

Normal approximation of the solution to the stochastic heat equation with Lévy noise

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Abstract

Given a sequence \dot{L}^ε of Lévy noises, we derive necessary and sufficient conditions in terms of their variances $\sigma^2(\varepsilon)$ such that the solution to the stochastic heat equation with noise $\sigma(\varepsilon)^{-1}\dot{L}^\varepsilon$ converges in law to the solution to the same equation with Gaussian noise. Our results apply to both equations with additive and multiplicative noise and hence lift the findings of S. Asmussen and J. Rosiński [*J. Appl. Probab.* **38** (2001) 482–493] and S. Cohen and J. Rosiński [*Bernoulli* **13** (2007) 195–210] for finite-dimensional Lévy processes to the infinite-dimensional setting without making distributional assumptions on the solutions such as infinite divisibility. One important ingredient of our proof is to characterize the solution to the limit equation by a sequence of martingale problems. To this end, it is crucial to view the solution processes both as random fields and as càdlàg processes with values in a Sobolev space of negative real order.

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1 Introduction

The importance of the Gaussian distribution in probability theory and its popularity in applications are manifested in the central limit theorem: The total effect of a large number of small independent contributions is approximately normal. Therefore, when physical systems governed by one or several equations are perturbed by white noise (where “white” means stationary and uncorrelated), it is frequently assumed that the noise is *Gaussian*.

For example, in his Saint-Flour lecture notes [27], J. B. Walsh discusses an application of parabolic stochastic PDEs to the modeling of neuron potentials. Subject to impulses arriving according to a marked Poisson point process (with mean 0 and atoms of size $\dot{L}(t, x)$ at time t and position x), the electrical potential $u(t, x)$ of the neuron, viewed as a thin cylinder of length, say, π , is then well described by the *stochastic cable equation*

$$\partial_t u(t, x) = \partial_{xx} u(t, x) - u(t, x) + \dot{L}(t, x), \quad (t, x) \in [0, T] \times [0, \pi], \quad (1.1)$$

with suitable boundary and initial conditions. Arguing that “[t]he impulses are generally small, and there are many of them, so that in fact \dot{L} is very nearly a white noise” ([27], p. 311), the author then approximates (1.1) by

$$\partial_t u(t, x) = \partial_{xx} u(t, x) - u(t, x) + \dot{W}(t, x), \quad (t, x) \in [0, T] \times [0, \pi], \quad (1.2)$$

where \dot{W} is a Gaussian space–time white noise.

But of course, the central limit theorem has limitations. In the absence of finite second moments, stable limits may arise; and if there are rare but large contributions, we may have a Poisson limit. In general, any *infinitely divisible distribution* can arise as a possible limit of compound Poisson laws; see Corollary 3.8 in [23]. This leads us to the following question: If we have a sequence of noises \dot{L}^ε as above where the atom sizes of \dot{L}^ε converge to 0 as $\varepsilon \rightarrow 0$, and if u^ε denotes the solution to (1.1) with noise $\sigma(\varepsilon)^{-1} \dot{L}^\varepsilon$ (where $\sigma^2(\varepsilon)$ is the variance of \dot{L}^ε), do we have convergence in distribution of u^ε to the solution u of (1.2) with the Gaussian noise \dot{W} ? A positive answer for this *normal approximation* is given in Theorem 7.10 in [27]: If the atoms of \dot{L}^ε are locally summable and the jump measure Q^ε of \dot{L}^ε satisfies

$$\frac{1}{\sigma^{2+\delta}(\varepsilon)} \int_{\mathbb{R}} |z|^{2+\delta} Q^\varepsilon(dz) \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (1.3)$$

for some $\delta > 0$, then u^ε converges in distribution to u .

The purpose of this work is to substantially generalize this result in two aspects. Given that (1.3) is sufficient but not necessary for $u^\varepsilon \xrightarrow{d} u$ (see Remark 2.3 below), our first contribution is to show that the *necessary and sufficient* condition for the normal approximation is

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sigma^2(\varepsilon)} \int_{|z| > \kappa \sigma(\varepsilon)} z^2 Q^\varepsilon(dz) = 0 \quad (1.4)$$

for all $\kappa > 0$. In fact, if L^ε (resp., W) is a Lévy process (resp., Brownian motion), the same condition was found to be necessary and sufficient for $\sigma(\varepsilon)^{-1} L^\varepsilon \xrightarrow{d} W$ in [8]. Somewhat surprisingly, in the special case of *small jump approximation*, that is, where $Q^\varepsilon(dz) = \mathbb{1}_{\{|z| \leq \varepsilon\}} Q(dz)$ and Q is a given Lévy measure, it was shown in [2] that condition (1.4) *fails* for prominent examples such as the compound Poisson or the gamma distribution. So in these cases, the small jump approximation is not true for Lévy processes and by our results, not true for stochastic PDEs, either.

Our second contribution is to consider equations with *multiplicative* noise. To our best knowledge, previous works on the normal approximation of stochastic PDEs with jumps have only considered the situation of *additive* noise; see, besides the mentioned results in [27], also [16, 26] (there is, of course, literature concerning approximation of multiplicative Gaussian white noise by *smoother* noises [4, 13], but these problems are very different in nature than the one considered here). The proofs in [16, 26, 27] (as well as those of [2, 8]) are based on characteristic functions and the Lévy–Khinchine formula for infinitely divisible distributions, which obviously do not generalize to the situation of multiplicative noise.

Instead, our approach will be to show that u^ε satisfies *martingale problems* which, assuming (1.4) only and not the stronger condition (1.3), have a limit with a unique solution. But this leads to several complications. In order to prove convergence of the associated martingales, we need some sort of uniformity in the time variable (as given, for example, by convergence in the Skorokhod topology). So taking simply the space $L^2([0, T] \times [0, \pi])$ to support the solutions u^ε and u will not be sufficient. This is why we will draw upon the results of [7] and view the solution u^ε (and also u) as a càdlàg process on $[0, T]$ with values in the Sobolev space H_{-r} for some $r > \frac{1}{2}$ (see Section 2.1 for a definition). In order to show tightness in that space with the Aldous criterion [1], we will use the factorization method from [9, 22] to obtain uniform bounds in time without taking moments of order higher than two. Another subtlety that arises in the analysis of the (semi-)martingales mentioned above is that their predictable characteristics are not given by a function of the former, which distinguishes our proof from the corresponding ones for (finite- or infinite-dimensional) stochastic differential equations in [11, 19, 20].

We shall also mention that the initial motivation in [2, 8] to study the normal approximation of Lévy processes comes from numerical simulation. Indeed, for stochastic PDEs as in (1.1) with multiplicative Lévy noise, the rate of convergence of a numerical scheme obtained by removing the small jumps of the noise is slower for noises with a high intensity of small jumps; see [5]. However, the results in [17] and [25] show that in the case of SDEs, an additional Gaussian approximation of the otherwise neglected small jumps improves the rate of convergence. We leave it to future research to examine to what extent this also holds for stochastic PDEs.

The remaining paper is organized as follows. In Section 2, we first describe in detail the considered equations and recall the definition of Sobolev spaces of real order in Section 2.1 before we state our main result, Theorem 2.1, in Section 2.2. Here we also explain the main steps and ideas behind the proof, whereas the details are given in Section 3.

2 Results

2.1 Preliminaries

Let $T > 0$ and consider on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ that satisfies the usual conditions, for any $\varepsilon > 0$, the *stochastic heat equation* on $[0, T] \times [0, \pi]$ with Dirichlet boundary conditions:

$$\begin{cases} \partial_t u^\varepsilon(t, x) = \partial_{xx} u^\varepsilon(t, x) + f(u^\varepsilon(t, x)) \frac{\dot{L}^\varepsilon(t, x)}{\sigma(\varepsilon)}, & (t, x) \in [0, T] \times [0, \pi], \\ u^\varepsilon(t, 0) = u^\varepsilon(t, \pi) = 0, & \text{for all } t \in [0, T], \\ u^\varepsilon(0, x) = 0, & \text{for all } x \in [0, \pi]. \end{cases} \quad (2.1)$$

The function $f: \mathbb{R} \rightarrow \mathbb{R}$ in equation (2.1) describes the *multiplicative* part of the noise and will be assumed to be a Lipschitz continuous function. Concerning the driving noise $\sigma(\varepsilon)^{-1} \dot{L}^\varepsilon$, we

assume that \dot{L}^ε is a pure-jump Lévy space–time white noise on $[0, T] \times [0, \pi]$ given by

$$L^\varepsilon(A) = \int_0^T \int_0^\pi \int_{\mathbb{R}} \mathbb{1}_A(t, x) z (\mu^\varepsilon - \nu^\varepsilon)(dt, dx, dz) \quad (2.2)$$

for all $A \in \mathcal{B}([0, T] \times [0, \pi])$. In this representation, μ^ε is a homogeneous Poisson random measure on $[0, T] \times [0, \pi] \times \mathbb{R}$ relative to the filtration \mathbf{F} , with intensity measure $\nu^\varepsilon = \text{Leb}_{[0, T] \times [0, \pi]} \otimes Q^\varepsilon$. Here Q^ε is a Lévy measure on \mathbb{R} , that is, $Q^\varepsilon(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge z^2) Q^\varepsilon(dz) < \infty$. We refer to Chapter II in [14] for the definition of stochastic integrals with respect to Poisson random measures. Furthermore, we assume that for all $\varepsilon > 0$,

$$0 < \sigma^2(\varepsilon) = \int_{\mathbb{R}} z^2 Q^\varepsilon(dz) < \infty. \quad (2.3)$$

Note that this integral is the variance of $L^\varepsilon([0, 1] \times [0, 1])$. In the special case where we have a single Poisson random measure μ having intensity measure $\nu = \text{Leb}_{[0, T] \times [0, \pi]} \otimes Q$, setting

$$Q^\varepsilon(A) = \int_{|z| \leq \varepsilon} \mathbb{1}_A(z) Q(dz), \quad A \in \mathcal{B}(\mathbb{R}), \quad \varepsilon > 0, \quad (2.4)$$

leads us to the case of *small jump approximation* considered in [2].

A predictable random field $u^\varepsilon = \{u^\varepsilon(t, x) \mid (t, x) \in [0, T] \times [0, \pi]\}$ is called a *mild solution* to (2.1) if for all $(t, x) \in [0, T] \times [0, \pi]$,

$$\begin{aligned} u^\varepsilon(t, x) &= \int_0^t \int_0^\pi G_{t-s}(x, y) \frac{f(u^\varepsilon(s, y))}{\sigma(\varepsilon)} L^\varepsilon(ds, dy) \\ &= \int_0^t \int_0^\pi \int_{\mathbb{R}} G_{t-s}(x, y) f(u^\varepsilon(s, y)) \frac{z}{\sigma(\varepsilon)} (\mu^\varepsilon - \nu^\varepsilon)(ds, dy, dz) \end{aligned} \quad (2.5)$$

\mathbb{P} -almost surely, where

$$G_t(x, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} \sin(kx) \sin(ky) e^{-k^2 t} \mathbb{1}_{\{t > 0\}}, \quad (2.6)$$

for $(t, x, y) \in [0, T] \times [0, \pi]^2$, is the Dirichlet Green's function of the heat operator on $[0, \pi]$.

The existence of a mild solution u^ε to (2.1) is guaranteed by Proposition 2.1 in [7] and it is, up to modifications, unique among all predictable random fields satisfying

$$\sup_{(t, x) \in [0, T] \times [0, \pi]} \mathbb{E}[|u^\varepsilon(t, x)|^p] < \infty \quad (2.7)$$

for any $0 < p < 3$.

In this paper, we want to examine when the *normal approximation* holds for u^ε , that is, when u^ε can be approximated in law by the mild solution u to the same stochastic heat equation as above, but driven by a Gaussian space–time white noise on $[0, T] \times [0, \pi]$:

$$\begin{cases} \partial_t u(t, x) = \partial_{xx} u(t, x) + f(u(t, x)) \dot{W}(t, x), & (t, x) \in [0, T] \times [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, & \text{for all } t \in [0, T], \\ u(0, x) = 0, & \text{for all } x \in [0, \pi]. \end{cases} \quad (2.8)$$

The driving noise \dot{W} is now a centered Gaussian random field $\{W(A) \mid A \in \mathcal{B}([0, T] \times [0, \pi])\}$ with covariance structure $\mathbb{E}[W(A)W(B)] = \text{Leb}_{[0, T] \times [0, \pi]}(A \cap B)$ for any measurable sets $A, B \subseteq$

$[0, T] \times [0, \pi]$. It is well-known (see, for example, Theorem 3.2 in [27]) that, up to modifications, equation (2.8) has a unique mild solution u satisfying the corresponding bound in (2.7) for all $p > 0$.

Throughout this work, we will look at the mild solutions u^ε and u from two different points of view. First, they are random elements in the function space $L^2([0, T] \times [0, \pi])$ as the uniform bound (2.7) shows. But then, as mentioned in the introduction, we will need stronger path regularity in the time variable for our proofs. This is why we shall consider u^ε and u also as stochastic processes with values in an infinite dimensional space, which we will describe in the following.

Consider for any $r > 0$, the fractional Sobolev space

$$H_r([0, \pi]) = \left\{ \phi \in L^2([0, \pi]) \mid \sum_{k=1}^{\infty} (1 + k^2)^r \langle \phi, \phi_k \rangle^2 < \infty \right\},$$

where $\phi_k(x) = \sqrt{2/\pi} \sin(kx)$, $k \in \mathbb{N}$, form an orthonormal basis of $L^2([0, \pi])$. This is a Hilbert space with scalar product

$$\langle f, g \rangle_r = \sum_{k=1}^{\infty} (1 + k^2)^r \langle f, \phi_k \rangle \langle g, \phi_k \rangle, \quad f, g \in H_r([0, \pi]),$$

and norm $\|\phi\|_r = \sqrt{\langle \phi, \phi \rangle_r}$ for $\phi \in H_r([0, \pi])$.

The topological dual $H_{-r}([0, \pi])$ of $H_r([0, \pi])$ is also a Hilbert space, whose dual norm $\|\cdot\|_{-r}$ can be expressed, by the Riesz representation theorem, as

$$\|\phi'\|_{-r}^2 = \sum_{k=1}^{\infty} \langle \phi', \phi_{r,k} \rangle^2 = \sum_{k=1}^{\infty} (1 + k^2)^{-r} \langle \phi', \phi_k \rangle^2, \quad \phi' \in H_{-r}([0, \pi]). \quad (2.9)$$

(Note that if φ_1, φ_2 are elements of the same L^2 -space, then $\langle \varphi_1, \varphi_2 \rangle$ will always denote the standard scalar product of that space. If ϕ is an element of a Hilbert space and ϕ' an element of its topological dual, then $\langle \phi', \phi \rangle$ will always denote the dual pairing of ϕ' with ϕ .)

Coming back to the mild solution to (2.1), if we identify u^ε with the process $(u_t^\varepsilon)_{t \leq T}$ where

$$u_t^\varepsilon : H_r([0, \pi]) \longrightarrow \mathbb{R}, \quad \phi \mapsto \langle u^\varepsilon(t, \cdot), \phi \rangle = \int_0^\pi u^\varepsilon(t, y) \phi(y) dy \quad (2.10)$$

for all $t \leq T$, then by Theorem 2.5 in [7], u^ε has a càdlàg modification in $H_{-r}([0, \pi])$ for any $r > 1/2$, which will be denoted by $\bar{u}^\varepsilon = (\bar{u}_t^\varepsilon)_{t \leq T}$ throughout this work. Similarly, by the identification (2.10) and Corollary 3.4 in [27], the mild solution u to (2.8) has a continuous modification \bar{u} in $H_{-r}([0, \pi])$ for each $r > 1/2$.

2.2 Main result

We now introduce the Cartesian product

$$\Omega^* = L^2([0, T] \times [0, \pi]) \times D([0, T], H_{-r}([0, \pi])), \quad (2.11)$$

with $r > 1/2$. Let d_1 denote the metric induced by the L^2 -norm on $L^2([0, T] \times [0, \pi])$ and d_2 be the Skorokhod metric on $D([0, T], H_{-r}([0, \pi]))$. We then equip Ω^* with the product metric

$$\tau((f_1, x_1), (f_2, x_2)) = d_1(f_1, f_2) + d_2(x_1, x_2) \quad (2.12)$$

for any $f_1, f_2 \in L^2([0, T] \times [0, \pi])$ and $x_1, x_2 \in D([0, T], H_{-r}([0, \pi]))$. The main result of this paper is the following limit theorem.

Theorem 2.1. *Assume that L^ε is given by (2.2) with a variance $\sigma^2(\varepsilon)$ that satisfies (2.3) for all $\varepsilon > 0$. Let u^ε be the $L^2([0, T] \times [0, \pi])$ -valued mild solution to the stochastic heat equation (2.1) driven by $\dot{L}^\varepsilon/\sigma(\varepsilon)$ and \bar{u}^ε be its càdlàg modification in $H_{-r}([0, \pi])$. Similarly, let u be the $L^2([0, T] \times [0, \pi])$ -valued mild solution to the stochastic heat equation (2.8) driven by \dot{W} and \bar{u} be its continuous modification in $H_{-r}([0, \pi])$.*

Furthermore, suppose that the Lipschitz function f satisfies $f(0) \neq 0$. Then, as $\varepsilon \rightarrow 0$,

$$(u^\varepsilon, \bar{u}^\varepsilon) \xrightarrow{d} (u, \bar{u}) \quad \text{in } (\Omega^*, \tau) \quad (2.13)$$

for all $r > 1/2$ if and only if (1.4) holds for all $\kappa > 0$.

Remark 2.2. We can generalize Theorem 2.1 to nonzero initial conditions. Assume that in both equations (2.1) and (2.8) we now have $u^\varepsilon(0, x) = u(0, x) = u_0(x)$ for all $x \in [0, \pi]$, where $u_0: [0, \pi] \rightarrow \mathbb{R}$ is a bounded continuous function with $u_0(0) = u_0(\pi) = 0$. Define

$$u_0(t, x) = \int_0^\pi G_t(x, y) u_0(y) dy, \quad (t, x) \in [0, T] \times [0, \pi].$$

Then Theorem 2.1 can be shown in a completely analogous manner if we assume that there exists $(t_0, x_0) \in [0, T] \times [0, \pi]$ such that $f(u_0(t_0, x_0)) \neq 0$ (instead of $f(0) \neq 0$), and this assumption is only needed for showing the necessity of (1.4). To be more precise, since the mild solution to (2.8) now satisfies

$$u(t, x) = u_0(t, x) + \int_0^t \int_0^\pi G_{t-s}(x, y) f(u(s, y)) W(ds, dy)$$

\mathbb{P} -almost surely, a similar argument as in Remark 3.14 shows that $\mathbb{P}(f(u(t_1, x_1)) \neq 0) > 0$ for some $(t_1, x_1) \in [0, T] \times [0, \pi]$ and hence, the expectation in (3.67) is nonzero.

Remark 2.3. Let us relate the two conditions (1.3) and (1.4) to each other. Using Hölder's and Chebyshev's inequalities, we see from the estimate

$$\begin{aligned} \frac{1}{\sigma^2(\varepsilon)} \int_{|z| > \kappa\sigma(\varepsilon)} z^2 Q^\varepsilon(dz) &\leq \frac{1}{\sigma^2(\varepsilon)} \left(\int_{\mathbb{R}} |z|^{2+\delta} Q^\varepsilon(dz) \right)^{\frac{2}{2+\delta}} Q^\varepsilon(\{|z| > \kappa\sigma(\varepsilon)\})^{\frac{\delta}{2+\delta}} \\ &\leq \frac{1}{\kappa^\delta \sigma^{2+\delta}(\varepsilon)} \int_{\mathbb{R}} |z|^{2+\delta} Q^\varepsilon(dz) \end{aligned}$$

that (1.3) implies (1.4).

The other implication is not true in general. For example, assume that Q^ε has density

$$q_\varepsilon(z) = \frac{1}{2|z|^2} \mathbb{1}_{\{|z| \leq \varepsilon\}} + \frac{\varepsilon^2}{2C|z|^3 \log(1+|z|)^2} \mathbb{1}_{\{|z| > 1\}}, \quad z \in \mathbb{R},$$

where $C = \int_1^\infty z^{-1} \log(1+z)^{-2} dz$. Then $\int_{\mathbb{R}} |z|^{2+\delta} Q^\varepsilon(dz) = \infty$ for every $\varepsilon, \delta > 0$, so condition (1.3) does not hold. But a direct calculation shows that $\sigma^2(\varepsilon) = \varepsilon + \varepsilon^2$. So given $\kappa > 0$, we have $1 > \kappa\sigma(\varepsilon) > \kappa\sqrt{\varepsilon} > \varepsilon$ for small values of ε , which implies (1.4) because

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sigma^2(\varepsilon)} \int_{|z| > \kappa\sigma(\varepsilon)} z^2 Q^\varepsilon(dz) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon + \varepsilon^2} \int_{|z| > \varepsilon} z^2 Q^\varepsilon(dz) = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^2}{\varepsilon + \varepsilon^2} = 0.$$

Proof of Theorem 2.1. We begin by showing that (1.4) implies the weak convergence (2.13). Since Ω^* is a metric space, we follow the classical scheme of first showing tightness and then uniqueness of the limiting distribution.

In Theorem 3.4, we show that $\{u^\varepsilon \mid \varepsilon > 0\}$ is tight in $L^2([0, T] \times [0, \pi])$ and in Theorem 3.7 that $\{\bar{u}^\varepsilon \mid \varepsilon > 0\}$ is tight in $D([0, T], H_{-r}([0, \pi]))$. By the subsequence principle, this immediately implies that the random elements $\{(u^\varepsilon, \bar{u}^\varepsilon) \mid \varepsilon > 0\}$ are tight in (Ω^*, τ) . As it turns out, no assumptions on the Lévy noise \dot{L}^ε other than the ones specified in (2.2) and (2.3) are needed for this tightness property.

As a consequence, we can apply Prokhorov's theorem, which provides for any sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ with $\varepsilon_k \rightarrow 0$, a subsequence $(\varepsilon_{k_l})_{l \in \mathbb{N}}$ such that $(u^{\varepsilon_{k_l}}, \bar{u}^{\varepsilon_{k_l}})_{l \in \mathbb{N}}$ converges weakly to some distribution on (Ω^*, τ) as $l \rightarrow \infty$. For notational simplicity, we will assume without loss of generality that the whole sequence $(u^{\varepsilon_k}, \bar{u}^{\varepsilon_k})_{k \in \mathbb{N}}$ converges weakly.

Since (Ω^*, τ) is a complete separable metric space, we can further apply Skorokhod's representation theorem (see Theorem 4.30 in [15]) and obtain random elements

$$(v^k, \bar{v}^k), (v, \bar{v}): (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) \longrightarrow (\Omega^*, \tau), \quad (2.14)$$

defined on a possibly different probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, satisfying the following properties:

$$\begin{aligned} (v^k, \bar{v}^k) &\stackrel{d}{=} (u^{\varepsilon_k}, \bar{u}^{\varepsilon_k}) \quad \text{for all } k \in \mathbb{N} \quad \text{and} \\ (v^k, \bar{v}^k)(\bar{\omega}) &\longrightarrow (v, \bar{v})(\bar{\omega}) \quad \text{in } (\Omega^*, \tau) \quad \text{as } k \rightarrow \infty \quad \text{for all } \bar{\omega} \in \bar{\Omega}. \end{aligned} \quad (2.15)$$

We will show that

$$(v, \bar{v}) \stackrel{d}{=} (u, \bar{u}),$$

which in turn implies (2.13). To do this, we first define a filtration $\bar{\mathbf{F}} = (\bar{\mathcal{F}}_t)_{t \leq T}$ on $\bar{\Omega}$ by setting

$$\bar{\mathcal{F}}_t = \bigcap_{u > t} \sigma(v^k(s, x), \bar{v}_s^k \mid 0 \leq x \leq \pi, k \in \mathbb{N}, s \leq u), \quad t \leq T. \quad (2.16)$$

We further define for $\xi \in \mathbb{R}$, $\phi \in C_c^\infty((0, \pi))$ and $t \leq T$,

$$\bar{B}_t = \int_0^t \langle v(s, \cdot), \phi'' \rangle ds, \quad \bar{C}_t = \int_0^t \int_0^\pi f^2(v(s, x)) \phi^2(x) ds dx, \quad \bar{A}_t = i\xi \bar{B}_t - \frac{1}{2} \xi^2 \bar{C}_t \quad (2.17)$$

as well as

$$\bar{M}_t = e^{i\xi \langle \bar{v}_t, \phi \rangle} - \int_0^t e^{i\xi \langle \bar{v}_s, \phi \rangle} \bar{A}(ds). \quad (2.18)$$

We will then show in Theorem 3.12 that, under assumption (1.4), the pair (v, \bar{v}) satisfies the following martingale problem. For all $\xi \in \mathbb{R}$ and $\phi \in C_c^\infty((0, \pi))$, the process $(\bar{M}_t)_{t \leq T}$ is a martingale with respect to $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{F}}, \bar{\mathbb{P}})$. Note that we are able to obtain this property only because in (2.15), we have $\bar{\omega}$ -wise convergence both in the Skorokhod topology and in $L^2([0, T] \times [0, \pi])$, which is the reason why we view the solutions to (2.1) and (2.8) as *pairs* in Ω^* .

Next, we will show in Theorem 3.13 that this martingale property in turn implies that there exists a Gaussian space-time white noise \tilde{W} on $[0, T] \times [0, \pi]$, possibly defined on a filtered extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{F}}, \tilde{\mathbb{P}})$ of $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{F}}, \bar{\mathbb{P}})$ such that, with probability one, the random field v is equal in $L^2([0, T] \times [0, \pi])$ to the mild solution \tilde{v} to the stochastic heat equation

$$\begin{cases} \partial_t \tilde{v}(t, x) = \partial_{xx} \tilde{v}(t, x) + f(\tilde{v}(t, x)) \tilde{W}(t, x), & (t, x) \in [0, T] \times [0, \pi], \\ \tilde{v}(t, 0) = \tilde{v}(t, \pi) = 0, & \text{for all } t \in [0, T], \\ \tilde{v}(0, x) = 0, & \text{for all } x \in [0, \pi], \end{cases} \quad (2.19)$$

and such that \bar{v} is indistinguishable from the continuous version in $H_{-r}([0, \pi])$ of \tilde{v} , which concludes the first part of the proof.

For the second part, under the assumption $f(0) \neq 0$, Theorem 3.15 directly shows that (2.13) implies (1.4). \square

3 Details of the proof

In the remainder of this work, the letter C will always denote a strictly positive constant whose value may change from line to line. Furthermore, by the Lipschitz continuity of the function f , there exists a positive constant K that we hold fixed from now on such that $|f(x)| \leq K|x| + |f(0)|$ for all $x \in \mathbb{R}$.

3.1 Tightness

We start with three lemmas that will provide uniform bounds in $\varepsilon > 0$ for the second moments of u^ε , which will be crucial for proving tightness of $\{(u^\varepsilon, \bar{u}^\varepsilon) \mid \varepsilon > 0\}$ in (Ω^*, τ) .

Lemma 3.1. *The family $\{u^\varepsilon \mid \varepsilon > 0\}$ of mild solutions to (2.1) satisfies*

$$\sup_{\varepsilon > 0} \sup_{(t,x) \in [0,T] \times [0,\pi]} \mathbb{E} \left[|u^\varepsilon(t,x)|^2 \right] < \infty \quad (3.1)$$

and this uniform bound only depends on the Lipschitz function f .

Proof. Using Itô's isometry and the definition (2.5) of u^ε , we have for fixed $\varepsilon > 0$ and $(t,x) \in [0, T] \times [0, \pi]$,

$$\begin{aligned} \mathbb{E} \left[|u^\varepsilon(t,x)|^2 \right] &= \mathbb{E} \left[\int_0^t \int_0^\pi \int_{\mathbb{R}} G_{t-s}^2(x,y) \frac{f^2(u^\varepsilon(s,y))}{\sigma^2(\varepsilon)} z^2 \nu^\varepsilon(ds, dy, dz) \right] \\ &= \mathbb{E} \left[\int_0^t \int_0^\pi G_{t-s}^2(x,y) f^2(u^\varepsilon(s,y)) ds dy \right] \left(\frac{1}{\sigma^2(\varepsilon)} \int_{\mathbb{R}} z^2 Q^\varepsilon(dz) \right) \\ &= \int_0^t \int_0^\pi G_{t-s}^2(x,y) \mathbb{E} \left[f^2(u^\varepsilon(s,y)) \right] ds dy. \end{aligned}$$

Using the Lipschitz continuity of f and the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we then obtain for $\varepsilon > 0$ and $(t,x) \in [0, T] \times [0, \pi]$,

$$\mathbb{E} \left[|u^\varepsilon(t,x)|^2 \right] \leq C \int_0^t \int_0^\pi G_{t-s}^2(x,y) \mathbb{E} \left[|u^\varepsilon(s,y)|^2 \right] ds dy + C \int_0^t \int_0^\pi G_{t-s}^2(x,y) ds dy. \quad (3.2)$$

Now in order to find a bound for $\mathbb{E} \left[|u^\varepsilon(t,x)|^2 \right]$ uniformly in t, x and ε , we will use a comparison principle for deterministic Volterra equations. By (B.5) in [3], there exists a constant $C > 0$ such that $|G_t(x,y)| \leq Cg_t(x-y)$ on $[0, T] \times [0, \pi]^2$, where

$$g_t(x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right) \mathbb{1}_{\{t>0\}} \quad (3.3)$$

is the heat kernel on \mathbb{R} . Since $\int_0^T \int_{\mathbb{R}} |g_t(x)|^q dt dx < \infty$ for all $q < 3$, we obtain

$$\sup_{(t,x) \in [0,T] \times [0,\pi]} \int_0^t \int_0^\pi G_{t-s}^2(x,y) ds dy \leq C \int_0^T \int_{\mathbb{R}} g_t^2(x) dt dx < \infty.$$

Recall from (2.7) that $\mathbb{E} \left[|u^\varepsilon(t, x)|^2 \right]$ is uniformly bounded in (t, x) for fixed $\varepsilon > 0$. Therefore, by Lemma 6.4 (2) and (3) in [6], the mild solution u^ε satisfies

$$\mathbb{E} \left[|u^\varepsilon(t, x)|^2 \right] \leq v(t, x)$$

for all $(t, x) \in [0, T] \times [0, \pi]$ and $\varepsilon > 0$, where v is the unique nonnegative solution of the deterministic Volterra equation

$$\begin{aligned} v(t, x) &= C \int_0^t \int_0^\pi G_{t-s}^2(x, y) v(s, y) \, ds \, dy \\ &\quad + C \int_0^t \int_0^\pi G_{t-s}^2(x, y) \, ds \, dy, \quad (t, x) \in [0, T] \times [0, \pi], \end{aligned}$$

and satisfies $\sup_{(t,x) \in [0,T] \times [0,\pi]} v(t, x) < \infty$. \square

The next lemma gives an alternative integral representation of u^ε and is an extension of the factorization method in [9] and [22].

Lemma 3.2. *For $\delta \in (0, 1/4)$ define*

$$Y_\delta^\varepsilon(t, x) = \int_0^t \int_0^\pi \frac{G_{t-s}(x, y)}{(t-s)^\delta} \frac{f(u^\varepsilon(s, y))}{\sigma(\varepsilon)} L^\varepsilon(ds, dy), \quad (t, x) \in [0, T] \times [0, \pi].$$

We then have

$$\sup_{\varepsilon > 0} \sup_{(t,x) \in [0,T] \times [0,\pi]} \mathbb{E} \left[|Y_\delta^\varepsilon(t, x)|^2 \right] < \infty \quad (3.4)$$

and for all $(t, x) \in [0, T] \times [0, \pi]$, the representation

$$u^\varepsilon(t, x) = \frac{\sin(\delta\pi)}{\pi} \int_0^t \int_0^\pi \frac{G_{t-s}(x, y)}{(t-s)^{1-\delta}} Y_\delta^\varepsilon(s, y) \, ds \, dy \quad (3.5)$$

holds \mathbb{P} -almost surely.

Proof. First, using Itô's isometry, the Lipschitz continuity of f and Lemma 3.1, we have

$$\begin{aligned} \mathbb{E} \left[|Y_\delta^\varepsilon(t, x)|^2 \right] &= \mathbb{E} \left[\int_0^t \int_0^\pi \frac{G_{t-s}^2(x, y)}{(t-s)^{2\delta}} f^2(u^\varepsilon(s, y)) \, ds \, dy \right] \left(\frac{1}{\sigma^2(\varepsilon)} \int_{\mathbb{R}} z^2 Q^\varepsilon(dz) \right) \\ &= \int_0^t \int_0^\pi \frac{G_{t-s}^2(x, y)}{(t-s)^{2\delta}} \mathbb{E} \left[f^2(u^\varepsilon(s, y)) \right] \, ds \, dy \leq C \int_0^t \int_0^\pi \frac{G_{t-s}^2(x, y)}{(t-s)^{2\delta}} \, ds \, dy. \end{aligned}$$

The last integral on the right-hand side is finite if $\delta < 1/4$. Indeed, by (B.5) in [3], it can be bounded by

$$C \int_0^t \int_0^\pi \frac{1}{(t-s)^{2\delta+1}} \exp\left(-\frac{|x-y|^2}{t-s}\right) \, ds \, dy = C \int_0^t \frac{1}{(t-s)^{2\delta+1/2}} \, ds = Ct^{-2\delta+1/2}.$$

The identity (3.5) follows in the same way as Lemma 5 in [22]. \square

Lemma 3.3. *The family $\{u^\varepsilon \mid \varepsilon > 0\}$ of mild solutions to (2.1) satisfies*

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_0^T \left(\int_0^\pi |u^\varepsilon(t, x)|^2 \, dx \right)^p \, dt \right)^{1/p} \right] < \infty \quad (3.6)$$

for all $p \in (1, 4/3)$.

Proof. We will use the integral representation (3.5) of Lemma 3.2. Fix $\delta \in (0, 1/4)$ and $t \in [0, T]$. Using Fubini's theorem, we have

$$\begin{aligned} & \int_0^\pi |u^\varepsilon(t, x)|^2 dx \\ &= C \int_0^\pi \left(\int_0^t \int_0^\pi \int_0^t \int_0^\pi \frac{G_{t-s}(x, y) G_{t-s'}(x, y')}{(t-s)^{1-\delta} (t-s')^{1-\delta}} Y_\delta^\varepsilon(s, y) Y_\delta^\varepsilon(s', y') dy' ds' dy ds \right) dx. \end{aligned}$$

By the semigroup property of the Green's function, the integral on the right-hand side is equal to

$$\begin{aligned} & \int_0^t \int_0^\pi \int_0^t \int_0^\pi \frac{G_{2t-s-s'}(y, y')}{(t-s)^{1-\delta} (t-s')^{1-\delta}} Y_\delta^\varepsilon(s, y) Y_\delta^\varepsilon(s', y') dy' ds' dy ds \\ &= 2 \int_0^t \int_0^\pi \int_0^s \int_0^\pi \frac{G_{2t-s-s'}(y, y')}{(t-s)^{1-\delta} (t-s')^{1-\delta}} Y_\delta^\varepsilon(s, y) Y_\delta^\varepsilon(s', y') dy' ds' dy ds. \end{aligned}$$

Using (B.5) in [3] and (3.3), we obtain

$$\int_0^\pi |u^\varepsilon(t, x)|^2 dx \leq C \int_0^t \int_0^\pi \int_0^s \int_0^\pi \frac{g_{2t-s-s'}(y-y')}{(t-s)^{1-\delta} (t-s')^{1-\delta}} |Y_\delta^\varepsilon(s, y) Y_\delta^\varepsilon(s', y')| dy' ds' dy ds.$$

Now let $p \in (1, 4/3)$, take the $\|\cdot\|_{L^p([0, T])}$ -norm of $t \mapsto \int_0^\pi |u^\varepsilon(t, x)|^2 dx$ and apply Minkowski's integral inequality to obtain

$$\begin{aligned} & \left(\int_0^T \left(\int_0^\pi |u^\varepsilon(t, x)|^2 dx \right)^p dt \right)^{1/p} \\ & \leq C \int_0^T \int_0^\pi \int_0^s \int_0^\pi \left(\int_0^T \left(\frac{g_{2t-s-s'}(y-y') \mathbb{1}_{\{s \leq t\}}}{(t-s)^{1-\delta} (t-s')^{1-\delta}} \right)^p dt \right)^{1/p} |Y_\delta^\varepsilon(s, y) Y_\delta^\varepsilon(s', y')| dy' ds' dy ds. \end{aligned}$$

Take expectation, use the Cauchy–Schwarz inequality and (3.4) to further obtain

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \left(\int_0^\pi |u^\varepsilon(t, x)|^2 dx \right)^p dt \right)^{1/p} \right] \\ & \leq C \int_0^T \int_0^\pi \int_0^s \int_0^\pi \left(\int_s^T \left(\frac{g_{2t-s-s'}(y-y')}{(t-s)^{1-\delta} (t-s')^{1-\delta}} \right)^p dt \right)^{1/p} dy' ds' dy ds. \end{aligned} \tag{3.7}$$

Note that the right-hand side of (3.7) does not depend on ε anymore. We now consider the integrand

$$\int_s^T \left(\frac{g_{2t-s-s'}(y-y')}{(t-s)^{1-\delta} (t-s')^{1-\delta}} \right)^p dt = \int_0^{T-s} \frac{g_{2t+s-s'}^p(y-y')}{t^{(1-\delta)p} (t+s-s')^{(1-\delta)p}} dt.$$

For fixed $x \in \mathbb{R}$, the maximum of the function $t \mapsto g_t(x)$ is $C/|x|$ for some C that is independent of x . Let $\eta \in (0, 1)$ and consider the estimate

$$g_t(x) = g_t(x)^{1-\eta} g_t(x)^\eta \leq C \frac{1}{|x|^{1-\eta}} \frac{1}{t^{\eta/2}}, \quad t > 0, \quad x \in \mathbb{R}.$$

Since $s' \leq s$, we obtain

$$\begin{aligned} & \left(\int_0^{T-s} \frac{g_{2t+s-s'}^p(y-y')}{t^{(1-\delta)p}(t+s-s')^{(1-\delta)p}} dt \right)^{1/p} \\ & \leq C \frac{1}{|y-y'|^{1-\eta}} \left(\int_0^T \frac{1}{t^{(1-\delta)p}(t+s-s')^{(1-\delta)p}(2t+s-s')^{p\eta/2}} dt \right)^{1/p} \\ & \leq C \frac{1}{|y-y'|^{1-\eta}} \left(\int_0^T \frac{1}{t^{((1-\delta)+\eta/2)p}(t+s-s')^{(1-\delta)p}} dt \right)^{1/p}. \end{aligned}$$

Moreover, the integral $\int_0^\pi \int_0^\pi |y-y'|^{\eta-1} dy' dy$ is finite because $\eta > 0$, and the expectation in (3.7) is now bounded by

$$C \int_0^T \int_0^s \left(\int_0^T \frac{1}{t^{((1-\delta)+\eta/2)p}(t+s-s')^{(1-\delta)p}} dt \right)^{1/p} ds' ds \quad (3.8)$$

for any $\delta \in (0, 1/4)$ and $\eta \in (0, 1)$. By assumption, $3/4 < 1/p < 1$ and $3/4 < (1-\delta) + \eta/2 < 3/2$. Hence, we can choose δ and η such that $(1-\delta) + \eta/2 < 1/p$. As a consequence, by Lemma 2 of Chapter 1 in [12], the estimate

$$\int_0^T \frac{1}{t^{((1-\delta)+\eta/2)p}(t+s-s')^{(1-\delta)p}} dt \leq C(s-s')^{1-((1-\delta)+\eta/2)p-(1-\delta)p}$$

holds for $0 \leq s' < s \leq T$. We have assumed that $((1-\delta) + \eta/2)p + (1-\delta)p > 1$ since otherwise, the last integral is bounded by some constant, which immediately implies that the expectation in (3.7) is uniformly bounded in ε .

Therefore, we further estimate the integral in (3.8) by

$$C \int_0^T \int_0^s (s-s')^{1/p-2(1-\delta)-\eta/2} ds' ds = C \int_0^T \int_0^s r^{1/p-2(1-\delta)-\eta/2} dr ds,$$

which is finite because our choice of δ and η implies $2(1-\delta) + \eta/2 - 1/p < 1$. This concludes the proof. \square

We can now proceed to showing tightness.

Theorem 3.4. *The family $\{u^\varepsilon \mid \varepsilon > 0\}$ of mild solutions to (2.1) is tight in the Hilbert space $L^2([0, T] \times [0, \pi])$.*

Proof. It is easy to see that the functions

$$\psi_{ij}(t, x) = \bar{\phi}_i(t) \phi_j(x), \quad (t, x) \in [0, T] \times [0, \pi],$$

where $\bar{\phi}_i(t) = \sqrt{2/T} \sin(it\pi/T)$ and $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$ for all $i, j \in \mathbb{N}$, form an orthonormal basis of $L^2([0, T] \times [0, \pi])$.

First, using the stochastic Fubini theorem (see, for example, Theorem 2.6 in [27]), we have for all $i, j \in \mathbb{N}$,

$$\begin{aligned} \langle u^\varepsilon, \psi_{ij} \rangle &= \int_0^T \int_0^\pi \left(\int_0^t \int_0^\pi G_{t-s}(x, y) \frac{f(u^\varepsilon(s, y))}{\sigma(\varepsilon)} L^\varepsilon(ds, dy) \right) \psi_{ij}(t, x) dt dx \\ &= \frac{1}{\sigma(\varepsilon)} \int_0^T \int_0^\pi f(u^\varepsilon(s, y)) \left(\int_s^T \int_0^\pi G_{t-s}(x, y) \psi_{ij}(t, x) dt dx \right) L^\varepsilon(ds, dy). \end{aligned} \quad (3.9)$$

Define for all $i, j \in \mathbb{N}$,

$$H_{ij}(s, y) = \int_s^T \int_0^\pi G_{t-s}(x, y) \psi_{ij}(t, x) dt dx, \quad (s, y) \in [0, T] \times [0, \pi].$$

Using Fubini's theorem, the expression (2.6) of the Green's function G and the orthogonal properties of ϕ_j , we obtain for all $(s, y) \in [0, T] \times [0, \pi]$,

$$H_{ij}(s, y) = \int_s^T \bar{\phi}_i(t) \left(\int_0^\pi G_{t-s}(x, y) \phi_j(x) dx \right) dt = \int_s^T \bar{\phi}_i(t) \phi_j(y) e^{-j^2(t-s)} dt. \quad (3.10)$$

Using the integral formula $\int (\sin ax) e^{bx} dx = (b \sin ax - a \cos ax) e^{bx} / (a^2 + b^2) + C$, we can further calculate

$$\begin{aligned} H_{ij}(s, y) &= \sqrt{\frac{2}{T}} \phi_j(y) e^{j^2 s} \int_s^T \sin\left(i \frac{\pi}{T} t\right) e^{-j^2 t} dt \\ &= \sqrt{\frac{2}{T}} \phi_j(y) \frac{1}{i^2 (\pi/T)^2 + j^4} \left(e^{-j^2(T-s)} i \frac{\pi}{T} (-1)^{i+1} + j^2 \sin\left(i \frac{\pi}{T} s\right) + i \frac{\pi}{T} \cos\left(i \frac{\pi}{T} s\right) \right) \\ &\leq C \left(\frac{i}{i^2 + j^4} + \frac{j^2}{i^2 + j^4} \right) \leq C \frac{1}{i + j^2} \quad \text{for all } i, j \in \mathbb{N}. \end{aligned}$$

For the L^2 -norm of H_{ij} , we then have

$$\int_0^T \int_0^\pi H_{ij}^2(s, y) ds dy \leq C \frac{1}{i^2 + j^4}$$

for all $i, j \in \mathbb{N}$. Since

$$\sum_{i,j=2}^\infty \frac{1}{i^2 + j^4} \leq \int_1^\infty \int_1^\infty \frac{1}{x^2 + y^4} dx dy = \int_1^\infty \frac{\arctan y^2}{y^2} dy < \infty,$$

we obtain from Lemma 3.1

$$\sum_{i,j=1}^\infty \sup_{\varepsilon > 0} \mathbb{E} \left[\langle u^\varepsilon, \psi_{ij} \rangle^2 \right] \leq C \sum_{i,j=1}^\infty \int_0^T \int_0^\pi H_{ij}^2(s, y) ds dy < \infty,$$

which implies that

$$\sup_{\varepsilon > 0} \mathbb{P} \left(\sum_{i,j \geq N} \langle u^\varepsilon, \psi_{ij} \rangle^2 > \delta \right) \leq \frac{1}{\delta^2} \sum_{i,j \geq N} \sup_{\varepsilon > 0} \mathbb{E} \left[\langle u^\varepsilon, \psi_{ij} \rangle^2 \right] \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

for all $\delta > 0$. Moreover, again by Lemma 3.1, we have

$$\begin{aligned} \sup_{\varepsilon > 0} \mathbb{P} \left(\sum_{i,j < N} \langle u^\varepsilon, \psi_{ij} \rangle^2 > \delta \right) &\leq \frac{1}{\delta^2} \sup_{\varepsilon > 0} \mathbb{E} \left[\sum_{i,j < N} \langle u^\varepsilon, \psi_{ij} \rangle^2 \right] \leq \frac{1}{\delta^2} \sup_{\varepsilon > 0} \mathbb{E} \left[\sum_{i,j=1}^\infty \langle u^\varepsilon, \psi_{ij} \rangle^2 \right] \\ &= \frac{1}{\delta^2} \sup_{\varepsilon > 0} \mathbb{E} \left[\int_0^T \int_0^\pi u^\varepsilon(t, x)^2 dt dx \right] \leq \frac{C}{\delta^2} \rightarrow 0 \end{aligned}$$

as $\delta \rightarrow \infty$ for all $N \in \mathbb{N}$. Therefore, we can conclude from Theorem 1 in [24] that $\{u^\varepsilon \mid \varepsilon > 0\}$ is tight in $L^2([0, T] \times [0, \pi])$. \square

The next two propositions will imply that $\{\bar{u}^\varepsilon \mid \varepsilon > 0\}$ is tight in $D([0, T], H_{-r}([0, \pi]))$.

Proposition 3.5. *The càdlàg processes $\{\bar{u}^\varepsilon \mid \varepsilon > 0\}$ satisfy the Aldous condition: Let $(\varepsilon_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ be sequences of positive numbers with $\varepsilon_n \rightarrow 0$ and $h_n \rightarrow 0$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, further let $\tau_n \in [0, T]$ be a stopping time with respect to the filtration generated by the stochastic process $(\bar{u}_t^{\varepsilon_n})_{t \leq T}$. Then we have for any $r > 1/2$,*

$$\mathbb{E} \left[\|\bar{u}_{\tau_n+h_n}^{\varepsilon_n} - \bar{u}_{\tau_n}^{\varepsilon_n}\|_{-r}^2 \right] \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Recall the expression of the dual norm $\|\cdot\|_{-r}$ in (2.9). We have

$$\|\bar{u}_{\tau_n+h_n}^{\varepsilon_n} - \bar{u}_{\tau_n}^{\varepsilon_n}\|_{-r}^2 = \sum_{k=1}^{\infty} (1+k^2)^{-r} \left(\langle \bar{u}_{\tau_n+h_n}^{\varepsilon_n}, \phi_k \rangle - \langle \bar{u}_{\tau_n}^{\varepsilon_n}, \phi_k \rangle \right)^2. \quad (3.11)$$

We will find a convenient semimartingale decomposition for the real-valued stochastic process $\langle \bar{u}^\varepsilon, \phi_k \rangle = (\langle \bar{u}_t^\varepsilon, \phi_k \rangle)_{t \leq T}$ for any $\varepsilon > 0$ and $k \in \mathbb{N}$ that will then allow us to estimate the expectation of the terms appearing in (3.11).

First, proceeding as in (3.9) and (3.10), we have for all $t \leq T$,

$$\begin{aligned} \langle u^\varepsilon(t, \cdot), \phi_k \rangle &= \int_0^\pi \left(\int_0^t \int_0^\pi G_{t-s}(x, y) \frac{f(u^\varepsilon(s, y))}{\sigma(\varepsilon)} L^\varepsilon(ds, dy) \right) \phi_k(x) dx \\ &= \frac{1}{\sigma(\varepsilon)} \int_0^t \int_0^\pi f(u^\varepsilon(s, y)) \phi_k(y) e^{-k^2(t-s)} L^\varepsilon(ds, dy). \end{aligned}$$

So if we define

$$X_t^{k,\varepsilon} = \int_0^t \int_0^\pi f(u^\varepsilon(s, y)) \phi_k(y) L^\varepsilon(ds, dy), \quad t \leq T,$$

for all $k \in \mathbb{N}$ and $\varepsilon > 0$, then

$$\frac{1}{\sigma(\varepsilon)} \int_0^t \int_0^\pi f(u^\varepsilon(s, y)) \phi_k(y) e^{-k^2(t-s)} L^\varepsilon(ds, dy) = \frac{1}{\sigma(\varepsilon)} \int_0^t e^{-k^2(t-s)} X_s^{k,\varepsilon} ds$$

for all $t \leq T$, where the last term is the Itô integral of the deterministic function $s \mapsto e^{-k^2(t-s)}$ against the square-integrable martingale $X^{k,\varepsilon}$. Because the integrand is a C^∞ -function, the integration by parts formula for semimartingales yields

$$\int_0^t e^{-k^2(t-s)} X_s^{k,\varepsilon} ds = X_t^{k,\varepsilon} - \int_0^t X_s^{k,\varepsilon} k^2 e^{-k^2(t-s)} ds.$$

Altogether we obtain the semimartingale decomposition

$$\begin{aligned} \langle u^\varepsilon(t, \cdot), \phi_k \rangle &= \int_0^t \int_0^\pi \frac{f(u^\varepsilon(s, y))}{\sigma(\varepsilon)} \phi_k(y) L^\varepsilon(ds, dy) \\ &\quad - \int_0^t \left(\int_0^s \int_0^\pi \frac{f(u^\varepsilon(r, y))}{\sigma(\varepsilon)} \phi_k(y) L^\varepsilon(dr, dy) \right) k^2 e^{-k^2(t-s)} ds \end{aligned} \quad (3.12)$$

for all $t \leq T$, $k \in \mathbb{N}$ and $\varepsilon > 0$.

Now the process $\langle \bar{u}^\varepsilon, \phi_k \rangle$ is the càdlàg version of $(\langle u^\varepsilon(t, \cdot), \phi_k \rangle)_{t \leq T}$, so we can infer that $\langle \bar{u}^\varepsilon, \phi_k \rangle$ and the right-hand side of (3.12) are indistinguishable since the latter is also càdlàg. Coming back to (3.11), we can now decompose

$$\langle \bar{u}_{\tau_n+h_n}^{\varepsilon_n}, \phi_k \rangle - \langle \bar{u}_{\tau_n}^{\varepsilon_n}, \phi_k \rangle = I_{k,n} + J_{k,n}^1 + J_{k,n}^2,$$

where

$$\begin{aligned} I_{k,n} &= \int_0^T \int_0^\pi \frac{f(u^{\varepsilon_n}(s, y))}{\sigma(\varepsilon_n)} \phi_k(y) \mathbb{1}_{(\tau_n, \tau_n+h_n]}(s) L^{\varepsilon_n}(ds, dy), \\ J_{k,n}^1 &= \int_0^{\tau_n} \left(\int_0^s \int_0^\pi \frac{f(u^{\varepsilon_n}(r, y))}{\sigma(\varepsilon_n)} \phi_k(y) L^{\varepsilon_n}(dr, dy) \right) k^2 \left(e^{-k^2(\tau_n-s)} - e^{-k^2(\tau_n+h_n-s)} \right) ds, \\ J_{k,n}^2 &= - \int_{\tau_n}^{\tau_n+h_n} \left(\int_0^s \int_0^\pi \frac{f(u^{\varepsilon_n}(r, y))}{\sigma(\varepsilon_n)} \phi_k(y) L^{\varepsilon_n}(dr, dy) \right) k^2 e^{-k^2(\tau_n+h_n-s)} ds \end{aligned}$$

for all $k, n \in \mathbb{N}$. We now gather some moment estimates for these three terms. First, for the martingale term $I_{k,n}$, we have by Itô's isometry

$$\begin{aligned} \mathbb{E}[I_{k,n}^2] &= \mathbb{E} \left[\int_0^T \int_0^\pi \int_{\mathbb{R}} f^2(u^{\varepsilon_n}(s, y)) \phi_k^2(y) \mathbb{1}_{(\tau_n, \tau_n+h_n]}(s) \frac{z^2}{\sigma^2(\varepsilon_n)} ds dy Q^{\varepsilon_n}(dz) \right] \\ &= \mathbb{E} \left[\int_0^T \int_0^\pi f^2(u^{\varepsilon_n}(s, y)) \phi_k^2(y) \mathbb{1}_{(\tau_n, \tau_n+h_n]}(s) ds dy \right]. \end{aligned} \quad (3.13)$$

Using the Lipschitz continuity of f , we can bound the last term in (3.13) by

$$C \mathbb{E} \left[\int_0^T \int_0^\pi u^{\varepsilon_n}(s, y)^2 \mathbb{1}_{(\tau_n, \tau_n+h_n]}(s) ds dy \right] + C \mathbb{E} \left[\int_0^T \int_0^\pi \mathbb{1}_{(\tau_n, \tau_n+h_n]}(s) ds dy \right]$$

for any $k \in \mathbb{N}$. The second term equals $C\pi h_n$, which converges to 0 as $n \rightarrow \infty$.

For the first term, choose $p \in (1, 4/3)$. Using Hölder's inequality and Lemma 3.3, we obtain

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \left(\int_0^\pi u^{\varepsilon_n}(s, y)^2 dy \right) \mathbb{1}_{(\tau_n, \tau_n+h_n]}(s) ds \right] \\ &\leq \mathbb{E} \left[\left(\int_0^T \left(\int_0^\pi u^{\varepsilon_n}(s, y)^2 dy \right)^p ds \right)^{1/p} \left(\int_0^T \mathbb{1}_{(\tau_n, \tau_n+h_n]}(s) ds \right)^{1-1/p} \right] \\ &\leq h_n^{1-1/p} \sup_{\varepsilon>0} \mathbb{E} \left[\left(\int_0^T \left(\int_0^\pi u^\varepsilon(s, y)^2 dy \right)^p ds \right)^{1/p} \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Altogether, this implies $\mathbb{E}[I_{k,n}^2] \rightarrow 0$ as $n \rightarrow \infty$ for all $k \in \mathbb{N}$.

Next, we have

$$\begin{aligned} \int_0^{\tau_n} k^2 \left(e^{-k^2(\tau_n-s)} - e^{-k^2(\tau_n+h_n-s)} \right) ds &= \left(1 - e^{-k^2 h_n} \right) \int_0^{\tau_n} k^2 e^{-k^2(\tau_n-s)} ds \\ &= \left(1 - e^{-k^2 h_n} \right) \left(1 - e^{-k^2 \tau_n} \right) \leq 1 - e^{-k^2 h_n}, \end{aligned}$$

and by Itô's isometry as well as Lemma 3.1,

$$\mathbb{E} \left[\left(\int_0^T \int_0^\pi \frac{f(u^{\varepsilon_n}(s, y))}{\sigma(\varepsilon_n)} \phi_k(y) L^{\varepsilon_n}(ds, dy) \right)^2 \right] = \int_0^T \int_0^\pi \mathbb{E} \left[f^2(u^{\varepsilon_n}(s, y)) \right] \phi_k^2(y) ds dy \leq C$$

for all $k, n \in \mathbb{N}$. Therefore, by Doob's inequality, $\mathbb{E}[(J_{k,n}^1)^2]$ is bounded by

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \leq T} \left| \int_0^s \int_0^\pi \frac{f(u^{\varepsilon_n}(r, y))}{\sigma(\varepsilon_n)} \phi_k(y) L^{\varepsilon_n}(dr, dy) \right|^2 \left(\int_0^{\tau_n} k^2 \left(e^{-k^2(\tau_n-s)} - e^{-k^2(\tau_n+h_n-s)} \right) ds \right)^2 \right] \\ & \leq \left(1 - e^{-k^2 h_n} \right)^2 \mathbb{E} \left[\sup_{s \leq T} \left| \int_0^s \int_0^\pi \frac{f(u^{\varepsilon_n}(r, y))}{\sigma(\varepsilon_n)} \phi_k(y) L^{\varepsilon_n}(dr, dy) \right|^2 \right] \\ & \leq C \left(1 - e^{-k^2 h_n} \right)^2 \longrightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for all $k \in \mathbb{N}$.

Finally, with $\int_{\tau_n}^{\tau_n+h_n} k^2 e^{-k^2(\tau_n+h_n-s)} ds = 1 - e^{-k^2 h_n}$ and similar calculations, $\mathbb{E}[(J_{k,n}^2)^2]$ is bounded by

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \leq T} \left| \int_0^s \int_0^\pi \frac{f(u^{\varepsilon_n}(r, y))}{\sigma(\varepsilon_n)} \phi_k(y) L^{\varepsilon_n}(dr, dy) \right|^2 \left(\int_{\tau_n}^{\tau_n+h_n} k^2 e^{-k^2(\tau_n+h_n-s)} ds \right)^2 \right] \\ & = \left(1 - e^{-k^2 h_n} \right)^2 \mathbb{E} \left[\sup_{s \leq T} \left| \int_0^s \int_0^\pi \frac{f(u^{\varepsilon_n}(r, y))}{\sigma(\varepsilon_n)} \phi_k(y) L^{\varepsilon_n}(dr, dy) \right|^2 \right] \\ & \leq C \left(1 - e^{-k^2 h_n} \right)^2 \longrightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for all $k \in \mathbb{N}$. As a consequence, recalling that $r > 1/2$ and thus $\sum_{k=1}^\infty (1+k^2)^{-r} < \infty$, we obtain by (3.11) and dominated convergence,

$$\begin{aligned} \mathbb{E} \left[\|\bar{u}_{\tau_n+h_n}^{\varepsilon_n} - \bar{u}_{\tau_n}^{\varepsilon_n}\|_{-r}^2 \right] &= \sum_{k=1}^\infty (1+k^2)^{-r} \mathbb{E} \left[\left(\langle \bar{u}_{\tau_n+h_n}^{\varepsilon_n}, \phi_k \rangle - \langle \bar{u}_{\tau_n}^{\varepsilon_n}, \phi_k \rangle \right)^2 \right] \\ &\leq 3 \sum_{k=1}^\infty (1+k^2)^{-r} \left(\mathbb{E} [I_{k,n}^2] + \mathbb{E} [(J_{k,n}^1)^2] + \mathbb{E} [(J_{k,n}^2)^2] \right) \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, which is the assertion of the proposition. \square

Proposition 3.6. *For any fixed $t \leq T$ and $r > 1/2$, the random elements $\{\bar{u}_t^\varepsilon \mid \varepsilon > 0\}$ are tight in $H_{-r}([0, \pi])$.*

Proof. Proceeding as in the proof of Proposition 3.5 and using Lemma 3.1, we have

$$\mathbb{E} \left[\langle \bar{u}_t^\varepsilon, \phi_k \rangle^2 \right] = \int_0^t \int_0^\pi \mathbb{E} \left[f^2(u^\varepsilon(s, y)) \right] \phi_k^2(y) e^{-2k^2(t-s)} ds dy \leq C \frac{1}{k^2} \left(1 - e^{-2k^2 t} \right)$$

for any $k \in \mathbb{N}$ and $t \leq T$. Hence, we have for all $q < 1/2$, $t \leq T$ and $\varepsilon > 0$,

$$\mathbb{E} \left[\|\bar{u}_t^\varepsilon\|_q^2 \right] = \sum_{k=1}^\infty \left(1 + k^2 \right)^q \mathbb{E} \left[\langle \bar{u}_t^\varepsilon, \phi_k \rangle^2 \right] \leq C \sum_{k=1}^\infty \left(1 + k^2 \right)^q \frac{1}{k^2} < \infty,$$

and thus $\bar{u}_t^\varepsilon \in H_q([0, \pi])$ \mathbb{P} -almost surely.

Because the penultimate term in the inequality above does not depend on ε , by Markov's inequality, we can further deduce

$$\lim_{\delta \rightarrow \infty} \sup_{\varepsilon > 0} \mathbb{P} \left(\|\bar{u}_t^\varepsilon\|_q > \delta \right) = 0 \quad (3.14)$$

for all $q < 1/2$ and $t \leq T$. Since the embeddings

$$H_q([0, \pi]) \hookrightarrow L^2([0, \pi]) \hookrightarrow H_{-r}([0, \pi])$$

are compact for $0 < q < 1/2 < r$ by Theorem 4.58 in [10], it follows that $\{\bar{u}_t^\varepsilon \mid \varepsilon > 0\}$ is tight in $H_{-r}([0, \pi])$ for any fixed $t \leq T$ and $r > 1/2$. \square

Theorem 3.7. *For any $r > 1/2$, the càdlàg modifications $\{\bar{u}^\varepsilon \mid \varepsilon > 0\}$ are tight in the Skorokhod space $D([0, T], H_{-r}([0, \pi]))$.*

Proof. By Theorem 6.8 in [27], this is a direct consequence of Propositions 3.5 and 3.6. \square

3.2 Characterization of the limit

After proving tightness in Section 3.1, our next goal is to characterize the limit distribution of weakly converging subsequences. Following the outline of the proof of Theorem 2.1, the first step is to show that under condition (1.4) on the Lévy measure Q^ε , the process \bar{M} in (2.18) is a martingale with respect to the filtration $\bar{\mathbf{F}}$ defined in (2.16). In order to achieve this result, which is Theorem 3.12 below, we prove that the pairs $(u^\varepsilon, \bar{u}^\varepsilon)$ satisfy related martingale problems (Theorem 3.8) and that these “converge” as $\varepsilon \rightarrow 0$ (Theorem 3.9).

Recall that for all test functions $\phi \in C_c^\infty((0, \pi))$ and fixed $t \leq T$,

$$\int_0^\pi u^\varepsilon(t, x)\phi(x) dx = \int_0^t \int_0^\pi u^\varepsilon(s, x)\phi''(x) ds dx + \int_0^t \int_0^\pi \frac{f(u^\varepsilon(s, x))}{\sigma(\varepsilon)}\phi(x) L^\varepsilon(ds, dx) \quad (3.15)$$

\mathbb{P} -almost surely. This follows, in a similar way to Theorem 3.2 in [27], from the fact that in our situation, we may apply the stochastic Fubini theorem; see, for example, Theorem 2.6 in [27].

Theorem 3.8. *For each $\varepsilon > 0$, the pair $(u^\varepsilon, \bar{u}^\varepsilon)$ where $u^\varepsilon \in L^2([0, T] \times [0, \pi])$ is the mild solution to the stochastic heat equation (2.1) and \bar{u}^ε is its càdlàg modification in $H_{-r}([0, \pi])$, with $r > 1/2$, satisfies the following martingale problem. For all $\xi \in \mathbb{R}$ and $\phi \in C_c^\infty((0, \pi))$, the complex-valued stochastic process*

$$\begin{aligned} M_t^\varepsilon &= e^{i\xi\langle \bar{u}_t^\varepsilon, \phi \rangle} - i\xi \int_0^t e^{i\xi\langle \bar{u}_s^\varepsilon, \phi \rangle} \langle u^\varepsilon(s, \cdot), \phi'' \rangle ds \\ &\quad - \int_0^t \int_0^\pi \int_{\mathbb{R}} e^{i\xi\langle \bar{u}_s^\varepsilon, \phi \rangle} \left(e^{i\xi \frac{f(u^\varepsilon(s, x))}{\sigma(\varepsilon)} \phi(x)z} - 1 - i\xi \frac{f(u^\varepsilon(s, x))}{\sigma(\varepsilon)} \phi(x)z \right) ds dx Q^\varepsilon(dz), \end{aligned} \quad (3.16)$$

with $t \leq T$, is a square-integrable \mathbf{F} -martingale with the uniform bound

$$\sup_{\varepsilon > 0} \sup_{t \leq T} \mathbb{E} \left[|M_t^\varepsilon|^2 \right] < \infty. \quad (3.17)$$

Proof. First, since \bar{u}^ε is the càdlàg version of u^ε , the stochastic process $\langle \bar{u}^\varepsilon, \phi \rangle$ is indistinguishable from the right-hand side of (3.15) for any $\phi \in C_c^\infty((0, \pi))$. This directly implies that $\langle \bar{u}^\varepsilon, \phi \rangle$ is an \mathbf{F} -semimartingale without continuous martingale part. Furthermore, one can easily verify that the \mathbf{F} -compensator of the jump measure μ_ϕ^ε of $\langle \bar{u}^\varepsilon, \phi \rangle$ (on $[0, T] \times \mathbb{R}$) is given by

$$\nu_\phi^\varepsilon(A) = \int_0^T \int_0^\pi \int_{\mathbb{R}} \mathbb{1}_A \left(t, \frac{f(u^\varepsilon(t, x))}{\sigma(\varepsilon)} \phi(x)z \right) dt dx Q^\varepsilon(dz), \quad A \in \mathcal{B}([0, T] \times \mathbb{R}). \quad (3.18)$$

As a consequence, using Itô's formula (see, for example, Theorem I.4.57 in [14]), (3.15) and the fact that $u^\varepsilon(0, x) = 0$, we have

$$\begin{aligned} e^{i\xi\langle\bar{u}_t^\varepsilon, \phi\rangle} &= 1 + i\xi \int_0^t e^{i\xi\langle\bar{u}_s^\varepsilon, \phi\rangle} \langle u^\varepsilon(s, \cdot), \phi'' \rangle ds + i\xi \int_0^t \int_0^\pi e^{i\xi\langle\bar{u}_{s-}^\varepsilon, \phi\rangle} \frac{f(u^\varepsilon(s, x))}{\sigma(\varepsilon)} \phi(x) L^\varepsilon(ds, dx) \\ &\quad + \int_0^t \int_{\mathbb{R}} e^{i\xi\langle\bar{u}_{s-}^\varepsilon, \phi\rangle} (e^{i\xi x} - 1 - i\xi x) \mu_\phi^\varepsilon(ds, dx), \end{aligned}$$

and therefore, by (3.16),

$$\begin{aligned} M_t^\varepsilon &= 1 + i\xi \int_0^t \int_0^\pi e^{i\xi\langle\bar{u}_{s-}^\varepsilon, \phi\rangle} \frac{f(u^\varepsilon(s, x))}{\sigma(\varepsilon)} \phi(x) L^\varepsilon(ds, dx) \\ &\quad + \int_0^t \int_{\mathbb{R}} e^{i\xi\langle\bar{u}_{s-}^\varepsilon, \phi\rangle} (e^{i\xi x} - 1 - i\xi x) (\mu_\phi^\varepsilon - \nu_\phi^\varepsilon)(ds, dx) \end{aligned} \tag{3.19}$$

for all $t \leq T$. The two integral processes on the right-hand side of (3.19) are square-integrable \mathbf{F} -martingales (for the second, this is implied by the elementary inequalities $(\cos(x) - 1)^2 \leq x^2$ and $(\sin(x) - x)^2 \leq 4x^2$ for all $x \in \mathbb{R}$, together with Lemma 3.1) and hence, this is also the case for M^ε . By Itô's isometry, we further obtain

$$\mathbb{E} \left[\left| \int_0^t \int_{\mathbb{R}} e^{i\xi\langle\bar{u}_s^\varepsilon, \phi\rangle} \frac{f(u^\varepsilon(s, x))}{\sigma(\varepsilon)} \phi(x) L^\varepsilon(ds, dx) \right|^2 \right] = \mathbb{E} \left[\int_0^t \int_0^\pi f^2(u^\varepsilon(s, x)) \phi^2(x) ds dx \right]$$

as well as

$$\begin{aligned} &\mathbb{E} \left[\left| \int_0^t \int_{\mathbb{R}} e^{i\xi\langle\bar{u}_s^\varepsilon, \phi\rangle} (e^{i\xi x} - 1 - i\xi x) (\mu_\phi^\varepsilon - \nu_\phi^\varepsilon)(ds, dx) \right|^2 \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}} |e^{i\xi x} - 1 - i\xi x|^2 \nu_\phi^\varepsilon(ds, dx) \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}} ((\cos(\xi x) - 1)^2 + (\sin(\xi x) - \xi x)^2) \nu_\phi^\varepsilon(ds, dx) \right] \end{aligned}$$

for all $t \leq T$. We estimate the last expectation, using the elementary inequalities given above as well as the definition of ν_ϕ^ε , by

$$\begin{aligned} 5\xi^2 \mathbb{E} \left[\int_0^t \int_{\mathbb{R}} x^2 \nu_\phi^\varepsilon(ds, dx) \right] &= 5\xi^2 \mathbb{E} \left[\int_0^t \int_0^\pi \frac{f^2(u^\varepsilon(s, x))}{\sigma^2(\varepsilon)} \phi^2(x) z^2 ds dx Q^\varepsilon(dz) \right] \\ &= 5\xi^2 \mathbb{E} \left[\int_0^t \int_0^\pi f^2(u^\varepsilon(s, x)) \phi^2(x) ds dx \right], \quad t \leq T. \end{aligned}$$

Altogether, we obtain (3.17) from (3.19), the Lipschitz continuity of f and Lemma 3.1. □

We now switch to the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{F}}, \bar{\mathbb{P}})$ from the Skorokhod construction in (2.14) and define the process \bar{M}^k in the same way as M^ε in (3.16), but with $(u^\varepsilon, \bar{u}^\varepsilon)$ and Q^ε replaced by (v^k, \bar{v}^k) in (2.15) and Q^{ε^k} , respectively.

Theorem 3.9. *Under (1.4), we have pointwise on $\bar{\Omega}$ for any $\xi \in \mathbb{R}$ and $\phi \in C_c^\infty((0, \pi))$,*

$$\bar{M}^k \longrightarrow \bar{M} \quad \text{as } k \rightarrow \infty$$

in the Skorokhod space $D([0, T], \mathbb{C})$, where \bar{M} is the process in (2.18).

In order to prove this result, we first rewrite \overline{M}^k in a more convenient form. For any fixed $k \in \mathbb{N}$, $\xi \in \mathbb{R}$ and $\phi \in C_c^\infty((0, \pi))$, let

$$\begin{aligned}\overline{v}^k(A) &= \int_0^T \int_0^\pi \int_{\mathbb{R}} \mathbb{1}_A \left(t, \frac{f(v^k(t, x))}{\sigma(\varepsilon_k)} \phi(x) z \right) dt dx Q^{\varepsilon_k}(dz), \\ \overline{B}_t^k &= \int_0^t \langle v^k(s, \cdot), \phi'' \rangle ds - \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| > 1\}} \overline{v}^k(ds, dx), \\ \overline{A}_t^k &= i\xi \overline{B}_t^k + \int_0^t \int_{\mathbb{R}} \left(e^{i\xi x} - 1 - i\xi x \mathbb{1}_{\{|x| \leq 1\}} \right) \overline{v}^k(ds, dx)\end{aligned}\tag{3.20}$$

for all $A \in \mathcal{B}([0, T] \times \mathbb{R})$ and $t \leq T$. Note that $(\overline{B}^k, 0, \overline{v}^k)$ are the predictable characteristics of the \overline{F} -semimartingale $\langle \overline{v}^k, \phi \rangle$ and that they are functions of the random field v^k and not of $\langle \overline{v}^k, \phi \rangle$ itself (which is another reason why we have adopted a dual view on the solutions to (2.1) and (2.8) as elements of Ω^*). The process \overline{M}^k introduced above can thus be written as

$$\overline{M}_t^k = e^{i\xi \langle \overline{v}_t^k, \phi \rangle} - \int_0^t e^{i\xi \langle \overline{v}_s^k, \phi \rangle} \overline{A}^k(ds), \quad t \leq T.\tag{3.21}$$

When we define the truncation functions

$$\varrho_h: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto x \mathbb{1}_{\{|x| \leq h\}}, \quad h > 0,\tag{3.22}$$

the key idea of the proof of Theorem 3.9 is to see that for fixed $\overline{\omega} \in \overline{\Omega}$, $t \leq T$ and $\phi \in C_c^\infty((0, \pi))$, the function \overline{A}_t^k in (3.20) (resp., \overline{A}_t in (2.17)) is the Lévy exponent of the infinitely divisible distribution η_k (resp., η) with characteristics $(\overline{B}_t^k, 0, \overline{v}^k([0, t] \times dx))$ (resp., $(\overline{B}_t, \overline{C}_t, 0)$) with respect to ϱ_1 . Then we can make use of the following result, which is the only place in this work where (1.4) will actually be needed.

Theorem 3.10. *If (1.4) holds, then for any $\phi \in C_c^\infty((0, \pi))$, $t \in [0, T]$ and $\overline{\omega} \in \overline{\Omega}$, we have*

$$\eta_k \xrightarrow{w} \eta \quad \text{as } k \rightarrow \infty.$$

Proof. As in the proof of Theorem 2.1 in [2], it suffices to show that

$$\begin{aligned}(i) \quad & \int_{|x| \leq h} x^2 \overline{v}^k([0, t] \times dx) \longrightarrow \overline{C}_t, \\ (ii) \quad & \overline{B}_t^k \longrightarrow \overline{B}_t, \\ (iii) \quad & \overline{v}^k([0, t] \times \{|x| > 1\}) \longrightarrow 0\end{aligned}\tag{3.23}$$

as $k \rightarrow \infty$ for all $h > 0$. Starting with (i), we have

$$\begin{aligned}& \int_{|x| \leq h} x^2 \overline{v}^k([0, t] \times dx) \\ &= \int_0^t \int_0^\pi f^2(v^k(s, x)) \phi^2(x) \frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \leq (h/|f(v^k(s, x))\phi(x)|)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) ds dx.\end{aligned}$$

We can ignore all points in the domain of integration where $|f(v^k(s, x))\phi(x)| = 0$. So if we let

$$\begin{aligned}I_h^k(s, x) &= \frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \leq (h/|f(v^k(s, x))\phi(x)|)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \quad \text{and} \\ \Sigma_h^k(s, x) &= 1 - I_h^k(s, x) = \frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| > (h/|f(v^k(s, x))\phi(x)|)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz)\end{aligned}\tag{3.24}$$

for any $(s, x) \in [0, T] \times [0, \pi]$, $h > 0$ and $k \in \mathbb{N}$, then, using the triangle inequality and the fact that $0 < I_h^k(s, x) \leq 1$, we obtain

$$\begin{aligned}
 & \left| \int_{|x| \leq h} x^2 \bar{\nu}^k([0, t] \times dx) - \bar{C}_t \right| \\
 & \leq \left| \int_0^t \int_0^\pi \left(f^2(v^k(s, x)) - f^2(v(s, x)) \right) \phi^2(x) I_h^k(s, x) ds dx \right| \\
 & \quad + \left| \int_0^t \int_0^\pi f^2(v(s, x)) \phi^2(x) \left(I_h^k(s, x) - 1 \right) ds dx \right| \\
 & \leq \int_0^t \int_0^\pi \left| f^2(v^k(s, x)) - f^2(v(s, x)) \right| \phi^2(x) ds dx + \int_0^t \int_0^\pi f^2(v(s, x)) \phi^2(x) \Sigma_h^k(s, x) ds dx.
 \end{aligned} \tag{3.25}$$

By the Lipschitz continuity of f and Hölder's inequality, we have for the first integral on the right-hand side of (3.25),

$$\begin{aligned}
 & \int_0^t \int_0^\pi \left| f^2(v^k(s, x)) - f^2(v(s, x)) \right| \phi^2(x) ds dx \\
 & \leq C \int_0^t \int_0^\pi \left| f(v^k(s, x)) - f(v(s, x)) \right| \left| f(v^k(s, x)) + f(v(s, x)) \right| ds dx \\
 & \leq C \left(\int_0^t \int_0^\pi \left(v^k(s, x) - v(s, x) \right)^2 ds dx \right)^{1/2} \left(\int_0^t \int_0^\pi \left(f(v^k(s, x)) + f(v(s, x)) \right)^2 ds dx \right)^{1/2}.
 \end{aligned} \tag{3.26}$$

By (2.15), $v^k \rightarrow v$ in $L^2([0, T] \times [0, \pi])$ pointwise on $\bar{\Omega}$. Hence, the sequence $(v^k)_{k \in \mathbb{N}}$ is bounded in $L^2([0, T] \times [0, \pi])$ and we have

$$\begin{aligned}
 & \int_0^t \int_0^\pi \left(f(v^k(s, x)) + f(v(s, x)) \right)^2 ds dx \\
 & \leq C \left(1 + \sup_{k \in \mathbb{N}} \int_0^t \int_0^\pi v^k(s, x)^2 ds dx + \int_0^t \int_0^\pi v(s, x)^2 ds dx \right) < \infty,
 \end{aligned} \tag{3.27}$$

which implies $\int_0^t \int_0^\pi \left| f^2(v^k(s, x)) - f^2(v(s, x)) \right| \phi^2(x) ds dx \rightarrow 0$ as $k \rightarrow \infty$.

The second integral in (3.25) is more difficult to handle. We decompose it into $I_1^{k,n} + I_2^{k,n,M} + I_3^{k,n,M}$, where

$$\begin{aligned}
 I_1^{k,n} &= \int_0^t \int_0^\pi f^2(v(s, x)) \Sigma_h^k(s, x) \mathbb{1}_{\{|v^k(s, x)| \leq n\}} ds dx, \\
 I_2^{k,n,M} &= \int_0^t \int_0^\pi f^2(v(s, x)) \Sigma_h^k(s, x) \mathbb{1}_{\{|v^k(s, x)| > n\}} \mathbb{1}_{\{|f(v(s, x))| \leq M\}} ds dx, \\
 I_3^{k,n,M} &= \int_0^t \int_0^\pi f^2(v(s, x)) \Sigma_h^k(s, x) \mathbb{1}_{\{|v^k(s, x)| > n\}} \mathbb{1}_{\{|f(v(s, x))| > M\}} ds dx
 \end{aligned}$$

for all $k, n, M \in \mathbb{N}$.

Again we will study each of these three integrals separately. On the set $\{|v^k(s, x)| \leq n\}$, we have $|f(v^k(s, x))\phi(x)| \leq (Kn + |f(0)|)\|\phi\|_\infty$ and therefore

$$\mathbb{1}_{\{|z| > (h/|f(v^k(s, x))\phi(x)|)\sigma(\varepsilon_k)\}} \leq \mathbb{1}_{\{|z| > (h/(Kn+|f(0)|)\|\phi\|_\infty)\sigma(\varepsilon_k)\}}.$$

Thus,

$$I_1^{k,n} \leq \int_0^t \int_0^\pi f^2(v(s,x)) \frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| > (h/(Kn+|f(0)|)\|\phi\|_\infty)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) ds dx.$$

Because the term $h/(Kn+|f(0)|)\|\phi\|_\infty$ does not depend on k , we can use condition (1.4), whence

$$\frac{1}{\sigma(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| > (h/(Kn+|f(0)|)\|\phi\|_\infty)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \longrightarrow 0 \quad \text{as } k \rightarrow \infty$$

for all $n \in \mathbb{N}$ and $h > 0$. Since $v \in L^2([0, T] \times [0, \pi])$, we obtain by dominated convergence that $I_1^{k,n} \longrightarrow 0$ as $k \rightarrow \infty$ for all $n \in \mathbb{N}$.

Next, we have by Chebyshev's inequality,

$$I_2^{k,n,M} \leq M^2 \int_0^t \int_0^\pi \mathbb{1}_{\{|v^k(s,x)| > n\}} ds dx \leq \frac{M^2}{n^2} \sup_{k \in \mathbb{N}} \int_0^t \int_0^\pi v^k(s,x)^2 ds dx,$$

which tends to 0 as $n \rightarrow \infty$, uniformly in k .

Finally, we have by dominated convergence,

$$I_3^{k,n,M} \leq \int_0^t \int_0^\pi f^2(v(s,x)) \mathbb{1}_{\{|f(v(s,x))| > M\}} ds dx \longrightarrow 0 \quad \text{as } M \rightarrow \infty,$$

uniformly in n and k . Altogether, we have just shown that the left-hand side of (3.25) converges to 0 as $k \rightarrow \infty$, which is condition (i) in (3.23).

The two other conditions will follow from our last calculations. Indeed, we have

$$|\overline{B}_t^k - \overline{B}_t| \leq \int_0^t \left| \langle v^k(s, \cdot), \phi'' \rangle - \langle v(s, \cdot), \phi'' \rangle \right| ds + \left| \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| > 1\}} \overline{\nu}^k(ds, dx) \right|,$$

where the first term vanishes because

$$\begin{aligned} \int_0^t \left| \langle v_s^k, \phi'' \rangle - \langle v_s, \phi'' \rangle \right| ds &\leq \int_0^t \int_0^\pi |v^k(s,x) - v(s,x)| |\phi''(x)| ds dx \\ &\leq C \left(\int_0^t \int_0^\pi (v^k(s,x) - v(s,x))^2 ds dx \right)^{1/2} \longrightarrow 0 \end{aligned} \quad (3.28)$$

as $k \rightarrow \infty$. Furthermore,

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| > 1\}} \overline{\nu}^k(ds, dx) &\leq \int_0^t \int_{\mathbb{R}} x^2 \mathbb{1}_{\{|x| > 1\}} \overline{\nu}^k(ds, dx) \\ &= \int_0^t \int_0^\pi f^2(v^k(s,x)) \phi^2(x) \Sigma_1^k(s,x) ds dx \\ &\leq \int_0^t \int_0^\pi |f^2(v^k(s,x)) - f^2(v(s,x))| \phi^2(x) ds dx \\ &\quad + \int_0^t \int_0^\pi f^2(v(s,x)) \phi^2(x) \Sigma_1^k(s,x) ds dx. \end{aligned} \quad (3.29)$$

These integrals are exactly the same as in the last line of (3.25), so we obtain $|\overline{B}_t^k - \overline{B}_t| \longrightarrow 0$, which is condition (ii). From this, condition (iii) follows immediately since we have

$$\overline{\nu}^k([0, t] \times \{|x| > 1\}) = \int_0^t \int_{\mathbb{R}} \mathbb{1}_{\{|x| > 1\}} \overline{\nu}^k(ds, dx) \leq \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| > 1\}} \overline{\nu}^k(ds, dx). \quad (3.30)$$

□

The following technical lemma is a direct consequence of Theorem 3.10 and will be crucial for proving Theorem 3.9 afterwards.

If $t \mapsto A_t$ is a function of locally finite variation, we denote by $\text{Var}(A)_t$ the total variation of the function A on the interval $[0, t]$. If A is complex-valued, we have $\text{Var}(A) = \text{Var}(\text{Re } A) + \text{Var}(\text{Im } A)$.

Lemma 3.11. *If (1.4) holds, then we have pointwise on $\bar{\Omega}$,*

$$\bar{A}_t^k \longrightarrow \bar{A}_t \quad \text{and} \quad \text{Var}(\bar{A}^k - \bar{A})_t \longrightarrow 0 \quad \text{as } k \rightarrow \infty$$

for any $\xi \in \mathbb{R}$, $\phi \in C_c^\infty((0, \pi))$ and $t \leq T$, where the processes \bar{A}^k and \bar{A} are defined in (3.20) and (2.17), respectively.

Proof. For fixed $\phi \in C_c^\infty((0, \pi))$, $t \in [0, T]$ and $\bar{\omega} \in \bar{\Omega}$, the infinitely divisible distributions η_k and η , defined before Theorem 3.10, have Lévy exponents \bar{A}_t^k and \bar{A}_t , respectively. By that theorem, $\eta_k \xrightarrow{w} \eta$ as $k \rightarrow \infty$. This immediately implies the first claim of the proposition (see, for example, Equation VII.2.6 in [14]).

For the second claim, we will need the truncation function

$$\vartheta(x) = \begin{cases} -1, & x < -1, \\ x, & |x| \leq 1, \\ 1, & x > 1. \end{cases}$$

The main difference between the function $\varrho_1(x) = x \mathbb{1}_{\{|x| \leq 1\}}$, used so far, and ϑ is that the latter is continuous. Since this property will be needed for technical reasons, we replace ϱ_1 by ϑ in the expression of \bar{A}_t^k in (3.20) and thus obtain

$$\begin{aligned} \bar{A}_t^k &= i\xi \left(\int_0^t \langle v^k(s, \cdot), \phi'' \rangle ds - \int_0^t \int_{\mathbb{R}} (x - \vartheta(x)) \bar{\nu}^k(ds, dx) \right) \\ &\quad + \int_0^t \int_{\mathbb{R}} (e^{i\xi x} - 1 - i\xi \vartheta(x)) \bar{\nu}^k(ds, dx). \end{aligned} \tag{3.31}$$

With (3.31) and (2.17), we then calculate

$$\begin{aligned} \text{Re}(\bar{A}_t^k - \bar{A}_t) &= \frac{1}{2} \xi^2 \int_0^t \int_0^\pi f^2(v(s, x)) \phi^2(x) ds dx + \int_{\mathbb{R}} (\cos(\xi x) - 1) \bar{\nu}^k([0, t] \times dx), \\ \text{Im}(\bar{A}_t^k - \bar{A}_t) &= \xi \left(\int_0^t \langle v^k(s, \cdot), \phi'' \rangle ds - \int_0^t \langle v(s, \cdot), \phi'' \rangle ds - \int_{\mathbb{R}} (x - \vartheta(x)) \bar{\nu}^k([0, t] \times dx) \right) \\ &\quad + \int_{\mathbb{R}} (\sin(\xi x) - \xi \vartheta(x)) \bar{\nu}^k([0, t] \times dx). \end{aligned} \tag{3.32}$$

Consequently,

$$\begin{aligned} \text{Var}(\text{Re}(\bar{A}^k - \bar{A}))_t &\leq \frac{1}{2} \xi^2 \int_0^t \int_0^\pi \left| f^2(v(s, x)) \phi^2(x) - \int_{\mathbb{R}} \vartheta^2 \left(\frac{f(v^k(s, x))}{\sigma(\varepsilon_k)} \phi(x) z \right) Q^{\varepsilon_k}(dz) \right| dx ds \\ &\quad + \int_{\mathbb{R}} \left| \cos(\xi x) - 1 + \frac{1}{2} \xi^2 \vartheta^2(x) \right| \bar{\nu}^k([0, t] \times dx). \end{aligned} \tag{3.33}$$

We will show that the two integrals above converge to 0 as $k \rightarrow \infty$.

The function $|\cos(\xi x) - 1 + \frac{1}{2}\xi^2\vartheta^2(x)|$ is bounded, continuous (because ϑ is) and $o(x^2)$ as $x \rightarrow 0$. Hence, by condition $[\delta_{1,3}]$ of Theorem VII.2.9 in [14], we can infer

$$\int_{\mathbb{R}} \left| \cos(\xi x) - 1 + \frac{1}{2}\xi^2\vartheta^2(x) \right| \bar{\nu}^k([0, t] \times dx) \longrightarrow 0 \quad \text{as } k \rightarrow \infty$$

from Theorem 3.10.

Consider now the first integral in (3.33), and notice that

$$\vartheta^2(x) = x^2 \mathbb{1}_{\{|x| \leq 1\}} + \mathbb{1}_{\{|x| > 1\}} \quad \text{for all } x \in \mathbb{R}.$$

Using the triangle inequality, we can therefore estimate this integral by

$$\begin{aligned} & \int_0^t \int_0^\pi \left| f^2(v(s, x))\phi^2(x) - \int_{\mathbb{R}} \left(f(v^k(s, x))\phi(x)z/\sigma(\varepsilon_k) \right)^2 \mathbb{1}_{\{|f(v^k(s, x))\phi(x)z|/\sigma(\varepsilon_k) \leq 1\}} Q^{\varepsilon_k}(dz) \right| dx ds \\ & + \int_{\mathbb{R}} \mathbb{1}_{\{|x| > 1\}} \bar{\nu}^k([0, t] \times dx). \end{aligned}$$

The second integral above converges to 0 as shown in (3.30), while the same holds for the first integral by (3.25) (set $h = 1$). Together with (3.33), we conclude that $\text{Var}(\text{Re}(\bar{A}^k - \bar{A}))_t \longrightarrow 0$ as $k \rightarrow \infty$.

It remains to show that $\text{Var}(\text{Im}(\bar{A}^k - \bar{A}))_t \longrightarrow 0$ as $k \rightarrow \infty$, which will be done in a similar manner as before. From (3.32), we have

$$\begin{aligned} \text{Var}(\text{Im}(\bar{A}^k - \bar{A}))_t & \leq |\xi| \int_0^t \left| \langle v^k(s, \cdot), \phi'' \rangle - \langle v(s, \cdot), \phi'' \rangle \right| ds + |\xi| \int_{\mathbb{R}} |x - \vartheta(x)| \bar{\nu}^k([0, t] \times dx) \\ & + \int_{\mathbb{R}} \left| \sin(\xi x) - \xi\vartheta(x) \right| \bar{\nu}^k([0, t] \times dx). \end{aligned}$$

The first integral on the right-hand side above converges to 0 by (3.28). Furthermore, since

$$\int_{\mathbb{R}} |x - \vartheta(x)| \bar{\nu}^k([0, t] \times dx) \leq 2 \int_{\mathbb{R}} |x|^2 \mathbb{1}_{\{|x| > 1\}} \bar{\nu}^k([0, t] \times dx),$$

also the second integral vanishes by (3.29). Finally, the function $|\sin(\xi x) - \xi\vartheta(x)|$ is bounded, continuous and $o(x^2)$ as $x \rightarrow 0$. Hence, we can again apply Theorem VII.2.9 in [14] in order to obtain

$$\int_{\mathbb{R}} \left| \sin(\xi x) - \xi\vartheta(x) \right| \bar{\nu}^k([0, t] \times dx) \longrightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We conclude that $\text{Var}(\text{Im}(\bar{A}^k - \bar{A}))_t \longrightarrow 0$, and altogether $\text{Var}(\bar{A}^k - \bar{A})_t \longrightarrow 0$ as $k \rightarrow \infty$. \square

Proof of Theorem 3.9. According to Proposition VI.1.23 in [14], because the function $t \mapsto \int_0^t \exp(i\xi \langle \bar{v}_s, \phi \rangle) \bar{A}(ds)$ is continuous, \bar{M}^k converges to \bar{M} in the Skorokhod topology for fixed $\bar{\omega} \in \bar{\Omega}$ if

$$e^{i\xi \langle \bar{v}^k, \phi \rangle} \longrightarrow e^{i\xi \langle \bar{v}, \phi \rangle} \quad \text{and} \quad \int_0^\cdot e^{i\xi \langle \bar{v}_s^k, \phi \rangle} \bar{A}^k(ds) \longrightarrow \int_0^\cdot e^{i\xi \langle \bar{v}_s, \phi \rangle} \bar{A}(ds)$$

in $D([0, T], \mathbb{C})$ as $k \rightarrow \infty$.

Using the definition of the Skorokhod topology, we can easily infer from the convergence of $(\bar{v}^k)_{k \in \mathbb{N}}$ to \bar{v} in $D([0, T], H_{-r}([0, \pi]))$ given in (2.15) that

$$\begin{aligned} \langle \bar{v}^k, \phi \rangle & \longrightarrow \langle \bar{v}, \phi \rangle \quad \text{in } D([0, T], \mathbb{R}) \quad \text{and} \\ e^{i\xi \langle \bar{v}^k, \phi \rangle} & \longrightarrow e^{i\xi \langle \bar{v}, \phi \rangle} \quad \text{in } D([0, T], \mathbb{C}) \end{aligned}$$

as $k \rightarrow \infty$ for all $\phi \in H_r([0, \pi])$.

Next, we have

$$\begin{aligned} & \sup_{t \leq T} \left| \int_0^t e^{i\xi \langle \bar{v}_s^k, \phi \rangle} \bar{A}^k(ds) - \int_0^t e^{i\xi \langle \bar{v}_s, \phi \rangle} \bar{A}(ds) \right| \\ & \leq \sup_{t \leq T} \left| \int_0^t e^{i\xi \langle \bar{v}_s^k, \phi \rangle} (\bar{A}^k - \bar{A})(ds) \right| + \sup_{t \leq T} \left| \int_0^t (e^{i\xi \langle \bar{v}_s^k, \phi \rangle} - e^{i\xi \langle \bar{v}_s, \phi \rangle}) \bar{A}(ds) \right| \\ & \leq \text{Var}(\bar{A}^k - \bar{A})_T + \int_0^T |e^{i\xi \langle \bar{v}_s^k, \phi \rangle} - e^{i\xi \langle \bar{v}_s, \phi \rangle}| \text{Var}(\bar{A})(ds). \end{aligned}$$

Lemma 3.11 then immediately gives us $\text{Var}(\bar{A}^k - \bar{A})_T \rightarrow 0$ as $k \rightarrow \infty$. In addition, the Skorokhod convergence of $e^{i\xi \langle \bar{v}_s^k, \phi \rangle}$ towards $e^{i\xi \langle \bar{v}_s, \phi \rangle}$ implies $e^{i\xi \langle \bar{v}_t^k, \phi \rangle} \rightarrow e^{i\xi \langle \bar{v}_t, \phi \rangle}$ for all continuity points of $e^{i\xi \langle \bar{v}, \phi \rangle}$; see, for example, VI.2.3 of [14]. Since a càdlàg function has at most countably many discontinuities, we have $e^{i\xi \langle \bar{v}_t^k, \phi \rangle} \rightarrow e^{i\xi \langle \bar{v}_t, \phi \rangle}$ for almost all $t \in [0, T]$. So dominated convergence implies that also the last term of the previous display converges to 0 as $k \rightarrow \infty$. \square

We have now gathered all the intermediate results needed for the following theorem.

Theorem 3.12. *If (1.4) holds, then (v, \bar{v}) in (2.15) satisfies the following martingale problem. For all $\xi \in \mathbb{R}$ and $\phi \in C_c^\infty((0, \pi))$, the process $(\bar{M}_t)_{t \leq T}$ defined in (2.18) is a martingale with respect to the filtration $\bar{\mathbf{F}}$ in (2.16).*

Furthermore, v has an $\bar{\mathbf{F}}$ -predictable modification and

$$\text{ess sup}_{(t,x) \in [0,T] \times [0,\pi]} \mathbb{E} \left[|v(t,x)|^2 \right] < \infty. \quad (3.34)$$

Finally, for almost all $t \in [0, T]$, $\bar{v}_t = \langle v(t, \cdot), \cdot \rangle$ as well as $\bar{v}_0 = 0$ holds with probability one.

Proof. By Theorem 3.8, for any $\xi \in \mathbb{R}$, $\phi \in C_c^\infty((0, \pi))$ and $k \in \mathbb{N}$, the process $M^{\varepsilon k}$ defined in (3.16) is a square-integrable \mathbf{F} -martingale. Moreover, as \bar{v}^k and v^k in (2.15) are adapted to the filtration $\bar{\mathbf{F}}$, the same holds for \bar{M}^k from (3.21) as well as \bar{v} , v and \bar{M} by a limit argument. Since \bar{M}^k has the same distribution as $M^{\varepsilon k}$ by (2.15), standard arguments now show that \bar{M}^k is an $\bar{\mathbf{F}}$ -martingale for all $\xi \in \mathbb{R}$, $\phi \in C_c^\infty((0, \pi))$ and $k \in \mathbb{N}$. This is the martingale problem satisfied by the pair (v^k, \bar{v}^k) .

Using Theorem 3.9, we have

$$\bar{M}^k(\bar{\omega}) \rightarrow \bar{M}(\bar{\omega}) \quad \text{in } D([0, T], \mathbb{C}) \quad (3.35)$$

as $k \rightarrow \infty$ for all $\bar{\omega} \in \bar{\Omega}$. This implies $\bar{M}_t^k(\bar{\omega}) \rightarrow \bar{M}_t(\bar{\omega})$ almost everywhere on $[0, T]$ for all $\bar{\omega} \in \bar{\Omega}$. Furthermore,

$$\mathbb{E} \left[|\bar{M}_t^k|^2 \right] = \mathbb{E} \left[|M_t^{\varepsilon k}|^2 \right] < \infty$$

uniformly in $k \in \mathbb{N}$ and $t \leq T$ by Theorem 3.8. Hence, again by standard arguments, we can deduce that \bar{M} is an $\bar{\mathbf{F}}$ -martingale as well for any $\xi \in \mathbb{R}$ and $\phi \in C_c^\infty((0, \pi))$.

Now we show the second part of the theorem. The convergence in (2.15) implies convergence in measure (with respect to the Lebesgue measure on $[0, T] \times [0, \pi]$) of $v^k(\bar{\omega})$ towards $v(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$. Hence, we have by dominated convergence,

$$\bar{\mathbb{P}} \otimes \text{Leb}_{[0,T] \times [0,\pi]} \left(|v^k - v| \geq \varepsilon \right) = \mathbb{E} \left[\int_0^T \int_0^\pi \mathbb{1}_{\{|v^k(\bar{\omega}, t, x) - v(\bar{\omega}, t, x)| \geq \varepsilon\}} dt dx \right] \rightarrow 0$$

as $k \rightarrow \infty$, and thus, v^k converges to v in $\bar{\mathbb{P}} \otimes \text{Leb}_{[0,T] \times [0,\pi]}$ -measure. Therefore, there exists a subsequence $(k_l)_{l \in \mathbb{N}}$ such that

$$v^{k_l} \longrightarrow v \quad \bar{\mathbb{P}} \otimes \text{Leb}_{[0,T] \times [0,\pi]} \text{-almost everywhere} \quad \text{as } l \rightarrow \infty, \quad (3.36)$$

and we will assume without loss of generality that (3.36) holds for the whole sequence. This in turn implies $v^k \longrightarrow v$ $\bar{\mathbb{P}}$ -almost surely as $k \rightarrow \infty$ for almost all $(t, x) \in [0, T] \times [0, \pi]$. Using Fatou's lemma, we obtain

$$\mathbb{E} \left[|v(t, x)|^2 \right] \leq \liminf_{k \rightarrow \infty} \mathbb{E} \left[|v^k(t, x)|^2 \right] \quad \text{Leb}_{[0,T] \times [0,\pi]} \text{-almost everywhere.} \quad (3.37)$$

Furthermore,

$$v^k(t, x) \stackrel{d}{=} u^{\varepsilon_k}(t, x) \quad \text{Leb}_{[0,T] \times [0,\pi]} \text{-almost everywhere,} \quad (3.38)$$

for all $k \in \mathbb{N}$, so (3.34) follows from Lemma 3.1. (In order to show (3.38), consider for $\alpha > 0$ the mollified random fields $J_\alpha v^k$ and $J_\alpha u^{\varepsilon_k}$ on $[0, T] \times [0, \pi]$, defined exactly as in (1.8) of Chapter 10 in [12]. Then (2.15) implies

$$(J_\alpha v^k)(t, x) \stackrel{d}{=} (J_\alpha u^{\varepsilon_k})(t, x) \quad (3.39)$$

for all $(t, x) \in [0, T] \times [0, \pi]$, $\alpha > 0$ and $k \in \mathbb{N}$. In addition, using Lemma 3 of Chapter 10 in [12], we have

$$J_\alpha v^k(\bar{\omega}) \longrightarrow v^k(\bar{\omega}) \quad \text{and} \quad J_\alpha u^{\varepsilon_k}(\omega) \longrightarrow u^{\varepsilon_k}(\omega) \quad \text{in } L^2([0, T] \times [0, \pi]) \quad \text{as } \alpha \rightarrow 0,$$

for all $k \in \mathbb{N}$, $\bar{\omega} \in \bar{\Omega}$ and $\omega \in \Omega$. As a consequence, we can find a sequence $(\alpha_l)_{l \in \mathbb{N}}$ converging to 0 such that

$$J_{\alpha_l} v^k(\bar{\omega}) \longrightarrow v^k(\bar{\omega}) \quad \text{and} \quad J_{\alpha_l} u^{\varepsilon_k}(\omega) \longrightarrow u^{\varepsilon_k}(\omega) \quad \text{Leb}_{[0,T] \times [0,\pi]} \text{-almost everywhere} \quad (3.40)$$

as $l \rightarrow \infty$ for all $k \in \mathbb{N}$, $\bar{\omega} \in \bar{\Omega}$ and $\omega \in \Omega$. So (3.38) follows from (3.39) and (3.40).

Next, u^ε is stochastically continuous by Theorem 4.7 in [6] and Lemma B.1 in [3]. This and (3.38) imply that v^k is also stochastically continuous. By a straightforward extension of Proposition 3.21 in [21] to two-parameter processes, each v^k has a predictable modification \tilde{v}^k . By (3.36), we have $\tilde{v}^k \longrightarrow v$ $\bar{\mathbb{P}} \otimes \text{Leb}_{[0,T] \times [0,\pi]}$ -almost everywhere, so v has a predictable modification as well.

Finally, the last statement is easy and we leave the details to the reader. \square

We can now finish the proof of the weak convergence (2.13). Indeed, the martingale problem stated in Theorem 3.12 and satisfied by (v, \bar{v}) in (2.15) will allow us to identify uniquely the distribution of (v, \bar{v}) (from now on we may and will assume that v is predictable).

Note that the next theorem holds independently of all our previous results.

Theorem 3.13. *On a filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{F}}, \bar{\mathbb{P}})$, let $v = \{v(t, x) \mid (t, x) \in [0, T] \times [0, \pi]\}$ be an $\bar{\mathbf{F}}$ -predictable random field and \bar{v} an $\bar{\mathbf{F}}$ -adapted càdlàg process in $H_{-r}([0, \pi])$, with $r > 1/2$. Assume that for almost all $t \in [0, T]$, $\bar{v}_t = \langle v(t, \cdot), \cdot \rangle$ as well as $\bar{v}_0 = 0$ holds $\bar{\mathbb{P}}$ -almost surely and that*

$$\text{ess sup}_{(t,x) \in [0,T] \times [0,\pi]} \mathbb{E} \left[|v(t, x)|^2 \right] < \infty. \quad (3.41)$$

In addition, assume that the pair (v, \bar{v}) satisfies the following martingale problem. For all $\xi \in \mathbb{R}$ and $\phi \in C_c^\infty((0, \pi))$, the process $(\bar{M}_t)_{t \leq T}$ defined via (2.17) and (2.18) is a local $\bar{\mathbf{F}}$ -martingale.

Then there exists a Gaussian space–time white noise \widetilde{W} on $[0, T] \times [0, \pi]$, possibly defined on a filtered extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{F}}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ such that, with probability one, v is equal in $L^2([0, T] \times [0, \pi])$ to the mild solution to the stochastic heat equation (2.8) with noise \widetilde{W} . Furthermore, \bar{v} is indistinguishable from the modification of the latter that is continuous in $H_{-r}([0, \pi])$.

Proof. The proof is inspired by Lemma 2.4 in [18]. First, Theorem II.2.42 in [14] shows that for any $\phi \in C_c^\infty((0, \pi))$, the stochastic process $\langle \bar{v}, \phi \rangle$ is an $\overline{\mathbf{F}}$ -semimartingale with first and second characteristic given by

$$t \mapsto \int_0^t \langle v(s, \cdot), \phi'' \rangle ds \quad \text{and} \quad t \mapsto \int_0^t \int_0^\pi f^2(v(s, x)) \phi^2(x) ds dx,$$

respectively. Furthermore, the third characteristic of $\langle \bar{v}, \phi \rangle$ equals 0, which implies that $\langle \bar{v}, \phi \rangle$ is continuous. As $\bar{v}_0 = 0$ $\overline{\mathbb{P}}$ -almost surely, its canonical decomposition is

$$\langle \bar{v}, \phi \rangle = \int_0^\cdot \langle v(s, \cdot), \phi'' \rangle ds + \langle \bar{v}, \phi \rangle^c,$$

where $\langle \bar{v}, \phi \rangle^c$ denotes the continuous martingale part of $\langle \bar{v}, \phi \rangle$. Since

$$\mathbb{E} \left[\int_0^T \int_0^\pi f^2(v(s, x)) \phi^2(x) ds dx \right] \leq C \left(\operatorname{ess\,sup}_{(s,x) \in [0,T] \times [0,\pi]} \mathbb{E} [|v(s, x)|^2] + 1 \right),$$

which is finite by assumption, the quadratic variation of $\langle \bar{v}, \phi \rangle^c$ is integrable, so

$$M_t(\phi) = \langle \bar{v}_t, \phi \rangle - \int_0^t \langle v(s, \cdot), \phi'' \rangle ds, \quad t \leq T, \tag{3.42}$$

is a continuous square-integrable $\overline{\mathbf{F}}$ -martingale with quadratic variation process

$$t \mapsto \int_0^t \int_0^\pi f^2(v(s, x)) \phi^2(x) ds dx, \tag{3.43}$$

for all $\phi \in C_c^\infty((0, \pi))$. The specifications (3.42) and (3.43) define an orthogonal martingale measure $\{M_t(A), t \in [0, T], A \in \mathcal{B}([0, \pi])\}$ relative to $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbf{F}}, \overline{\mathbb{P}})$, in the sense of Chapter 2 in [27], with covariation measure

$$Q_M(A \times B \times [s, t]) = \int_s^t \int_{A \cap B} f^2(v(r, x)) dr dx \tag{3.44}$$

for all $A, B \in \mathcal{B}([0, \pi])$.

Now let $(\Omega', \mathcal{F}', \mathbf{F}', \mathbb{P}')$ be another filtered probability space on which a Gaussian space–time white noise W' on $[0, T] \times [0, \pi]$ is defined. Set

$$\widetilde{\Omega} = \overline{\Omega} \times \Omega', \quad \widetilde{\mathcal{F}} = \overline{\mathcal{F}} \otimes \mathcal{F}', \quad \widetilde{\mathcal{F}}_t = \bigcap_{s>t} \overline{\mathcal{F}}_s \otimes \mathcal{F}'_s, \quad \widetilde{\mathbb{P}} = \overline{\mathbb{P}} \otimes \mathbb{P}',$$

and extend the random measures M and W' as well as the random elements \bar{v} and v to $\widetilde{\Omega}$ in the standard way so that on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{F}}, \widetilde{\mathbb{P}})$, W' is independent of (\bar{v}, v) and thus of M . In addition, on this extension, M is still an orthogonal martingale measure satisfying (3.42) and (3.44) by Lemma II.7.3 in [14]. Define

$$\begin{aligned} \widetilde{W}_t(\phi) &= \int_0^t \int_0^\pi \frac{1}{f(v(s, x))} \mathbb{1}_{\{f^2(v(s, x)) \neq 0\}} \phi(x) M(ds, dx) \\ &\quad + \int_0^t \int_0^\pi \mathbb{1}_{\{f^2(v(s, x)) = 0\}} \phi(x) W'(ds, dx) \end{aligned}$$

for all $t \leq T$ and $\phi \in C_c^\infty((0, \pi))$. As before, this defines a martingale measure $\{\widetilde{W}_t(A), t \in [0, T], A \in \mathcal{B}([0, \pi])\}$ relative to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{F}}, \widetilde{\mathbb{P}})$.

Since M and W' are independent, we have from (3.44),

$$\begin{aligned} Q_{\widetilde{W}}(A \times B \times [s, t]) &= \int_s^t \int_{A \cap B} \frac{1}{f^2(v(r, x))} \mathbb{1}_{\{f^2(v(r, x)) \neq 0\}} \phi^2(x) f^2(v(r, x)) \, dr \, dx \\ &\quad + \int_s^t \int_{A \cap B} \mathbb{1}_{\{f^2(v(r, x)) = 0\}} \phi^2(x) \, dr \, dx \\ &= \int_s^t \int_{A \cap B} \phi^2(x) \, dr \, dx \end{aligned}$$

for all $A, B \in \mathcal{B}([0, \pi])$. Therefore, it follows from Proposition 2.1 in [27] that \widetilde{W} is orthogonal and from Proposition 2.10 in [27] that the martingale measure \widetilde{W} is a Gaussian space–time white noise on $[0, T] \times [0, \pi]$ with respect to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{F}}, \widetilde{\mathbb{P}})$. Moreover, we have

$$\begin{aligned} \int_0^t \int_0^\pi f(v(s, x)) \phi(x) \widetilde{W}(ds, dx) &= \int_0^t \int_0^\pi f(v(s, x)) \frac{1}{f(v(s, x))} \mathbb{1}_{\{f^2(v(s, x)) \neq 0\}} \phi(x) M(ds, dx) \\ &\quad + \int_0^t \int_0^\pi f(v(s, x)) \mathbb{1}_{\{f^2(v(s, x)) = 0\}} \phi(x) W'(ds, dx) \\ &= \int_0^t \int_0^\pi \mathbb{1}_{\{f^2(v(s, x)) \neq 0\}} \phi(x) M(ds, dx). \end{aligned} \tag{3.45}$$

Since, by (3.44),

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^T \int_0^\pi \mathbb{1}_{\{f^2(v(s, x)) \neq 0\}} \phi(x) M(ds, dx) - \int_0^T \int_0^\pi \phi(x) M(ds, dx) \right)^2 \right] \\ &= \mathbb{E} \left[\left(\int_0^T \int_0^\pi \mathbb{1}_{\{f^2(v(s, x)) = 0\}} \phi(x) M(ds, dx) \right)^2 \right] \\ &= \mathbb{E} \left[\int_0^T \int_0^\pi \int_0^\pi \phi(x) \mathbb{1}_{\{f^2(v(s, x)) = 0\}} \phi(y) \mathbb{1}_{\{f^2(v(s, y)) = 0\}} Q_M(ds, dx, dy) \right] \\ &= \mathbb{E} \left[\int_0^T \int_0^\pi \phi^2(x) \mathbb{1}_{\{f^2(v(s, x)) = 0\}} f^2(v(s, x)) \, dx \, ds \right] = 0, \end{aligned}$$

the $\widetilde{\mathbf{F}}$ -martingales $t \mapsto \int_0^t \int_0^\pi \mathbb{1}_{\{f^2(v(s, x)) \neq 0\}} \phi(x) M(ds, dx)$ and $t \mapsto \int_0^t \int_0^\pi \phi(x) M(ds, dx)$ are indistinguishable. This implies, together with (3.42) and (3.45), that we have for any $\phi \in C_c^\infty((0, \pi))$,

$$\int_0^t \int_0^\pi f(v(s, x)) \phi(x) \widetilde{W}(ds, dx) = M_t(\phi) = \langle \bar{v}_t, \phi \rangle - \int_0^t \langle v(s, \cdot), \phi'' \rangle \, ds, \quad t \leq T, \tag{3.46}$$

$\widetilde{\mathbb{P}}$ -almost surely. By assumption, the equality in (3.46) holds also $\widetilde{\mathbb{P}}$ -almost surely for almost all $t \leq T$ if we replace $\langle \bar{v}_t, \phi \rangle$ with $\langle v(t, \cdot), \phi \rangle$. This and the assumption (3.41) imply, by the proof of Theorem 3.2 in [27], that we have

$$v(t, x) = \int_0^t \int_0^\pi G_{t-s}(x, y) f(v(s, y)) \widetilde{W}(ds, dy) \quad \widetilde{\mathbb{P}}\text{-almost surely} \tag{3.47}$$

for almost all $(t, x) \in [0, T] \times [0, \pi]$, i.e., v satisfies the mild formulation of (2.19) almost everywhere. Now let \tilde{v} be a mild solution to (2.19). Again by Theorem 3.2 in [27] and its proof, we can infer that $\tilde{\mathbb{P}}$ -almost surely, v and \tilde{v} are equal almost everywhere and hence, in $L^2([0, T] \times [0, \pi])$.

Finally, let \hat{v} be the continuous modification in $H_{-r}([0, \pi])$ of \tilde{v} , which we obtain from Corollary 3.4 in [27]. By (3.47), $\hat{v}_t = \langle v(t, \cdot), \cdot \rangle = \bar{v}_t$ $\tilde{\mathbb{P}}$ -almost surely for almost all $t \leq T$, and therefore, because \hat{v} is continuous and \bar{v} càdlàg, these two processes are indistinguishable. \square

3.3 Necessity of the condition (1.4)

Remark 3.14. Suppose that the Lipschitz function f satisfies $f(0) \neq 0$. Then there must be $(t_1, x_1) \in [0, T] \times [0, \pi]$ such that $\mathbb{P}(f(u(t_1, x_1)) \neq 0) > 0$, where u is the mild solution to (2.8). Indeed, if we had $f(u(t, x)) = 0$ \mathbb{P} -almost surely for all (t, x) , it would imply $u = 0$ everywhere on $[0, T] \times [0, \pi]$ by equation (2.8). This in turn would imply $f(0) = 0$, which contradicts the assumption.

Theorem 3.15. *Assume that $f(0) \neq 0$. In the setting of Theorem 2.1, if (2.13) holds, then we have (1.4) for all $\kappa > 0$.*

Proof. If (2.13) holds, we can use Skorokhod's representation theorem as in the first part of the proof of Theorem 2.1 and obtain for any sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ converging to 0, random elements

$$(v^k, \bar{v}^k), (v, \bar{v}) : (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) \longrightarrow (\Omega^*, \tau)$$

on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ possibly different from $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfy (2.15). Of course, we now have

$$(v, \bar{v}) \stackrel{d}{=} (u, \bar{u}).$$

Consider the same filtration $\bar{\mathbf{F}} = (\bar{\mathcal{F}}_t)_{t \leq T}$ on $\bar{\Omega}$ as in (2.16). For fixed $\phi \in C_c^\infty((0, \pi))$, define the $\bar{\mathbf{F}}$ -adapted processes

$$\begin{aligned} \bar{X}_t^k &= \langle \bar{v}_t^k, \phi \rangle - \int_0^t \int_0^\pi v^k(s, x) \phi''(x) \, ds \, dx, \\ \bar{X}_t &= \langle \bar{v}_t, \phi \rangle - \int_0^t \int_0^\pi v(s, x) \phi''(x) \, ds \, dx \end{aligned} \tag{3.48}$$

for all $k \in \mathbb{N}$ and $t \leq T$. It is straightforward to infer from (2.15) that pointwise on $\bar{\Omega}$,

$$\bar{X}^k \longrightarrow \bar{X} \quad \text{in } D([0, T], \mathbb{R}) \quad \text{as } k \rightarrow \infty. \tag{3.49}$$

Furthermore, by (2.15) and (3.48), \bar{X}^k and \bar{X} have the same distribution as the square-integrable $\bar{\mathbf{F}}$ -martingales

$$t \mapsto \int_0^t \int_0^\pi \frac{f(u^{\varepsilon_k}(s, x))}{\sigma(\varepsilon_k)} \phi(x) L^{\varepsilon_k}(ds, dx) \quad \text{and} \quad t \mapsto \int_0^t \int_0^\pi f(u(s, x)) \phi(x) W(ds, dx),$$

respectively, and therefore, by standard arguments, we can deduce that \bar{X}^k and \bar{X} are $\bar{\mathbf{F}}$ -martingales and that \bar{X} is continuous.

Recall the truncation function ϱ_h introduced in (3.22). Using Theorem II.2.21 in [14], we can further infer that the $\bar{\mathbf{F}}$ -semimartingale characteristics of \bar{X}^k and \bar{X} , relative to ϱ_h for a fixed

but arbitrary $h > 0$, are given by $(\overline{B}^{k,h}, 0, \overline{\nu}^k)$ and $(0, \overline{C}, 0)$, respectively, where $\overline{\nu}^k$ is defined as in (3.20), \overline{C} is defined as in (2.17) and

$$\overline{B}_t^{k,h} = - \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x|>h\}} \overline{\nu}^k(ds, dx), \quad t \leq T. \quad (3.50)$$

Define $\overline{X}^k(\varrho_h) = \overline{X}^k - \sum_{s \leq \cdot} \Delta \overline{X}_s^k \mathbb{1}_{\{|\Delta \overline{X}_s^k|>h\}}$ for all $k \in \mathbb{N}$. Then we have, by definition of the first characteristic,

$$\overline{X}^k(\varrho_h) = \overline{M}^{k,h} + \overline{B}^{k,h}, \quad (3.51)$$

where $\overline{M}^{k,h}$ is a local $\overline{\mathbf{F}}$ -martingale.

Now since \overline{X} is continuous, Proposition VI.2.7 in [14] and (3.49) imply that $\overline{\omega}$ -wise,

$$\overline{X}^k(\varrho_h) \longrightarrow \overline{X} \quad \text{in } D([0, T], \mathbb{R}) \quad \text{as } k \rightarrow \infty. \quad (3.52)$$

We also have

$$\overline{\nu}^k([0, t] \times \{|x| > a\}) \xrightarrow{\overline{\mathbb{P}}} 0 \quad \text{as } k \rightarrow \infty \quad (3.53)$$

for any $t \leq T$ and $a > 0$ by Proposition VI.3.26 and Lemma VI.4.22 in [14]. Therefore, there exists a subsequence of $(\overline{\nu}^k([0, T] \times \{|x| > h\}))_{k \in \mathbb{N}}$ converging $\overline{\mathbb{P}}$ -almost surely to 0. For the sake of clarity, assume without loss of generality that this holds for the whole sequence. Applying the Cauchy–Schwarz inequality to $\overline{B}^{k,h}$ in (3.50), we further deduce that

$$\begin{aligned} \sup_{t \leq T} |\overline{B}_t^{k,h}|^2 &\leq \left(\int_0^T \int_0^\pi \int_{\mathbb{R}} \frac{f^2(v^k(t, x))}{\sigma^2(\varepsilon_k)} \phi^2(x) z^2 dt dx Q^{\varepsilon_k}(dz) \right) \overline{\nu}^k([0, T] \times \{|x| > h\}) \\ &\leq C \overline{\nu}^k([0, T] \times \{|x| > h\}) \left(1 + \sup_{k \in \mathbb{N}} \int_0^T \int_0^\pi v^k(t, x)^2 dt dx \right) \end{aligned}$$

and the last term converges $\overline{\mathbb{P}}$ -almost surely to 0 (note that the supremum is finite because $v^k \rightarrow v$ in $L^2([0, T] \times [0, \pi])$). This implies

$$\overline{B}^{k,h} \longrightarrow 0 \quad \text{in } D([0, T], \mathbb{R}) \quad \text{as } k \rightarrow \infty \quad (3.54)$$

$\overline{\mathbb{P}}$ -almost surely. Using Proposition VI.1.23 in [14], (3.51), (3.52) and (3.54), we obtain

$$\overline{M}^{k,h} \longrightarrow \overline{X} \quad \text{in } D([0, T], \mathbb{R}) \quad \text{as } k \rightarrow \infty$$

as well as

$$(\overline{M}^{k,h}, -2\overline{M}^{k,h}, (\overline{M}^{k,h})^2) \longrightarrow (\overline{X}, -2\overline{X}, \overline{X}^2) \quad \text{in } D([0, T], \mathbb{R}^3) \quad \text{as } k \rightarrow \infty \quad (3.55)$$

$\overline{\mathbb{P}}$ -almost surely. Since the jumps of $\overline{M}^{k,h}$ are uniformly bounded by h , we can apply Proposition VI.6.13 in [14] on the sequence $(\overline{M}^{k,h})_{k \in \mathbb{N}}$ and then Theorem VI.6.22 (c) in [14] on the processes in (3.55) in order to obtain

$$\left(\overline{M}^{k,h}, -2\overline{M}^{k,h}, (\overline{M}^{k,h})^2, -2 \int_0^\cdot \overline{M}_s^{k,h} \overline{M}^{k,h}(ds) \right) \xrightarrow{\overline{\mathbb{P}}} \left(\overline{X}, -2\overline{X}, \overline{X}^2, -2 \int_0^\cdot \overline{X}_s \overline{X}(ds) \right)$$

in $D([0, T], \mathbb{R}^4)$ as $k \rightarrow \infty$. By definition of the quadratic variation, we can therefore deduce that

$$(\overline{M}^{k,h}, [\overline{M}^{k,h}, \overline{M}^{k,h}]) \xrightarrow{\overline{\mathbb{P}}} (\overline{X}, \overline{C}) \quad \text{in } D([0, T], \mathbb{R}^2) \quad \text{as } k \rightarrow \infty. \quad (3.56)$$

Denoting by $\bar{\mu}^k$ the jump measure of \bar{X}^k , we have, since $\bar{B}^{k,h}$ is continuous,

$$[\bar{M}^{k,h}, \bar{M}^{k,h}]_t = \int_0^t \int_{\mathbb{R}} x^2 \mathbb{1}_{\{|x| \leq h\}} \bar{\mu}^k(ds, dx), \quad t \leq T. \quad (3.57)$$

Now denote for any $k \in \mathbb{N}$,

$$\begin{aligned} \tilde{C}_t^k &= \int_0^t \int_{\mathbb{R}} x^2 \mathbb{1}_{\{|x| \leq h\}} \bar{\nu}^k(ds, dx) \quad \text{and} \\ \bar{Y}_t^k &= [\bar{M}^{k,h}, \bar{M}^{k,h}]_t - \tilde{C}_t^k = \int_0^t \int_{\mathbb{R}} x^2 \mathbb{1}_{\{|x| \leq h\}} (\bar{\mu}^k - \bar{\nu}^k)(ds, dx). \end{aligned} \quad (3.58)$$

Then \bar{Y}^k is a square-integrable $\bar{\mathbf{F}}$ -martingale with $|\Delta \bar{Y}^k| \leq h$, and for any bounded stopping time T , we have, by the optional stopping theorem, $\mathbb{E}[(\bar{Y}_T^k)^2] \leq \mathbb{E}[[\bar{Y}^k, \bar{Y}^k]_T]$. Therefore, by Lenglart's inequality (see Lemma I.3.30 in [14]), we obtain for all $\delta > 0$ and $\eta > 0$,

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t} |\bar{Y}_s^k|^2 \geq \delta\right) &\leq \frac{1}{\delta} \left(\eta + \mathbb{E}\left[\sup_{s \leq t} \Delta[\bar{Y}^k, \bar{Y}^k]_s\right] \right) + \mathbb{P}([\bar{Y}^k, \bar{Y}^k]_t \geq \eta) \\ &\leq 2\frac{\eta}{\delta} + \left(\frac{h}{\delta} + 1\right) \mathbb{P}([\bar{Y}^k, \bar{Y}^k]_t \geq \eta). \end{aligned} \quad (3.59)$$

By (3.57), we have

$$[\bar{Y}^k, \bar{Y}^k]_t = \int_0^t \int_{\mathbb{R}} x^4 \mathbb{1}_{\{|x| \leq h\}} \bar{\mu}^k(ds, dx) \leq \left(\sup_{s \leq t} |\Delta[\bar{M}^{k,h}, \bar{M}^{k,h}]_s|\right) [\bar{M}^{k,h}, \bar{M}^{k,h}]_t.$$

Moreover, because $[\bar{M}^{k,h}, \bar{M}^{k,h}]_t \xrightarrow{\bar{\mathbb{P}}} \bar{C}_t$ by (3.56) and $\sup_{s \leq t} |\Delta[\bar{M}^{k,h}, \bar{M}^{k,h}]_s| \xrightarrow{\bar{\mathbb{P}}} 0$ by Proposition VI.3.26 (iii) in [14], we deduce from the inequality above that $[\bar{Y}^k, \bar{Y}^k]_t \xrightarrow{\bar{\mathbb{P}}} 0$ and, by (3.59), that

$$\sup_{s \leq t} |\bar{Y}_s^k| \xrightarrow{\bar{\mathbb{P}}} 0 \quad \text{as } k \rightarrow \infty \quad (3.60)$$

for all $t \leq T$. Finally, combine (3.56), (3.58) and (3.60) to see that

$$\tilde{C}_t^k = \int_0^t \int_{\mathbb{R}} x^2 \mathbb{1}_{\{|x| \leq h\}} \bar{\nu}^k(ds, dx) \xrightarrow{\bar{\mathbb{P}}} \bar{C}_t \quad \text{as } k \rightarrow \infty \quad (3.61)$$

for all $t \leq T$ and $h > 0$. Taking a subsequence if necessary, we will from now on assume that the convergence in (3.61) holds even $\bar{\mathbb{P}}$ -almost surely.

Recall now the definition of $\Sigma_h^k(s, x)$ in (3.24) and that, because $v^k \rightarrow v$ in $L^2([0, T] \times [0, \pi])$, we have $\int_0^t \int_0^\pi |f^2(v^k(s, x)) - f^2(v(s, x))| \phi^2(x) ds dx \rightarrow 0$ as $k \rightarrow \infty$; see the calculations in (3.26) and (3.27). Together with (3.61) this implies $\bar{\mathbb{P}}$ -almost surely,

$$\int_0^T \int_0^\pi f^2(v(s, x)) \phi^2(x) \Sigma_h^k(s, x) ds dx \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (3.62)$$

for all $h > 0$ and $\phi \in C_c^\infty((0, \pi))$ by a similar calculation as in (3.25) (note that the first inequality there becomes an equality if $|\cdot|$ is replaced by (\cdot) throughout).

Now on the set $\{|f(v^k(s, x))\phi(x)| \geq \delta\}$, where $\delta > 0$, we have

$$\mathbb{1}_{\{|z| \geq (h/|f(v^k(s, x))\phi(x)|)\sigma(\varepsilon_k)\}} \geq \mathbb{1}_{\{|z| \geq (h/\delta)\sigma(\varepsilon_k)\}},$$

and thus

$$\Sigma_h^k(s, x) \geq \frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \geq (h/\delta)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz).$$

Therefore, as a consequence of (3.62), we obtain $\bar{\mathbb{P}}$ -almost surely,

$$\begin{aligned} & \frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \geq (h/\delta)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \\ & \times \int_0^T \int_0^\pi f^2(v(s, x)) \phi^2(x) \mathbb{1}_{\{|f(v^k(s, x))\phi(x)| \geq \delta, |f(v(s, x))\phi(x)| > \delta\}} ds dx \longrightarrow 0 \end{aligned} \quad (3.63)$$

as $k \rightarrow \infty$ for all $h > 0$, $\delta > 0$ and $\phi \in C_c^\infty((0, \pi))$.

We have seen in (3.36) that we can assume (perhaps for a subsequence) that

$$v^k \longrightarrow v \quad \text{as } k \rightarrow \infty \quad \bar{\mathbb{P}} \otimes \text{Leb}_{[0, T] \times [0, \pi]} \text{-almost everywhere,}$$

which implies, by dominated convergence and continuity of f ,

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_0^\pi f^2(v(s, x)) \phi^2(x) \mathbb{1}_{\{|f(v^k(s, x))\phi(x)| \geq \delta, |f(v(s, x))\phi(x)| > \delta\}} ds dx \right] \\ & \longrightarrow \mathbb{E} \left[\int_0^T \int_0^\pi f^2(v(s, x)) \phi^2(x) \mathbb{1}_{\{|f(v(s, x))\phi(x)| > \delta\}} ds dx \right] \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (3.64)$$

So from (2.15), (3.63) and (3.64), we deduce that

$$\begin{aligned} & \frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \geq (h/\delta)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \\ & \times \mathbb{E} \left[\int_0^T \int_0^\pi f^2(u(s, x)) \phi^2(x) \mathbb{1}_{\{|f(u(s, x))\phi(x)| > \delta\}} ds dx \right] \longrightarrow 0 \quad \text{as } k \rightarrow \infty \end{aligned} \quad (3.65)$$

for all $h > 0$, $\delta > 0$ and $\phi \in C_c^\infty((0, \pi))$. Moreover,

$$\begin{aligned} & \left(\frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \geq (h/\delta)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \right) \mathbb{E} \left[\int_0^T \int_0^\pi f^2(u(s, x)) \phi^2(x) ds dx \right] \\ & \leq \left(\frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \geq (h/\delta)\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \right) \\ & \times \mathbb{E} \left[\int_0^T \int_0^\pi f^2(u(s, x)) \phi^2(x) \mathbb{1}_{\{|f(u(s, x))\phi(x)| > \delta\}} ds dx \right] + T\pi\delta^2. \end{aligned} \quad (3.66)$$

So if we choose $h = \kappa\delta$ with $\kappa > 0$ arbitrary, then by (3.65), the first term on the right-hand side of (3.66) converges to 0 as $k \rightarrow \infty$ for all $\kappa > 0$ and $\delta > 0$. The second term does not depend on k nor h and converges to 0 as $\delta \rightarrow 0$. This implies

$$\left(\frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \geq \kappa\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \right) \mathbb{E} \left[\int_0^T \int_0^\pi f^2(u(s, x)) \phi^2(x) ds dx \right] \longrightarrow 0 \quad (3.67)$$

as $k \rightarrow \infty$ for all $\kappa > 0$. Since $f(0) \neq 0$, there exists $(t_1, x_1) \in [0, T] \times [0, \pi]$ such that $\mathbb{E}[f^2(u(t_1, x_1))] > 0$ by Remark 3.14. Moreover, the mild solution u is continuous in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, which follows from the proof of Corollary 3.4 in [27]. We can thus infer that the expectation in (3.67) is not 0 and we obtain

$$\frac{1}{\sigma^2(\varepsilon_k)} \int_{\mathbb{R}} z^2 \mathbb{1}_{\{|z| \geq \kappa\sigma(\varepsilon_k)\}} Q^{\varepsilon_k}(dz) \longrightarrow 0 \quad \text{as } k \rightarrow \infty$$

for all $\kappa > 0$, which is exactly (1.4). \square

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