}. \quad (15)$$

As reported in [6],  $(A, B)$  is stabilizable if formula (15) is available. In conclusion, the stabilizability of  $(A, B)$  is necessary for stationary bipartite consensusability of the opinion dynamics (1).

The system matrix in (7), in spirit similar to (14), has the spectrum

$$\text{sp}\{\mathbf{I}_N \otimes A - \tilde{L} \otimes BK\} = \cup_{\lambda \in \text{sp}\{\tilde{\mathcal{L}}\} \cup \{\alpha\}} \text{sp}\{A - \lambda BK\}.$$

Based upon the fact that  $\text{sp}\{A - \lambda BK\} \subseteq \mathbb{C}_{<0}$  for all  $\lambda \in \text{sp}\{\tilde{\mathcal{L}}\}$ , the spectral property of matrix  $A - \alpha BK$  primarily characterizes the dynamical behavior of the system (8). Then, the statement that  $\tilde{x}$  converges to a bounded constant non-zero vector  $\mathbf{1} \otimes v$  for any initial conditions, indicates that  $A - \alpha BK \in \mathbb{M}$  and all the marginally stable eigenvalues of matrix  $A - \alpha BK$  are 0. Therefore, we complete the proof.  $\square$

*Remark 3.* The most noticeable point of Theorem 2 is no explicit requirement of SB graphs for stationary bipartite consensus of opinion dynamics. Nevertheless, one still can extract the SB condition from the equality relation (6) but far more than that. Structural balance theory [16] states that for an unweighted graph being not exactly SB, the least number of edges that must be changed of sign can be used to compute a distance to exact structural balance (i.e., a measure of the amount of structural unbalance in the network.). This statement has an immediate extension in weighted graphs [17]. In specific, we can argue that the quantity  $\sum_i \alpha_i = \sum_{i,j} (|\tilde{w}_{ij}| - \tilde{w}_{ij})$  can be treated as an unbalance metric which measures the distance

to a desired SB structure specified by  $\rho = [\rho_1, \dots, \rho_N]^T$ . Since such a metric captures the collective effect of unbalance, the index  $\alpha_i$  distinguishes the individual contribution of node  $i$  to disrupt the global structural balance w.r.t.  $\rho$ . Hence, the relation (6) reads that all agents contribute an equal impact on network unbalance. In the special case of  $\alpha_i = 0$  for all  $i \in \mathbb{V}$ , all nodes exhibit a local structural balance, and the network entails a natural SB structure: the community splits into two hostile camps, individuals with the same sign of  $\rho_i$  come from the same camp, the social ties inside each fraction are cooperative, whereas the interrelations cross fractions are competitive. Hence, Theorem 2 reveals an appealing, and previously unexplored, relationship between SB bipartition of topology and state clustering of opinion dynamics. In particular, the scenario  $\alpha > 0$  meaning the mismatch of the bipartition pattern between opinion organization and network structure, mirrors real-world phenomena, especially, in political and commercial voting. The U.S. House Vote [18] in the two-party congress serves as a typical example. Although the underlying interconnection topology of Republicans and Democrats exhibits a natural SB graph, representatives from the same party would not always vote for the same (pros or cons, usually) and the vote result is highly dependent on the specific content of the bills.

Until now, the discussion is carried out in terms of sufficient and necessary conditions, respectively. In particular, the necessary conditions in a generic circumstance are missing. In the following theorem, we make a step towards filling this gap.

**Theorem 3.** *Let the system (1) with non-zero initial conditions be of a non-Hurwitz matrix  $A$  and satisfy*

$$\rho_i \rho_j = d_i d_j, \quad \forall i, j \in \mathbb{V}. \quad (16)$$

*The opinion dynamics (1) evolving over a SB graph  $\mathcal{G}$  is bipartite consensusable w.r.t.  $\mathbb{U}$ , if and only if the following conditions hold:*

- i)  $\mathcal{G}$  is quasi-strongly connected.
- ii).  $A \in \mathbb{M}$  and  $(A, B)$  is stabilizable.

*Proof.* Sufficiency is an immediate result from Theorem 1. Here we need only to prove the necessity.

The assembly line of Theorem 2 exposes that a QSC graph is necessary. The relation (16) results in  $\alpha_i = \sum_{j=1}^N (|\tilde{w}_{ij}| - \tilde{w}_{ij}) = 0$  for  $i \in \mathbb{V}$ , implying  $\alpha = 0$ . Meanwhile, the dynamics (7) becomes

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) - BK \sum_{j=1}^N (|\tilde{w}_{ij}| \tilde{x}_i(t) - \tilde{w}_{ij} \tilde{x}_j(t)),$$

where  $\tilde{w}_{ij} = d_i d_j w_{ij} \geq 0$  due to the structural balance of the graph. Consequently, the auxiliary system (9) here reduces to

$$\dot{\xi}_i = A\xi_i - BK \sum_{j=1}^N [(\tilde{w}_{1j} - \tilde{w}_{ij})\xi_j + \tilde{w}_{ij}\xi_i], \quad (17)$$

where  $\xi_i = \tilde{x}_1 - \tilde{x}_i$  and  $i = 2, \dots, N$ . The compact form of (17) can be given by

$$\dot{\xi}(t) = (\mathbf{I}_{N-1} \otimes A - \overline{\mathcal{L}} \otimes BK)\xi(t),$$

where  $\overline{\mathcal{L}} := \overline{L}_{22} - \mathbf{1}_{N-1} \overline{L}_{12}$  upon the matrix partition

$$\overline{L} = DLD = \begin{bmatrix} \sum_{j=1}^N \tilde{w}_{1j} & \overline{L}_{12} \\ \overline{L}_{21} & \overline{L}_{22} \end{bmatrix}. \quad (18)$$

The remainder of the proof mimics the procedure of Theorem 2 which entails the stabilizability of  $(A, B)$  and  $A \in \mathbb{M}$  as a result. Noteworthily, by revisiting the similarity transformation (13) with the Laplacian matrix  $\overline{L}$  of the associated unsigned graph  $\overline{\mathcal{G}}$  which is QSC,  $\overline{\mathcal{L}}$  inherits all nonzero eigenvalues of  $\overline{L}$ .  $\square$

## B. Neutralizability

Besides bipartition of opinions, either unanimous or opposite behaviors, among individuals, sometimes people tend to hold neutral opinions for certain political or economic reasons in social activities. This phenomenon motivates us to discuss the neutralizability of opinion dynamics.

The answer to neutralizability of systems (1) with a Hurwitz matrix  $A$ , is affirmative, where  $K = \mathbf{O}$  is an intuitive option. In this scenario, individuals, who are totally closed to the social influence, persist neutral attitudes to all topics. Thus, we restrict ourselves to the case of non-Hurwitz  $A$  while dealing with neutralizability.

**Theorem 4.** *Given a graph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$ , the system (1) with non-Hurwitz matrix  $A$  is neutralizable w.r.t. to  $\mathbb{U}$ , if and only if the following conditions are satisfied,*

- i)  $(A, B)$  is stabilizable.
- ii)  $\mathcal{G}$  is neither structurally balanced nor contains an in-isolated structurally balanced subgraph.

*Proof.* Consider the compact form of the system (1) as below

$$\dot{x}(t) = (\mathbf{I}_N \otimes A - L \otimes BK)x(t), \quad (19)$$

from which, one can easily obtain

$$\text{sp}\{\mathbf{I}_N \otimes A - L \otimes BK\} = \cup_{\lambda \in \text{sp}\{L\}} \text{sp}\{A - \lambda BK\}.$$

Therefore, the neutralizability problem can be transformed to show  $\text{sp}\{A - \lambda BK\} \subseteq \mathbb{C}_{<0}$  for all  $\lambda \in \text{sp}\{L\}$ .

*Sufficiency:* Since  $\mathcal{G}$  is not SB and contains no isolated SB subgraphs, it is already known that  $0 \notin \text{sp}\{L\}$ . In addition, the stabilizability of  $(A, B)$  always allows one to find a matrix  $K$  to such that the matrix  $A - \lambda BK, \forall \lambda \in \text{sp}\{L\}$  is stable, thereby evidencing the neutralizability of the compact system (19) w.r.t.  $\mathbb{U}$ .

*Necessity:* The background that the system (19) is neutralizable w.r.t.  $\mathbb{U}$  amounts to  $\text{sp}\{A - \lambda BK\} \subseteq \mathbb{C}_{<0}$  for all  $\lambda \in \text{sp}\{L\}$ . We posit that graph  $\mathcal{G}$  is SB or contains an in-isolated SB subgraph, which leads to  $0 \in \text{sp}\{L\}$ . The condition  $\text{sp}\{A - \lambda BK\} \subseteq \mathbb{C}_{<0}$  for  $\lambda = 0$  means  $A$  is a stable matrix which is contradictory to the fact that  $A$  is not Hurwitz. Therefore, we derive the necessity of condition ii). In reference to condition i), we can adopt the same method presented in [6] to display that  $(A, B)$  is stabilizable.  $\square$

## IV. HETEROGENEOUS OPINION DYNAMICS

Hitherto we only investigate modulus consensusability of opinion forming for agents with identical systems, input and feedback matrices. In the real world, individuals even living in the same house may be tremendously different in educational background, life history, personal preference, etc., and all these discrepancies, in turn, influence the decision-making of each agent. Therefore, heterogeneity is of great significance and necessity in practice.

With the illustration of individual heterogeneity in mind, we consider the following dynamics for the opinion-forming process,

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad i \in \mathbb{V}, \quad (20)$$

where  $A_i \in \mathbb{R}^{n \times n}$  and  $B_i \in \mathbb{R}^{n \times m}$  are system and input matrices, respectively. Correspondingly, the feedback control is chosen as

$$u_i(t) = -K_i \sum_{(j,i) \in \mathbb{E}} |w_{ij}| (x_i(t) - \text{sgn}(w_{ij}) x_j), \quad \forall i \in \mathbb{V},$$

and the admissible protocol set in the heterogeneous case becomes

$$\widehat{\mathbb{U}} = \{u(t) \in \mathbb{R}^{mN} \mid u_i(t) = -K_i \sum_{j=1}^N |w_{ij}| (x_i - \text{sgn}(w_{ij}) x_j), \quad \forall t > 0, K_i \in \mathbb{R}^{m \times n}, i = 1, \dots, N\}.$$

Aside from the same social interpretations as in the homogeneous setting, the subscripts associated to the matrices  $A_i$ ,  $B_i$  and  $K_i$  emphasize individual diversity. The compact form of the closed-loop system (20) reads

$$\dot{x}(t) = [\mathbf{A} - \mathbf{B}(L \otimes \mathbf{I}_n)]x(t), \quad (21)$$

where  $\mathbf{A} = \text{diag}(A_1, \dots, A_N)$  and  $\mathbf{B} = \text{diag}(B_1 K_1, \dots, B_N K_N)$ .

### A. Bipartite Consensusability

Although the involvement of system heterogeneity in opinion formation opens up the possibility of richer social psychological findings, considerable challenges arise naturally in their theoretical and empirical exploration. As a starting point, we study the necessary conditions for bipartite consensusability of heterogeneous dynamics. To save cliché, we assume that  $(A_i, B_i)$  of the individual systems is stabilizable for  $i = 1, \dots, N$ .

**Theorem 5.** *Let the relation (16) hold. The stabilizable systems (20) evolving on a SB graph  $\mathcal{G}$  is bipartite consensusable w.r.t.  $\hat{\mathbf{U}}$ , only if both of the following conditions are satisfied:*

- i).  $\mathcal{G}$  is quasi-strongly connected.
- ii). *There exist a positive integer  $q \leq n$ , a sequence of matrices  $Q_i \in \mathbb{R}^{n \times q}$  of full column rank and a diagonalizable matrix  $S \in \mathbb{R}^{q \times q}$  satisfying  $\text{sp}\{S\} \in \mathbb{R}$  such that*

$$A_i Q_i = Q_i S, \quad \forall i \in \mathbb{V}. \quad (22)$$

*Proof.* The proof of condition i) is an immediate result of Theorem 2.

As for condition ii), the bipartite consensusability of systems (20) amounts to the accessibility of the following systems to consensus,

$$\dot{\tilde{x}}(t) = [\mathbf{A} - \mathbf{B}(\bar{L} \otimes \mathbf{I}_n)]\tilde{x}(t), \quad (23)$$

where  $\tilde{x} = [\tilde{x}_1^\top, \dots, \tilde{x}_N^\top]^\top$  with  $\tilde{x}_i = \rho_i x_i$ . The matrix  $\bar{L}$  is defined in (18) and is, thanks to the implication  $\rho_i \rho_j w_{ij} = d_i d_j w_{ij}$  in (16), the Laplacian matrix of the unsigned graph associated to  $\mathcal{G}$ . Therefore, the dynamics (23) achieving consensus  $\lim_{t \rightarrow \infty} \tilde{x}_i - \tilde{x}_j = \mathbf{0}$  for all  $i, j \in \mathbb{V}$  infers the existence of an asymptotically attractive invariant subspace  $\tilde{\mathbb{X}}$  on which  $\tilde{x}_i = \tilde{x}_j$ . By noticing  $\mathbf{1} \subseteq \ker \bar{L}$ , the dynamical behavior of the closed-loop system (23) on  $\tilde{\mathbb{X}}$  is governed by the dynamics  $\dot{\tilde{x}}(t) = \mathbf{A}\tilde{x}(t)$ . The requirement that the uniformly bounded  $\tilde{x}_i$  has no asymptotically stable equilibrium set, allows  $\tilde{\mathbb{X}}$  to contain marginally stable modes with dimension  $q \in \mathbb{N}_{>0}$  and  $q \leq n$ . Especially, one can construct a diagonalizable matrix  $S \in \mathbb{R}^{q \times q}$  which possesses  $q$  marginally stable eigenvalues and a block matrix  $\mathbf{Q} := [Q_1^\top, \dots, Q_N^\top]^\top \in \mathbb{R}^{nN \times q}$  whose columns span the subspace  $\tilde{\mathbb{X}}$ , such that  $\mathbf{A}\mathbf{Q} = \mathbf{Q}S$  from which the equalities (22) emerge and  $\text{sp}\{S\} \subset \mathbb{R}$  can be inferred.  $\square$

It should be emphasized that  $\text{sp}\{S\} \subseteq \bigcap_{i \in \mathbb{V}} \{\text{sp}\{A_i\} \cap \mathbb{R}\}$ , which implies matrix  $S$  lumps together some (or all) of those marginally stable modes that individual system matrices commonly share. In other words, the existence of one or more common pure imaginary eigenvalues in the spectrum of matrix  $A_i$  is essentially important to bipartite consensusability of opinion dynamics. This finding is reminiscent of the real-life situation that certain basic social norms serve a baseline to reach an opinion consensus among diverse social actors.

Note that the statements shown in Theorem 5 are a necessary but not sufficient condition. This is because, for example, when  $A_i$  for all  $i \in \mathbb{V}$  share a common eigenvalue  $\lambda_c$  which associates with an identical eigenspace, i.e., there exists  $\mu \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  such that  $A_i \mu = \lambda_c \mu$ , then one can obtain

$$(\mathbf{A} - \mathbf{B}(\bar{L} \otimes \mathbf{I}_n))\mu = \mathbf{A}\mu - \mathbf{B}(\bar{L} \otimes \mathbf{I}_n)(\mathbf{1}_N \otimes \mu) = \lambda_c \mu, \quad (24)$$

where  $\mu = [\mu^\top, \dots, \mu^\top]^\top \in \mathbb{R}^{nN}$ . Thus,  $\lambda_c$  is an eigenvalue of the system matrix of the dynamics (23) with corresponding eigenvector  $\mu$ . That is to say, the mode associated with  $\lambda_c$  would appear in the solution of equation (23). If  $\lambda_c$  is a defective pure imaginary eigenvalue or  $\lambda_c \in \mathbb{C}_{>0}$ , individual opinions may not converge to bipartite consensus, even go to infinity.

To avoid the occurrence of these undesired scenarios, we impose some specifications on the system matrix  $A_i$  and then provide sufficient conditions for bipartite consensusability of heterogeneous opinion dynamics.

**Theorem 6.** *Consider the stabilizable system (20) with non-zero initial conditions forming opinions on a graph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$ . Suppose  $A_i$  satisfies the similarity transformation*

$$T_i^{-1} A_i T_i = \text{diag}(S, \hat{S}, J_i), \quad \forall i \in \mathbb{V}, \quad (25)$$

where  $S \in \mathbb{R}^{k \times k}$  and  $\hat{S} \in \mathbb{R}^{(q-k) \times (q-k)}$  with positive integers  $k$  and  $q$  satisfying  $k \leq q \leq n$  and  $S := \text{diag}(S, \hat{S}) \in \mathbb{R}^{q \times q}$  is a diagonalizable matrix in real Jordan normal form satisfying  $\text{sp}\{S\} \subseteq \mathbb{R}$ ;  $J_i \in \mathbb{R}^{(n-q) \times (n-q)}$  is a stable matrix. Moreover,  $T_i := [Q, Q_i, P_i] \in \mathbb{R}^{n \times n}$  is a nonsingular matrix wherein  $Q_i := [Q, Q_i] \in \mathbb{R}^{n \times q}$  of full column rank with  $Q \in \mathbb{R}^{n \times k}$  and  $Q_i \in \mathbb{R}^{n \times (q-k)}$  and  $P_i \in \mathbb{R}^{n \times (n-q)}$ .

If the following conditions hold,

- i).  $\mathcal{G}$  is structurally balanced and quasi-strongly connected.
- ii). Matrix  $\Xi$  is Hurwitz, where

$$\Xi = \text{diag}(J_1, A_2, \dots, A_N) - \begin{bmatrix} [\mathbf{0} \ \mathbf{I}_{n-q}]T_1^{-1} & \mathbf{0} & \dots & \mathbf{0} \\ -[Q_2 \ \mathbf{0}]T_1^{-1} & \mathbf{I}_n & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ -[Q_N \ \mathbf{0}]T_1^{-1} & \mathbf{0} & \dots & \mathbf{I}_n \end{bmatrix} \times \mathbf{B}(\bar{L} \otimes \mathbf{I}_n) \text{diag}(P_1, \mathbf{I}_n, \dots, \mathbf{I}_n),$$

then the system (20) is bipartite consensusable w.r.t.  $\hat{\mathbf{U}}$ . Moreover, the system (20) is consensusable if  $\mathcal{G}$  is an unsigned graph while polarizable if  $\mathcal{G}$  is a signed graph.

*Proof.* Since  $\mathcal{G}$  is SB, one can denote  $\bar{x}_i(t) = d_i x_i(t)$  for all  $i \in \mathbb{V}$ , and obtain a closed-loop system as below

$$\dot{\bar{x}}_i(t) = A_i \bar{x}_i(t) - B_i K_i \sum_{j=1}^N \bar{w}_{ij} (\bar{x}_i(t) - \bar{x}_j(t)), \quad \forall i \in \mathbb{V}. \quad (26)$$

With the help of the notation  $z_i := T_i^{-1} \bar{x}_i$  and the matrix transformation (25), it is not difficult to attain

$$\dot{z}_i = \text{diag}(S, J_i) z_i - T_i^{-1} B_i K_i \sum_{j=1}^N \bar{w}_{ij} (T_i z_i - T_j z_j), \quad \forall i \in \mathbb{V}.$$

For the convenience of further derivation, the following decompositions are needed

$$z_1(t) = \begin{bmatrix} z_{1q} \\ z_{1p} \end{bmatrix}, \quad T_i^{-1} B_i K_i = \begin{bmatrix} F_{iq} \\ F_{ip} \end{bmatrix}, \quad \forall i \in \mathbb{V},$$

where  $z_{1q}(t)$  and  $z_{1p}(t)$  are the first  $q$  rows and the last  $(n-q)$  rows of  $z_1(t)$ , respectively.  $F_{iq}$  and  $F_{ip}$  are analogously defined. After denoting  $\hat{z}_i(t) := z_i(t) - [z_{1q}^\top(t), \mathbf{0}_p^\top]^\top$  and  $e_i(t) = \sum_{j=1}^N \bar{w}_{ij} (\bar{x}_i(t) - \bar{x}_j(t))$ ,  $\forall i \in \mathbb{V}$ , the opinion dynamics (26) for  $i = 2, 3, \dots, N$  can be rewritten as

$$\begin{aligned} \dot{z}_{1p}(t) &= J_1 z_{1p}(t) - F_{1p} e_1(t) \\ \dot{\hat{z}}_i(t) &= \text{diag}(S, J_i) \hat{z}_i - \begin{bmatrix} F_{iq} \\ F_{ip} \end{bmatrix} e_i(t) + \begin{bmatrix} F_{1q} \\ \mathbf{0} \end{bmatrix} e_1(t). \end{aligned} \quad (27)$$

Let  $\zeta(t) := [z_{1p}^\top(t), \hat{z}_2^\top(t), \dots, \hat{z}_N^\top(t)]^\top$  to be the aggregation of variables. The compact form of (27) can be written as

$$\dot{\zeta}(t) = \hat{\Xi}\zeta(t), \quad (28)$$

where

$$\hat{\Xi} = \text{diag}(J_1, \mathcal{S}, J_2, \dots, \mathcal{S}, J_N) - \begin{bmatrix} F_{1p} & \mathbf{O} & \dots & \mathbf{O} \\ -[F_{1q}^\top \ \mathbf{O}]^\top & [F_{2q}^\top \ F_{2p}^\top]^\top & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ -[F_{1q}^\top \ \mathbf{O}]^\top & \mathbf{O} & \dots & [F_{Nq}^\top \ F_{Np}^\top]^\top \end{bmatrix} \times (\bar{L} \otimes \mathbf{I}_n) \text{diag}(P_1, T_2, \dots, T_N).$$

The observation that  $\hat{\Xi}$  is similar to the Hurwitz matrix  $\Xi$  in condition ii) can be obtained from the transformation below

$$\Xi = \text{diag}(\mathbf{I}_{n-q}, T_2, \dots, T_N) \hat{\Xi} \text{diag}(\mathbf{I}_{n-q}, T_2^{-1}, \dots, T_N^{-1}),$$

which guarantees that  $\hat{\Xi}$  is also Hurwitz and  $\lim_{t \rightarrow \infty} \zeta(t) = \mathbf{0}$ . Then the difference between  $\bar{x}_1$  and  $\bar{x}_i$ ,  $i = 2, 3, \dots, N$  needs to be inspected and thus, we denote  $\delta(t) = [\bar{x}_1^\top - \bar{x}_2^\top, \dots, \bar{x}_1^\top - \bar{x}_N^\top]^\top$  and one can obtain

$$\delta(t) = \Gamma\zeta(t), \quad (29)$$

where

$$\Gamma = \begin{bmatrix} P_1 & -T_2 & \dots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ P_1 & \mathbf{O} & \dots & -T_N \end{bmatrix}.$$

The asymptotic stability of (28) yields that  $\lim_{t \rightarrow \infty} \delta(t) = \mathbf{0}$  in (29), which implies that (26) reaches consensus. Since the eigenvalues in  $\text{sp}\{S\}$  have the same eigenspace  $Q$ , in reference to (24), one can explore  $\text{sp}\{S\} \subseteq \text{sp}\{\mathbf{A} - \mathbf{B}(\bar{L} \otimes \mathbf{I}_n)\}$ . That is to say, the matrix  $\mathbf{A} - \mathbf{B}(\bar{L} \otimes \mathbf{I}_n)$  contains at least one eigenvalue on imaginary axis, which is commonly shared by all  $A_i$  with  $i \in \mathbb{V}$ . As the equation (25) stated that  $\mathcal{S}$  is a diagonalizable matrix with all eigenvalues on the imaginary axis and  $J_i$  is stable, we can further display that the matrix  $\mathbf{A}$  is marginally stable. Namely, for any non-zero initial conditions, the limit behavior of  $\bar{x}_i(t)$  is bounded and does not converge to  $\mathbf{0}$ ,  $\forall i \in \mathbb{V}$ . Finally, if  $d_i = 1$ ,  $\forall i \in \mathbb{V}$ , which implies  $\bar{L} = L$ , the sufficient conditions for consensusability can be also derived.  $\square$

We have to admit that the sufficient condition is not generally applicable due to the strict requirements on individual opinion dynamics and the network topology. However, bipartite consensusability w.r.t. the static protocol shows an interesting insight into the difficulty of individuals to (separately) reach an agreement. The heterogeneity rooted in different aspects, e.g., languages, ages, genders, political beliefs, is the major obstacle. Alternatively, dynamic protocols based on self-learning are more promising for bipartite consensusability of heterogeneous opinion dynamics, which will be our future work.

### B. Neutralizability

In the homogeneous case, we conclude that structural balance is an obstacle for public opinions to achieve neutralization. However, according to the compact form (21), the impact of  $L$  is extraordinarily weakened because of the diversity of  $A_i$ ,  $B_i$ , and  $K_i$ . It implies that neutralization is much easier to establish in heterogeneous case. Here only a necessary condition is provided as an insight of the relation between neutralizability and network topology. Note that the assumption in the homogeneous case is then naturally extended as  $A_i$  is not Hurwitz for all  $i \in \mathbb{V}$ .

**Theorem 7.** Consider a strongly connected signed graph  $\mathcal{G} = (\mathbb{V}, \mathbb{E}, W)$  and the matrix transformation (25) holds. The stabilizable

dynamics (20) is neutralizable w.r.t.  $\hat{\mathbf{U}}$  only if  $\mathcal{G}$  is not structurally balanced.

*Proof.* Taking the compact form of the heterogeneous opinion dynamics in (21) into consideration, one can obtain  $\text{sp}\{\mathbf{A} - \mathbf{B}(L \otimes \mathbf{I}_n)\} \in \mathbb{C}_{<0}$ , if (20) is neutralized. Then we prove Theorem 7 by contradiction. If  $\mathcal{G}$  is SB, according to (24) and in conjugation with (25), the modes associated with the eigenvalues in  $\mathcal{S}$  appear in the solution of (21) which yields that neutralization is negated. Thus the opposite is true and we complete the proof.  $\square$

Based on the above results and analysis, one may notice that the heterogeneity of the opinion dynamics and the weak reliability of neutralizability on network topology are the major reasons for incapability of forming valuable sufficient and necessary conditions. Intuitively, it is a primary requirement that the dynamics share some common eigenvalues to achieve modulus consensus, which is also true for bipartite consensus. However, neutralization is peculiar because the target is the origin which means that the common eigenvalues of all the agent dynamics are not necessary. Moreover, the diversity of parameter matrices in the dynamics leads to multitudes of ways to reach neutralization. As a consequence of the aforementioned facts, sufficient and necessary condition for modulus consensusability cannot be expected.

## V. NUMERICAL EXAMPLE

In this section, several simulations serve to demonstrate the main results. Figure 2 shows a paradigmatic network consisting of 6 individuals. Evidently, the underlying weighted graph is SB and QSC.

Consider a homogeneous opinion dynamics in (1) with a state input  $A = [0, 0, 0; 0, 0, 1; 0, -1, 0]$  and an input matrix  $B = [1, 1, 1]^\top$ . It is apparent that  $(A, B)$  is stabilizable and a feedback-gain matrix  $K = [1, 1.41, 0]$  can be derived according to the algorithm given in [6]. As is shown in Figure 3(a), individual opinions achieve asymptotically polarization wherein the steady-state opinions on the second and third issues are non-stationary, but they are stationary on the first issue.

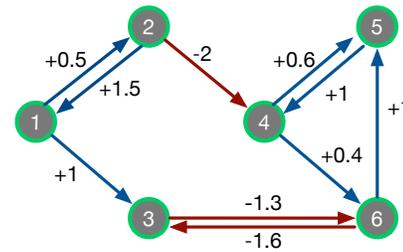


Fig. 2: An example of SB and QSC graph.

Over the same SB graph in Figure 2, we consider the non-identical subsystems dynamics (20) whose state and input matrices are given respectively by

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.5 & 0.25 & -0.5 \\ 0 & -1 & 0 \\ 0 & -0.5 & 0 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & -2 \\ -2 & 0 & -2 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ -3.5 & -0.25 & -3.5 \end{bmatrix}, & A_4 &= \begin{bmatrix} -2 & 1 & -2 \\ 0 & -5 & 0 \\ 0 & -2.5 & 0 \end{bmatrix}, \\ A_5 &= \begin{bmatrix} 0 & 0 & 0 \\ -13 & -1.5 & -13 \\ -8 & 0 & -8 \end{bmatrix}, & A_6 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -3 & 1 & -3 \end{bmatrix}, \\ B_1 &= [1, 2, 3]^\top, & B_2 &= [-1, 3, 1]^\top, & B_3 &= [1, 1, -7]^\top, \\ B_4 &= [1, -3, 1]^\top, & B_5 &= [-2, 1, 1]^\top, & B_6 &= [1, 1, -1]^\top. \end{aligned} \quad (30)$$

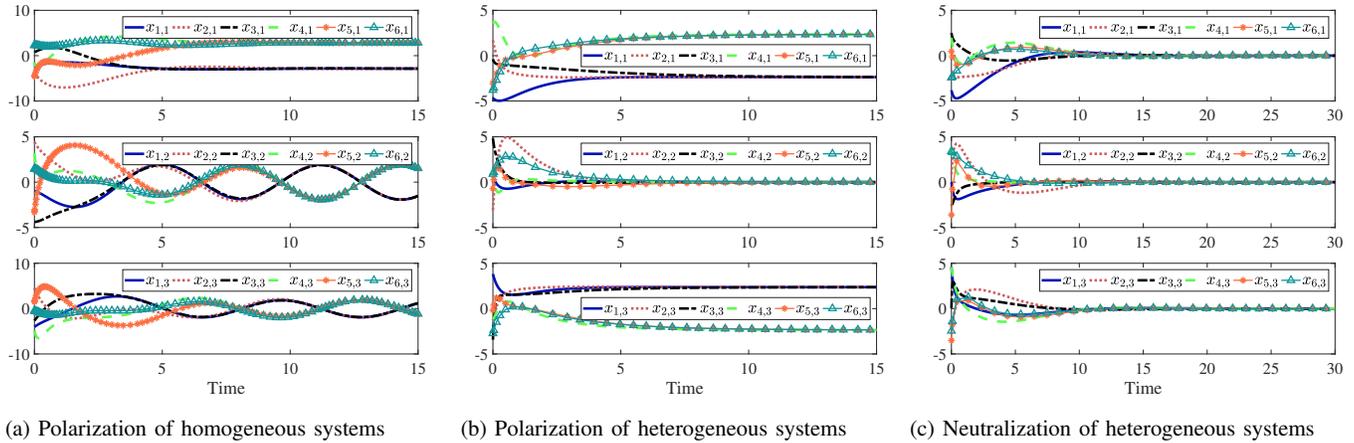


Fig. 3: Opinion formation processes on competitive networks

Note that  $A_i$  for  $i = 1, \dots, 6$  shares the same marginally stable eigenvalue 0 w.r.t. the same eigenvector  $[1, 0, -1]^T$  but possesses different stable eigenvalues. By choosing  $K_1 = [-0.1437, -0.0853, 1.2705]$ ,  $K_2 = [-1.2283, 0.4789, 0.1860]$ ,  $K_3 = [0.8217, 0.0701, -0.5925]$ ,  $K_4 = [-0.1401, -0.7124, 1.2742]$ ,  $K_5 = [-1.5199, 0.2548, -0.1057]$  and  $K_6 = [1.2190, 0.1231, -0.1952]$ , the bipartite consensusability conditions of Theorem 6 are satisfied. It can be observed from Figure 3(b) that the opinions of individuals reach polarity on the first and third topics, while neutrality on the second topic, which is allowable from Definition 1.

Finally, after adding an edge of weight +1 from node 4 to node 1, we obtain a structurally unbalanced graph which has no in-isolated SB subgraph and test the heterogeneous dynamics (30) on this modified network. Figure 3(c) shows the opinion formation process of heterogeneous dynamics over such network achieves asymptotically neutralization.

### CONCLUSION

This technical note investigates modulus consensusability (polarizability, consensusability, and neutralizability) of opinion-forming protocols in competitive networks. Criteria to test those accessibility-like issues in terms of sufficient and/or necessary conditions are our major contributions. Specifically, the joint effect of the algebraic constraints of the system dynamics and topological properties of the interaction structure is underlined in the examination of modulus consensusability. Different from sufficient conditions, we demonstrate that the structural balance of quasi-strongly connected social networks is not necessary for polarizability. With the emphasis on the individual heterogeneity, the investigation of modulus consensusability of systems with non-identical dynamics displays that the existence of commonality in eigenvalues and eigenvectors of system dynamics is of central importance to develop consensus and polarization protocol.

Future work focuses on modulus consensusability of outputs in the case of time-varying networks and discrete-time opinion dynamics with respect to output feedback protocols and dynamic protocols.

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