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Hedging and Valuation of Contingent Guarantees

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To Conny, for all her patience and support.
And to Xaver, for the joy he brings into our lives.

Abstract

This doctoral thesis studies the problem of hedging and valuing modern guarantee concepts in unit-linked life insurance, where the guaranteed amount at the policy's maturity grows contingent on the performance of the underlying investment fund. In contrast to standard hedging and valuation problems, the fund serves as both the underlying security and the replicating portfolio, rendering existing approaches from mathematical finance inadequate.

Using an extension of the classical portfolio insurance framework, the problem of hedging these contingent guarantees is transformed into an associated fixed-point problem, whose solution leads to a set of derivatives super-replicating the guaranteed amount. Sufficient conditions for the existence of such hedging derivatives are derived in the settings of complete and incomplete financial market models. Moreover, efficient numerical methods for constructing and pricing these derivatives are developed, implemented, and tested.

The presented hedging and valuation framework allows to investigate the quantitative and qualitative characteristics of a broad class of life insurance liabilities and can also be applied for the fair valuation and hedging of traditional participating life insurance policies, which currently rely on approximation methods.

Zusammenfassung

Untersuchungsgegenstand der vorliegenden Dissertationsschrift sind moderne Garantiekonzepte im Bereich der fondsgebundenen Lebensversicherung, bei denen das Garantiekapital zum Vertragsende basierend auf der Entwicklung eines zugrundeliegenden Fonds über die Laufzeit der Police wächst. Die zentrale Fragestellung befasst sich mit der Replikation und Bewertung solcher bedingter Garantien, bei denen der Fonds gleichzeitig als Basiswert und als replizierendes Portfolio fungiert. Standardmethoden der Finanzmathematik zur Replikation und Bewertung bedingter Auszahlungsprofile, wie zum Beispiel von Aktienoptionen, sind somit nicht ohne Weiteres anwendbar.

Basierend auf einer Erweiterung des klassischen Konzepts der Portfolioabsicherung, kann das Problem der Replikation und Bewertung bedingter Garantien in ein verwandtes Fixpunktproblem umgewandelt werden. Die Lösung dieses Fixpunktproblems führt zu einer Klasse von Absicherungsderivaten, welche in der Lage sind, den Garantiewert zum Vertragsende zu gewährleisten. Neben hinreichenden Bedingungen für die Existenz solcher Derivate werden auch effiziente numerische Verfahren für deren Konstruktion und Bewertung hergeleitet.

Der entwickelte Replikations- und Bewertungsansatz für bedingte Garantien erlaubt es, eine Vielzahl an Verbindlichkeiten im Bereich der Lebensversicherung, einschließlich klassischer nicht fondsgebundener Policen, auf ihre quantitativen und qualitativen Eigenschaften hin zu untersuchen.

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Copyright and Authorship Statement

Parts of this thesis have been quoted verbatim from the following two publications:

- Bienek and Scherer (2018),
  Hedging and Valuation of Contingent Guarantees,
  Working paper, in review;

- Bienek and Scherer (2019),
  Valuation of Contingent Guarantees using Least-Squares Monte Carlo,
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In both cases, the leading/first author is also the author of this thesis. Figures and tables are reprinted with permission.

Contents

1 Introduction

New supervisory regimes, such as the European *Solvency II* framework (European Union, 2015), and persistent low interest rates have forced the life insurance market into a phase of radical transformation. The mandatory market-consistent valuation of liabilities and risk-based capital requirements are exposing insurers' vulnerability to changes in the market environment, which has led to increased regulatory pressure on insurance companies to adequately manage the (financial) risks associated with their policies. Insurers are reacting to these developments by completely revising their product portfolio and traditional participating life insurance policies, which contain valuable minimum return guarantees, have become the main victim of this process. Many companies have already ceased the issuance of these contracts and some are even unwinding their existing policy portfolios (see, e.g., Welt, 2018).

Modern insurance products, in which policyholders usually bear (most of) the investment risk, are gradually replacing traditional participating contracts. Unit-linked policies, whose performance is linked to an underlying investment fund, are a particularly popular example. In order to make these products more attractive for risk-adverse clients, insurers are offering unit-linked policies utilizing so-called *portfolio insurance strategies* (Graf et al., 2012; Gatzert and Schmeiser, 2013). These asset allocation strategies are designed to guarantee a fixed minimum capital at the policy's maturity while also allowing the policyholder to participate in favorable market developments via a controlled exposure to risky asset classes.

This thesis studies a novel guarantee concept, where the guaranteed amount at maturity is not fixed, but grows contingent on the performance of the underlying investment fund. While portfolio insurance strategies for fixed guarantees, such as the *constant proportion portfolio insurance* (CPPI) and the *option based portfolio insurance* (OBPI), are well studied and understood (see, e.g., Bertrand and Prigent, 2005; Balder and Mahayni, 2010), there exists no method to construct super-replicating strategies for these *contingent guarantees.*

In contrast to the standard problem of hedging and pricing contingent claims (e.g. vanilla stock options), where the payoff depends on some externally given asset, here, the investment fund serves as both the underlying security and the hedging portfolio: Different investment strategies (i.e. hedging strategies) will lead to different guarantee levels (i.e. payoffs) at the policy's expiration. This renders existing approaches from mathematical finance inadequate and prompts the development of a distinct hedging and valuation framework for contingent guarantees, which is the main contribution of this thesis.

Ch. 2 provides a brief overview of the necessary mathematical toolbox, namely fixed-point theory, which is used extensively throughout this thesis. The mathematical model and basic characteristics of a contingent guarantee are introduced in Ch. 3 alongside some examples, an overview of the related literature, and the central research questions.

Ch. 4 studies the problem of hedging and valuing contingent guarantees in a complete financial market (Sec. 4.1). Based on an extension of the classical portfolio insurance framework (Sec. 4.2), the problem of constructing hedging strategies for contingent guarantees is transformed into an associated fixed-point problem whose solution leads to a set of hedging derivatives super-replicating the guaranteed amount at the policy's maturity (Sec. 4.3). This transformation allows to establish sufficient conditions for the existence of portfolio insurance strategies by invoking suitable fixed-point theorems. Moreover, a numerical routine to construct hedging derivatives based on a fixed-point iteration is developed and tested (Sec. 4.4).

Ch. 5 turns the attention to incomplete financial markets (Sec. 5.1) and adapts the fixed-point problem derived in Ch. 4 accordingly (Sec. 5.2). Sufficient conditions for the existence of attainable hedging derivatives are derived (Sec. 5.2.1) and a special class of static portfolio insurance strategies is introduced (Sec. 5.2.2). A numerical case study investigates the cost of static super-replication and compares results with the complete market setting (Sec. 5.3).

Finally, Ch. 6 develops an efficient numerical routine for the valuation of contingent guarantees using an adaptation of the *least-squares Monte Carlo* (LSMC) approach for American option pricing problems (Sec. 6.3). By approximating the fixed-point problem of Ch. 4 in two successive stages, the resulting pricing algorithm can be applied to a wide range of realistic and large-scale valuation problems and provides a significant improvement in performance compared to the numerical methods introduced in Sec. 4.4 and 5.3. The quality and behavior of the approximation is investigated in a numerical case study (Sec. 6.4).

Although modern life insurance offerings serve as the primary motivation for studying contingent guarantees, the exposition of the following chapters rely by no means on an insurance context. Indeed, the term 'unit-linked policy' is used quite freely throughout this thesis and should be regarded as synonymous to any financial product, where clients' funds are pooled in an investment vehicle and invested on their behalf by an investment management company (often an insurance company or one of its subsidiaries). In particular, the focus lies exclusively on the financial aspects of these kinds of products. Insurance specific aspects, such as mortality or lapse, are not considered.

Contribution of this Thesis

The problem of hedging and valuing traditional participating policies, which can be considered as special types of contingent guarantees (see Sec. 3.5), has been studied quite extensively in the related literature (see Sec. 3.4). However, to the best of the author's knowledge, there does not yet exist a consistent approach to solve the problem of super-replicating and pricing contingent guarantees in general. The present thesis aims at filling this gap. Its main contribution to the existing literature can be summarized as follows:

1. The abstract concept of a contingent guarantee is placed on a firm mathematical ground and the problem of hedging and valuing these types of financial liabilities is embedded into the classical theory of risk-neutral pricing.

2. The problem of hedging contingent guarantees is transformed into an associated fixed-point problem, which completely characterizes the set of super-replicating strategies.

3. Sufficient conditions for the existence of hedging strategies are derived and the quantitative and qualitative properties of different contingent guarantees are investigated.

4. Efficient numerical routines for pricing contingent guarantees and for constructing corresponding hedging strategies are developed, implemented, and tested.

2 A Primer on Fixed-Point Theory

Many of the technical results of the following chapters rely on an application of a suitable fixed-point theorem, which makes fixed-point theory the mathematical 'workhorse' of this thesis. The purpose of this chapter is to provide a (very) brief introduction to the topic with a clear focus on results that can be exploited for the problem of hedging and valuing contingent guarantees. A thorough treatment of fixed-point theory together with applications can be found in Smart (1980).

A *fixed-point* of a function $f : A \to A$, which maps a set A into itself, is any point $x \in A$ with $x = f(x)$, i.e. any point that is mapped to itself by f. More generally, if $f : A \to 2^A$ is a set-valued function, which maps A to its power set 2^A, a fixed-point of f is any point $x \in A$ with $x \in f(x)$.

Fixed-point theory studies the question under what sufficient conditions on f and A fixed-points exist and how they can be constructed or approximated. It has found widespread application in both theoretical and applied mathematics. For example, in the context of ordinary differential equations, Banach's theorem (Prop. 2.1.2) can be used to prove the existence and uniqueness of solutions to sufficiently regular initial value problems (Smart, 1980, Sec. 1.3). On the other hand, Brouwer's theorem (Prop. 2.2.1) is frequently applied in mathematical economics to prove the existence of equilibria and has been famously utilized in game theory by Nobel laureate John Nash (Franklin, 2002, Sec. 3).

Theorems within the field of fixed-point theory are generally grouped into three categories: Metric theorems, topological theorems, and order-theoretical theorems. The central results of each category are presented below. Note, however, that the lines separating these groups are blurry and that classifying fixed-point theorems is less straightforward than might be suggested by the following sections. For example, there exist results that combine metric and order-theoretic considerations (Jachymski, 2001).

2.1 Metric Fixed-Point Theorems

Metric fixed-point theorems are based on tools from the general theory of metric spaces. They usually place quite strong conditions on the function f – namely the contraction property in Def. 2.1.1 or variants thereof – and comparatively weak conditions on the set A.

The foundational result of metric fixed-point theory is Banach's theorem (Prop. 2.1.2). It is valid on any complete metric space, but is stated here in a more narrow form for closed subsets of *Banach spaces*, i.e. complete normed vector spaces $(\mathcal{B}, \|\cdot\|)$, which are of course also complete metric spaces with the induced metric $d : \mathcal{B} \times \mathcal{B}$, $(x, y) \mapsto \| x - y \|$. Most spaces encountered in applications and certainly all vector spaces considered in this thesis are Banach spaces, including the real space $\mathbb{R}^n$, $n \in \mathbb{N}$, when endowed with an arbitrary norm, and the space of square-integrable random variables $\mathcal{L}^2$ when endowed with the corresponding $\mathcal{L}^2$-norm (see App. 6.E).

Definition 2.1.1 (Contraction) *Let $(\mathcal{B}, \|\cdot\|)$ be a normed vector space, $A \subseteq \mathcal{B}$ closed, and $f : A \to A$. If there exists a constant $\Lambda \in [0, 1)$, such that*

$$\| f(x) - f(y) \| \;\leq\; \Lambda \, \| x - y \|$$

for all $x, y \in A$, then f is called a contraction.

Proposition 2.1.2 (Banach's Theorem, Smart (1980, Theo. 1.2.2)) *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $A \subseteq \mathcal{B}$ closed, and $f : A \to A$ a contraction. Then there exists a unique fixed-point $\hat{x} \in A$ with $\hat{x} = f(\hat{x})$. Moreover, the sequence $\{_{(u)}x\}_{u \in \mathbb{N}_0} \subseteq A$ defined by*

$$_{(u+1)}x \;:=\; f\big(_{(u)}x\big) \,, \quad u \in \mathbb{N}_0 \,, \tag{2.1.1}$$

converges to $\hat{x}$ for any starting value $_{(0)}x \in A$.

Banach's theorem is one of the few results within the realm of fixed-point theory, which guarantees the uniqueness of fixed-points and whose proof is constructive: If f is a contraction, its fixed-point can be approximated using the simple iterative procedure in (2.1.1). Note that this result is independent of the norm $\|\cdot\|$, in the sense that it suffices to show that f is contracting for an arbitrary norm on $\mathcal{B}$. Unfortunately, this contraction property is generally quite hard to prove.

An extension of Banach's theorem for set-valued functions has been derived by Nadler (1969). It requires the definition of a metric on the power set 2^A or rather a properly chosen subset thereof. For a normed vector space $(\mathcal{B}, \|\cdot\|)$ and a closed subset $A \subseteq \mathcal{B}$, let $\mathrm{NCB}(A) \subset 2^A$ be the set of non-empty, closed and bounded subsets of A, i.e.

$$\mathrm{NCB}(A) \;:=\; \{\, B \in 2^A \;:\; B \neq \emptyset,\, B \text{ is closed},\, B \text{ is bounded} \,\} \quad ,$$

and, for $\epsilon > 0$, let

$$\mathrm{N}(\epsilon, B) \;:=\; \{\, x \in A \;:\; \exists\, y \in B,\, \|\, x - y \,\| < \epsilon \,\}$$

denote the ϵ-neighborhood of a set $B \in \mathrm{NCB}(A)$. Then, the (*generalized*) *Hausdorff metric* h is defined by

$$h(B, C) \;:=\; \inf\{\, \epsilon > 0 \;:\; B \subseteq \mathrm{N}(\epsilon, C),\, C \subseteq \mathrm{N}(\epsilon, B) \,\}$$

for $B, C \in \mathrm{NCB}(A)$ (Nadler, 1969). Note that h implicitly depends on the norm $\|\cdot\|$ of the underlying Banach space $(\mathcal{B}, \|\cdot\|)$.

Intuitively, if $h(B, C)$ is small for $B, C \in \mathrm{NCB}(A)$, then every point of the set B lies 'close' to some point of C and vice versa. More generally, h gives the smallest 'amount' either set has to be increased by to include the other.

Definition 2.1.3 (Set-Valued Contraction) *Let $(\mathcal{B}, \|\cdot\|)$ be a normed vector space, $A \subseteq \mathcal{B}$ closed, and $f : A \to \mathrm{NCB}(A)$. If there exists a constant $\Lambda \in [0, 1)$, such that*

$$h(f(x), f(y)) \;\leq\; \Lambda \,\|\, x - y \,\|$$

for all $x, y \in A$, then f is called a (set-valued) contraction.

Roughly speaking, a (single-valued) function $f : A \to A$ will be a contraction, if a change in the function's input results in a 'comparatively smaller' change of the function's output. The same intuition underlies the definition above: A set-valued function $f : A \to \mathrm{NCB}(A)$ will be a contraction, if the 'distance' between the sets $f(x)$ and $f(y)$ is smaller than the distance between the inputs $x, y \in A$.

Proposition 2.1.4 (Nadler's Theorem, Nadler (1969, Theo. 5)) *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $A \subseteq \mathcal{B}$ closed, and $f : A \to \mathrm{NCB}(A)$ a contraction. Then there exists a fixed-point $\hat{x} \in A$ with $\hat{x} \in f(\hat{x})$.*

Note that, in contrast to the original result of Banach, the theorem above does not guarantee the uniqueness of a fixed-point and also does not provide a method to construct or approximate fixed-points.

2.2 Topological Fixed-Point Theorems

As the name suggests, topological fixed-point theorems are based on results and notions from topology. In contrast to metric fixed-point theorems, they usually place rather weak conditions on f, but comparatively strong conditions on A. The foundational result of topological fixed-point theory is Brouwer's theorem (Prop. 2.2.1), which has been extended to set-valued functions by Kakutani (Prop. 2.2.2).

Proposition 2.2.1 (Brouwer's Theorem, Smart (1980, Theo. 1.2.2)) *Let $n \in \mathbb{N}$, $A \subset \mathbb{R}^n$ non-empty, compact and convex, and $f : A \to A$ continuous. Then there exists a fixed-point $\hat{x} \in A$ with $\hat{x} = f(\hat{x})$.*

Proposition 2.2.2 (Kakutani's Theorem, Franklin (2002, Sec. 3.6)) *Let $n \in \mathbb{N}$ and $A \subset \mathbb{R}^n$ non-empty, compact and convex. Moreover, let $f : A \to 2^A$, be such that $f(x)$ is non-empty and convex for all $x \in A$ and such that the graph of f, i.e. the set $\{(x, y) : x \in A,\ y \in f(x)\} \subset \mathbb{R}^n \times \mathbb{R}^n$, is closed. Then there exists a fixed-point $\hat{x} \in A$ with $\hat{x} \in f(\hat{x})$.*

The fixed-point theorems of Brouwer and Kakutani require the underlying vector space to be of finite dimension and thus cannot be applied to general function spaces, such as the space of square-integrable random variables $\mathcal{L}^2$ (see App. 6.E). Although Brouwer's theorem has been generalized to infinite-dimensional normed vector spaces – a famous result known as Schauder's theorem (Smart, 1980, Theo. 2.3.7) – the characterization of compact sets in these spaces is by no means trivial and will not be discussed in this thesis.

2.3 Order-Theoretical Fixed-Point Theorems

Order-theoretical fixed-point theorems are based on notions from the theory of ordered sets and often employ only elementary set theory and logic. The arguably most famous result within this group is Tarski's theorem (Prop. 2.3.2), which is based on the notion of a *complete lattice*, i.e. a partially ordered set $(A, \leq)$, such that for every subset $B \subseteq A$, there exists a *greatest lower bound* and a *least upper bound* in A (see below).

Instead of stating the result in its full generality, it is reproduced here in the form it is used in this thesis. In particular, let $n \in \mathbb{N}$ and consider the real space $\mathbb{R}^n$ with the usual partial order given by the component-wise vector comparison '$\leq$', i.e.

$$x \;\leq\; y \quad :\Leftrightarrow \quad x_i \;\leq\; y_i \;\; \forall\, i = 1, \ldots, n$$

for $x, y \in \mathbb{R}^n$. Then $(A, \leq)$, where A is of the form

$$A \;:=\; \{\, x \in \mathbb{R}^n \;:\; a \leq x \leq b \,\}$$

for some $a, b \in \mathbb{R}^n$ with $a \leq b$, is a complete lattice. Indeed, for every subset $B \subseteq A$, there exists a greatest lower bound $x^{\text{glb}} \in A$ of B, which is given by the component-wise infimum $x_i^{\text{glb}} = \inf_{y \,\in\, B} y_i$ for $i = 1, \ldots, n$. Analogously, there exists a least upper bound $x^{\text{lub}} \in A$ of B, which is given by the component-wise supremum $x_i^{\text{lub}} = \sup_{y \,\in\, B} y_i$ for $i = 1, \ldots, n$.

Definition 2.3.1 (Monotone Function) *Let $n, m \in \mathbb{N}$, $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ and $f : A \to B$. If $f(x) \leq f(y)$ for all $x, y \in A$ with $x \leq y$, then f is called monotone.*

Proposition 2.3.2 (Tarski's Theorem, Tarski (1955, Theo. 1)) *Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}^n$ with $a \leq b$, and $A := \{\, x \in \mathbb{R}^n : a \leq x \leq b \,\}$. Moreover, let $f : A \to A$ be monotone. Then there exists a fixed-point $\hat{x} \in A$ with $\hat{x} = f(\hat{x})$. Moreover, there exists a 'least' fixed-point $x^{\downarrow} \in A$ with $x^{\downarrow} = f(x^{\downarrow})$ and the property that*

$$f(x) \;\leq\; x \quad \Rightarrow \quad x^{\downarrow} \;\leq\; x$$

for all $x \in A$. Similarly, there exists a 'greatest' fixed-point $x^{\uparrow} \in A$ with $x^{\uparrow} = f(x^{\uparrow})$ and the property that

$$x \;\leq\; f(x) \quad \Rightarrow \quad x \;\leq\; x^{\uparrow}$$

for all $x \in A$.

A remarkable consequence of Tarski's theorem is the existence and characterization of the least fixed-point $x^{\downarrow}$ of f: It is the 'smallest' element $x \in A$, such that $f(x) \leq x$, or, more precisely, it is the greatest lower bound of the set $\{\, x : f(x) \leq x \,\} \subseteq A$. Intriguingly, if f is additionally semi-continuous in the sense of the following theorem, then $x^{\downarrow}$ can be approximated by the iterative procedure of Banach's theorem.

Proposition 2.3.3 (Kleene's Theorem, Jachymski (2001, Theo. 4.1)) *Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}^n$ with $a \leq b$, and $A := \{\, x \in \mathbb{R}^n : a \leq x \leq b \,\}$. Moreover, let $f : A \to A$ be monotone and such that*

$$\lim_{u \to \infty} f\big({}_{(u)}x\big) \;=\; f\big(\lim_{u \to \infty} {}_{(u)}x\,\big)$$

for all ascending sequences $\{{}_{(u)}x\}_{u \,\in\, \mathbb{N}_0} \subset A$ with ${}_{(u)}x \;\leq\; {}_{(u+1)}x$. Then the sequence defined in (2.1.1) *started at ${}_{(0)}x = a$ converges to the least fixed-point $x^{\downarrow}$ of f.*

Similar to the previous theorems, Tarski's theorem has been extended to set-valued functions as well (see, e.g., Smithson, 1971). However, the required assumptions seem to be rarely justified in the context of contingent guarantees and these results are therefore of little practical relevance for the hedging and valuation problem considered in this thesis.

3 Contingent Guarantees

This chapter introduces the basic characteristics of a contingent guarantee and its mathematical model, which forms the cornerstone of this thesis. The central research questions alongside some examples of contingent guarantees are presented as well. The exposition is kept as general as possible and without reference to a specific financial market model.

3.1 The Mathematical Model

The economic narrative throughout this thesis is that of an investment fund that can invest freely in a frictionless financial market, in which all economic activity takes place on an ordered set of trading time points $\mathcal{T} \subset [0, \infty)$ with $0 \in \mathcal{T}$. Depending on the financial market model, $\mathcal{T}$ can take one of two forms: It is either a discrete finite set (Ch. 4 and 5) or a compact interval (Ch. 6). The fund's non-negative *net asset value* (NAV), i.e. the value of one share, is denoted by the process $X = \{X(t)\}_{t \in \mathcal{T}}$.

The fund's management company promises that the NAV at some predefined future time point will exceed a lower threshold consisting of

- a predefined fixed amount, and
- a variable amount that increases contingent on the evolution of the NAV according to a predefined mechanism.

This promise should not be understood as a simple declaration of intent, but as a legally binding contractual guarantee.

Definition 3.1.1 (Lock-In Mechanisms and Contingent Guarantees) *Let $N \in \mathbb{N}$ and $\bar{T} = \{\bar{T}_n\}_{n=0}^N \subseteq \mathcal{T}$ a set of ordered time points with $\bar{T}_0 = 0$ and $\bar{T}_{n-1} < \bar{T}_n$ for $n = 1, \dots, N$. Moreover, let $\bar{L} = \{\bar{L}_n\}_{n=1}^N$ be a family of non-negative measurable functions $\bar{L}_n : \mathbb{R}^{n+1} \to [0, \infty)$. Then $\bar{T}$ is called a set of lock-in time points, $\bar{L}$ is called lock-in mechanism, and, together with a fixed guarantee $\bar{F} \in [0, \infty)$, the tuple $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ is called contingent guarantee. The terminal guaranteed amount of a contingent guarantee $\bar{G}$ at time $\bar{T}_N$ is given by*

$$\bar{F} \; + \; \sum_{n=1}^{N} \bar{L}_n\big(\big\{X(\bar{T}_i)\big\}_{i=0}^{n}\big) \quad . \tag{3.1.1}$$

By virtue of the lock-in mechanism, the guaranteed 'payoff' to investors in (3.1.1) depends on the evolution of the fund's NAV at the lock-in time points $\bar{T}$, which in most practical applications correspond to equidistant yearly time points. Clearly, the fund management aims at choosing an investment strategy, such that $X(\bar{T}_N)$ does not fall short of this amount, i.e. such that the contingent guarantee is 'hedged' or 'super-replicated'.

The terminal time point $\bar{T}_N$ can be interpreted as the 'maturity' of $\bar{G}$. The financial guarantee only holds at this maturity, i.e. the fund's NAV may well fall below the (currently) guaranteed amount

$$\bar{F} \; + \sum_{\{\, n \,:\, \bar{T}_n \,\leq\, t \,\}} \bar{L}_n\big(\big\{X(\bar{T}_i)\big\}_{i=0}^{n}\big) \tag{3.1.2}$$

at any time $t \in \mathcal{T}$ with $t < \bar{T}_N$.

Standing Assumption *It is assumed throughout this thesis that $\bar{T}_N = \max \mathcal{T}$ for all contingent guarantees $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$. In economic terms, the terminal lock-in time point $\bar{T}_N$ is also the time of liquidation of the underlying investment fund, when the fund's assets are sold and the proceeds are distributed to its shareholders. This assumption does not affect the generality of the results of the following chapters, but serves merely to ease the notation and exposition.*

The guarantee structure defined above should be regarded as a special feature of the investment fund rather than of the linked life insurance policy. A premium payment at a given time $t \in \mathcal{T}$, which results in the purchase of (a fraction of) a share in the fund for the amount $X(t)$, is then associated to the promise that $X(\bar{T}_N)$ will not fall short of the currently guaranteed amount in (3.1.2) plus the future lock-in

$$\sum_{\{\, n \,:\, \bar{T}_n \,>\, t \,\}} \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) \quad ,$$

which is still unknown.

All characteristics of a contingent guarantee, i.e. the lock-in time points $\bar{T}$, the fixed guaranteed amount $\bar{F}$, and the lock-in mechanism $\bar{L}$, are agreed upon before the initial time $t = 0$. They remain unaltered throughout the lifetime of the investment fund (and thereby also throughout the lifetime of the life insurance policy linked to the investment fund).

3.2 The Hedging and Valuation Problem

The purpose of this thesis is to provide answers to the following questions.

HEDGING. Given a contingent guarantee, how should the fund management invest in order to 'super-replicate' the guaranteed amount (3.1.1) at maturity? What are sufficient conditions to ensure that a contingent guarantee can be hedged?

VALUATION. How can the 'value' of a contingent guarantee be defined? What are the major risk factors impacting this value? How can it be computed efficiently?

The reason for studying the problem of *hedging* a contingent guarantee is rather straightforward: The capability to effectively offset financial risks using an appropriate investment strategy directly affects a financial institution's profit margin and its ability to meet future obligations. Devising an investment policy that enables an insurer or investment manager to meet its liabilities is therefore in the clear interest of both the company and its customers.

The motivation for studying the problem of *valuation* is slightly more subtle: Modern regulatory and accounting regimes, such as Solvency II and *IFRS 17* (International Accounting Standards Board, 2017), require financial institutions to value their liabilities in a market-consistent (i.e. risk-neutral) manner. In the context of contingent guarantees, this implies that an insurer or an investment management company should be able to determine the fair value of these financial obligations.

Certainly, by the classical theory of no-arbitrage pricing, hedging and valuation are actually 'two sides of the same coin': The value of a financial claim is equal to the minimal initial capital required to (super-)replicate said claim with a (self-financing) investment strategy. This interconnection of hedging and valuation is even more pronounced for contingent guarantees, as the hedging strategy directly affects the terminal guaranteed amount in (3.1.1). From the perspective of the investment manager, the investment fund serves as both the underlying security and the replicating portfolio, which makes this hedging and valuation problem highly non-standard.

3.3 Connection to Variable Annuities

Contingent guarantees are closely linked to so-called *variable annuities*, which combine an investment into a fund X (or a portfolio of funds) with a guarantee scheme similar to the one described in Def. 3.1.1 (see, e.g., Bauer et al., 2008). The major difference between variable annuities and the setup described in Sec. 3.1 is that the insurance company hedges the liability arising from a variable annuity externally (i.e. independent of the fund X) and charges its clients an explicit 'guarantee fee' for this service. More precisely, the company hedges the terminal shortfall

$$\max\left\{ \left(\bar{F} + \sum_{n=1}^{N} \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) \right) - X(\bar{T}_N),\ 0 \right\} \tag{3.3.1}$$

either by directly buying a derivative with this payoff from another market participant, which will rarely be possible, or by replicating the payoff through an appropriate investment strategy.

The main difficulties in implementing a hedge for the payoff in (3.3.1) are

- finding a suitable set of liquid proxy instruments for the fund X, which might be controlled by a third-party investment company, and

- finding a cost-efficient and effective replicating strategy so as to gain a competitive advantage.

This naturally leads back to the question, if it is possible to structure the fund X in such a way, that the payoff in (3.3.1) becomes 'cheap' and 'easy'. After all, the fund itself is its best proxy and internalizing (part of) the hedge into the fund might lead to a reduction in costs and (operational) risks. For this reason, the following chapters also provide valuable insights for the future development and management of variable annuities.

3.4 Related Literature

The related literature, which concerns itself mostly with the asset and liability management of traditional participating life insurance policies (see Sec. 3.5), often disregards the deep interconnection between hedging and valuation of contingent guarantees. Three main approaches to construct (approximate) hedging strategies for a contingent guarantee can be distinguished.

Exogenous Approaches

Exogenous approaches fix an investment strategy and thereby treat the underlying portfolio, i.e. the investment fund X, as an exogenously given asset (see, e.g., Grosen and Jørgensen, 2000; Hansen and Miltersen, 2002; Bauer et al., 2006). Using standard valuation techniques, such as Monte Carlo simulation, these approaches then derive the risk-neutral value of the terminal guaranteed amount in (3.1.1). While a 'good' hedging strategy will result in a 'small' value of the terminal shortfall in (3.3.1), the exogenous approach offers no clear recipe on how to construct such an investment strategy other than by brute-force search.

Moreover, an apparent drawback of this valuation technique – at least from a technical perspective – is the following: It completely decouples hedging and valuation, in the sense that the initially chosen investment strategy is most likely not equal to a replicating strategy for the terminal guaranteed amount it generates. In other words, a replicating strategy for the guaranteed amount would result in a different NAV process (i.e. different from the initial exogenously fixed process X) and thereby also in a different guaranteed amount itself. Thus, the initial capital necessary to replicate the terminal guaranteed amount can by no means be considered equal to the fair value of the contingent guarantee.

Despite these shortcomings, the exogenous approach is currently prescribed for the valuation of participating life insurance policies under the Solvency II regulatory regime (European Union, 2015) and is used extensively in practice, where it is appreciated for its flexibility and ease of use. Intriguingly, an adaptation of the exogenous approach allows the construction of actual hedging strategies for contingent guarantees (see Sec. 4.3.6).

Endogenous Approaches

Endogenous approaches use multi-period stochastic programming to find an optimal investment strategy according to some selection criterion, such as minimum shortfall probability or maximum utility (see, e.g., Consiglio et al., 2006, 2008, 2015). The path-dependency and structure of contingent guarantees result in large-scale non-linear optimization problems that require substantial simplifications to be computationally feasible. Additionally, to reduce the complexity, all former references only consider static asset allocation strategies that are fixed at time $t = 0$ and remain unaltered until time $t = \bar{T}_N$, i.e. until the maturity of the contingent guarantee.

Depending on the selection criterion, the resulting investment policies will not be hedging strategies in the classical sense of mathematical finance, as they will usually not lead to an almost-sure super-replication of their self-induced terminal guaranteed amount. Moreover, endogenous approaches do not provide a meaningful way of defining the value of a contingent guarantee.

Under certain conditions, the construction of static super-replicating strategies for contingent guarantees is indeed possible (see Sec. 5.2.2), but the initial cost of implementing these investment policies is usually prohibitively high (see Sec. 5.3).

Genuine Hedging Approaches

Among the few to fully recognize the interconnection between hedging and valuation of contingent guarantees and to present a genuine hedging approach are Kleinow and Willder (2007) and Kleinow (2009). They consider a participating life insurance policy with an implicit lock-in mechanism of the form

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) \;=\; \left(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{X(\bar{T}_j)\}_{j=0}^{i}\big)\right) W\!\left(\frac{X(\bar{T}_n)}{X(\bar{T}_{n-1})}\right)$$

for $n = 1, \ldots, N$, where $W : [0, \infty) \to [0, \infty)$ satisfies certain technical conditions (see Sec. 4.3.4). The guaranteed amount (3.1.2) at time $\bar{T}_n$ is then given by

$$\bar{F} \prod_{i=1}^{n} \left[W\!\left(\frac{X(\bar{T}_n)}{X(\bar{T}_{n-1})}\right) \;+\; 1 \right]$$

for $n = 0, \ldots, N$. The function W can be thought as a 'participation function' that calculates the (relative) increase in the guaranteed amount based on the fund's achieved relative return (see also Ex. 3.5.5). Using repeated backwards induction, Kleinow (2009) constructs an NAV process X with

$$X(\bar{T}_N) \;=\; \bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) \quad , \tag{3.4.1}$$

i.e. such that the terminal guaranteed amount is perfectly hedged. A replicating strategy for the NAV process is then a proper hedging strategy for the contingent guarantee. Unfortunately, the inductive construction approach critically depends on the special multiplicative structure of $\bar{L}$ above and cannot be easily adapted to more general types of lock-in mechanisms.

3.5 Examples

Contingent guarantees are common components of investment funds underlying unit-linked life insurance policies or related investment offerings. The following list presents some popular examples.

Example 3.5.1 (Threshold Lock-In) *The threshold lock-in mechanism mimics the payoff profile of a call option. More precisely, the lock-in is equal to a fraction* $\alpha \in [0,1]$ *of the NAV above a fixed threshold* $\bar{X} \in [0,\infty)$*, i.e.*

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; \alpha\Big[\,X(\bar{T}_n)\;-\;\bar{X}\,\Big]^+$$

for $n = 1,\ldots,N$*, where* $[\,x\,]^+ := \max\{x,0\}$ *is the positive part of* $x \in \mathbb{R}$*. A 'digital' alternative of the lock-in mechanism above is given by*

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; \begin{cases} \Lambda & , \textit{ if } X(\bar{T}_n) \geq \bar{X}, \\ 0 & , \textit{ else}, \end{cases} \tag{3.5.1}$$

for $n = 1,\ldots,N$ *with* $\Lambda \in [0,\infty)$*.*

Example 3.5.2 (Take-Profit Lock-In) *The purpose of a 'take-profit' lock-in mechanism is to secure part of the fund's investment profits. For example, the fund management might choose (or rather might be forced by contractual agreements) to secure a fraction* $\alpha \in [0,1]$ *of the investment gain by moving funds from risky into secure assets. This rationale can be modeled by the lock-in mechanism*

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; \alpha\Big[\,X(\bar{T}_n)\;-\;X(\bar{T}_{n-1})\,\Big]^+ \tag{3.5.2}$$

for $n = 1,\ldots,N$*. Alternatively, the 'investment gain' could be regarded as the excess of the current NAV over the currently guaranteed amount, i.e.*

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; \alpha\left[\,X(\bar{T}_n)\;-\underbrace{\left(\bar{F}\;+\;\sum_{i=1}^{n-1}\bar{L}_i\big(\{X(\bar{T}_j)\}_{j=0}^i\big)\right)}_{\textit{guaranteed amount at time } \bar{T}_n}\right]^+ \quad , \tag{3.5.3}$$

or as the excess of the current NAV over the average NAV, i.e.

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; \alpha\left[\,X(\bar{T}_n)\;-\;\frac{1}{n+1}\sum_{i=0}^{n}X(\bar{T}_i)\right]^+ \tag{3.5.4}$$

for $n = 1,\ldots,N$*.*

Example 3.5.3 (Peak-Level Lock-In, Graf et al. (2012)) *A 'peak-level' or 'high watermark' lock-in mechanism is designed to guarantee a terminal NAV greater or equal to a fraction* $\alpha \in [0,1]$ *of the highest NAV recorded at the predefined dates* $\bar{T}$ *over the lifetime of the fund. This is a contingent guarantee* $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ *with the lock-in mechanism* $\bar{L}$ *given by*

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) \;=\; \left[\,\alpha\, X(\bar{T}_n) \;-\underbrace{\left(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{X(\bar{T}_j)\}_{j=0}^{i}\big)\right)}_{=\max\{\,\bar{F},\,\alpha\, X(\bar{T}_1),\,\dots,\,\alpha\, X(\bar{T}_{n-1})\,\}}\,\right]^{+}$$

for $n = 1, \dots, N$*. The terminal guaranteed amount in* (3.1.1) *is then given by*

$$\max\big\{\,\bar{F},\,\alpha\, X(\bar{T}_1),\,\dots,\,\alpha\, X(\bar{T}_N)\,\big\} \quad .$$

An exemplary evolution of the guaranteed amount in (3.1.2) for the take-profit and peak-level lock-in mechanisms is depicted in Fig. 3.1, which can be found at the end of this section. For the take-profit lock-in mechanisms (3.5.2) and (3.5.4), the guaranteed amount in (3.1.2) might increase even if it already exceeds the current NAV. The take-profit lock-in (3.5.3) and the peak-level lock-in in Ex. 3.5.3 generally behave very similar, except that the former leads to a positive lock-in any time the current NAV exceeds the guaranteed amount even by a minuscule amount.

Contingent guarantees are also implicit components of traditional participating (non-unit-linked) life insurance policies, in which the insurance company pools its clients' funds and invests them in the financial market. In addition to receiving a minimum guaranteed return, policyholders participate in the company's investment surplus, i.e. the capital gain exceeding the guaranteed return, by some crediting mechanism that is often mandated by law (Bauer et al., 2006).

These crediting mechanisms are usually based on the accounting values (i.e. 'book values') of the company's assets instead of the mark-to-market values and often allow for some management discretion over the precise amount of bonus that is credited to policyholders. Devising a formal model for these crediting mechanisms is therefore quite challenging. Two examples from the related literature can be found below.

Example 3.5.4 (German Participating Life Insurance Policy, Bauer et al. (2006)) *A German participating life insurance policy with upfront premium $P \in [0,\infty)$, guaranteed interest rate $\bar{r} \in [0,\infty)$, and maturity $\bar{T}_N$, guarantees its holder a fixed terminal amount of $\bar{F} = P\,e^{\bar{r}\,\bar{T}_N}$. Furthermore, the policyholder receives (at least) 90% of the company's (yearly) investment surplus exceeding the guaranteed interest payments on technical reserves, which are the currently guaranteed amount discounted at $\bar{r}$ (Bundesrepublik Deutschland, 2016). Any surplus is also compounded at the guaranteed interest rate. This corresponds to the rather complex lock-in mechanism $\bar{L}$ given by*

$$\begin{aligned}\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) &= e^{\bar{r}\,(\bar{T}_N-\bar{T}_n)}\Bigg[90\%\,\big(X(\bar{T}_n)-X(\bar{T}_{n-1})\big)\\ &\qquad -\underbrace{(e^{\bar{r}}-1)\,\underbrace{e^{-\bar{r}\,(\bar{T}_N-\bar{T}_{n-1})}\left(\bar{F}+\sum_{i=1}^{n-1}\bar{L}_i\big(\{X(\bar{T}_j)\}_{j=0}^{i}\big)\right)}_{\text{technical reserves at time } \bar{T}_{n-1}}}_{\text{guaranteed interest for the period } [\bar{T}_{n-1},\bar{T}_n]}\Bigg]^{+}\end{aligned}$$

for $n = 1,\ldots,N$.

Example 3.5.5 (Generalized Participating Life Insurance Policy, Kleinow (2009)) *A generalized (and somewhat simplified) version of a participating life insurance policy can be modeled by the lock-in mechanism*

$$\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) = \left(\bar{F}+\sum_{i=1}^{n-1}\bar{L}_i\big(\{X(\bar{T}_j)\}_{j=0}^{i}\big)\right)\left[\max\left\{e^{\bar{r}},\left(\frac{X(\bar{T}_n)}{X(\bar{T}_{n-1})}\right)^{\delta}\right\}-1\right]$$

for $n = 1,\ldots,N$, where $\bar{r} \in [0,\infty)$ is the 'guaranteed rate', $\delta \in (0,1)$ is the 'participation rate', and the lock-in time points $\bar{T}$ are evenly spaced at yearly intervals. The guaranteed amount at time $\bar{T}_n$ is then given by

$$\bar{F}\prod_{i=1}^{n}\max\left\{e^{\bar{r}},\left(\frac{X(\bar{T}_n)}{X(\bar{T}_{n-1})}\right)^{\delta}\right\}$$

for $n = 1,\ldots,N$.

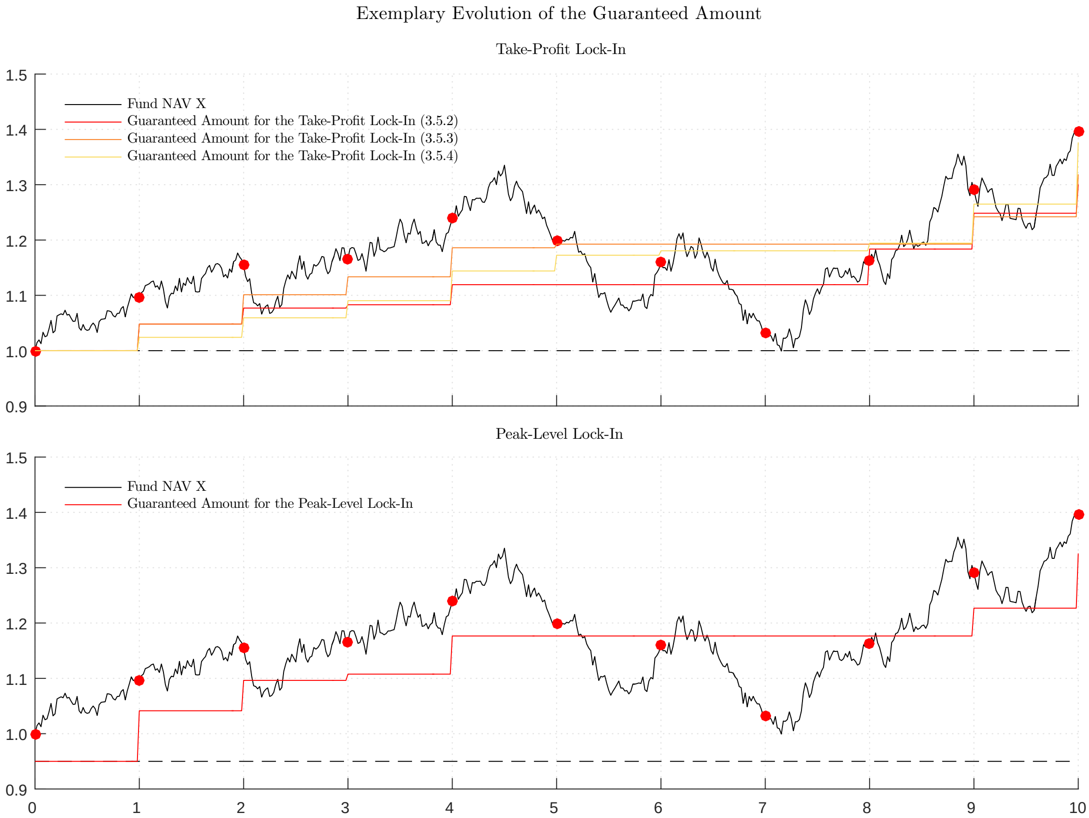

Figure 3.1: An exemplary trajectory of an investment fund with the take-profit and peak-level lock-in mechanisms $\bar{L}$ in Ex. 3.5.2 (top) and Ex. 3.5.3 (bottom), where the lock-in rate α is set to 50% in all cases. The fund's NAV (black line, y-axis) is recorded at equidistant yearly intervals $\{\bar{T}_n\}_{n=0}^{10}$ with $\bar{T}_n = n$ (red dots, x-axis). The dashed black line shows the fixed guarantee $\bar{F} = 1$, while the colored solid lines depict the evolution of the guaranteed amount in (3.1.2).

4 Hedging and Valuation in Complete Markets

Mathematical finance is perfectly capable of hedging and pricing claims whose payoffs are contingent on some externally given underlying asset (e.g. vanilla stock options). In fact, in a complete financial market, hedging and valuation of such derivatives can be considered as two separate tasks:

- First, the value process of the contingent claim is calculated as the risk-neutral (conditional) expectation of the discounted payoff;

- Second, a hedging strategy is constructed by replicating this value process.

For a contingent guarantee, however, the hedging portfolio is simultaneously the underlying security, in the sense that different investment strategies (i.e. hedging strategies) will lead to different terminal guaranteed amounts (i.e. payoffs). Hedging and valuation are thus deeply interconnected and cannot be separated as easily as for standard contingent claims.

This chapter is based on Bienek and Scherer (2018) and presents an extension of the classical portfolio insurance framework that allows to transform the problem of hedging contingent guarantees into an associated fixed-point problem, whose solution leads to a set of derivatives super-replicating the guaranteed amount. Sufficient conditions for the existence of such hedging derivatives are established and a numerical routine to construct them is developed. All proofs are postponed to App. 4.C.

4.1 Financial Market Model

The results of this chapter are based on a discrete-time and -state financial market model. In particular, the set of trading time points $\mathcal{T}$ is given by a discrete ordered set, i.e. $\mathcal{T} = \{t_0, t_1, \ldots, t_M\}$ with $M \in \mathbb{N}$, $t_0 = 0$, and $t_m \in (t_{m-1}, \infty)$ for $m = 1, \ldots, M$. The time span (in years) between two time points, which is denoted by $\Delta \in (0, \infty)$, is assumed to be constant for simplicity. From a practical point of view, $\mathcal{T}$ will usually correspond to equidistant monthly or weekly time points and t_M is the (known) time of liquidation of the investment fund.

Financial risk is modeled by a discrete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with (finite) sample space $\Omega = \{\omega_1, \ldots, \omega_K\}$, $K \in \mathbb{N}$, filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{t \in \mathcal{T}}$ and *physical probability measure* $\mathbb{P}$. Furthermore, the following simplifying assumptions are made: $\mathbb{P}\big(\{\omega_k\}\big) > 0$ for $k = 1, \ldots, K$, $\mathcal{F}(0) = \{\, \Omega, \emptyset \,\}$, and $\mathcal{F}(t_M) = \mathcal{F} = 2^{\Omega}$.

A real-valued random variable Y on $(\Omega, \mathcal{F}, \mathbb{P})$ allows for two mathematical interpretations: It is a function $Y : \Omega \to \mathbb{R}$, but also a vector $Y \in \mathbb{R}^K$. Throughout this chapter (and this thesis), both of these representations are used interchangeably with the correct interpretation given implicitly by the context. All (in)equalities involving random variables are meant to hold $\mathbb{P}$-almost surely, i.e. for all $\omega \in \Omega$ as $\mathbb{P}(\{\omega\}) > 0$ by assumption.

The financial market is arbitrage-free and complete, and contains $1 + M + D$, $D \in \mathbb{N}_0$, securities:

- A riskless *bank account* $B = \{B(t)\}_{t \in \mathcal{T}}$;
- A riskless *zero-coupon bond* $P^\tau = \{P^\tau(t)\}_{t \in \mathcal{T}}$ for each maturity $\tau = t_1, \ldots, t_M$;
- *Risky assets* $S^d = \{S^d(t)\}_{t \in \mathcal{T}}$, $d = 1, \ldots, D$, which might include stocks, real estate, commodities, and defaultable bonds.

These securities do not pay any coupons or dividends and their price processes are positive, finite, and adapted to the filtration $\mathbb{F}$. Note that the underlying probability space of any discrete-time complete financial market is finite (Föllmer and Schied, 2016, Theo. 5.37).

The bank account B is given by $B(t_m) = B(t_{m-1})\, e^{r(t_m)\Delta}$ for $m = 1, \ldots, M$ with $B(0) = 1$. Here, $r = \{r(t)\}_{t=t_1}^{t_M}$ is $\mathbb{F}$-predictable, with $r(t_m)$ the riskless (*spot*) *interest rate* for borrowing and lending over the period $[t_{m-1}, t_m)$. The riskless zero-coupon bond P^τ pays 1 at its maturity τ with certainty, i.e. $P^\tau(\tau) = 1$, and ceases to exist thereafter.

The probability measure $\mathbb{P}$ dictates the behavior of real-world asset prices. Since the financial market is free of arbitrage and complete, there exists a unique (*spot*) *pricing measure* $\mathbb{Q} \sim \mathbb{P}$, such that asset prices discounted by the bank account B are $\mathbb{Q}$-martingales (Pliska, 1997, (4.18)). More precisely,

$$\frac{P^\tau(t)}{B(t)} = \mathbb{E}_{\mathbb{Q}}\left[\, \frac{P^\tau(s)}{B(s)} \,\middle|\, \mathcal{F}(t) \right] \quad \text{and} \quad \frac{S^d(t)}{B(t)} = \mathbb{E}_{\mathbb{Q}}\left[\, \frac{S^d(s)}{B(s)} \,\middle|\, \mathcal{F}(t) \right]$$

for $\tau \geq s \geq t$ and $d = 1, \ldots, D$, where $\mathbb{E}_{\mathbb{Q}}$ is the (conditional) expectation under $\mathbb{Q}$.

Frequently, it will be more convenient to consider expectations under the so-called *forward pricing measure* $\mathbb{Q}^{t_M} \sim \mathbb{Q}$, which is defined by the Radon–Nikodým derivative

$$\frac{\mathrm{d}\mathbb{Q}^{t_M}}{\mathrm{d}\mathbb{Q}} \;=\; \frac{B(0)}{P^{t_M}(0)} \frac{P^{t_M}(t_M)}{B(t_M)} \tag{4.1.1}$$

and corresponds to a *change of numéraire* from the bank account B to the zero-coupon bond P^{t_M} (Pliska, 1997, Sec. 6.4).

Lemma 4.1.1 (Pliska (1997, (6.34))) *For any real-valued random variable $Y : \Omega \to \mathbb{R}$ and time points $t, s \in \mathcal{T}$ with $t \leq s$,*

$$B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\frac{Y}{B(s)} \,\middle|\, \mathcal{F}(t) \right] \;=\; P^{t_M}(t)\,\mathbb{E}_{\mathbb{Q}^{t_M}}\left[\frac{Y}{P^{t_M}(s)} \,\middle|\, \mathcal{F}(t) \right] \quad .$$

In particular, asset prices discounted by the zero-coupon bond P^{t_M} are $\mathbb{Q}^{t_M}$-martingales.

At any time $t \in \mathcal{T}$, the fund management may decide on the amount of money to invest in each security, but must do so without anticipation of the future and such that the fund's NAV stays non-negative (i.e. such that investors cannot lose more than their initial investment).

Definition 4.1.2 (Investment Strategies) *An investment strategy is an $\mathbb{F}$-predictable process*

$$\pi = \Big\{ \big(\, \pi_0(t),\, \pi_1(t),\, \ldots,\, \pi_M(t),\, \pi_{M+1}(t),\, \ldots,\, \pi_{M+D}(t)\,\big) \Big\}_{t=t_1}^{t_M} \quad ,$$

where $\pi(t_m)$ gives the amounts invested into the different assets after a potential reallocation at time t_{m-1}. Specifically, π_0 is the amount held in the bank account B, π_1 through π_M the amounts invested into the zero-coupon bonds P^{t_1} through P^{t_M}, and π_{M+1} through π_{M+D} the amounts invested into the risky assets S^1 through S^D, respectively. Negative values imply short-selling of the corresponding asset and $\pi_m(t) = 0$ for $t > t_m$, $m = 1, \ldots, M$. The fund's NAV, which will be denoted by X^π to emphasize the dependence on the investment strategy π, is then given by

$$X^\pi(t_{m-1}) \;=\; \pi_0(t_m) \;+\; \sum_{i=m}^{M+D} \pi_i(t_m) \tag{4.1.2}$$

for $m = 1, \ldots, M$. Moreover, any investment strategy is assumed to possess the following two properties.

Admissibility: *The NAV stays non-negative, i.e. $X^\pi(t) \geq 0$ for all $t \in \mathcal{T}$.*

Self-Financeability: *The NAV evolves according to*

$$\begin{aligned} \Delta X^\pi(t_m) &= \pi_0(t_m)\,\frac{\Delta B(t_m)}{B(t_{m-1})} + \sum_{i=m}^{M} \pi_i(t_m)\,\frac{\Delta P^{t_i}(t_m)}{P^{t_i}(t_{m-1})} + \sum_{d=1}^{D} \pi_{M+d}(t_m)\,\frac{\Delta S^d(t_m)}{S^d(t_{m-1})}\,, \\ X^\pi(0) &= x_0 \geq 0 \end{aligned} \tag{4.1.3}$$

for $m = 1, \ldots, M$, where $\Delta Z(t_m) := Z(t_m) - Z(t_{m-1})$ denotes the absolute increments of a process Z.

The set of all investment strategies π is denoted by Π.

Self-financeability implies that no capital is injected into or withdrawn after time $t = 0$. Thus, capital gains and losses stem purely from changes in asset prices and the fund management's investment decisions. This property might seem improbable for the asset allocation strategy of an investment fund, as shares can usually be newly issued or redeemed at any time (i.e. the fund might experience frequent capital in- and outflows). Here, however, X^π models the value of a share that is issued at time $t = 0$ and held until time $t = t_M$.

Lemma 4.1.3 (Pliska (1997, (3.21))) *For a given investment strategy $\pi \in \Pi$, the NAV X^π discounted by the bond price P^{t_M} is a $\mathbb{Q}^{t_M}$-martingale, i.e.*

$$\frac{X^\pi(t)}{P^{t_M}(t)} = \mathbb{E}_{\mathbb{Q}^{t_M}}\left[\frac{X^\pi(s)}{P^{t_M}(s)} \,\middle|\, \mathcal{F}(t) \right]$$

for all $t, s \in \mathcal{T}$ with $t \leq s$.

Since the financial market is complete, every $\mathcal{F}(s)$-measurable non-negative payoff $Y : \Omega \to [0, \infty)$ at some time $s \in \mathcal{T}$ is attainable, i.e. it can be replicated by an investment strategy $\pi \in \Pi$ with

$$X^\pi(t) = B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\frac{Y}{B(s)} \,\middle|\, \mathcal{F}(t) \right]$$

for all $t \leq s$ (see, e.g., Pliska, 1997, (4.1)). In particular, $X^\pi(s) = Y$ and $X^\pi(t)$ is the 'value' or 'price' of Y at time t, which is independent of the chosen replicating strategy. This attainability property technically also holds for negative payoffs, however the corresponding replicating strategy will then not satisfy the criterion of admissibility.

4.2 Portfolio Insurance

Hedging strategies for simple fixed guarantees $\bar{F}$, i.e. contingent guarantees $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ with $\bar{L}_n \equiv 0$ for $n = 1, \ldots, N$, are well studied and understood. In fact, these investment strategies have become a category of their own and are commonly referred to as portfolio insurance strategies.

In simple terms, portfolio insurance strategies are designed to achieve a terminal NAV above some predefined lower threshold, while also allowing the investor to participate in favorable market developments via a controlled exposure to risky asset classes. The most prominent representatives of this class of asset allocation strategies are the CPPI and the OBPI, which are briefly introduced in Ex. 4.2.3 and 4.2.4. This section extends the classical portfolio insurance framework to accommodate contingent guarantees and derives the fixed-point relation that is central to the construction of hedging strategies in Sec. 4.3.

Definition 4.2.1 (Portfolio Insurance Strategies) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee. An investment strategy $\pi \in \Pi$ with*

$$X^\pi(\bar{T}_N) \;\geq\; \bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big)$$

is called portfolio insurance strategy or hedging strategy for $\bar{G}$. In other words, a portfolio insurance strategy is super-replicating its 'self-induced' terminal guaranteed amount. The set of all portfolio insurance strategies for $\bar{G}$ is denoted by $\Pi^{\bar{G}}$.

For a given contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ and a portfolio insurance strategy $\pi \in \Pi^{\bar{G}}$,

$$\begin{aligned} \frac{X^\pi(t)}{P^{\bar{T}_N}(t)} \;&=\; \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\frac{X^\pi(\bar{T}_N)}{P^{\bar{T}_N}(\bar{T}_N)} \,\middle|\, \mathcal{F}(t) \right] \\ &=\; \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \,\middle|\, \mathcal{F}(t) \right] \end{aligned} \tag{4.2.1}$$

for $t \in \mathcal{T}$, where the first equality stems from the martingale property of the compounded NAV $\{X^\pi(t) \,/\, P^{\bar{T}_N}(t)\}_{t\in\mathcal{T}}$ (Lem. 4.1.3) and

$$\bar{E}^{\bar{G},\pi} \;:=\; X^\pi(\bar{T}_N) \;-\; \left(\bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \right) \;\geq\; 0$$

is the non-negative *terminal excess* over the guaranteed amount. This directly leads to the fixed-point relationship

$$X^{\pi}(\bar{T}_n) \;=\; P^{\bar{T}_N}(\bar{T}_n)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\bar{F} \;+\; \sum_{i=1}^{N}\bar{L}_n\big(\{X^{\pi}(\bar{T}_j)\}_{j=0}^{i}\big) \;+\; \bar{E}^{\bar{G},\pi}\;\middle|\;\mathcal{F}(\bar{T}_n)\right] \tag{4.2.2}$$

for $n = 0,\dots,N$, such that each random variable $X^{\pi}(\bar{T}_n)$ is equal to 'a function of' the complete family $\{X^{\pi}(\bar{T}_n)\}_{n=1}^{N}$.

Given that the NAV process of any portfolio insurance strategy satisfies the fixed-point relationship in (4.2.2), one could ask the question if the converse is true as well: Given a family of random variables satisfying (4.2.2), is it possible to construct a corresponding portfolio insurance strategy? This is indeed the case and the Martingale Method of Sec. 4.3 is based on this key insight.

However, before continuing with the introduction of the Martingale Method, it is worth further investigating the structure of portfolio insurance strategies using the classical portfolio insurance framework, which is adapted to the setting of contingent guarantees below.

Definition 4.2.2 (Portfolio Processes) *For a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ and investment strategy $\pi \in \Pi$, the variable guarantee is the non-decreasing and $\mathbb{F}$-predictable process $V^{\bar{G},\pi} = \{V^{\bar{G},\pi}(t)\}_{t\in\mathcal{T}}$ given by*

$$V^{\bar{G},\pi}(t) \;:=\; \sum_{\{\,n\,:\,\bar{T}_n < t\,\}} \bar{L}_n\big(\{X^{\pi}(\bar{T}_i)\}_{i=0}^{n}\big)$$

for $t \in \mathcal{T}$, such that $V^{\bar{G},\pi}(t)$ is the sum of the 'lock-in' prior to time t. Moreover, the guarantee $G^{\bar{G},\pi} = \{G^{\bar{G},\pi}(t)\}_{t\in\mathcal{T}}$ is defined as the (risk-neutral) value of the guaranteed amount at time t, i.e.

$$G^{\bar{G},\pi}(t) \;:=\; B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\,\frac{\bar{F} + V^{\bar{G},\pi}(t)}{B(\bar{T}_N)}\;\middle|\;\mathcal{F}(t)\right] \tag{4.2.3}$$

for $t \in \mathcal{T}$. The process $C^{\bar{G},\pi} = \{C^{\bar{G},\pi}(t)\}_{t\in\mathcal{T}}$ defined by

$$C^{\bar{G},\pi}(t) \;:=\; X^{\pi}(t) \;-\; G^{\bar{G},\pi}(t) \tag{4.2.4}$$

for $t \in \mathcal{T}$, gives the excess of the NAV X^{π} over the guarantee $G^{\bar{G},\pi}$ and is called the cushion.

The variable guarantee $V^{\bar{G},\pi}$ is constant in the intervals $(T_{n-1}, T_n]$, $n = 1,\dots,N$, and increases by the lock-in immediately after each lock-in time point, i.e.

$$\Delta V^{\bar{G},\pi}(t) \;=\; \begin{cases} \bar{L}_n\big(\{X^{\pi}(\bar{T}_i)\}_{i=0}^{n}\big) & ,\ \text{if } t = \min\{\,t\in\mathcal{T} \,:\, t > \bar{T}_n\,\},\ n = 1,\dots,N-1, \\ 0 & ,\ \text{else,} \end{cases}$$

$$V^{\bar{G},\pi}(0) \;=\; 0$$

for $t \in \mathcal{T} \setminus \{0\}$. Moreover, from (4.2.3),

$$G^{\bar{G},\pi}(t) \;=\; \mathbb{E}_{\mathbb{Q}}\left[\, \frac{B(t)}{B(\bar{T}_N)} \,\middle|\, \mathcal{F}(t) \right] \big(\bar{F} + V^{\bar{G},\pi}(t)\big) \;=\; P^{\bar{T}_N}(t)\,\big(\bar{F} + V^{\bar{G},\pi}(t)\big)$$

and thus, with (4.2.1),

$$\begin{aligned} X^{\pi}(t) \;&=\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^{\pi}(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \,\middle|\, \mathcal{F}(t) \right] \\ &=\; G^{\bar{G},\pi}(t) \;+\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \sum_{\{\, n \,:\, \bar{T}_n \geq t \,\}} \bar{L}_n\big(\{X^{\pi}(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \,\middle|\, \mathcal{F}(t) \right] \end{aligned}$$

for $t \in \mathcal{T}$. In particular, by Def. 4.2.2,

$$C^{\bar{G},\pi}(t) \;=\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \sum_{\{\, n \,:\, \bar{T}_n \geq t \,\}} \bar{L}_n\big(\{X^{\pi}(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \,\middle|\, \mathcal{F}(t) \right] \tag{4.2.5}$$

for $t \in \mathcal{T}$.

With (4.2.5) the cushion at any time $t \in \mathcal{T}$ is equal to the current (risk-neutral) value of the future lock-in and the terminal excess. Moreover, for a portfolio insurance strategy $\pi \in \Pi^{\bar{G}}$, the terminal excess is by definition non-negative, i.e. $\bar{E}^{\bar{G},\pi} \geq 0$, such that $C^{\bar{G},\pi}(t) \geq 0$ for $t \in \mathcal{T}$.

In other words, the NAV process corresponding to a portfolio insurance strategy can be split into an 'insurance' or 'backward-looking' part equal to a long position of $\bar{F} + V^{\bar{G},\pi}$ zero-coupon bonds $P^{\bar{T}_N}$ (i.e. the guarantee $G^{\bar{G},\pi}$) and a 'speculative' or 'forward-looking' part consisting of the non-negative excess (i.e. the cushion $C^{\bar{G},\pi}$).

This partition of a portfolio insurance strategy can be made explicit by using (4.1.2) and plugging

$$\begin{aligned} \pi_0(t_m) \;&=\; \left(X^{\pi}(t_{m-1}) - \sum_{i=m}^{M+D} \pi_i(t_m) \right) \\ &=\; \left(G^{\bar{G},\pi}(t_{m-1}) + C^{\bar{G},\pi}(t_{m-1}) - \sum_{i=m}^{M+D} \pi_i(t_m) \right) \end{aligned}$$

into (4.1.3), such that

$$\begin{aligned} \Delta X^{\pi}(t_m) \;=\;& \left(G^{\bar{G},\pi}(t_{m-1}) + C^{\bar{G},\pi}(t_{m-1}) - \sum_{i=m}^{M+D} \pi_i(t_m) \right) \frac{\Delta B(t_m)}{B(t_{m-1})} \\ &+\; \sum_{i=m}^{M} \pi_i(t_m)\,\frac{\Delta P^{t_i}(t_m)}{P^{t_i}(t_{m-1})} \;+\; \sum_{d=1}^{D} \pi_{M+d}(t_m)\,\frac{\Delta S^d(t_m)}{S^d(t_{m-1})} \end{aligned}$$

$$
\begin{aligned}
&= G^{\bar{G},\pi}(t_{m-1}) \frac{\Delta P^{\bar{T}_N}(t_m)}{P^{\bar{T}_N}(t_{m-1})} \\
&\quad + \tilde{\pi}_0(t_m) \frac{\Delta B(t_m)}{B(t_{m-1})} + \sum_{i=m}^{M} \tilde{\pi}_i(t_m) \frac{\Delta P^{t_i}(t_m)}{P^{t_i}(t_{m-1})} + \sum_{d=1}^{D} \tilde{\pi}_{M+d}(t_m) \frac{\Delta S^d(t_m)}{S^d(t_{m-1})}
\end{aligned}
$$

for $m = 1, \ldots, M$. The $\mathbb{F}$-predictable process $\tilde{\pi} = \{\tilde{\pi}(t)\}_{t=t_1}^{t_m}$ is given by

$$
\begin{aligned}
\tilde{\pi}_M(t_m) &:= \pi_M(t_m) - G^{\bar{G},\pi}(t_{m-1}) , \\
\tilde{\pi}_d(t_m) &:= \pi_d(t_m) \quad \text{for } d \in \{1, \ldots, M-1, M+1, \ldots, D\}, \text{ and} \\
\tilde{\pi}_0(t_m) &:= \pi_0(t_m) = C^{\bar{G},\pi}(t_m) - \sum_{i=m}^{M+D} \tilde{\pi}_i(t_m)
\end{aligned}
\tag{4.2.6}
$$

for $m = 1, \ldots, M$ (recall that $\bar{T}_N = t_M$). This directly leads to

$$
\begin{aligned}
\Delta C^{\bar{G},\pi}(t_m) &= \tilde{\pi}_0(t_m) \frac{\Delta B(t_m)}{B(t_{m-1})} + \sum_{i=m}^{M} \tilde{\pi}_i(t_m) \frac{\Delta P^{t_i}(t_m)}{P^{t_i}(t_{m-1})} \\
&\quad + \sum_{d=1}^{D} \tilde{\pi}_{M+d}(t_m) \frac{\Delta S^d(t_m)}{S^d(t_{m-1})} - P^{\bar{T}_N}(t_m) \Delta V^{\bar{G},\pi}(t_m)
\end{aligned}
\tag{4.2.7}
$$

$$
\Delta G^{\bar{G},\pi}(t_m) = G^{\bar{G},\pi}(t_{m-1}) \frac{\Delta P^{\bar{T}_N}(t_m)}{P^{\bar{T}_N}(t_{m-1})} + P^{\bar{T}_N}(t_m) \Delta V^{\bar{G},\pi}(t_m) \tag{4.2.8}
$$

for $m = 1, \ldots, M$.

At each lock-in time point – or rather immediately thereafter – the lock-in mechanism 'shifts' funds from the cushion $C^{\bar{G},\pi}$ to the guarantee $G^{\bar{G},\pi}$ in the amount of the discounted lock-in, which can be observed in (4.2.7) and (4.2.8). In fact, the dynamics in (4.2.7) might remind the experienced reader of the general theory of optimal investment and consumption (Pliska, 1997, Sec. 5). Indeed, the lock-in mechanism could be thought of as a consumption process on the cushion $C^{\bar{G},\pi}$.

With (4.2.6), π and $\tilde{\pi}$ can be used interchangeably. In fact, the only difference between both processes is that π contains an additional long position of the $\bar{T}_N$-maturity zero-coupon bond in the amount of $\bar{F} + V^{\bar{G},\pi}$ (i.e. the currently guaranteed amount). Moreover, the process $\tilde{\pi}$ can be interpreted as an investment strategy with the cushion $C^{\bar{G},\pi}$ as its corresponding 'NAV process.' One might even be inclined to write $C^{\bar{G},\pi} \equiv X^{\tilde{\pi}}$. However, in contrast to Def. 4.1.2, $\tilde{\pi}$ will generally only be self-financing in the intervals $(\bar{T}_n, \bar{T}_{n+1}]$, $n = 0, \ldots, N-1$, because of the 'consumption' caused by the lock-in mechanism (compare (4.2.7) and (4.1.3)).

Note that the guarantee $G^{\bar{G},\pi}$ at the lock-in time points is given by

$$G^{\bar{G},\pi}(\bar{T}_n) \;=\; \begin{cases} P^{\bar{T}_n}(\bar{T}_n)\,\bar{F} & \text{, if } n = 0,1, \\ P^{\bar{T}_n}(\bar{T}_n)\left(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{G^{\bar{G},\pi}(\bar{T}_j) + C^{\bar{G},\pi}(\bar{T}_j)\}_{j=0}^{i}\big)\right) & \text{, else,} \end{cases}$$

such that $G^{\bar{G},\pi}(\bar{T}_n)$ can be regarded as a function of the random variables $\{C^{\bar{G},\pi}(\bar{T}_i)\}_{i=0}^{n-1}$. Put in simple terms, the guarantee process at the lock-in time points can be reconstructed from (the history of) the cushion process at the lock-in time points.

With former observation, (4.2.5) defines a fixed-point relationship for the family of random variables $\{C^{\bar{G},\pi}(\bar{T}_n)\}_{n=0}^{N}$. Indeed,

$$C^{\bar{G},\pi}(\bar{T}_n) \;=\; P^{\bar{T}_N}(\bar{T}_n)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\;\sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G^{\bar{G},\pi}(\bar{T}_j) + C^{\bar{G},\pi}(\bar{T}_j)\}_{j=0}^{i}\big) \;+\; \bar{E}^{\bar{G},\pi}\;\middle|\;\mathcal{F}(\bar{T}_n)\right] \tag{4.2.9}$$

for $n = 0, \ldots, N$, such that each random variable $C^{\bar{G},\pi}(\bar{T}_n)$ is equal to 'a function of' the complete family $\{C^{\bar{G},\pi}(\bar{T}_n)\}_{n=0}^{N}$.

The fixed-point relationships (4.2.9) and (4.2.2) are completely equivalent, since the NAV process of a portfolio insurance strategy defines its own cushion process and vice versa. The Martingale Method is developed using (4.2.9) rather than (4.2.2), as the former allows for a more vivid economic interpretation.

Two classical examples of portfolio insurance strategies for simple fixed guarantees $\bar{F}$, i.e. contingent guarantees $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ with $\bar{L}_n \equiv 0$ for $n = 1, \ldots, N$, can be found below.

Example 4.2.3 (Classical Portfolio Insurance: CPPI) *The CPPI of Black and Jones (1987) and Perold and Sharpe (1988) is a dynamic portfolio insurance strategy, where the amount invested into the risky assets is a constant proportion $\beta \in [0, \infty)$ of the cushion, i.e.*

$$\sum_{d=1}^{D} \pi_{M+d}(t_m) \;=\; \sum_{d=1}^{D} \tilde{\pi}_{M+d}(t_m) \;=\; \beta\, C^{\bar{G},\pi}(t_{m-1})$$

for all $m = 1, \ldots, M$. The multiple β must be chosen, such that the cushion remains non-negative. The remaining funds are invested into the riskless zero-coupon bond $P^{\bar{T}_N}$, such that $\pi_M(t_m) = X^{\pi}(t_{m-1}) - \beta\, C^{\bar{G},\pi}(t_{m-1})$, or equivalently $\tilde{\pi}_M(t_m) = (1-\beta)\, C^{\bar{G},\pi}(t_{m-1})$, and $\pi_0(t_m) = \ldots = \pi_{M-1}(t_m) = 0$ for all $m = 1, \ldots, M$.

Example 4.2.4 (Classical Portfolio Insurance: OBPI) *The OBPI of Leland and Rubinstein (1998) is a static portfolio insurance strategy, where a long position in a portfolio of risky securities is insured using a protective (basket) put option with strike $\bar{F} \in [0, \infty)$ and maturity $\bar{T}_N$. By put-*

call-parity this is equivalent to a long position in $\bar{F}$ zero-coupon bonds $P^{\bar{T}_N}$ and a (basket) call option with the same parameters as the put (Föllmer and Schied, 2016, Sec. 1.3). In the complete financial market of Sec. 4.1 $\tilde{\pi}$ is then simply the call's replicating strategy.

4.3 The Martingale Method

The direct construction of a portfolio insurance strategy for a contingent guarantee is generally a daunting task, because there is no direct relationship between the terminal guaranteed amount in (3.1.1) and an investment strategy $\pi \in \Pi$, which could be exploited. However, the common link between both is the NAV process X^π and, with the fixed-point relation in (4.2.9), it immediately transpires that the problem of hedging contingent guarantees should be solved in the domain of portfolio processes first, rather than directly on the level of investment strategies.

This is precisely the key insight of the *Martingale Method*:

- First, a family of random variables $\{C^{\bar{G},\pi}(\bar{T}_n)\}_{n=0}^N$ satisfying the fixed-point relation in (4.2.9) is constructed. These random variables can be interpreted as (the risk-neutral value process of) a 'hedging derivative' that pays the discounted lock-in at the time points $\{\bar{T}_n\}_{n=0}^N$ and that is thereby super-replicating the terminal guaranteed amount of the contingent guarantee.

- Once such a derivative has been constructed, it is then replicated by an investment strategy π (or rather $\tilde{\pi}$), which will always exist in the complete financial market of Sec. 4.1.

A similar two step process underlies the eponymous Martingale Method in portfolio optimization, where, in a first step, a payoff corresponding to the optimal terminal wealth is derived, and, in a second step, a hedging strategy for this payoff is constructed (Pliska, 1997, Sec. 5.2).

For a set of lock-in time points $\bar{T}$, let $\mathcal{V}^{\bar{T}} := [0,\infty)^{K\times(N+1)}$ be the non-negative orthant of the vector space $\mathbb{R}^{K\times(N+1)}$. Moreover, for a contingent guarantee $\bar{G} = (\bar{T},\bar{F},\bar{L})$ and a family of non-negative random variables $C = \{C_n\}_{n=0}^N \in \mathcal{V}^{\bar{T}}$, let $G^{\bar{G},C} = \{G_n^{\bar{G},C}\}_{n=0}^N \in \mathcal{V}^{\bar{T}}$ be defined iteratively by

$$G_n^{\bar{G},C} \;:=\; P^{\bar{T}_N}(\bar{T}_n)\left(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{G_j^{\bar{G},C} + C_j\}_{j=0}^{i}\big)\right)$$

for $n = 0,\ldots,N$. Then $G^{\bar{G},C}$ is the guarantee of Def. 4.2.2 at the lock-in time points $\bar{T}$, but with the cushion process replaced by the random variables C.

Definition 4.3.1 (Budget Function) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee and $\bar{E} \in [0, \infty)^K$ a (targeted) non-negative terminal excess. The function $H^{\bar{G}}_{\bar{E}} : \mathcal{V}^{\bar{T}} \to \mathcal{V}^{\bar{T}}$ defined by*

$$H^{\bar{G}}_{\bar{E}}(C) \;:=\; \left\{ P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \;+\; \bar{E} \;\middle|\; \mathcal{F}(\bar{T}_n) \right] \right\}_{n=0}^{N} \tag{4.3.1}$$

for $C \in \mathcal{V}^{\bar{T}}$ is called budget function.

Definition 4.3.2 (Hedging Derivatives) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee. The set of fixed-points of $H^{\bar{G}}_{\bar{E}}$ with $\bar{E} \in [0, \infty)^K$ is denoted by $\mathcal{C}^{\bar{G}}$. More precisely,*

$$\mathcal{C}^{\bar{G}} \;:=\; \left\{ C \in \mathcal{V}^{\bar{T}} \;:\; \exists\, \bar{E} \in [0, \infty)^K \text{ with } C = H^{\bar{G}}_{\bar{E}}(C) \right\} \quad .$$

An element $C \in \mathcal{C}^{\bar{G}}$ is called hedging derivative.

The budget function derives its name from the fact that it gives the risk-neutral value of the future lock-in and the terminal excess at each lock-in time point. If $\pi \in \Pi^{\bar{G}}$ is a portfolio insurance strategy, then (4.2.9) implies $\{C^{\bar{G},\pi}_{\bar{T}_n}\}_{n=0}^{N} \in \mathcal{C}^{\bar{G}}$. On the other hand, given a family of random variables $C \in \mathcal{V}^{\bar{T}}$ satisfying (4.2.9), it is possible to construct a corresponding portfolio insurance strategy.

Lemma 4.3.3 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee, $\bar{E} \in [0, \infty)^K$ a terminal excess, and $C \in \mathcal{C}^{\bar{G}}$ a fixed-point of the budget function $H^{\bar{G}}_{\bar{E}}$, i.e. $C = H^{\bar{G}}_{\bar{E}}(C)$. Moreover, let $\pi \in \Pi$ be a replicating strategy for the corresponding terminal guaranteed amount, i.e.*

$$X^{\pi}(t) \;=\; P^{\bar{T}_N}(t)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{G^{\bar{G},C}_i + C_i\}_{i=0}^{n}\big) \;+\; \bar{E} \;\middle|\; \mathcal{F}(t) \right] \tag{4.3.2}$$

for $t \in \mathcal{T}$. Then,

$$G^{\bar{G},\pi}(\bar{T}_n) \;=\; G^{\bar{G},C}_n \quad \text{and} \quad C^{\bar{G},\pi}(\bar{T}_n) \;=\; C_n \quad ,$$

for $n = 0, \ldots, N$, $\bar{E}^{\bar{G},\pi} = \bar{E}$, and thus $\pi \in \Pi^{\bar{G}}$.

With Lem. 4.3.3, the problem of constructing portfolio insurance strategies for a contingent guarantee is equivalent to the problem of constructing fixed-points of the budget function: 'Every portfolio insurance strategy corresponds to a fixed-point and every fixed-point corresponds to at least one portfolio insurance strategy.'

For all $C = \{C_n\}_{n=0}^N \in \mathcal{C}^{\bar{G}}$, the random variables C_n are by definition $\mathcal{F}(\bar{T}_n)$-measurable for $n = 0, \dots, N$. Moreover, using the tower property of conditional expectations (Durrett, 2010, Theo. 5.1.6), the fixed-point condition $C = H_{\bar{E}}^{\bar{G}}(C)$, i.e.

$$C_n \;=\; P^{\bar{T}_N}(\bar{T}_n)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G_j^{\bar{G},C} + C_j\}_{j=0}^{i}\big) \;+\; \bar{E}\;\middle|\;\mathcal{F}(\bar{T}_n)\right]$$

for $n = 0, \dots, N$, can be rewritten as

$$C_n \;=\; P^{\bar{T}_N}(\bar{T}_n)\,\bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{n}\big) \;+\; P^{\bar{T}_N}(\bar{T}_n)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\frac{C_{n+1}}{P^{\bar{T}_N}(\bar{T}_{n+1})}\;\middle|\;\mathcal{F}(\bar{T}_n)\right] \tag{4.3.3a}$$

for $n = 1, \dots, N-1$,

$$C_0 \;=\; P^{\bar{T}_N}(0)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\frac{C_1}{P^{\bar{T}_N}(\bar{T}_1)}\,\right] \tag{4.3.3b}$$

and

$$C_N \;=\; \bar{L}_N\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{N}\big) \;+\; \bar{E} \quad . \tag{4.3.3c}$$

This representation yields another vivid economic interpretation: C is (the risk-neutral value process of) a derivative that pays the discounted lock-in $P^{\bar{T}_N}(\bar{T}_n)\,\bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{n}\big)$ at time $\bar{T}_n$, for $n = 1, \dots, N$, and additionally the corresponding terminal excess $\bar{E}$ at time $\bar{T}_N$. The constant C_0 is then the price of this derivative.

If the fund's initial asset allocation at time $t = 0$ consists of such a hedging derivative and $\bar{F}$ zero-coupon bonds $P^{\bar{T}_N}$ – which requires the initial capital $P^{\bar{T}_N}(0)\,\bar{F} + C_0$ – then the fund management can pursue a simple portfolio insurance strategy: At each of the lock-in time points $\bar{T}_1$ through $\bar{T}_{N-1}$ the payoffs $P^{\bar{T}_N}(\bar{T}_n)\,\bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{n}\big)$ of the hedging derivative are invested into the zero-coupon bond $P^{\bar{T}_N}$. Indeed, this investment policy results in the same NAV process as a replicating strategy for the terminal guaranteed amount (see Lem. 4.3.3).

Moreover, if π is a replicating strategy as in (4.3.2), then the process $\tilde{\pi}$ in (4.2.6) is simply a replicating strategy for the hedging derivative C. More precisely, in each of the intervals $(\bar{T}_n, \bar{T}_{n+1}]$, $\{\tilde{\pi}(t)\}_{\bar{T}_n < t \le \bar{T}_{n+1}}$ is a replicating strategy of C_{n+1}, such that

$$C^{\bar{G},\pi}(t) \;=\; X^{\tilde{\pi}}(t) \;=\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\frac{C_{n+1}}{P^{\bar{T}_N}(\bar{T}_{n+1})}\;\middle|\;\mathcal{F}(t)\right]$$

for $\bar{T}_n < t \le \bar{T}_{n+1}$, $n = 0, \dots, N-1$.

Definition 4.3.4 (Viability and Value of a Contingent Guarantee) *A contingent guarantee $\bar{G}$ with $\mathcal{C}^{\bar{G}} \neq \emptyset$ is called viable. Moreover, the value of $\bar{G}$ is defined as*

$$\Phi^{\bar{G}} \;:=\; P^{\bar{T}_N}(0)\,\bar{F} \;+\; \inf\Big\{\, C_0 \;:\; C \in \mathcal{C}^{\bar{G}} \,\Big\} \tag{4.3.4}$$

with the convention $\Phi^{\bar{G}} := \infty$ if $\bar{G}$ is not viable.

$\Phi^{\bar{G}}$ gives the minimum initial capital x_0 required to super-replicate the terminal guaranteed amount of $\bar{G}$ with a self-financing investment strategy (if the infimum in (4.3.4) is attained by a hedging derivative $C \in \mathcal{C}^{\bar{G}}$). Def. 4.3.4 is thereby consistent with the classical notion of the value of a contingent claim in mathematical finance.

The Martingale Method treats the terminal excess as a control variable, in the sense that the fund management chooses $\bar{E}$ first and then aims to construct a hedging derivative (or rather a portfolio insurance strategy) producing precisely this terminal excess. Note that for $\bar{E}^1, \bar{E}^2 \in [0,\infty)^K$ with corresponding hedging derivatives $C^1, C^2 \in \mathcal{C}^{\bar{G}}$, such that $C^1 = H^{\bar{G}}_{\bar{E}^1}(C^1)$ and $C^2 = H^{\bar{G}}_{\bar{E}^2}(C^2)$,

$$\bar{E}^1 \;\leq\; \bar{E}^2 \quad\not\Rightarrow\quad \sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C^1} + C_i^1\}_{i=0}^{n}\big) \;+\; \bar{E}^1 \;\leq\; \sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C^2} + C_i^2\}_{i=0}^{n}\big) \;+\; \bar{E}^2 \quad,$$

as the total lock-in produced by C^1 could exceed the total lock-in produced by C^2 depending on the structure of the lock-in mechanism.

Similarly, $\bar{E}^1 \leq \bar{E}^2$ does not necessarily imply $C_0^1 \leq C_0^2$, i.e. the hedging derivative C^1 might be more expensive even though the hedging derivative C_2 produces a larger terminal excess. Moreover, the infimum in (4.3.4) must not necessarily be attained by a hedging derivative producing a zero terminal excess, i.e. by $C \in \mathcal{C}^{\bar{G}}$ with $C = H_0^{\bar{G}}(C)$ – although numerical experiments suggests that this is the case in most practical applications.

From the fund management's perspective, there are two reasonable choices for the terminal excess:

- The fund management might aim to hedge a contingent guarantee $\bar{G}$ in a manner, such that the terminal guaranteed amount is perfectly replicated without any excess, i.e. $\bar{E} = 0$. In this case the cushion only needs to finance the future lock-in (see (4.2.5)), which should lead to a small required initial capital.

- Another option would be a call-like excess $\bar{E} = \max\{S^d(\bar{T}_N) - \bar{S},\, 0\}$, where $\bar{S} \in (0,\infty)$ and $d \in \{1,\ldots,D\}$. This will provide a controlled exposure to the risky asset classes for further upside potential.

The following sections present sufficient conditions for the viability of a contingent guarantee, all of which are based on an application of a fixed-point theorem. In fact, the discrete (and finite) probabilistic setup in Sec. 4.1 has been chosen precisely for this reason: To make full use of the rich mathematical toolbox of fixed-point theory. Only few of the results of this chapter can be transferred to the continuous financial market model of Ch. 6.

4.3.1 Continuous and Capped Lock-In Mechanisms

Definition 4.3.5 (Continuous Lock-In Mechanisms) *A lock-in mechanism $\bar{L} = \{\bar{L}_n\}_{n=1}^{N}$ with $\bar{L}_n$ continuous for $n = 1, \dots, N$ is called continuous.*

From a practical point of view, continuity is a rather weak condition on lock-in mechanisms: All lock-in mechanisms presented in Sec. 3.5 except for the digital threshold lock-in in (3.5.1) have this property. A straightforward approach to ensure the viability of a contingent guarantee with a continuous lock-in mechanism is to outright bound – or 'cap' – the lock-in functions by some constant (see Def. 4.3.6). Contingent guarantees with this type of lock-in mechanism can be hedged with any desired fixed guarantee $\bar{F}$ and for any targeted terminal excess $\bar{E}$. Mathematically, this result is due to Brouwer's theorem (Prop. 2.2.1).

Definition 4.3.6 (Capped Lock-In Mechanisms) *Let $\bar{L} = \{\bar{L}_n\}_{n=1}^{N}$ be a lock-in mechanism with $\bar{L}_n$ bounded for $n = 1, \dots, N$, i.e. there exists $\Lambda \in [0, \infty)$ with $\bar{L}_n(x) \leq \Lambda$ for all $x \in \mathbb{R}^{n+1}$ and $n = 1, \dots, N$. Then $\bar{L}$ is called capped.*

Proposition 4.3.7 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ continuous and capped, and $\bar{E} \in [0, \infty)^K$. Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H_{\bar{E}}^{\bar{G}}(C)$. In particular, $\bar{G}$ is viable.*

Lemma 4.3.8 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ continuous. If $\bar{G}$ is viable, then the infimum in (4.3.4) is attained, i.e. there exists $C \in \mathcal{C}^{\bar{G}}$ with $\Phi^{\bar{G}} = P^{\bar{T}_N}(0)\, \bar{F} + C_0$.*

Example 4.3.9 (Capped Take-Profit Lock-In) *Let $\bar{T}$ be a set of lock-in time points, $\bar{F} \in [0, \infty)$ a fixed guaranteed amount, and consider the lock-in mechanism $\bar{L}$ given by*

$$\bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;=\; \min\big\{\, \alpha \left[X^\pi(\bar{T}_n) - X^\pi(\bar{T}_{n-1})\right]^+,\; \beta\, \bar{F} \,\big\}$$

for all $n = 1, \dots, N$, where $\alpha \in [0, 1]$ and $\beta \in [0, \infty)$. Then the lock-in at time $\bar{T}_n$ is a fraction of the investment gain over the previous period, but not more than a fixed fraction of the initial *guaranteed amount ($\Lambda = \beta\,\bar{F}$ in Def. 4.3.6). Alternatively, one could set*

$$\bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^n\big) \;=\; \min\Big\{\alpha\left[X^\pi(\bar{T}_n) - X^\pi(\bar{T}_{n-1})\right]^+,\; \underbrace{\beta\Big(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{X^\pi(\bar{T}_j)\}_{j=0}^i\big)\Big)}_{\le\,\beta\,(1+\beta)^{n-1}\,\bar{F}}\Big\}$$

with the interpretation that the lock-in does not exceed a fixed fraction of the currently guaranteed amount ($\Lambda = \beta\,(1+\beta)^{N-1}\,\bar{F}$ in Def. 4.3.6).

4.3.2 Monotone and Capped Lock-In Mechanisms

Definition 4.3.10 (Monotone Lock-In Mechanisms) *Let $\bar{L} = \{\bar{L}_n\}_{n=1}^N$ be a lock-in mechanism. If $\bar{L}_n$ is monotone (Def. 2.3.1) for $n = 1, \ldots, N$, i.e. $\bar{L}_n(x) \le \bar{L}_n(y)$ for all $x, y \in \mathbb{R}^{n+1}$ with $x \le y$, then $\bar{L}$ is called monotone.*

In contrast to continuity, monotonicity is a rather exotic property: Out of the examples in Sec. 3.5, only the threshold lock-in mechanisms in Ex. 3.5.1 are monotone. Again, the viability of a contingent guarantee with a monotone lock-in mechanism can be assured by capping the lock-in mechanism. Moreover, contingent guarantees with such a lock-in mechanism can be hedged with any desired fixed guarantee $\bar{F}$ and for any targeted terminal excess $\bar{E}$. In particular, this implies that the digital threshold mechanism in (3.5.1) is viable for any $\bar{X}, \Lambda \in [0, \infty)$. The mathematical basis for this result is given by Tarski's theorem (Prop. 2.3.2).

Proposition 4.3.11 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ monotone and capped, and $\bar{E} \in [0, \infty)^K$. Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H^{\bar{G}}_{\bar{E}}(C)$. In particular, $\bar{G}$ is viable.*

By Kleene's theorem (Prop. 2.3.3), if the lock-in mechanism is additionally continuous, a hedging derivative $C \in \mathcal{C}^{\bar{G}}$ corresponding to a given terminal excess $\bar{E} \in [0, \infty)^K$ can be constructed from a fixed-point iteration.

Proposition 4.3.12 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ continuous, monotone, and capped. Then, for all $\bar{E} \in [0, \infty)^K$, the sequence $\{{}_{(u)}C\}_{u \in \mathbb{N}_0} \subset \mathcal{V}^{\bar{T}}$ defined by ${}_{(u+1)}C := H^{\bar{G}}_{\bar{E}}({}_{(u)}C)$, ${}_{(0)}C = 0$, converges to a fixed-point $C \in \mathcal{V}^{\bar{T}}$ with $C = H^{\bar{G}}_{\bar{E}}(C)$. This hedging derivative is then the cheapest among all derivatives producing the terminal excess $\bar{E}$, i.e. $C_0 \le \tilde{C}_0$ for all $\tilde{C} \in \mathcal{C}^{\bar{G}}$ with $\tilde{C} = H^{\bar{G}}_{\bar{E}}(\tilde{C})$.*

For a contingent guarantee with a monotone and capped lock-in mechanism, the value $\Phi^{\bar{G}}$ is attained by a hedging derivative $C \in \mathcal{C}^{\bar{G}}$ with $C = H_0^{\bar{G}}(C)$, i.e. a hedging derivative that produces no terminal excess.

Lemma 4.3.13 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ monotone and capped. Then there exists $C \in \mathcal{C}^{\bar{G}}$ with $\Phi^{\bar{G}} = P^{\bar{T}_N}(0)\,\bar{F} + C_0$ and $C = H_0^{\bar{G}}(C)$.*

The properties of continuity and monotonicity can be combined in a useful hedging result that is applicable to continuous lock-in mechanisms, which are not bound by a constant as in Def. 4.3.6, but by another continuous and monotone lock-in mechanism (see Ex. 4.3.20).

Lemma 4.3.14 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ of the form*

$$\bar{L}_n(x) \;=\; \min\left\{\bar{L}_n^1(x),\, \bar{L}_n^2(x)\right\}$$

for all $x \in \mathbb{R}^{n+1}$ and $n = 1, \ldots, N$, where the two lock-in mechanisms $\bar{L}^1 = \{\bar{L}_n^1\}_{n=1}^N$ and $\bar{L}^2 = \{\bar{L}_n^2\}_{n=1}^N$ are continuous and $\bar{L}^2$ is additionally monotone. Moreover, let $\bar{E} \in [0,\infty)^K$ and assume that there exists $C^2 \in \mathcal{V}^{\bar{T}}$ with $C^2 = H_{\bar{E}}^{(\bar{T},\bar{F},\bar{L}^2)}(C^2)$. Then there also exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H_{\bar{E}}^{\bar{G}}(C)$ and $C_n \le C_n^2$ for $n = 0, \ldots, N$. In particular, $\bar{G}$ is viable.

4.3.3 Contracting Lock-In Mechanisms

Definition 4.3.15 (Contracting Lock-In Mechanisms) *Let $\bar{T}$ be a set of lock-in time points and $\bar{L}$ a corresponding lock-in mechanism. If $H_0^{(\bar{T},0,\bar{L})}$ is a contraction (Def. 2.1.1), i.e. there exists $\Lambda \in [0,1)$ and a norm $\|\cdot\|$ on $\mathbb{R}^{K\times(N+1)}$ with*

$$\left\| \, H_0^{(\bar{T},0,\bar{L})}(C^1) \;-\; H_0^{(\bar{T},0,\bar{L})}(C^2) \, \right\| \;\le\; \Lambda \left\| \, C^1 \;-\; C^2 \, \right\| \tag{4.3.5}$$

for all $C^1, C^2 \in \mathcal{V}^{\bar{T}}$, then the pair $(\bar{T}, \bar{L})$ is called contracting.

Whereas the results of Sec. 4.3.1 and 4.3.2 place conditions on the lock-in mechanism only (continuity (Def. 4.3.5), boundedness (Def. 4.3.6), monotonicity (Def. 4.3.10)), the contraction property above additionally involves the corresponding lock-in time points $\bar{T}$. The reason for this is straightforward: The contraction property in (4.3.5) will generally depend on both the lock-in mechanism and the distribution of the bond price $P^{\bar{T}_N}$ at the lock-in time points. In other words, a lock-in mechanism $\bar{L}$ that results in a contracting budget function for a particular choice $\bar{T}$ might fail to do so for a different choice of lock-in time points.

With Banach's theorem (Prop. 2.1.2), a contingent guarantee with a contracting pair $(\bar{T}, \bar{L})$ can be hedged with any desired fixed guarantee $\bar{F}$ and for any targeted terminal excess $\bar{E}$.

Proposition 4.3.16 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $(\bar{T}, \bar{L})$ contracting, and $\bar{E} \in [0,\infty)^K$. Then there exists a unique $C \in \mathcal{V}^{\bar{T}}$ with $C = H_{\bar{E}}^{\bar{G}}(C)$ and the sequence $\{{}_{(u)}C\}_{u \in \mathbb{N}_0} \subset \mathcal{V}^{\bar{T}}$ defined by ${}_{(u+1)}C := H_{\bar{E}}^{\bar{G}}({}_{(u)}C)$ converges to C for any starting value ${}_{(0)}C \in \mathcal{V}^{\bar{T}}$. In particular, $\bar{G}$ is viable.*

With Prop. 4.3.16, the hedging derivative $C \in \mathcal{C}^{\bar{G}}$ corresponding to a given terminal excess $\bar{E}$ is unique and can be constructed from a fixed-point iteration.

In many practical applications the lock-in mechanism $\bar{L}$ is given by a family of Lipschitz continuous functions (see Ex. 3.5.2). Intuitively, the budget function will be contracting, if the corresponding Lipschitz constants are small enough, i.e. if the lock-in $\bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^n\big)$ does not grow too fast with the (history of the) NAV $\{X^\pi(\bar{T}_i)\}_{i=0}^n$. The sufficient condition (4.3.6) below is based on precisely this rationale, but tends to be too strong for practical purposes.

Lemma 4.3.17 *Let $\bar{T}$ be a set of lock-in time points and $\bar{L} = \{\bar{L}_n\}_{n=1}^N$ a lock-in mechanism such that, for $n = 1, \ldots, N$, there exists a family $\{\Lambda_{n,i}\}_{i=0}^n \in [0,\infty)^{n+1}$ with*

$$\left|\, \bar{L}_n\big(\{x_i\}_{i=0}^n\big) - \bar{L}_n\big(\{y_i\}_{i=0}^n\big) \,\right| \;\le\; \sum_{i=0}^n \Lambda_{n,i} \,|\, x_i - y_i \,|$$

for $\{x_i\}_{i=0}^n, \{y_i\}_{i=0}^n \in [0,\infty)^{n+1}$. Furthermore, let $\{Z_n\}_{n=1}^N \in [0,\infty)^N$ be defined iteratively by

$$Z_n \;:=\; \sum_{i=0}^n \Lambda_{n,i} \left(1 \;+\; \hat{P}_i \sum_{j=1}^{i-1} Z_j \right)$$

for $n = 1, \ldots, N$ with $\hat{P}_n := \max_{k=1}^K P^{\bar{T}_N}(\bar{T}_n; \omega_k)$. If

$$\max_{n=0}^N \; \hat{P}_n \left(\sum_{i=n\vee 1}^N Z_i \right) \;<\; 1 \quad , \tag{4.3.6}$$

then $(\bar{T}, \bar{L})$ is contracting. The norm, in which $H_0^{(\bar{T},0,\bar{L})}$ is contracting, is given by

$$\|\, Y \,\|_{\max} := \max_{n=0}^N \max_{k=1}^K \big|\, Y_n(\omega_k) \,\big| \tag{4.3.7}$$

for $Y = \{Y_n\}_{n=0}^N \in \mathbb{R}^{K\times(N+1)}$ and the corresponding Lipschitz constant is given in (4.3.6).

Contracting lock-in mechanisms have the convenient property that the hedging derivative is a continuous function of the terminal excess. In fact, by the uniqueness property of Prop. 4.3.16, the terminal excess $\bar{E}$ can be used to 'parameterize' the set of hedging derivatives.

Lemma 4.3.18 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $(\bar{T}, \bar{L})$ contracting. Moreover, for $\bar{E} \in [0,\infty)^K$, let $C^{\bar{E}} \in \mathcal{C}^{\bar{G}}$ be the (unique) hedging derivative with $C^{\bar{E}} = H^{\bar{G}}_{\bar{E}}(C^{\bar{E}})$. Then the map $\bar{E} \mapsto C^{\bar{E}}$ is Lipschitz continuous.*

Corollary 4.3.19 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $(\bar{T}, \bar{L})$ contracting. Then there exists $C \in \mathcal{C}^{\bar{G}}$ with $\Phi^{\bar{G}} = P^{\bar{T}_N}(0)\,\bar{F} + C_0$.*

The continuity of $\bar{E} \mapsto C^{\bar{E}}$ implies that, for all $x_0 \geq \Phi^{\bar{G}}$, there exists $\bar{E} \in [0,\infty)^K$ and a corresponding hedging derivative $C^{\bar{E}} \in \mathcal{C}^{\bar{G}}$ with $x_0 = P^{\bar{T}_N}(0)\,\bar{F} + C^{\bar{E}}_0$, because $\bar{E} \mapsto C^{\bar{E}}_0$ is unbounded above. In economic terms, the fund management will always be able to hedge a contingent guarantee with a contracting lock-in mechanism as long as the NAV is equal to or above the lower threshold $\Phi^{\bar{G}}$. This is not necessarily the case for non-contracting lock-in mechanisms, which then leads to a peculiar problem: 'Too much' capital can put the fund at risk of falling short of the terminal guaranteed amount, because an increase in the NAV might result in an even larger increase in the (value of the) lock-in.

Example 4.3.20 (Contracting Take-Profit Lock-In) *Instead of bounding the lock-in by a constant as in Ex. 4.3.9, one could also bound it by a (continuous and monotone) contracting lock-in mechanism. For example, let $\bar{T}$ be a set of lock-in time points, $\alpha \in [0,1]$, $0 \leq \beta \ll 1$, and consider the lock-in mechanism $\bar{L}$ given by*

$$\bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^n\big) \;=\; \min\big\{\, \alpha\,\big[X^\pi(\bar{T}_n) - X^\pi(\bar{T}_{n-1})\big]^+,\; \beta\, X^\pi(\bar{T}_n)\,\big\}$$

for all $n = 1, \ldots, N$. Then the lock-in at time $\bar{T}_n$ is a fraction of the investment gain over the previous period, but not more than a fixed fraction of the current NAV. If β is chosen sufficiently small, then the resulting contingent guarantee will be viable by Lem. 4.3.14.

4.3.4 The Sufficient Conditions of Kleinow (2009)

Kleinow and Willder (2007) and Kleinow (2009) consider the problem of hedging a generalized participating life insurance policy with an implicit lock-in mechanism of the form

$$\bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^n\big) \;=\; \left(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{X^\pi(\bar{T}_j)\}_{j=0}^i\big)\right) W\!\left(\frac{X^\pi(\bar{T}_n)}{X^\pi(\bar{T}_{n-1})}\right) \qquad (4.3.8)$$

for $n = 1, \ldots, N$ (see also Ex. 3.5.5), where $W : [0, \infty) \to [0, \infty)$ satisfies the following technical conditions:

(A) W is continuous;

(B) W is monotone increasing, i.e. $W(y_1) \leq W(y_2)$ for $y_1 \leq y_2$;

(C) $\frac{y}{W(y)+1}$ is strictly monotone increasing, i.e. $\frac{y_1}{W(y_1)+1} < \frac{y_2}{W(y_2)+1}$ for $y_1 < y_2$;

(D) $\lim_{y\to\infty} \frac{y}{W(y)+1} = \infty$.

In order for the lock-in mechanism above to be well-defined, the NAV X must necessarily stay positive at the lock-in time points, i.e. $X^\pi(\bar{T}_n) > 0$ for $n = 0, \ldots, N-1$. This can be achieved by the restriction $\bar{F} > 0$ (a similar assumption is made by Kleinow (2009)), such that $X^\pi(t) \geq G^{\bar{G},\pi}(t) > 0$ for all $t \in \mathcal{T}$. Under the conditions above, Kleinow (2009) uses repeated backward induction to construct an NAV process X^π that (almost-surely) super-replicates its terminal guaranteed amount.

The guaranteed amount (3.1.2) at time $\bar{T}_n$ is given by

$$\bar{F} \prod_{i=1}^{n} \left[W\left(\frac{X(\bar{T}_n)}{X(\bar{T}_{n-1})} \right) + 1 \right]$$

for $n = 0, \ldots, N$. Handling this 'multiplicative' structure with the 'additive' definition of lock-in used in this thesis is somewhat cumbersome. In particular, for these types of lock-in mechanisms, it is more convenient to regard the targeted terminal excess not as the difference between the terminal NAV $X(\bar{T}_N)$ and the terminal guaranteed amount, but rather as the 'exceedance ratio'

$$\frac{X(\bar{T}_N)}{\bar{F} + \sum_{n=1}^{N} \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big)} \geq 1 \quad ,$$

as is implicitly also done by Kleinow (2009). The following proposition states the hedging result of Kleinow (2009) for the case where this ratio is equal to 1 (i.e. $\bar{E} = 0$).

Proposition 4.3.21 (Kleinow (2009, Lem. 1 and Theo. 2)) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{F} > 0$, $\bar{L}$ as in* (4.3.8)*, and W such that conditions (A) and (D) above are satisfied. Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H_0^{\bar{G}}(C)$. In particular, $\bar{G}$ is viable.*

4.3.5 Hedging with Interest Rate Derivatives

So far, a hedging derivative $C \in \mathcal{C}^{\bar{G}}$ was assumed to be measurable with respect to the market filtration $\mathbb{F}$. This allows for arbitrarily complex payoffs that might depend on the complete variety of assets available in the market. A reasonable question to ask is, whether there exist particularly 'simple' derivatives. For example, the fund management might – for various practical reasons – prefer interest rate derivatives over more complex multi-asset instruments to hedge a contingent guarantee $\bar{G}$. Luckily, the Martingale Method allows to control the complexity of the resulting hedging derivatives by means of the filtration in (4.3.1).

This section examines conditions, under which the set $\mathcal{C}^{\bar{G}}$ contains hedging derivatives that are dependent on interest rates, or, more precisely, whose payoffs are measurable functions of the (history of the) bond price $P^{\bar{T}_N}$.

For a set of lock-in time points $\bar{T} = \{\bar{T}_i\}_{i=0}^N$, let $\{\mathcal{P}_n^{\bar{T}}\}_{n=0}^N$ be the filtration defined by $\mathcal{P}_n^{\bar{T}} := \sigma\big(\{P^{\bar{T}_N}(\bar{T}_i)\}_{i=0}^n\big) \subseteq \mathcal{F}(\bar{T}_n)$ for $n = 0, \dots, N$, i.e. $\mathcal{P}_n^{\bar{T}}$ is the σ-algebra generated by the path of the bond price $P^{\bar{T}_N}$ over the lock-in time points $\{\bar{T}_i\}_{i=0}^n$. Note that $\mathcal{P}_N^{\bar{T}} = \mathcal{P}_{N-1}^{\bar{T}}$, because $\sigma\big(P^{\bar{T}_N}(\bar{T}_N)\big) = \{\,\Omega, \emptyset\,\}$.

Furthermore, for a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ and a terminal excess $\bar{E} \in [0,\infty)^K$, let $H_{\bar{E},\mathcal{P}}^{\bar{G}} : \mathcal{V}^{\bar{T}} \to \mathcal{V}^{\bar{T}}$ be defined by

$$H_{\bar{E},\mathcal{P}}^{\bar{G}}(C) \;:=\; \left\{\, P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\big\{G_j^{\bar{G},C} + C_j\big\}_{j=0}^{i}\big) \;+\; \bar{E} \,\middle|\, \mathcal{P}_n^{\bar{T}} \,\right] \right\}_{n=0}^{N} \tag{4.3.9}$$

for $C \in \mathcal{V}^{\bar{T}}$, which is the budget function of Def. 4.3.1, but with the market filtration $\mathbb{F}$ replaced by the bond price filtration $\mathcal{P}^{\bar{T}}$.

Fixed-points of $H_{\bar{E},\mathcal{P}}^{\bar{G}}$, i.e. $C \in \mathcal{V}^{\bar{T}}$ with $C = H_{\bar{E},\mathcal{P}}^{\bar{G}}(C)$, have the convenient property that the corresponding random variables C_n are $\mathcal{P}_n^{\bar{T}}$-measurable for $n = 0, \dots, N$, i.e. they are measurable functions of the history of the bond price $P^{\bar{T}_N}$. Therefore, such fixed-points might be interpreted as (path-dependent) interest rate derivatives. This interpretation however, is only valid if $C \in \mathcal{C}^{\bar{G}}$, i.e. if C is indeed a proper hedging derivative.

Lemma 4.3.22 *Let $\bar{G}$ be a contingent guarantee, $\bar{E} \in [0,\infty)^K$ and $C \in \mathcal{V}^{\bar{T}}$ a fixed-point of $H_{\bar{E},\mathcal{P}}^{\bar{G}}$. If the bond price $P^{\bar{T}_N}$ is Markov (w.r.t. $\mathbb{F}$, see Def. 4.D.2), then C is also a fixed-point of $H_{\bar{E}}^{\bar{G}}$, i.e. $C \in \mathcal{C}^{\bar{G}}$. In particular, C is a path-dependent interest rate derivative with terminal excess $\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\big[\,\bar{E}\,\big|\,\mathcal{P}_N^{\bar{T}}\,\big]$.*

The result above can be generalized to models, where the bond price $P^{\bar{T}_N}$ is not Markovian itself, but is a component of a higher dimensional Markov process, say $\{Z(t)\}_{t\in\mathcal{T}}$. The correct choice of the family $\{\mathcal{P}_n^{\bar{T}}\}_{n=0}^N$ is then given by $\mathcal{P}_n^{\bar{T}} = \sigma\big(\{Z(\bar{T}_i)\}_{i=0}^n\big)$ for $n = 0, \ldots, N$.

Furthermore, Lem. 4.3.22 has an important ramification for market models with Markovian bond prices (which is the case for most common interest rate models): $H^{\bar{G}}_{\bar{E},\mathcal{P}}$ requires the evaluation of conditional expectations on a much coarser filtration than $H^{\bar{G}}_{\bar{E}}$, which significantly reduces the computational burden for the numerical construction of hedging derivatives (see Sec. 4.4 and App. 4.B).

The remaining question is under what conditions there exist fixed-points of $H^{\bar{G}}_{\bar{E},\mathcal{P}}$, i.e. under what conditions it is possible to hedge a contingent guarantee $\bar{G}$ using a (path-dependent) interest rate derivative. The results of Sec. 4.3.1, 4.3.2, and 4.3.4 can be transferred to the setting above without further ado. Indeed, the proofs of Prop. 4.3.7, 4.3.11, and 4.3.21 do not rely on the choice of filtration, and one thus immediately obtains the following results, whose proofs are omitted.

Lemma 4.3.23 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ continuous and capped, and $\bar{E} \in [0,\infty)^K$. Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H^{\bar{G}}_{\bar{E},\mathcal{P}}(C)$.*

Lemma 4.3.24 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ monotone and capped, and $\bar{E} \in [0,\infty)^K$. Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H^{\bar{G}}_{\bar{E},\mathcal{P}}(C)$.*

Lemma 4.3.25 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{F} > 0$, $\bar{L}$ as in* (4.3.8)*, and W such that conditions (A) and (D) of Sec. 4.3.4 are satisfied. Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H^{\bar{G}}_{0,\mathcal{P}}(C)$.*

On the other hand, the contraction property of Def. 4.3.15 depends on the choice of the filtration: For a given norm $\|\cdot\|$ on $\mathbb{R}^{K\times(N+1)}$, a set of lock-in time points $\bar{T}$, and a lock-in mechanism $\bar{L}$, the budget function $H_0^{(\bar{T},0,\bar{L})}$ might be a contraction, while $H_{0,\mathcal{P}}^{(\bar{T},0,\bar{L})}$ is not (and vice versa). However, there exists a broad class of norms (see Def. 4.3.26 below), under which the contraction property of $H_0^{(\bar{T},0,\bar{L})}$ is indeed inherited by $H_{0,\mathcal{P}}^{(\bar{T},0,\bar{L})}$.

Definition 4.3.26 (Contraction Norms) *Let the p-norm of a real-valued random variable $Y : \Omega \to \mathbb{R}$ be defined by*

$$\|Y\|_p := \begin{cases} \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\big[\,|Y|^p\,\big]^{\frac{1}{p}} & , \text{ for } p \in [1,\infty), \\ \max\limits_{k=1}^{K} |\,Y(\omega_k)\,| & , \text{ for } p = \infty. \end{cases}$$

Moreover, for a set of lock-in time points $\bar{T}$, let $\mathcal{N}^{\bar{T}}$ be the class of norms on $\mathbb{R}^{K\times(N+1)}$ which are *of the form*

$$\| Y \| \; = \; \big\| \left(\, \|Y_0\|_{p_0}, \, \dots, \, \|Y_N\|_{p_N} \, \right)^\top \big\|_\star \tag{4.3.10}$$

for $Y = \{Y_n\}_{n=0}^N \in \mathbb{R}^{K \times (N+1)}$*, where* $\{p_n\}_{n=0}^N \in [1,\infty]^{N+1}$ *and* $\| \cdot \|_\star$ *is any monotone norm on* $\mathbb{R}^{N+1}$*, i.e. a norm with* $\|x\|_\star \leq \|y\|_\star$ *for all* $x, y \in [0,\infty)^{N+1}$ *with* $x \leq y$*.*

Lemma 4.3.27 *Let* $\bar{T}$ *be a set of lock-in time points. An element* $\| \cdot \| \in \mathcal{N}^{\bar{T}}$ *is a norm on* $\mathbb{R}^{K \times (N+1)}$*. Moreover,*

$$\Big\| \left\{ \, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\, Y_n \, \middle| \, \mathcal{P}_n^{\bar{T}} \, \right] \, \right\}_{n=0}^N \Big\| \; \leq \; \Big\| \left\{ \, Y_n \, \right\}_{n=0}^N \Big\|$$

for all $\| \cdot \| \in \mathcal{N}^{\bar{T}}$ *and* $Y = \{Y_n\}_{n=0}^N \in \mathbb{R}^{K \times (N+1)}$*. In particular,*

$$\Big\| \; H_{0,\mathcal{P}}^{(\bar{T},0,\bar{L})}(C^1) \; - \; H_{0,\mathcal{P}}^{(\bar{T},0,\bar{L})}(C^2) \; \Big\| \; \leq \; \Big\| \; H_0^{(\bar{T},0,\bar{L})}(C^1) \; - \; H_0^{(\bar{T},0,\bar{L})}(C^2) \; \Big\|$$

for all $\| \cdot \| \in \mathcal{N}^{\bar{T}}$*, all* $C^1, C^2 \in \mathcal{V}^{\bar{T}}$*, and all lock-in mechanisms* $\bar{L}$*.*

With Lem. 4.3.27,

$$H_0^{(\bar{T},0,\bar{L})} \text{ is a contraction} \quad \Rightarrow \quad H_{\bar{E},\mathcal{P}}^{(\bar{T},0,\bar{L})} \text{ is a contraction}$$

for the normed vector spaces $(\mathbb{R}^{K \times (N+1)}, \| \cdot \|)$, where $\| \cdot \| \in \mathcal{N}^{\bar{T}}$. This inheritance relationship is particularly important for interest rate models with Markovian bond prices: As every fixed-point of $H_{\bar{E},\mathcal{P}}^{\bar{G}}$ is also a fixed-point of $H_{\bar{E}}^{\bar{G}}$ (see Lem. 4.3.22), the uniqueness of fixed-points for a contracting pair $(\bar{T}, \bar{L})$ implies that the hedging derivative for a $\mathcal{P}_N^{\bar{T}}$-measurable terminal excess $\bar{E} \in [0,\infty)^K$ is then always a path-dependent interest rate derivative. This holds in particular for the derivative $C^0 \in \mathcal{C}^{\bar{G}}$ corresponding to the constant terminal excess $\bar{E} = 0$. Numerical experiments suggest that C^0 satisfies $\Phi^{\bar{G}} = P^{\bar{T}_N}(0)\,\bar{F} + C_0^0$ for most contracting lock-in mechanisms.

The results of this section strongly suggest that contingent guarantees, including traditional participating life insurance liabilities (Ex. 3.5.4 and 3.5.5), should be regarded as (path-dependent) interest rate derivatives. This stands in sharp contrast to the prescribed exogenous valuation approach under the Solvency II regulatory regime, which involves the simulation of complex multi-asset investment strategies (see Sec. 3.4).

Moreover, regulatory frameworks often define standard stress scenarios to assess the solvency of an insurance company. Solvency II for example places considerable emphasis on stress scenarios for risky asset classes such as defaultable bonds, stocks, and real estate, but defines only two scenarios related to (riskless) interest rates: an upward and a downward shift of the riskless term structure (European Union, 2015). Other risk factors related to interest rates, such as interest rate volatility, should be considered as the main threat to an insurance company's ability to meet its liabilities.

4.3.6 The Portfolio Fixed-Point Problem

The construction of portfolio insurance strategies for a contingent guarantee $\bar{G}$ using the Martingale Method is based on a two-step procedure: First, a hedging derivative is constructed by solving the fixed-point problem $C = H_{\bar{E}}^{\bar{G}}(C)$, for some $\bar{E} \in [0,\infty)^K$. Second, a portfolio insurance strategy is built by replicating the corresponding terminal guaranteed amount (Lem. 4.3.3). As shown below, both construction steps can actually be combined into a single fixed-point problem.

Let the (set-valued) map $\hat{\pi} : [0,\infty)^K \to 2^\Pi$ be such that, for a payoff $Y \in [0,\infty)^K$ at time t_M, $\hat{\pi}(Y)$ is the non-empty set of replicating strategies of Y, i.e. the set of investment strategies $\pi \in \Pi$ with

$$X^\pi(t) \;=\; B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\frac{Y}{B(t_M)}\,\middle|\,\mathcal{F}(t)\right] \tag{4.3.11}$$

for all $t \in \mathcal{T}$. While there might be multiple replicating strategies for a given payoff Y, its value process is uniquely determined by the risk-neutral valuation formula in (4.3.11), such that X^π is constant in $\pi \in \hat{\pi}(Y)$. If the replicating strategy for a given payoff Y is unique, then $\hat{\pi}(Y)$ is a singleton set.

Moreover, for a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ and terminal excess $\bar{E} \in [0,\infty)^K$, define the (set-valued) *replication function* $R_{\bar{E}}^{\bar{G}} : \Pi \to 2^\Pi$ by

$$R_{\bar{E}}^{\bar{G}}(\pi) \;:=\; \hat{\pi}\left(\bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}\right)$$

for $\pi \in \Pi$. In other, words $R_{\bar{E}}^{\bar{G}}$ maps an investment strategy $\pi \in \Pi$ to the set of investment strategies replicating the corresponding terminal guaranteed amount.

For a portfolio insurance strategy $\pi \in \Pi^{\bar{G}}$,

$$X^\pi(\bar{T}_N) \;=\; \bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi}$$

with $\bar{E}^{\bar{G},\pi} \geq 0$ (by definition), such that π is a replicating strategy for the payoff on the right-hand side. Thus, π is a fixed-point of the replication function, in the sense that $\pi \in R_{\bar{E}^{\bar{G},\pi}}^{\bar{G}}(\pi)$. On the other hand, for $\bar{E} \in [0,\infty)^K$, any fixed-point $\pi \in \Pi$ with $\pi \in R_{\bar{E}}^{\bar{G}}(\pi)$ is super-replicating its 'self-induced' terminal guaranteed amount with $\bar{E}^{\bar{G},\pi} = \bar{E}$, such that $\pi \in \Pi^{\bar{G}}$. Altogether, the problem of constructing portfolio insurance strategies is equivalent to the construction of fixed-points of the replication function.

Assuming that there exists some selection function sel : $2^{\Pi} \to \Pi$, which chooses an investment strategy $\mathrm{sel}(A) \in A$ out of a given subset $A \subseteq \Pi$, $A \neq \emptyset$, according to some selection criteria, one might be inclined to run a fixed-point iteration using the replication function. More precisely, one could consider the sequence of investment strategies $\{{}_{(u)}\pi\}_{u \in \mathbb{N}_0}$ defined by

$$ {}_{(u+1)}\pi \;=\; \mathrm{sel}\big(R_{\bar{E}}^{\bar{G}}({}_{(u)}\pi)\big) \tag{4.3.12}$$

for $u \in \mathbb{N}_0$ and study its behavior for $u \to \infty$.

Intriguingly, the exogenous approach presented in Sec. 3.4 can be regarded as the first iteration of such an iterative procedure. Starting with a given investment strategy ${}_{(0)}\pi \in \Pi$, the exogenous approach uses Monte Carlo techniques to simulate and value the terminal guaranteed amount

$$\bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^{{}_{(0)}\pi}(\bar{T}_i)\}_{i=0}^{n}\big)$$

(in this setting the terminal excess $\bar{E}$ is usually set to zero). If one were to construct a replicating strategy ${}_{(1)}\pi \in R_0^{\bar{G}}({}_{(0)}\pi)$ for the payoff above and then reapply the exogenous approach with ${}_{(1)}\pi$, this would result precisely in the iteration scheme described in (4.3.12). Of course, a rigorous analysis of the convergence behavior of this iteration will require the proper embedding of the set Π into a surrounding Banach space and technical assumptions on the replication function (e.g. the contraction property in Def. 2.1.1).

There exists a rich mathematical literature concerned with fixed-point problems involving set-valued maps, however the corresponding results tend to be non-constructive (see Ch. 2). Moreover, since there is no straightforward relationship between the characteristics of a contingent guarantee $\bar{G}$ and the properties of replicating strategies for its terminal guaranteed amount, it will generally be quite difficult to investigate the structure and properties of the replication function.

4.4 Numerical Case Study

Banach's theorem (Prop. 2.1.2) and Kleene's theorem (Prop. 2.3.3) are among the few constructive results within the realm of fixed-point theory. In the context of hedging contingent guarantees, they give conditions on $\bar{G}$ under which a hedging derivative can be constructed by a fixed-point iteration using the budget function (see Prop. 4.3.12 and 4.3.16).

Implementing this fixed-point iteration requires the evaluation of conditional expectations, which can usually be done efficiently in discrete financial market models with Markovian asset prices. In fact, under these preconditions, the results of Sec. 4.3.5, specifically Lem. 4.3.22, allow for a significant reduction in complexity: It might be possible to construct an interest rate hedging derivative $C \in \mathcal{C}^{\bar{G}}$, where C_n is a function of the bond price $P^{\bar{T}_N}$ at the lock-in time steps $\{\bar{T}_i\}_{i=0}^n$. In particular, C_n does not depend on the value of $P^{\bar{T}_N}$ at intermediate time steps and it suffices to consider a financial market with $D = 0$, i.e. the asset universe consists of the bank account and the riskless zero-coupon bonds only.

4.4.1 Implementation

For the numerical case study, the fixed-point iteration

$$ {}_{(u+1)}C \;=\; H^{\bar{G}}_{\bar{E},\mathcal{P}}\big({}_{(u)}C\,\big)\,, \quad u \in \mathbb{N}_0\,, \tag{4.4.1} $$

is implemented using the Hull–White trinomial tree (see App. 4.A, Hull and White (1993)). This is a discrete-time and -state approximation of the continuous Hull–White model, in which interest rates are normally distributed and mean-reverting (see App. 6.A, Hull and White (1990)). The trinomial tree has two parameters $\kappa, \sigma \in (0, \infty)$, where κ is the *speed of mean-reversion* and σ the *volatility* of the spot rate r in Sec. 4.1. Moreover, the tree reproduces a given initial term structure $\{P^{t_m}(0)\}_{m=1}^M$.

As discussed below, the construction of the forward pricing measure $\mathbb{Q}^{\bar{T}_N}$ in lattice models is computationally challenging (see also Pliska, 1997, p. 226). Therefore, using (4.3.3) and a change of numéraire, the fixed-point iteration in (4.4.1) is rewritten as follows:

$$ \begin{aligned} {}_{(u+1)}C_n \;&= P^{\bar{T}_N}(\bar{T}_n)\,\bar{L}_n\big(\{G_i^{\bar{G},{}_{(u)}C} + {}_{(u)}C_i\}_{i=0}^n\big) \\ &\quad +\; \mathbb{E}_{\mathbb{Q}}\left[\, \frac{B(\bar{T}_n)}{B(\bar{T}_{n+1})}\;{}_{(u+1)}C_{n+1} \,\middle|\, \mathcal{P}_n^{\bar{T}} \right] \end{aligned} \tag{4.4.2a} $$

for $n = 1, \ldots, N-1$,

$$ {}_{(u+1)}C_0 \;=\; \mathbb{E}_{\mathbb{Q}}\left[\, \frac{1}{B(\bar{T}_1)}\;{}_{(u+1)}C_1 \right] \tag{4.4.2b} $$

and

$$ {}_{(u+1)}C_N \;=\; \bar{L}_N\big(\{G_i^{\bar{G},{}_{(u)}C} + {}_{(u)}C_i\}_{i=0}^N\big) \;+\; \bar{E} \tag{4.4.2c} $$

for $u \in \mathbb{N}_0$. Note that, for each iteration cycle u, the random variables ${}_{(u)}C_n$ must be updated backwards in time starting at $\bar{T}_N$.

The complete fixed-point iteration for the Hull–White trinomial tree is specified in App. 4.B (Alg. 1). The algorithm requires an initial guess ${}_{(0)}C$ and runs until one of the following two stopping criteria is satisfied:

1. The norm of the difference of two consecutive iterates ${}_{(u)}C$ and ${}_{(u+1)}C$ is less than a prespecified tolerance tol $\in (0,\infty)$;

2. The number of iterations has reached a prespecified maximum $u_\infty \in \mathbb{N}$.

In case 2 the algorithm did not converge, which, however, does not necessarily imply that $\mathcal{C}_{\mathcal{P}}^{\bar{G}} = \emptyset$. In case 1 the algorithm constructs, or rather approximates, an interest rate derivative $C^\star \in \mathcal{C}_{\mathcal{P}}^{\bar{G}}$ with terminal excess $\bar{E} = 0$. The extension of the fixed-point algorithm to a non-negative (and $\mathcal{P}_N^{\bar{T}}$-measurable) terminal excess is straightforward, but this will generally result in an increase of the required initial capital.

Alg. 1 will converge, if the sufficient conditions of Prop. 4.3.12 are met, or if $H_{\bar{E},\mathcal{P}}^{\bar{G}}$ is a contraction (Def. 2.1.1), i.e. there exits a Lipschitz constant $\Lambda \in [0,1)$ and a norm $\|\cdot\|$ on $\mathbb{R}^{K\times(N+1)}$ with

$$\| H_{\bar{E},\mathcal{P}}^{\bar{G}}(C^1) - H_{\bar{E},\mathcal{P}}^{\bar{G}}(C^2) \| \leq \Lambda \| C^1 - C^2 \| \tag{4.4.3}$$

for all $C^1, C^2 \in \mathcal{V}^{\bar{T}}$ (cf. Prop. 4.3.16). In this case,

$$\| {}_{(u+1)}C - C^\star \| \leq \Lambda \| {}_{(u)}C - C^\star \| \tag{4.4.4}$$

for $u \in \mathbb{N}_0$, where $C^\star \in \mathcal{C}_{\mathcal{P}}^{\bar{G}}$ is the unique fixed-point of $H_{\bar{E},\mathcal{P}}^{\bar{G}}$.

The Hull–White (trinomial) model is one of the most widely applied interest rate models in practice and comparably easy to implement and calibrate. However, it might not be the best choice to hedge and value contingent guarantees, as it can produce quite extreme negative interest rate scenarios (see Fig. 4.9 in App. 4.A). These extreme scenarios might cause the fixed-point iteration in (4.4.1) to diverge or even cause the contingent guarantee to be not viable altogether. Term structure models which feature a lower bound for interest rates, such as the CIR(++) model (Brigo and Mercurio, 2006, Sec. 3.9), might allow to construct hedging derivatives even when the Hull–White model fails to do so.

4.4.2 Results

The fixed-point algorithm is applied to different contingent guarantees $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$, where, throughout, $\bar{F} = 1$ and $\bar{T}_n = n$ for $n = 0, \ldots, N$ (i.e. the lock-in time points are evenly spaced at yearly intervals). The tree is discretized with four quarterly time steps between the lock-in time points, i.e. $\Delta = \frac{1}{4}$ and $\mathcal{T} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots, \bar{T}_N\}$.

The algorithm is implemented in Matlab on a computer with a 3.2 GHz CPU and 16 GB RAM, and its parameters are set to tol $= 10^{-6}$, $u_\infty = 2500$, and ${}_{(0)}C = 0$. The convergence criterion in step 17 of Alg. 1 is evaluated using the maximum-norm in (4.3.7).

For the lock-in mechanism $\bar{L}$, the following choices from Ex. 3.5.2 are considered:

$$
\begin{aligned}
&\text{Lock-In I:} && \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^n\big) \;=\; \alpha\Big[\, X^\pi(\bar{T}_n) \;-\; X^\pi(\bar{T}_{n-1}) \,\Big]^+, \\
&\text{Lock-In II:} && \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^n\big) \;=\; \alpha\Big[\, X^\pi(\bar{T}_n) \;-\; \Big(1 \;+\; \textstyle\sum_{i=1}^{n-1} \bar{L}_i\big(\{X^\pi(\bar{T}_j)\}_{j=0}^i\big)\Big)\Big]^+, \\
&\text{Lock-In III:} && \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^n\big) \;=\; \alpha\Big[\, X^\pi(\bar{T}_n) \;-\; \tfrac{1}{n+1}\textstyle\sum_{i=0}^n X^\pi(\bar{T}_i) \,\Big]^+,
\end{aligned}
$$

for $n = 1, \ldots, N$, where $\alpha \in [0, 1]$.

The Impact of Guarantee Parameters

To study the impact of the parameters of the contingent guarantees on the value of the (approximated) hedging derivative $C^\star$, N is varied from 5 to 8 and the 'lock-in rate' α from 10% to 50%. The model parameters are set to $\kappa = 0.1000$, $\sigma = 0.0100$, and $P^{t_m}(0) = 1$ for all $m = 1, \ldots, M$. Results are reported in Tab. 4.1, which can be found at the end of this section.

As expected, the price $C_0^\star$ for all three lock-in mechanisms increases monotonically in both N and α. The same is true for the number of iterations needed to satisfy the convergence criterion. This is due to the fact that the convergence speed of the fixed-point iteration is linked to the Lipschitz constant Λ in (4.4.3) with the convergence decelerating for Λ increasing towards 1 (see (4.4.4)).

Tab. 4.1 suggests that the increase in Λ, i.e. the deceleration of the convergence, is significantly more pronounced for lock-in mechanisms Lock-In II and III, where the lock-in depends on the complete history of portfolio values, than for the 'myopic' Lock-In I. This effect also manifests itself in the sufficient condition (4.3.6) of Lem. 4.3.17.

Note that the computational costs of a single iteration increase dramatically in N, i.e. in the number of possible paths of the bond price $P^{\bar{T}_N}$. For this very reason, the construction of the forward pricing measure in lattice models quickly becomes infeasible: The evaluation of the Radon–Nikodým derivative in (4.1.1) requires the computation of the (strongly path-dependent) bank account B.

Fig. 4.1 through 4.3 show the $\mathbb{Q}$-distribution of the total lock-in

$$\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C^\star} + C_i^\star\}_{i=0}^n\big) \tag{4.4.5}$$

for the different lock-in mechanisms and different choices of N.

For lock-in mechanisms Lock-In I and III (Fig. 4.1 and 4.3) the shape of the distribution roughly resembles a log-normal distribution with the total lock-in of mechanism Lock-In III being slightly more (right-)skewed. Note that the maximum realized values for Lock-In III are significantly higher and increase much stronger with N. This behavior is commonly observed for lock-in mechanisms, where the lock-in depends on a long history of portfolio values (see also the discussion of the convergence behavior above).

For lock-in mechanism Lock-In II (Fig. 4.2) the distribution is much more erratic and does not resemble any known distribution. The same behavior can be observed for other lock-in mechanisms, which calculate the lock-in based on a comparison of the portfolio value (i.e. the discounted guaranteed amount) with the (undiscounted) guaranteed amount. Similar to lock-in mechanism Lock-In II, these mechanisms often lead to quite extreme realizations of the total lock-in.

These extreme realizations also become apparent when examining certain spot rate scenarios from the Hull–White trinomial model. Fig. 4.4 through 4.7 show the evolution of portfolio processes corresponding to the hedging derivative $C^\star$ for the three different lock-in mechanisms. In negative interest rate scenarios (Fig. 4.4 and Fig. 4.6), when the guarantee G^C (i.e. the cost of hedging the currently guaranteed amount) is already high, Lock-In II may lead to additional large increases in the guaranteed amount. In times of rising interest rates (Fig. 4.5 and Fig. 4.6), all three mechanisms lead to a subdued lock-in.

The Impact of Market Parameters

In order to assess the impact of the current market environment on the value of the hedging derivative $C^\star$, three different initial yield curves are considered:

Curve A: $r^{t_m}(0) := -\frac{1}{t_m} \ln P^{t_m}(0) = 0.0050$ for all $m = 1, \ldots, M$, such that $P^6(0) = 0.9753$;

Curve B: $r^{t_m}(0) = 0.0000$ for all $m = 1, \ldots, M$, such that $P^6(0) = 1.0000$

Curve C: $r^{t_m}(0) = -0.0050$ for all $m = 1, \ldots, M$, such that $P^6(0) = 1.0253$.

Moreover, the spot rate volatility σ is varied from 0.0080 to 0.0120. The remaining parameters are set to $\kappa = 0.1000$, $N = 6$, and $\alpha = 30\%$. Results are reported in Tab. 4.2.

For all three lock-in mechanisms, an increase of the interest rate volatility σ results in an increase of the price $C_0^\star$ of the hedging derivative, because larger movements of the bond price $P^{\bar{T}_N}$ (generally) result in a larger lock-in.

While the value $P^{\bar{T}_N}(0)\,\bar{F}$ of the fixed guarantee $\bar{F}$ increases with decreasing market interest rates $r^{t_m}(0)$, the price $C_0^\star$ decreases for Lock-In I and III. The 'pull-to-par effect' of the bond price $P^{\bar{T}_N}$ implies that, for a positive initial yield $r^{\bar{T}_N}(0)$, the guarantee $G^{\bar{G},C}$ will (generally) increase over time and thereby generate a positive lock-in. The opposite is the case for a negative initial yield.

For Lock-In II the price $C_0^\star$ behaves differently: A decrease in market interest rates (i.e. an increase in the bond price $P^{\bar{T}_N}$) results in an increase in the difference between the guarantee $G^{\bar{G},C}$ and the guaranteed amount $\bar{F} + V^{\bar{G},C}$, such that the lock-in (generally) increases as well.

Construction of Portfolio Insurance Strategies

Given the hedging derivative $C^\star$, the construction of a corresponding portfolio insurance strategy, i.e. the construction of an investment strategy $\pi \in \Pi$ that replicates the value process of the terminal guaranteed amount

$$X^\star(t) \;:=\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C^\star} + C_i^\star\}_{i=0}^{n}\big)\;\middle|\; \mathcal{F}(t)\right]$$

for $t \in \mathcal{T}$ (see Lem. 4.3.3), is a matter of solving a system of linear equations. Specifically, from (4.1.3), one obtains the condition

$$X^\star(t_m) \;=\; \sum_{i=m}^{M} \pi_i(t_m)\,\frac{P^{t_i}(t_m)}{P^{t_i}(t_{m-1})} \tag{4.4.6}$$

for $m = 1, \ldots, M$. Note that $\frac{B(t_m)}{B(t_{m-1})} = \frac{P^{t_m}(t_m)}{P^{t_m}(t_{m-1})} = e^{r(t_m)\Delta}$ for $m = 1, \ldots, M$, i.e. an investment into the bank account yields the same return as an investment into the shortest-maturity zero-coupon bond (the bank account is a redundant asset).

In the Hull–White trinomial model, $X^\star(t_m)$ may take up to three different values conditional on the information at time t_{m-1} (corresponding to the 'upward', 'middle', and 'downward' movement of the spot rate r, see App. 4.A). At each node of the trinomial tree, one therefore obtains a system of three linear equations from (4.4.6), which can always solved, if there are at least three different zero-coupon bonds to invest in.

Since $\mathcal{P}^{\bar{T}}_{N} = \mathcal{P}^{\bar{T}}_{N-1}$, the fund NAV at the terminal time $\bar{T}_N = t_M$ is $\mathcal{P}^{\bar{T}}_{N-1}$-measurable by construction of the interest rate derivative $C^\star$ (as a fixed-point of $H^{\bar{G}}_{\mathcal{P}}$). In particular, from time $\bar{T}_{N-1}$ onward, the value of $X^\star(\bar{T}_N)$ is known with certainty and thus replicated by a simple static long position in the terminal maturity zero-coupon bond $P^{\bar{T}_N}$.

The only case, which could prove problematic, is $t_{M-1} = \bar{T}_{N-1}$: There are only two zero-coupon bonds $P^{t_{M-1}}$ and P^{t_M} at time t_{M-2} to replicate the three different states of $X^\star(t_{M-1})$. A simple way to circumvent this difficulty is to introduce a new zero-coupon bond by using a finer discretization between the last two lock-in time points $\bar{T}_{N-1}$ and $\bar{T}_N = t_M$, such that $\bar{T}_{N-1} \leq t_{M-2}$.

Tab. 4.3 shows a portfolio insurance strategy $\pi \in \Pi^{\bar{G}}$ obtained from $C^\star$ in the small-scale trinomial model depicted in Fig. 4.8. The model parameters are set to $\Delta = \frac{1}{2}$ (biannual discretization), $M = 6$ ($t_M = 3$), $\kappa = 0.1000$, $\sigma = 0.0100$, and $P^{t_m}(0) = 1$ for all $m = 1, \ldots, M$. As before, the lock-in time points are evenly spaced at yearly intervals, i.e. $\bar{T}_n = n$ for $n = 0, \ldots, 3$, and the fixed guaranteed amount is set to $\bar{F} = 1$. Lock-In I with $\alpha = 50\%$ is used as the lock-in mechanism.

The interested reader may check that π indeed satisfies (4.4.6) and thereby correctly replicates the (value process of) terminal guaranteed amount. Additionally, Tab. 4.4 shows how π is obtained as the solution of a linear equation system.

As is often the case when replicating exotic derivatives in discrete models, π leads to extremely leveraged allocations: For each of the time steps $t_0 = 0$ through $t_3 = \frac{3}{2}$, π contains a short position in one of the zero-coupon bonds, which is a large multiple of the NAV $X^\pi \equiv X^\star$. In order to prevent extreme short (and long) positions, one should consider to include liquid derivatives, such as call and put options on bonds, into the replicating portfolio. Theses derivatives usually have an 'intrinsic' leverage effect with respect to their underlying. Another approach for constructing (semi-)replicating strategies, called *delta-hedging*, is introduced in Sec. 6.4.2.

Lock-In Mechanism	α	N = 5	6	7	8
Lock-In I (excess over previous NAV)	10%	0.0045 (6)	0.0062 (6)	0.0082 (7)	0.0104 (7)
	30%	0.0140 (12)	0.0197 (13)	0.0262 (15)	0.0335 (17)
	50%	0.0250 (23)	0.0356 (29)	0.0479 (36)	0.0620 (55)
Lock-In II (excess over guaranteed amount)	10%	0.0055 (12)	0.0091 (15)	0.0138 (18)	0.0200 (23)
	30%	0.0325 (51)	0.0642 (95)	0.1235 (231)	–
	50%	0.1822 (690)	–	–	–
Lock-In III (excess over average NAV)	10%	0.0030 (7)	0.0047 (8)	0.0067 (9)	0.0092 (10)
	30%	0.0097 (19)	0.0155 (31)	0.0233 (66)	–
	50%	0.0198 (1713)	–	–	–
No. of Paths		5.90×10^4	5.31×10^5	4.78×10^6	4.30×10^7
Duration (Sec.)		0.18	1.43	14.20	124.36

Table 4.1: The price $C_0^\star$ for different values of N and α. Cases, in which Alg. 1 does not converge, are marked by '–'. The number of iterations is reported in parenthesis. The bottom rows give the total number of paths $\sum_{k=-K_{m_N}}^{K_{m_N}} |\mathcal{L}_k^N|$ of the bond price $P^{\bar{T}_N}$ up to time $\bar{T}_N$ (see (4.B.1)) and the average duration of a single iteration of the algorithm in seconds. Results have been rounded.

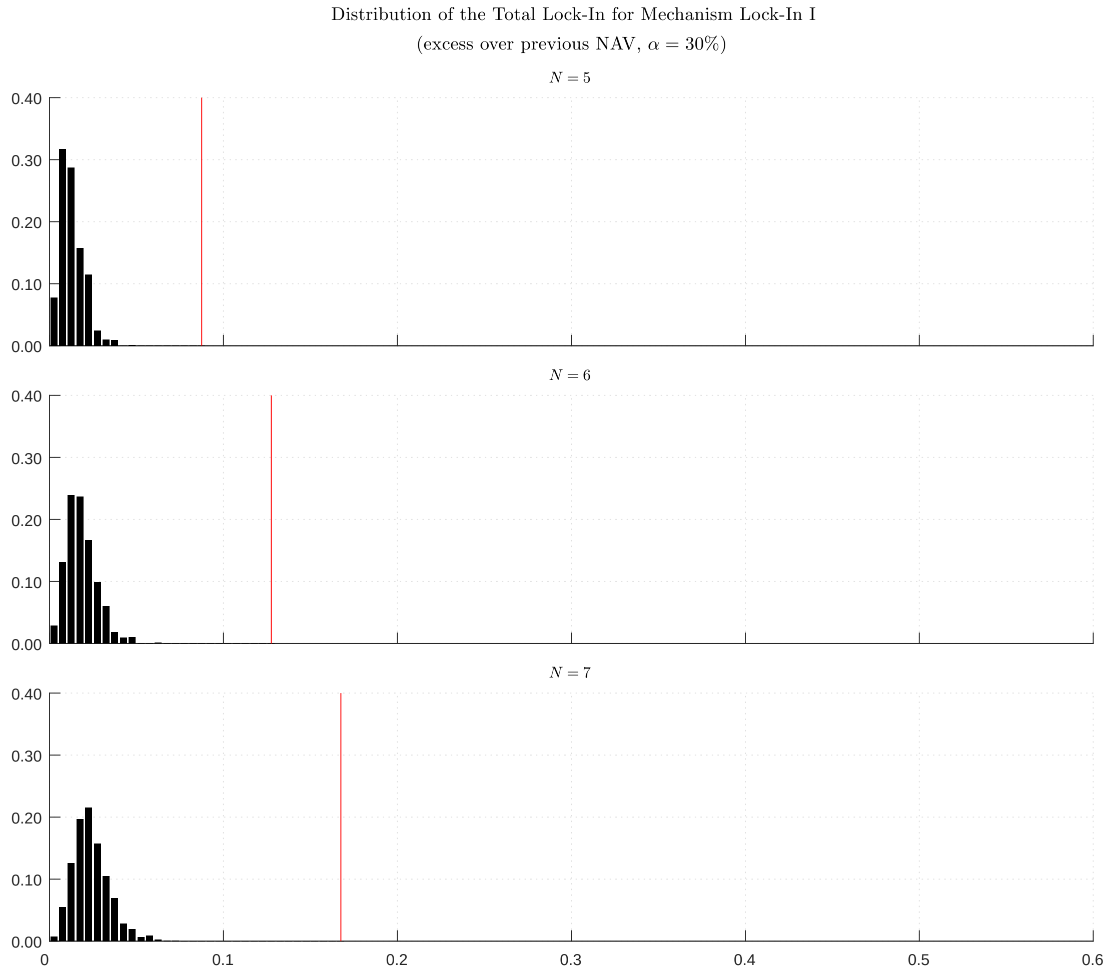

Figure 4.1: Distribution of the total lock-in (4.4.5) under $\mathbb{Q}$ for lock-in mechanism Lock-In I with the lock-in rate set to $\alpha = 30\%$, and the number of lock-in time points set to $N = 5$ (top), $N = 6$ (center), and $N = 7$ (bottom). The maximum realized values (marked by red lines) are given by 0.0875 ($N = 5$), 0.1275 ($N = 6$), and 0.1675 ($N = 7$).

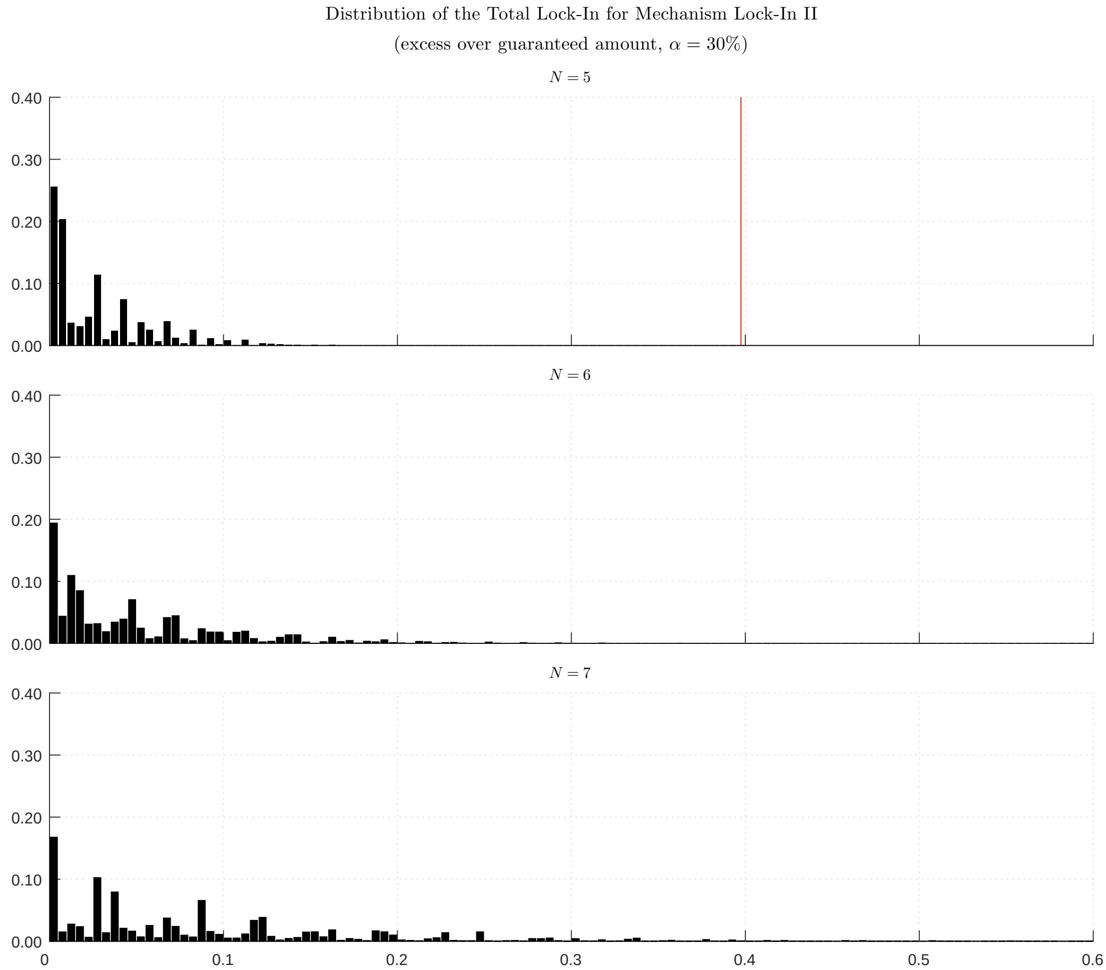

Figure 4.2: Distribution of the total lock-in (4.4.5) under $\mathbb{Q}$ for lock-in mechanism Lock-In II with the lock-in rate set to $\alpha = 30\%$, and the number of lock-in time points set to $N = 5$ (top), $N = 6$ (center), and $N = 7$ (bottom). The maximum realized values (marked by red lines, where they do not exceed the scale) are given by 0.3975 ($N = 5$), 1.0475 ($N = 6$), and 4.4525 ($N = 7$).

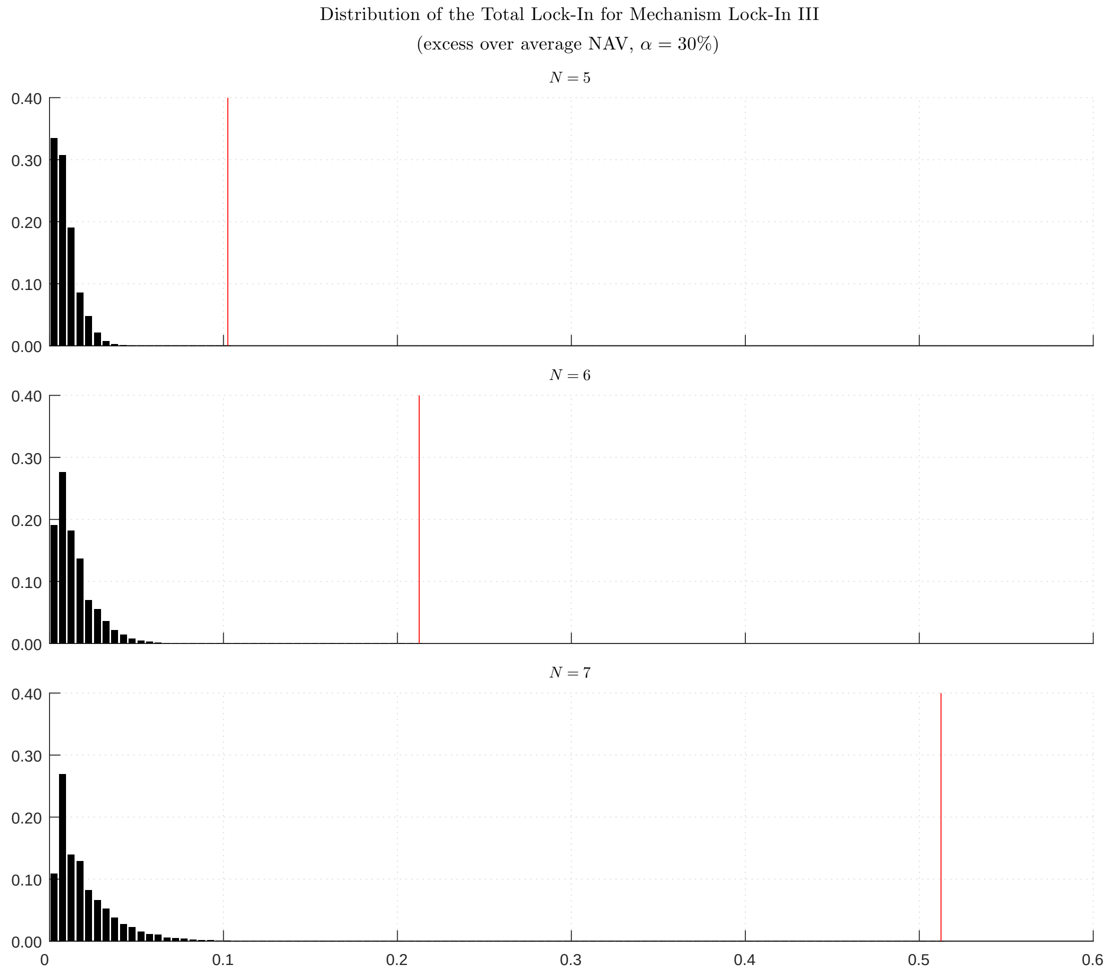

Figure 4.3: Distribution of the total lock-in (4.4.5) under $\mathbb{Q}$ for lock-in mechanism Lock-In III with the lock-in rate set to $\alpha = 30\%$, and the number of lock-in time points set to $N = 5$ (top), $N = 6$ (center), and $N = 7$ (bottom). The maximum realized values (marked by red lines) are given by 0.1025 ($N = 5$), 0.2125 ($N = 6$), and 0.5125 ($N = 7$).

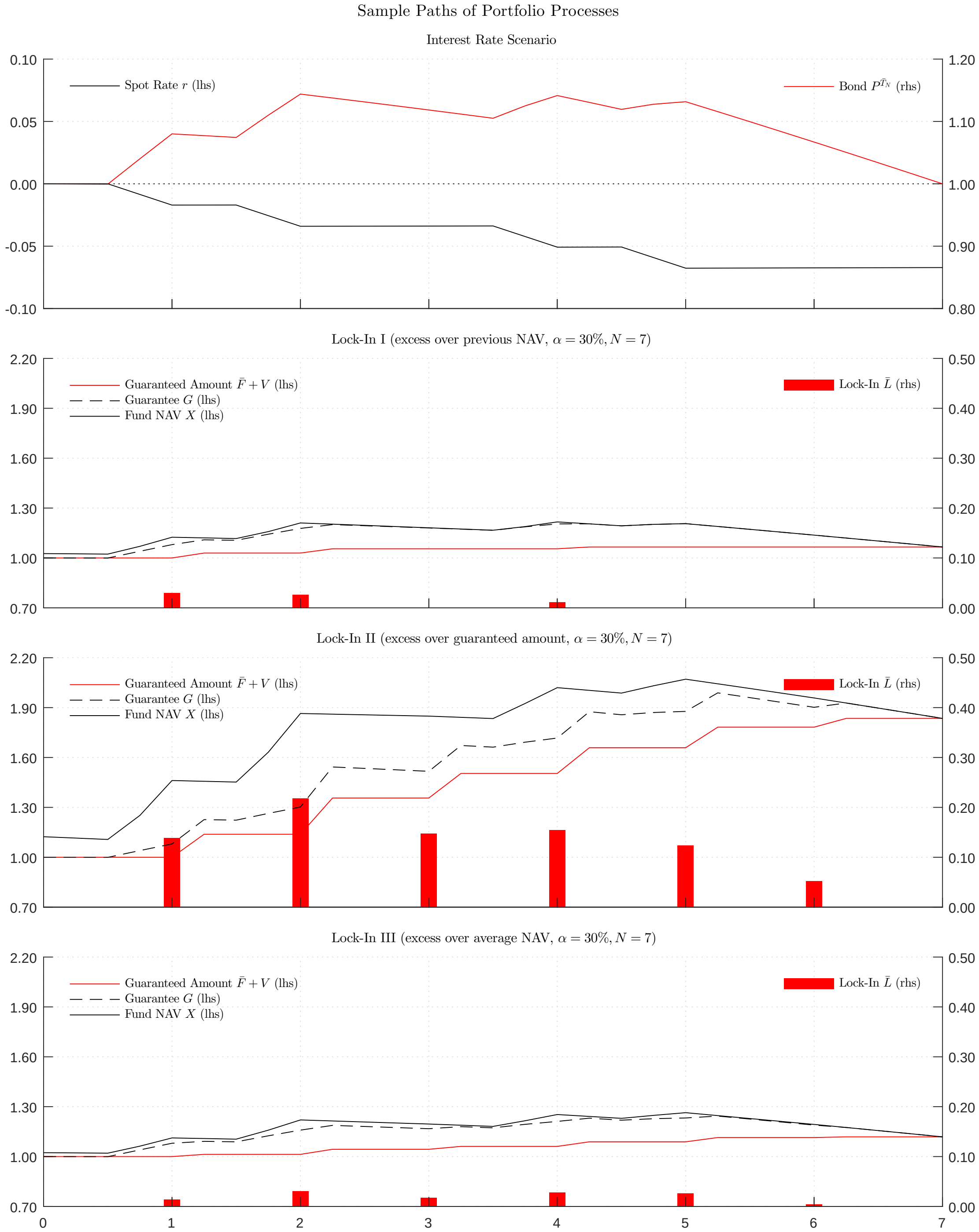

Figure 4.4: The portfolio processes and lock-in corresponding to the hedging derivative $C^\star$ for a given spot rate scenario from the Hull–White trinomial tree. The tree is discretized with four quarterly time steps between the lock-in time points, i.e. $\Delta = \frac{1}{4}$, and the lock-in time points are evenly spaced a yearly intervals, i.e. $\bar{T}_n = n$ for $n = 0, \ldots, 7$ (x-axis). The fixed guaranteed amount is set to $\bar{F} = 1$ and the lock-in rate is set to $\alpha = 30\%$.

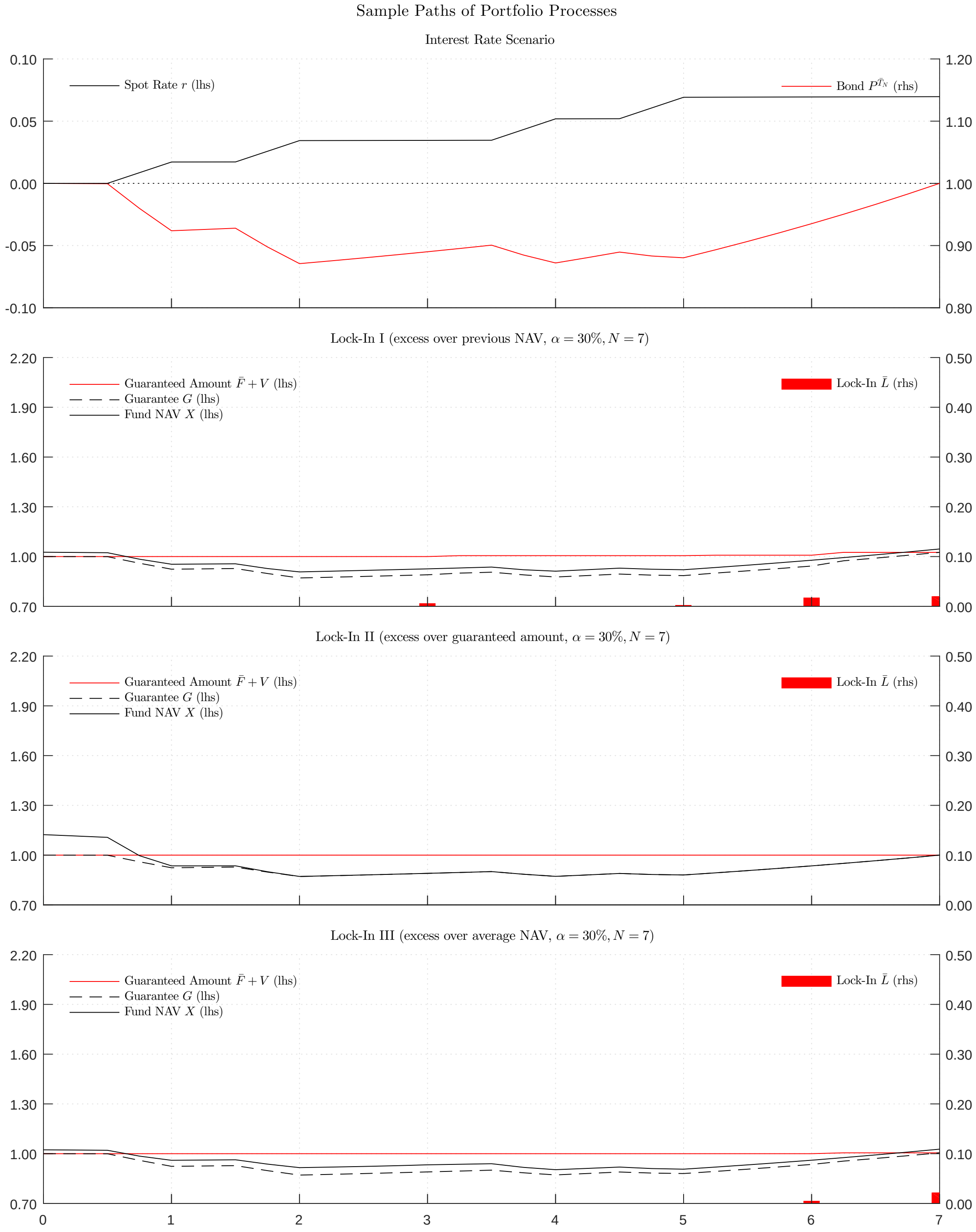

Figure 4.5: The portfolio processes and lock-in corresponding to the hedging derivative $C^\star$ for a given spot rate scenario from the Hull–White trinomial tree. The tree is discretized with four quarterly time steps between the lock-in time points, i.e. $\Delta = \frac{1}{4}$, and the lock-in time points are evenly spaced a yearly intervals, i.e. $\bar{T}_n = n$ for $n = 0, \ldots, 7$ (x-axis). The fixed guaranteed amount is set to $\bar{F} = 1$ and the lock-in rate is set to $\alpha = 30\%$.

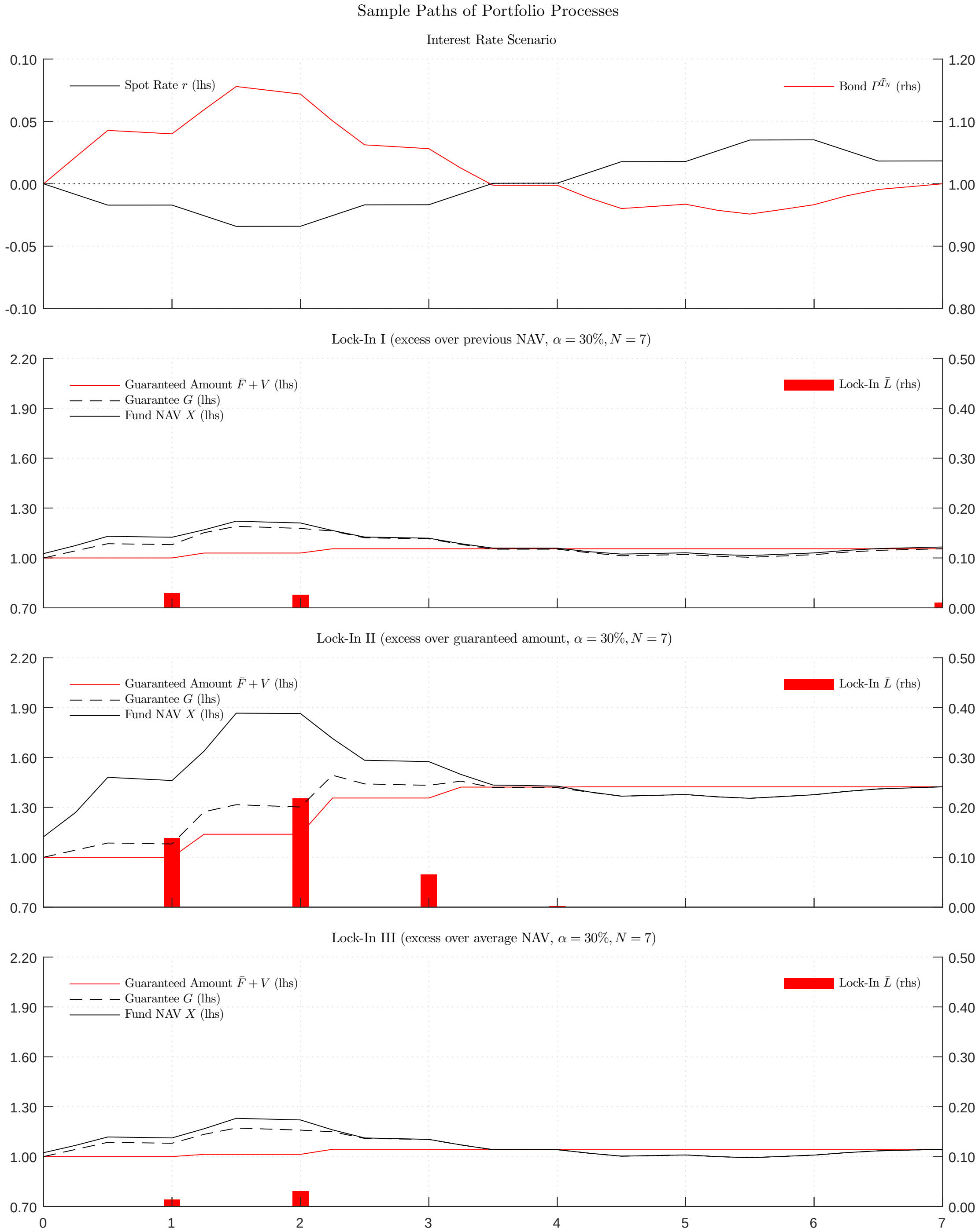

Figure 4.6: The portfolio processes and lock-in corresponding to the hedging derivative $C^\star$ for a given spot rate scenario from the Hull–White trinomial tree. The tree is discretized with four quarterly time steps between the lock-in time points, i.e. $\Delta = \frac{1}{4}$, and the lock-in time points are evenly spaced a yearly intervals, i.e. $\bar{T}_n = n$ for $n = 0, \ldots, 7$ (x-axis). The fixed guaranteed amount is set to $\bar{F} = 1$ and the lock-in rate is set to $\alpha = 30\%$.

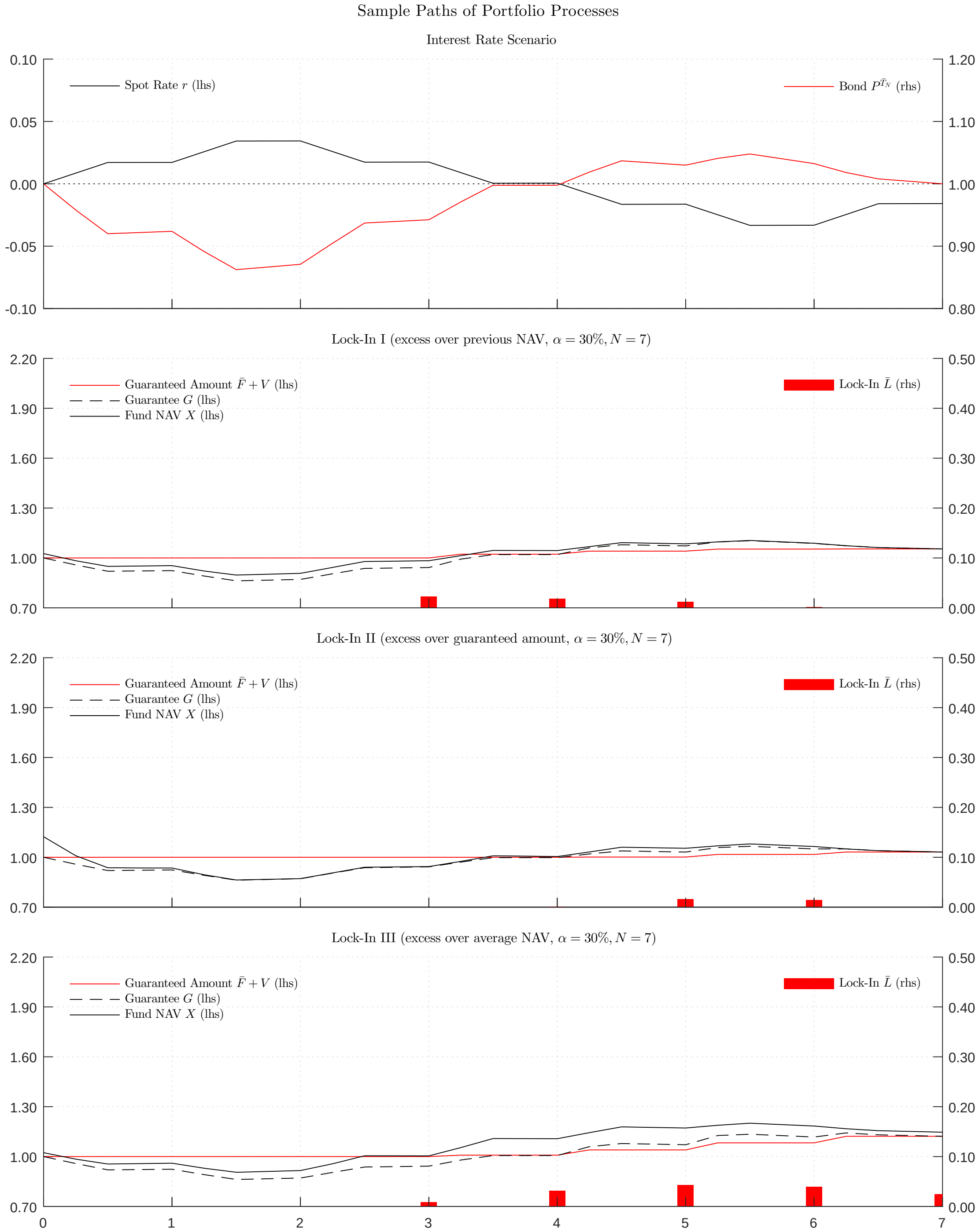

Figure 4.7: The portfolio processes and lock-in corresponding to the hedging derivative $C^{\star}$ for a given spot rate scenario from the Hull–White trinomial tree. The tree is discretized with four quarterly time steps between the lock-in time points, i.e. $\Delta = \frac{1}{4}$, and the lock-in time points are evenly spaced a yearly intervals, i.e. $\bar{T}_n = n$ for $n = 0, \ldots, 7$ (x-axis). The fixed guaranteed amount is set to $\bar{F} = 1$ and the lock-in rate is set to $\alpha = 30\%$.

Lock-In Mechanism	Curve	σ: 0.0080	0.0090	0.0100	0.0110	0.0120
Lock-In I (excess over previous NAV)	A (50 bp)	0.0192	0.0211	0.0230	0.0250	0.0270
	B (0 bp)	0.0156	0.0177	0.0197	0.0218	0.0238
	C (-50 bp)	0.0126	0.0147	0.0168	0.0189	0.0211
Lock-In II (excess over guaranteed amount)	A (50 bp)	0.0213	0.0269	0.0327	0.0388	0.0452
	B (0 bp)	0.0493	0.0566	0.0642	0.0722	0.0805
	C (-50 bp)	0.0976	0.1057	0.1142	0.1232	0.1330
Lock-In III (excess over average NAV)	A (50 bp)	0.0212	0.0225	0.0239	0.0253	0.0267
	B (0 bp)	0.0123	0.0139	0.0155	0.0171	0.0187
	C (-50 bp)	0.0069	0.0083	0.0098	0.0114	0.0129

Table 4.2: The price $C_0^\star$ for different initial yield curves and interest rate volatilities σ. Results have been rounded.

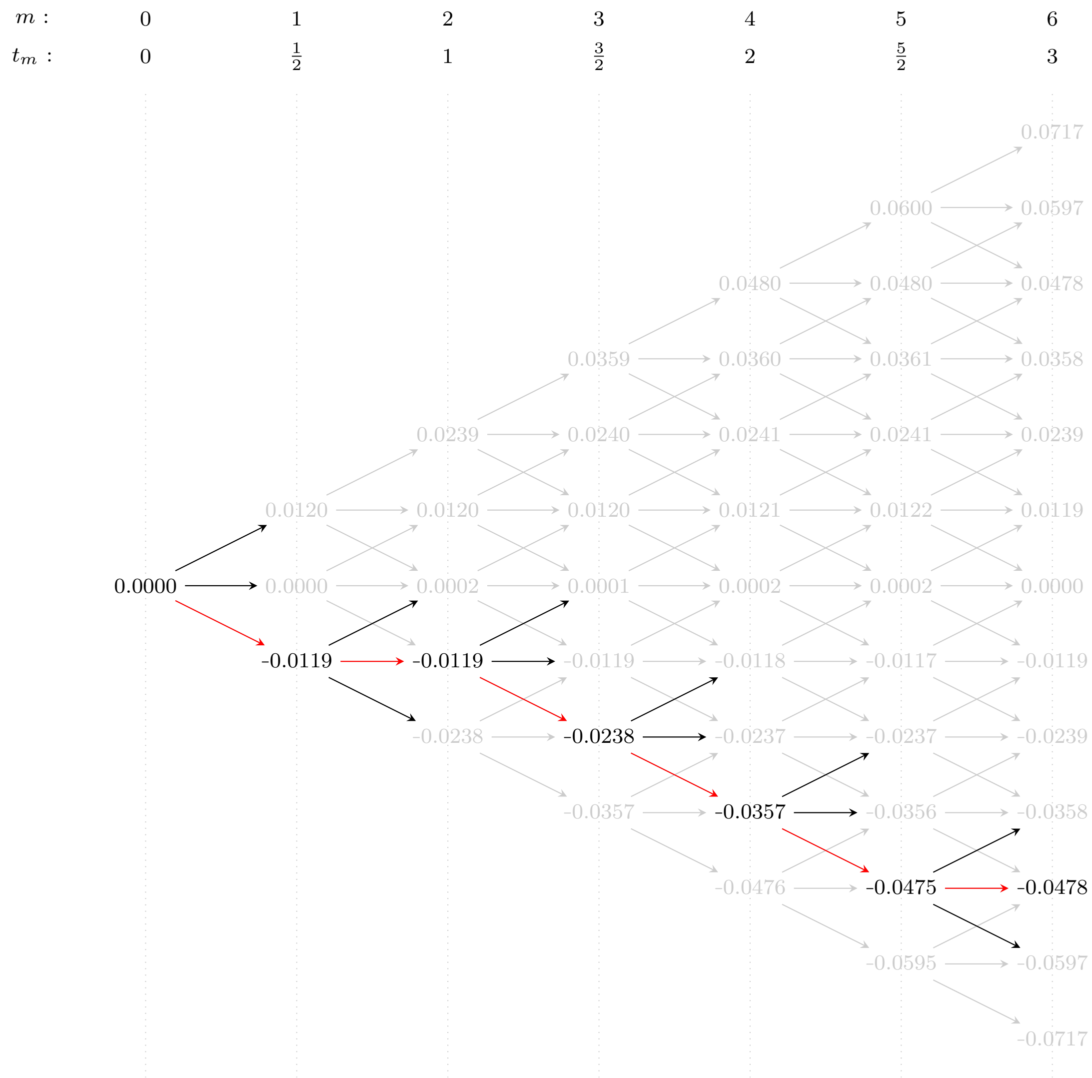

Figure 4.8: The Hull–White trinomial tree with $\Delta = \frac{1}{2}$ (biannual discretization) and $M = 6$ time steps ($t_M = 3$). The model parameters are set to $\kappa = 0.1000$, $\sigma = 0.0100$, and $P^{t_m}(0) = 1$ for all $m = 1, \ldots, M$. Shown are the values of the spot rate r at the corresponding nodes. Each node has three children as indicated by the arrows. Red arrows represent a realized path of the spot rate, for which a corresponding portfolio insurance strategy is given in Tab. 4.3.

m		0	1	2	3	4	5	6
t_m		0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$r(t_{m+1})$		0.0000	-0.0119	-0.0119	-0.0238	-0.0357	-0.0475	-0.0478
$X^\star(t_m)$		1.0092	1.0437	1.0370	1.0510	1.0635	1.0518	1.0271
$P^\tau(t_m)$	τ							
	$\frac{1}{2}$	1.000000	1.000000					
	1	1.000000	1.005986	1.000000				
	$\frac{3}{2}$	1.000000	1.011701	1.005969	1.000000			
	2	1.000000	1.017159	1.011658	1.011970	1.000000		
	$\frac{5}{2}$	1.000000	1.022367	1.017081	1.023459	1.017999	1.000000	
	3	1.000000	1.027339	1.022248	1.034480	1.035382	1.024058	1.000000
$\pi^\tau(t_{m+1})$	τ							
	$\frac{1}{2}$	43.036864						
	1	–	51.176012					
	$\frac{3}{2}$	-75.680384	–	28.909257				
	2	–	-109.331998	-44.518051	179.520427			
	$\frac{5}{2}$	–	–	–	-369.241732	–		
	3	33.652746	59.199662	16.645840	190.772271	1.063457	1.051826	
$\bar{L}$				0.0139		0.0132		0.0000
$V^{\bar{G},\pi}(t_m)$		0.0000	0.0000	0.0000	0.0139	0.0139	0.0271	0.0271
$G^{\bar{G},\pi}(t_m)$		1.0000	1.0273	1.0222	1.0489	1.0498	1.0518	1.0271
$C^{\bar{G},\pi}(t_m)$		0.0092	0.0163	0.0148	0.0021	0.0137	0.0000	0.0000

Table 4.3: The portfolio insurance strategy π obtained by replicating the value process $X^\star$ of the terminal guaranteed amount produced by $C^\star$ for the sample path depicted in Fig. 4.8. The trinomial model has been build with $M = 6$ time steps and a discretization of $\Delta = \frac{1}{2}$. The lock-in time points are set to $\bar{T}_n = n$ for $n = 0, \dots, 3$, the fixed guaranteed amount is set to $\bar{F} = 1$, and Lock-In I with $\alpha = 50\%$ is used for the lock-in mechanism. At each time point t_m, $m = 0, \dots, 3$, the investment strategy π invests in three different zero-coupon bonds thereby replicating the three possible values of the NAV at the following time step (see also Tab. 4.4). Results have been rounded.

t_m		1	$\frac{3}{2}$		
NODE		(2, -1)	(3, 0)	(3, -1)	(3, -2)
$r(t_{m+1})$		-0.0119	0.0001	-0.0119	-0.0238
$X^\star(t_m)$		1.0370	1.0147	1.0312	1.0510
$P^\tau(t_m)$	τ				
	$\frac{1}{2}$				
	1	1.000000			
	$\frac{3}{2}$	1.005969	1.000000	1.000000	1.000000
	2	1.011658	0.999951	1.005943	1.011970
	$\frac{5}{2}$	1.017081	0.999875	1.011598	1.023459
	3	1.022248	0.999776	1.016980	1.034480
$\frac{P^\tau(t_m)}{P^\tau(t_{m-1})}$	τ				
	$\frac{1}{2}$				
	1				
	$\frac{3}{2}$		0.994066	0.994066	0.994066
	2		0.988428	0.994351	1.000308
	$\frac{5}{2}$		0.983083	0.994609	1.006271
	3		0.978017	0.994847	1.011966
$\pi^\tau(t_{m+1})$	τ				
	$\frac{1}{2}$				
	1				
	$\frac{3}{2}$	28.909257			
	2	-44.518051			
	$\frac{5}{2}$	–			
	3	16.645840			

Table 4.4: The portfolio insurance strategy π of Tab. 4.3 at the node $(2,-1)$ ($m = 2$, $t_m = 1$). Note that π correctly replicates the value process $X^\star$ at all three child nodes $(3,0)$ (up), $(3,-1)$ (middle), and $(3,-2)$ (down), i.e. π solves $X^\star(t_m) = \sum_{i=m}^{M} \pi_i(t_m) \frac{P^{t_i}(t_m)}{P^{t_i}(t_{m-1})}$. Results have been rounded.

Appendices

4.A The Hull–White Trinomial Tree

The Hull and White (1993) trinomial tree is a discrete-time and -state approximation of the continuous Hull–White model (see App. 6.A, Hull and White (1990)). The following construction is based on Brigo and Mercurio (2006, Sec. 3.3.3).

Let $\kappa, \sigma, \Delta \in (0, \infty)$ be given, where κ is the speed of mean-reversion, σ the volatility, and Δ the (constant) distance between two time points (in years). The spot interest rate r of Sec. 4.1 is modeled as the sum of a stochastic process $\{z_m\}_{m=1}^M$ and a deterministic shift $\{\phi_m\}_{m=1}^M$. Tree nodes are denoted by (m, k), where $m = 0, \ldots, M$ represents the time point t_m and $k = -K_m, \ldots, K_m$ is the state index with $K_m \in \mathbb{N}_0$. Moreover, for $m = 1, \ldots, M$, $z_{m,k} = k\,\Delta z$ denotes the value of the process z at the node $(m-1, k)$, where the step size is set to $\Delta z = \sqrt{3}\,V$ with

$$V := \sqrt{\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\,\Delta}\right)} \quad .$$

The tree starts with the single node $(0, 0)$ and each node (m, k) has three children $(m+1, j_k+1)$, $(m+1, j_k)$, and $(m+1, j_k-1)$ at the subsequent time step, where the middle child is chosen such that j_k is the nearest integer to $k\,e^{-\kappa\,\Delta}$. This fully determines the geometry of the tree.

The conditional $\mathbb{Q}$-probabilities q_{k,j_k+1}, q_{k,j_k}, and q_{k,j_k-1} of moving from node (m, k) to one of its children $(m+1, j_k+1)$, $(m+1, j_k)$, and $(m+1, j_k-1)$ respectively, are set to

$$\begin{aligned} q_{k,j_k+1} &= \frac{1}{6} + \frac{\eta_k^2}{6\,V^2} + \frac{\eta_k}{2\sqrt{3}\,V} \quad , \\ q_{k,j_k} &= \frac{2}{3} - \frac{\eta_k^2}{3\,V^2} \quad , \\ q_{k,j_k-1} &= \frac{1}{6} + \frac{\eta_k^2}{6\,V^2} - \frac{\eta_k}{2\sqrt{3}\,V} \quad , \end{aligned}$$

where $\eta_k := \Delta z\,(k\,e^{-\kappa\,\Delta} - j_k)$. The remaining conditional probabilities $q_{k,j}$ for $j \notin \{j_k+1, j_k, j_k-1\}$ are zero.

For the lattice to correctly reproduce the initial market term structure, the values $z_{m,k}$ need to be displaced by the deterministic shift ϕ. Let ϕ_m denote the shift applied to the nodes $(m-1,\cdot)$ and $Q_{m,k}$ the present value of a payment of 1 if node (m,k) is reached ($Q_{m,k}$ is a so-called 'Arrow-Debreu' price).

With $\phi_1 = -\ln\big(P^{t_1}(0)\big)$ and $Q_{0,0}=1$, the remaining values of ϕ are calculated iteratively by

$$Q_{m,k} \;=\; \sum_{j=-K_{m-1}}^{K_{m-1}} Q_{m-1,j}\; q_{j,k}\; e^{-(\phi_m\,+\,z_{m,j})\Delta}$$

and

$$\phi_{m+1} \;=\; \frac{1}{\Delta}\ln\left(\frac{\sum_{k=-K_m}^{K_m} Q_{m,k}\; e^{-z_{m+1,k}\,\Delta}}{P^{t_{m+1}}(0)}\right)$$

for $m=1,\ldots,M-1$. Altogether, $r_{m,k}=\phi_m+z_{m,k}$ is the riskless spot interest rate in state k and for the interval $[t_{m-1},t_m)$ (see Sec. 4.1).

Considering that for the fixed-point iteration, only the nodes at the lock-in time points $\bar{T}$ are relevant, let $m_n\in\{0,\ldots,N\}$ be such that $t_{m_n}=\bar{T}_n$ for $n=0,\ldots,N$. Depending on the difference between the time grids $\mathcal{T}$ and $\bar{T}$, a node (m_n,k) may have more than three children at the subsequent lock-in time step $\bar{T}_{n+1}$.

The evaluation of the (conditional) expectations in (4.4.2) requires the discounted conditional probabilities $\tilde{q}^{n,\hat{n}}_{k,j}$ of moving from node (m_n,k) to node $(m_{\hat{n}},j)$, $\hat{n}>n$. They are calculated as follows: Let

$$\mathcal{L}^{n,\hat{n}}_{k,j} \;\subseteq\; \{\,k\,\}\; \times_{i=m_n+1}^{m_{\hat{n}}-1} \{\,-K_i,\,\ldots,\,K_i\,\}\;\times\;\{\,j\,\} \tag{4.A.1}$$

be the collection of possible paths from node (m_n,k) to node $(m_{\hat{n}},j)$, where a vector

$$\big(k,\,l_2,\,l_3,\,\ldots,\,l_{m_{\hat{n}}-m_n},\,j\big)\;\in\;\mathcal{L}^{n,\hat{n}}_{k,j}$$

represents the path over the nodes

$$(m_n,k),\,(m_n+1,l_2),\,(m_n+2,l_3),\,\ldots,\,(m_{\hat{n}}-1,l_{m_{\hat{n}}-m_n}),\,(m_{\hat{n}},j)\quad .$$

Then,

$$\tilde{q}^{n,\hat{n}}_{k,j} \;=\; \sum_{l\,\in\,\mathcal{L}^{n,\hat{n}}_{k,j}}\left(\prod_{i=1}^{m_{\hat{n}}-m_n} p_{l_i,l_{i+1}}\; e^{-(\phi_{m_n+i}\,+\,z_{m_n+i,l_i})\Delta}\right)\quad . \tag{4.A.2}$$

Note that $\tilde{q}^{n,\hat{n}}_{k,j}$ can also be interpreted as a 'conditional' Arrow-Debreu price. Moreover, the bond price $P^{\bar{T}_N}$ at node (m_n, k) is given by

$$P_{n,k} = \sum_{j=-K_M}^{K_M} \tilde{q}_{k,j}^{n,M} \quad .$$

The distribution of interest rates under $\mathbb{Q}$ (approximatively normal) can be seen in Fig. 4.9 at the end of this section. An exemplary trinomial lattice is depicted in Fig. 4.10.

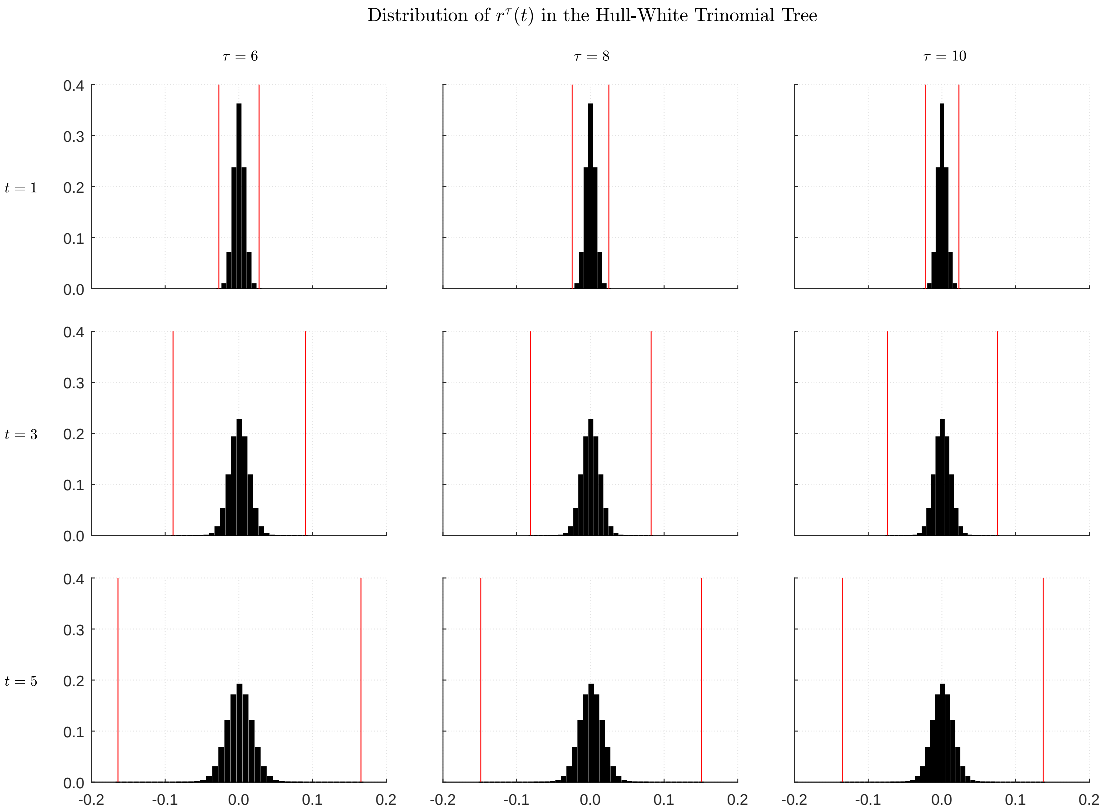

Figure 4.9: Distribution of interest rates $r^\tau(t) := -\frac{1}{(\tau - t)} \ln P^\tau(t)$ for $t = 1, 3, 5$ and $\tau = 6, 8, 10$ under $\mathbb{Q}$ in the Hull–White trinomial tree. The tree is discretized with quarterly time steps, i.e. $\Delta = \frac{1}{4}$ and $\mathcal{T} = \left\{0, \frac{1}{4}, \frac{2}{4}, \dots\right\}$, and the initial term structure is set to $P^\tau(0) = 1$ for $\tau \in \mathcal{T} \setminus \{0\}$. Moreover, $\kappa = 0.1000$ and $\sigma = 0.0100$. The red lines mark the maximum and minimum values.

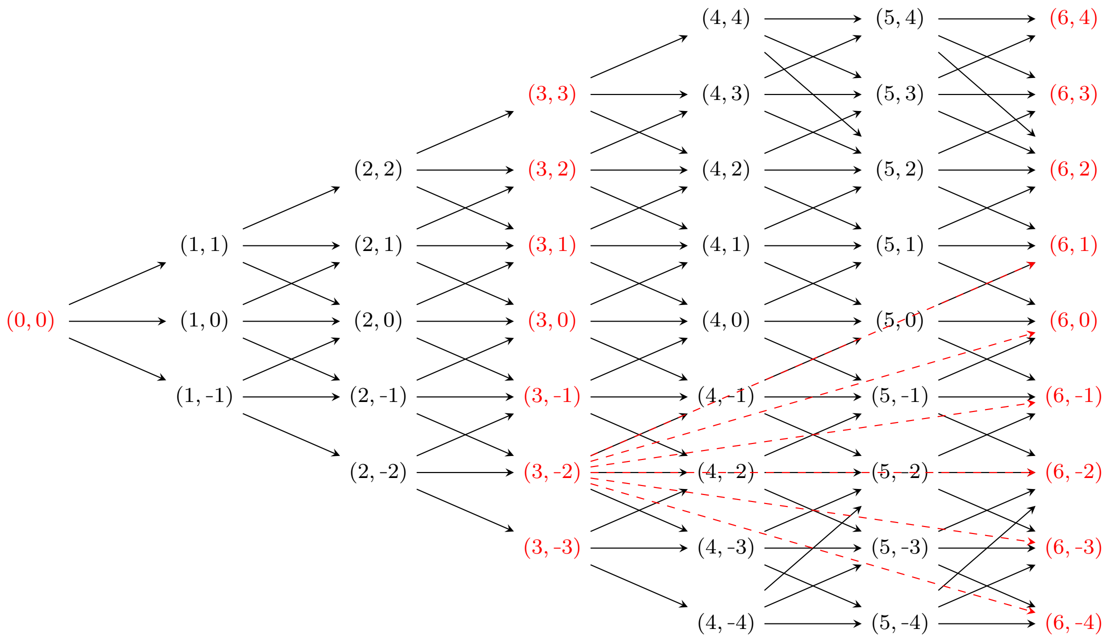

Figure 4.10: An exemplary Hull–White trinomial lattice with $M = 6$, $N = 2$, $\bar{T}_1 = t_3$, and $\bar{T}_2 = t_6$. The black solid arrows represent positive conditional probabilities $p_{k,j}$. Nodes at the lock-in time points are marked in red. The red dotted arrows represent the positive discounted conditional probabilities $\tilde{p}^{1,2}_{-2,k}$ of moving from node $(m_1, -2)$ to the nodes (m_2, k), $k = -4, \ldots, 1$. The set of paths in (4.A.1) from node $(m_1, 2)$ to node $(m_2, 4)$ is given by $\mathcal{L}^{1,2}_{2,4} = \{\, (2, 3, 4, 4),\, (2, 3, 3, 4),\, (2, 2, 3, 4) \,\}$.

4.B The Fixed-Point Algorithm

Given the availability of the discounted conditional probabilities under $\mathbb{Q}$ in the trinomial model (see (4.A.2)) the fixed-point iteration in (4.4.2) can be implemented in a straightforward manner to construct a hedging derivative $C \in \mathcal{C}^{\bar{G}}$.

Analogous to (4.A.1), let

$$\mathcal{L}_k^n \subseteq \{0\} \times_{i=1}^{n-1} \{-K_{m_i}, \ldots, K_{m_i}\} \times \{k\} \tag{4.B.1}$$

be the collection of possible paths from the initial node $(0,0)$ to the node (m_n, k), where only nodes at the lock-in time points $\{T_n\}_{n=0}^{N}$ are considered. Moreover, for a time point $\bar{T}_n$, a state k, a path $l \in \mathcal{L}_n^k$, and an iteration cycle $u \in \mathbb{N}_0$ let ${}_{(u)}C_{n,k}^l$ and ${}_{(u)}G_{n,k}^l$ denote C_n and $G_n^{\bar{G},C}$, respectively. With this notation, the complete fixed-point algorithm is specified in Alg. 1.

Algorithm 1 The Fixed-Point Algorithm for the Hull–White Trinomial Model

---

Require: $\{{}_{(0)}C_{n,k}^l\}_{n,k,l}$ ▷ Initial guess
Require: $u_\infty \in \mathbb{N}$ ▷ Maximum number of iterations
Require: $\text{tol} \in (0, \infty)$ ▷ Convergence tolerance

$u \leftarrow 0$ ▷ Iteration counter
$\varepsilon \leftarrow \infty$ ▷ Norm of the difference between two iterates ${}_{(u)}C$ and ${}_{(u+1)}C$

while $(u < u_\infty)$ and $(\varepsilon > \text{tol})$ **do**

  for $k \leftarrow -K_{m_N}$ to K_{m_N} **do** ▷ Update C_N
    for all $l \in \mathcal{L}_k^N$ **do**
      ${}_{(u+1)}C_{N,k}^l \leftarrow \bar{L}_N\left(\left\{{}_{(u)}G_{i,l_{i+1}}^{(l_1,\ldots,l_{i+1})} + {}_{(u)}C_{i,l_{i+1}}^{(l_1,\ldots,l_{i+1})}\right\}_{i=0}^{N}\right)$
    end for
  end for

  for $n \leftarrow N-1$ to 1 **do** ▷ Update C_n, $n = N-1, \ldots, 1$
    for $k \leftarrow -K_{m_n}$ to K_{m_n} **do**
      for all $l \in \mathcal{L}_k^n$ **do**
        $${}_{(u+1)}C_{n,k}^l \leftarrow P_{n,k}^N \, \bar{L}_n\left(\left\{{}_{(u)}G_{i,l_{i+1}}^{(l_1,\ldots,l_{i+1})} + {}_{(u)}C_{i,l_{i+1}}^{(l_1,\ldots,l_{i+1})}\right\}_{i=0}^{n}\right) + \sum_{j=-K_{T_{n+1}}}^{K_{T_{n+1}}} \tilde{q}_{k,j}^{n,n+1} \; {}_{(u+1)}C_{n+1,j}^{(l_1,\ldots,l_{n+1},j)}$$
      end for
    end for
  end for

  ${}_{(u+1)}C_{0,0}^{(0)} \leftarrow \sum_{j=-K_{m_1}}^{K_{m_1}} \tilde{q}_{0,j}^{0,1} \; {}_{(u+1)}C_{1,j}^{(0,j)}$ ▷ Update C_0

  $\varepsilon \leftarrow \|\, {}_{(u+1)}C - {}_{(u)}C \,\|$
  $u \leftarrow u + 1$

end while

4.C Proofs

Proof of Lem. 4.3.3. Since $C = H^{\bar{G}}_{\bar{E}}(C)$,

$$\begin{aligned} X^\pi(\bar{T}_n) \;&=\; P^{\bar{T}_N}(\bar{T}_n)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\bar{F} \;+\; \sum_{i=n\vee 1}^{N} \bar{L}_n\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \;+\; \bar{E}\;\middle|\; \mathcal{F}(\bar{T}_n)\,\right] \\ &=\; P^{\bar{T}_N}(\bar{T}_n)\left(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big)\right) \\ &\qquad\qquad +\; P^{\bar{T}_N}(\bar{T}_n)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\;\sum_{i=n\vee 1}^{N} \bar{L}_n\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \;+\; \bar{E}\;\middle|\; \mathcal{F}(\bar{T}_n)\,\right] \\ &=\; G^{\bar{G},C}_n \;+\; C_n \end{aligned}$$

for $n = 0, \ldots, N$. From

$$X^\pi(\bar{T}_n) \;=\; G^{\bar{G},\pi}(\bar{T}_n) \;+\; C^{\bar{G},\pi}(\bar{T}_n)$$

for $n = 0, \ldots, N$, and

$$G^{\bar{G},\pi}(\bar{T}_0) \;=\; G^{\bar{G},C}_0 \;=\; P^{\bar{T}_N}(0)\,\bar{F} \quad\text{and}\quad G^{\bar{G},\pi}(\bar{T}_1) \;=\; G^{\bar{G},C}_1 \;=\; P^{\bar{T}_N}(\bar{T}_1)\,\bar{F} \quad ,$$

one thus immediately obtains $C^{\bar{G},\pi}(\bar{T}_0) = C_0$ and $C^{\bar{G},\pi}(\bar{T}_1) = C_1$. Since $C^{\bar{G},\pi}(\bar{T}_i) = C_i$ for $i < n$ implies $G^{\bar{G},\pi}(\bar{T}_n) = G^{\bar{G},C}_n$, the assertion for the remaining time points follows by induction over $n = 2, \ldots, N$. Altogether, the lock-in $\bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big)$ caused by the replicating strategy π is precisely equal to $\bar{L}_n\big(\{G^{\bar{G},C}_i + C_i\}_{i=0}^{n}\big)$ for $n = 1, \ldots, N$, such that

$$X^\pi(\bar{T}_N) \;=\; \bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E} \quad .$$

Thus, $\bar{E}^{\bar{G},\pi} = \bar{E}$ and consequently also $\pi \in \Pi^{\bar{G}}$. □

Proof of Prop. 4.3.7. Let $\hat{P} \coloneqq \max_{n=0}^{N}\max_{k=1}^{K} P^{\bar{T}_N}(\bar{T}_n, \omega_k)$, $\hat{E} \coloneqq \max_{k=1}^{K} \bar{E}(\omega_k)$, and $\Lambda \in [0, \infty)$ as in Def. 4.3.6. Moreover, let A be the non-empty, convex, and compact subset of $\mathbb{R}^{K\times(N+1)}$ defined as

$$A \coloneqq \Big\{\, C \in \mathcal{V}^{\bar{T}} \;:\; C_{k,n} \;\le\; \hat{P}\,(N\,\Lambda \;+\; \hat{E}) \;\; \forall\, k = 1, \ldots, K,\; n = 0, \ldots, N \,\Big\} \quad . \tag{4.C.1}$$

By assumption, $H^{\bar{G}}_{\bar{E}}$ is continuous and maps A into itself. By Brouwer's theorem (Prop. 2.2.1) there exists a fixed-point $C \in A$ of $H^{\bar{G}}_{\bar{E}}$. □

Proof of Lem. 4.3.8. For $C \in \mathcal{V}^{\bar{T}}$, let $\bar{E}^{\bar{G},C} := C_N - \bar{L}_N\big(\{G_n^{\bar{G},C} + C_n\}_{n=0}^N\big) \in \mathbb{R}^K$ be the corresponding terminal excess. Moreover, let $\mathcal{V}^\star := \big\{\, C \in \mathcal{V}^{\bar{T}} : \bar{E}^{\bar{G},C} \geq 0 \,\big\}$ be the closed subset of families C producing a non-negative terminal excess, and $F : \mathcal{V}^\star \to \mathbb{R}^{K\times(N+1)}, C \mapsto C - H^{\bar{G}}_{\bar{E}^{\bar{G},C}}(C)$. Then F is continuous and $\mathcal{C}^{\bar{G}} = F^{-1}(\{0\})$. Altogether, $\mathcal{C}^{\bar{G}}$ is closed and the infimum in (4.3.4) is indeed a minimum. □

Proof of Prop. 4.3.11. Let A be as in (4.C.1). The monotonicity of the lock-in mechanism immediately implies that the guarantee $G^{\bar{G},C}$ is monotone in C, i.e. $G^{\bar{G},C^1} \leq G^{\bar{G},C^2}$ for $C^1, C^2 \in \mathcal{V}^{\bar{T}}$ with $C^1 \leq C^2$, where '$\leq$' denotes the usual component-wise vector comparison on $\mathbb{R}^{K\times(N+1)}$ (see Sec. 2.3). The same is then also true for the budget function $H^{\bar{G}}_{\bar{E}}$, which maps A into itself by assumption. By Tarski's theorem (Prop. 2.3.2) there exists a fixed-point $C \in A$ of $H^{\bar{G}}_{\bar{E}}$. □

Proof of Prop. 4.3.12. The proof works completely analogous to the proof of Prop. 4.3.11 and additionally invokes Kleene's theorem (Prop. 2.3.3). □

Proof of Lem. 4.3.13. By the proof of Prop. 4.3.11 and Tarski's theorem (Prop. 2.3.2) there exists a 'least' fixed-point $C^0 \in A$ of $H^{\bar{G}}_0$ with the property that $H^{\bar{G}}_0(C) \leq C$ implies $C^0 \leq C$ for all $C \in A$, where A is given in (4.C.1) and '$\leq$' denotes the usual component-wise vector comparison on $\mathbb{R}^{K\times(N+1)}$ (see Sec. 2.3). In particular, for $\bar{E} \in [0,\infty)^K$, $H^{\bar{G}}_{\bar{E}}(C) = C \Rightarrow H^{\bar{G}}_0(C) \leq C \Rightarrow C^0 \leq C$. Altogether, $C^0 \leq C$ for all $C \in \mathcal{C}^{\bar{G}}$. □

Proof of Lem. 4.3.14. Let $\bar{G}^2 = (\bar{T}, \bar{F}, \bar{L}^2)$ and $A := \big\{\, C \in \mathcal{V}^{\bar{T}} : C \leq C^2 \,\big\}$, where '$\leq$' denotes the usual component-wise vector comparison on $\mathbb{R}^{K\times(N+1)}$ (see Sec. 2.3). Then A is a non-empty, convex, and compact subset of $\mathbb{R}^{K\times(N+1)}$. By the monotonicity of $\bar{L}^2$, $H^{\bar{G}}_{\bar{E}}(C) \leq H^{\bar{G}^2}_{\bar{E}}(C) \leq H^{\bar{G}^2}_{\bar{E}}(C^2) = C^2$ for all $C \in A$, such that $H^{\bar{G}}_{\bar{E}}$ maps A into itself. By Brouwer's theorem (Prop. 2.2.1) there exists a fixed-point $C \in A$ of $H^{\bar{G}}_{\bar{E}}$. □

Proof of Prop. 4.3.16. Let $C^1, C^2 \in \mathcal{V}^{\bar{T}}$. Then,

$$\begin{aligned}
\| \, H_{\bar{E}}^{(\bar{T},\bar{F},\bar{L})}(C^1) \; - \; H_{\bar{E}}^{(\bar{T},\bar{F},\bar{L})}(C^2) \, \| \; &= \; \| \, H_0^{(\bar{T},\bar{F},\bar{L})}(C^1) \; - \; H_0^{(\bar{T},\bar{F},\bar{L})}(C^2) \, \| \\
&= \; \| \, H_0^{(\bar{T},0,\bar{L})}(\tilde{C}^1) \; - \; H_0^{(\bar{T},0,\bar{L})}(\tilde{C}^2) \, \| \\
&\leq \; \Lambda \, \| \, \tilde{C}^1 \; - \; \tilde{C}^2 \, \| \\
&= \; \Lambda \, \| \, C^1 \; - \; C^2 \, \| \quad ,
\end{aligned}$$

where $\tilde{C}^i := \big\{\, C_n^i + P^{\bar{T}_N}(\bar{T}_n)\, \bar{F} \,\big\}_{n=0}^N \in \mathcal{V}^{\bar{T}}$ for $i = 1,2$ and $\Lambda \in [0,1)$ as in Def. 4.3.15. In particular, the budget function $H^{\bar{G}}_{\bar{E}}$ is a contraction (Def. 2.1.1) on $\mathcal{V}^{\bar{T}}$. The rest of the assertion is then a straightforward application of Banach's theorem (Prop. 2.1.2). □

Proof of Prop. 4.3.17. Let $C^1, C^2 \in \mathcal{V}^{\bar{T}}$. The assertion is proved by showing that $H_0^{(\bar{T},0,\bar{L})}$ is a contraction in the maximum-norm $\|\cdot\|_{\max}$ defined in (4.3.7).

Step 1: By induction,

$$\begin{aligned} &\max_{k=1}^{K} \left| \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\bar{L}_n\big(\{G_i^{\bar{G},C^1} + C_i^1\}_{i=0}^n\big) - \bar{L}_n\big(\{G_i^{\bar{G},C^2} + C_i^2\}_{i=0}^n\big) \,\middle|\, \mathcal{G} \right](\omega_k) \right| \\ &\quad \le Z_n \left\| C^1 - C^2 \right\|_{\max} \end{aligned} \tag{4.C.2}$$

for all $n = 1\ldots, N$, and any (sub-)σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. Indeed, for $n = 1$,

$$\begin{aligned} &\max_{k=1}^{K} \left| \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\bar{L}_1\big(\{G_i^{\bar{G},C^1} + C_i^1\}_{i=0}^1\big) - \bar{L}_1\big(\{G_i^{\bar{G},C^2} + C_i^2\}_{i=0}^1\big) \,\middle|\, \mathcal{G} \right](\omega_k) \right| \\ &\quad \le \sum_{i=0}^{1} M_{1,i} \max_{k=1}^{K} \left| C_i^1(\omega_k) - C_i^2(\omega_k) \right| \le Z_1 \left\| C^1 - C^2 \right\|_{\max} . \end{aligned}$$

Now, let $\hat{n} \in \{2, \ldots, N\}$ and assume (4.C.2) holds for all $n < \hat{n}$. Then,

$$\begin{aligned} &\max_{k=1}^{K} \left| \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\bar{L}_{\hat{n}}\big(\{G_i^{\bar{G},C^1} + C_i^1\}_{i=0}^{\hat{n}}\big) - \bar{L}_{\hat{n}}\big(\{G_i^{\bar{G},C^2} + C_i^2\}_{i=0}^{\hat{n}}\big) \,\middle|\, \mathcal{G} \right](\omega_k) \right| \\ &\quad \le \sum_{i=0}^{\hat{n}} M_{\hat{n},i} \left(\max_{k=1}^{K} \left| C_i^1(\omega_k) - C_i^2(\omega_k) \right| + \max_{k=1}^{K} \left| G_i^{\bar{G},C^1}(\omega_k) - G_i^{\bar{G},C^2}(\omega_k) \right| \right) \\ &\quad \le \sum_{i=0}^{\hat{n}} M_{\hat{n},i} \left(1 + \hat{P}_i \sum_{j=1}^{i-1} Z_j \right) \left\| C^1 - C^2 \right\|_{\max} . \end{aligned}$$

Step 2: From (4.C.2),

$$\begin{aligned} &\max_{k=1}^{K} \left| P^{\bar{T}_N}(\bar{T}_n, \omega_k)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G_j^{\bar{G},C^1} + C_j^1\}_{j=0}^i\big) \right.\right. \\ &\qquad\qquad \left.\left. - \bar{L}_i\big(\{G_j^{\bar{G},C^2} + C_j^2\}_{j=0}^i\big) \,\middle|\, \mathcal{F}(\bar{T}_n) \right](\omega_k) \right| \\ &\quad \le \hat{P}_n \left(\sum_{i=n\vee 1}^{N} Z_i \right) \left\| C^1 - C^2 \right\|_{\max} , \end{aligned}$$

such that

$$\left\| H_0^{(\bar{T},0,\bar{L})}(C^1) - H_0^{(\bar{T},0,\bar{L})}(C^2) \right\|_{\max} \le \underbrace{\left[\max_{n=0}^{N} \hat{P}_n \left(\sum_{i=n\vee 1}^{N} Z_i \right) \right]}_{<\, 1 \text{ by assumption}} \left\| C^1 - C^2 \right\|_{\max} ,$$

which concludes the proof. □

Proof of Lem. 4.3.18. For $\bar{E} \in [0,\infty)^K$, let $\mathcal{E}^{\bar{E}} := \big\{ P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \big[\, \bar{E} \,\big|\, \mathcal{F}(\bar{T}_n) \,\big] \big\}_{n=0}^{N}$ which is linear and thereby Lipschitz continuous in $\bar{E}$. Then, for all $\bar{E}^1, \bar{E}^2 \in [0,\infty)^K$,

$$
\begin{aligned}
\Big\| \, C^{\bar{E}_1} \; - \; C^{\bar{E}_2} \, \Big\| &= \Big\| \, H^{\bar{G}}_{\bar{E}_1}\big(C^{\bar{E}_1}\big) \; - \; H^{\bar{G}}_{\bar{E}_2}\big(C^{\bar{E}_2}\big) \, \Big\| \\
&= \Big\| \, \Big(H^{\bar{G}}_0\big(C^{\bar{E}_1}\big) \; + \; \mathcal{E}^{\bar{E}_1} \Big) \; - \; \Big(H^{\bar{G}}_0\big(C^{\bar{E}_2}\big) \; + \; \mathcal{E}^{\bar{E}_2} \Big) \, \Big\| \\
&\leq \Big\| \, H^{\bar{G}}_0\big(C^{\bar{E}_1}\big) \; - \; H^{\bar{G}}_0\big(C^{\bar{E}_2}\big) \, \Big\| \; + \; \Big\| \, \mathcal{E}^{\bar{E}_1} - \mathcal{E}^{\bar{E}_2} \, \Big\| \\
&\leq \Lambda \, \Big\| \, C^{\bar{E}_1} \; - \; C^{\bar{E}_2} \, \Big\| \; + \; \Big\| \, \mathcal{E}^{\bar{E}_1} - \mathcal{E}^{\bar{E}_2} \, \Big\| \quad ,
\end{aligned}
$$

where $\Lambda \in [0,1)$ is the Lipschitz constant in Def. 4.3.15. Altogether, the map $\mathcal{E}^{\bar{E}} \mapsto \mathcal{C}^{\bar{E}}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{1-\Lambda}$ and $\bar{E} \mapsto C^{\bar{E}}$ is the composition of two Lipschitz continuous functions. □

Proof of Cor. 4.3.19. For $\bar{E} \in [0,\infty)^K$, let $C^{\bar{E}} \in \mathcal{C}^{\bar{G}}$ be the (unique) hedging derivative with $C^{\bar{E}} = H^{\bar{G}}_{\bar{E}}(C^{\bar{E}})$, and consider the continuous map

$$
\bar{E} \;\mapsto\; C^{\bar{E}}_0 \;=\; P^{\bar{T}_N}(\bar{T}_0)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\; \sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C^{\bar{E}}} + C_i^{\bar{E}}\}_{i=0}^{n}\big) \; + \; \bar{E} \; \right] \quad .
$$

Since the lock-in $\bar{L}_n$, $n = 1,\ldots,N$, is bounded below by zero, there exists $\Lambda \in [0,\infty)$ with $C^{\bar{E}}_0 \geq C^0_0$ for all $\bar{E} \geq \Lambda$. In particular, the map above attains its minimum at some point $\bar{E}^\star$ of the compact set $\{\, \bar{E} \in [0,\infty)^K \;:\; \bar{E} \leq \Lambda \,\}$. But then, $\Phi^{\bar{G}} = P^{\bar{T}_N}(0)\, \bar{F} + C^{\bar{E}^\star}_0$. □

Proof of Lem. 4.3.22. This is a direct consequence of Def. 4.D.2. More precisely,

$$
\begin{aligned}
& P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\; \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G_j^{\bar{G},C} + C_j\}_{j=0}^{i}\big) \; + \; \bar{E} \;\middle|\; \mathcal{F}(\bar{T}_n) \right] \\
&\quad = \; P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\; \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G_j^{\bar{G},C} + C_j\}_{j=0}^{i}\big) \; + \; \bar{E} \;\middle|\; \mathcal{P}^{\bar{T}}_n \right] \\
&\quad = \; C_n
\end{aligned}
$$

for $n = 0,\ldots,N$, where the first equality stems from the Markov property of $P^{\bar{T}_N}$ (Def. 4.D.2) and the second equality from the fixed-point property $C = H^{\bar{G}}_{\bar{E},\mathcal{P}}(C)$. □

Proof of Lem. 4.3.27. Let $\|\cdot\| \in \mathcal{N}^{\bar{T}}$ and $Y^1, Y^2 \in \mathbb{R}^{K\times(N+1)}$. Then,

$$\begin{aligned} \| Y^1 + Y^2 \| &= \| (\|Y_0^1 + Y_0^2\|_{p_0}, \ldots, \|Y_N^1 + Y_N^2\|_{p_N})^\top \|_\star \\ &\leq \| (\|Y_0^1\|_{p_0} + \|Y_0^2\|_{p_0}, \ldots, \|Y_N^1\|_{p_N} + \|Y_N^2\|_{p_N})^\top \|_\star \\ &\leq \| Y^1 \| + \| Y^2 \| \quad , \end{aligned}$$

where the first inequality stems from the monotonicity of $\|\cdot\|_\star$. The remaining properties of a norm, namely absolute homogeneity and positive definiteness, follow directly from the definition in (4.3.10). Furthermore, from Jensen's inequality (Prop. 4.D.1),

$$\left| \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[Y \,\middle|\, \mathcal{P}_n^{\bar{T}} \right] \right|^p \leq \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[|Y|^p \,\middle|\, \mathcal{P}_n^{\bar{T}} \right]$$

for all random variables $Y \in \mathbb{R}^K$, $n = 0, \ldots, N$, and $p \in [1, \infty)$, such that $\| \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} [Y \,|\, \mathcal{P}_n^{\bar{T}}] \|_p \leq \|Y\|_p$. The analogous result holds for $p = \infty$. The rest of the assertion then follows from the monotonicity of $\|\cdot\|_\star$. The inequality

$$\left\| H_{0,\mathcal{P}}^{(\bar{T},0,\bar{L})}(C^1) - H_{0,\mathcal{P}}^{(\bar{T},0,\bar{L})}(C^2) \right\| \leq \left\| H_0^{(\bar{T},0,\bar{L})}(C^1) - H_0^{(\bar{T},0,\bar{L})}(C^2) \right\|$$

is now just a straightforward consequence of the linearity and tower property of conditional expectations (Durrett, 2010, Theo. 5.1.6). □

4.D Auxiliary Definitions and Results

Proposition 4.D.1 (Jensen's Inequality for Conditional Expectations, Musiela and Rutkowski (2005, Lem. A.1.3)) *Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function and $Y : \Omega \to \mathbb{R}$ a real-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $g(Y)$ is $\mathbb{P}$-integrable. Then, for any (sub-)σ-algebra $\mathcal{G} \subseteq \mathcal{F}$,*

$$g\big(\mathbb{E}_{\mathbb{P}}\big[\, Y \,\big|\, \mathcal{G} \,\big]\big) \;\leq\; \mathbb{E}_{\mathbb{P}}\big[\, g(Y) \,\big|\, \mathcal{G} \,\big] \quad .$$

Definition 4.D.2 (Markov Property, Musiela and Rutkowski (2005, Sec. 11.2.1)) *Let $\mathcal{I} \subseteq [0, \infty)$ be an ordered index set and $\{Y(t)\}_{t \in \mathcal{I}}$ a real-valued adapted process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \in \mathcal{I}}, \mathbb{P})$. If, for any bounded measurable function $g : \mathbb{R} \to \mathbb{R}$ and $s, t \in \mathcal{I}$ with $t \leq s$,*

$$\mathbb{E}_{\mathbb{P}}\big[\, g(Y(s)) \,\big|\, \mathcal{F}(t) \,\big] \;=\; \mathbb{E}_{\mathbb{P}}\big[\, g(Y(s)) \,\big|\, Y(t) \,\big] \quad , \tag{4.D.1}$$

then Y is said to possess the Markov property (with respect to $\{\mathcal{F}(t)\}_{t \in \mathcal{I}}$).

5 Hedging and Valuation in Incomplete Markets

In a complete financial market the Martingale Method introduced in Sec. 4.3 allows to split the problem of hedging contingent guarantees into two subproblems:

- First, a hedging derivative that super-replicates the terminal guaranteed amount is constructed.
- Second, a portfolio insurance strategy is constructed by replicating the corresponding terminal guaranteed amount (Lem. 4.3.3).

If the financial market is incomplete, then not every payoff is attainable by an investment strategy. In this case, the first step above requires additional analysis, in order to ensure that the resulting hedging derivative can indeed be replicated.

This chapter explores the problem of hedging and valuing contingent guarantees in incomplete markets. Sufficient conditions for the existence of attainable hedging derivatives and a numerical routine to construct them are derived. The notation introduced in Ch. 4 remains valid for this chapter as well and all proofs are postponed to App. 5.A.

5.1 Financial Market Model

The results of this chapter are based on a discrete-time and -state financial market, which is largely equivalent to the model of Sec. 4.1. In particular, the set of trading time points is again given by $\mathcal{T} = \{t_0, t_1, \dots, t_M\}$ with $M \in \mathbb{N}$, $t_0 = 0$, and $t_m \in (t_{m-1}, \infty)$ for $m = 1, \dots, M$, and financial risk is modeled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with sample space $\Omega = \{\omega_1, \dots, \omega_K\}$, $K \in \mathbb{N}$, filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{t \in \mathcal{T}}$ and physical probability measure $\mathbb{P}$. Furthermore, the same simplifying assumptions are made: $\mathbb{P}\big(\{\omega_k\}\big) > 0$ for $k = 1, \dots, K$, $\mathcal{F}(0) = \{\, \Omega, \emptyset \,\}$, and $\mathcal{F}(t_M) = \mathcal{F} = 2^{\Omega}$. All (in)equalities involving random variables are meant to hold $\mathbb{P}$-almost surely, i.e. for all $\omega \in \Omega$.

The financial market is arbitrage-free and contains the same $1 + M + D$, $D \in \mathbb{N}_0$, securities as in Sec. 4.1:

- A riskless bank account $B = \{B(t)\}_{t \in \mathcal{T}}$ with $B(0) = 1$;
- A riskless zero-coupon bond $P^\tau = \{P^\tau(t)\}_{t \in \mathcal{T}}$ for each maturity $\tau = t_1, \ldots, t_M$;
- Risky assets $S^d = \{S^d(t)\}_{t \in \mathcal{T}}$, $d = 1, \ldots, D$, which do not pay any coupons or dividends.

Again, price processes are positive, finite, and adapted to the filtration $\mathbb{F}$. Investment strategies are defined as in Def. 4.1.2.

Definition 5.1.1 (Set of (Forward) Pricing Measures) *Let $\mathbb{M}$ be the set of probability measures $\mathbb{Q} : \Omega \to (0, 1]$ on $(\Omega, \mathcal{F})$ with $\mathbb{Q} \sim \mathbb{P}$ and such that asset prices discounted by the bank account B are $\mathbb{Q}$-martingales. Moreover, let $\mathbb{M}^{t_M}$ be the set of probability measures $\mathbb{Q}^{t_M} : \Omega \to (0, 1]$ on $(\Omega, \mathcal{F})$, which are given by*

$$\frac{\mathrm{d}\mathbb{Q}^{t_M}}{\mathrm{d}\mathbb{Q}} \;=\; \frac{B(0)}{P^{t_M}(0)} \frac{P^{t_M}(t_M)}{B(t_M)}$$

for some $\mathbb{Q} \in \mathbb{M}$. Note that for $\mathbb{Q} \in \mathbb{M}$ and its corresponding forward measure $\mathbb{Q}^{t_M} \in \mathbb{M}^{t_M}$ the identity in Lem. 4.1.1 holds.

In contrast to the market model of Sec. 4.1, the spot pricing measure is not uniquely determined anymore: $|\mathbb{M}| > 1$ and thus $|\mathbb{M}^{t_M}| > 1$. The financial market is thereby incomplete, i.e. not every payoff can be replicated by an investment strategy (Pliska, 1997, (4.18)). Incompleteness arises most commonly in market models, where there are more 'sources of randomness' than traded assets, such as stochastic volatility models or models with jumps in asset prices (Staum, 2007). Frictions, such as transaction costs or portfolio constraints, may also cause a financial market to be incomplete.

Definition 5.1.2 (Attainable Payoffs) *A payoff $Y \in [0, \infty)^K$ at time t_M is called attainable, if there exists a replicating investment strategy $\pi \in \Pi$ with $X^\pi(t_M) = Y$.*

Lemma 5.1.3 (Föllmer and Schied (2016, Theo. 5.25 and 5.32)) *Let $Y \in [0, \infty)^K$ be an attainable payoff at time t_M and $\pi \in \Pi$ a corresponding replicating strategy. Then,*

$$X^\pi(t) \;=\; B(t)\, \mathbb{E}_{\mathbb{Q}}\left[\frac{Y}{B(s)} \,\middle|\, \mathcal{F}(t) \right]$$

for all $t \in \mathcal{T}$. In particular, the (conditional) expectation on the right-hand side above is constant in $\mathbb{Q} \in \mathbb{M}$ and independent of π. The opposite result holds as well: If $Y \in [0, \infty)^K$ is a payoff at *time t_M, such that*

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{Y}{B(t_M)}\right]$$

is constant in $\mathbb{Q} \in \mathbb{M}$, then Y is attainable.

Note that any non-negative linear combination $\lambda^1 Y^1 + \lambda^2 Y^2$ of two attainable payoffs Y^1 and Y^2 (at time t_M) with $\lambda^1, \lambda^2 \in [0, \infty)$ is again an attainable payoff (at time t_M). Moreover, with Lem. 5.1.3, the value of an attainable payoff can be calculated using the standard risk-neutral valuation formula from Sec. 4.1. This is not the case for a non-attainable payoff, however, it is still possible to give an upper bound on the price for which this payoff should trade in the market.

Definition 5.1.4 (Ask Price) *Let $Y \in [0, \infty)^K$ be a payoff (not necessarily attainable) at time t_M. Then,*

$$\text{ask}(Y) \;:=\; \sup_{\mathbb{Q} \in \mathbb{M}} \mathbb{E}_{\mathbb{Q}}\left[\frac{Y}{B(t_M)}\right] \tag{5.1.1}$$

is called the ask price of Y. Note that, if Y is attainable, the expected value above is constant in $\mathbb{Q} \in \mathbb{M}$, such that the ask price is just the usual risk-neutral value of Y.

Lemma 5.1.5 (Föllmer and Schied (2016, Cor. 7.15 and 7.18)) *Let $Y \in [0, \infty)^K$ be a payoff (not necessarily attainable) at time t_M. Then* (5.1.1) *is finite and gives the minimum initial capital necessary to super-replicate Y. More precisely, there exists an investment strategy $\hat{\pi} \in \Pi$ with*

$$X^{\hat{\pi}}(0) \;=\; \text{ask}(Y) \quad \textit{and} \quad X^{\hat{\pi}}(t_M) \;\geq\; Y \quad .$$

Moreover, for any investment strategy $\pi \in \Pi$ with $X^{\pi}(0) < X^{\hat{\pi}}(0)$ there exists $\omega \in \Omega$ with $X^{\pi}(t_m, \omega) < Y(\omega)$.

For any non-attainable payoff Y at time t_M, a super-replicating strategy $\pi \in \Pi$ will – by definition – result in a non-negative excess over Y, which is positive with positive probability (i.e. there exists $\omega \in \Omega$ with $X^{\pi}(t_M, \omega) > Y(\omega)$). Thus, the ask price in Def. 5.1.4 is not an arbitrage-free price, but rather the minimum amount an investor would charge for being short Y, i.e. for having to pay Y at time t_M. If a derivative with the non-attainable payoff Y would actually trade in the market for its ask price, this will result in an immediate arbitrage opportunity: One could sell the derivative for its ask price and use the proceeds to enter into the super-replicating strategy π (see also Föllmer and Schied, 2016, Theo. 5.32).

In the financial market introduced above, trading is still assumed to be frictionless. In particular, the fund management is not bound by any constraints and may enter into an arbitrary large long position of the terminal maturity zero-coupon bond P^{t_M}, which implies that any constant payoff at time t_M is attainable.

5.2 The Martingale Method in Incomplete Markets

The Martingale Method developed in Sec. 4.3 can be adapted to the setting of the incomplete financial market of Sec. 5.1, by placing an additional requirement on the fixed-points of the budget function. Indeed, for a given contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$, terminal excess $\bar{E} \in [0,\infty)^K$, and $C \in \mathcal{V}^{\bar{T}}$, the budget function $H^{\bar{G}}_{\bar{E}}(C)$ (Def. 4.3.1) might not even be well-defined, as the conditional expectations in (4.3.1) could take different values for different choices of the forward pricing measure $\mathbb{Q}^{\bar{T}_N} \in \mathbb{M}^{\bar{T}_N}$. With Lem. 5.1.3, it immediately transpires that C needs to satisfy an attainability criterion, in order for the expression '$H^{\bar{G}}_{\bar{E}}(C)$' to make sense.

This criterion clearly manifests itself, when considering a portfolio insurance strategy $\pi \in \Pi^{\bar{G}}$ (Def. 4.2.1) for a given contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$, since

$$X^\pi(\bar{T}_N) \;=\; \bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \quad ,$$

such that π is a replicating strategy for the payoff on the right-hand side, which is – by definition – attainable. From Lem. 5.1.3,

$$\begin{aligned} X^\pi(t) \;&=\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\,\bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \;\middle|\; \mathcal{F}(t)\right] \\ &=\; P^{\bar{T}_N}(t)\,\big(\bar{F} + V^{\bar{G},\pi}(t)\big) \;+\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{\{\,n\,:\,\bar{T}_n \geq t\,\}} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \;\middle|\; \mathcal{F}(t)\right] \\ &=\; G^{\bar{G},\pi}(t) \;+\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{\{\,n\,:\,\bar{T}_n \geq t\,\}} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \;\middle|\; \mathcal{F}(t)\right] \end{aligned}$$

and thus

$$C^{\bar{G},\pi}(t) \;=\; P^{\bar{T}_N}(t)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{\{\,n\,:\,T_n \geq t\,\}} \bar{L}_n\big(\{X^\pi(\bar{T}_i)\}_{i=0}^{n}\big) \;+\; \bar{E}^{\bar{G},\pi} \;\middle|\; \mathcal{F}(t)\right]$$

for all $t \in \mathcal{T}$ and all $\mathbb{Q}^{\bar{T}_N} \in \mathbb{M}^{\bar{T}_N}$. One thereby obtains the familiar fixed-point relation (4.2.9) for the family of random variables $\{C^{\bar{G},\pi}(\bar{T}_n)\}_{n=0}^{N}$ with the additional observation that the corresponding terminal guaranteed amount

$$\bar{F} \; + \; \sum_{n=1}^{N} \bar{L}_n\big(\big\{G^{\bar{G},\pi}(\bar{T}_i) + C^{\bar{G},\pi}(\bar{T}_i)\big\}_{i=0}^{n}\big) \; + \; \bar{E}^{\bar{G},\pi}$$

is an attainable payoff in the sense of Def. 5.1.2.

On the other hand, if, for a given family of non-negative random variables $C \in \mathcal{V}^{\bar{T}}$, the terminal excess $\bar{E} \in [0,\infty)^K$ is chosen, such that

$$\sum_{n=1}^{N} \bar{L}_n\big(\big\{G_i^{\bar{G},C} + C_i\big\}_{i=0}^{n}\big) \; + \; \bar{E} \tag{5.2.1}$$

and thereby also

$$\bar{F} \; + \; \sum_{n=1}^{N} \bar{L}_n\big(\big\{G_i^{\bar{G},C} + C_i\big\}_{i=0}^{n}\big) \; + \; \bar{E} \tag{5.2.2}$$

is attainable, then the risk-neutral value of this payoff at time $\bar{T}_n$, i.e.

$$\begin{aligned} & P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\bar{F} \; + \; \sum_{n=1}^{N} \bar{L}_n\big(\big\{G_i^{\bar{G},C} + C_i\big\}_{i=0}^{n}\big) \; + \; \bar{E} \;\middle|\; \mathcal{F}(\bar{T}_n) \right] \\ & \quad = \; P^{\bar{T}_N}(\bar{T}_n) \left(\bar{F} \; + \; \sum_{i=1}^{n-1} \bar{L}_i\big(\big\{G_j^{\bar{G},C} + C_j\big\}_{j=0}^{i}\big) \right) \\ & \qquad\qquad + \; P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{i=n\vee 1}^{N} \bar{L}_n\big(\big\{G_j^{\bar{G},C} + C_j\big\}_{j=0}^{i}\big) \; + \; \bar{E} \;\middle|\; \mathcal{F}(\bar{T}_n) \right] \\ & \quad = \; G_n^{\bar{G},C} \; + \; P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{i=n\vee 1}^{N} \bar{L}_i\big(\big\{G_j^{\bar{G},C} + C_j\big\}_{j=0}^{i}\big) \; + \; \bar{E} \;\middle|\; \mathcal{F}(\bar{T}_n) \right] \quad , \end{aligned}$$

is constant in $\mathbb{Q}^{\bar{T}_N} \in \mathbb{M}^{\bar{T}_N}$ for $n = 0, \ldots, N$. In particular, the expression '$H_{\bar{E}}^{\bar{G}}(C)$' is well-defined for this choice of C and $\bar{E}$.

By the same arguments as in Sec. 4.3, a fixed-point $C \in \mathcal{V}^{\bar{T}}$ of $H_{\bar{E}}^{\bar{G}}$, such that the total lock-in plus the terminal excess in (5.2.1) is attainable, can be interpreted as (the risk-neutral value process of) a hedging derivative, which pays the discounted lock-in $P^{\bar{T}_N}(\bar{T}_n)\, \bar{L}_n\big(\big\{G_i^{\bar{G},C} + C_i\big\}_{i=0}^{n}\big)$ at time $\bar{T}_n$, for $n = 1, \ldots, N$, and additionally the excess $\bar{E}$ at time $\bar{T}_N$. Moreover, a replicating strategy for the (attainable) payoff in (5.2.2) is also a portfolio insurance strategy for $\bar{G}$ (see Lem. 4.3.3).

Definition 5.2.1 (Attainable Hedging Derivatives, cf. Def. 4.3.2) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee. The set of attainable hedging derivatives for $\bar{G}$ is defined by*

$$\hat{\mathcal{C}}^{\bar{G}} := \Bigg\{ C \in \mathcal{V}^{\bar{T}} \;:\; \exists\, \bar{E} \in [0,\infty)^K \;\; \textit{with} \;\; C = H^{\bar{G}}_{\bar{E}}(C) \;\; \textit{and} \\ \sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{n}\big) + \bar{E} \;\; \textit{is attainable} \Bigg\} \;.$$

Definition 5.2.2 (Viability and Value of a Contingent Guarantee in an Incomplete Financial Market, cf. Def. 4.3.4) *In an incomplete financial market, a contingent guarantee $\bar{G}$ is called viable, if $\hat{\mathcal{C}}^{\bar{G}} \neq \emptyset$. Moreover, the value of $\bar{G}$ is defined as*

$$\hat{\Phi}^{\bar{G}} := P^{\bar{T}_N}(0)\,\bar{F} \;+\; \inf\Big\{ C_0 \;:\; C \in \hat{\mathcal{C}}^{\bar{G}} \Big\} \tag{5.2.3}$$

with the convention $\hat{\Phi}^{\bar{G}} := \infty$ if $\bar{G}$ is not viable.

As in the complete financial market setup, the value $\hat{\Phi}^{\bar{G}}$ gives the minimum initial capital necessary to super-replicate the terminal guaranteed amount of $\bar{G}$ with a self-financing investment strategy. The crucial question is, under what sufficient conditions fixed-points of the budget function, such that (5.2.1) is attainable, exist and how they can be constructed.

5.2.1 Minimal Super-Replication

The question of the existence of attainable hedging derivatives can be answered using the classical theory of 'minimal' super-replication (MSR) in incomplete markets introduced in Lem. 5.1.5 and the fixed-point theorems for set-valued functions of Ch. 2.

For a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ and $C \in \mathcal{V}^{\bar{T}}$, consider the set

$$\bar{E}^{\bar{G}}_{\mathrm{MSR}}(C) := \Bigg\{ \bar{E} \in [0,\infty)^K \;:\; \exists\, \pi \in \Pi \;\; \text{with} \\ \bar{E} \;=\; X^{\pi}(\bar{T}_N) \;-\; \sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{n}\big) \tag{5.2.4} \\ \text{and} \;\; X^{\pi}(0) \;=\; \text{ask}\left(\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{n}\big) \right) \Bigg\} \;,$$

which contains the non-negative terminal excesses of all minimal super-replicating strategies of the total lock-in produced by the family C. Note that $\bar{E}^{\bar{G}}_{\mathrm{MSR}}(C) \neq \emptyset$ for all $C \in \mathcal{V}^{\bar{T}}$ by Lem. 5.1.5.

Definition 5.2.3 (Minimal Super-Replicating Budget Function, cf. Def. 4.3.1) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee. The set-valued function $H^{\bar{G}}_{MSR} : \mathcal{V}^{\bar{T}} \to 2^{\mathcal{V}^{\bar{T}}}$ defined by*

$$H^{\bar{G}}_{MSR}(C) \;:=\; \Big\{ \, H^{\bar{G}}_{\bar{E}}(C) \;:\; \bar{E} \in \bar{E}^{\bar{G}}_{MSR}(C) \, \Big\}$$

for $C \in \mathcal{V}^{\bar{T}}$ is called MSR budget function.

For a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ and $C \in \mathcal{V}^{\bar{T}}$ the expression '$H^{\bar{G}}_{\bar{E}}(C)$' is well-defined for all $\bar{E} \in \bar{E}^{\bar{G}}_{\text{MSR}}(C)$, since (5.2.1) is then attainable by some (minimal) super-replicating strategy $\pi \in \Pi$. In particular, the MSR budget function is well-defined. Moreover, if C is a fixed-point of the MSR budget function, i.e. $C \in H^{\bar{G}}_{\text{MSR}}(C)$, then C is – by definition – also a fixed-point of the budget function $H^{\bar{G}}_{\bar{E}}$ for some $\bar{E} \geq 0$, such that $C \in \hat{\mathcal{C}}^{\bar{G}}$.

Technically, the set $\bar{E}^{\bar{G}}_{\text{MSR}}$ in (5.2.4) does not need to be constrained to minimal super-replicating strategies, but could contain the (non-negative) terminal excesses of just any super-replicating strategy. Under this alternative definition, a fixed-point $C \in H^{\bar{G}}_{\text{MSR}}(C)$ would still be an attainable hedging derivative. The reason for considering only minimal super-replicating strategies is given by the following lemma, which – together with other suitable conditions – allows to apply Kakutani's theorem (Prop. 2.2.2) and Nadler's theorem (Prop. 2.1.4) to answer the question of existence of attainable hedging derivatives.

Lemma 5.2.4 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee. Then, for all $C \in \mathcal{V}^{\bar{T}}$, $H^{\bar{G}}_{MSR}(C)$ is non-empty, compact, and convex.*

Proposition 5.2.5 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ continuous (Def. 4.3.5) and capped (Def. 4.3.6). Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C \in H^{\bar{G}}_{MSR}(C)$. In particular, $\bar{G}$ is viable.*

Proposition 5.2.6 *Let $\bar{T}$ be a set of lock-in time points and $\bar{L}$ a corresponding lock-in mechanism, such that the MSR budget function $H^{(\bar{T},0,\bar{L})}_{MSR}$ is contracting (Def. 2.1.3), i.e. there exists $\Lambda \in [0, 1)$ with*

$$h\Big(\, H^{(\bar{T},0,\bar{L})}_{MSR}(C^1), \; H^{(\bar{T},0,\bar{L})}_{MSR}(C^2) \, \Big) \;\leq\; \Lambda \, \| \, C^1 \;-\; C^2 \, \| \tag{5.2.5}$$

for all $C^1, C^2 \in \mathcal{V}^{\bar{T}}$, where $\| \cdot \|$ is a norm on $\mathbb{R}^{K \times (N+1)}$ and h denotes the corresponding Hausdorff metric (see Sec. 2.1). Then, for all fixed guarantees $\bar{F} \in [0, \infty)$, there exists $C \in \mathcal{V}^{\bar{T}}$ with $C \in H^{\bar{G}}_{MSR}(C)$, where $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$. In particular, $\bar{G}$ is viable.

As in the setting of a complete financial market, the contraction condition of the 'metric' viability result (Prop. 5.2.6) is generally quite hard to prove. The fact that this condition cannot even be tested numerically – e.g. using a fixed-point iteration – further aggravates this situation. Again, the heuristic argument that the lock-in $\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big)$ does not 'grow faster' than the (history of the) NAV $\{X(\bar{T}_i)\}_{i=0}^n$ must suffice.

Note that, with the viability results above, one has no longer control over the resulting terminal excess, as was the case in the complete market setup. For a given fixed-point $C \in H_{\mathrm{MSR}}^{\bar{G}}(C)$ of the MSR budget function, the terminal excess of a corresponding portfolio insurance is now given implicitly by $\bar{E} \in \bar{E}_{\mathrm{MSR}}^{\bar{G}}(C)$ with $C = H_{\bar{E}}^{\bar{G}}(C)$. Moreover, the results of this section are non-constructive and thus of rather theoretical nature.

5.2.2 Static Super-Replication

An intuitive and particularly transparent approach to actually construct attainable hedging derivatives is to consider a family of conveniently parameterized investment strategies, whose terminal portfolio values are – by definition – attainable payoffs. The question is then to identify portfolio insurance strategies out of this set of investment policies, or, more precisely, to identify which of those investment policies are solutions of the portfolio fixed-point problem of Sec. 4.3.6.

This section presents sufficient conditions for the existence of attainable hedging derivatives, which can be replicated by a simple static long position in the terminal zero-coupon bond $P^{\bar{T}_N}$. The resulting portfolio insurance strategies are then static super-replicating (SSR) strategies for the terminal guaranteed amount.

For a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ and a family of random variables $C \in \mathcal{V}^{\bar{T}}$, consider a 'buy-and-hold' strategy of

$$\bar{F} \;+\; \Lambda_{\mathrm{SSR}}^{\bar{G}}(C) \;:=\; \bar{F} \;+\; \left\| \;\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big)\;\right\|_\infty \tag{5.2.6}$$

zero-coupon bonds $P^{\bar{T}_N}$, where

$$\|Y\|_\infty \;=\; \max_{k=1}^{K} |\,Y(\omega_k)\,|$$

is the ∞-norm of a random variable $Y : \Omega \to \mathbb{R}$ (Def. 4.3.26).

The NAV process corresponding to this static investment policy, which is denoted by $\pi^{\mathrm{SSR}}(C) \in \Pi$, is then given by

$$X^{\pi^{\mathrm{SSR}}(C)}(t) \;=\; P^{\bar{T}_N}(t)\,\big(\bar{F} + \Lambda_{\mathrm{SSR}}^{\bar{G}}(C)\big)$$

for $t \in \mathcal{T}$, which results in the terminal excess

$$
\begin{aligned}
\bar{E}^{\bar{G},\pi^{\text{SSR}}(C)} &= X^{\pi^{\text{SSR}}(C)}(\bar{T}_N) - \left(\bar{F} + \sum_{n=1}^{N} \bar{L}_n\big(\{X^{\pi^{\text{SSR}}(C)}(\bar{T}_i)\}_{i=0}^{n}\big) \right) \\
&= P^{\bar{T}_N}(\bar{T}_N)\big(\bar{F} + \Lambda^{\bar{G}}_{\text{SSR}}(C)\big) - \left(\bar{F} + \sum_{n=1}^{N} \bar{L}_n\big(\{X^{\pi^{\text{SSR}}(C)}(\bar{T}_i)\}_{i=0}^{n}\big) \right) \\
&= \Lambda^{\bar{G}}_{\text{SSR}}(C) - \sum_{n=1}^{N} \bar{L}_n\big(\{X^{\pi^{\text{SSR}}(C)}(\bar{T}_i)\}_{i=0}^{n}\big) \quad .
\end{aligned}
$$

For a given $C \in \mathcal{V}^{\bar{T}}$, the terminal excess $\bar{E}^{\bar{G},\pi^{\text{SSR}}(C)}$ is not necessarily non-negative, such that generally $\pi^{\text{SSR}}(C) \notin \Pi^{\bar{G}}$.

If, however, $C \in \mathcal{V}^{\bar{T}}$ is chosen, such that $C = H^{\bar{G}}_{\bar{E}^{\bar{G},\pi^{\text{SSR}}(C)}}(C)$, which implies

$$
G^{\bar{G},\pi^{\text{SSR}}(C)}(\bar{T}_n) = G^{\bar{G},C}_n \quad \text{and} \quad C^{\bar{G},\pi^{\text{SSR}}(C)}(\bar{T}_n) = C_n
$$

for $n = 0, \ldots, N$ by Lem. 4.3.3, then

$$
\begin{aligned}
\bar{E}^{\bar{G},\pi^{\text{SSR}}(C)} &= \Lambda^{\bar{G}}_{\text{SSR}}(C) - \sum_{n=1}^{N} \bar{L}_n\big(\{G^{\bar{G},\pi^{\text{SSR}}(C)}(\bar{T}_i) + C^{\bar{G},\pi^{\text{SSR}}(C)}(\bar{T}_i)\}_{i=0}^{n}\big) \\
&= \Lambda^{\bar{G}}_{\text{SSR}}(C) - \sum_{n=1}^{N} \bar{L}_n\big(\{G^{\bar{G},C}_i + C_i\}_{i=0}^{i}\big)
\end{aligned}
\tag{5.2.7}
$$

is non-negative by definition of $\Lambda^{\bar{G}}_{\text{SSR}}$ in (5.2.6).

Definition 5.2.7 (Static Super-Replicating Budget Function, cf. Def. 4.3.1) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee. The function $H^{\bar{G}}_{SSR} : \mathcal{V}^{\bar{T}} \to \mathcal{V}^{\bar{T}}$ defined by*

$$
\begin{aligned}
H^{\bar{G}}_{SSR}(C) &:= H^{\bar{G}}_{\bar{E}^{\bar{G},\pi^{SSR}(C)}}(C) \\
&= \Bigg\{ P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\Bigg[\sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \\
&\qquad + \left(\Lambda^{\bar{G}}_{SSR}(C) - \sum_{i=1}^{N} \bar{L}_i\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \right) \Bigg| \mathcal{F}(\bar{T}_n) \Bigg] \Bigg\}_{n=0}^{N} \\
&= \left\{ P^{\bar{T}_N}(\bar{T}_n) \left(\Lambda^{\bar{G}}_{SSR}(C) - \sum_{i=1}^{n-1} \bar{L}_i\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \right) \right\}_{n=0}^{N}
\end{aligned}
$$

is called SSR budget function.

For a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$, a fixed-point $C = H^{\bar{G}}_{\mathrm{SSR}}(C)$ of the SSR budget function is – by definition – also a fixed-point of the budget function $H^{\bar{G}}_{\bar{E}}$ with $\bar{E} = \bar{E}^{\bar{G},\pi^{\mathrm{SSR}}(C)} \geq 0$. Moreover, the total lock-in plus the terminal excess in (5.2.1), i.e.

$$\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{n=0}^{N}\big) \; + \; \bar{E} \; = \; \Lambda^{\bar{G}}_{\mathrm{SSR}}(C) \quad ,$$

is attainable by a simple static long position of $\Lambda^{\bar{G}}_{\mathrm{SSR}}(C)$ zero-coupon bonds $P^{\bar{T}_N}$. Altogether, C is an attainable hedging derivative, i.e. $C \in \hat{\mathcal{C}}^{\bar{G}}$, which corresponds to the static portfolio insurance strategy $\pi^{\mathrm{SSR}}(C) \in \Pi^{\bar{G}}$. Moreover, any fixed-point $C = \{C_n\}_{n=0}^{N}$ of $H^{\bar{G}}_{\mathrm{SSR}}$ is actually a (path-dependent) interest rate derivative, since C_n is $\sigma\big(\{P^{\bar{T}_N}(\bar{T}_i)\}_{i=0}^{n}\big)$-measurable for $n = 0, \ldots, N$ (see Sec. 4.3.5).

Similar to the results of Sec. 4.3.1 and 4.3.3, which give sufficient conditions for the viability of a contingent guarantee in a complete financial market, sufficient conditions for the existence of fixed-points of the SSR budget function can be derived from Brouwer's theorem (Prop.2.2.1) and Banach's theorem (Prop.2.1.2).

Proposition 5.2.8 (cf. Prop. 4.3.7) *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee with $\bar{L}$ continuous (Def. 4.3.5) and capped (Def. 4.3.6). Then there exists $C \in \mathcal{V}^{\bar{T}}$ with $C = H^{\bar{G}}_{SSR}(C)$. In particular, $\bar{G}$ is viable.*

Proposition 5.2.9 (cf. Prop. 4.3.16) *Let $\bar{T}$ be a set of lock-in time points and $\bar{L}$ a corresponding lock-in mechanism, such that the SSR budget function $H^{(\bar{T},0,\bar{L})}_{SSR}$ is contracting (Def. 2.1.1), i.e. there exists $\Lambda \in [0, 1)$ with*

$$\| \, H^{(\bar{T},0,\bar{L})}_{SSR}(C^1) \; - \; H^{(\bar{T},0,\bar{L})}_{SSR}(C^2) \, \| \; \leq \; \Lambda \, \| \, C^1 \; - \; C^2 \, \|$$

for all $C^1, C^2 \in \mathcal{V}^{\bar{T}}$. Then, for all fixed guarantees $\bar{F} \in [0, \infty)$, there exists a unique $C \in \mathcal{V}^{\bar{T}}$ with $C \in H^{\bar{G}}_{MSR}(C)$, where $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$, and the sequence $\{{}_{(u)}C\}_{u \in \mathbb{N}_0} \subset \mathcal{V}^{\bar{T}}$ defined by ${}_{(u+1)}C := H^{\bar{G}}_{SSR}({}_{(u)}C)$ converges to C for any starting value ${}_{(0)}C \in \mathcal{V}^{\bar{T}}$. In particular, $\bar{G}$ is viable.

The simple and robust nature of static portfolio insurance strategies makes them quite appealing from a practical point of view: They do not require any rebalancing of the fund's assets and thereby reduce operational costs and risks. Moreover, the market for (riskless zero-)coupon bonds is highly liquid. Unsurprisingly, these benefits of SSR strategies come at an increased initial cost for implementing the corresponding hedging strategy (see Sec. 5.3).

With the definition of $\Lambda^{\bar{G}}_{\mathrm{SSR}}$ in (5.2.6), SSR strategies can be thought of as 'worst-case' portfolio insurance strategies, in the sense that they assume the maximum lock-in to occur with absolute certainty. In many incomplete financial market models, the ask price of common payoffs is precisely equal to such a worst-case bound. For example, in stochastic volatility models or models with jumps in asset prices, the minimal super-replicating strategy of a call option is given by a static long position in the underlying (see, e.g., Frey and Sin, 1999; Eberlein and Jacod, 1997). It is reasonable to assume that this is also the case for most contingent guarantees, which would then imply $H^{\bar{G}}_{\text{SSR}}(C) \in H^{\bar{G}}_{\text{MSR}}(C)$ for all $C \in \mathcal{V}^{\bar{T}}$.

The usual remedy to this rather unsatisfying situation is to loosen the hedging condition in Def. 4.2.1 from $\mathbb{P}$-a.s., i.e. for all $\omega \in \Omega$, to some subset of Ω containing the 'relevant' or 'most likely' outcomes. This approach then leads to a partial hedging problem (see, e.g., Föllmer and Schied, 2016, Sec. 8).

5.3 Numerical Case Study

With Banach's theorem (Prop. 2.1.2), a static super-replicating strategy for a contingent guarantee can be constructed with a fixed-point iteration using the SSR budget function. In contrast to the fixed-point iteration developed in Sec. 4.4 (App. 4.B), an evaluation of the SSR budget function does not require the computation of conditional expectations, such that – at least technically – one is no longer constrained to interest rate models with Markovian bond prices, in order for the resulting algorithm to be computationally feasible.

The purpose of this section is to implement a fixed-point iteration using the SSR budget function and to compare the required initial capital of the resulting static hedging strategies with the results obtained in Sec. 4.4.

5.3.1 Implementation

Analogous to Sec. 4.4, the iteration

$$ {}_{(u+1)}C \;=\; H^{\bar{G}}_{\text{SSR}}({}_{(u)}C)\,, \quad u \in \mathbb{N}_0\,, \tag{5.3.1} $$

is implemented in the Hull–White trinomial model (see App. 4.A, Hull and White (1993)). The details are given in Alg. 2 (App. 5.B).

The algorithm requires an initial guess ${}_{(0)}C$ and runs until one of the following two stopping criteria is satisfied:

1. The norm of the difference of two consecutive iterates ${}_{(u)}C$ and ${}_{(u+1)}C$ is less than a prespecified tolerance $\text{tol} \in (0, \infty)$;

2. The number of iterations has reached a prespecified maximum $u_\infty \in \mathbb{N}$.

In case 2 the algorithm did not converge, which, of course, does not necessarily imply that there does not exist a fixed-point of $H_{\text{SSR}}^{\bar{G}}$. In case 1 the algorithm constructs, or rather approximates, a hedging derivative $C^{\text{SSR}} \in \hat{\mathcal{C}}^{\bar{G}}$ with $C^{\text{SSR}} = H_{\text{SSR}}^{\bar{G}}(C^{\text{SSR}})$. The corresponding SSR strategy $\pi^{\text{SSR}}(C^{\text{SSR}}) \in \Pi^{\bar{G}}$ is then given by investing $P^{\bar{T}_N}(0)\,\bar{F} + C_0^{\text{SSR}} = P^{\bar{T}_N}(0)\left(\bar{F} + \Lambda_{\text{SSR}}^{\bar{G}}(C^{\text{SSR}})\right)$ into the terminal maturity zero-coupon bond $P^{\bar{T}_N}$ at time $t = 0$. A sufficient condition for the convergence of Alg. 2 is the contraction property of $H_{\text{SSR}}^{\bar{G}}$ in Prop. 5.2.9.

5.3.2 Results

The fixed-point algorithm is applied to the same contingent guarantees $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$, which were already considered in Sec. 4.4, i.e. $\bar{F} = 1$, $\bar{T}_n = n$ for $n = 0, \dots, N$, and $\bar{L}$ given by one of the following:

$$\text{Lock-In I:} \quad \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) = \alpha \Big[\, X(\bar{T}_n) - X(\bar{T}_{n-1}) \,\Big]^+,$$

$$\text{Lock-In II:} \quad \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) = \alpha \Big[\, X(\bar{T}_n) - \Big(1 + \textstyle\sum_{i=1}^{n-1} \bar{L}_i\big(\{X(\bar{T}_j)\}_{j=0}^{i}\big)\Big)\Big]^+,$$

$$\text{Lock-In III:} \quad \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) = \alpha \Big[\, X(\bar{T}_n) - \tfrac{1}{n+1} \textstyle\sum_{i=0}^{n} X(\bar{T}_i) \,\Big]^+,$$

for $n = 1, \dots, N$, where $\alpha \in [0, 1]$.

Again, the trinomial tree is discretized with four quarterly time steps between the lock-in time points. The algorithm is implemented in MATLAB on a computer with a 3.2 GHz CPU and 16 GB RAM, and the parameters are set to $\text{tol} = 10^{-6}$, $u_\infty = 2500$, and ${}_{(0)}C = 0$. The convergence criterion in step 17 of Alg. 2 is evaluated using the maximum-norm in (4.3.7).

The Impact of Guarantee Parameters

To study the impact of the parameters of the contingent guarantee on the cost of implementing a static hedging strategy, N is varied from 5 to 8 and the 'lock-in rate' α from 10% to 50%. The model parameters are set to $\kappa = 0.1000$, $\sigma = 0.0100$, and $P^{t_m}(0) = 1$ for all $m = 1, \dots, M$. Results are reported in Tab. 5.1, which can be found at the end of this section.

As expected, the price C_0^{SSR} for all three lock-in mechanisms increases monotonically in both N and α. Moreover, a comparison with Tab. 4.1 shows that C_0^{SSR} is always higher than the cost $C_0^{\star}$ of the hedging derivative corresponding to a zero terminal excess. In some cases, the cost increases more than 10-fold and amounts to a large fraction of the cost $P^{\bar{T}_N}(0)\,\bar{F} = 1$ of the fixed guaranteed amount.

Note that there are cases, where iteration (5.3.1) using the SSR budget function $H^{\bar{G}}_{\mathrm{SSR}}$ converges, but not iteration (4.4.1) using the budget function $H^{\bar{G}}_{0,\mathcal{P}}$ (and vice versa). Moreover, the number of iterations needed to satisfy the convergence criterion is generally lower for the SSR iteration using Alg. 2.

The numerical results suggest that the construction of an SSR portfolio insurance strategy is generally practical for lock-in mechanisms, which calculate the lock-in based on a comparison of portfolio values, as Lock-In I and III. In contrast, lock-in mechanisms, which calculate the lock-in based on a comparison of the portfolio value with the (undiscounted) guaranteed amount, as Lock-In II, will often not admit an SSR strategy. Indeed, the numerical results of Sec. 4.4.2 already suggest that latter mechanisms can lead to quite extreme scenarios of the total lock-in (see Fig. 4.2).

Fig. 5.1 and 5.2 show the $\mathbb{Q}$-distribution of the total lock-in

$$\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C^{\mathrm{SSR}}} + C_i^{\mathrm{SSR}}\}_{i=0}^{n}\big) \tag{5.3.2}$$

for lock-in mechanisms Lock-In I and III, and different choices of N.

When comparing Fig. 4.1 and 4.3 with Fig. 5.1 and 5.2 an interesting observation can be made: the maximum realized value of the total lock-in (5.3.2) generated by C^{SSR} is actually lower than the maximum realized value of the total lock-in (4.4.5) generated by $C^{\star}$. This effect is easily explained by the fact that, for both lock-in mechanisms, the lock-in $\bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big)$ is decreasing in $\{X(\bar{T}_i)\}_{i=0}^{n-1}$. In other words, higher initial portfolio values (as they are required for the SSR strategies) will generally lead to a lower lock-in at later time points (at least for the lock-in mechanisms considered in this case study).

The Impact of Market Parameters

The numerical case study is closed with an assessment of the impact of the market environment on the required initial capital to implement a static portfolio insurance strategy. For this purpose, the same three initial yield curves as in Sec. 4.4.2 are considered:

Curve A: $r^{t_m}(0) = -\frac{1}{t_m}\ln P^{t_m}(0) = 0.0050$ for all $m = 1, \ldots, M$, such that $P^6(0)\,\bar{F} = 0.9753$;

Curve B: $r^{t_m}(0) = 0.0000$ for all $m = 1, \ldots, M$, such that $P^6(0)\,\bar{F} = 1.0000$

Curve C: $r^{t_m}(0) = -0.0050$ for all $m = 1, \ldots, M$, such that $P^6(0)\,\bar{F} = 1.0253$.

Moreover, the spot rate volatility σ is varied from 0.0080 to 0.0120. The remaining parameters are set to $\kappa = 0.1000$, $N = 6$, and $\alpha = 30\%$. Results are reported in Tab. 5.2.

As for $C_0^\star$, an increase of the interest rate volatility σ results in an increase of C_0^{SSR}, because larger movements of bond price $P^{\bar{T}_N}$ will (generally) result in a larger lock-in.

Moreover, for lock-in mechanisms Lock-In I and III, the price C_0^{SSR} also generally decreases with decreasing market interest rates $r^{t_m}(0)$. However, the relative effect of a decreasing initial yield curve is much weaker for static portfolio insurance strategies and quickly fades with increasing interest rate volatility. For mechanism Lock-In I and $\sigma = 0.0120$, the effect actually reverses.

Lock-In Mechanism	α	N = 5	6	7	8
Lock-In I (excess over previous NAV)	10%	0.0210 (5)	0.0317 (5)	0.0448 (6)	0.0609 (6)
	30%	0.0659 (7)	0.1015 (9)	0.1476 (12)	0.2080 (13)
	50%	0.1149 (12)	0.1814 (12)	0.2729 (19)	0.4025 (22)
Lock-In II (excess over guaranteed amount)	10%	0.0955 (9)	0.1949 (11)	0.3800 (14)	0.7521 (19)
	30%	1.1135 (41)	–	–	–
	50%	–	–	–	–
Lock-In III (excess over average NAV)	10%	0.0239 (5)	0.0428 (5)	0.0721 (7)	0.1082 (7)
	30%	0.0755 (9)	0.1402 (10)	0.2526 (12)	0.4140 (22)
	50%	0.1324 (13)	0.2579 (16)	0.5063 (30)	–

Table 5.1: The price C_0^{SSR} for different values of N and α. Cases, in which Alg. 2 does not converge, are marked by '–'. The number of iterations is reported in parenthesis. Results have been rounded.

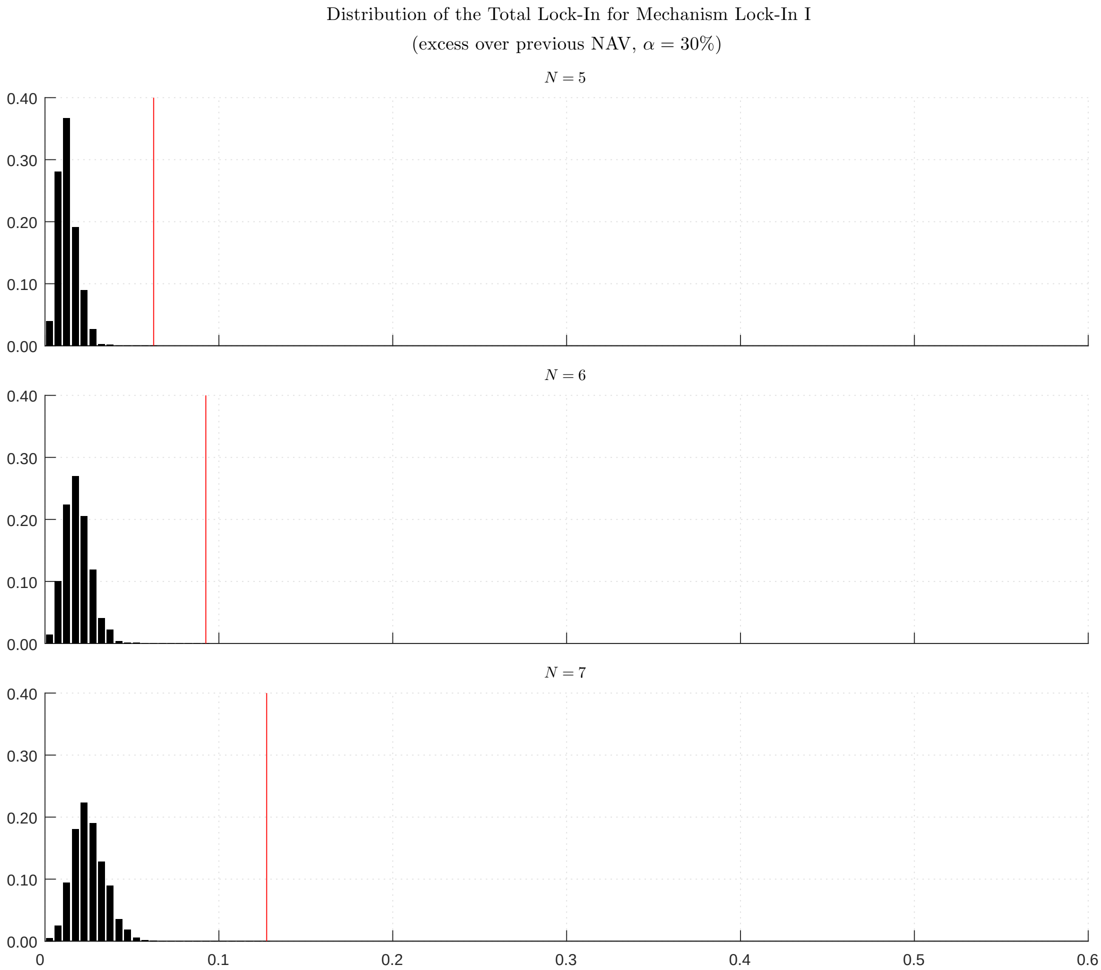

Figure 5.1: Distribution of the total lock-in (5.3.2) under $\mathbb{Q}$ for lock-in mechanism Lock-In I with the lock-in rate set to $\alpha = 30\%$, and the number of lock-in time points set to $N = 5$ (top), $N = 6$ (center), and $N = 7$ (bottom). The maximum realized values (marked by dashed red lines) are given by 0.0625 ($N = 5$), 0.0925 ($N = 6$), and 0.1275 ($N = 7$).

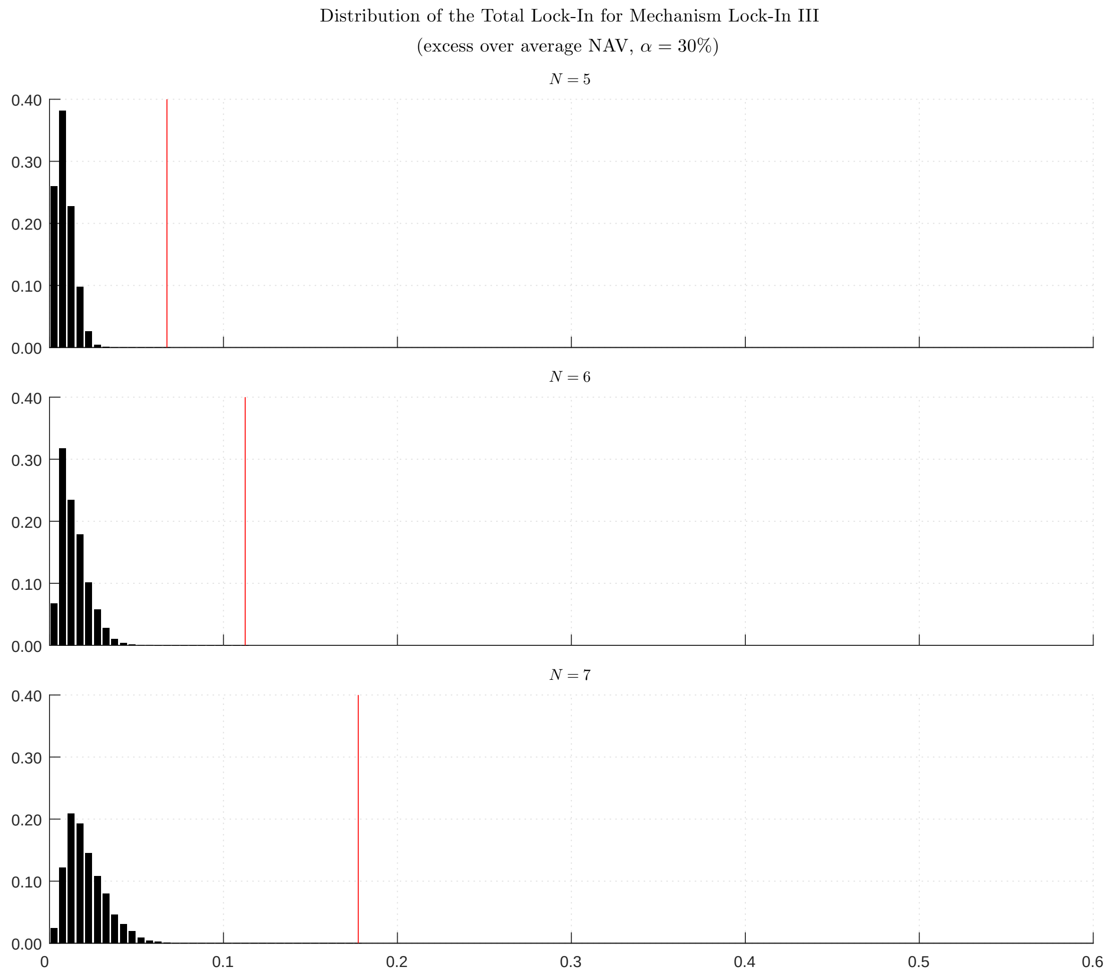

Figure 5.2: Distribution of the total lock-in (5.3.2) under $\mathbb{Q}$ for lock-in mechanism Lock-In III with the lock-in rate set to $\alpha = 30\%$, and the number of lock-in time points set to $N = 5$ (top), $N = 6$ (center), and $N = 7$ (bottom). The maximum realized values (marked by dashed red lines) are given by 0.0675 ($N = 5$), 0.1125 ($N = 6$), and 0.1775 ($N = 7$).

Lock-In Mechanism	Curve	σ				
		0.0080	0.0090	0.0100	0.0110	0.0120
Lock-In I (excess over previous NAV)	A (50 bp)	0.0788	0.0901	0.1020	0.1145	0.1276
	B (0 bp)	0.0773	0.0891	0.1015	0.1144	0.1281
	C (-50 bp)	0.0755	0.0877	0.1006	0.1140	0.1282
Lock-In III (excess over average NAV)	A (50 bp)	0.1109	0.1265	0.1436	0.1617	0.1809
	B (0 bp)	0.1060	0.1226	0.1402	0.1589	0.1786
	C (-50 bp)	0.1011	0.1182	0.1363	0.1555	0.1758

Table 5.2: The price C_0^{SSR} for different initial yield curves and interest rate volatilities σ. Results have been rounded.

Appendices

5.A Proofs

Proof of Lem. 5.2.4. Let $C \in \mathcal{V}^{\bar{T}}$ be fixed. The assertion is proved in four steps.
Step 1: $H^{\bar{G}}_{\text{MSR}}(C)$ is non-empty. This is a direct consequence of Lem. 5.1.5, which states that $\bar{E}^{\bar{G}}_{\text{MSR}}(C)$ and thereby also $H^{\bar{G}}_{\text{MSR}}(C)$ is non-empty.
Step 2: $H^{\bar{G}}_{\text{MSR}}(C)$ is bounded. Recall the definition of $\Lambda^{\bar{G}}_{\text{SSR}}(C)$ in (5.2.6). Since the constant payoff $\Lambda^{\bar{G}}_{\text{SSR}}(C)$ is attainable (by a simple static long position in the zero-coupon bond $P^{\bar{T}_N}$), one obtains

$$\Lambda^{\bar{G}}_{\text{SSR}}(C) \;\geq\; \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \sum_{n=1}^{N} \bar{L}_n\big(\{G^{\bar{G},C}_i + C_i\}_{i=0}^{n}\big) \;+\; \bar{E} \,\right]$$

for all $\bar{E} \in \bar{E}^{\bar{G}}_{\text{MSR}}(C)$ and $\mathbb{Q}^{\bar{T}_N} \in \mathbb{M}^{\bar{T}_N}$ by the minimality of the super-replicating strategies generating the set $\bar{E}^{\bar{G}}_{\text{MSR}}(C)$. In particular, $\bar{E}^{\bar{G}}_{\text{MSR}}(C)$ and thereby also $H^{\bar{G}}_{\text{MSR}}(C)$ is bounded.
Step 3: $H^{\bar{G}}_{\text{MSR}}(C)$ is closed. Let $\{{}_{(u)}\bar{E}\}_{u\in\mathbb{N}_0} \subseteq \bar{E}^{\bar{G}}_{\text{MSR}}(C)$, such that $\{H^{\bar{G}}_{{}_{(u)}\bar{E}}(C)\}_{u\in\mathbb{N}_0} \subseteq H^{\bar{G}}_{\text{MSR}}(C)$ is a converging sequence with $H^{\bar{G}}_{{}_{(u)}\bar{E}}(C) \to H^\star = \{H^\star_n\}_{n=0}^N \in \mathcal{V}^{\bar{T}}$ for $u \to \infty$. Then, by definition of the budget function $H^{\bar{G}}_{{}_{(u)}\bar{E}}$ (Def. 4.3.1),

$$\bar{L}_N\big(\{G^{\bar{G},C}_n + C_n\}_{n=0}^{N}\big) \;+\; {}_{(u)}\bar{E} \quad \xrightarrow{u\to\infty} \quad H^\star_N$$

and thus

$${}_{(u)}\bar{E} \quad \xrightarrow{u\to\infty} \quad H^\star_N \;-\; \bar{L}_N\big(\{G^{\bar{G},C}_n + C_n\}_{n=0}^{N}\big) \;=:\; \bar{E}^\star \;\in [0,\infty)^K \quad ,$$

such that

$$\begin{aligned} &P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \;+\; {}_{(u)}\bar{E} \,\middle|\, \mathcal{F}(\bar{T}_n) \right] \\ &\xrightarrow{u\to\infty} \quad P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G^{\bar{G},C}_j + C_j\}_{j=0}^{i}\big) \;+\; \bar{E}^\star \,\middle|\, \mathcal{F}(\bar{T}_n) \right] \;=\; H^\star_n \end{aligned}$$

for $n = 0, \ldots, N$ and all $\mathbb{Q}^{\bar{T}_N} \in \mathbb{M}^{\bar{T}_N}$. In particular, $H^\star = H^{\bar{G}}_{\bar{E}^\star}(C)$. Moreover,

$$\begin{aligned} \operatorname{ask}\left(\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big)\right) &= P^{\bar{T}_N}(\bar{T}_0)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big) \;+\; {}_{(u)}\bar{E}\right] \\ &= P^{\bar{T}_N}(\bar{T}_0)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big) \;+\; \bar{E}^\star\right] \end{aligned}$$

for all $u \in \mathbb{N}_0$ and all $\mathbb{Q}^{\bar{T}_N} \in \mathbb{M}^{\bar{T}_N}$ by definition of $\bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$. With Lem. 5.1.3,

$$\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big) \;+\; \bar{E}^\star$$

is attainable and thus $\bar{E}^\star \in \bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$.

Step 4: $H^{\bar{G}}_{\mathrm{MSR}}(C)$ is convex. Let $\bar{E}^1, \bar{E}^2 \in \bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$ and $\lambda \in [0,1]$. Then,

$$\lambda\, H^{\bar{G}}_{\bar{E}^1}(C) \;+\; (1-\lambda)\, H^{\bar{G}}_{\bar{E}^2}(C) \;=\; H^{\bar{G}}_{\lambda \bar{E}^1 + (1-\lambda)\bar{E}^2}(C)$$

and, by definition of $\bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$,

$$\begin{aligned} &P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big) \;+\; \lambda\,\bar{E}^1 \;+\; (1-\lambda)\,\bar{E}^2\right] \\ &\quad= \lambda\, P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big) \;+\; \bar{E}^1\right] \\ &\qquad\quad+\; (1-\lambda)\, P^{\bar{T}_N}(\bar{T}_n)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big) \;+\; \bar{E}^2\right] \\ &\quad= \operatorname{ask}\left(\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big)\right) \end{aligned}$$

for all $\mathbb{Q}^{\bar{T}_N} \in \mathbb{M}^{\bar{T}_N}$. With Lem. 5.1.3, $\lambda\,\bar{E}^1 + (1-\lambda)\,\bar{E}^2 \in \bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$ and thus also $\lambda\, H^{\bar{G}}_{\bar{E}^1}(C) + (1-\lambda)\, H^{\bar{G}}_{\bar{E}^2}(C) \in H^{\bar{G}}_{\mathrm{MSR}}(C)$. □

Proof of Prop. 5.2.5. Let $\hat{P} := \max_{n=0}^{N} \max_{k=1}^{K} P^{\bar{T}_N}(\bar{T}_n, \omega_k)$, $\Lambda \in [0,\infty)$ as in Def. 4.3.6 and recall the definition of $\Lambda^{\bar{G}}_{\mathrm{SSR}}(C)$ in (5.2.6). Then,

$$N\,\Lambda \;\geq\; \Lambda^{\bar{G}}_{\mathrm{SSR}}(C) \;\geq\; \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^n\big) \;+\; \bar{E}\right]$$

for all $C \in \mathcal{V}^{\bar{T}_N}$ and $\bar{E} \in \bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$, since $\Lambda^{\bar{G}}_{\mathrm{SSR}}(C)$ is an attainable payoff and any $\bar{E}$ corresponds to a minimal super-replicating strategy of the total lock-in produced by C. In particular, there exists an upper bound $\hat{\Lambda} \in [0, \infty)$, such that

$$\hat{\Lambda} \;\geq\; \sum_{n=1}^{N} \bar{L}_n\big(\{G_i^{\bar{G},C} + C_i\}_{i=0}^{n}\big) \;+\; \bar{E}$$

for all $C \in \mathcal{V}^{\bar{T}}$ and $\bar{E} \in \bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$. Altogether, $H^{\bar{G}}_{\mathrm{MSR}}$ maps the non-empty, compact and convex set

$$A \;:=\; \big\{\; C \in \mathcal{V}^{\bar{T}} \;:\; C_{k,n} \;\leq\; \hat{P}\,\hat{\Lambda} \;\;\forall\, k = 1, \ldots, K,\; n = 0, \ldots, N \;\big\}$$

to its power set 2^A. Moreover, with Lem. 5.2.4, $H^{\bar{G}}_{\mathrm{MSR}}(C)$ is non-empty and convex for all $C \in \mathcal{V}^{\bar{T}}$, and it remains to show that the graph of $H^{\bar{G}}_{\mathrm{MSR}}$ is closed. Let $\{{}_{(u)}C\}_{u \in \mathbb{N}_0} \subseteq A$ and ${}_{(u)}\bar{E} \in \bar{E}^{\bar{G}}_{\mathrm{MSR}}({}_{(u)}C)$, such that ${}_{(u)}C \to C^\star$ and $H^{\bar{G}}_{{}_{(u)}\bar{E}}({}_{(u)}C) \to H^\star = \{H^\star_n\}_{n=0}^{N} \in \mathcal{V}^{\bar{T}}$ for $u \to \infty$. Then, by definition of the budget function $H^{\bar{G}}_{{}_{(u)}\bar{E}}$ (Def. 4.3.1),

$$\bar{L}_N\big(\{G_n^{\bar{G},{}_{(u)}C} + {}_{(u)}C_n\}_{n=0}^{N}\big) \;+\; {}_{(u)}\bar{E} \quad \xrightarrow{u\to\infty} \quad H^\star_N$$

and thus,

$${}_{(u)}\bar{E} \quad \xrightarrow{u\to\infty} \quad H^\star_N \;-\; \bar{L}_N\big(\{G_n^{\bar{G},C^\star} + C^\star_n\}_{n=0}^{N}\big) \;=:\; \bar{E}^\star \;\in [0,\infty)^K \quad ,$$

since $\bar{L}$ is continuous. By the same arguments as in step 3 of the proof of Lem. 5.2.4, one has $\bar{E}^\star \in \bar{E}^{\bar{G}}_{\mathrm{MSR}}(C^\star)$ and $H^\star = H^{\bar{G}}_{\bar{E}^\star}(C^\star) \in H^{\bar{G}}_{\mathrm{MSR}}(C^\star)$. Altogether, there exists a fixed-point $C \in A$ with $C \in H^{\bar{G}}_{\mathrm{MSR}}(C)$ by Kakutani's theorem (Prop. 2.2.2). □

Proof of Prop. 5.2.6. Let $C^1, C^2 \in \mathcal{V}^{\bar{T}}$. Then,

$$\begin{aligned} h\Big(\, H^{(\bar{T},\bar{F},\bar{L})}_{\mathrm{MSR}}(C^1),\; H^{(\bar{T},\bar{F},\bar{L})}_{\mathrm{MSR}}(C^2) \,\Big) \;&=\; h\Big(\, H^{(\bar{T},0,\bar{L})}_{\mathrm{MSR}}(\tilde{C}^1),\; H^{(\bar{T},0,\bar{L})}_{\mathrm{MSR}}(\tilde{C}^2) \,\Big) \\ &\leq\; \Lambda \,\|\, \tilde{C}^1 \;-\; \tilde{C}^2 \,\| \\ &=\; \Lambda \,\|\, C^1 \;-\; C^2 \,\| \quad , \end{aligned}$$

where $\tilde{C}^i := \big\{\, C^i_n + P^{\bar{T}_N}(\bar{T}_n)\,\bar{F} \,\big\}_{n=0}^{N} \in \mathcal{V}^{\bar{T}}$ for $i = 1, 2$. In particular, the MSR budget function $H^{\bar{G}}_{\mathrm{MSR}}$ is a contraction (Def. 2.1.3) on $\mathcal{V}^{\bar{T}}$. The rest of the assertion is then a straightforward application of Nadler's theorem (Prop. 2.1.4). □

Proof of Prop. 5.2.8. Let $\hat{P} := \max_{n=0}^{N} \max_{k=1}^{K} P^{\bar{T}_N}(\bar{T}_n, \omega_k)$, $\Lambda \in [0, \infty)$ as in Def. 4.3.6, and A be the non-empty, convex, and compact subset of $\mathbb{R}^{K \times (N+1)}$ defined as

$$A \;:=\; \left\{\; C \in \mathcal{V}^{\bar{T}} \;:\; C_{k,n} \;\leq\; \hat{P}\, N\, \Lambda \;\; \forall\, k = 1, \ldots, K,\; n = 0, \ldots, N \;\right\} \quad .$$

Then, $\Lambda^{\bar{G}}_{\mathrm{SSR}}(C) \leq N\Lambda$ for all $C \in \mathcal{V}^{\bar{T}}$. Moreover, since the ∞-norm is continuous, so is $\Lambda^{\bar{G}}_{\mathrm{SSR}}$. Altogether, $H^{\bar{G}}_{\mathrm{SSR}}$ is continuous and maps A into itself. By Brouwer's theorem (Prop. 2.2.1) there exists a fixed-point $C \in A$ of $H^{\bar{G}}_{\mathrm{SSR}}$. □

Proof of Prop. 5.2.9. Let $C^1, C^2 \in \mathcal{V}^{\bar{T}}$. Then,

$$\begin{aligned} \left\| \, H^{(\bar{T},\bar{F},\bar{L})}_{\mathrm{SSR}}(C^1) \;-\; H^{(\bar{T},\bar{F},\bar{L})}_{\mathrm{SSR}}(C^2) \, \right\| \;&=\; \left\| \, H^{(\bar{T},0,\bar{L})}_{\mathrm{SSR}}(\tilde{C}^1) \;-\; H^{(\bar{T},0,\bar{L})}_{\mathrm{SSR}}(\tilde{C}^2) \, \right\| \\ &\leq\; \Lambda \left\| \, \tilde{C}^1 \;-\; \tilde{C}^2 \, \right\| \\ &=\; \Lambda \left\| \, C^1 \;-\; C^2 \, \right\| \quad , \end{aligned}$$

where $\tilde{C}^i := \left\{ C^i_n + P^{\bar{T}_N}(\bar{T}_n)\, \bar{F} \right\}_{n=0}^{N} \in \mathcal{V}^{\bar{T}}$ for $i = 1, 2$. In particular, the SSR budget function $H^{\bar{G}}_{\mathrm{SSR}}$ is a contraction (Def. 2.1.1) on $\mathcal{V}^{\bar{T}}$. The rest of the assertion is then a straightforward application of Banach's theorem (Prop. 2.1.2). □

5.B The SSR Fixed-Point Algorithm

Using the notation introduced in Sec. 4.A and 4.B, the implementation of the SSR fixed-point iteration (5.3.1) for the Hull–White trinomial model is specified in Alg. 2.

Algorithm 2 The SSR Fixed-Point Algorithm for the Hull–White Trinomial Model

Require: $\left\{ {}_{(0)}C^{l}_{n,k} \right\}_{n,k,l}$ ▷ Initial guess
Require: $u_\infty \in \mathbb{N}$ ▷ Maximum number of iterations
Require: $\text{tol} \in (0, \infty)$ ▷ Convergence tolerance

$u \leftarrow 0$ ▷ Iteration counter
$\varepsilon \leftarrow \infty$ ▷ Norm of the difference between two iterates ${}_{(u)}C$ and ${}_{(u+1)}C$

while $(u < u_\infty)$ and $(\varepsilon > \text{tol})$ **do**

  $\Lambda^{\bar{G}}_{\text{SSR}} \leftarrow 0$

  for $k \leftarrow -K_{m_N}$ to K_{m_N} **do** ▷ Determine $\Lambda^{\bar{G}}_{\text{SSR}}({}_{(u)}C)$
    for all $l \in \mathcal{L}^N_k$ **do**
      $\Lambda^{\bar{G}}_{\text{SSR}} \leftarrow \max \left\{ \Lambda^{\bar{G}}_{\text{SSR}}, \sum_{n=1}^{N} \bar{L}_n \left(\left\{ {}_{(u)}G^{(l_1,\ldots,l_{i+1})}_{i,l_{i+1}} + {}_{(u)}C^{(l_1,\ldots,l_{i+1})}_{i,l_{i+1}} \right\}_{i=0}^{n} \right) \right\}$
    end for
  end for

  for $n \leftarrow 1$ to N **do** ▷ Update C_n, $n = 0, \ldots, N$
    for $k \leftarrow -K_{m_n}$ to K_{m_n} **do**
      for all $l \in \mathcal{L}^n_k$ **do**
        ${}_{(u+1)}C^{l}_{n,k} \leftarrow P^{N}_{n,k} \left(\Lambda^{\bar{G}}_{\text{SSR}} - \sum_{i=1}^{n-1} \bar{L}_i \left(\left\{ {}_{(u)}G^{(l_1,\ldots,l_{j+1})}_{j,l_{j+1}} + {}_{(u)}C^{(l_1,\ldots,l_{j+1})}_{j,l_{j+1}} \right\}_{j=0}^{i} \right) \right)$
      end for
    end for
  end for

  $\varepsilon \leftarrow \left\| {}_{(u+1)}C - {}_{(u)}C \right\|$
  $u \leftarrow u + 1$

end while

6 Monte Carlo Methods

The numerical construction of a hedging derivative for a contingent guarantee $\bar{G}$ presented in Sec. 4.4 requires the evaluation of conditional expectations of highly path-dependent payoffs. In financial market models with a finite state space, such as the Hull–White trinomial model (App. 4.A, Hull and White (1993)), conditional expectations can usually be calculated in a straightforward manner and at acceptable computational costs. The path-dependency of contingent guarantees, however, causes the numerical routine to quickly exceed available computational resources. Consequently, only small-scale and coarsely discretized problems can be solved efficiently and in a timely manner.

A similar challenge is encountered in the valuation of derivatives with early exercise features, whose payoffs are path-dependent or depend on multiple underlying assets. For these cases, the LSMC method of Longstaff and Schwartz (2001) has emerged as a popular alternative to lattice-based valuation frameworks. This chapter is based on Bienek and Scherer (2019) and presents an adaptation of the LSMC approach to the fixed-point problem derived in Ch. 4 and proves the convergence of the resulting pricing procedure. All proofs are postponed to App. 6.C.

6.1 Financial Market Model

In contrast to Ch. 4, the results of this chapter are based on a fairly general financial market model. In particular, asset prices are allowed to fluctuate in an almost arbitrary fashion and, given a fixed terminal time horizon $T \in (0, \infty)$, the set of trading time points $\mathcal{T}$ from Ch. 3 may either be a discrete finite set as in Sec. 4.1, i.e. $\mathcal{T} = \{0, t_1, t_2, \dots, T\}$, or the continuous interval $\mathcal{T} = [0, T]$. Recall that $\bar{T}_N = T$ for all sets of lock-in time points $\bar{T} = \{\bar{T}_n\}_{n=0}^{N} \subseteq \mathcal{T}$.

As before, financial risk is modeled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^T)$ with forward pricing measure $\mathbb{Q}^T$ and market filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{t \in \mathcal{T}}$. All (in)equalities involving random variables are understood to hold $\mathbb{Q}^T$-a.s. and $\mathcal{F}(0) = \{\Omega, \emptyset\}$.

The financial market is assumed to be arbitrage-free and complete, and contains two primary traded securities:

- A riskless *bank account* $B = \{B(t)\}_{t \in \mathcal{T}}$;
- A riskless *zero-coupon bond* $P^\tau = \{P^\tau(t)\}_{t \in [0,\tau] \cap \mathcal{T}}$ for each maturity $\tau \in \mathcal{T} \setminus \{0\}$ that pays 1 at time τ and ceases to exist thereafter.

Bond prices are assumed to be positive, $\mathbb{F}$-adapted, square-integrable (i.e. $\mathbb{E}_{\mathbb{Q}^T}\left[P^\tau(t)^2\right] < \infty$ for all $t, \tau \in \mathcal{T}$ with $t \leq \tau$), and Markov processes (see Def. 4.D.2) with state space $\mathcal{P} \subseteq (0, \infty)$. As in Sec. 4.1, the financial market might feature additional (risky) assets, such as stocks, commodities or real estate. However, these assets will not be relevant in the following, as the focus lies exclusively on the construction of interest rate hedging derivatives.

The forward pricing measure $\mathbb{Q}^T$ is the (unique) probability measure, such that asset prices discounted by the bond P^T are $\mathbb{Q}^T$-martingales. Since the financial market is complete, the (risk-neutral) price process of a random payoff $Y : \Omega \to \mathbb{R}$ at time $s \in \mathcal{T}$ (Y is $\mathcal{F}(s)$-measurable and $\frac{Y}{P^T(s)}$ is $\mathbb{Q}^T$-integrable) is given by

$$P^T(t)\, \mathbb{E}_{\mathbb{Q}^T}\left[\frac{Y}{P^T(s)} \,\middle|\, \mathcal{F}(t) \right]$$

for $t \leq s \leq T$.

The fund's NAV $X = \{X(t)\}_{t \in \mathcal{T}}$ is assumed to be square-integrable and controlled by a self-financing investment strategy, such that $\{\, X(t)/P^T(t) \,\}_{t \in \mathcal{T}}$ is again a $\mathbb{Q}^T$-martingale. Moreover, in case bond prices are continuously distributed, an additional assumption on the lock-in mechanisms of contingent guarantees is needed.

Assumption 6.1.1 *Let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a contingent guarantee. Throughout this chapter, one of the following is assumed:*

- *Bond prices are bounded above (interest rates are bounded below), i.e. $\sup \mathcal{P} < \infty$, and the lock-in mechanism $\bar{L} = \{\bar{L}_n\}_{n=1}^N$ satisfies the linear growth condition*

  $$\bar{L}_n(x) \;\leq\; \Lambda \left(1 \,+\, \sum_{i=0}^{n} |\, x_i \,| \right)$$

  for all $x = \{x_i\}_{i=0}^n \in \mathbb{R}^{n+1}$ and $n = 1, \ldots, N$, where $\Lambda \in [0, \infty)$;
- *The lock-in mechanism $\bar{L} = \{\bar{L}_n\}_{n=1}^N$ is capped (see Def. 4.3.6), i.e. there exists $\Lambda \in [0, \infty)$ with $\bar{L}_n(x) \leq \Lambda$ for all $x \in \mathbb{R}^{n+1}$ and $n = 1, \ldots, N$.*

6.2 The Fixed-Point Problem

The portfolio insurance framework of Sec. 4.2 readily extends to the more general financial market setup above. In particular, the definition of portfolio processes is virtually identical and repeated here only for the sake of completeness.

Definition 6.2.1 (Portfolio Processes, cf. Def. 4.2.2) *For a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ the variable guarantee is the non-decreasing and $\mathbb{F}$-predictable process $V^{\bar{G}} = \{V^{\bar{G}}(t)\}_{t \in \mathcal{T}}$ given by*

$$V^{\bar{G}}(t) \;:=\; \sum_{\{\, n \,:\, \bar{T}_n < t \,\}} \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big)$$

for $t \in \mathcal{T}$, such that $V^{\bar{G}}(t)$ is the sum of the 'lock-in' prior to time t. Moreover, the guarantee $G^{\bar{G}} = \{G^{\bar{G}}(t)\}_{t \in \mathcal{T}}$ is defined as the (risk-neutral) value of the guaranteed amount at time t, i.e.

$$G^{\bar{G}}(t) \;:=\; P^{T}(t)\,\mathbb{E}_{\mathbb{Q}^T}\left[\, \frac{\bar{F} + V^{\bar{G}}(t)}{P^T(\bar{T}_N)} \,\middle|\, \mathcal{F}(t) \right] \;=\; P^T(t)\,\big(\bar{F} + V^{\bar{G}}(t)\big) \tag{6.2.1}$$

for $t \in \mathcal{T}$. The process $C^{\bar{G}} = \{C^{\bar{G}}(t)\}_{t \in \mathcal{T}}$ defined by

$$C^{\bar{G}}(t) \;:=\; X(t) \;-\; G^{\bar{G}}(t) \tag{6.2.2}$$

for $t \in \mathcal{T}$, gives the excess of the NAV X over the guarantee $G^{\bar{G}}$ and is called the cushion.

By the same arguments as in Sec. 4.2 (and also Sec. 5.2), one arrives at the following fixed-point relationship:

$$C^{\bar{G}}(\bar{T}_n) \;=\; P^{\bar{T}_N}(\bar{T}_n)\,\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}}\left[\, \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{G^{\bar{G}}(\bar{T}_j) + C^{\bar{G}}(\bar{T}_j)\}_{j=0}^{i}\big) \;+\; \bar{E}^{\bar{G}} \,\middle|\, \mathcal{F}(\bar{T}_n) \right] \tag{6.2.3}$$

for $n = 0, \ldots, N$, where

$$\begin{aligned} \bar{E}^{\bar{G}} \;&:=\; C^{\bar{G}}(\bar{T}_N) \;-\; \bar{L}_N\big(\{X(\bar{T}_n)\}_{n=0}^{N}\big) \\ &=\; X(\bar{T}_N) \;-\; \left(\; \bar{F} \;+\; \sum_{n=1}^{N} \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^{n}\big) \right) \quad . \end{aligned}$$

Again, a family $\{C^{\bar{G}}(\bar{T}_n)\}_{n=0}^{N}$ satisfying (6.2.3) can be interpreted as (the risk-neutral value process of) a hedging derivative for the contingent guarantee $\bar{G}$.

The aim of this chapter is to determine the initial value $C^{\bar{G}}(0)$ of a hedging derivative in cases where the number of lock-in time points N is large. To simplify the hedging problem, two assumptions are made:

1. The hedging derivative produces no terminal excess, i.e. $\bar{E}^{\bar{G}} = 0$;

2. The hedging derivative is a path-dependent interest rate derivative, i.e. the 'market sigma-algebra' $\mathcal{F}(\bar{T}_n)$ in (6.2.3) is replaced by the coarser sigma-algebra $\sigma(\{P^T(\bar{T}_i)\}_{i=0}^n)$.

The mathematical justification for the second assumption above is the Markov property of the bond price P^T (see Sec. 4.3.5). The first assumption serves merely to ease the exposition and the results of this chapter readily extend to the more general case, where the terminal excess is non-negative and $\sigma(\{P^T(\bar{T}_n)\}_{n=0}^N)$-measurable.

To ease the notation, let $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ be a fixed contingent guarantee for the remainder of this chapter. For $n = 0, \ldots, N$, let $P_n := P^T(\bar{T}_n)$, $P_{[n]} := (P_0, \ldots, P_n)$, and $\mathcal{L}^2(\sigma(P_{[n]}))$ the Hilbert space of all $\sigma(P_{[n]})$-measurable and square-integrable random variables $Y : \Omega \to \mathbb{R}$ with $\| Y \|_{\mathcal{L}^2} = \mathbb{E}_{\mathbb{Q}^T}[\, Y^2 \,]^{\frac{1}{2}} < \infty$ (see App. 6.E).

Furthermore, let $\mathcal{V}$ be the Cartesian product

$$\mathcal{V} := \Big\{ \, C = \{C_n\}_{n=0}^N \; : \; C_n \in \mathcal{L}^2(\sigma(P_{[n]})) \text{ for } n = 0, \ldots, N \, \Big\} \quad ,$$

which is equipped with a norm $\| \cdot \|$ given by

$$\| C \| := \Big\| \, \{ \| C_n \|_{\mathcal{L}^2} \}_{n=0}^N \, \Big\|_\star \tag{6.2.4}$$

for $C \in \mathcal{V}$, where $\| \cdot \|_\star$ is an arbitrary monotone norm on $\mathbb{R}^{N+1}$.

Lemma 6.2.2 *$\mathcal{V}$ is a Banach space.*

For $C = \{C_n\}_{n=0}^N \in \mathcal{V}$, let

$$V_n^C := \sum_{i=1}^{n-1} \bar{L}_i\big(\{G_j^C + C_j\}_{j=0}^i\big) \; , \quad G_n^C := P_n\left(\bar{F} + V_n^C\right) \; , \quad \text{and} \quad X_n^C := G_n^C + C_n$$

for $n = 0, \ldots, N$ (cf. Def. 6.2.1). Then, by Ass. 6.1.1, $G^C = \{G_n^C\}_{n=0}^N \in \mathcal{V}$ and thereby also $X^C = \{X_n^C\}_{n=0}^N \in \mathcal{V}$.

Definition 6.2.3 (Budget Function, cf. Def. 4.3.1) *The function $H : \mathcal{V} \to \mathcal{V}$ defined by*

$$H(C) \;:=\; \left\{ \mathbb{E}_{\mathbb{Q}^T}\left[P_n \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{X_j^C\}_{j=0}^{i}\big) \;\middle|\; \sigma(P_{[n]}) \right] \right\}_{n=0}^{N} \tag{6.2.5}$$

for all $C \in \mathcal{V}$, is called budget function.

Assumption 6.2.4 *H is a contraction with Lipschitz constant $\Lambda \in [0,1)$, i.e.*

$$\| \, H(C^1) \; - \; H(C^2) \, \| \; \leq \; \Lambda \, \| \, C^1 \; - \; C^2 \, \|$$

for all $C^1, C^2 \in \mathcal{V}$.

In many practical applications the lock-in mechanism $\bar{L}$ is given by a family of Lipschitz continuous functions (see Ex. 3.5.2). Put simply, Ass. 6.2.4 holds if the corresponding Lipschitz constants are small enough, i.e. if the guaranteed amount does not grow too fast with the NAV (cf. Sec. 4.3.3).

With Lem. 6.2.2 and Banach's theorem (Prop. 2.1.2), Ass. 6.2.4 implies that there exists a unique fixed-point $C^\star \in \mathcal{V}$ with $C^\star = H(C^\star)$ and that the sequence $\{{}_{(u)}C\}_{u \in \mathbb{N}_0}$ defined by

$${}_{(u+1)}C \;=\; H({}_{(u)}C) \;, \quad u \in \mathbb{N}_0 \;, \tag{6.2.6}$$

converges (in norm) to $C^\star$ for an arbitrary starting value ${}_{(0)}C \in \mathcal{V}$. Note that the non-negativity of the lock-in implies $C_n^\star \geq 0$ for $n = 0, \ldots, N$.

6.3 Least-Squares Monte Carlo

The implementation of the fixed-point iteration in (6.2.6) requires the evaluation of conditional expectations of highly path-dependent payoffs, namely the lock-in $\bar{L}_n\big(\{X_i^C\}_{i=0}^{n}\big)$. In lattice-based financial market models conditional probability distributions are readily available, such that conditional expectations can be calculated efficiently and in a straightforward manner. However, these kinds of models are inherently unsuited for pricing path-dependent derivatives.

Monte Carlo based valuation frameworks are naturally able to handle path-dependencies, however the ability to calculate conditional expectations is lost. A common approach to overcome this problem is to approximate conditional expectations using the Monte Carlo samples themselves. The LSMC method of Longstaff and Schwartz (2001), which is adapted to the problem of pricing contingent guarantees in the following, is arguably the most popular representative of this approximation method.

The technical analysis of this section closely follows the results of Clément et al. (2002), who prove the convergence of the LSMC approach for American option pricing problems and divide the LSMC procedure into two stages:

1. In the *first-stage approximation* conditional expectations are replaced by orthogonal projections of the (path-dependent) payoffs onto a set of basis functions;

2. In the *second-stage approximation* these projections are approximated using Monte Carlo samples and a cross-sectional least-squares regression.

This section provides convergence results for both approximation stages, which together establish the convergence of the LSMC algorithm presented in Sec. 6.3.

6.3.1 First-Stage Approximation

The first-stage approximation consists of replacing the conditional expectations in (6.2.5) by an orthogonal projection onto the linear space generated by a finite number of basis functions of the vectors $P_{[n]}$.

For $n = 1, \ldots, N-1$, consider a fixed sequence $e_n = \{e_{n,m}\}_{m=1}^{\infty}$ of measurable functions $e_{n,m} : \mathcal{P}^{n+1} \to \mathbb{R}$. A popular choice is to build these functions from polynomial families, such as monomials or the Laguerre, Hermite, Legendre, or Chebyshev polynomial families (see, e.g., Hochstrasser, 1964). The sequence e_n is then constructed by applying these polynomials to measurable functions of the vector $P_{[n]}$. In Sec. 6.4, for example, monomials with varying degree are applied to the bond prices P_1 through P_n.

Assumption 6.3.1 *For $n = 1, \ldots, N-1$, the sequence $\{e_{n,m}(P_{[n]})\}_{m=1}^{\infty}$ is (A) linearly independent and (B) complete in $\mathcal{L}^2(\sigma(P_{[n]}))$. More precisely:*

(A) For $M \geq 1$ and $\lambda \in \mathbb{R}^M$, $\sum_{m=1}^{M} \lambda_m \, e_{n,m}(P_{[n]}) = 0$ (a.s.) implies $\lambda_m = 0$ for all $m = 1, \ldots, M$;

(B) $\overline{Span}\big(\{e_{n,m}(P_{[n]})\}_{m=1}^{\infty}\big) = \mathcal{L}^2(\sigma(P_{[n]}))$ (see Def. 6.E.2).

For $n = 1, \ldots, N-1$, $M \geq 1$ and $p \in \mathcal{P}^{n+1}$, let $e_n^M(p)$ denote the vector $\big(e_{n,1}(p), \ldots, e_{n,M}(p)\big)^\top \in \mathbb{R}^M$ and $\mathcal{Q}_n^M$ the orthogonal projection from $\mathcal{L}^2(\sigma(P_{[n]}))$ onto the linear span of $\{e_{n,m}(P_{[n]})\}_{m=1}^M$ (see Sec. 6.E), i.e.

$$\mathcal{Q}_n^M(Y) \;:=\; \operatorname*{arg\,min}_{\hat{Y} \,\in\, S_n^M} \big\| \, Y \;-\; \hat{Y} \, \big\|_{\mathcal{L}^2}^2$$

for $Y \in \mathcal{L}^2(\sigma(P_{[n]}))$, where $S_n^M := \mathrm{Span}\{\, e_{n,1}(P_{[n]}),\, \ldots,\, e_{n,M}(P_{[n]})\,\}$. With Ass. 6.3.1 (A), $\mathcal{Q}_n^M(Y)$ can be given an explicit representation.

Lemma 6.3.2 *Let $M \geq 1$ and $n \in \{1, \ldots, N-1\}$. Then,*

$$\mathcal{Q}_n^M(Y) \;=\; \alpha_n^M(Y)^\top e_n^M(P_{[n]}) \tag{6.3.1}$$

for $Y \in \mathcal{L}^2(\sigma(P_{[n]}))$, where $\alpha_n^M(Y) \in \mathbb{R}^M$ is given by

$$\alpha_n^M(Y) \;:=\; \big(A_n^M\big)^{-1} \, \mathbb{E}_{\mathbb{Q}^T}\Big[\; Y \, e_n^M(P_{[n]}) \;\Big]$$

and

$$A_n^M \;:=\; \mathbb{E}_{\mathbb{Q}^T}\Big[\, e_n^M(P_{[n]}) \, e_n^M(P_{[n]})^\top \,\Big] \;\in \mathbb{R}^{M \times M} \quad .$$

Definition 6.3.3 (First-Stage Approximation) *For $M \geq 1$, the first-stage approximation $H^M : \mathcal{V} \to \mathcal{V}$ of the budget function is given by*

$$H^M(C) \;:=\; \left\{ \; \mathcal{Q}_n^M\!\left(\; P_n \sum_{i=n\vee 1}^{N} \bar{L}_i\big(\{X_j^C\}_{j=0}^i\big) \right) \right\}_{n=0}^N$$

for $C \in \mathcal{V}$, where $\mathcal{Q}_0^M \equiv \mathbb{E}_{\mathbb{Q}^T}$ and $\mathcal{Q}_N^M \equiv id$.

Assumption 6.3.4 *There exists $\bar{M} \geq 1$, such that, for all $M \geq \bar{M}$, H^M is a contraction with Lipschitz constant $\Lambda^M \in [0, 1)$ (Def. 2.1.1), i.e.*

$$\big\| \, H^M(C^1) \;-\; H^M(C^2) \, \big\| \;\leq\; \Lambda^M \, \big\| \, C^1 \;-\; C^2 \, \big\|$$

for all $C^1, C^2 \in \mathcal{V}$. Moreover, $\limsup_{M\to\infty} \Lambda^M < 1$.

By Ass. 6.3.4 and Banach's theorem (Prop. 2.1.2), there exists a unique fixed-point $C^M \in \mathcal{V}$ of H^M for $M \geq \bar{M}$. The first convergence result of this section establishes the convergence (in norm) of these approximate derivatives C^M towards the true derivative $C^\star$, which, in particular, implies $C_0^M \to C_0^\star$ for $M \to \infty$.

Proposition 6.3.5

$$C^M \quad \xrightarrow[M\to\infty]{\mathcal{V}} \quad C^\star \quad .$$

Assumption 6.3.4 above might seem strong and indeed there are no straightforward sufficient conditions to guarantee that it holds. However, any attempt to prove the convergence of C^M towards $C^\star$ necessarily requires that the approximate derivatives C^M exist in the first place. In that sense, Ass. 6.3.4 should simply be regarded as a sufficient condition for the existence (and uniqueness) of C^M and for the convergence of the LSMC fixed-point iteration developed below.

6.3.2 Second-Stage Approximation

In the second stage, H^M is approximated using $K \in \mathbb{N}$, $K \geq M$, independent Monte Carlo samples of the bond price path $P_{[N]}$, which are denoted by $\{P_{k,n}\}_{n=0}^N$ for $k = 1, \dots, K$, and a cross-sectional least-squares regression.

For $n = 1, \dots, N-1$, let $\mathcal{P}_n^{M,K}$ be the set of matrices $p \in \mathcal{P}^{K\times(n+1)}$ such that the vectors

$$\begin{pmatrix} e_{n,1}(p_{1,[n]}) \\ \vdots \\ e_{n,1}(p_{K,[n]}) \end{pmatrix}, \dots, \begin{pmatrix} e_{n,M}(p_{1,[n]}) \\ \vdots \\ e_{n,M}(p_{K,[n]}) \end{pmatrix}$$

are linearly independent, where $p_{k,[i]} := \big(p_{k,0}, \ \dots, \ p_{k,i} \big)$ for $0 \leq i \leq n$. Furthermore, to ease the notation, let

$$p_{[K],n} \ := \ \big(p_{1,n}, \ \dots, \ p_{K,n} \big)^\top \ \in \mathcal{P}^K \quad \text{and} \quad p_{[K],[i]} \ := \ \big\{ p_{k,i} \big\}_{\substack{k=1,\dots,K \\ j=0,\dots,i}} \ \in \mathcal{P}^{K\times(i+1)}$$

for $p \in \mathcal{P}_n^{M,K}$ and $0 \leq i \leq n$. Note that the matrix of bond price samples $P_{[K],[n]}$ is (almost-surely) an element of $\mathcal{P}_n^{M,K}$ by Ass. 6.3.1 (A).

For $n = 1, \dots, N-1$, let $\alpha_n^{M,K} : \mathbb{R}^K \times \mathcal{P}_n^{M,K} \to \mathbb{R}^M$ denote the least-squares estimator

$$(y, p) \ \mapsto \ \operatorname*{arg\,min}_{\alpha \in \mathbb{R}^M} \ \sum_{k=1}^K \Big(\ y_k \ - \ \alpha^\top e_n^M(p_{k,[n]}) \Big)^2$$

and let $\mathcal{Q}_n^{M,K} : \mathbb{R}^K \times \mathcal{P}_n^{M,K} \to \mathbb{R}^K$ be defined by

$$\mathcal{Q}_n^{M,K}(y,p) \;:=\; \begin{pmatrix} \alpha_n^{M,K}(y,p)^\top \, e_n^M(p_{1,[n]}) \\ \dots \\ \alpha_n^{M,K}(y,p)^\top \, e_n^M(p_{K,[n]}) \end{pmatrix} .$$

The map $\mathcal{Q}_n^{M,K}$ can be regarded as the 'sample counterpart' of $\mathcal{Q}_n^M$ in (6.3.1). Moreover, the least-square estimator $\alpha_n^{M,K}$ has the explicit representation

$$\alpha_n^{M,K}(y,p) \;=\; \Big(A_n^{M,K}(p)\Big)^{-1} \left(\frac{1}{K} \sum_{k=1}^K \Big(y_k \, e_n^M(p_{k,[n]}) \Big) \right) ,$$

where

$$A_n^{M,K}(p) \;:=\; \left\{ \frac{1}{K} \sum_{k=1}^K e_{n,m}(p_{k,[n]}) \, e_{n,\ell}(p_{k,[n]}) \right\}_{1 \le m,\, \ell \le M} \in \mathbb{R}^{M \times M} ,$$

which is indeed non-singular by definition of the set $\mathcal{P}_n^{M,K}$.

Definition 6.3.6 (Second-Stage Approximation) *For $M \geq 1$ and $K \geq M$, the second-stage approximation $H^{M,K} : \mathbb{R}^{K\times(N+1)} \times \mathcal{P}_N^{M,K} \to \mathbb{R}^{K\times(N+1)}$ of the budget function is given by*

$$H^{M,K}(c,p) \;:=\; \left\{ \mathcal{Q}_n^{M,K}\left(\left\{ p_{k,n} \left(\sum_{i=n\vee 1}^N \bar{L}_i\big(\{g_{k,j}^{c,p} + c_{k,j}\}_{j=0}^i\big) \right) \right\}_{k=1}^K ,\, p_{[K],[n]} \right) \right\}_{n=0}^N ,$$

where $g_{k,n}^{c,p}$ is the 'sample guarantee' defined by

$$g_{k,n}^{c,p} \;:=\; p_{k,n} \left(\bar{F} \;+\; \sum_{i=1}^{n-1} \bar{L}_i\big(\{g_{k,j}^{c,p} + c_{k,j}\}_{j=0}^i\big) \right)$$

for $k = 1, \dots, K$ and $n = 0, \dots, N$. Moreover, $\mathcal{Q}_0^{M,K}(y,p) := \Big(\frac{1}{K} \sum_{k=1}^K y_k\Big) \mathbb{1}_K$ and $\mathcal{Q}_N^{M,K}(y,p) := y$ with $\mathbb{1}_K = (1, \dots, 1)^\top \in \mathbb{R}^K$.

Whereas the first-stage approximation H^M (and the budget function H itself) are functions on the rather abstract space $\mathcal{V}$, the second-stage approximation operates on the vector space of matrices $\mathbb{R}^{K\times(N+1)}$. The central idea behind the LSMC approach in Sec. 6.4 is to run a fixed-point iteration on $\mathbb{R}^{K\times(N+1)}$ using $H^{M,K}(\,\cdot\,, P_{[K],[N]})$ and the set of Monte Carlo samples $P_{[K],[N]}$ of the bond price P^T. Under the following assumption, this iteration will converge and the resulting fixed-point will allow to approximate the initial value C_0^M.

Assumption 6.3.7 *For all $M \geq \bar{M}$ there exists $\bar{K}_M \geq M$, such that, for all $K \geq \bar{K}_M$, the second-stage approximation $H^{M,K}(\,\cdot\,, P_{[K],[N]})$ using the Monte Carlo samples $P_{[K],[N]}$ is almost-surely a contraction (Def. 2.1.1). More precisely, there exists a Lipschitz constant $\Lambda^{M,K} \in [0,1)$ and a norm $\|\cdot\|_{M,K}$ on $\mathbb{R}^{K\times(N+1)}$ with*

$$\left\| H^{M,K}(c^1, P_{[K],[N]}) - H^{M,K}(c^2, P_{[K],[N]}) \right\|_{M,K} \leq \Lambda^{M,K} \left\| c^1 - c^2 \right\|_{M,K} \quad (a.s.)$$

for all $c^1, c^2 \in \mathbb{R}^{K\times(N+1)}$. Moreover, $\limsup_{K\to\infty} \Lambda^{M,K} < 1$ for all $M \geq \bar{M}$.

By Ass. 6.3.7 and Banach's theorem (Prop. 2.1.2), there exists a unique fixed-point $C^{M,K} \in \mathbb{R}^{K\times(N+1)}$ of $H^{M,K}(\,\cdot\,, P_{[K],[N]})$ for $M \geq \bar{M}$ and $K \geq \bar{K}_M$.

Let $\hat{C}^{M,K} \in \mathbb{R}^{K\times(N+1)}$ be the sampled approximate derivative C^M, which is obtained from $\hat{C}^{M,K}_{k,n} = C^M_n(P_{k,[n]})$ for $k = 1,\ldots,K$ and $n = 0,\ldots,N$. Indeed, the random variables C^M_n of the approximate derivative $C^M = \{C^M_n\}_{n=0}^N$ are, by definition, measurable functions of $P_{[n]}$, such that, for each sample path of the bond price P^T, there exists an associated 'sample path' of C^M. More precisely,

$$\begin{aligned}
\hat{C}^{M,K}_{k,0} &= P_{k,0}\, \mathbb{E}\left[\sum_{n=1}^{N} \bar{L}_n\big(\{X_i^{C^M}\}_{i=0}^{n}\big)\right], \\
\hat{C}^{M,K}_{k,n} &= \alpha_n^M \left(P_n \sum_{i=n}^{N} \bar{L}_i\big(\{X_j^{C^M}\}_{j=0}^{i}\big)\right)^{\top} e_n^M(P_{k,[n]}), \text{ for } n = 1,\ldots,N-1, \\
\hat{C}^{M,K}_{k,N} &= \bar{L}_N\big(\{\hat{X}^{M,K}_{k,n}\}_{n=0}^{N}\big)
\end{aligned}$$

for $k = 1,\ldots,K$, where

$$\hat{X}^{M,K}_{k,n} := X_n^{C^M}(P_{k,[n]}) = g_{k,n}^{\hat{C}^{M,K}, P_{[K],[N]}} + \hat{C}^{M,K}_{k,n}$$

for $k = 1,\ldots,K$ and $n = 0,\ldots,N$.

The second convergence result of this section establishes the (almost sure) convergence of the sample fixed-points $C^{M,K}$ towards the sampled approximate derivative $\hat{C}^{M,K}$, which, in particular, implies $C^{M,K}_{\cdot,0} \to C^M_0$ for $K \to \infty$.

Proposition 6.3.8 *Let $M \geq \bar{M}$ be fixed. Then,*

$$C^{M,K} \xrightarrow[K\to\infty]{a.s.} \hat{C}^{M,K} \,.$$

6.3.3 Limitations

The convergence results of this section are based on rather strong technical assumptions on the first- and second-stage approximations of the budget function, whose validity will depend on the particular choice of the lock-in mechanism, the basis functions, and even the interest rate model. Generally, the quality of the approximation of the LSMC method (see (6.4.2) below) can only be assessed ex post by a comparison with other available valuation methods and through numerical experiments.

Moreover, while providing an efficient means to derive the value of a contingent guarantee, the proposed LSMC approach does not offer a straightforward method to construct a corresponding hedging strategy other than by *delta-* or *cross-hedging*, i.e. by constructing a (static) portfolio of securities which exhibits similar price sensitivities to movements in the underlying risk factors (e.g. the term structure of interest rates, interest rate volatility, etc.) as the hedging derivative (see Sec. 6.4.2 and Johnson (1960)). A possible remedy to this problem might be given by the *replicating portfolio* approach, which uses a regression-based approximation of payoffs rather than conditional expectations, and thereby directly yields an approximate hedging portfolio (for a detailed comparison of both methods see, e.g., Pelsser and Schweizer, 2016).

6.4 Numerical Case Study

With Prop. 6.3.5 and 6.3.8, the initial capital $C_0^\star$ of the hedging derivative $C^\star$ can be approximated by generating a set of Monte Carlo samples $P_{[K],[N]}$ of the bond price P^T (under the forward pricing measure $\mathbb{Q}^T$) and running a fixed-point iteration using the second-stage approximation $H^{M,K}(\,\cdot\,, P_{[K],[N]})$:

$$_{(u+1)}C^{M,K} \;=\; H^{M,K}\big({}_{(u)}C^{M,K},\, P_{[K],[N]}\big)\;, \quad u \in \mathbb{N}_0 \quad . \tag{6.4.1}$$

Under Ass. 6.3.4 and 6.3.7, this iteration will converge to a fixed-point $C^{M,K} \in \mathbb{R}^{K\times(N+1)}$ with

$$C^{M,K}_{\cdot,0} \quad \xrightarrow[M,K\to\infty]{} \quad C_0^\star \quad . \tag{6.4.2}$$

The purpose of this section is to investigate the behavior and quality of the approximation above for different contingent guarantees $\bar{G}$, basis functions $\{e_{n,m}\}$, and numbers of scenarios K.

6.4.1 Implementation

The implementation of the LSMC fixed-point iteration is detailed in Alg. 3 (App. 6.B). The algorithm requires a set of Monte Carlo samples $P_{[K],[N]}$ and an initial guess ${}_{(0)}C^{M,K} \in \mathbb{R}^{K\times(N+1)}$, and runs until either the difference of two consecutive iterates ${}_{(u)}C^{M,K}$ and ${}_{(u+1)}C^{M,K}$ is less than a prespecified tolerance tol $\in (0,\infty)$, or the number of iterations has reached a prespecified maximum $u_\infty \in \mathbb{N}$.

The LSMC fixed-point iteration can be applied independently from the underlying asset price model, in the sense that the algorithm can be run on any arbitrary set of Monte Carlo samples $P_{[K],[N]}$ regardless of how its has been generated. In order to assess the quality of the approximation in (6.4.2), two different LSMC approaches are implemented: Approach LSMC I applies Alg. 3 to Monte Carlo samples obtained from the continuous Hull–White model (see App. 6.A), whereas approach LSMC II is based on samples from the corresponding trinomial model (a discrete approximation of the continuous model, see App. 4.A). This allows to benchmark the LSMC method against the fixed-point iteration using the Hull–White trinomial tree in Sec. 4.4 (henceforth called TREE approach).

6.4.2 Results

Approaches LSMC I and LSMC II are applied to different contingent guarantees $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$, where, throughout, $\bar{F} = 1$ and $\bar{T}_n = n$ for $n = 0, \dots, N$ (i.e. the lock-in time points are evenly spaced at yearly intervals). For the lock-in mechanism $\bar{L}$, the same choices as in Sec. 4.4.2 are considered:

$$\text{LOCK-IN I:} \quad \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; 30\% \Big[\, X(\bar{T}_n) \;-\; X(\bar{T}_{n-1}) \,\Big]^+,$$

$$\text{LOCK-IN II:} \quad \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; 30\% \Big[\, X(\bar{T}_n) \;-\; \Big(\bar{F} \;+\; \textstyle\sum_{i=1}^{n-1} \bar{L}_i\big(\{X(\bar{T}_j)\}_{j=0}^i\big)\Big)\Big]^+,$$

$$\text{LOCK-IN III:} \quad \bar{L}_n\big(\{X(\bar{T}_i)\}_{i=0}^n\big) \;=\; 30\% \Big[\, X(\bar{T}_n) \;-\; \tfrac{1}{n+1} \textstyle\sum_{i=0}^n X(\bar{T}_i) \,\Big]^+,$$

for $n = 1, \dots, N$.

Alg. 3 is implemented in MATLAB on a computer with a 3.2 GHz CPU and 16 GB RAM. The parameters are set to tol $= 10^{-6}$, $u_\infty = 2500$, ${}_{(0)}C^{M,K} = 0$, and the convergence criterion in step 14 is evaluated using the maximum-norm in (4.3.7). The parameters of the Hull–White model are set to $\kappa = 0.1000$, $\sigma = 0.0100$, and $P^\tau(0) = 1$ for all $\tau \in \mathcal{T} \setminus \{0\}$ (unless stated otherwise). Bond prices are sampled using antithetic variates.

Choice of Basis Functions

In order to assess the impact of the choice of basis functions $\{e_{n,m}\}$ on the quality of the approximation (6.4.2), two different sets of basis functions are considered: Set I_d consists of all monomials up to degree $d \in \mathbb{N}$ of the bond price P_n, whereas set II_d aims at capturing the path-dependency of lock-in mechanisms and consists of all monomials up to degree d of the bond price history $P_{[n]}$ (see Tab. 6.1 at the end of this section). The number of samples is set to $K = 10^6$ and the number of lock-in time points to $N = 7$ (i.e. 6 regression steps). Results are reported in Tab. 6.2.

For lock-in mechanism Lock-In I, both sets of basis functions yield similar prices, which is not surprising: The lock-in at time $\bar{T}_{n+1}$ and thereafter is independent of the NAV at times $\bar{T}_1, \ldots, \bar{T}_{n-1}$, such that the history of the bond price is indeed redundant. This is not the case for lock-in mechanisms Lock-In II and III: The lock-in at any time point depends on the complete history of the NAV, such that the sets I_d are clearly insufficient. Henceforth, set I_2 is used for lock-in mechanism Lock-In I, set II_3 for lock-in mechanism Lock-In II, and set II_2 for lock-in mechanism Lock-In III. Results remain the same, when using the Laguerre polynomial family suggested by Longstaff and Schwartz (2001) instead of monomials (see App. 6.F).

Note that the path-dependency of a lock-in mechanism may start to unfold only for higher values of N. For example, testing the different choices of basis functions with $N \leq 6$ suggests that the two sets I_3 and II_3 perform equally well for lock-in mechanism Lock-In II, which is certainly not true for $N \geq 7$. Of course, simply increasing N might not always be an option as one is bound by the computational constraints of the benchmark method (in this case approach Tree). Deciding on the right choice of basis functions might therefore require more extensive numerical tests, which also incorporate varying model parameters κ, σ, and Δ.

Generally speaking, the LSMC approach is easy to implement and yields robust results for such lock-in mechanisms, where the lock-in is based on a comparison of portfolio values, such as Lock-In I and III. Lock-in mechanisms, which calculate the lock-in based on a comparison of the portfolio value (i.e. the discounted guaranteed amount) with the (undiscounted) guaranteed amount, such as Lock-In II, require a somewhat more careful consideration of the basis functions and model parameters (see also the discussion in Sec. 4.4.2 and 5.3.2).

Method Comparison

Tab. 6.3 reports the approximate derivative prices obtained from the different valuation approaches and for different choices of N and Δ, where Δ is the discretization of the trinomial model (see App. 4.A). The number of Monte Carlo samples is set to $K = 10^6$.

For a given discretization Δ, the prices obtained from the methods Tree and Lsmc II are very close: The average relative error across all combinations of N and Δ and across the three lock-in mechanisms is less than 0.20%. Moreover, as expected, both prices approach the Lsmc I price as Δ decreases, i.e. as the discretization becomes finer.

The LSMC methods generally require fewer iterations and lead to a very significant gain in performance: For $N = 6$ and $\Delta = \frac{1}{12}$ a single iteration of the Tree approach takes about 10 minutes, whereas approaches Lsmc I and II require less than 10 seconds.

Monte Carlo Error

As is characteristic for Monte Carlo based valuation approaches, the standard deviation of the approximate price decreases proportional to one over the square root of the number of scenarios K (see Fig. 6.1).

Delta-Hedging

The LSMC fixed-point iteration is first and foremost an efficient numerical routine to determine the price of a hedging derivative of a given contingent guarantee. However, by using a simple finite difference scheme, it can also be used to construct a delta-hedge for this derivative and thereby approximate an actual portfolio insurance strategy.

Delta measures the sensitivity of the price of a derivative (e.g. a vanilla call option on a stock) towards changes in the price of the underlying (e.g. the stock). Mathematically, it is the partial derivative of the instrument's price with respect to the price of the underlying asset. The central idea behind delta-hedging is to construct a portfolio consisting of liquidly traded instruments (e.g. the stock and a bank account), such that, at each point in time, the portfolio's delta coincides with the derivative's delta. Any movements in the price of the underlying will then cause (approximately) the same changes in both the value of the derivative and the value of the portfolio. In that sense, a delta-hedge allows to (approximately) replicate the value process of the derivative.

For complex derivatives, it is often not possible to derive their delta in closed-form. In this case, finite difference methods can be applied to approximate delta by repricing the derivative with slightly perturbed input parameters (see, e.g., Glasserman, 2003, Sec. 7.1). Furthermore, for interest rate derivatives, the underlying is not a single asset, but rather the complete (initial) term structure $\tau \mapsto r^\tau(0) = -\frac{1}{\tau} \ln P^\tau(0)$, such that one needs to consider changes in the prices of a multitude of assets. This gives rise to several different notions of 'delta' for interest rate derivatives, some of which are discussed in Benhamou and Nodelman (2002).

The arguably most simple definition of delta, which also underlies the concept of the *duration* of bonds, is to assume an upward or downward parallel shift of the entire term structure. In other words, one values the derivative under both the original term structure $\tau \mapsto r^\tau(0)$ and the shifted term structures $\tau \mapsto r^\tau(0) \pm \beta$, where β is usually between 1 and 10 basis points.

The impact of such parallel shifts on the value of hedging derivatives has already been investigated in Sec. 4.4.2 (Tab. 4.2). In the following, a similar analysis using the three curves

BASE: $r^\tau(0) = 0.0000$ for all $\tau \in \mathcal{T} \setminus \{0\}$,

UP: $r^\tau(0) = 0.0010$ for all $\tau \in \mathcal{T} \setminus \{0\}$,

DOWN: $r^\tau(0) = -0.0010$ for all $\tau \in \mathcal{T} \setminus \{0\}$,

is conducted. The currently observed market term structure is assumed to be represented by the BASE curve. The UP and DOWN curves represent a 10 basis point upward and downward parallel shift, respectively.

Using the curves above, the delta of the hedging derivative can be approximated using the *central-difference estimator*

$$\delta_C := \frac{C_{\text{UP}} - C_{\text{DOWN}}}{2 \cdot 0.0010}$$

(Glasserman, 2003, Sec. 7.1.1), where C_i is the approximate derivative price under term structure i ($i \in \{\text{BASE}, \text{UP}, \text{DOWN}\}$). For a given number of lock-in time points N, a delta-hedging portfolio consisting of the two zero-coupon bonds $P^{\bar{T}_1}$ and $P^{\bar{T}_N}$ can then be constructed by solving

$$\begin{pmatrix} C_{\text{BASE}} \\ \delta_C \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\bar{T}_1 & -\bar{T}_N \end{pmatrix} \begin{pmatrix} \pi^1 \\ \pi^N \end{pmatrix} , \tag{6.4.3}$$

where π^1 and π^N are the amounts invested into $P^{\bar{T}_1}$ and $P^{\bar{T}_N}$, respectively.

The first line of the equation system in (6.4.3) guarantees that the initial value of the hedging portfolio coincides with the (approximate) value of the hedging derivative (note that $P^\tau(0) = 1$ for all $\tau \in \mathcal{T} \setminus \{0\}$ in the BASE scenario). The second line ensures that the delta of the hedging portfolio coincides with the delta of the hedging derivative. Indeed, the delta (duration) of the zero-coupon bond P^τ is given by

$$\frac{\mathrm{d}}{\mathrm{d}r} e^{-r\tau} \Big|_{r = r^\tau(0)} = -\tau\, e^{-r\tau} \Big|_{r = r^\tau(0)} = -\tau$$

for $\tau \in \mathcal{T} \setminus \{0\}$, where the last equality stems from the fact that $r^\tau(0) = 0.0000$ in the BASE scenario.

Note that one must not necessarily choose $P^{\bar{T}_1}$ and $P^{\bar{T}_N}$ for the delta-hedge. Any other choice of two (distinct) zero-coupon bonds is just as valid. However, using bonds with similar maturities usually leads to more leveraged allocations (with a large short position in one of the bonds).

Tab. 6.4 reports approximate derivative prices, deltas, and delta-hedging portfolios for lock-in mechanism LOCK-IN I and different choices of N. Prices are obtained from approach LSMC I with the number of Monte Carlo samples set to $K = 10^6$.

For a given number of lock-in time points N, the derivative's delta is quite low (in absolute terms) compared to the delta of the $\bar{T}_N$-maturity zero-coupon bond. For $N = 5$ and $N = 10$ an increase in interest rates leads to an increase in the price of the derivative (the derivative's delta is positive), which can be explained by the pull-to-par effect of bond prices (see also the discussion in Sec. 4.4.2). For $N = 15$ and $N = 20$ on the other hand, the delta is negative, which suggests that for longer maturities the effect of an increased (initial) discount factor plays a more dominant role.

Note that the delta-hedging portfolios are much less leveraged than the replicating portfolio derived in Sec. 4.4.2 (Tab. 4.3). In fact, for $N = 15$ and $N = 20$, the delta-hedge contains only long positions. This makes delta-hedging a practical alternative for the 'full' replication presented in Sec. 4.4.2, if one is comfortable with the shortfall risk arising from discretely hedging only parallel shifts of the term structure.

To allow for more realistic perturbations of the term structure than the parallel shift approach, which assumes that all rates of the term structure move by exactly the same amount (up or down), an approach using multiple 'factors' $\beta_i = \{\beta_i^\tau\}_{\tau \in \mathcal{T}}$, which are usually derived from a principle component analysis, might be a better choice. By repricing the derivative using the perturbed term structures $\tau \mapsto r^\tau(0) \pm \beta_i^\tau$, one then obtains several deltas (one for each factor β_i). For many interest rate markets, three factors have been found to explain a sufficiently large part of the variance of the term structure (see, e.g., Filipović, 2009, Sec. 3.4).

Another important aspect in constructing delta-hedging portfolios is model choice: As a one-factor model for the sport rate, the Hull–White model is able to produce only a fairly limited range of perturbations of the term structure (r^{τ_1} and r^{τ_2} are linearly dependent and thereby perfectly correlated; see, e.g., Brigo and Mercurio, 2006, Sec. 4.1). A popular way to achieve a 'decorrelation' of rates and to allow for more realistic movements of the term structure is to add more stochastic factors as is done in the G2++ model (a two-factor extension of the Hull–White model; see Brigo and Mercurio, 2006, Sec. 4.2).

Moreover, movements of the term structure are certainly not the only risks one needs to consider when constructing hedging portfolios. In particular, changing market expectations of future volatility might have a strong impact on the value of a derivative. This is commonly referred to as *vega*-risk. In order to construct a vega-hedging portfolio, the same finite difference approach as above can be applied (instead of shifting the term structure, one perturbs the 'implied'/model volatility σ). However, the corresponding hedging portfolio will then also include other (liquid) interest rate derivatives, such as (bond) call and put options.

Set	Description	Basis Functions
I_d	Monomials of the current bond price up to degree d	$\{ 1, P_n, (P_n)^2, \dots, (P_n)^d \}$
II_d	Monomials of the current and past bond prices up to degree d	$\{ 1, P_1, (P_1)^2, \dots, (P_1)^d, P_2, (P_2)^2, \dots, (P_2)^d, \dots, P_n, (P_n)^2, \dots, (P_n)^d \}$

Table 6.1: Different choices for the sets of basis functions $\{e_{n,m}\}_{m=1}^{M}$. Note that the number of basis functions M is implicitly given by the degree $d \in \mathbb{N}$.

Lock-In Mechanism	Benchmark Price	Basis Func.	Degree d: 1	2	3	4
Lock-In I (excess over previous NAV)	0.0262	I_d	0.0257	**0.0261**	0.0261	0.0262
		II_d	0.0257	0.0261	0.0261	0.0261
Lock-In II (excess over guaranteed amount)	0.1235	I_d	0.1237	0.1239	0.1230	0.1229
		II_d	0.1241	0.1246	**0.1235**	0.1235
Lock-In III (excess over average NAV)	0.0233	I_d	0.0219	0.0225	0.0225	0.0225
		II_d	0.0230	**0.0234**	0.0234	0.0234

Table 6.2: The mean approximate price $C_{\cdot,0}^{M,K}$ obtained from 100 independent runs of method LSMC II for the two sets of basis functions I_d and II_d in Tab. 6.2, and different choices of the degree d. The trinomial model is discretized with 4 quarterly time steps per year ($\Delta = \frac{1}{4}$, see App. 4.A). The benchmark prices are obtained from method TREE (see Tab. 4.1, $N = 7$ and $\alpha = 30\%$). Bold entries mark the chosen sets of basis functions. Results have been rounded.

Lock-In Mechanism	Method	Δ	Number of Lock-In Time Points N: 5	6	7
Lock-In I (excess over previous NAV)	Tree	$\frac{1}{2}$	0.0139 (10)	0.0196 (11)	0.0260 (12)
		$\frac{1}{4}$	0.0140 (12)	0.0197 (13)	0.0262 (15)
		$\frac{1}{12}$	0.0139 (15)	0.0196 (18)	–
	Lsmc II	$\frac{1}{2}$	0.0139 (12)	0.0195 (14)	0.0259 (14)
		$\frac{1}{4}$	0.0140 (12)	0.0196 (14)	0.0261 (15)
		$\frac{1}{12}$	0.0139 (13)	0.0196 (14)	0.0260 (15)
	Lsmc I		0.0138 (13)	0.0194 (14)	0.0258 (15)
Lock-In II (excess over guaranteed amount)	Tree	$\frac{1}{2}$	0.0321 (44)	0.0643 (75)	0.1251 (146)
		$\frac{1}{4}$	0.0325 (51)	0.0642 (95)	0.1235 (231)
		$\frac{1}{12}$	0.0324 (70)	0.0635 (198)	–
	Lsmc II	$\frac{1}{2}$	0.0323 (17)	0.0645 (23)	0.1252 (34)
		$\frac{1}{4}$	0.0326 (19)	0.0643 (25)	0.1235 (36)
		$\frac{1}{12}$	0.0325 (20)	0.0635 (26)	0.1222 (37)
	Lsmc I		0.0323 (20)	0.0634 (26)	0.1216 (37)
Lock-In III (excess over average NAV)	Tree	$\frac{1}{2}$	0.0098 (18)	0.0156 (28)	0.0236 (52)
		$\frac{1}{4}$	0.0097 (19)	0.0155 (31)	0.0233 (66)
		$\frac{1}{12}$	0.0097 (22)	0.0154 (38)	–
	Lsmc II	$\frac{1}{2}$	0.0097 (9)	0.0156 (10)	0.0236 (12)
		$\frac{1}{4}$	0.0097 (9)	0.0155 (10)	0.0234 (12)
		$\frac{1}{12}$	0.0097 (9)	0.0154 (10)	0.0232 (12)
	Lsmc I		0.0096 (9)	0.0153 (10)	0.0231 (12)

Table 6.3: Approximate derivative prices $C^{M,K}_{\cdot,0}$ obtained from the approaches Lsmc I, Lsmc II, and Tree. The values reported for the LSMC methods are the mean from 100 independent runs. The (mean) number of iterations is reported in parenthesis. Note that the Tree method is computationally infeasible for $N = 7$ and $\Delta = \frac{1}{12}$. Results have been rounded.

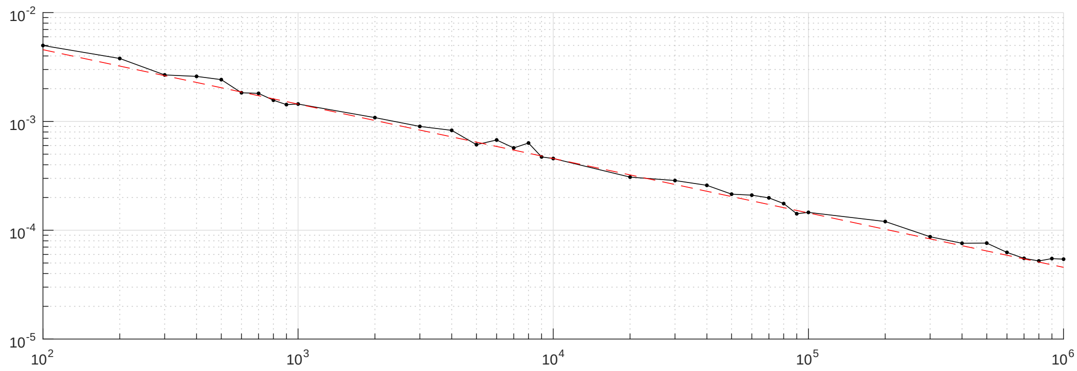

Figure 6.1: The standard deviation of $C_{\cdot,0}^{M,K}$ (y-axis, log-scale) from 100 independent runs of method LSMC I for different numbers of scenarios K (x-axis, log-scale), $N = 15$, and lock-in mechanism LOCK-IN I. The dashed red line runs proportional to $1/\sqrt{K}$.

	NUMBER OF LOCK-IN TIME POINTS N			
	5	10	15	20
C_{BASE}	0.0138	0.0495	0.1036	0.1777
C_{UP}	0.0145	0.0502	0.1034	0.1751
C_{DOWN}	0.0132	0.0488	0.1038	0.1805
δ_C	0.6105	0.6841	-0.2196	-2.7346
π^1	0.169937	0.131010	0.095327	0.043177
π^N	-0.156097	-0.081509	0.008285	0.134571

Table 6.4: Approximate derivative prices obtained from approach LSMC I for different choices of N and for the three initial yield curves BASE, UP, and DOWN (reported are the mean values from 100 independent runs). The central-difference estimator δ_C serves as an approximation of the derivative's delta. The delta-hedging portfolios (π^1, π^N) are obtained by solving (6.4.3). Results have been rounded.

Appendices

6.A The Hull–White Model

The interest rate model of Hull and White (1990) is a continuous-time and -state model for the (riskless) spot rate $\{r(t)\}_{t \in \mathcal{T}}$. The following construction is based on Brigo and Mercurio (2006, Sec. 3.3) and starts under the (spot) pricing measure $\mathbb{Q}$.

Let $\mathcal{T} = [0, T]$ and $W^{\mathbb{Q}} = \{W^{\mathbb{Q}}(t)\}_{t \in \mathcal{T}}$ a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} = \{\mathcal{F}(t)\}_{t \in \mathcal{T}}$ is the $\mathbb{Q}$-augmentation of the natural filtration of $W^{\mathbb{Q}}$.

The spot rate is modeled as the sum $r(t) := \phi(t) + z(t)$ with the deterministic function $\phi : \mathcal{T} \to \mathbb{R}$ chosen so as to exactly fit the initial market term structure (see below) and the Ornstein–Uhlenbeck process $z = \{z(t)\}_{t \in \mathcal{T}}$ evolving according to

$$\mathrm{d}z(t) \;=\; -\kappa\, z(t)\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}W^{\mathbb{Q}}(t), \quad z(0) \;=\; 0, \tag{6.A.1}$$

where $\kappa \in (0, \infty)$ is the speed of mean-reversion and $\sigma \in (0, \infty)$ the volatility.

The bank account B is then given by

$$B(t) \;=\; e^{\int_0^t r(s)\,\mathrm{d}s} \;=\; e^{\int_0^t \phi(s)\,\mathrm{d}s \,+\, \int_0^t z(s)\,\mathrm{d}s}$$

for $t \in \mathcal{T}$ and the riskless zero-coupon bond P^τ that pays 1 at its maturity $\tau \in (0, T]$ is given by

$$P^\tau(t) \;=\; \mathbb{E}_{\mathbb{Q}}\Big[\, e^{-\int_t^\tau \phi(s)\,\mathrm{d}s \,-\, \int_t^\tau z(s)\,\mathrm{d}s} \;\Big|\; \mathcal{F}(t) \,\Big] \;=\; D(t,\tau)\, e^{-\int_t^\tau \phi(s)\,\mathrm{d}s \,-\, z(t)\,E(t,\tau)}$$

for $t \in [0, \tau]$, where

$$D(t,\tau) \;:=\; e^{\frac{\sigma^2}{2\kappa^2}((\tau-t) \,-\, E(t,\tau)) \,-\, \frac{\sigma^2}{4\kappa}\,E(t,\tau)^2}$$

and

$$E(t,\tau) \;:=\; \frac{1}{\kappa}\left(1 - e^{-\kappa\,(\tau-t)}\right) \quad .$$

The function ϕ is chosen so that the model reproduces the observed initial market term structure $\{P^\tau(0)\}_{\tau\in(0,T]}$, which implies

$$\int_t^\tau \phi(s)\,\mathrm{d}s \;=\; -\ln\left(\frac{P^\tau(0)}{P^t(0)}\,\frac{D(0,t)}{D(0,\tau)}\right)$$

for $t\in[0,\tau]$ and thus

$$\phi(t) \;=\; f^t(0) \;+\; \frac{\sigma^2}{2\kappa^2}\left(1-e^{-\kappa\,t}\right)^2 \quad ,$$

where $f^\tau(t) := -\frac{\partial}{\partial\tau}\ln P^\tau(t)$ (see also Brigo and Mercurio, 2006, Sec. 3.8).

The LSMC approach of Sec. 6.3 requires samples of the bond price P^T under the forward pricing measure $\mathbb{Q}^T$, which is defined by the Radon–Nikodým derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}(t)} \;=\; \frac{B(0)}{P^T(0)}\,\frac{P^T(t)}{B(t)} \;=\; e^{-\frac{1}{2}\int_0^t \sigma^2\,E(s,T)^2\,\mathrm{d}s\;-\;\int_0^t \sigma\,E(s,T)\,\mathrm{d}W^{\mathbb{Q}}(s)}$$

(cf. Sec. 4.1), such that, by Girsanov's theorem (Prop. 6.D.2), the process $W^{\mathbb{Q}^T} = \{W^{\mathbb{Q}^T}(t)\}_{t\in\mathcal{T}}$ defined by

$$W^{\mathbb{Q}^T}(t) \;:=\; W^{\mathbb{Q}}(t) \;+\; \int_0^t \sigma\,E(s,T)\,\mathrm{d}s$$

for $t\in\mathcal{T}$ is a Brownian motion under $\mathbb{Q}^T$.

The dynamics of z in (6.A.1) then change to

$$\mathrm{d}z(t) \;=\; -\big(\sigma^2\,E(t,T)+\kappa\,z(t)\big)\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}W^{\mathbb{Q}^T}(t), \quad z(0)\;=\;0,$$

such that z is normally distributed under $\mathbb{Q}^T$ with

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^T}\big[\,z(s)\,\big|\,\mathcal{F}(t)\,\big] \;&=\; z(t)\,e^{-\kappa\,(s-t)} \;-\; \frac{\sigma^2}{\kappa^2}\left(1-e^{-\kappa\,(s-t)}\right) \;+\; \frac{\sigma^2}{2\kappa^2}\left(e^{-\kappa\,(T-s)}-e^{-\kappa\,(T+s-2t)}\right)\\
\mathbb{V}_{\mathbb{Q}^T}\big[\,z(s)\,\big|\,\mathcal{F}(t)\,\big] \;&=\; \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa\,(s-t)}\right)
\end{aligned}$$

for $t\le s$, where $\mathbb{V}_{\mathbb{Q}^T}$ is the (conditional) variance under $\mathbb{Q}^T$.

The process z can thus easily be simulated under $\mathbb{Q}^T$, to obtain $K\in\mathbb{N}$ different sample trajectories $\{z_{k,n}\}_{n=0}^N$, $k=1,\ldots,K$, where $z_{k,n}$ denotes the value of $z(\bar{T}_n)$ in the k-th sample path. The bond price sample paths for the second-stage approximation in Sec. 6.3.2 are then given by

$$P_{k,n} \;=\; P^{\bar{T}_N}(0)\,\frac{D(\bar{T}_n,\bar{T}_N)}{D(0,\bar{T}_N)}\,e^{-\,z_{k,n}\,E(\bar{T}_n,\bar{T}_N)}$$

for $k=1,\ldots,K$ and $n=0,\ldots,N$.

6.B The LSMC Fixed-Point Algorithm

Under Ass. 6.3.7, the approximate price $C_{\cdot,0}^{M,K}$ can be obtained from a fixed-point iteration using the second-stage approximation $H^{M,K}$ (see (6.4.1)). For the implementation of this fixed-point iteration, a slight change in the specification of the budget function in Def. 6.2.3 has proven to be useful for increasing the speed of convergence:

$$\begin{aligned} H(C) \;=\; \Bigg\{ & P_0\,\mathbb{E}_{\mathbb{Q}^T}\Bigg[\,\sum_{n=1}^{N}\bar{L}_n\big(\{X_i^C\}_{i=0}^{n}\big)\Bigg]\,, \\ & \Bigg\{ P_n\,\bar{L}_n\big(\{X_i^C\}_{i=0}^{n}\big) \;+\; \mathbb{E}_{\mathbb{Q}^T}\Bigg[\,P_n \sum_{i=n+1}^{N}\bar{L}_i\big(\{X_j^C\}_{j=0}^{i}\big)\,\Bigg|\,\sigma(P_{[n]})\Bigg]\Bigg\}_{n=1}^{N-1}\,, \\ & P_N\,\bar{L}_N\big(\{X_n^C\}_{n=0}^{N}\big)\Bigg\} \end{aligned} \tag{6.B.1}$$

for $C \in \mathcal{V}$, i.e. the $\sigma(P_{[n]})$-measurable discounted lock-in $P_n\,\bar{L}_n\big(\{X_i^C\}_{i=0}^{n}\big)$ is separated from the conditional expectation.

The first-stage approximation in Def. 6.3.3 is then given by

$$\begin{aligned} H^M(C) \;=\; \Bigg\{ & P_0\,\mathbb{E}_{\mathbb{Q}^T}\Bigg[\,\sum_{n=1}^{N}\bar{L}_n\big(\{X_i^C\}_{i=0}^{n}\big)\Bigg]\,, \\ & \Bigg\{ P_n\,\bar{L}_n\big(\{X_i^C\}_{i=0}^{n}\big) \;+\; \mathcal{Q}_n^M\Bigg(\,P_n \sum_{i=n+1}^{N}\bar{L}_i\big(\{X_j^C\}_{j=0}^{i}\big)\Bigg)\Bigg\}_{n=1}^{N-1}\,, \\ & P_N\,\bar{L}_N\big(\{X_n^C\}_{n=0}^{N}\big)\Bigg\} \end{aligned}$$

for $M \geq 1$ and $C \in \mathcal{V}$, and the second-stage approximation in Def. 6.3.6 changes analogously.

The fixed-point iteration in (6.4.1) becomes

$$_{(u+1)}C_{k,n}^{M,K} \;=\; P_{k,n}\,\bar{L}_n\Bigg(\Bigg\{\, g_{k,i}^{{}_{(u)}C^{M,K},P_{[K],[N]}} + {}_{(u)}C_{k,i}^{M,K}\,\Bigg\}_{i=0}^{n}\Bigg) \;+\; \hat{\alpha}_n^\top e_n^M(P_{k,[n]})$$

for $n = 1, \ldots, N-1$,

$$_{(u+1)}C_{k,0}^{M,K} \;=\; \frac{P_{k,0}}{K}\sum_{\ell=1}^{K}\sum_{i=1}^{N}\bar{L}_i\Bigg(\Bigg\{\, g_{\ell,j}^{{}_{(u)}C^{M,K},\,P_{[K],[N]}} \;+\; {}_{(u)}C_{\ell,j}^{M,K}\,\Bigg\}_{j=0}^{i}\Bigg)\quad,$$

and

$$_{(u+1)}C_{k,N}^{M,K} \;=\; \bar{L}_N\Bigg(\Bigg\{\, g_{k,n}^{{}_{(u)}C^{M,K},\,P_{[K],[N]}} \;+\; {}_{(u)}C_{k,n}^{M,K}\,\Bigg\}_{n=0}^{N}\Bigg)$$

for $k = 1, \dots, K$, where

$$\hat{\alpha}_n \;:=\; \alpha_n^{M,K}\left(\left\{P_{k,n}\sum_{i=n+1}^{N}\bar{L}_i\left(\left\{\, g_{k,j}^{{}_{(u)}C^{M,K},\,P_{[K],[N]}} \;+\; {}_{(u)}C_{k,j}^{M,K}\right\}_{j=0}^{i}\right)\right\}_{k=1}^{K},\, P_{[K],[n]}\right) \quad .$$

The convergence results of this chapter remain valid under the alternative specification (6.B.1) of the budget function. The original specification of H in Def. 6.2.3 has been chosen merely to ease the notational burden. The complete fixed-point iteration is detailed in Alg. 3.

Algorithm 3 The LSMC fixed-point algorithm

Require: $P_{[K],[N]}$ ▷ Monte Carlo samples
Require: ${}_{(0)}C^{M,K}$ ▷ Initial guess
Require: $u_\infty \in \mathbb{N}$ ▷ Maximum number of iterations
Require: $\text{tol} \in (0, \infty)$ ▷ Convergence tolerance

$u \leftarrow 0$ ▷ Iteration counter
$\varepsilon \leftarrow \infty$ ▷ Norm of the difference of two iterates ${}_{(u)}C$ and ${}_{(u+1)}C$

while $(u < u_\infty)$ and $(\varepsilon > \text{tol})$ **do**
  ${}_{(u+1)}C_{\cdot,0}^{M,K} \leftarrow P_{\cdot,0}\,\frac{1}{K}\sum_{k=1}^{K}\left(\sum_{n=1}^{N}\bar{L}_n\left(\left\{\, g_{k,i}^{{}_{(u)}C^{M,K},\,P_{[K],[N]}} \;+\; {}_{(u)}C_{k,i}^{M,K}\right\}_{i=0}^{n}\right)\right)$ ▷ Update $C_{\cdot,0}^{M,K}$
  for $n \leftarrow 1$ to $N-1$ **do**
    $\hat{\alpha}_n \leftarrow \alpha_n^{M,K}\left(\left\{P_{k,n}\sum_{i=n+1}^{N}\bar{L}_i\left(\left\{\, g_{k,j}^{{}_{(u)}C^{M,K},\,P_{[K],[N]}} \;+\; {}_{(u)}C_{k,j}^{M,K}\right\}_{j=0}^{i}\right)\right\}_{k=1}^{K},\, P_{[K],[n]}\right)$ ▷ LS estimate
    for $k \leftarrow 1$ to K **do**
      ${}_{(u+1)}C_{k,n}^{M,K} \leftarrow P_{k,n}\,\bar{L}_n\left(\left\{\, g_{k,i}^{{}_{(u)}C^{M,K},P_{[K],[N]}} + {}_{(u)}C_{k,i}^{M,K}\right\}_{i=0}^{n}\right) \;+\; \hat{\alpha}_n^\top e_n^M(P_{k,[n]})$ ▷ Update $C_{\cdot,n}^{M,K}$
    end for
  end for
  for $k \leftarrow 1$ to K **do**
    ${}_{(u+1)}C_{k,N}^{M,K} \leftarrow \bar{L}_N\left(\left\{\, g_{k,n}^{{}_{(u)}C^{M,K},\,P_{[K],[N]}} \;+\; {}_{(u)}C_{k,n}^{M,K}\right\}_{n=0}^{N}\right)$ ▷ Update $C_{\cdot,N}^{M,K}$
  end for
  $\varepsilon \leftarrow \left\| \, {}_{(u+1)}C^{M,K} \;-\; {}_{(u)}C^{M,K} \, \right\|$
  $u \leftarrow u + 1$
end while

6.C Proofs

Proof of Lem. 6.2.2. By the same arguments as in the proof of Lem. 4.3.27, $\|\cdot\|$ defined in (6.2.4) is indeed a norm on $\mathcal{V}$.
Let $\{{}_{(u)}Y\}_{u\in\mathbb{N}} \subset \mathcal{V}$ be a Cauchy sequence and $\Lambda^1, \Lambda^2 \in (0,\infty)$ constants such that

$$\Lambda^1 \max_{n=0}^{N} |\, y_n \,| \;\le\; \|\, y \,\|_\star \;\le\; \Lambda^2 \max_{n=0}^{N} |\, y_n \,|$$

for all $y = \{y_n\}_{n=0}^N \in \mathbb{R}^{N+1}$. Then, for all $\epsilon > 0$, there exists $U \ge 1$ with

$$\epsilon \;>\; \frac{1}{\Lambda^1}\,\|\, {}_{(u)}Y \;-\; {}_{(\ell)}Y \,\| \;\ge\; \max_{n=0}^{N} \big\|\, {}_{(u)}Y_n \;-\; {}_{(\ell)}Y_n \,\big\|_{\mathcal{L}^2}$$

for all $u, \ell > U$. In particular, for $n = 0, \dots, N$, $\{{}_{(u)}Y_n\}_{u\in\mathbb{N}}$ is a Cauchy sequence that converges to a random variable $\hat{Y}_n \in \mathcal{L}^2(\sigma(P_{[n]}))$. From

$$\|\, {}_{(u)}Y \;-\; \hat{Y} \,\| \;\le\; \Lambda^2 \max_{n=0}^{N} \big\|\, {}_{(u)}Y_n \;-\; \hat{Y}_n \,\big\|_{\mathcal{L}^2} \;\xrightarrow{u\to\infty}\; 0$$

it follows that $\{{}_{(u)}Y\}_{u\in\mathbb{N}}$ converges to $\hat{Y} = \{\hat{Y}_n\}_{n=0}^N \in \mathcal{V}$. □

Proof of Lem. 6.3.2. By the definition of $\mathcal{Q}_n^M$ and S_n^M,

$$\mathcal{Q}_n^M(Y) \;=\; \left(\operatorname*{arg\,min}_{\alpha\,\in\,\mathbb{R}^M} \; \mathbb{E}_{\mathbb{Q}^T}\Big[\big(Y \;-\; \alpha^\top e_n^M(P_{[n]}) \big)^2 \Big] \right)^{\!\top} e_n^M(P_{[n]}) \quad .$$

The first-order condition for the minimization problem above is given by

$$\begin{aligned} & -2\, \mathbb{E}_{\mathbb{Q}^T}\Big[e_n^M(P_{[n]}) \big(Y \;-\; \alpha^\top e_n^M(P_{[n]}) \big) \Big] \;=\; 0 \\ \Leftrightarrow \quad & \mathbb{E}_{\mathbb{Q}^T}\Big[e_n^M(P_{[n]})\, e_n^M(P_{[n]})^\top \Big]\, \alpha \;=\; \mathbb{E}_{\mathbb{Q}^T}\Big[e_n^M(P_{[n]})\, Y \Big] \quad , \end{aligned}$$

which is solved by $\alpha = \alpha_n^M(Y)$. Note that

$$\alpha^\top A_n^M\, \alpha \;=\; \big\|\, \alpha^\top e_n^M(P_{[n]}) \,\big\|_{\mathcal{L}^2}^2 \;>\; 0$$

for all $\alpha \in \mathbb{R}^M \backslash \{0\}$ by Ass. 6.3.1 (A), such that A_n^M is positive definite and thereby invertible. □

Proof of Prop. 6.3.5. With Ass. 6.3.4 and $C^M = H^M(C^M)$,

$$\begin{aligned} \| C^M - C^\star \| &\le \| C^M - H^M(C^\star) \| + \| H^M(C^\star) - C^\star \| \\ &\le \Lambda^M \| C^M - C^\star \| + \| H^M(C^\star) - C^\star \| \quad , \end{aligned}$$

such that

$$\| C^M - C^\star \| \le \frac{1}{1-\Lambda^M} \| H^M(C^\star) - C^\star \|$$

for all $M \ge \bar{M}$. Let $Z = \{Z_n\}_{n=0}^N := H^M(C^\star) - C^\star \in \mathcal{V}$. Then, $Z_0 = 0$ and $Z_N = 0$ by the definition of H^M and the fixed-point property of $C^\star$. Furthermore,

$$Z_n = \mathcal{Q}_n^M\left(P_n \sum_{i=n}^N \bar{L}_i\big(\{X_j^{C^\star}\}_{j=0}^i\big) \right) - \mathbb{E}\left[P_n \sum_{i=n}^N \bar{L}_i\big(\{X_j^{C^\star}\}_{j=0}^i\big) \,\middle|\, \sigma(P_{[n]}) \right]$$

for $n = 1, \ldots, N-1$. With Ass. 6.3.1 (B) the right-hand side above converges to zero (in $\mathcal{L}^2$) for $M \to \infty$ and thus, together with $\limsup_{M\to\infty}(1-\Lambda^M)^{-1} < \infty$,

$$\| C^M - C^\star \| \xrightarrow{M\to\infty} 0 \quad ,$$

which is what was claimed. □

Proof of Prop. 6.3.8. Analogously to the proof of Prop. 6.3.5,

$$\| C^{M,K} - \hat{C}^{M,K} \|_{M,K} \le \frac{1}{1-\Lambda^{M,K}} \| H^{M,K}(\hat{C}^{M,K}, P_{[K],[N]}) - \hat{C}^{M,K} \|_{M,K}$$

for $K \ge \bar{K}_M$ by Ass. 6.3.7. Let $Z := H^{M,K}(\hat{C}^{M,K}, P_{[K],[N]}) - \hat{C}^{M,K}$, such that Z is a random variable taking values in $\mathbb{R}^{K\times(N+1)}$. Then, $Z_{k,N} = 0$ for $k = 1, \ldots, K$ by definition of $H^{M,K}$ and

$$Z_{k,0} = \frac{1}{K} \sum_{k=1}^K \left(P_{0,k} \sum_{n=1}^N \bar{L}_n\big(\{\hat{X}_{k,i}^{M,K}\}_{i=0}^n\big) \right) - \mathbb{E}\left[P_0 \sum_{n=1}^N \bar{L}_n\big(\{X_i^{C^M}\}_{i=0}^n\big) \right]$$

for $k = 1, \ldots, K$. By the strong law of large numbers (Prop. 6.D.1), $Z_{k,0} \xrightarrow[K\to\infty]{\text{a.s.}} 0$ for $k = 1, \ldots, K$. On the other hand,

$$\begin{aligned} Z_{k,n} = \Bigg(\alpha_n^{M,K}\left(\left\{ P_{k,n} \sum_{i=n}^N \bar{L}_i\big(\{\hat{X}_{k,j}^{M,K}\}_{j=0}^i\big) \right\}_{k=1}^K, P_{[K],[n]} \right) \\ - \alpha_n^M\left(P_n \sum_{i=n}^N \bar{L}_i\big(\{X_j^{C^M}\}_{j=0}^i\big) \right) \Bigg)^\top e_n^M(P_{k,[n]}) \end{aligned}$$

for $k = 1, \ldots, K$ and $n = 1, \ldots, N-1$. Again, by the strong law of large numbers (Prop. 6.D.1),

$A_n^{M,K}(P_{[K],[n]}) \xrightarrow[K\to\infty]{\text{a.s.}} A_n^M$ and

$$\frac{1}{K}\sum_{k=1}^{K}\left[\left(P_{k,n}\sum_{i=n}^{N}\bar{L}_i\big(\{\hat{X}_{k,j}^{M,K}\}_{j=0}^{i}\big)\right)e_n^M(P_{k,[n]})\right] \xrightarrow[K\to\infty]{\text{a.s.}} \mathbb{E}\left[\left(P_n\sum_{i=n}^{N}\bar{L}_i\big(\{X_j^{C^M}\}_{j=0}^{i}\big)\right)e_n^M(P_{[n]})\right] \quad ,$$

such that $Z_{k,n} \xrightarrow[K\to\infty]{\text{a.s.}} 0$ for $k=1,\dots,K$ and $n=1,\dots,N-1$. □

6.D Auxiliary Definitions and Results

Proposition 6.D.1 (Strong Law of Large Numbers, Durrett (2010, Theo. 2.4.1.)) *Let* $\{Y_k\}_{k=1}^{\infty}$ *be a sequence of i.i.d. random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\mathbb{E}_{\mathbb{P}}\big[|Y_1|\big] < \infty$*. Then,*

$$\frac{1}{K}\sum_{k=1}^{K} Y_k \xrightarrow[K\to\infty]{a.s.} \mathbb{E}_{\mathbb{P}}\big[\, Y_1 \,\big] \quad .$$

Proposition 6.D.2 (Girsanov's Theorem, Musiela and Rutkowski (2005, Theo. A.15.1)) *Let* $T \in (0,\infty)$*,* $\mathcal{T} = [0,T]$*, and* $W = \{W(t)\}_{t\in\mathcal{T}}$ *a Brownian motion on a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, where* $\mathbb{F} = \{\mathcal{F}(t)\}_{t\in\mathcal{T}}$ *and* $\mathcal{F}(T) = \mathcal{F}$*. Moreover, let* $\gamma = \{\gamma(t)\}_{t\in\mathcal{T}}$ *be an* $\mathbb{F}$*-progressively measurable (real-valued) process with*

$$\mathbb{E}_{\mathbb{P}}\Big[e^{-\frac{1}{2}\int_0^T \gamma(s)^2\,\mathrm{d}s \;-\; \int_0^T \gamma(s)\,\mathrm{d}W(s)}\Big] \;=\; 1$$

and define the probability measure $\mathbb{P}^\star$ *on* $(\Omega, \mathcal{F})$ *by the Radon–Nikodým derivative*

$$\frac{\mathrm{d}\mathbb{P}^\star}{\mathrm{d}\mathbb{P}} \;=\; e^{-\frac{1}{2}\int_0^T \gamma(s)^2\,\mathrm{d}s \;-\; \int_0^T \gamma(s)\,\mathrm{d}W(s)} \quad .$$

Then the process $W^\star = \{W^\star(t)\}_{t\in\mathcal{T}}$*, defined by*

$$W^\star(t) \;:=\; W(t) \;+\; \int_0^t \gamma(s)\,\mathrm{d}t$$

for $t \in \mathcal{T}$*, is a Brownian motion on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\star)$*.*

6.E The Hilbert Space $\mathcal{L}^2$ and Orthogonal Projections

The convergence results presented in this chapter are based on basic properties of Hilbert spaces, which are briefly summarized here for the special case of the Hilbert space $\mathcal{L}^2$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ a (sub-)σ-algebra. The vector space of $\mathcal{G}$-measurable random variables $Y : \Omega \to \mathbb{R}$ with $\mathbb{E}_{\mathbb{P}}[\, Y^2 \,] < \infty$ is denoted by $\mathcal{L}^2(\mathcal{G})$, where two random variables $Y^1, Y^2 \in \mathcal{L}^2(\mathcal{G})$ are identified if $Y^1 = Y^2$ $\mathbb{P}$-a.s. (as always $\mathbb{E}_{\mathbb{P}}$ is the expectation under $\mathbb{P}$).

$\mathcal{L}^2(\mathcal{G})$ can be endowed with an inner product $\langle \cdot\, , \cdot \rangle : \mathcal{L}^2(\mathcal{G}) \times \mathcal{L}^2(\mathcal{G}) \to \mathbb{R}$ defined by

$$\langle\, Y_1,\ Y_2 \,\rangle \ := \ \mathbb{E}_{\mathbb{P}}[\, Y_1\, Y_2 \,]$$

for $Y_1, Y_2 \in \mathcal{L}^2(\mathcal{G})$, which then makes it a Hilbert space (Young, 1988, Def. 3.4). The norm induced by $\langle \cdot\, , \cdot \rangle$ is precisely the 2-norm introduced in Def. 4.3.26, i.e.

$$\|\, Y \,\|_{\mathcal{L}^2} \ = \ \langle\, Y,\ Y \,\rangle^{\frac{1}{2}} \ = \ \mathbb{E}_{\mathbb{P}}[\, Y^2 \,]^{\frac{1}{2}} \ < \ \infty$$

for all $Y \in \mathcal{L}^2(\mathcal{G})$.

An important feature of Hilbert spaces is given by Hilbert's projection theorem (Young, 1988, Theo. 3.8), which states that for every non-empty closed convex set $A \subseteq \mathcal{L}^2(\mathcal{G})$ and $Y \in \mathcal{L}^2(\mathcal{G})$, there exists a unique *orthogonal projection* $\hat{Y} \in A$ *of* Y *onto* A with

$$\|\, \hat{Y} \ - \ Y \,\|_{\mathcal{L}^2} \ = \ \min_{Z \,\in\, A} \|\, Z \ - \ Y \,\|_{\mathcal{L}^2} \quad .$$

If A is a closed linear subspace of $\mathcal{L}^2(\mathcal{G})$, e.g. $A = \mathcal{L}^2(\mathcal{P}) \subseteq \mathcal{L}^2(\mathcal{G})$ with $\mathcal{P} \subseteq \mathcal{G}$ a (sub-)σ-algebra, then a necessary and sufficient condition for $\hat{Y}$ to attain the minimum above is given by

$$\langle\, \hat{Y} \ - \ Y,\ Z \,\rangle \ = \ 0$$

for all $Z \in A$ (Young, 1988, Theo. 4.24).

Intriguingly, for $Y \in \mathcal{L}^2(\mathcal{G})$ and $\mathcal{P} \subseteq \mathcal{G}$ a (sub-)σ-algebra, the conditional expectation $\mathbb{E}_{\mathbb{P}}\big[\, Y \,|\, \mathcal{P} \,\big]$, which is $\mathcal{P}$-measurable (by definition) and square-integrable (by Jensen's inequality (Prop. 4.D.1)), is precisely the orthogonal projection of Y onto $\mathcal{L}^2(\mathcal{P})$. Indeed, for all $Z \in \mathcal{L}^2(\mathcal{P})$,

$$\mathbb{E}_{\mathbb{P}}\big[\, Z\, Y \,\big] \ = \ \mathbb{E}_{\mathbb{P}}\Big[\, \mathbb{E}_{\mathbb{P}}\big[\, Z\, Y \,|\, \mathcal{P} \,\big] \,\Big] \ = \ \mathbb{E}_{\mathbb{P}}\Big[\, Z\, \mathbb{E}_{\mathbb{P}}\big[\, Y \,|\, \mathcal{P} \,\big] \,\Big] \quad ,$$

such that

$$\mathbb{E}_{\mathbb{P}}\Big[\ Z\, Y \ - \ Z\, \mathbb{E}_{\mathbb{P}}\big[\, Y \,|\, \mathcal{P} \,\big] \ \Big] \ = \ \langle\, Y \ - \ \mathbb{E}_{\mathbb{P}}\big[\, Y \,|\, \mathcal{P} \,\big],\ Z \,\rangle \ = \ 0 \quad .$$

Definition 6.E.1 (Orthonormality) *Let $\mathcal{P} \subseteq \mathcal{G}$ be a (sub-)σ-algebra. A sequence of random variables $\{Y_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^2(\mathcal{P})$ with*

$$\mathbb{E}_{\mathbb{P}}\big[\, Y_n \, Y_{\hat{n}} \,\big] \;=\; \begin{cases} 1 & , \textit{ for } n = \hat{n}, \\ 0 & , \textit{ else}, \end{cases}$$

is called orthonormal.

Definition 6.E.2 (Completeness) *Let $\mathcal{P} \subseteq \mathcal{G}$ be a (sub-)σ-algebra. A sequence of random variables $\{Y_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^2(\mathcal{P})$, such that*

$$\overline{Span}(\{Y_n\}_{n \in \mathbb{N}}) \;=\; \mathcal{L}^2(\mathcal{P})$$

is called complete (in $\mathcal{L}^2(\mathcal{P})$). Here, $\overline{Span}(\{Y_n\}_{n \in \mathbb{N}})$ is the closure of the linear span

$$Span(\{Y_n\}_{n \in \mathbb{N}}) \;=\; \left\{ \sum_{n=1}^{N} \lambda_n \, Y_n \;:\; N \in \mathbb{N},\; \lambda \in \mathbb{R}^N \right\} \quad ,$$

which contains all (finite) linear combinations of elements of $\{Y_n\}_{n \in \mathbb{N}}$.

The concepts of orthonormality and completeness are essential to the theory of Hilbert spaces, as they allow to give a 'coordinate expression' for orthogonal projections. More precisely, for a (sub-)σ-algebra $\mathcal{P} \subseteq \mathcal{G}$, an orthonormal sequence $\{Y_n\}_{n \in \mathbb{N}} \subseteq \mathcal{L}^2(\mathcal{P})$, and $N \in \mathbb{N}$, the projection of $Z \in \mathcal{L}^2(\mathcal{G})$ onto the linear span

$$\mathrm{Span}\big(\{Y_n\}_{n=1}^{N}\big) \;=\; \left\{ \sum_{n=1}^{N} \lambda_n \, Y_n \;:\; \lambda \in \mathbb{R}^N \right\}$$

is given by

$$\sum_{n=1}^{N} \langle\, Z,\; Y_n \,\rangle \, Y_n$$

(Young, 1988, Theo. 4.6). If $\{Y_n\}_{n \in \mathbb{N}}$ is additionally complete in $\mathcal{L}^2(\mathcal{P})$, then

$$\sum_{n=1}^{N} \langle\, Z,\; Y_n \,\rangle \, Y_n \quad \xrightarrow{N \to \infty} \quad \sum_{n=1}^{\infty} \langle\, Z,\; Y_n \,\rangle \, Y_n \;=\; \mathbb{E}_{\mathbb{P}}\big[\, Z \,|\, \mathcal{P} \,\big]$$

for all $Z \in \mathcal{L}^2(\mathcal{G})$ (Young, 1988, Theo. 4.14). In other words, the conditional expectation $\mathbb{E}_{\mathbb{P}}\big[\, Y \,|\, \mathcal{P} \,\big]$ can be approximated by orthogonal projections onto the subspaces spanned by an orthonormal and complete sequence in $\mathcal{L}^2(\mathcal{P})$. Using the *Gram-Schmidt procedure* (Young, 1988, p. 42) the same then also holds true for any linearly independent and complete sequence in $\mathcal{L}^2(\mathcal{P})$.

6.F Additional Numerical Results

Longstaff and Schwartz (2001) suggest to build the set of basis functions using the (weighted) Laguerre polynomials given by

$$W_d(x) \;:=\; e^{-\frac{1}{2}x}\,\frac{e^x}{d!}\,\frac{\mathrm{d}^d}{\mathrm{d}x^d}\big(x^d\,e^{-x}\big)$$

with degree $d \in \mathbb{N}$.

The numerical results of Sec. 6.4.2 remain valid when using this polynomial family instead of monomials. In particular, the results presented in Tab. 6.2 are virtually the same when using the sets of basis functions in Tab. 6.5 below (see Tab. 6.6).

Set	Description	Basis Functions
$\mathrm{I}_d^\star$	Laguerre polynomials of the current bond price up to degree d	$\Big\{ 1, W_1(P_n), \ldots, W_d(P_n) \Big\}$
$\mathrm{II}_d^\star$	Laguerre polynomials of the current and past bond prices up to degree d	$\Big\{ 1, W_1(P_1), \ldots, W_d(P_1), W_1(P_2), \ldots, W_d(P_2), \ldots, W_1(P_n), \ldots, W_d(P_n) \Big\}$

Table 6.5: Different choices for the sets of basis functions $\{e_{n,m}\}_{m=1}^{M}$ using the (weighted) Laguerre polynomials instead of monomials (cf. Tab. 6.1).

Lock-In Mechanism	Benchmark Price	Basis Func.	Degree d			
			1	2	3	4
Lock-In I (excess over previous NAV)	0.0262	$\mathrm{I}_d^\star$	0.0257	0.0261	0.0261	0.0262
		$\mathrm{II}_d^\star$	0.0257	0.0261	0.0261	0.0262
Lock-In II (excess over guaranteed amount)	0.1235	$\mathrm{I}_d^\star$	0.1238	0.1239	0.1230	0.1229
		$\mathrm{II}_d^\star$	0.1241	0.1246	0.1235	0.1235
Lock-In III (excess over average NAV)	0.0233	$\mathrm{I}_d^\star$	0.0219	0.0225	0.0225	0.0225
		$\mathrm{II}_d^\star$	0.0230	0.0234	0.0234	0.0234

Table 6.6: The mean approximate price $C_{\cdot,0}^{M,K}$ obtained from 100 independent runs of method Lsmc II for the two sets of basis functions $\mathrm{I}_d^\star$ and $\mathrm{II}_d^\star$ in Tab. 6.5, and different choices of the degree d. Model parameters are chosen as in Tab. 6.2. Results have been rounded.

7 Conclusion

With the advent of modern supervisory regimes and the proliferation of new guarantee concepts in unit-linked life insurance, the problem of hedging and valuing contingent guarantees has attracted considerable attention from academics, practitioners, and regulators. The unique structure of these types of financial liabilities make them a highly non-standard object of study in mathematical finance, which defies simple attempts to apply classical hedging and valuation approaches. This thesis develops a unified hedging and valuation framework for contingent guarantees and provides the first steps towards answering the questions raised in Sec. 3.2.

Hedging: How should the fund management invest in order to super-replicate the guaranteed amount (3.1.1) of a contingent guarantee $\bar{G}$?

By suitably extending the classical portfolio insurance framework (Sec. 4.2), the problem of super-replicating the terminal guaranteed amount of $\bar{G}$ can be transformed into the associated fixed-point problem $C = H^{\bar{G}}_{\bar{E}}(C)$, where the non-negative terminal excess $\bar{E}$ must be such that the combined payoff of guaranteed amount and excess is attainable (complete markets: Sec. 4.3, incomplete markets: Sec. 5.2). A solution to this fixed-point problem can be interpreted as the value process of a hedging derivative, which pays the discounted lock-in and the terminal excess.

If the desired hedging derivative is offered by another market participant, the fund management could simply buy this derivative and the initial guarantee $P^{\bar{T}_N}(0)\,\bar{F}$. Where this is not possible, the investment management should pursue a corresponding portfolio insurance strategy, which can be obtained from a replicating strategy for the hedging derivative (see Lem. 4.3.3 and Sec. 4.4.2). From a practical perspective, a delta-hedging approach might already suffice (see Sec. 6.4.2).

Hedging: What are sufficient conditions to ensure that a contingent guarantee can be hedged?

Sufficient conditions for the existence of hedging strategies – or rather hedging derivatives – can be derived from suitable fixed-point theorems. For example, Brouwer's theorem (Prop. 2.2.1) and its extension by Kakutani (Prop. 2.2.2) imply the existence of hedging derivatives for continuous and capped lock-in mechanisms in complete (Sec. 4.3.1) and incomplete financial markets (Sec. 5.2.1).

Similarly, Tarski's theorem (Prop. 2.3.2) shows that contingent guarantees with monotone and capped lock-in mechanisms can always be hedged in a complete financial market (Sec. 4.3.2).

Sufficient conditions based on the 'metric' fixed-point theorems of Banach (Prop. 2.1.2, Sec. 4.3.3) and Nadler (Prop. 2.1.4, Sec. 5.2.1) are less straightforward to verify. They rely on the heuristic argument that the terminal guaranteed amount does not grow 'too fast' with the fund's NAV. Fortunately, many lock-in mechanisms are structured in a way, such that this 'soft' growth condition can be satisfied by a suitable parameterization (see e.g. the take-profit mechanisms in Ex. 3.5.2).

Valuation: How can the 'value' of a contingent guarantee $\bar{G}$ be defined?

The Martingale Method developed in Sec. 4.3 allows to define the value of a contingent guarantee as the cost of the cheapest hedging derivative for $\bar{G}$, or, equivalently, as the minimum initial capital necessary to super-replicate the terminal guaranteed amount of $\bar{G}$ with a self-financing investment strategy (Def. 4.3.4 and 5.2.2). This definition is consistent with the classical notion of the value of a contingent claim in mathematical finance.

Valuation: What are the major risk factors impacting the value of $\bar{G}$?

The value of a contingent guarantee will often be attained by an interest rate hedging derivative (see Sec. 4.3.5). Risks associated with (riskless) interest rates, such as shifts in the term structure and interest rate volatility, should thus be considered as the main factors impacting the value of a contingent guarantee.

However, one of the peculiar consequences of the structure of a contingent guarantee is that its quantitative and qualitative characteristics are closely linked to the underlying investment strategy. In that sense, each hedging derivative defines its own value – namely its initial cost – and also its own risk profile of the contingent guarantee $\bar{G}$. This is particularly relevant for regulatory frameworks, such as Solvency II, that require insurance companies to value their liabilities under given stress-scenarios in order to assess their ability to meet future financial obligations: If the insurance company chooses to utilize an investment strategy, which significantly differs from the hedging strategy underpinning the valuation of these liabilities, then these 'stress tests' have very limited predictive power.

Valuation: How can the value of a contingent guarantee be calculated efficiently?

With Banach's theorem (Prop. 2.1.2) and Kleene's theorem (Prop. 2.3.3), a hedging derivative can be approximated using a fixed-point iteration. In finite-dimensional market models, this iteration can be implemented in a straightforward manner (Sec. 4.4), but the resulting algorithm suffers greatly from the curse of dimensionality. An adaptation of the LSMC approach (Sec. 6.3) allows to overcome this challenge and provides the means to solve even large-scale valuation problems in a timely manner.

7.1 Extensions

The portfolio insurance framework of Sec. 4.2 and the Martingale Method introduced in Sec. 4.3 allow for some straightforward extensions and generalizations, some of which are summarized in the following.

Stochastic Initial Guarantee

For example, the fixed guaranteed amount $\bar{F}$ of a contingent guarantee $\bar{G} = (\bar{T}, \bar{F}, \bar{L})$ must not necessarily be constant, but could be any non-negative attainable payoff. In the (discrete) complete financial market of Sec. 4.1, this would imply $\bar{F} \in [0, \infty)^K$. For any investment strategy $\pi \in \Pi$, the guarantee $G^{\bar{G},\pi}$ in Def. 4.2.2, is then given by

$$G^{\bar{G},\pi}(t) \;=\; P^{\bar{T}_N}(t) \left(\mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\bar{F} \,\middle|\, \mathcal{F}(t) \right] \;+\; V^{\bar{G},\pi}(t) \right)$$

and the value $\Phi^{\bar{G}}$ of $\bar{G}$ in Def. 4.3.4 changes to

$$\Phi^{\bar{G}} \;=\; P^{\bar{T}_N}(0)\, \mathbb{E}_{\mathbb{Q}^{\bar{T}_N}} \left[\bar{F} \right] \;+\; \inf \left\{ C_0 \;:\; C \in \mathcal{C}^{\bar{G}} \right\} \quad .$$

All results remain valid under this stochastic specification of $\bar{F}$.

An example of a stochastic initial guaranteed amount is given by periodically compounding some given notional $F \in [0, \infty)$, e.g.

$$\bar{F} \;=\; F \prod_{n=1}^{N} \frac{1}{P^{\bar{T}_n}(\bar{T}_{n-1})} \quad ,$$

which lends itself to the economic interpretation that an investment into the fund X yields at least the return of a recurring money market investment (over the same period and with notional F).

Interest Rate Sensitive Lock-In

A similar extension to the stochastic initial guaranteed amount above is to allow for a stochastic lock-in by incorporating the interest rate environment. For example, a lock-in mechanism

$\bar{L} = \{\bar{L}_n\}_{n=1}^N$ could be defined as a family of non-negative measurable functions $\bar{L}_n : \mathbb{R}^{n+1} \times (0,\infty)^{n+1} \to [0,\infty)$ with the lock-in at time $\bar{T}_n$ given by

$$\bar{L}_n\Big(\big\{X(\bar{T}_i)\big\}_{i=0}^n , \big\{P^{\bar{T}_N}(\bar{T}_i)\big\}_{i=0}^n \Big) \quad ,$$

i.e. by a function of the history of the fund NAV X and the bond price $P^{\bar{T}_N}$. The terminal guaranteed amount of $\bar{G}$ in (3.1.1) then changes to

$$\bar{F} + \sum_{n=1}^N \bar{L}_n\Big(\big\{X(\bar{T}_i)\big\}_{i=0}^n , \big\{P^{\bar{T}_N}(\bar{T}_i)\big\}_{i=0}^n \Big) \quad . \tag{7.1.1}$$

Under the specification of $\bar{L}$ above, the definitions of a continuous, capped, and monotone lock-in mechanism readily generalize as follows: $\bar{L}$ is called continuous/capped/monotone, if, for $n = 1,\ldots,N$ and any given $p \in (0,\infty)^{n+1}$, the lock-in $\bar{L}_n(\,\cdot\,,p)$ is continuous/capped/monotone in the sense of Def. 4.3.5/4.3.6/4.3.10. In particular, the hedging results based on Brouwer's theorem (Prop. 2.2.1, Sec. 4.3.1) and Tarski's theorem (Prop. 2.3.2, Sec. 4.3.2) hold under these alternative definitions.

Similarly, a hedging result based on Banach's theorem (Prop. 2.1.2, Sec. 4.3.3) can be obtained under the condition that the budget function using this interest rate sensitive lock-in is contracting (Def. 2.1.1). In fact, a particularly useful application of the generalization above is precisely to make this condition hold: By reducing the lock-in in low (negative) interest rate scenarios, the lock-in is essentially 'dampened', which prevents the terminal guaranteed amount from growing excessively.

Example (Dampened Take-Profit Lock-In) *A dampened version of the take-profit lock-in mechanism in* (3.5.2) *is given by*

$$\bar{L}_n\Big(\big\{X(\bar{T}_i)\big\}_{i=0}^n , \big\{P^{\bar{T}_N}(\bar{T}_i)\big\}_{i=0}^n \Big) = \alpha \left[X(\bar{T}_n) - X(\bar{T}_{n-1})\right]^+ \min\left\{1,\ \frac{\bar{P}}{P^{\bar{T}_N}(\bar{T}_n)}\right\}$$

for $n = 1,\ldots,N$, where $\alpha \in [0,1]$ and $\bar{P} \in (0,\infty)$. In this case, if $P^{\bar{T}_N}$ exceeds the given threshold $\bar{P}$, the lock-in is reduced to offset the impact of an increased discount factor.

Negative Lock-In

Another straightforward generalization is to allow for a reduction of the terminal guaranteed amount by means of a negative lock-in (although this could be considered to contradict the notion of a guarantee). More precisely, the lock-in functions $\bar{L}_n$ of Def. 3.1.1 could be defined to take values on the complete real line $\mathbb{R}$ instead of just $[0,\infty)$ with the additional constraint that

$$\bar{L}_n(x) \ \geq \ -\Big(\bar{F} \ + \ \sum_{i=1}^{n-1} \bar{L}_i\big(\{x_j\}_{j=0}^{i}\big)\Big)$$

for $n = 1,\dots,N$ and $x = \{x_i\}_{i=0}^{n} \in \mathbb{R}^{n+1}$.

The terminal guaranteed amount in (3.1.1) may then fall to zero, but cannot become negative. In particular the NAV process corresponding to a replicating strategy of the terminal guaranteed amount is still non-negative.

With a negative lock-in, the cushion may become negative as well, such that $\mathcal{V}^{\bar{T}}$ is no longer equal to the non-negative orthant $[0,\infty)^{K\times(N+1)}$, but rather to the set

$$\left\{ C = \{C_n\}_{n=0}^{N} \in \mathbb{R}^{K\times(N+1)} \ : \ C_n \ \geq \ -G_n^{\bar{G},C} \text{ for } n = 0,\dots,N \right\} \quad ,$$

which now depends on all characteristics of the contingent guarantee $\bar{G}$ and not just the lock-in time points $\bar{T}$. The hedging results obtained in Sec. 4.3.1 through 4.3.3 remain valid.

7.2 Future Research

The results of this thesis can only be considered as a first step towards a full understanding of contingent guarantees. Several questions concerning the hedging and valuation of these financial liabilities are still open and prompt further investigation.

First and foremost, a deeper exploration of fixed-point theory might reveal other useful and applicable theorems apart from the results presented in Ch. 2. In particular, other constructive fixed-point theorems may give rise to new numerical methods for the construction of hedging derivatives and for the valuation contingent guarantees. New fixed-point theorems might also allow to relax some of the rather strong sufficient conditions presented in Sec. 4.3.

Although the existence of portfolio insurance strategies is theoretically implied by the existence of (attainable) hedging derivatives, an actual construction method for these investment strategies requires additional research. One possible approach could be to adapt the replicating portfolio method (see, e.g., Pelsser and Schweizer, 2016), which would not only allow to derive the value of a contingent guarantee, but simultaneously also a corresponding hedging strategy.

Classical portfolio insurance strategies – in particular the CPPI of Ex. 4.2.3 – are known to suffer from several drawbacks, such as 'cash-lock' (the risk of the cushion falling to zero and the fund thus being fully and permanently invested into the guarantee; see, e.g., Balder and Mahayni (2009)) or 'gap-risk' (the risk of falling short of the terminal guaranteed amount, because asset prices jump unexpectedly; see, e.g., Cont and Tankov (2009)). It is still unclear, what role the lock-in mechanism plays in mitigating or exaggerating these risks. A case study examining the practicality of portfolio insurance strategies under real market conditions might reveal desirable properties of lock-in mechanisms.

Throughout this thesis lock-in mechanisms are more or less considered as a given and irrefutable property of the investment fund. A fair question to ask is, which lock-in mechanisms can be considered 'optimal' from the perspective of an investor, or, conversely, under what conditions and behavioral considerations an investor actually profits from a particular lock-in mechanism. Expected utility theory (see, e.g., Pliska, 1997, Sec. 5) may provide answers in this regard.

Other endeavors, such as the construction of partial hedging strategies (see, e.g., Föllmer and Schied, 2016, Sec. 8) or the construction of hedging strategies under portfolio constraints (see, e.g., Cvitanic and Karatzas, 1992), both of which are particularly compelling from a practical perspective, might prove fruitful as well.

Abbreviations and Symbols

Ch.	Chapter.
Sec.	Section.
App.	Appendix.
Def.	Definition.
Lem.	Lemma.
Pro.	Proposition.
Theo.	Theorem.
Cor.	Corollary.
Ex.	Example.
Ass.	Assumption.
Alg.	Algorithm.
Tab.	Table.
Fig.	Figure.
CPPI	Constant proportion portfolio insurance (Ex. 4.2.3).
OBPI	Option based portfolio insurance (Ex. 4.2.4).
NAV	Net asset value.
$\mathbb{N}$	The natural numbers excluding 0.
$\mathbb{N}_0$	The natural numbers including 0.
$\mathbb{R}$	The real numbers.
2^A	The power set (i.e. the set of all subsets) of a set A.
max	The maximum.
min	The minimum.
sup	The supremum.

$\inf$	The infimum.
$\limsup$	The limes superior.
ask	The ask price of a payoff (Def. 5.1.4).
$\mathcal{T} \subset [0, \infty)$	The ordered set of trading time points (Sec. 3.1, 4.1, 5.1, and 6.1).
$X = \{X(t)\}_{t \in \mathcal{T}}$	The fund NAV.
$\bar{G} = (\bar{T}, \bar{F}, \bar{L})$	A contingent guarantee (Def. 3.1.1).
$\bar{T} = \{\bar{T}_n\}_{n=0}^N$	The set of lock-in time points of a contingent guarantee (Def. 3.1.1).
$\bar{F} \in [0, \infty)$	The fixed guaranteed amount of a contingent guarantee (Def. 3.1.1).
$\bar{L} = \{\bar{L}_n\}_{n=0}^N$	The lock-in mechanism of a contingent guarantee (Def. 3.1.1).
Ω	The sample space of a probability space.
$K \in \mathbb{N}$	The number of samples in a discrete sample space $\Omega = \{\omega_1, \ldots, \omega_K\}$. Alternatively, the number of Monte Carlo samples used in the LSMC approach (Sec. 6.3).
$M \in \mathbb{N}$	The number of time points in a discrete financial market (Sec. 4.1 and 5.1). Alternatively, the number of basis functions used in the LSMC approach (Sec. 6.3).
$D \in \mathbb{N}_0$	The number of risky assets (Sec. 4.1 and 5.1).
Δ	The constant time difference in years between two time steps in a discrete financial market (Sec. 4.1, see also App. 4.A).
$\mathbb{P}$	The physical probability measure describing the real-world behavior of asset prices (Sec. 4.1 and 5.1). More generally, any probability measure.
$\mathbb{Q}$	The risk-neutral pricing measure with numéraire B (Sec. 4.1 and 5.1).
$\mathbb{Q}^\tau$	The forward pricing measure with numéraire P^τ (Sec. 4.1, 5.1, and 6.1).
$\mathbb{F} = \{\mathcal{F}(t)\}_{t \in \mathcal{T}}$	The market filtration (Sec. 4.1, 5.1, and 6.1).
$\mathbb{E}_\mathbb{P}$	The (conditional) expectation under a measure $\mathbb{P}$.
$B = \{B(t)\}_{t \in \mathcal{T}}$	The price process of the bank account (Sec. 4.1, 5.1, and 6.1).
$P^\tau = \{P^\tau(t)\}_{t \leq \tau}$	The price process of the riskless zero-coupon bond with maturity $\tau \in \mathcal{T} \setminus \{0\}$ (Sec. 4.1, 5.1, and 6.1).

$S^d = \{S^d(t)\}_{t\in\mathcal{T}}$	The price process of the d-th risky asset (Sec. 4.1 and 5.1).
$X^\pi = \{X^\pi(t)\}_{t\in\mathcal{T}}$	The fund NAV process corresponding to a given investment strategy $\pi \in \Pi$ (Def. 4.1.2).
Π	The set of investment strategies (Def. 4.1.2).
$\Pi^{\bar{G}}$	The set of portfolio insurance strategies for the contingent guarantee $\bar{G}$ (Def. 4.2.1).
$V^{\bar{G},\pi} = \{V^{\bar{G},\pi}(t)\}_{t\in\mathcal{T}}$	The variable guaranteed amount corresponding to the contingent guarantee $\bar{G}$ and the investment strategy $\pi \in \Pi$ (Def. 4.2.2).
$G^{\bar{G},\pi} = \{G^{\bar{G},\pi}(t)\}_{t\in\mathcal{T}}$	The guarantee process corresponding to the contingent guarantee $\bar{G}$ and the investment strategy $\pi \in \Pi$ (Def. 4.2.2).
$C^{\bar{G},\pi} = \{C^{\bar{G},\pi}(t)\}_{t\in\mathcal{T}}$	The cushion process corresponding to the contingent guarantee $\bar{G}$ and the investment strategy $\pi \in \Pi$ (Def. 4.2.2).
$\mathcal{V}^{\bar{T}}$	The non-negative orthant $[0,\infty)^{K\times(N+1)}$ (Sec.4.3).
$C = \{C_n\}_{n=0}^N$	A family of random variables either from $\mathcal{V}^{\bar{T}}$ (Sec. 4.3 and 5.2) or $\mathcal{V}$ (Sec. 6.2).
$G^{\bar{G},C} = \{G_n^{\bar{G},C}\}_{n=0}^N$	The guarantee corresponding to the contingent guarantee $\bar{G}$ and the family $C \in \mathcal{V}^{\bar{T}}$ (Sec. 4.3).
$\bar{E}^{\bar{G},\pi}$	The terminal excess corresponding to the contingent guarantee $\bar{G}$ and the investment strategy $\pi \in \Pi$ (Sec. 4.2).
$\bar{E} \in [0,\infty)^K$	A non-negative terminal excess.
$\bar{E}^{\bar{G}}_{\mathrm{MSR}}(C)$	The set of non-negative terminal excesses of all minimal super-replicating strategies of the total lock-in produced by the family $C \in \mathcal{V}^{\bar{T}}$ (Sec. 5.2.1).
$H^{\bar{G}}_{\bar{E}}$	The budget function corresponding to the contingent guarantee $\bar{G}$ and the terminal excess $\bar{E}$.
$\mathcal{C}^{\bar{G}}$, $\hat{\mathcal{C}}^{\bar{G}}$	The set of (attainable) hedging derivatives (Def. 4.3.2 and 5.2.1).
$\Phi^{\bar{G}}$, $\hat{\Phi}^{\bar{G}}$	The value of a contingent guarantee (Def. 4.3.4 and 5.2.2).
$H^{\bar{G}}_{\bar{E},\mathcal{P}}$	The budget function corresponding to the contingent guarantee $\bar{G}$ and the terminal excess $\bar{E}$ with the market filtration $\mathcal{F}$ replaced by the filtration $\mathcal{P}^{\bar{T}}$ of bond prices (Sec. 4.3.5).
$\mathcal{C}^{\bar{G}}_{\mathcal{P}}$	The set of interest rate hedging derivatives (Sec. 4.3.5).

$H^{\bar{G}}_{\mathrm{MSR}}$	The minimal super-replicating budget function corresponding to the contingent guarantee $\bar{G}$ (Def. 5.2.3).
$H^{\bar{G}}_{\mathrm{SSR}}$	The static super-replicating budget function corresponding to the contingent guarantee $\bar{G}$ (Def. 5.2.7).
$\Lambda^{\bar{G}}_{\mathrm{SSR}}(C)$	The maximum lock-in produced by the family $C \in \mathcal{V}^{\bar{T}}$ for a given contingent guarantee $\bar{G}$ (see (5.2.6)).
$\mathcal{L}^2$	The space of square-integrable random variables (App. 6.E).
$\mathcal{V}$	The cartesian product of $\mathcal{L}^2$-spaces (Sec. 6.2).
$V^{\bar{G}} = \{V^{\bar{G}}(t)\}_{t \in \mathcal{T}}$	The variable guaranteed amount corresponding to the contingent guarantee $\bar{G}$ (Def. 6.2.1).
$G^{\bar{G}} = \{G^{\bar{G}}(t)\}_{t \in \mathcal{T}}$	The guarantee process corresponding to the contingent guarantee $\bar{G}$ (Def. 6.2.1).
$C^{\bar{G}} = \{C^{\bar{G}}(t)\}_{t \in \mathcal{T}}$	The cushion process corresponding to the contingent guarantee $\bar{G}$ (Def. 6.2.1).
$V^C = \{V^C_n\}_{n=0}^N \in \mathcal{V}$	The variable guaranteed amount corresponding to the family $C \in \mathcal{V}$ (Sec. 6.2).
$G^C = \{G^C_n\}_{n=0}^N \in \mathcal{V}$	The guarantee corresponding to the family $C \in \mathcal{V}$ (Sec. 6.2).
$X^C = \{X^C_n\}_{n=0}^N \in \mathcal{V}$	The NAV process corresponding to the family $C \in \mathcal{V}$ (Sec. 6.2).
$\{e_{n,m}\}_{m=1}^{\infty}$	The set of basis functions used in the LSMC approach (Sec. 6.3).
H^M	The first-stage approximation of the budget function (Def. 6.3.3).
$H^{M,K}$	The second-stage approximation of the budget function (Def. 6.3.6).
$C^M \in \mathcal{V}$	The (unique) fixed-point of the first-stage approximation H^M (Sec. 6.3.1).
$C^{M,K} \in \mathbb{R}^{K \times (N+1)}$	The (unique) fixed-point of the second-stage approximation $H^{M,K}$ (Sec. 6.3.1).

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