Fakultät für Mathematik
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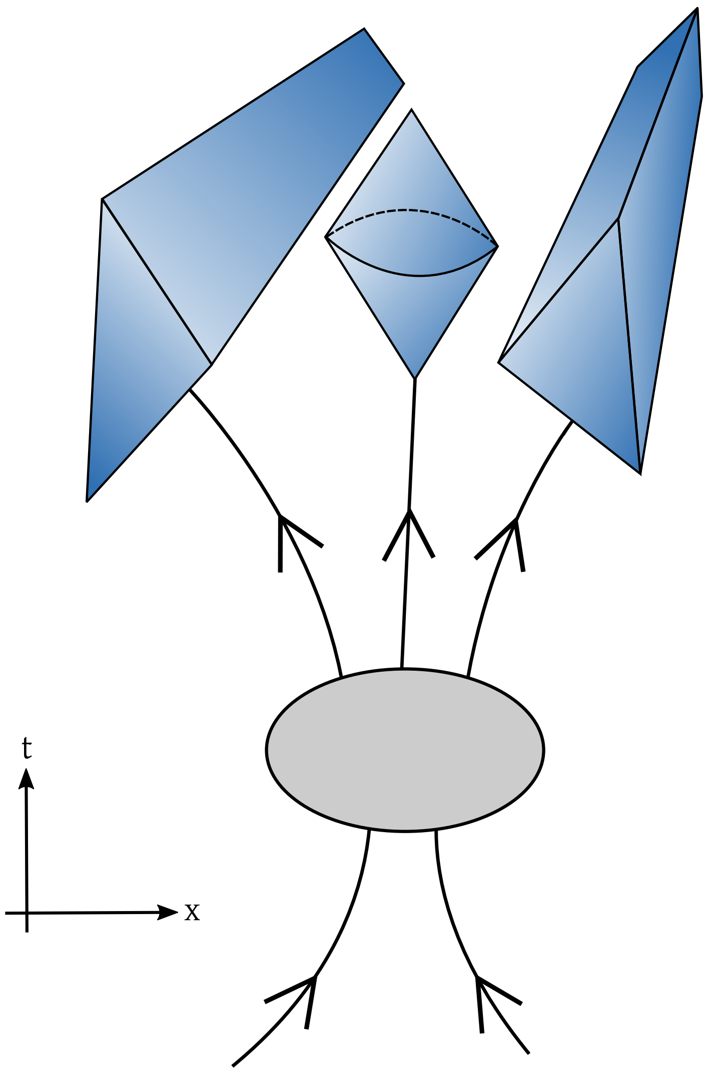
Non-local methods in Haag-Ruelle scattering theory

Maximilian Duell



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Maximilian Duell

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Vorsitzende:
Prof. Dr. Simone Warzel

Prüfer der Dissertation:
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Abstract

The main aim of the present thesis is the rigorous scattering-theoretic analysis of quantum field theory models, which are beyond the scope of the classical Haag-Ruelle theory. In the first part we consider wedge-local quantum field theories, which have been intensely studied in the recent literature. In the general wedge-local case only two-particle scattering states have been constructed and states of three or more particles have not been considered physically meaningful for geometrical reasons. In the present work we prove the convergence and Fock structure of velocity-ordered scattering states with an arbitrary number of particles relying only on the mass gaps and wedge duality. The development of wedge-local N-particle scattering theory also provides the means for future studies of asymptotic completeness in various recently constructed interacting wedge-local models. Our wedge-local investigations conclude with the definition of multi-particle wave operators and scattering data, and an analysis of the asymptotic action of space-time symmetries in general wedge-local quantum field theories.

In the second part we consider the scattering problem in presence of massless particles. In this case the scattering theoretic analysis becomes technically more challenging already in local quantum field theory. We construct Haag-Ruelle scattering states of massive Wigner particles via non-local Reeh-Schlieder vacuum correlation effects, developing the required non-equal-time commutator and clustering estimates, together with suitable energy norm bounds. We expect that this method applies for example to neutral particles such as hydrogen atoms as described within quantum electrodynamics. Our strategy complements the previously used approach of I. Herbst involving spectral regularity conditions, whose physical status is not clear. The required strengthened form of the Reeh-Schlieder property has been verified in simple non-interacting models. The status of this condition in the context of the Herbst regularity condition and in integrable quantum field theories is also discussed.

Zusammenfassung

Ziel dieser Arbeit ist die streutheoretische Untersuchung von Modellen der Quantenfeldtheorie, welche bei mathematisch strenger Herangehensweise jenseits der Anwendbarkeit der klassischen Haag-Ruelle Theorie liegen. Im ersten Teil betrachten wir Keil-lokale Quantenfeldtheorien, die in den letzten Jahren in der Literatur intensiv erforscht wurden. In solchen Theorien wurden jedoch nur zwei-Teilchen Streuzustände konstruiert. Die Existenz und physikalische Bedeutung von Zustände mit drei oder mehr Teilchen konnte angesichts Keil-geometrischer Einschränkungen bisher nicht geklärt werden. In der vorliegenden Arbeit beweisen wir die Konvergenz und Fock-Struktur von geschwindigkeitsgeordneten Streuzuständen mit beliebiger Teilchenzahl allein auf Basis der Massen-Lücken und Keil-Dualität in allgemeinen Keil-lokalen Theorien. Die Entwicklung der Keil-lokalen N-Teilchen Streutheorie ermöglicht auch zukünftige Untersuchungen der asymptotischen Vollständigkeit verschiedener wechselwirkender Keil-lokaler Modelle, die in jüngerer Zeit konstruiert wurden. Unsere Keil-lokalen Betrachtungen schließen mit der Definition von Wellenoperatoren und Streudaten und einer allgemeinen Analyse der asymptotischen Wirkung der Raumzeit-Symmetrien in Keil-lokalen Quantenfeldtheorien.

Im zweiten Teil betrachten wir das Streuproblem bei Anwesenheit masseloser Teilchen. In diesem Fall ist die streutheoretische Analyse bereits in der lokalen Quantenfeldtheorie technisch anspruchsvoller. Wir konstruieren Haag-Ruelle Streuzustände massiver Teilchen mittels spezifischer nicht-lokaler Reeh-Schlieder Vakuumkorrelationseffekte und entwickeln sowohl die erforderlichen asynchronen Kommutator- und Clustering-Abschätzungen, als auch geeignete Energienormschranken. Wir erwarten, dass die Methode bei ungeladenen Teilchen, beispielsweise dem Wasserstoffatom im Rahmen der Quantenelektrodynamik, anwendbar ist. Sie ist komplementär zu der von I. Herbst entwickelten Herangehensweise mittels spektraler Regularitätsbedingungen, auf der alle vorherigen Konstruktionen für Teilchen mit eingebetteter Massenschale basieren, obgleich deren physikalische Interpretation noch nicht geklärt werden konnte. Die verwendete starke Form der Reeh-Schlieder Bedingung wurde in einfachen nicht-wechselwirkenden Modellen verifiziert, und wir diskutieren Varianten dieser Annahme für integrable Modelle der Quantenfeldtheorie und im Kontext der Herbst'schen Spektralbedingung.

Contents

1 Introduction

1.1 Scientific context and motivations for wedge-local quantum field theory

Quantum field theory (QFT) is the standard framework of theoretical physics to describe nature at shortest distance scales and high energies. These regimes are accessible via scattering experiments, such as the current generation of the LHC (Large Hadron Collider) at CERN. Theoretical physicists have devised successful computational schemes based on quantum field theory such as perturbation theory or lattice discretizations, which provide deep explanations and predictions for various experimental results in this context. On the other hand there exist only very few non-perturbative QFT models which have been established with full mathematical rigour, and these examples are quite far from experimental reality. This is a rather unsatisfactory situation considering that almost a century has passed since the inception of the quantum theory of fields.

The most prominent rigorously studied non-perturbative interacting QFT models are simplified theories defined on lower-dimensional space-times [GJ], where the high-energy behaviour of quantum fields is less singular. This has demonstrated that the fundamental objective of QFT, the unification of quantum mechanics and special relativity, is mathematically consistent with non-trivial interactions. However for the various interacting QFT models relevant in high-energy physics, there are presently no rigorous constructions with full mathematical control [Sum12]. The mathematical challenges are so formidable, that we are in fact lacking any rigorously established local and non-perturbative QFT models which display interactions on four-dimensional Minkowski space-time.[1]

In more recent works of Grosse, Lechner, Buchholz, and Summers, novel construction methods have been developed, which yield non-trivial relativistic QFT-like models for space-times of arbitrary dimension, including the physical case $d = 3 + 1$ [GL07; BS08; BLS11]. These models are non-perturbative, but the physical notion of localization in bounded space-time regions is not manifest in the construction and there is some evidence against strict locality of the models in dimension $d > 1 + 1$ [BLS11]. A distinguishing feature of this approach is that an important weakened localization concept is available: there exist numerous quantum mechanical observables which are localized in certain unbounded *wedge*-shaped space-time regions. Due to the weaker notion of localization, the current knowledge about physical applications and interpretation of such wedge-local models is still very limited. Yet it has been verified that these models are interacting in the sense of *scattering theory*, but due to geometrical limitations arising from wedge locality this claim was so far restricted to two-particle scattering processes [GL07; BS08].

[1] Concrete lattice or perturbative schemes used for physical predictions can indeed be put onto rigorous footings, but typically important physical properties are lost in these approaches. For example causality and relativistic symmetry are broken in lattice models, and in perturbation theory the construction of physical states and their Hilbert-space structure is problematic. Recent examples for such mathematical endeavours are the perturbative AQFT approach [FR15], and the non-perturbative construction of lattice QCD [GR17].

1.2 The scattering problem in wedge-local QFT

The main objective of this thesis is to establish and study multi-particle scattering theory for general relativistic QFT models, with special emphasis on the modern wedge-local perspective. Scattering theory in general is concerned with the qualitative and quantitative behaviour of a given dynamics, considering the limits of large times (and large distances). In mathematical scattering theory the basic task is to establish the relation of a given *interacting dynamics* to a suitable *comparison dynamics* at asymptotic times $\tau \to \pm\infty$. This limiting regime is practically relevant for physics, as scattering occurs in the standard experimental situation that a specified configuration of non-interacting particles is prepared in the far past to collide in a fixed space-time region, where the physical interactions determine the configurations of particles that can be measured in the far future. Another important physical aspect of scattering situations is that they should feature interactions which are small in magnitude and typically take place only within a small spatial region during a short time frame. For many approximative computations in physics it is relevant that outside of the interaction region, one can describe the time evolution by a comparison dynamics which is usually non-interacting or explicitly solvable [RS3, Sec. XI.1].

The above scattering-theoretic perspective can also be seen as motivation and justification for perturbative methods, where the interactions are treated as small correction to some non-interacting model. More importantly, from the conceptual and axiomatic perspective, the rigorous interpretation of interacting QFT in terms of particles is in fact obtained via scattering theory[2]. For these reasons the study of scattering has deep roots within the development of QFT [LSZ55; Ha58; Ru62; Hep65]. On the other hand, the first application of Haag-Ruelle theory in a general wedge-local setting is more recent and the conventional approach yields only two-particle scattering states [BBS01; GL07; BS08]. Our main contribution is a well-defined wedge-local scattering theory for scattering states composed of arbitrarily many massive particles. Another interesting problem we will consider is the scattering theory for models with *embedded* mass shells. This situation can appear if a given theory describes massless particles, or multiple massive particles. While the purely massless situation is under complete control by the results of Buchholz [Bu77], it is possible to approach the massive embedded case via different strategies [Hrb71; Dy05; Hrd13; DH14; Du17] and some mathematically interesting open questions remain.

The physical significance of such scattering-theoretic results lie in their implications for the experimental interpretation of a given QFT model: if well-defined scattering states can be constructed, they may be used to define wave operators and the scattering matrix. The S-matrix is defined via the scalar product between outgoing and incoming scattering states, and interactions lead to deviation of S from the identity map. Therefore the S-matrix provides the basic criterion to distinguish interacting and non-interacting theories. Further the S-matrix also plays an important role in applications. Namely, it yields the quantitative scattering amplitudes, scattering probabilities, and scattering cross sections as they are observed in experiments [IZ, Ch. 5].

After successfully establishing a scattering theory for a class of models, many new questions with important physical motivations arise. For example the asymptotic data constrain the possible mathematical structure of a model, and vice versa any space-time or internal symmetries present in a theory are expected to yield corresponding symmetries of the scattering data. In this regard we will investigate and make more precise earlier claims that wedge-local models can have broken Lorentz symmetry on the level of the scattering states, even if the model itself

[2] More precisely, particles can also arise in other limiting regimes such as the short-distance renormalization group analysis [BV95; BV98], cf. the discussions in [Bu93].

is Poincaré covariant. This effect is reflected in a corresponding non-invariance of the S-matrix, which has been demonstrated in [GL07; BS08].

But even in models where the scattering data can be worked out explicitly, it is not guaranteed that the asymptotic analysis captures all physical aspects of the interacting dynamics. Here a deep and technically very difficult, but conceptually basic, question is whether there exist states which cannot be uniquely described by specifying their asymptotic particle content. This property of *asymptotic completeness* (AC) is well understood in non-relativistic quantum mechanics [DeG1], but remains largely open in relativistic quantum field theory [CD82; Le08; DyG13]. It may be regarded as the physical interpretation of AC that the evolution by the dynamics under consideration will, after sufficiently long time, disintegrate any state of the interacting system into suitable "elementary constituents" (particles). This picture aligns particularly well with the experiences and expectations gained over the years from the study of free and perturbative quantum field theory. Yet a sufficiently general mathematical theory of asymptotic completeness in non-perturbative interacting quantum field theories apparently still needs to be developed [Bu93; Dy10]. Wedge-local models can help shed some light on such questions by providing rigorous testing grounds and pedagogically valuable examples and counterexamples. In addition there are various mathematically more tractable and more explicitly describable wedge-local interacting models, many of them related to integrable quantum field theories. The first full asymptotic completeness results for a family of local interacting relativistic quantum field theories has in fact been achieved only recently by means of wedge-local operator-algebraic methods [Le08].

Finally let us note that in asymptotically complete models all information about the interacting system is in principle accessible from scattering experiments. On the other hand asymptotic completeness does not imply that the interacting dynamics is already fixed by the asymptotic data. This uniqueness question falls into the domain of the *Inverse Scattering Problem*, which has also received past and recent attention in QFT [BF77; Le08; Tan12; AL17], but which will not be treated here. These are some examples how the scattering theoretic perspective leads to various interesting further physical and mathematical questions in quantum field theory, which remain to be answered.

1.3 Wedge-locality in quantum field theory

1.3.1 Operator-algebraic formulation of local and wedge-local QFT

The mathematical settings of our investigations are the Haag-Kastler framework [HK64; A; Ha] and its wedge-local variant [Bor92; BLS11]. They provide an axiomatic approach to relativistic quantum theory emphasizing the algebraic structure of physical observables and describe their space-time localization. A central notion in this formulation is the basic physical principle of *locality*, which states that measurements or more generally physical operations which are carried out in separated space-time regions must be independent.

The algebraic viewpoint is mathematically based on the work of Stone and von Neumann, and builds upon the basic postulate that physical observables generate a unital $*$-algebra $\mathfrak{A}$. In the operator-algebraic formulation, the concept of localization of observables $A \in \mathfrak{A}$ corresponds to an additional algebraic structure

$$\mathfrak{A} \supset \bigcup_{\mathcal{O}\in\mathbf{Reg}} \mathfrak{A}(\mathcal{O}) =: \mathfrak{A}_{\text{loc}}, \tag{1.3.1}$$

where **Reg** denotes an admissible family of space-time regions $\mathcal{O} \subset \mathbb{R}^{s+1}$ and for each $\mathcal{O} \in \mathbf{Reg}$ there is an associated $*$-algebra $\mathfrak{A}(\mathcal{O})$, whose elements are interpreted as localizable in $\mathcal{O} \in \mathbf{Reg}$. This physical interpretation also relies on the axioms of **Isotony**,

$$\forall \mathcal{O}_1, \mathcal{O}_2 \in \mathbf{Reg} : \mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2), \tag{HK1}$$

and **Locality**,

$$\forall \mathcal{O}_1, \mathcal{O}_2 \in \mathbf{Reg} : \mathcal{O}_1 \subset \mathcal{O}_2' \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'. \tag{HK2}$$

Here $\mathfrak{A}(\mathcal{O}_2)' := \{B \in \mathfrak{A} : [A, B] := AB - BA = 0 \ \forall A \in \mathfrak{A}(\mathcal{O}_2)\}$ denotes the commutant of $\mathfrak{A}(\mathcal{O}_2)$ relative to $\mathfrak{A}$ and the *causal complement* $\mathcal{O}'$ (more precisely: the *open causal complement*) of $\mathcal{O} \subset \mathbb{R}^{s+1}$ is defined as

$$\mathcal{O}' := \{y \in \mathbb{R}^{s+1} : \sup_{x \in \mathcal{O}} d_M(x, y) < 0\}, \tag{1.3.2}$$

with respect to the Minkowski causal distance of two space-time points $x_k = (t_k, \mathbf{x}_k) \in \mathbb{R}^d$, $(k = 1, 2)$,

$$d_M(x_1, x_2) := (t_2 - t_1)^2 - (\mathbf{x}_2 - \mathbf{x}_1)^2. \tag{1.3.3}$$

Further it is of interest to keep the dimension of space-time $d := s + 1$ general, $s \in \mathbb{N}$. This makes it possible to apply our results also for the analysis of lower-dimensional constructive efforts and displays the independence of our scattering-theoretic investigations regarding the space-time dimension.

At this point let us briefly remark that QFTs describe physical systems with an infinite number of degrees of freedom, so that any meaningful mathematical analysis will require that the localizable algebras $\mathfrak{A}(\mathcal{O})$ and the global algebra $\mathfrak{A}$ are topological $*$-algebras and complete. We adopt the standard setting, where $\mathfrak{A}(\mathcal{O})$ are taken to be von Neumann algebras. In the present context and in most concrete examples there is a distinguished representation on a Hilbert space $\mathscr{H}$ such that $\mathfrak{A}(\mathcal{O}) \subset \mathrm{B}(\mathscr{H})$ are *-subalgebras which are closed with respect to the weak operator topology, i.e., they are jointly represented as concrete von Neumann subalgebras of $\mathrm{B}(\mathscr{H})$.

For a *strictly local* theory one demands that its physical content is completely accessible from observations within bounded regions. At the present level one may take $\mathfrak{A}(\mathcal{O})$ to be defined a priori only for regions from

$$\mathbf{Reg}_b := \{\mathcal{O} \in \mathbb{R}^{s+1} : \mathcal{O} \text{ open and bounded}\}. \tag{1.3.4}$$

Additionally the algebra of all observables $\mathfrak{A}$ is then taken as the quasilocal algebra $\overline{\mathfrak{A}_{\mathrm{loc}}}$, where the closure is taken with respect to the C^*-norm topology of the inductive limit $\cup_{\mathcal{O} \in \mathbf{Reg}} \mathfrak{A}(\mathcal{O})$. If necessary or convenient it is also possible to make more restrictive choices for **Reg**. For example a further restriction to contractible regions is appropriate in theories with (global) gauge symmetries [Ha, Secs. III.3.3, III.4.2]. Extending a given net $\mathfrak{A}$ to a larger class of regions $\mathbf{Reg}_{\mathrm{ext}} \supset \mathbf{Reg}$ can then be accomplished e.g. by setting for $\mathcal{O} \in \mathbf{Reg}_{\mathrm{ext}}$

$$\mathfrak{A}_{\mathrm{ext}}(\mathcal{O}) := \Big(\bigcup_{\substack{\mathcal{O}_1 \in \mathbf{Reg}, \\ \text{s.t. } \mathcal{O}_1 \subset \mathcal{O}}} \mathfrak{A}(\mathcal{O}_1) \Big)''. \tag{1.3.5}$$

Mathematically the above construction also makes sense for unbounded regions. An important class of unbounded regions in quantum field theory is given by the Rindler wedges.

Definition 1.3.1 (wedge regions)**.** *The family of* wedge regions *is defined as the orbit*

$$\mathbf{Reg}_W(\mathbb{R}^d) := \{\lambda\mathcal{W}_\mathrm{r} \subset \mathbb{R}^d : \lambda \in \mathcal{P}\}, \tag{1.3.6}$$

under the canonical action of the Poincaré group $\mathcal{P} = \mathbb{R}^{s+1} \rtimes \mathcal{L}$ of the standard Rindler wedge[3]

$$\mathcal{W}_\mathrm{r} := \{x \in \mathbb{R}^d : |x^0| < x^1\}. \tag{1.3.7}$$

Remark 1.3.2. For $d > 1+1$ one may restrict λ in the definition of $\mathbf{Reg}_W$ to the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{s+1} \rtimes \mathcal{L}_+^\uparrow$. The case $d = 2$ is exceptional with

$$\mathbf{Reg}_W(\mathbb{R}^2) := \{\pm\mathcal{W}_\mathrm{r} + a \subset \mathbb{R}^2 : a \in \mathbb{R}^2\}. \tag{1.3.8}$$

For the present introduction let us just note two simple geometric advantages of the wedge-local perspective. The first point concerns space-time symmetries and their action on **Reg** and its regions. For the case of flat Minkowski space as considered in the present thesis, the geometric space-time symmetry is given by the Poincaré transformations $\lambda = (a, \Lambda) \in \mathcal{P} = \mathbb{R}^{s+1} \rtimes \mathcal{L}$, $\lambda x := \Lambda x + a$. The Lorentz group $\mathcal{L} = O(1, s) \subset \mathrm{GL}_{s+1}(\mathbb{R})$ contains in particular spatial rotations, space-, and time-inversions, and the standard Lorentz boosts

$$\Lambda^\beta x = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & \dots & 0 \\ \sinh\beta & \cosh\beta & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} x^0 \\ x^1 \\ \\ \vdots \\ \\ x^s \end{pmatrix} \quad \text{for } \beta \in \mathbb{R}. \tag{1.3.9}$$

It is easily seen that $\mathbf{Reg}_b$ is invariant under the action of the Poincaré group $\mathcal{P}$ and similarly $\mathbf{Reg}_W$ is invariant under $\mathcal{P}$ by construction. Mathematically it is further natural to consider space-time regions which are highly symmetric with respect to the action of the Poincaré group themselves. The standard double cones

$$\mathscr{C}_R := \{(t, \mathbf{x}) \in \mathbb{R}^{s+1} : |t| + |\mathbf{x}| < R\}, \quad (R > 0), \tag{1.3.10}$$

are an important example of a family of bounded regions which is invariant under the full rotation group $\mathrm{SO}(s) \subset \mathcal{L}$ and which is causally complete in the sense that $\mathscr{C}_R'' = \mathscr{C}_R$. It is clear that a given bounded region $\mathcal{O} \in \mathbf{Reg}_b$ can at best be invariant under a subgroup of the Poincaré group, which in particular cannot contain any translations or boosts. Any wedge-region $\mathcal{W} \in \mathbf{Reg}_W$ on the other hand is mapped into itself by a semigroup of translations and a subgroup of boosts.

The second geometrical motivation for considering wedge regions is their symmetry under causal complements. Namely, for $\mathcal{W} \in \mathbf{Reg}_W$ we have that $\mathcal{W}' = (\lambda\mathcal{W}_\mathrm{r})' = (\Lambda\mathcal{W}_\mathrm{r} + a)' = -\Lambda\mathcal{W}_\mathrm{r} + a$ is congruent to $\mathcal{W}$. Bounded regions $\mathcal{O} \in \mathbf{Reg}_b$ however have unbounded $\mathcal{O}'$, so that a family of bounded regions such as $\mathbf{Reg}_b$ is never closed under causal complements. We summarize:

Proposition 1.3.3 (elementary properties of $\mathbf{Reg}_W$)**.** *Let $\mathcal{W} \in \mathbf{Reg}_W$. Then*

(i) $\mathcal{W} = \mathcal{W}''$,

[3] In the literature $\mathcal{W}_\mathrm{r}$ is also sometimes called the *right wedge, reference wedge,* [GL07] or *standard wedge* [BS08].

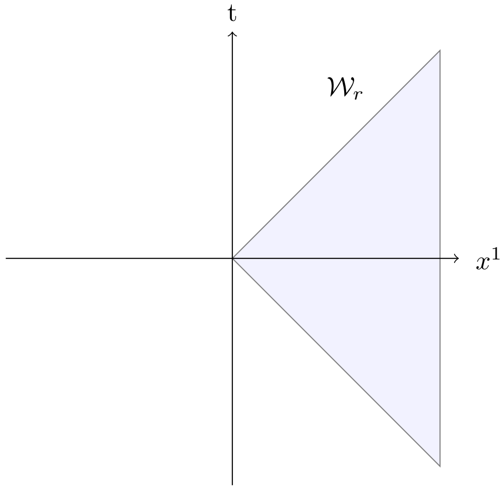

Figure 1.1: The reference wedge $\mathcal{W}_\mathrm{r} \subset \mathbb{R}^{s+1}$ in the (t, x^1)-plane.

(ii) $\mathcal{W}' \in \mathbf{Reg}_W$,[4]

(iii) $\lambda\mathcal{W} \in \mathbf{Reg}_W$ *for all* $\lambda \in \mathcal{P}$.

(iv) The standard double-cones $\mathscr{C}_R$, *(*$R > 0$*), and their Poincaré transforms can be obtained as intersections of wedges. Further, unions of double cones generate* $\mathbf{Reg}_b$.

(v) The stabilizer of any $\mathcal{W} \in \mathbf{Reg}_W$ *contains a one-parameter subgroup of Poincaré boosts,*

$$\mathrm{Stab}_{\mathcal{P}_+^\uparrow} \mathcal{W} \cong SO(1,1)^\uparrow \times SO(s-2).$$

The above geometrical symmetries of wedge regions have been of importance for various fundamental structural theorems and constructive results, cf. Sections 4.1 and 4.2. These provide further motivation to consider the general operator-algebraic framework for wedge-local quantum field theory. In this case observables associated to wedge regions are collected by a family of von Neumann algebras

$$\mathbf{Reg}_W \ni \mathcal{W} \longmapsto \mathfrak{A}(\mathcal{W}). \tag{1.3.11}$$

In the wedge-local case there is no unique inductive limit of the family $(\mathfrak{A}(\mathcal{W}))_{\mathcal{W}\in\mathbf{Reg}_W}$ on which a covariant action of the full space-time symmetry group can be defined.[5] This leads us to demand that jointly $\mathfrak{A}(\mathcal{W}) \subset \mathrm{B}(\mathscr{H})$ for all $\mathcal{W} \in \mathbf{Reg}_W$ on a separable Hilbert space $\mathscr{H}$.

So far we described the basic algebraic structure of observables. Proceeding towards the scattering theoretic analysis we will require to have the space-time translation symmetry at our disposal: it is assumed that translations are represented on $\mathscr{H}$ by a strongly-continuous group of unitaries $\mathbb{R}^{s+1} \ni (t, \mathbf{x}) \longmapsto U(t, \mathbf{x})$. By the Stone-Naimark-Ambrose-Godement (SNAG) Theorem [RS1, Thm. VIII.12] we can write the translation group in terms of self-adjoint strongly

[4] In many references the *causal complement* and the notation $\mathcal{O}'$ denotes the set $\{x \in \mathbb{R}^d : d_M(x,y) < 0\ \forall y \in \mathcal{O}\}$. This is distinct from our definition and less convenient in the wedge-local context as it leads to causal complements of open sets being closed and vice versa.

[5] The non-uniqueness of inductive limits is also reflected in the structure of KMS states of higher-dimensional wedge-local theories. Namely, Lechner and Schlemmer have observed that fibers associated to different wedges decouple in KMS representations of the covariant Grosse-Lechner model [LS16].

commuting generators as $U(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$, where H denotes the Hamiltonian and $\boldsymbol{P}$ the total momentum operator. The joint spectral resolution of the energy-momentum operators $(H, \boldsymbol{P})$ by projection-operator-valued measures will be denoted by $\Delta \longmapsto E_{(H,\boldsymbol{P})}(\Delta) = E(\Delta)$ for Borel sets $\Delta \subset \mathbb{R}^{s+1}$. The spectral condition demands that the support of E is contained in the closed forward lightcone $\bar{V}^+ := \{(\omega, \mathbf{p}) \in \mathbb{R}^{s+1} : \omega \geq |\mathbf{p}|\}$.

The *vacuum state* is a distinguished normalized vector $\Omega \in \mathscr{H}$ which is invariant under translations,

$$U(t, \mathbf{x})\Omega = \Omega \; \forall (t, \mathbf{x}) \in \mathbb{R}^{s+1}. \tag{1.3.12}$$

Wedge algebras must further contain sufficiently many non-trivial operators to allow a meaningful analysis of the physical content of the theory. This is enforced by the requirement that Ω is cyclic for any $\mathfrak{A}(\mathcal{W})$, i.e. $\overline{\mathfrak{A}(\mathcal{W})\Omega} = \mathscr{H}$. At the same time cyclicity together with locality (HK2) also restricts the size of the wedge algebras, as these properties imply that Ω is also separating for $\mathfrak{A}(\mathcal{W}) \subset \mathfrak{A}(\mathcal{W}')'$.

Additionally space-time translations are required to act geometrically on the family of wedge algebras. Namely for any $A \in \mathfrak{A}(\mathcal{W})$ and $x \in \mathbb{R}^s$,

$$\alpha_x(A) = U(x)AU(x)^* \in \mathfrak{A}(\mathcal{W} + x), \tag{1.3.13}$$

defines an isomorphism of the two von Neumann algebras $\mathfrak{A}(\mathcal{W})$ and $\mathfrak{A}(\mathcal{W} + x)$.

An analogous requirement can be imposed for a representation of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow \ni \lambda \longmapsto U(\lambda)$ on $\mathscr{H}$, such that

$$\alpha_\lambda(\mathfrak{A}(\mathcal{W})) := U(\lambda)\mathfrak{A}(\mathcal{W})U(\lambda)^* = \mathfrak{A}(\lambda\mathcal{W}). \tag{1.3.14}$$

This identity suggests that we can also define a wedge-local theory by specifying the von Neumann algebra $\mathfrak{M} = \mathfrak{A}(\mathcal{W})$ for only one reference wedge $\mathcal{W} \in \mathbf{Reg}_W$ together with $U(\lambda)$. The isotony and locality assumptions then become the compatibility conditions of a *causal Borchers triple* $(\mathfrak{M}, U, \Omega)$,

$$\begin{aligned} \alpha_\lambda(\mathfrak{M}) &\subset \mathfrak{M}, \quad \forall \lambda \in \mathcal{P}_+^\uparrow, \text{ s.t. } \lambda\mathcal{W} \subset \mathcal{W}, \\ \alpha_\lambda(\mathfrak{M}) &\subset \mathfrak{M}', \quad \forall \lambda \in \mathcal{P}_+^\uparrow, \text{ s.t. } \lambda\mathcal{W} \subset \mathcal{W}'. \end{aligned} \tag{1.3.15}$$

see [BLS11, Sec. 4]. Here we can see that the formulation of the theory in terms of the family of wedge algebras contains some redundancies if Poincaré transformations are available. As the latter are not necessary for the scattering-theoretic analysis, we prefer the description in terms of the full family of wedge algebras $\mathcal{W} \longmapsto \mathfrak{A}(\mathcal{W})$. This formulation also appears to be more helpful for guiding intuition when discussing the Haag-Ruelle construction of scattering states.

For scattering theory we further require uniqueness of the vacuum state (up to a phase), the standard Haag-Ruelle mass gap assumption on the energy-momentum spectrum, and wedge duality, which is a strengthened form of wedge locality. We conclude this section with the condensed definition of the wedge-local framework.

Definition 1.3.4 (wedge-local QFT)**.** *We consider a quadruple* $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$*, where*

- $\mathscr{H}$ *is a separable Hilbert space,*
- $\Omega \in \mathscr{H}$ *a distinguished unit vector (the "vacuum"),*
- $\mathfrak{A}(\mathcal{W}) \subset \mathrm{B}(\mathscr{H})$ *a family of von Neumann algebras labeled by wedge regions* $\mathcal{W} \in \mathbf{Reg}_W$*, and*

- $\alpha_\lambda(A) := U(\lambda)AU(\lambda)^*$ *denotes the action of Poincaré transformations* $\lambda = (x, \Lambda) \in \mathcal{P}_+^\uparrow$ *on observables* $A \in \mathfrak{A}(\mathcal{W})$, $\mathcal{W} \in \mathbf{Reg}_W$, *which are implemented by a strongly continuous unitary group* $\lambda \longmapsto U(\lambda)$.

  The energy-momentum operators *are defined as the generators of the translation subgroup* $U(x) = U(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$ *and their joint POVM-resolution will be denoted by* $\Delta \longmapsto E(\Delta) := E_{(H,\boldsymbol{P})}(\Delta)$.

These mathematical objects are constrained by the following conditions for any choice of wedge regions $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2 \in \mathbf{Reg}_W$,

$$\textbf{\textit{Isotony}} \quad \mathfrak{A}(\mathcal{W}_1) \subset \mathfrak{A}(\mathcal{W}_2) \text{ for } \mathcal{W}_1 \subset \mathcal{W}_2, \tag{HK1}$$

$$\textbf{\textit{Locality}} \quad \mathfrak{A}(\mathcal{W}_1) \subset \mathfrak{A}(\mathcal{W}_2)' \text{ for } \mathcal{W}_1 \subset \mathcal{W}_2', \tag{HK2}$$

$$\textbf{\textit{Wedge-Duality}} \quad \mathfrak{A}(\mathcal{W}') = \mathfrak{A}(\mathcal{W})', \tag{HK2$^\sharp$}$$

$$\textbf{\textit{Translation-Covariance}} \quad \alpha_x(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}(\mathcal{W} + x), \quad x \in \mathbb{R}^d, \tag{HK3}$$

$$\textbf{\textit{Poincaré-Covariance}} \quad \alpha_\lambda(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}(\lambda\mathcal{W}), \quad \lambda \in \mathcal{P}_+^\uparrow, \tag{HK3$^\sharp$}$$

where $\mathfrak{A}(\mathcal{W})' := \{B \in \mathrm{B}(\mathscr{H}) : [A, B] = 0 \;\forall B \in \mathfrak{A}(\mathcal{W})\}$ *is the commutant of* $\mathfrak{A}(\mathcal{W})$ *relative to* $\mathrm{B}(\mathscr{H})$. *Secondly, we note the representation-theoretic properties*

$$\textbf{\textit{Uniqueness of }}\boldsymbol{\Omega} \quad E(\{0\})\mathscr{H} = \mathbb{C}\Omega, \tag{HK4}$$

$$\textbf{\textit{Cyclicity of }}\boldsymbol{\Omega} \quad \overline{\mathfrak{A}(\mathcal{W})\Omega} = \mathscr{H}, \tag{HK5}$$

$$\textbf{\textit{Spectral Condition}} \quad \operatorname{supp} E \subset \bar{V}^+, \tag{HK6}$$

$$\textbf{\textit{Mass Gap}} \quad H_m \subset \operatorname{supp} E \subset \{0\} \cup H_m \cup \bar{H}_M \subset \bar{V}^+, \tag{HK6$^\sharp$}$$

for some $M > m > 0$, *where* $H_m := \{(\omega_m(\mathbf{p}), \mathbf{p}) \in \mathbb{R}^{s+1} : \mathbf{p} \in \mathbb{R}^s\}$, $\omega_m(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m^2}$, *is the (positive) hyperboloid of mass* $m > 0$ *and* $\bar{H}_M := \{(\omega, \mathbf{p}) \in \mathbb{R}^{s+1} : \mathbf{p} \in \mathbb{R}^s, \omega \geq \omega_M(\mathbf{p})\}$ *denotes the convex hull of* H_M. *A* wedge-local QFT *is a quadruple* $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$ *satisfying the basic postulates (HK1)–(HK6).*

Definition 1.3.5 (local QFT)**.** *The definition of a local QFT in the sense of Haag-Kastler is analogous. Instead of a wedge-local family, the local algebras form a net of von Neumann algebras* $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O}) \subset \mathrm{B}(\mathscr{H})$ *labeled by* $\mathcal{O} \in \mathbf{Reg}_b$. *A local QFT is a quadruple* $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$ *with* α, $\mathscr{H}$, *and* Ω *as in* (1.3.4), *satisfying analogous basic assumptions (HK1)–(HK6) where all wedges have been replaced by bounded open regions* $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \in \mathbf{Reg}_b$.

The constructions of two simple models which can be accommodated by the wedge-local and local framework as given by Definitions 1.3.4 and 1.3.5 are sketched in Section 1.3.2. One of the first general structural results on wedge-local theories is due to Borchers [Bor92]. Earlier works assumed that there is an underlying local theory, e.g. Bisognano and Wichmann study the wedge algebras specifically constructed from a Wightman quantum field theory [BW75].

Let us note that the wedge-local setting is mathematically more general than the local Haag-Kastler setting in the sense that it is possible to construct a canonical wedge-local theory from a local QFT. The opposite passage is much more challenging and requires deep mathematical methods, see [BDL90; BL04; Le06; Le15] and the concluding remarks in [BS08]. We will return to these points in Sections 4, where it is explained how the wedge-local perspective has so far proven itself beneficial for various structural and constructive efforts in quantum field theory.

Our present scattering theoretic analysis for wedge-local models requires the Haag-Ruelle mass gap assumption (HK6$^\sharp$). It will be stated explicitly which of our arguments additionally require wedge duality (HK2$^\sharp$), Poincaré symmetry (HK3$^\sharp$), or when specific consequences of these assumptions will be used.

1.3.2 Two basic examples

The construction of wedge-local models is a non-trivial mathematical problem, even though wedge-locality is a less stringent constraint when compared to strict locality. The most direct way to obtain a wedge-local theory with methods familiar from theoretical physics makes use of the standard free scalar field ϕ_0 and certain modified fields ϕ_Q, respectively, on the bosonic Fock space $\mathscr{H} = \Gamma(L^2(\mathbb{R}^s))$.

Example 1.3.6. For $\mathcal{W} \in \mathbf{Reg}_W$ we define

$$\mathfrak{A}(\mathcal{W}) := \overline{\operatorname{span}\{\mathrm{e}^{\mathrm{i}\phi_0(f)} : f \in \mathscr{S}(\mathbb{R}^d, \mathbb{R}),\ \operatorname{supp} f \subset \mathcal{W}\}}^{\text{w.o.t.}} \subset \mathrm{B}(\mathscr{H}). \tag{1.3.16}$$

Let further $\Omega \in \mathscr{H}$ denote the Fock vacuum and $\alpha_x(A) := U(x)AU(x)^*$, where $U(x) = \Gamma(U_1(x))$, $(x \in \mathbb{R}^d)$, is defined as the second quantization of the one-particle space-time translations acting on $\mathscr{H}_1 = L^2(\mathbb{R}^s)$ via

$$(U_1(x)\Psi)(\mathbf{k}) := \mathrm{e}^{\mathrm{i}\omega_m(\mathbf{k})t - \mathrm{i}\mathbf{k}\cdot\mathbf{x}}\Psi(\mathbf{k}). \tag{1.3.17}$$

Then $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$ defines a wedge-local quantum field theory.[6] This construction is standard and works analogously when replacing the wedges $\mathcal{W}$ by bounded space-time regions $\mathcal{O} \in \mathbf{Reg}_b$, so that we actually obtain a strictly local quantum field theory.

This standard example gives the wedge-local and local algebras for the scalar free field. A more modern construction without reference to quantum fields can be achieved with the method of standard subspaces [BGL02].

Another interesting class of wedge-local models has been constructed and studied by Grosse and Lechner [GL07]. Below we will follow the presentation of [GL07] using the compact but a priori formal notation of operator-valued distributions. More elaborate constructions using operator-algebraic and oscillatory integral methods have been worked out by Buchholz, Lechner, and Summers [BLS11], and Lechner [Le12]. The basis of the original construction of Grosse and Lechner [GL07] is a deformed CCR-algebra defined by

$$\begin{aligned} a(Q,\mathbf{p})a(Q,\mathbf{p}') &= \mathrm{e}^{-\mathrm{i}p\cdot Qp'}a(Q,\mathbf{p}')a(Q,\mathbf{p}), \\ a(Q,\mathbf{p})a^*(Q,\mathbf{p}') &= \mathrm{e}^{\mathrm{i}p\cdot Qp'}a^*(Q,\mathbf{p}')a(Q,\mathbf{p}) + \delta^{(s)}(\mathbf{p}-\mathbf{p}'), \end{aligned} \tag{1.3.18}$$

where $p = (\omega_m(\mathbf{p}), \mathbf{p}) \in \mathbb{R}^d$, $\omega_m(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$, and similarly for p'. The deformation depends on a matrix parameter $Q \in \mathbb{R}^{d\times d}$. For consistency of (1.3.18) with their adjoint relations Q is required to be antisymmetric, $p \cdot Qp' = -p' \cdot Qp$, with respect to the Minkowski scalar product $p \cdot q := p^0q^0 - \mathbf{p}\cdot\mathbf{q}$. Such deformed CCR-algebras also appear in the context of QFT models on non-commutative space-times, as explained in [GL07]. The operator-valued field distributions are then defined by

$$\phi(Q,x) := \int \frac{\mathrm{d}^s k}{(2\pi)^{s/2}\omega_m(\mathbf{k})^{1/2}} \left(a^*(Q,\mathbf{k})\mathrm{e}^{\mathrm{i}k_\mu x^\mu} + a(Q,\mathbf{k})\mathrm{e}^{-\mathrm{i}k_\mu x^\mu}\right). \tag{1.3.19}$$

Grosse and Lechner have shown that $\phi(Q,f)$ define a wedge-local quantum field theory in the sense of a wedge-local generalization of the Wightman framework, if the deformation

[6] A similar construction can also be carried out for general Wightman fields ϕ satisfying energy-bounds $\left\|\phi(f)(1+H)^{-N}\right\| < \infty$ for all $f \in \mathscr{S}(\mathbb{R}^d, \mathbb{R})$ and some $N > 0$. The latter are used for establishing bounded functions of $\phi(f)$ satisfying locality, see e.g. [Bu90b].

parameters Q are suitably chosen. More precisely, (1.3.18) can be realized on the bosonic Fock space via

$$a(Q,\mathbf{p}) = \mathrm{e}^{\frac{\mathrm{i}}{2} p\cdot QP} a(\mathbf{p}), \tag{1.3.20}$$

where $a(\mathbf{p})$ denote the usual bosonic annihilators and $P = (H, \boldsymbol{P})$. Covariance requires

$$Q = Q_{\mathcal{W}_\mathrm{r}} := \begin{pmatrix} 0 & \kappa & 0 & \cdots & 0 \\ \kappa & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{1.3.21}$$

with $\kappa \in \mathbb{R}$.[7] For general centered $\mathcal{W} = \Lambda_\mathcal{W}\mathcal{W}_\mathrm{r}$ one further sets $Q_\mathcal{W} := \Lambda_\mathcal{W} Q \Lambda_\mathcal{W}^{-1}$. Then $\phi_\mathcal{W}(x) := \phi(Q_\mathcal{W}, x)$ are wedge local and Poincaré covariant [GL07]. Explicitly, we have in a distributional sense on a dense domain $D \subset \mathscr{H}$

$$\begin{aligned} &[\phi_{\mathcal{W}'}(x), \phi_\mathcal{W}(y)] = 0, \text{ for any centered wedge } \mathcal{W} \text{ and all } y - x \in \mathcal{W}, \\ &U(\Lambda, x)\phi_\mathcal{W}(y)U(\Lambda, x)^* = \phi_{\Lambda\mathcal{W}}(\Lambda y + x), \end{aligned} \tag{1.3.22}$$

where $U(\Lambda, x)$ denotes the scalar unitary representation of the Poincaré group on the bosonic Fock space.

In this model, the wedge algebras can be defined by a similar construction as in Example 1.3.6, additionally taking into account the wedge-dependence of the field,

$$\mathfrak{A}(\mathcal{W}) := \overline{\operatorname{span}\{\mathrm{e}^{\mathrm{i}\phi_\mathcal{W}(f)} : f \in \mathscr{S}(\mathbb{R}^d, \mathbb{R}),\ \operatorname{supp} f \subset \mathcal{W}\}}^{\text{w.o.t.}}. \tag{1.3.23}$$

Then together with the Fock vacuum Ω and denoting by $\alpha_\lambda(A) := U(\lambda)AU(\lambda)^*$ the adjoint action of $U(\lambda) = U(\Lambda, x)$, $(\lambda = (\Lambda, x) \in \mathcal{P}_+^\uparrow)$, a wedge-local quantum field theory $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$ is obtained [Le12, Prop. 5.3].

1.4 Preliminaries on Haag-Ruelle scattering theory

Haag-Ruelle theory provides a general construction of asymptotic scattering states in quantum field theory [Ha58; Ru62]. In local quantum field theories it yields a justification for the LSZ-reduction formulae [Hep65], which are used in most perturbative computations of scattering data and collision cross sections. In the wedge-local context, scattering theory is at an earlier stage of development. Until now only the construction of two-particle scattering states via wedge-localized operators has been studied, see the previous works [BBS01; GL07; BS08].

The underlying physical intuition of scattering theory is that asymptotic states should resemble separating configurations of non-interacting one-particle states. Thereby the interacting models acquire a corpuscular interpretation at large times. Mathematically the construction yields a Fock-space structure of states resembling the intrinsic particle structure of free QFT models.

We will focus in the following on the scattering theory based on the Wigner particle concept, which has an analogous formulation in the wedge-local setting of Definition 1.3.4. Similarly as in the local setting, the information about the Wigner one-particle structure of a given wedge-local

[7] For $s = 3$ the exceptional form $Q = \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{pmatrix}$ is also admissible, with $\kappa, \eta \in \mathbb{R}$ [GL07].

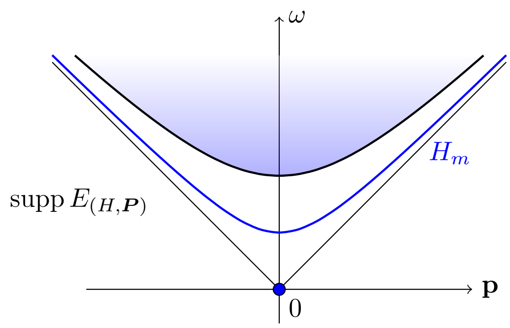

Figure 1.2: Typical energy-momentum spectrum compatible with the mass gap condition (HK6$^\sharp$).

model is obtained from the spectral analysis of the energy-momentum operators $(H, \boldsymbol{P})$. For our present investigations we further restrict ourselves to the vacuum sector, so that the pure one-particle states can be described by vectors in the vacuum Hilbert space.

Definition 1.4.1 (Wigner particle)**.** *A* one-particle state *of mass $m \geq 0$ in the sense of Wigner is an eigenvector $\Psi_1 \in \mathscr{H}$ of the relativistic* mass *operator $M := \sqrt{H^2 - \boldsymbol{P}^2}$ with eigenvalue m. In the vacuum representation, the subspace of Wigner* one-particle states *is defined as*

$$\mathscr{H}_1 := \bigcup_{m \geq 0} E(H_m)\mathscr{H}, \tag{1.4.1}$$

where $E(H_m) = E_{(H,\boldsymbol{P})}(H_m)$ denotes the energy-momentum spectral projection onto the mass hyperboloid $H_m := \{(\omega_m(\mathbf{p}), \mathbf{p}),\ \mathbf{p} \in \mathbb{R}^s\}$.

An important insight of the Haag-Ruelle construction is that the basic operation of space-time translations together with locality of observables suffice to develop a mathematically rigorous multi-particle scattering theory. Following the Haag-Ruelle prescription, we require as input a wedge-localizable operator $A \in \mathfrak{A}(\mathcal{W}) \subset \mathrm{B}(\mathscr{H})$ such that $E(H_m)A\Omega \neq 0$. The creation operator approximants are then defined by "smearing" of A with respect to space-time translations and a suitable weight function. We recall that the translates of A will be written as $\alpha_x(A) = \alpha_{(t,\mathbf{x})}(A) = U(t,\mathbf{x})AU(t,\mathbf{x})^*$, $x = (t, \mathbf{x}) \in \mathbb{R}^{s+1}$.

Definition 1.4.2 (Haag-Ruelle creation operator approximants)**.** *With $A \in \mathfrak{A}(\mathcal{W})$, $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$, and f a regular positive-energy Klein-Gordon solution we set for $\tau \in \mathbb{R}$*

$$B := A(\chi) = \int \mathrm{d}^{s+1}x\ \chi(x)\alpha_x(A), \tag{1.4.2}$$

$$\mathcal{B}_\tau(f) := \int \mathrm{d}^s x\ f(\tau, \mathbf{x})\alpha_{(\tau,\mathbf{x})}(B). \tag{1.4.3}$$

Here, a *regular positive-energy* Klein-Gordon solution f is defined as having an integral representation in terms of its wave packet $\tilde{f} \in C_c^\infty(\mathbb{R}^s)$,

$$f(t, \mathbf{x}) = \int \frac{\mathrm{d}^s k}{(2\pi)^s} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_m(\mathbf{k})t}\ \tilde{f}(\mathbf{k}), \qquad \omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}. \tag{1.4.4}$$

In local quantum field theories the existence of asymptotic limits $\tau \to \pm\infty$ and their role as asymptotic creation operators have been established by Haag and Ruelle [Ha58; Ru62].

Theorem 1.4.3 (Haag-Ruelle)**.** *Consider a local quantum field theory satisfying the mass gap assumption* (HK6$^\sharp$) *for the particle mass $m > 0$. Let $A_1, \ldots, A_n \in \mathfrak{A}(\mathcal{O})$, f_k regular positive-energy Klein-Gordon solutions of mass m with disjointly supported wave packets, $(k = 1, \ldots, n)$, and let $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ be an admissible*[8] *Haag-Ruelle auxiliary function. Then the scattering state approximants*

$$\Psi_\tau := \mathcal{B}_{1\,\tau}(f_1)\mathcal{B}_{2\,\tau}(f_2)\ldots\mathcal{B}_{n\,\tau}(f_n)\Omega, \ (\tau \in \mathbb{R}) \tag{1.4.5}$$

converge in norm for $\tau \to \pm\infty$ and are independent of χ. The scalar products of any two outgoing or incoming scattering states

$$\Psi^\pm := \lim_{\tau\to\pm\infty} \mathcal{B}_{1\,\tau}(f_1)\mathcal{B}_{2\,\tau}(f_2)\ldots\mathcal{B}_{n\,\tau}(f_n)\Omega, \tag{1.4.6}$$

$$\Psi'^\pm := \lim_{\tau\to\pm\infty} \mathcal{B}'_{1\,\tau}(f'_1)\mathcal{B}'_{2\,\tau}(f'_2)\ldots\mathcal{B}'_{n'\,\tau}(f'_{n'})\Omega \tag{1.4.7}$$

can be obtained from the Fock formula

$$\langle\Psi^+, \Psi'^+\rangle = \delta_{nn'} \sum_{\pi\in\mathfrak{S}_n} \prod_{k=1}^{n} \lim_{\tau\to\infty} \langle\mathcal{B}_{k\,\tau}(f_k)\Omega, \mathcal{B}'_{\pi(k)\,\tau}(f'_{\pi(k)})\Omega\rangle, \tag{1.4.8}$$

and similarly for incoming states. The limit on the right side can be dropped, as $\mathcal{B}_{k\,\tau}(f_k)\Omega$ do not depend on τ by construction.

Proof. See [Ha58; Ru62; Hep66] for the original approach via clustering estimates. A proof using more modern energy-momentum transfer arguments is given in [Dy14, Sec. 2.1]. ☐

To conclude this section let us briefly discuss the basic Haag-Ruelle argument for proving convergence of scattering states in non-technical terms to explain the distinction between the local and the wedge-local situations. We begin by recalling that the spatial smearing with a Klein-Gordon solution f can be understood as a comparison dynamics in the scattering-theoretic sense. The χ-smearing on the other hand is used to solve the one-particle problem. Namely, we can arrange for particles with isolated mass shells that $B\Omega \in \mathscr{H}_1$ for suitable choices of $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$. Then the secondary Klein-Gordon smearing precisely cancels the time evolution $\alpha_\tau(B)\Omega = \mathrm{e}^{\mathrm{i}H\tau}B\Omega = \mathrm{e}^{\mathrm{i}\omega_m(\boldsymbol{P})\tau}B\Omega$, and we obtain

$$\partial_\tau \mathcal{B}_\tau(f)\Omega = 0. \tag{1.4.9}$$

Here only the mass-gap assumption, spectral calculus, and translation-invariance of the vacuum enter. Hence at the one-particle level the same argument works in both local and wedge-local theories, and convergence of one-particle states is trivial.

The difference between the local and wedge-local setting appears when we move on to the existence argument for multi-particle states. The Haag-Ruelle convergence proof is based on

[8] More precisely, admissibility of $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ requires $\hat{\chi}$ to be compactly supported in a neighbourhood of the mass shell, such that

(1) $\hat{\chi}(\omega_m(\mathbf{k}), \mathbf{k}) = (2\pi)^{-(s+1)/2}$ for all $\mathbf{k} \in \operatorname{supp} \tilde{f}_j$, $(1 \le j \le n)$,

(2) $\operatorname{supp} \hat{\chi} \cap \operatorname{supp} E_{(H,\boldsymbol{P})} \subset H_m$,

(3) $(\operatorname{supp} \hat{\chi} - \operatorname{supp} \hat{\chi}) \cap \operatorname{supp} E_{(H,\boldsymbol{P})} = \{0\}$.

Via spectral calculus and spectral transfer relations, these properties imply that $B = A(\chi)$ create one-particle states from the vacuum and satisfy a clustering property, as explained in Sections 3.1 and 3.2.

Cook's method,

$$\|\Psi_{t_2} - \Psi_{t_1}\| = \left\| \int_{t_1}^{t_2} \mathrm{d}\tau\, \partial_\tau \Psi_\tau \right\| \leq \int_{t_1}^{t_2} \mathrm{d}\tau\ \|\partial_\tau \Psi_\tau\| \,. \tag{1.4.10}$$

This elementary estimate shows that integrability of $\|\partial_\tau \Psi_\tau\|$ when taking the limit $t_2 \to \infty$ is a sufficient criterion for the existence of scattering states. For the two-particle Haag-Ruelle approximants we expand the derivative,

$$\begin{aligned}\partial_\tau \Psi_\tau &= \partial_\tau \mathcal{B}_{1\,\tau}(f_1)\mathcal{B}_{2\,\tau}(f_2)\Omega \\ &= (\partial_\tau \mathcal{B}_{1\,\tau}(f_1))\mathcal{B}_{2\,\tau}(f_2)\Omega + \mathcal{B}_{1\,\tau}(f_1)\partial_\tau \mathcal{B}_{2\,\tau}(f_2)\Omega,\end{aligned} \tag{1.4.11}$$

and the second term vanishes immediately due to (1.4.9). For estimating the first term we would like to make use of the available localization information. This can be accomplished by commuting the creation operator approximants,

$$(\partial_\tau \mathcal{B}_{1\,\tau}(f_1))\mathcal{B}_{2\,\tau}(f_2)\Omega = \mathcal{B}_{2\,\tau}(f_2)(\partial_\tau \mathcal{B}_{1\,\tau}(f_1))\Omega + [\partial_\tau \mathcal{B}_{1\,\tau}(f_1), \mathcal{B}_{2\,\tau}(f_2)]\Omega, \tag{1.4.12}$$

where the first term vanishes again due to (1.4.9). Altogether we can estimate

$$\|\partial_\tau \Psi_\tau\| = \|[\partial_\tau \mathcal{B}_{1\,\tau}(f_1), \mathcal{B}_{2\,\tau}(f_2)]\Omega\| \leq \|[\partial_\tau \mathcal{B}_{1\,\tau}(f_1), \mathcal{B}_{2\,\tau}(f_2)]\| \,. \tag{1.4.13}$$

The commutator can be bounded from above using locality and propagation-geometrical estimates for the Klein-Gordon solutions. At this point one can see the important difference between local and wedge-local theories:

- In local theories the creation-operator approximants are defined from local $A_k \in \mathfrak{A}(\mathcal{O})$, $\mathcal{O} \in \mathbf{Reg}_b$. Then the norm of the commutator (1.4.13) for creation operator approximants with disjointly supported regular wave packets $\tilde{f}_k \in C_c^\infty(\mathbb{R}^s)$ is rapidly decreasing. That is, for any $N \in \mathbb{N}$ there exist $C_N > 0$ (depending on f_k and χ) such that

$$\|[\partial_\tau \mathcal{B}_{1\,\tau}(f_1), \mathcal{B}_{2\,\tau}(f_2)]\| \leq \frac{C_N}{(1+|\tau|)^N} \quad \forall \tau \in \mathbb{R}, \tag{1.4.14}$$

  and analogously without the τ-derivative acting. This implies the required integrability in (1.4.10) and thus the convergence of two-particle scattering states with disjointly supported wave packets. In the local setting the convergence argument extends directly to states with arbitrarily many particles.

- In a wedge-local setting the same arguments can be applied for the two-particle case, but the geometrical situation is much more restrictive. Firstly, choosing a wedge-local $A_1 \in \mathfrak{A}(\mathcal{W})$, we can derive an analogous commutator estimate from wedge-locality of the theory only if $A_2 \in \mathfrak{A}(\mathcal{W}^\perp)$ with $\mathcal{W}^\perp := \mathcal{W}' + y$ for some $y \in \mathbb{R}^{s+1}$. In particular we cannot choose A_2 from a slightly rotated wedge, and similarly we cannot take $A_2 = A_1$, even though this is admissible in (1.4.14).

  Secondly, even if the restriction $A_1 \in \mathfrak{A}(\mathcal{W})$, $A_2 \in \mathfrak{A}(\mathcal{W}^\perp)$ is honored, the wedge-local analog of (1.4.14) requires additionally more restrictive support conditions to be imposed on the wave packets $\tilde{f}_k$. For example, if f and $f^\perp$ describe wave packets propagating into the directions of respective wedges $\mathcal{W} = \mathcal{W}_\mathrm{r}$ and $\mathcal{W}^\perp = \mathcal{W}_\mathrm{r}{}' = -\mathcal{W}_\mathrm{r}$ for $\tau \to \infty$ (e.g.

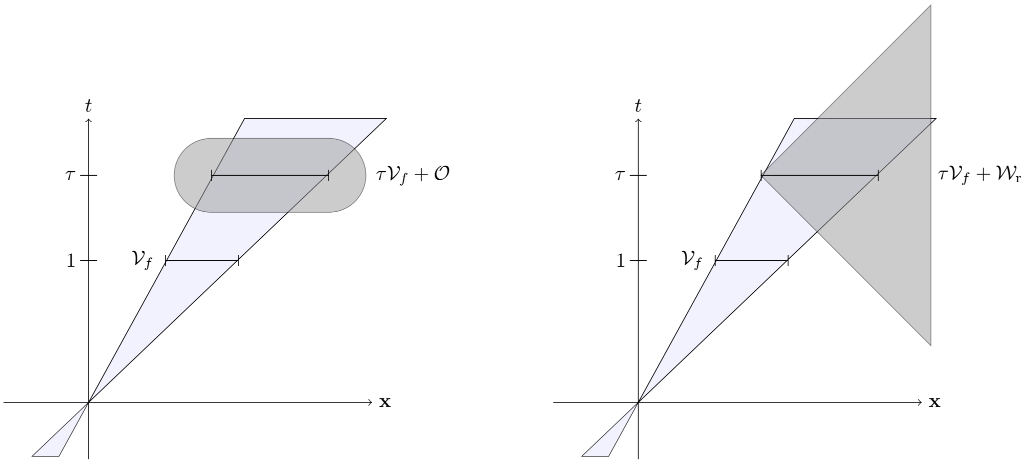

Figure 1.3: Propagation region spanned by the classical velocity supports $\mathcal{V}_f$ and approximate localization of generating operators for one-particle states $\Psi_1 = \mathcal{B}_\tau(f)\Omega = \tilde{f}(\boldsymbol{P})E(H_m)A\Omega$ for fixed $\tau \in \mathbb{R}$ (schematically), comparing the cases of local $A \in \mathfrak{A}(\mathcal{O})$ (left), and wedge-local $A \in \mathfrak{A}(\mathcal{W}_\mathrm{r})$ (right).

supp $\tilde{f} \subset \{\mathbf{k} = (k^1, \ldots, k^s) \in \mathbb{R}^s, k^1 > 0\}$, supp $\tilde{f}^\perp \subset \{\mathbf{k} \in \mathbb{R}^s, k^1 < 0\}$), we obtain

$$\left\| \left[\partial_\tau \mathcal{B}_{1\tau}(f), \mathcal{B}_{2\tau}(f^\perp) \right] \right\| \leq \frac{C_N}{(1+\tau)^N}, \ \forall \tau > 0. \tag{1.4.15}$$

Here we see, thirdly, that wedge geometry leads to a distinction between the outgoing regime $\tau \to +\infty$ and the incoming regime $\tau \to -\infty$. The analogous commutator estimate for the latter requires the opposite geometrical propagation for wave packets. For example, if f and $f^\perp$ are kept as above, we have

$$\left\| \left[\partial_\tau \mathcal{B}_{1\tau}(f^\perp), \mathcal{B}_{2\tau}(f) \right] \right\| \leq \frac{C_N}{(1+|\tau|)^N}, \ \forall \tau < 0. \tag{1.4.16}$$

Summarizing, we obtain for $A := A_1 \in \mathfrak{A}(\mathcal{W})$, $A^\perp := A_2 \in \mathfrak{A}(\mathcal{W}^\perp)$ and suitably propagating wave packets f, $f^\perp$ existence of the limits [BBS01]

$$\begin{aligned} \Psi^+ &:= \lim_{\tau \to +\infty} \mathcal{B}_\tau(f)\mathcal{B}^\perp_\tau(f^\perp)\Omega, \\ \Psi^- &:= \lim_{\tau \to -\infty} \mathcal{B}_\tau(f^\perp)\mathcal{B}^\perp_\tau(f)\Omega. \end{aligned} \tag{1.4.17}$$

In this thesis we investigate in detail two specific situations, where these standard arguments fail. Firstly, in wedge-local theories the Haag-Ruelle theorem (1.4.3) has so far been proven only for $n = 2$ using the arguments sketched above [BBS01; Le03; GL07; BS08]. With three or more particles the geometric limitations of wedge geometry become an obstruction for constructing corresponding scattering states.

Secondly, the present method does not apply when considering *embedded mass shells*. As a simple example for the latter situation one can think of a quantum field theory describing two stable Wigner particles of masses $m_1 > 0$, $m_2 > 2m_1$. If the lighter mass shell H_{m_1} is isolated in the energy-momentum spectrum, we can use Theorem 1.4.3 to construct corresponding scattering states $\Psi_1^\pm$. Now, when the spectral analysis is performed to investigate the neighbourhood of the mass shell H_{m_2} of the second particle, one finds that it is embedded in the continuous energy-momentum spectrum consisting of the scattering states $\Psi_1^{\pm}$. Then the static χ-smearing procedure (1.4.2) is in general no longer sufficient for solving the one-particle problem. On the technical level the standard approach is to introduce a τ-dependent χ-smearing, which corresponds in to an ergodic τ-averaging in configuration space [Dy05]. This leads to a norm-convergent solution of the one-particle problem, but it is not sufficient to establish convergence of multi-particle states in the massive case. So far convergence in the massive embedded case has only been proven under additional assumptions, e.g. on the spectral background present in the vicinity of the mass shell of vectors $A\Omega$, $A \in \mathfrak{A}(\mathcal{O})$ [Hrb71; Dy05; Hrd13; DH14]. In the attached publication [Du17] we developed an alternative approach which does not require τ-dependent χ-smearing. Instead we make use of a strengthened form of the Reeh-Schlieder property.

2 Summary of results

This section provides an overview of the main results that were obtained during the research period for this thesis work. The results have been published in the papers [Du17; Du18], which are the research articles serving as the core publications for this thesis. In the following we will be emphasizing the central ideas and arguments. The full technical details from these papers will not be reproduced here.

2.1 Wedge-local N-particle scattering theory

We begin with a summary of results of [Du18] concerning multi-particle scattering theory for general wedge-local theories. While two-particle scattering states have been studied in wedge-local theories [BBS01; Le03; GL07; BS08], a construction of scattering states with more than two particles has so far not been achieved in the literature. First indications that N-particle scattering states with $N \geq 3$ are also meaningful in wedge-local theories can be found in the thesis of Lechner [Le06]. Namely it is shown in [Le06, Ch. 6] that velocity-ordered N-particle scattering states constructed via conventional Haag-Ruelle theory in certain two-dimensional integrable local QFT models can be re-expressed in terms of wedge-local fields. In our analysis the comparison to standard Haag-Ruelle theory is not required, and N-particle scattering states are rigorously constructed directly in the general wedge-local framework.

2.1.1 Wedge-local N-particle Haag-Ruelle theorem

To formulate the wedge-local N-particle Haag-Ruelle theorem let us consider the scattering state approximants

$$\Psi_\tau := \mathcal{B}_{1\,\tau}(f_1)\mathcal{B}_{2\,\tau}(f_2)\ldots\mathcal{B}_{n\,\tau}(f_n)\Omega. \tag{2.1.1}$$

For $B_k := A_k(\chi)$ defined on the basis of wedge-local $A_k \in \mathfrak{A}(\mathcal{W})$ for some fixed $\mathcal{W} \in \mathbf{Reg}_W$ and regular positive-energy Klein-Gordon solutions f_k, $(1 \leq k \leq n)$, the conventional Haag-Ruelle method cannot be used to justify the convergence of Ψ_τ, or their interpretation as particle states via the Fock structure. Our approach yields convergence, Fock structure and covariance properties of Ψ_τ for $\tau \to \pm\infty$ for suitable geometrical propagation configurations of the Klein-Gordon solutions f_k.

From standard non-stationary-phase estimates we obtain that f_k are rapidly decreasing outside the semi-classical propagation region[1] defined by

$$\Upsilon_{f_k} := \{\kappa(\omega_m(\mathbf{k}), \mathbf{k}) \in \mathbb{R}^{s+1} : \mathbf{k} \in \operatorname{supp} \tilde{f}_k,\ \kappa \in \mathbb{R}\}. \tag{2.1.3}$$

[1] The non-stationary phase argument from [RS3, App. 1 to XI.3] actually leads to the rapid decay estimate

$$|f_k(t, \mathbf{x})| \leq \frac{C_N^\delta}{(1 + |\tau| + |\mathbf{x}|)^N} \text{ for all } (t, \mathbf{x}) \in \mathbb{R}^{s+1} \setminus \Upsilon_{f_k}^{[\delta]} \tag{2.1.2}$$

on the complement of any δ-extended propagation region $\Upsilon_{f_k}^{[\delta]} := \{\kappa(\omega_m(\mathbf{k}), \mathbf{k}) \in \mathbb{R}^{s+1} : \mathbf{k} \in (\operatorname{supp} \tilde{f}_k)^\delta,\ \kappa \in \mathbb{R}\}$, for $\delta > 0$, where $(\operatorname{supp} \tilde{f}_k)^\delta := \operatorname{supp} \tilde{f}_k + B_\delta(0) = \{\mathbf{k}' \in \mathbb{R}^s : \exists \mathbf{k} \in \operatorname{supp} \tilde{f}_k, |\mathbf{k}' - \mathbf{k}| < \delta\}$. For specification of the propagation geometry the non-extended regions Υ_{f_k} suffice by compactness of $\operatorname{supp} \tilde{f}_k$.

The localization and commutation properties of the creation-operator approximants $\mathcal{B}_{k\tau}(f_k)$ can be conveniently described by introducing the velocity supports

$$\mathcal{V}_{f_k} := \{(1, \mathbf{k}/\omega_m(\mathbf{k})) : \mathbf{k} \in \operatorname{supp} \tilde{f}_k\} = \Upsilon_{f_k} \cap \{(t, \mathbf{x}) \in \mathbb{R}^{s+1} : t = 1\}. \tag{2.1.4}$$

Due to the wedge-localization the disjointness of velocity supports is no longer sufficient for asymptotic vanishing of the commutators of the $\mathcal{B}_{k\tau}(f_k)$ and corresponding opposite localized $\mathcal{B}^{\perp}_{k\tau}(f_k)$ and has to be strengthened to ordering of velocity supports with respect to the *precursor relation* [BBS01] defined with respect to a given wedge $\mathcal{W} \in \mathbf{Reg}_W$ by

$$\mathcal{V}_1 \prec_{\mathcal{W}} \mathcal{V}_2 :\Longleftrightarrow \mathcal{V}_2 - \mathcal{V}_1 \subset \mathcal{W}_{\mathrm{c}}, \tag{2.1.5}$$

where $\mathcal{V}_k \subset \mathbb{R}^{s+1}$ and $\mathcal{W}_{\mathrm{c}} := \Lambda_{\mathcal{W}} \mathcal{W}_{\mathrm{r}}$ denotes the *centering* of the given wedge $\mathcal{W} = \Lambda_{\mathcal{W}} \mathcal{W}_{\mathrm{r}} + x_{\mathcal{W}} \in \mathbf{Reg}_W$. In the present section we make use of the standard creation-operator approximants from Definition 1.4.2. While this definition is simple and notationally convenient, it is based on singling out a distinguished family of Lorentz frames. In space-time dimension $d \geq 2+1$ this also leads to a preference for localizing wedges $\mathcal{W} \in \mathbf{Reg}_W$ which are *upright* in the sense that their edges are contained in constant-time hyperplanes $T_\tau := \{(\tau, \mathbf{x}) \in \mathbb{R}^{s+1} : \mathbf{x} \in \mathbb{R}^s\}$ (equivalently, $\mathcal{W} = R\mathcal{W}_{\mathrm{r}} + x$ for some spatial rotation $R \in \mathcal{L}$, $x \in \mathbb{R}^{s+1}$). Having established the necessary notation we can now state our main result on the existence and Fock structure of ordered scattering states in the wedge-local setting.

Theorem 2.1.1. *Consider a wedge-local QFT with mass gap (HK6$^\sharp$), fix a wedge $\mathcal{W}$ and let $\Psi^{(1)}_k \in \mathscr{H}_1$ $(1 \leq k \leq n)$ be single-particle vectors isolated from the remaining energy-momentum spectrum which satisfy the swapping relation $\Psi^{(1)}_k = E(H_m) A_k \Omega = E(H_m) A^{\perp}_k \Omega$, for some $A_k \in \mathfrak{A}(\mathcal{W})$, $A^{\perp}_k \in \mathfrak{A}(\mathcal{W}^{\perp})$. Let further $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ be an admissible Haag-Ruelle auxiliary function supported in a sufficiently small neighbourhood of the mass shell H_m (cf. Lemma 3.2.4).*

(i) For any family of regular positive-energy Klein-Gordon solutions f_k satisfying

$$\mathcal{V}_{f_n} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}} \prec_{\mathcal{W}} \ldots \prec_{\mathcal{W}} \mathcal{V}_{f_1}, \tag{2.1.6}$$

the scattering state approximants

$$\Psi_\tau := \mathcal{B}_{1\tau}(f_1) \mathcal{B}_{2\tau}(f_2) \ldots \mathcal{B}_{n\tau}(f_n) \Omega, \quad (\tau \in \mathbb{R}), \tag{2.1.7}$$

converge in norm for $\tau \to \infty$.

(ii) Let $\Psi^+_n := \lim_{\tau\to\infty} \Psi_\tau$, $\Psi'^+_{n'} := \lim_{\tau\to\infty} \Psi'_\tau$ be scattering states as in (i), constructed from operators localizable with respect to the same wedge $\mathcal{W}$. Then for upright $\mathcal{W}$ their scalar products can be computed using the Fock identity

$$\left\langle \Psi^+_n, \Psi'^+_{n'} \right\rangle = \delta_{nn'} \prod_{k=1}^{n} \left\langle \mathcal{B}_{k\tau}(f_k)\Omega, \mathcal{B}'_{k\tau}(f'_k)\Omega \right\rangle, \tag{2.1.8}$$

where the right-hand side is independent of τ.

Analogous statements hold for the convergence and Fock structure of any two incoming scattering states ($\tau \to -\infty$) defined using the reversed ordering of wave packets

$$\mathcal{V}_{f_n} \succ_{\mathcal{W}} \mathcal{V}_{f_{n-1}} \succ_{\mathcal{W}} \ldots \succ_{\mathcal{W}} \mathcal{V}_{f_1}. \tag{2.1.9}$$

Let us now explain the essential ingredient in the proof and the central idea in this thesis, which is the applicability and implications of the *wedge-swapping symmetry* of states for the construction of multi-particle scattering states. We will say that a vector $\Psi \in \mathscr{H}$ satisfies the *swapping property* with respect to $\mathcal{W}$ if there exists $A \in \mathfrak{A}(\mathcal{W})$ and $A^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$ with $\mathcal{W}^\perp = \mathcal{W}' + y$ for some $y \in \mathbb{R}^{s+1}$, such that

$$\Psi = A\Omega = A^\perp\Omega. \tag{2.1.10}$$

For the scattering theory we require an analogous swapping symmetry for one-particle states, and we will say that the one-particle vectors $\Psi_k^{(1)} \in E(H_m)\mathscr{H}$, $(k = 1, \dots, n)$, are *swappable* with respect to $\mathcal{W}$ if

$$\Psi_k^{(1)} = E(H_m)A_k\Omega = E(H_m)A_k^\perp\Omega, \quad A_k \in \mathfrak{A}(\mathcal{W}),\ A_k^\perp \in \mathfrak{A}(\mathcal{W}^\perp). \tag{2.1.11}$$

We note that one-particle vectors satisfying (2.1.11) can be easily obtained from swappable vectors Ψ satisfying (2.1.10) and $E(H_m)\Psi \neq 0$.

The swapping partners $A_k^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, $\mathcal{W}^\perp = \mathcal{W}' + x$, themselves do not appear in the definition of the scattering states Ψ_n^+. Yet the existence of swapping partners is essential for proving the convergence and Fock structure. The admissibility of overlap in (2.1.11) (that is, $\mathcal{W}^\perp = \mathcal{W}' + x$, $x \in \mathcal{W}$, or equivalently $\mathcal{W}^\perp \cap \mathcal{W} \neq \emptyset$) leads to trivial realizability of swapping in local QFT. Yet overlap of the wedges is not necessary, and for general wedge-local models it has been pointed out by Buchholz that the swapping symmetries (2.1.11), (2.1.10), can be established by using wedge-duality (HK2$^\sharp$) and Tomita-Takesaki theory for the case of "touching" wedges $\mathcal{W}^\perp = \mathcal{W}'$ [Bu17].

In the remainder of this section the construction of three particle scattering states will be discussed more explicitly. In the wedge-local setting they are exemplary, as we can use them to both illustrate the obstruction in the conventional approach and the basic proof idea of using swapping symmetry. Further the three-particle convergence proof already requires the essential manipulations and estimations which enter in the general inductive proof.

We now consider a scattering state approximant $\Psi_\tau := \mathcal{B}_{1\tau}(f_1)\mathcal{B}_{2\tau}(f_2)\mathcal{B}_{3\tau}(f_3)\Omega$, defined in terms of $B_k := A_k(\chi)$, $A_k \in \mathfrak{A}(\mathcal{W}_k)$, where suitable localization wedges $\mathcal{W}_k$ have to be determined. Inspection of the conventional Haag-Ruelle argument then shows that we run into the mentioned geometrical obstruction: abbreviating $\mathcal{B}_{k\tau} := \mathcal{B}_{k\tau}(f_k)$, $\dot{\mathcal{B}}_{k\tau} := \partial_\tau\mathcal{B}_{k\tau}(f_k)$, and following the traditional approach, we obtain from the Cook estimate (1.4.10) that

$$\begin{aligned}\|\Psi_{t_2} - \Psi_{t_1}\| &\leq \int_{t_1}^{t_2} \mathrm{d}\tau\ \|\partial_\tau\Psi_\tau\| \leq \int_{t_1}^{t_2} \mathrm{d}\tau\ \left\|\dot{\mathcal{B}}_{1\tau}\mathcal{B}_{2\tau}\mathcal{B}_{3\tau}\Omega\right\| + \left\|\mathcal{B}_{1\tau}\dot{\mathcal{B}}_{2\tau}\mathcal{B}_{3\tau}\Omega\right\| + \left\|\mathcal{B}_{1\tau}\mathcal{B}_{2\tau}\dot{\mathcal{B}}_{3\tau}\Omega\right\| \\ &\leq \int_{t_1}^{t_2} \mathrm{d}\tau\ \left\|[\dot{\mathcal{B}}_{1\tau}, \mathcal{B}_{2\tau}]\mathcal{B}_{3\tau}\Omega\right\| + \left\|\mathcal{B}_{2\tau}[\dot{\mathcal{B}}_{1\tau}, \mathcal{B}_{3\tau}]\Omega\right\| + \left\|\mathcal{B}_{1\tau}[\dot{\mathcal{B}}_{2\tau}, \mathcal{B}_{3\tau}]\Omega\right\|,\end{aligned} \tag{2.1.12}$$

where we used that terms involving $\dot{\mathcal{B}}_{k\tau}\Omega = 0$ vanish. Now we expand into operator norms and from the basic estimate $\|\mathcal{B}_{k\tau}(f_k)\| \leq \|B_k\|\,\|f_{k\tau}\|_{L^1(\mathbb{R}^s)} \leq C_k(1+|\tau|^{s/2})$, where $f_{k\tau}(\mathbf{x}) := f_k(\tau, \mathbf{x})$ we obtain

$$\|\Psi_{t_2} - \Psi_{t_1}\| \leq C\int_{t_1}^{t_2} \mathrm{d}\tau\ |\tau|^{s/2}\left(\|[\dot{\mathcal{B}}_{2\tau}, \mathcal{B}_{3\tau}]\| + \|[\dot{\mathcal{B}}_{1\tau}, \mathcal{B}_{2\tau}]\| + \|[\dot{\mathcal{B}}_{1\tau}, \mathcal{B}_{3\tau}]\|\right). \tag{2.1.13}$$

After this estimation we see that it has become impossible to assign localizing wedges $\mathcal{W}_k$, $(k = 1, 2, 3)$, in such a way that integrability in (2.1.13) can be established on general grounds. The problem is of a simple geometrical nature, as illustrated by Figure 2.1: if we take $A_1 \in \mathfrak{A}(\mathcal{W})$

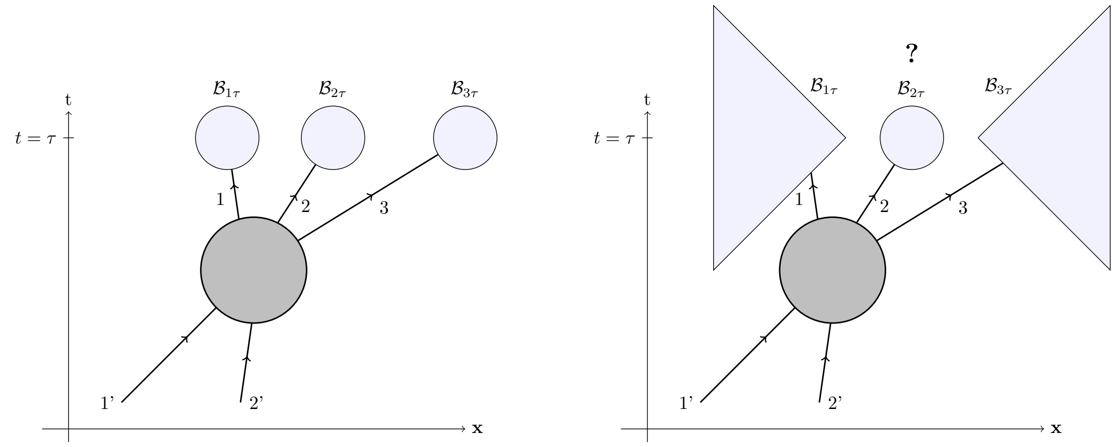

Figure 2.1: Geometrical obstruction due to wedge-localization for constructing 3-particle scattering states (right) compared to the local setting (left): at most two wedge regions can be mutually space-like separated.

for some $\mathcal{W}$, we require $A_3 \in \mathfrak{A}(\mathcal{W}_3)$ in a spacelike separated wedge or more generally in a translate $\mathcal{W}^\perp := \mathcal{W}' + x$ for some $x \in \mathbb{R}^{s+1}$. But now there are no wedge regions $\mathcal{W}_2 \in \mathbf{Reg}_W$ which are spacelike separated to both $\mathcal{W}$ and $\mathcal{W}^\perp$. The two-particle case is not affected by this obstruction, as we can take e.g. $A_1 \in \mathfrak{A}(\mathcal{W})$ and $A_2 \in \mathfrak{A}(\mathcal{W}')$ and impose $\mathcal{V}_{f_2} \prec_\mathcal{W} \mathcal{V}_{f_1}$ for $\tau \to +\infty$. Clearly, this opposite localization ansatz already described in (1.4.17) based on the conventional Haag-Ruelle argument has no obvious generalisation to three or more particles.

Let us now address the resolution of the geometrical puzzle encountered in Figure 2.1, and explain the basic idea of the wedge-local Haag-Ruelle argument for the three particle case. In contrast to the two-particle case and as stated in Theorem 2.1.3 we will fix a common wedge $\mathcal{W}$ for localizing $A_k \in \mathfrak{A}(\mathcal{W})$, $(1 \le k \le 3)$. Letting f_k be regular positive-energy Klein-Gordon solutions satisfying

$$\mathcal{V}_{f_3} \prec_\mathcal{W} \mathcal{V}_{f_2} \prec_\mathcal{W} \mathcal{V}_{f_1}, \tag{2.1.14}$$

we set $B_k := A_k(\chi)$ with admissible $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ and define $\mathcal{B}_{k\tau}(f_k)$ as before. We recall that due to the mass-gaps the one-particle problem is solved by $\mathcal{B}_{k\tau}(f_k)\Omega$, $(k = 1, 2, 3)$, at any finite τ, i.e.

$$\mathcal{B}_{k\tau}(f_k)\Omega = \tilde{f}_k(\boldsymbol{P})E(H_m)A_k\Omega = \tilde{f}_k(\boldsymbol{P})\Psi_k^{(1)} =: \Psi_k'^{(1)} \quad \forall \tau \in \mathbb{R} \tag{2.1.15}$$

with $A_k \in \mathfrak{A}(\mathcal{W})$. At this point the swapping property (2.1.11) provides for each $\Psi_k'^{(1)}$ an additional oppositely wedge-localized operator $A_k^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$ such that $E(H_m)A_k\Omega = E(H_m)A_k^\perp\Omega$. This yields for each $\Psi_k'^{(1)}$ similarly an additional oppositely localized Haag-Ruelle operator $\mathcal{B}_{k\tau}^\perp(f_k)$, such that

$$\mathcal{B}_{k\tau}^\perp(f_k)\Omega = \tilde{f}_k(\boldsymbol{P})E(H_m)A_k^\perp\Omega = \tilde{f}_k(\boldsymbol{P})E(H_m)A_k\Omega = \Psi_k'^{(1)}. \tag{2.1.16}$$

For these Haag-Ruelle operators we obtain from $\mathcal{V}_{f_3} \prec_\mathcal{W} \mathcal{V}_{f_2} \prec_\mathcal{W} \mathcal{V}_{f_1}$ together with $A_k \in \mathfrak{A}(\mathcal{W})$, $A_k^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, the ordered outgoing commutator estimates

$$\left.\begin{aligned} \left\| \left[\partial_\tau \mathcal{B}_{j\tau}(f_j), \mathcal{B}_{k\tau}^\perp(f_k)\right] \right\| &\le C_N(1+\tau)^{-N} \\ \left\| \left[\mathcal{B}_{j\tau}(f_j), \mathcal{B}_{k\tau}^\perp(f_k)\right] \right\| &\le C_N'(1+\tau)^{-N} \end{aligned}\right\} \quad \text{for } 1 \le j < k \le 3 \text{ and } \tau > 0. \tag{2.1.17}$$

With these preparations let us now consider the expansion of the Cook integrand $\|\partial_\tau \Psi_\tau\|$ from (1.4.10) for $\Psi_\tau := \mathcal{B}_{1\tau}(f_1)\mathcal{B}_{2\tau}(f_2)\mathcal{B}_{3\tau}(f_3)\Omega$. The derivative again splits into three terms, yielding

$$\begin{aligned}\|\partial_\tau \Psi_\tau\| \leq &\|(\partial_\tau \mathcal{B}_{1\tau}(f_1))\mathcal{B}_{2\tau}(f_2)\mathcal{B}_{3\tau}(f_3)\Omega\| \\ &+ \|\mathcal{B}_{1\tau}(f_1)(\partial_\tau \mathcal{B}_{2\tau}(f_2))\mathcal{B}_{3\tau}(f_3)\Omega\| \\ &+ \|\mathcal{B}_{1\tau}(f_1)\mathcal{B}_{2\tau}(f_2)\partial_\tau \mathcal{B}_{3\tau}(f_3)\Omega\|\,. \end{aligned} \tag{2.1.18}$$

Here the third term $\mathcal{B}_{1\tau}(f_1)\mathcal{B}_{2\tau}(f_2)(\partial_\tau \mathcal{B}_{3\tau}(f_3))\Omega$ already vanishes due to the τ-independence of $\mathcal{B}_{3\tau}(f_3)\Omega$. Instead of attempting to commute these operators as in the conventional Haag-Ruelle argument, we shall estimate the remaining terms by first applying the swapping symmetry. This gives

$$\begin{aligned}\mathcal{B}_{1\tau}(f_1)(\partial_\tau \mathcal{B}_{2\tau}(f_2))\mathcal{B}_{3\tau}(f_3)\Omega &= \mathcal{B}_{1\tau}(f_1)(\partial_\tau \mathcal{B}_{2\tau}(f_2))\mathcal{B}^{\perp}_{3\tau}(f_3)\Omega \\ &= \mathcal{B}_{1\tau}(f_1)\left[\partial_\tau \mathcal{B}_{2\tau}(f_2), \mathcal{B}^{\perp}_{3\tau}(f_3)\right]\Omega \\ &\quad + \mathcal{B}_{1\tau}(f_1)\mathcal{B}^{\perp}_{3\tau}(f_3)\partial_\tau \mathcal{B}_{2\tau}(f_2)\Omega \end{aligned} \tag{2.1.19}$$

where we may drop the second term due to $\partial_\tau \mathcal{B}_{2\tau}(f_2)\Omega = 0$. The first term is treated analogously with an additional step of swapping,

$$\begin{aligned}(\partial_\tau \mathcal{B}_{1\tau}(f_1))\mathcal{B}_{2\tau}(f_2)\mathcal{B}_{3\tau}(f_3)\Omega &= (\partial_\tau \mathcal{B}_{1\tau}(f_1))\mathcal{B}_{2\tau}(f_2)\mathcal{B}^{\perp}_{3\tau}(f_3)\Omega \\ &= (\partial_\tau \mathcal{B}_{1\tau}(f_1))\left[\mathcal{B}_{2\tau}(f_2), \mathcal{B}^{\perp}_{3\tau}(f_3)\right]\Omega \\ &\quad + \left[\partial_\tau \mathcal{B}_{1\tau}(f_1), \mathcal{B}^{\perp}_{3\tau}(f_3)\right]\mathcal{B}_{2\tau}(f_2)\Omega \\ &\quad + \mathcal{B}^{\perp}_{3\tau}(f_3)(\partial_\tau \mathcal{B}_{1\tau}(f_1))\mathcal{B}_{2\tau}(f_2)\Omega, \end{aligned} \tag{2.1.20}$$

where the last summand can be rewritten as before,

$$\mathcal{B}^{\perp}_{3\tau}(f_3)(\partial_\tau \mathcal{B}_{1\tau}(f_1))\mathcal{B}_{2\tau}(f_2)\Omega = \mathcal{B}^{\perp}_{3\tau}(f_3)\left[\partial_\tau \mathcal{B}_{1\tau}(f_1), \mathcal{B}^{\perp}_{2\tau}(f_2)\right]\Omega. \tag{2.1.21}$$

Collecting the terms of (2.1.19) and (2.1.20) we obtain with some $C > 0$ that for all $\tau > 0$

$$\begin{aligned}\|\partial_\tau \Psi_\tau\| &\leq C\,|\tau|^{s/2} \cdot \Big(\left\| \left[\partial_\tau \mathcal{B}_{2\tau}(f_2), \mathcal{B}^{\perp}_{3\tau}(f_3)\right] \right\| + \left\| \left[\mathcal{B}_{2\tau}(f_2), \mathcal{B}^{\perp}_{3\tau}(f_3)\right] \right\| \\ &\qquad + \left\| \left[\partial_\tau \mathcal{B}_{1\tau}(f_1), \mathcal{B}^{\perp}_{3\tau}(f_3)\right] \right\| + \left\| \left[\partial_\tau \mathcal{B}_{1\tau}(f_1), \mathcal{B}^{\perp}_{2\tau}(f_2)\right] \right\| \Big) \\ &\leq C_N \tau^{-N}. \end{aligned} \tag{2.1.22}$$

Here we applied the standard estimate $\|\mathcal{B}_{k\tau}(f_k)\| \leq C\,|\tau|^{s/2}$, and in the last step we used that all operator pairs are correctly ordered to obtain rapid decay from the outgoing ordered commutator estimates (2.1.17). Now the convergence of the outgoing 3-particle scattering-state approximant Ψ_τ follows as usual from Cook's method.

The Fock structure of such scattering states is established on the basis of analogous swapping arguments, assuming that the localization wedges of the two states under consideration are compatible. An additional geometrical argument is required to obtain outgoing ordered commutators (2.1.17) also across two families of $\mathcal{B}_{k\tau}(f_k)$, $\mathcal{B}'_{k\tau}(f'_k)$ with their swapping partners $\mathcal{B}^{\perp}_{k\tau}(f_k)$, $\mathcal{B}'^{\perp}_{k\tau}(f'_k)$. For this, one can make use of the fact that the precursor relation $\prec_{\mathcal{W}}$ behaves almost like a total order for upright wedges $\mathcal{W}$ when restricted to velocity supports. Finally let us note that for this reason the one-particle state matrix elements with additional permutations are indeed absent from the Fock relation (2.1.8), cf. (2.2.10).

2.1.2 Scattering theory with general wedge and reference frames

For wedge-local theories the Lorentz covariance property (HK3$^\sharp$) is a basic physical requirement, which becomes particularly stringent in higher dimensions $d > 1+1$. Yet, the Lorentz invariance of the S-matrix may fail as shown in various wedge-local constructions [GL07; BLS11]. For a model-independent study of this effect we generalize the above discussion, which was based on a distinguished reference frame. A clear resolution of this problem specific to the higher-dimensional ($s \geq 2$) wedge-local context consists in passing to Haag-Ruelle operators $\mathcal{B}_\tau^\Lambda(f)$, which are adapted to Lorentz frames specified by a boost $\Lambda \in \mathcal{L}_+^\uparrow$. This also strengthens the construction so that scattering states and their Fock structure can be obtained from $A_k \in \mathfrak{A}(\mathcal{W})$ for any wedge $\mathcal{W} = \Lambda\mathcal{W}_\mathrm{r} + x$, without requiring $\mathcal{W}$ to be upright.

Definition 2.1.2 (adapted Haag-Ruelle operators)**.** *Fix a wedge $\mathcal{W} \in \mathbf{Reg}_W$, $A \in \mathfrak{A}(\mathcal{W})$, and let $B := A(\chi)$, with $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ as before. For any regular positive-energy Klein-Gordon solution f, and $\tau \in \mathbb{R}$ we define*

$$\mathcal{B}_\tau^\Lambda(f) := \int \mathrm{d}^s x \; f(\Lambda(\tau, \mathbf{x}))\alpha_{(\Lambda(\tau,\mathbf{x}))}(B), \tag{2.1.23}$$

where $\Lambda \in \mathcal{L}^*(\mathcal{W}) := \{\Lambda \in \mathcal{L}_+^\uparrow : \Lambda\mathcal{W}_\mathrm{r} = \mathcal{W}_\mathrm{c}\}$ or more generally $\Lambda \in \mathcal{L}_+^\uparrow$.

The approximate space-time localization of $\mathcal{B}_\tau^\Lambda(f)$ is again relevant for obtaining outgoing ordered commutator estimates and for specifying the propagation geometry. For (2.1.23) it is described by adapted velocity supports

$$\mathcal{V}_f^\Lambda := (\Lambda T_1) \cap \Upsilon_f, \tag{2.1.24}$$

for $T_1 := \{(1, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^s\}$, and where Υ_f denotes the propagation region of f.

Theorem 2.1.3 (Wedge-local N-particle Haag-Ruelle theorem)**.** *Let $\Lambda \in \mathcal{L}_+^\uparrow$ and $\Psi_j^{(1)} = E(H_m)A_j\Omega = E(H_m)A_j^\perp\Omega$ with $A_j \in \mathfrak{A}(\mathcal{W})$, $A_j^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$ and let $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ be an admissible auxiliary function (supported in a sufficiently small neighbourhood of the isolated mass shell).*

(i) For regular positive-energy Klein-Gordon solutions f_j satisfying the outgoing ordering condition

$$\mathcal{V}_{f_n}^\Lambda \prec_\mathcal{W} \mathcal{V}_{f_{n-1}}^\Lambda \prec_\mathcal{W} \ldots \prec_\mathcal{W} \mathcal{V}_{f_1}^\Lambda, \tag{2.1.25}$$

the scattering state approximants $\Psi_n^\Lambda(\tau) := \mathcal{B}_{1\tau}^\Lambda(f_1)\mathcal{B}_{2\tau}^\Lambda(f_2)\ldots\mathcal{B}_{n\tau}^\Lambda(f_n)\Omega$ converge in norm for $\tau \to \infty$.

(ii) For $\Lambda \in \mathcal{L}^*(\mathcal{W})$ the scalar products of any two outgoing scattering states

$$\Psi_n^{+,\Lambda} := \lim_{\tau\to\infty} \mathcal{B}_{1\tau}^\Lambda(f_1)\ldots\mathcal{B}_{n\tau}^\Lambda(f_n)\Omega, \tag{2.1.26}$$

$$\Psi_{n'}^{\prime+,\Lambda} := \lim_{\tau\to\infty} \mathcal{B}_{1\tau}^{\prime\Lambda}(f_1')\ldots\mathcal{B}_{n'\tau}^{\prime\Lambda}(f_{n'}')\Omega, \tag{2.1.27}$$

constructed w.r.t. the same wedge $\mathcal{W}$ satisfy

$$\left\langle \Psi_n^{+,\Lambda}, \Psi_{n'}^{\prime+,\Lambda} \right\rangle = \delta_{nn'} \prod_{j=1}^n \left\langle \mathcal{B}_{j\tau}^\Lambda(f_j)\Omega, \mathcal{B}_{j\tau}^{\prime\Lambda}(f_j')\Omega \right\rangle. \tag{2.1.28}$$

Analogous statements hold for incoming scattering states assuming opposite ordering of wave packets (while preserving the order of applying the creation-operator approximants to the vacuum).

2.1.3 Wedge-local scattering data and covariance

On the basis of Theorem 2.1.3 and following arguments familiar from the local setting, it is possible define wave operators and a multi-particle S-matrix. In [Du18] we proposed a specialized wedge-local formulation, with the purpose of making the wave-packet and operator ordering assumptions more transparent for future investigations of wedge-local scattering data. In particular it provides a simple explanation for the previously observed Lorentz covariance breaking effects at the level of scattering states, and we established the precise residual Lorentz covariance of the scattering data, which must be present in all wedge local theories satisfying Lorentz-covariance (HK3$^\sharp$). We begin by introducing general ordered Fock spaces, on which the wave operators will be defined.

Definition 2.1.4. *The ordered tensor products over one-particle Hilbert space $\mathscr{H}_1$ with respect to a given partial order $\prec$ on $\mathscr{H}_1$ are defined as closure $\otimes^n_\prec \mathscr{H}_1 := \overline{\hat{\otimes}^n_\prec \mathscr{H}_1}$ of the finite linear spans*

$$\hat{\otimes}^n_\prec \mathscr{H}_1 := \mathrm{span}\{\Psi_1^1 \otimes \ldots \otimes \Psi_1^n : \Psi_1^k \in \mathscr{H}_1, \Psi_1^1 \prec \Psi_1^2 \prec \ldots \prec \Psi_1^n\}. \tag{2.1.29}$$

Using the conventions $\hat{\otimes}^0_\prec \mathscr{H}_1 := \mathbb{C}\Omega$, $\hat{\otimes}^1_\prec \mathscr{H}_1 := \mathscr{H}_1$, we obtain corresponding ordered Fock spaces $\Gamma^\prec(\mathscr{H}_1) := \bigoplus_{n=0}^\infty \otimes^n_\prec \mathscr{H}_1$. The subspace of finite linear combinations of ordered tensor product vectors with $\Psi_1^k \in \mathscr{H}_1'$ where $\mathscr{H}_1'$ is any subset or subspace of $\mathscr{H}_1$ shall be denoted by $\Gamma_0^\prec(\mathscr{H}_1') := \hat{\bigoplus}_{n=0}^\infty \hat{\otimes}^n_\prec \mathscr{H}_1'$.

Of particular importance for the applicability of Theorem 2.1.3 are the velocity-ordered Fock spaces spanned by swappable one-particle vectors, and we recall that our present results require their mass shells to be isolated (HK6$^\sharp$).

Definition 2.1.5 (swappable one-particle subspace and vectors of bounded energy)**.**

$$\begin{aligned} \mathscr{H}_1^{\mathcal{W}} &:= \{\Psi_1 \in \mathscr{H}_1 : \Psi_1 \textit{ swappable w.r.t. } \mathcal{W} + x \textit{ for some } x \in \mathbb{R}^d\}, \\ \mathscr{H}_{1c}^{\mathcal{W}} &:= \{\tilde{f}(\boldsymbol{P})\Psi_1 : \Psi_1 \in \mathscr{H}_1^{\mathcal{W}},\ \tilde{f} \in C_c^\infty(\mathbb{R}^s)\}. \end{aligned} \tag{2.1.30}$$

Let us note that due to $\overline{\mathscr{H}_{1c}^{\mathcal{W}}} = \overline{\mathscr{H}_1^{\mathcal{W}}}$ we have density in $\mathscr{H}_1$ if wedge-duality (HK2$^\sharp$) is satisfied.[2] The velocity ordering with respect to the opening directions of a given wedge $\mathcal{W}$ can be lifted from Klein-Gordon solutions to one-particle vectors $\Psi_1 \in \mathscr{H}_1$ via the energy-momentum spectral measure $E_{(H,\boldsymbol{P})}(\Delta)$, ($\Delta \subset \mathbb{R}^{s+1}$ Borel).

Definition 2.1.6. *The propagation region, adapted velocity supports, and the precursor ordering of one-particle vectors $\Psi_1, \Psi_1' \in \mathscr{H}_1$ are defined by*

$$\Upsilon_{\Psi_1} := \{t \cdot (\omega, \mathbf{k}) : (\omega, \mathbf{k}) \in \mathrm{supp}(E_{(H,\boldsymbol{P})}\Psi_1),\ t \in \mathbb{R}\},$$

$$\mathcal{V}^\Lambda_{\Psi_1} := \Upsilon_{\Psi_1} \cap \Lambda T_1, \qquad T_1 := \{(1, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^s\}, \tag{2.1.31}$$

$$\Psi_1' \prec_{\mathcal{W}} \Psi_1 :\Longleftrightarrow \mathcal{V}^\Lambda_{\Psi_1'} \prec_{\mathcal{W}} \mathcal{V}^\Lambda_{\Psi_1}, \tag{2.1.32}$$

with some $\Lambda \in \mathcal{L}^*(\mathcal{W})$.

[2] The introduction of $\mathscr{H}_{1c}^{\mathcal{W}}$ is required for technical reasons. Namely, we cannot expect $\mathscr{H}_1^{\mathcal{W}}$ to contain vectors which have compact energy-momentum spectrum [BBS01, Lem. 3.4]. For the same analyticity reasons $\mathscr{H}_{1c}^{\mathcal{W}}$ is neither a linear space, nor contained in $\mathscr{H}_1^{\mathcal{W}}$. One may introduce the linear span $\tilde{\mathscr{H}}_{1c}^{\mathcal{W}}$ of $\mathscr{H}_{1c}^{\mathcal{W}}$, but clearly $\Gamma_0^{\prec_{\mathcal{W}}}(\tilde{\mathscr{H}}_{1c}^{\mathcal{W}}) = \Gamma_0^{\prec_{\mathcal{W}}}(\mathscr{H}_{1c}^{\mathcal{W}})$.

The well-definedness of (2.1.32) with respect to the choice $\Lambda \in \mathcal{L}^*(\mathcal{W})$ follows from a corresponding property of the relativistic velocity-transformation law. With the above definitions it is possible to express the scattering-theoretic content of Theorem 2.1.3 at the level of Møller-type wave operators.

Definition 2.1.7 (wave operators). *For any given centered wedge $\mathcal{W}$ we define*

$$\mathbb{W}^+_{\mathcal{W}} : \begin{cases} \Gamma_0^{\succ_{\mathcal{W}}}(\mathscr{H}_{1c}^{\mathcal{W}}) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau\to\infty} \mathcal{B}^\Lambda_{1\tau}(f_1)\ldots\mathcal{B}^\Lambda_{n\tau}(f_n)\Omega, \end{cases} \tag{2.1.33}$$

$$\mathbb{W}^-_{\mathcal{W}} : \begin{cases} \Gamma_0^{\prec_{\mathcal{W}}}(\mathscr{H}_{1c}^{\mathcal{W}}) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau\to-\infty} \mathcal{B}^\Lambda_{1\tau}(f_1)\ldots\mathcal{B}^\Lambda_{n\tau}(f_n)\Omega. \end{cases} \tag{2.1.34}$$

Here the creation operators $\mathcal{B}^\Lambda_{k\tau}(f_k)$ are chosen such that $\mathcal{B}^\Lambda_{k\tau}(f_k)\Omega = \Psi_1^k$ with $\Lambda \in \mathcal{L}^*(\mathcal{W})$. Such operators with $B_k = A_k(\chi)$ for swappable $A_k \in \mathfrak{A}(\mathcal{W})$ can be obtained due to $\Psi_1^k \in \mathscr{H}_{1c}^{\mathcal{W}}$ by definition. In two dimensions we take any $\Lambda \in \mathcal{L}_+^\uparrow = \mathcal{L}^*(\mathcal{W}_{\mathrm{r}})$ also for $\mathcal{W} = \mathcal{W}_{\mathrm{r}}{}^\perp$.[3]

The well-definedness and covariance properties of $\mathbb{W}^\pm_{\mathcal{W}}$ are non-trivial consequences of the Fock structure (Theorem 2.1.3 (ii)) and a technical covariance result which concerns the dependence on the reference frame specified by Λ. If Lorentz transformations are implementable in the given wedge-local theory (HK3$^\sharp$), we obtain as a corollary of these technical considerations also the precise covariance relation obeyed by the scattering states.

Theorem 2.1.8. *Assuming wedge-duality (HK2$^\sharp$), the wave operators (2.1.33), (2.1.34) are well-defined and extend to bounded linear isometries $\mathbb{W}^+_{\mathcal{W}} : \Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1) \longrightarrow \mathscr{H}$, and $\mathbb{W}^-_{\mathcal{W}} : \Gamma^{\prec_{\mathcal{W}}}(\mathscr{H}_1) \longrightarrow \mathscr{H}$. For $\lambda = (a,\Lambda) \in \mathcal{P}_+^\uparrow$ we have further*

$$\mathbb{W}^\pm_{\mathcal{W}+a} = \mathbb{W}^\pm_{\mathcal{W}},$$
$$U(\lambda)\mathbb{W}^\pm_{\mathcal{W}} = \mathbb{W}^\pm_{\Lambda\mathcal{W}} U_0(\lambda),$$

where $U_0(\lambda)$ denotes the unitary representation of the Poincaré group on the unsymmetrized Fock space over $\mathscr{H}_1$ restricted to $\Gamma^{\succ_{\mathcal{W}}/\prec_{\mathcal{W}}}(\mathscr{H}_1)$. Thereby the wave operators $\mathbb{W}^\pm_{\mathcal{W}}$ depend on the localization wedge $\mathcal{W}$ only modulo translations, and it suffices to consider $\mathbb{W}^\pm_{\mathcal{W}}$ for centered wedges $\mathcal{W}_c \in \mathbf{Reg}_W$.

Most notably, a dependence of the wave operators $\mathbb{W}^\pm_{\mathcal{W}}$ on the localization wedge of reference $\mathcal{W}$ cannot be completely ruled out in the general wedge-local setting. Such dependences are confirmed by computations of Grosse-Lechner [GL07] and Buchholz-Summers [BS08] in a class of recently constructed higher-dimensional wedge-local QFT models. If we seriously take into account this possible wedge-dependence of the wave operators, a systematic study of the scattering data also has to contain the specification of the reference wedges.

Definition 2.1.9 (S-matrix and wedge-transition maps). *The scattering data in the wedge-local setting are the S-matrices defined by the matrix elements between final and initial states*

$$S^{\mathcal{W}_{\mathrm{f}},\mathcal{W}_{\mathrm{i}}}_{\mathrm{f\,i}} := (\mathbb{W}^+_{\mathcal{W}_{\mathrm{f}}})^*\mathbb{W}^-_{\mathcal{W}_{\mathrm{i}}},$$

[3] The two-dimensional case is exceptional as $\mathcal{L}_+^\uparrow$ acts non-transitively on $\mathbf{Reg}_W$, and in particular $\mathcal{L}^*(\mathcal{W}_{\mathrm{r}}{}') = \emptyset$. In this case one can easily define $\mathbb{W}_{\mathcal{W}_{\mathrm{r}}{}^\perp}$ by taking $\Lambda = \mathbb{1}$, but due to swapping this does not produce new scattering states which cannot be obtained by $\mathbb{W}_{\mathcal{W}_{\mathrm{r}}}$.

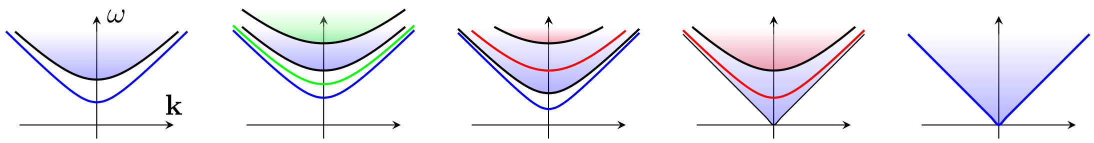

Figure 2.2: Energy-momentum spectra in QFTs with structurally different Wigner-particle contents. In (a) and (b) all mass shells are isolated, whereas the remaining examples contain *embedded* mass shells. The results of Haag and Ruelle [Ha58; Ru62] address (a), (b), and the massless case (e) has been solved by Buchholz [Bu77]. Aside from superselection theory, the massive embedded cases have first been treated for (c) in [Hrb71], and (d) in [Dy05].

where $\mathcal{W}_\mathrm{f}, \mathcal{W}_\mathrm{i} \in \mathbf{Reg}_W$ *are centered wedges. Additionally we define the wedge-transition maps*

$$S^{\mathcal{W}',\mathcal{W}}_{\mathrm{f\,f}} := (\mathbb{W}^+_{\mathcal{W}'})^* \mathbb{W}^+_{\mathcal{W}}, \quad S^{\mathcal{W}',\mathcal{W}}_{\mathrm{i\,i}} := (\mathbb{W}^-_{\mathcal{W}'})^* \mathbb{W}^-_{\mathcal{W}}, \tag{2.1.35}$$

depending on centered wedges $\mathcal{W}, \mathcal{W}'$*.*

With the explicit wedge-dependences we obtain also a clear description of the admissible asymptotic Poincaré-symmetry breaking in wedge-local theories. If we consider the problem from the opposite direction we can in fact restore Poincaré covariance of the scattering data by including the correct transformation of the wedge-dependences of S-matrices and wedge-transition maps.

Theorem 2.1.10. *The wedge-local* S*-matrices are Poincaré-covariant in the sense that for* $\lambda = (a, \Lambda) \in \mathcal{P}^\uparrow_+$ *we have*

$$U_0(\lambda) S^{\mathcal{W}_\mathrm{f},\mathcal{W}_\mathrm{i}}_{\mathrm{f\,i}} U_0(\lambda)^* = S^{\Lambda\mathcal{W}_\mathrm{f},\Lambda\mathcal{W}_\mathrm{i}}_{\mathrm{f\,i}}, \tag{2.1.36}$$

and similarly the wedge transition maps (2.1.35) *satisfy*

$$U_0(\lambda) S^{\mathcal{W},\mathcal{W}'}_{\mathrm{f\,f}} U_0(\lambda)^* = S^{\Lambda\mathcal{W},\Lambda\mathcal{W}'}_{\mathrm{f\,f}}, \tag{2.1.37}$$

$$U_0(\lambda) S^{\mathcal{W},\mathcal{W}'}_{\mathrm{i\,i}} U_0(\lambda)^* = S^{\Lambda\mathcal{W},\Lambda\mathcal{W}'}_{\mathrm{i\,i}}. \tag{2.1.38}$$

If the wave operators are asymptotically complete (i.e. have dense range in $\mathscr{H}$*) it is possible to explicitly compute the influence of the wedge-localization choices via transition identities such as* $S^{\mathcal{W}_\mathrm{f},\mathcal{W}_\mathrm{i}}_{\mathrm{f\,i}} = S^{\mathcal{W}_\mathrm{f},\mathcal{W}'_\mathrm{f}}_{\mathrm{f\,f}} S^{\mathcal{W}'_\mathrm{f},\mathcal{W}'_\mathrm{i}}_{\mathrm{f\,i}} S^{\mathcal{W}'_\mathrm{i},\mathcal{W}_\mathrm{i}}_{\mathrm{i\,i}}$*.*

2.2 Scattering theory via Reeh-Schlieder non-local correlations

We now consider the scattering problem in presence of massless particles. On the technical level our wedge-local constructions from Sections 2.1.1–2.1.3 are substantially based on the Haag-Ruelle mass gap assumption (HK6$^\sharp$). Yet in models describing massless particles (HK6$^\sharp$) this assumption is necessarily violated, as can be seen in Figure 2.2 (d), (e).

In local QFT models the scattering theory of massless particles is under complete mathematical control [Bu75b; Bu75a; Bu77; DH14; AD17] and there exist various results also for massive particles with embedded mass shells [Hrb71; Dy05; Hrd13]. It is an interesting question whether these presently available methods can be applied in the wedge-local setting to provide an extension of the constructions from Sections 2.1.1–2.1.3 to models describing massless particles or for massive particles with embedded mass shells. We will consider the scattering problem for

Wigner particles, noting that depending on the physical context the more general infraparticle concept may be necessary [BPS91].[4]

In the scattering theoretic analysis of particles with embedded mass shells we are faced with two challenges, which require additional technical considerations. Broadly speaking they are associated to the absence of the upper mass gap above embedded mass shells as in Figure 2.2 (c)–(e), and absence of the lower mass gap separating the vacuum from the remaining energy-momentum spectrum as in Figure 2.2 (d)–(e), respectively.

Firstly, the presence of spectral background leads to $\partial_\tau \mathcal{B}_{k\tau}(f_k)\Omega \neq 0$ and there are no currently known general arguments which show that the one-particle problem can be solved using operators from the quasi-local or from a quasi-wedge-local algebra. The standard strategy adopts a τ-dependent χ-smearing in the definition of the creation-operator approximants. While this construction allows to restore integrability of $\|\partial_\tau \mathcal{B}_{k\tau}(f_k)\Omega\|$, the Cook convergence argument additionally requires decay of the commutators $\|[\mathcal{B}_{k\tau}(f_k), \mathcal{B}_{j\tau}(f_j)]\|$ and depending on the proof strategy and the strength of the commutator decay also additional energy bounds [Dy05] or clustering properties [Hrb71]. For the method of time-dependent smearing the uncertainty principle leads to a tension between the one-particle integrability and the commutator decay due to

$$A(\chi_\tau)\Omega = (2\pi)^{(s+1)/2}\hat{\chi}_\tau(H, \boldsymbol{P})A\Omega. \tag{2.2.1}$$

In the smearing method, $\hat{\chi}_\tau \in \mathscr{S}(\mathbb{R}^{s+1})$ are constructed pointwise convergent with shrinking supports, e.g. $\hat{\chi}_\tau(\omega, \mathbf{k}) \longrightarrow \mathbb{1}_{\Delta_m}(\omega, \mathbf{k})$ for $\tau \to \pm\infty$, where $\mathbb{1}_{\Delta_m}$ is the characteristic function of a compact subset of the mass shell. Then the uncertainty principle implies that the improved localization of $\hat{\chi}_\tau(\omega, \mathbf{p})$ in energy-momentum space as required for better one-particle integrability leads to delocalization of $\chi_\tau(t, \mathbf{x})$ in configuration space. This in turn enters in the decay estimates for commutators, which also have to be sufficiently decaying for the success of Cook's method. In the literature this tension between localization in configuration space and localization in energy-momentum space is usually resolved by imposing additional spectral conditions on $A\Omega$ [Hrb71; Dy05; Hrd13; DH14] for suitable $A \in \mathfrak{A}_{\mathrm{loc}}$.

For our investigations we would like to avoid the delocalization of creation-operators in the solution of the one-particle problem resulting from the smearing strategy (2.2.1). Similarly we do not want to impose assumptions on the action of individual operators $A \in \mathfrak{A}_{\mathrm{loc}}$. In the scattering theory of massless particles such additional assumptions are not required, as has been shown by Buchholz [Bu75a; Bu77].

There is in fact a well-known model-independent feature of relativistic quantum field theories, by which the one-particle problem can be solved without delocalization in the sense of (2.2.1): in local quantum field theories the Reeh-Schlieder property states that the vacuum vector Ω is cyclic for the algebras $\mathfrak{A}(\mathcal{O})$ associated to bounded space-time regions $\mathcal{O} \in \mathbf{Reg}_b$, that is

$$\overline{\mathfrak{A}(\mathcal{O})\Omega} = \mathscr{H}. \tag{2.2.2}$$

In particular given any one-particle vector $\Psi_1 \in E(H_m)\mathscr{H}$, there exists a family of operators $A_\beta \in \mathfrak{A}(\mathcal{O})$, ($\beta > 0$), such that

$$\lim_{\beta \to 0} A_\beta \Omega = \Psi_1. \tag{2.2.3}$$

[4] A physical example for infraparticles are electrically charged particles in Quantum Electrodynamics. On the other hand electrically neutral particles such as hydrogen atoms are within the scope of the classical Wigner particle concept. The scattering theoretic analysis is even more challenging for infraparticles, see e.g. [Pi05; AD17; Dy17; CD18]. In quantum electrodynamics the nonexistence of eigenvalues of the mass operator for charged particles can be derived as a consequence of the Gauss' law [Bu86].

For wedge-local theories we can analogously find a family of operators $A_\beta \in \mathfrak{A}(\mathcal{W})$, $(\beta > 0)$, for any given $\mathcal{W} \in \mathbf{Reg}_W$.[5] We will say that $(A_\beta)_{\beta>0}$ is a local, respectively wedge-local, *Reeh-Schlieder family* for Ψ_1.

Let us begin with a preliminary definition for a Haag-Ruelle-type creation operator approximants which make use of Reeh-Schlieder effects for solving the one-particle problem by setting

$$\mathcal{A}_\tau(f) := \int \mathrm{d}^s x\, f(\tau, \mathbf{x}) A_{\beta(\tau)}(\tau, \mathbf{x}). \tag{2.2.4}$$

Here $\beta(\tau)$ denotes a scaling function satisfying $\lim_{\tau\to\pm\infty} \beta(\tau) = 0$. For now we have to leave open the choice of the scaling function, as we will need to adapt it to both the family A_β and to properties of the scattering states to be constructed. For concreteness we note that a specific but sufficiently flexible choice which will become relevant in the following is given by the power law scalings $\beta(\tau) := |\tau|^{-\mu}$, $(\mu > 0)$.

The creation-operator approximants $\mathcal{A}_\tau(f)$ satisfy the two mentioned requirements of the Cook method of having good localization properties and fast convergence in the one-particle case. Yet, we have so far not been able to establish a scattering theory directly on the basis of the operators $\mathcal{A}_\tau(f)$. This is due to the fact that $\mathcal{A}_\tau(f)$ is in general expected to become unbounded in the asymptotic limits $\tau \to \pm\infty$ and this has to be taken into account for $N \geq 2$ particles when computing the Cook derivative, e.g. for $N = 2$

$$\partial_\tau \Psi_\tau = \mathcal{A}_{1\tau}(f_1)\partial_\tau \mathcal{A}_{2\tau}(f_2)\Omega + (\partial_\tau \mathcal{A}_{1\tau}(f_1))\mathcal{A}_{2\tau}(f_2)\Omega. \tag{2.2.5}$$

The modern method for estimating such terms in local QFT uses the energy-bounds established by Buchholz [Bu90a; Dy05], whereas the traditional approach is based on space-like clustering properties of vacuum expectation values [Ha58; Ru62; Hrb71]. In the present context the latter amount to writing

$$\begin{aligned} \|\mathcal{A}_{1\tau}(f_1)\partial_\tau \mathcal{A}_{2\tau}(f_2)\Omega\|^2 &= \langle \Omega, (\partial_\tau \mathcal{A}_{2\tau}(f_2)^*)\mathcal{A}_{1\tau}(f_1)^*\mathcal{A}_{1\tau}(f_1)\partial_\tau \mathcal{A}_{2\tau}(f_2)\Omega\rangle \\ &= \langle \Omega, \mathcal{A}_{1\tau}(f_1)^*\mathcal{A}_{1\tau}(f_1)(\partial_\tau \mathcal{A}_{2\tau}(f_2)^*)\partial_\tau \mathcal{A}_{2\tau}(f_2)\Omega\rangle \\ &\quad + (\text{commutators}). \end{aligned} \tag{2.2.6}$$

For suitably propagating wave packets the two product operators $C_{1\tau} := \mathcal{A}_{1\tau}(f_1)^*\mathcal{A}_{1\tau}(f_1)$ and $C_{2\tau} := (\partial_\tau \mathcal{A}_{2\tau}(f_2)^*)\partial_\tau \mathcal{A}_{2\tau}(f_2)$ will have space-like separating space-time localization when taking $\tau \to \infty$. In local QFT with and without lower mass gap model-independent clustering estimates have been established for such vacuum expectation values [AHR62]. These imply that in the limit of large space-like separation, i.e. for $\tau \to \infty$ we have (up to error terms depending on $\|C_{k\tau}\| + \|[H, C_{k\tau}]\|$)

$$\langle \Omega, C_{1\tau}C_{2\tau}\Omega\rangle - \langle \Omega, C_{1\tau}\Omega\rangle \cdot \langle \Omega, C_{2\tau}\Omega\rangle = \langle \Omega, C_{1\tau}E_\Omega^\perp C_{2\tau}\Omega\rangle \longrightarrow 0. \tag{2.2.7}$$

This brings us to the second challenge posed by the presence of massless particles, namely absence of the lower mass gap and the associated question of validity of clustering estimates. The clustering estimates derived in [Fre85; AHR62] apply in the wedge-local setting with lower mass

[5] The original Reeh-Schlieder theorem established cyclicity of Ω for local polynomial field algebras in Wightman quantum field theories [RS61] [SW00, Thm. 4.2]. The Reeh-Schlieder argument can also be used to show cyclicity of Ω in the Haag-Kastler setting for domains $\mathcal{O} \subset \mathbb{R}^{s+1}$, which are such that any bounded region $K \in \mathbf{Reg}_b$ can be translated into $\mathcal{O}$ [A, Sec. 4.7, Rem. 2]. By this argument Ω is always cyclic for wedge algebras $\mathfrak{A}(\mathcal{W})$. Cyclicity of Ω also for the local algebras $\mathfrak{A}(\mathcal{O})$ is not automatic, but can be derived from other properties such as weak additivity, see e.g. [A, Sec. 4.7] or the introductory discussions in [Bu90b].

gap. To our knowledge there are presently no model-independent clustering estimates available in the general wedge-local framework which do not require the lower mass gap assumption, i.e. which are compatible with the presence of massless particles. Yet all presently known rigorous scattering theoretic constructions in quantum field theory require space-like clustering properties or the more stringent lower mass gap assumption. Hence the case without mass gaps appears presently out of reach in the wedge-local setting.

In the following we will consider the scattering problem in absence of mass gaps for *strictly local* framework from Definition 1.3.5. Then the conventional clustering estimates, and energy bounds from [Bu90a] are at our disposal. Further we will consider the case of four-dimensional Minkowski space-time as the clustering estimates without lower mass gap have been established in this context [AHR62].

So far we have argued that the Reeh-Schlieder property can be used to accelerate the convergence in the one-particle problem and improve the localization of creation-operator approximants for particles with embedded mass shells. Both of these achievements are helpful for proving convergence of multi-particle states. At first sight they appear to be suitable for resolving the commutation versus integrability tension in Haag-Ruelle theory without introducing the Herbst spectral condition. Yet the detailed inspection of the Cook convergence proof reveals additional requirements. While these have been rather straighforwardly satisfied in the smearing-approach (2.2.1), they do require special attention in the present Reeh-Schlieder context.

(1) We require differentiability of $\mathcal{B}_{k\tau}(f_k)$ with respect to τ in the uniform operator topology. In the smearing method this can be easily obtained by choosing χ_τ with smooth τ dependence. On the other hand the differentiability of Reeh-Schlieder families A_β as a function of $\beta > 0$ is unclear due to the non-constructive nature of the Reeh-Schlieder argument.

(2) The use of clustering and energy-bound arguments as in [Hrb71; Dy05] to estimate terms such as $\|\mathcal{A}_{1\tau}(f_1)\partial_\tau\mathcal{A}_{2\tau}(f_2)\Omega\|$ also produces specific error terms. In the Reeh-Schlieder context we expect these to be less well-behaved, but controllable relative to $\|A_{\beta(\tau)}\|$. Yet the construction via the Reeh-Schlieder property yields convergence only for $A_\beta\Omega$, and the norms $\|A_\beta^*\Omega\|$ and $\|A_\beta\|$ are in general expected to diverge as $\beta \to 0$.

We address (1) by a discretized Cook argument, requiring additionally that the norm growth of $\|A_\beta\|$ mentioned in point (2) is not too strong as $\beta \to 0$. Using the methods we developed in [Du17], it turns out that any polynomial growth $\|A_\beta\| \leq \beta^{-\gamma}$ is sufficient for the construction of multi-particle states. In particular, the exponents $\gamma > 0$ may be arbitrarily large for this purpose.

Definition 2.2.1 (Reeh-Schlieder operator family)**.** *A family of observables $(A_\beta)_{\beta>0}$ localized in some fixed bounded region $\mathcal{O}$ which satisfies for all sufficiently small $\beta > 0$*

$$\|A_\beta\Omega - \Psi\| \leq \beta, \qquad \|A_\beta\| \leq \beta^{-\gamma}. \tag{2.2.8}$$

will be called a Reeh-Schlieder family *of* degree $\gamma \geq 0$ *for $\Psi \in \mathscr{H}$. We further define the* Reeh-Schlieder degree $\gamma_{\mathrm{RS}} \in [0, \infty]$ *of a vector Ψ as the infimum over all $\gamma \geq 0$ for which there exists a corresponding Reeh-Schlieder family for Ψ of degree γ.*

One particle states $\Psi = \Psi_1 \in \mathscr{H}_1$ with $\gamma_{\mathrm{RS}}(\Psi_1) < \infty$ may be regarded as a sharpened Wigner particle concept. Reeh-Schlieder families A_β for one-particle states $\Psi_1 \in E(H_m)\mathscr{H}$ are easy to obtain in free QFT models, but the status of this strengthened Reeh-Schlieder condition in interacting theories is at present mostly unclear, see Section 5.2.

In the construction of scattering states for one-particle vectors $\Psi_1 \in \mathscr{H}_1$ with $\gamma_{\mathrm{RS}}(\Psi_1) < \infty$, the technical challenge consists in handling of the divergent norms of A_β as $\beta \to 0$. For technical reasons we will have to retain the χ-smearing in the definition of the creation-operator approximants. In the present approach it plays a different role compared to (2.2.1), namely to assure the applicability of energy bounds.

Definition 2.2.2 (Reeh-Schlieder creation-operator approximants)**.** *Let $A_\beta \in \mathfrak{A}(\mathcal{O})$ be a Reeh-Schlieder family for $\Psi_1 \in E(H_m)\mathscr{H}$, $m \geq 0$. Fixing $\hat{\chi} \in C_c^\infty(\mathbb{R}^4 \setminus \bar{V}^-)$ we set $B_\beta := A_\beta(\chi)$ and for $\tau \in \mathbb{R} \setminus \{0\}$ and a regular positive-energy Klein-Gordon solution f of the same mass m we define*

$$\mathcal{B}_\tau := \int \mathrm{d}^s x\, f(\tau, \mathbf{x}) B_{\beta(\tau)}(\tau, \mathbf{x}), \tag{2.2.9}$$

where $\beta(\tau) := |\tau|^{-\mu}$ for some fixed $\mu > 0$.

The main result of our work [Du17] is a Haag-Ruelle theorem based on the Reeh-Schlieder effect with a temperate norm growth as in (2.2.8). Its conclusions apply both for massive and massless particles and allow a uniform treatment of these two cases.

Theorem 2.2.3 (Haag-Ruelle theorem based on strengthend Reeh-Schlieder property)**.** *Let $A_{1\beta}$, ..., $A_{n\beta}$ be Reeh-Schlieder families of finite degree less than some $\gamma > 0$, let $f_1, \ldots, f_n$ be regular positive-energy Klein-Gordon solutions with disjoint velocity supports, and take a scaling exponent $\mu \in (0, \frac{\kappa}{4n\gamma})$ (here $\kappa = 3/2$ for $m > 0$ and $\kappa = 1 - \epsilon$ with some $\epsilon \in (0,1]$ when $m \geq 0$). Then*

(i) $\Psi_\tau := \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega$ *is convergent in norm as $\tau \to \pm\infty$.*

(ii) The limit is independent of the choice of μ, $A_{k\beta}$ and f_k within the specified restrictions, as long as the associated operators $\mathcal{B}'_{k\tau}$ create from the vacuum the same family of single-particle states $\Psi_k^{(1)} = \lim_{\tau\to\infty} \mathcal{B}_{k\tau}\Omega$.

(iii) Scalar products of any two outgoing scattering states of the above form are given by

$$\left\langle \Psi^+, \Psi'^+ \right\rangle = \delta_{nn'} \sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^{n} \left\langle \Psi_k^{(1)}, \Psi'^{(1)}_{\pi(k)} \right\rangle, \tag{2.2.10}$$

and similarly for incoming states.

Let us now consider the convergence proof for two-particle states $\Psi_\tau = \mathcal{B}_{1\tau}\mathcal{B}_{2\tau}\Omega$ as an example to illustrate how the Reeh-Schlieder norm divergence $\|A_\beta\| \to \infty$ enters in the limits $\tau \to \infty$ and $\beta \to 0$. This also allows to motivate and explain the technical tools which were developed in [Du17] for treating this problem. Firstly we should use a discrete analog of *Cook's method*,

$$\|\Psi_{\tau_N} - \Psi_{\tau_0}\| \leq \sum_{k=0}^{N-1} \left\|\Psi_{\tau_{k+1}} - \Psi_{\tau_k}\right\|, \tag{2.2.11}$$

where $(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}_{>0}$ is monotone increasing with $\tau_N \longrightarrow \infty$ for $N \to \infty$.[6] Let us note here that A. Pizzo has previously constructed asymptotic states for charged infraparticles in the nonrelativistic setting of the massless Nelson model [Pi05] via a similar discretized Cook

[6] Here we consider only the outgoing case $\tau \to +\infty$. The incoming case $\tau \to -\infty$ can be treated analogously.

argument. The first application of such discretizations in the relativistic context appears to be our work in [Du13; Du17].

One motivation for using the discretized Cook method is the mentioned unclear differentiability status of $\beta \longmapsto A_\beta$. A second and perhaps more interesting reason for studying the discretized Cook argument is that it allows to explore precisely how far one can safely move away from the conventional Haag-Ruelle-Cook argument towards the strategy taken by D. Buchholz in the scattering theory for massless particles. Similarly as in [Bu75a; Bu77], the computation (2.2.11) can be used to reduce the convergence of Ψ_τ to the one-particle problem.

It will be our goal to establish estimates on $\left\|\Psi_{\tau_{k+1}} - \Psi_{\tau_k}\right\|$, which are sufficiently strong to obtain existence of the limit $N \to \infty$ on the right-hand side of (2.2.11). Analogously to the conventional Cook argument (1.4.10), (1.4.11), we write

$$\begin{aligned}\|\Psi_{\tau_2} - \Psi_{\tau_1}\| &= \|\mathcal{B}_{1\tau_2}\mathcal{B}_{2\tau_2}\Omega - \mathcal{B}_{1\tau_1}\mathcal{B}_{2\tau_1}\Omega\| \\ &\leq \|(\mathcal{B}_{1\tau_2} - \mathcal{B}_{1\tau_1})\mathcal{B}_{2\tau_2}\Omega\| + \|\mathcal{B}_{1\tau_1}(\mathcal{B}_{2\tau_2} - \mathcal{B}_{2\tau_1})\Omega\| \\ &\leq \|[\mathcal{B}_{1\tau_2} - \mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}]\Omega\| \\ &\qquad + \|\mathcal{B}_{2\tau_2}(\mathcal{B}_{1\tau_2} - \mathcal{B}_{1\tau_1})\Omega\| + \|\mathcal{B}_{1\tau_1}(\mathcal{B}_{2\tau_2} - \mathcal{B}_{2\tau_1})\Omega\| \,. \end{aligned} \tag{2.2.12}$$

It is intuitively plausible that the equal-time commutator estimates used in standard Haag-Ruelle theory extend also to non-equal-time commutators $\|[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}]\|$ under the restriction that $|\tau_2 - \tau_1|$ is sufficiently small. Already rapid decay $\left\|[\mathcal{B}_{1\tau_k}, \mathcal{B}_{2\tau_{k+1}}]\right\| \leq C_N\, |\tau_k|^{-N}$ for equally spaced $\tau_k = \tau_0 + k\delta$ for some $\delta > 0$, $k \in \mathbb{N}$, would be sufficient for summability for $N \to \infty$ of these commutator terms inserted into (2.2.11). Yet, in the relativistic setting there is further room for improvement, and we shall return to this point in a moment.

Let us now consider the terms from (2.2.12) involving one-particle differences. Using the basic norm estimate

$$\|\mathcal{B}_{1\tau_1}\| \leq \int \mathrm{d}^s x \left\|f_1(\tau_1, \mathbf{x})\; B_{\beta(\tau_1)}(\tau_1, \mathbf{x})\right\| \leq \left\|A_{\beta(\tau_1)}\right\| \|\chi\|_{L^1(\mathbb{R}^{s+1})}\, \|f_{1\tau_1}\|_{L^1(\mathbb{R}^s)} \tag{2.2.13}$$

with $\|A_{\beta(\tau_1)}\| \leq \beta(\tau_1)^{-\gamma} \leq |\tau_1|^{\gamma\mu}$, and $\|f_{1\tau_1}\|_{L^1(\mathbb{R}^s)} \leq C_{f_1}(1 + |\tau_1|^{s/2})$, we can estimate for large enough $\tau_2 > \tau_1 > 0$ that

$$\begin{aligned}\|\mathcal{B}_{1\tau_1}(\mathcal{B}_{2\tau_2} - \mathcal{B}_{2\tau_1})\Omega\| &\leq \|\mathcal{B}_{1\tau_1}\| \left\|\mathcal{B}_{2\tau_2}\Omega - \Psi_2^{(1)}\right\| + \left\|\Psi_2^{(1)} - \mathcal{B}_{2\tau_1}\Omega\right\| \\ &\leq C(1 + |\tau_1|^{s/2}) \cdot \beta(\tau_1)^{-\gamma_1} \cdot (\beta(\tau_2) + \beta(\tau_1)) \\ &\leq C'\, |\tau_1|^{s/2} \cdot \beta(\tau_1)^{-\gamma_1} \cdot \beta(\tau_1) \leq C'\, |\tau_1|^{s/2 - \mu(1-\gamma_1)} \,, \end{aligned} \tag{2.2.14}$$

where $\Psi_2^{(1)} := \lim_{\tau\to\infty} \mathcal{B}_{2\tau}\Omega$, γ_1 denotes the Reeh-Schlieder degree of the family $(A_{1\beta})_{\beta>0}$, and where we also used monotone decrease of $\beta(\tau) = |\tau|^{-\mu}$ for $\tau > 0$. Therefore, if the Reeh-Schlieder degree satisfies $\gamma_1 < 1$ it is easy to obtain convergence in (2.2.11) simply by choosing a sufficiently large scaling $\mu > 0$ and equidistant $\tau_k := \tau_0 + k\delta$.

However, we expect that the assumption $\gamma < 1$ is too strong for interacting QFT, and we will conclude this section with a brief description of our strategy to improve on the estimates (2.2.11), (2.2.14), such that the convergence of scattering states can also be established for $\gamma > 1$. We shall explain the various technical estimates for the general n-particle approximants $\Psi_\tau^n = \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega$ and then briefly discuss the application to the two-particle case.

For $\gamma > 1$ the χ-smearing in Definition 2.2.2 becomes relevant, as it restricts the energy-momentum transfer of the creation-operator approximants. From this it follows that for any number of particles k, the $\Psi_\tau^k = \mathcal{B}_{1\tau} \ldots \mathcal{B}_{k\tau}\Omega$ have finite energy-momentum for any τ, i.e.

$$\Psi_\tau^k \in E(\operatorname{supp}\hat{\chi}_1 + \ldots + \operatorname{supp}\hat{\chi}_k)\mathscr{H} \subset E(\Delta)\mathscr{H} \tag{2.2.15}$$

for some large enough compact set $\Delta \subset \mathbb{R}^{s+1}$ independent of $1 \le k \le n$.

This allows us to address the problematic polynomially growing term $|\tau_1|^{s/2}$ in (2.2.14), whose speed of divergence cannot be directly controlled via the choice of the scaling $\mu > 0$. To achieve this we replace the basic norm estimate (2.2.13) with an energy-bound derived using the method of Buchholz [Bu90a]. In contrast to energy-bounds from [Dy05; Hrd13; DH14; AD17] the estimates in the present context are no longer uniform in τ as the bound inherits the norm growth of the Reeh-Schlieder family, which is controllable via the choice of scaling $\tau \longmapsto \beta(\tau)$.

Lemma 2.2.4 (Energy bounds)**.** *Without further restrictions on the families of operators A_β and $A_{k\beta} \in \mathfrak{A}(\mathcal{O})$, we have for any compact $\Delta \subset \mathbb{R}^{s+1}$,*

$$\|\mathcal{B}_\tau E(\Delta)\| \le C \left\|A_{\beta(\tau)}\right\|, \tag{2.2.16}$$

$$\|\mathcal{B}_{1\tau_1} \ldots \mathcal{B}_{n\tau_n} E(\Delta)\| \le C \prod_{k=1}^{n} \left\|A_{k\beta(\tau_k)}\right\|, \tag{2.2.17}$$

where the constant C depends on Δ, $\mathcal{O}$, $\operatorname{supp}\hat{\chi}$, the number of operators n, and the corresponding wave packets $\tilde{f}$, $\tilde{f}_k$, but it is independent of τ.

For the two-particle case with energy-bound estimation, $\|\mathcal{B}_{1\tau_1}(\mathcal{B}_{2\tau_2} - \mathcal{B}_{2\tau_1})\Omega\| \le C\,|\tau_1|^{-\mu(1-\gamma_1)}$, and hence we still require $\gamma_1 < 1$. Yet we see that the result has improved: for fixed scaling $\mu > 0$ we obtain faster convergence, and we also have achieved admissibility of a larger range of scalings μ.

A further convergence improvement can be obtained from a closer investigation of the admissible time discretizations. With regard to the latter there are two basic observations when considering the expansion

$$\sum_{k=0}^{N-1} \left\|\Psi_{\tau_{k+1}} - \Psi_{\tau_k}\right\| \le \sum_{k=0}^{N-1} \Big(\left\|[\mathcal{B}_{1\tau_{k+1}}, \mathcal{B}_{2\tau_{k+1}}]\right\| + \left\|[\mathcal{B}_{1\tau_k}, \mathcal{B}_{2\tau_{k+1}}]\right\| + \left\|\mathcal{B}_{2\tau_{k+1}}(\mathcal{B}_{1\tau_{k+1}} - \mathcal{B}_{1\tau_k})\Omega\right\| + \left\|\mathcal{B}_{1\tau_k}(\mathcal{B}_{2\tau_{k+1}} - \mathcal{B}_{2\tau_k})\Omega\right\| \Big). \tag{2.2.18}$$

Firstly, if we are given fixed $\tau' > \tau_0 > 0$, the estimation term obtained from summation over the one-particle terms ($\tau_N = \tau'$)

$$\sum_{k=0}^{N-1} \Big(\left\|\mathcal{B}_{2\tau_{k+1}}(\mathcal{B}_{1\tau_{k+1}} - \mathcal{B}_{1\tau_k})\Omega\right\| + \left\|\mathcal{B}_{1\tau_k}(\mathcal{B}_{2\tau_{k+1}} - \mathcal{B}_{2\tau_k})\Omega\right\| \Big) \tag{2.2.19}$$

appears to become stronger if we reduce the number of discretization steps to be summed over. In other words we would like to increase the spacing $\delta > 0$ of the time steps $\tau_k := \tau_0 + k\delta$ as far as possible. Secondly, the commutator terms in (2.2.18) exhibit the exact opposite behaviour. This resembles the situation encountered in the smearing approach, cf. the discussion below (2.2.1). Yet here we have more refined control, as we can adjust the time discretization independently of the one-particle convergence speed.

Remark 2.2.5 (vanishing Reeh-Schlieder degree). To take this idea to the extreme let us suppose that $\|\mathcal{B}_{k\tau} E(\Delta)\| \le C$ uniformly in τ, as we could write via energy-bounds (2.2.16) for the case of vanishing degree $\gamma = 0$. Then for fixed spacing δ, the summability in (2.2.19) only depends on the speed of convergence of $\mathcal{B}_{k\tau}\Omega \to \Psi_k^{(1)}$. Further, if we could choose the trivial spacing with $\delta = \tau' - \tau_0$ (i.e. $N = 1$ and $\tau_1 = \tau'$), summability in (2.2.19) would be trivial and independent of the speed of convergence in the one-particle problem. In the massive case the presence of the non-equal-time commutators $\left\|[\mathcal{B}_{1\tau_k}, \mathcal{B}_{2\tau_{k+1}}]\right\|$ in (2.2.18) prevents us from deriving a meaningful estimate on (2.2.18) for arbitrarily large $\tau_1 > \tau_0$. In the massless case the situation is different, as the commutant of a suitably constructed creation-operator family $(\mathcal{B}_\tau)_{\tau>0}$ is non-trivial due to Huygens' principle [Bu75a; Bu77].

Regarding the optimal choice of the discretization we see that an investigation of non-equal-time commutators is warranted. We have established a rapid decay estimate for the case of disjoint velocity supports, with τ_k constrained to linearly growing intervals

$$\tau_1, \tau_2 \in [\tau, (1+\rho)\tau] \tag{2.2.20}$$

where $\tau > 0$, and $\rho > 0$ is a constant, which depends on the geometry and on the separation of velocity supports as illustrated in Figure 2.3.

Lemma 2.2.6 (non-equal-time commutator estimate)**.** *Let $A_{k\beta}$, $(k = 1, 2)$, be Reeh-Schlieder families of finite degree, take regular Klein-Gordon solutions f_k with disjoint velocity supports and assume a fixed scaling $\beta(\tau) = |\tau|^{-\mu}$, $\mu > 0$. Then there exists $\rho > 0$ and for any $N \in \mathbb{N}$ a constant $C_N > 0$, such that for arbitrary $\tau \in \mathbb{R} \setminus \{0\}$ and all τ_1, τ_2 from the corresponding interval spanned by τ and $\tau + \rho\tau$,*

$$\|[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}]\| \leq C_N (1 + |\tau|)^{-N}. \tag{2.2.21}$$

By means of Lemma 2.2.6 it follows that the relativistic discretized Cook method admits a geometric time-spacing $\tau_k := (1+\rho)^k \tau_0$, with $\rho > 0$ depending on the separation of the velocity supports, and some $\tau_0 > 0$, as recognized in [Du13]. Returning to our benchmark case of degree $\gamma < 1$, we now obtain convergence for any choice of the scaling $\mu > 0$.

With this improved estimation strategy it finally becomes possible to address the convergence issue also for $\gamma \geq 1$. Here we use clustering estimates from [AHR62] which give space-like decay of vacuum correlations that can compensate the norm growth $\left\|A_{\beta(\tau)}\right\| \leq \beta(\tau)^{-\gamma}$ for sufficiently small scaling $\mu > 0$, and which yield corresponding clustering identities for creation-operator approximants. A convenient formulation for proving our main result (Theorem 2.2.3) admits multiple pairs $\mathcal{B}^*_{k\tau_k}\mathcal{B}_{k\tau_k}$ at distinct times τ_k, which are not restricted relative to each other.

Lemma 2.2.7 (multi-operator clustering)**.** *For $\tau_1, \ldots, \tau_n \in \mathbb{R} \setminus \{0\}$ denote by $|\tau_{\min}|$ and $|\tau_{\max}|$ the minimum and maximum of absolute values $|\tau_k|$, $(1 \leq k \leq n)$, respectively. Then for large enough $\tau_{\min}$ we have*

$$\left\| E_\Omega^\perp \left(\prod_{k=1}^{n} \mathcal{B}^*_{k\tau_k}\mathcal{B}_{k\tau_k} \right) \Omega \right\| \leq C\, |\tau_{\max}|^{2n\gamma\mu} \cdot |\tau_{\min}|^{-\kappa/2}\,. \tag{2.2.22}$$

The constant C is independent of the τ_k, but depends on the number of creation-annihilation-operator approximant pairs n, wave packets, Reeh-Schlieder families, and the smearing χ, and the cluster coefficient κ is as in Theorem 2.2.3.

An important step for proving Lemma 2.2.7 is the case $n = 1$, which is obtained by refining the clustering results from [Dy05] to admit the Reeh-Schlieder operator dependence and norm growth. Here it is important that the τ_k agree within each creation-annihilation-operator approximant pair. In particular, the clustering estimate of [AHR62] by itself is not suitable for

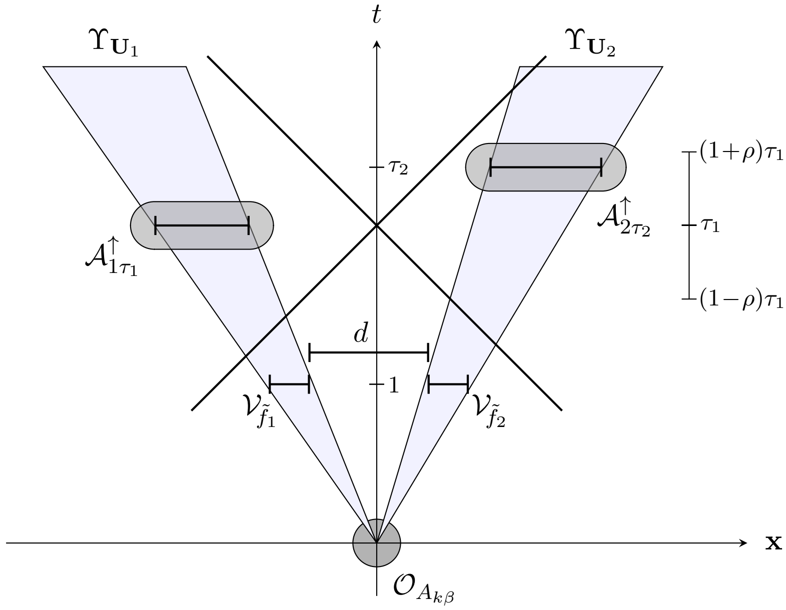

Figure 2.3: Localization regions of asymptotically dominant parts $\mathcal{A}^{\uparrow}_{k\tau_k}$ with disjoint velocity supports and $\tau_1 \neq \tau_2$ (schematically; a separating pair of wedges is indicated, restricting $|\tau_2 - \tau_1|$).

the geometric τ_k-spacing. This is also what prevents us from applying Lemma 2.2.7 to estimate the norm differences $\left\|\Psi_{\tau_{k+1}} - \Psi_{\tau_k}\right\|$ directly. Yet by combining Lemma 2.2.7 with energy bounds and the commutator estimates, we obtain that the multi-particle difference terms in the discrete Cook approach can indeed be bounded by the corresponding one-particle differences.

Proposition 2.2.8 (local difference estimate)**.** *Consider two families of creation operators $\mathcal{B}_{k\tau} := \mathcal{B}_{k\tau}(f_k)$, $\mathcal{B}'_{k\tau} := \mathcal{B}'_{k\tau}(f'_k)$, $(1 \leq k \leq n)$ with disjoint velocity supports within each family and such that the velocity supports of f_k and f'_j are mutually disjoint for $k \neq j$. Then there exists $\rho > 0$ and some sufficiently small scaling coefficient $\mu > 0$, so that with certain constants $C_1, C_2 > 0$ and for sufficiently large $|\tau| > 0$ and any subsequent choice of τ_1, τ_2 from the interval spanned by τ and $(1+\rho)\tau$, we have*

$$\left\|\Psi_{\tau_2} - \Psi'_{\tau_1}\right\| \leq C_1 \sum_{k=1}^{n} \left\|\mathcal{B}_{k\tau_2}\Omega - \mathcal{B}'_{k\tau_1}\Omega\right\| + C_2\, |\tau|^{n\gamma\mu - \kappa/4}\,. \tag{2.2.23}$$

Here $\kappa > 0$ is as in Theorem 2.2.3, and n denotes the number of particles.

In fact, this proposition collects all technical estimates for the convergence proof of Theorem 2.2.3, and provides the desired reduction to the one-particle convergence within the specified τ_k-restriction.

3 Mathematical methods

Beyond the challenges faced in the construction and analysis of non-perturbative physical QFT models, quantum field theory also remains an inspiring source of fascinating and deep mathematics. In this section we will give a brief overview of the general mathematical methods, on which our constructions and investigations rely.

3.1 Spectral theory of strongly-continuous unitary groups

The Wigner particle concept is defined in terms of the infinitesimal generators $G = (H, \boldsymbol{P})$ of space-time translations. In the QFT setting, these generators are unbounded and only defined on their respective domains. From this perspective it is mathematically not automatic that objects such as the relativistic mass operator

$$M := \sqrt{H^2 - P_1^2 - \ldots - P_s^2} \tag{3.1.1}$$

are well defined. The existence of the generators, and the domain and self-adjointness properties required to define (3.1.1) can be derived from the description of space-time translations as a strongly continuous group of unitary operators

Definition 3.1.1. *A map $\mathbb{R}^n \ni x \longmapsto U(x) \in \mathrm{B}(\mathscr{H})$ defines a strongly continuous n-parameter unitary group if for all $x, x' \in \mathbb{R}^n$,*

$$\begin{aligned} U(x)U(x') &= U(x + x'), \\ U(x)^* &= U(x)^{-1} = U(-x), \qquad (3.1.2) \\ \lim_{x \to x'} U(x)\Psi &= U(x')\Psi \;\forall \Psi \in \mathscr{H}. \qquad (3.1.3) \end{aligned}$$

Theorem 3.1.2 (Stone-Naimark-Ambrose-Godement (SNAG), [RS1] Thm. VIII.12)**.** *Every strongly-continuous n-parameter unitary group $\mathbb{R}^n \ni x \longmapsto U(x)$ can be uniquely represented by a projection-operator valued measure $E(\Delta)$ defined on Borel sets $\Delta \subset \mathbb{R}^n$ via*

$$U(x) = \int \mathrm{d}E(k) \exp\left(\mathrm{i} \sum_{j=1}^{n} x_j k_j\right), \; x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{3.1.4}$$

The spectral measure $E(\Delta)$ yields a decomposition of vectors $\Psi \in \mathscr{H}$ into their spectral components. Equivalently one can define infinitesimal generators on a dense domain of vectors Ψ by

$$G_j \Psi = \lim_{\epsilon \to 0} \frac{U(\epsilon e_j) - \mathbb{1}}{\mathrm{i}\epsilon} \Psi = \int \mathrm{d}E(k)\, k_j \Psi, \tag{3.1.5}$$

where $(e_j)_{1 \le j \le n}$ denotes the canonical basis in $\mathbb{R}^n$. In general the generators G_j are unbounded operators, which are strongly commuting and essentially self-adjoint on a common U-invariant dense domain of vectors D_G. The joint spectral calculus of the G_j for Borel measurable

$f : \mathbb{R}^n \longrightarrow \mathbb{C}$ is conventionally written as

$$f(G) = f(G_1, \dots, G_n) := \int \mathrm{d}E(k) f(k), \tag{3.1.6}$$

so that in particular $U(x) = \mathrm{e}^{\mathrm{i}G\cdot x}$.

Theorem 3.1.3 (spectral theorem for strongly commuting families of self-adjoint operators [RS1])**.** *Let $G = (G_1, \dots G_n)$ be a family of strongly commuting self-adjoint operators.*

(i) The joint spectral calculus (3.1.6) provides a continuous homomorphism from the $*$-algebra of bounded Borel measurable functions $g : \mathbb{R}^n \longrightarrow \mathbb{C}$ to the von Neumann algebra generated by $G_1, \dots, G_n$. That is,

$$\begin{aligned} g_1(G) + \lambda g_2(G) &= (g_1 + \lambda g_2)(G), \\ g(G)^* &= g^*(G), \\ g_1(G) g_2(G) &= (g_1 \cdot g_2)(G), \\ \|g(G)\| &\leq \|g\|_\infty . \end{aligned}$$

(ii) If $g_j : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a bounded family of measurable functions which converge pointwise to $g(x) := \lim_{j\to\infty} g_j(x)$, then $g_j(G)$ converges to $g(G)$ with respect to the strong operator topology.

The generators G_k are often also physically significant as they can be interpreted as quantities that are conserved under the action of $U(x)$. When considering a local or wedge-local quantum field theory on Minkowski space-time of dimension $d := s+1$, the generators of the representation of space-time translations are most conveniently defined using the relativistic convention $U(x) = U(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}Ht - \mathrm{i}\boldsymbol{P}\cdot\mathbf{x}}$, $x = (t, \mathbf{x}) \in \mathbb{R}^{s+1}$, to obtain the energy-momentum operators $(H, \boldsymbol{P}) = (H, P_1, \dots P_s)$. We recall that one-particle states $\Psi_1 \in \mathscr{H}_1$ are eigenvectors of the Klein-Gordon operator $M^2 = H^2 - \boldsymbol{P}^2$. Clearly the eigenspaces for fixed mass $m > 0$, can be equivalently written using the spectral measure as $E_{(H,\boldsymbol{P})}(H_m)\mathscr{H}$, where $H_m := \{(\omega_m(\mathbf{k}), \mathbf{k}) \in \mathbb{R}^{s+1} : \mathbf{k} \in \mathbb{R}^s\}$ and $\omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}$.

Let us conclude this section by discussing the application of the multivariate spectral calculus in Haag-Ruelle theory. There it provides a natural method for solving the one-particle problem by an explicit construction in terms of local or wedge-local operators.

Lemma 3.1.4 (basic Haag-Ruelle separation lemma)**.** *Consider a wedge-local quantum field theory whose energy-momentum spectrum contains an isolated mass shell of mass $m > 0$, i.e. $\mathscr{H}_1 = E_{(H,\boldsymbol{P})}(H_m)\mathscr{H} \neq \{0\}$. For any wedge $\mathcal{W}$ we consider vectors of the form*

$$\Psi_1 = g(H, \boldsymbol{P}) A\Omega \tag{3.1.7}$$

with $A \in \mathfrak{A}(\mathcal{W})$ and $g \in C_c^\infty(\mathbb{R}^{s+1})$. Then for $\operatorname{supp} g \cap (\operatorname{supp} E_{(H,\boldsymbol{P})} \setminus H_m) = \emptyset$ we have $\Psi_1 \in \mathscr{H}_1$ and the subspace spanned by such Ψ_1 is dense in $\mathscr{H}_1$.

Proof. Let $\Psi_1' \in \mathscr{H}_1$ be a given one-particle state and let $\epsilon > 0$. Then by cyclicity of Ω for $\mathfrak{A}(\mathcal{W})$ there exists $A \in \mathfrak{A}(\mathcal{W})$ s.t. $\|A\Omega - \Psi_1'\| \leq \epsilon/2$. Next we can approximate Ψ_1' to arbitrary precision by vectors of finite-energy momentum of the form $g(H, \boldsymbol{P})\Psi_1'$ with $g \in C_c^\infty(\mathbb{R}^{s+1})$ satisfying $\operatorname{supp} g \cap (\operatorname{supp} E_{(H,\boldsymbol{P})} \setminus H_m) = \emptyset$ and $g \leq 1$. Here we make use of the fact that we can separate any compact subset $\Delta \subset H_m$ from the remaining energy-momentum spectrum $\operatorname{supp} E_{(H,\boldsymbol{P})} \setminus H_m$ by such $g = g_\Delta \in C_c^\infty(\mathbb{R}^{s+1})$ satisfying $g_\Delta(p) = 1$ for $p \in \Delta$ and $g_\Delta(p) = 0$

on supp $E_{(H,\boldsymbol{P})} \setminus H_m$, as can be shown by a mollification argument. The claims now follow from the spectral calculus and Theorem 3.1.3: firstly we conclude from the support properties of g and $E_{(H,\boldsymbol{P})}$ that

$$\begin{aligned} g(H,\boldsymbol{P})A\Omega &= \int \mathrm{d}E(\omega,\mathbf{p})\, g(\omega,\mathbf{p})A\Omega \\ &= \int \mathrm{d}E(\omega,\mathbf{p})\, \mathbb{1}_{\operatorname{supp} E_{(H,\boldsymbol{P})}\cap \operatorname{supp} g}(\omega,\mathbf{p})\, g(\omega,\mathbf{p})A\Omega \\ &= \int \mathrm{d}E(\omega,\mathbf{p})\, \mathbb{1}_{H_m}(\omega,\mathbf{p}) g(\omega,\mathbf{p})A\Omega \\ &= (\mathbb{1}_{H_m}\cdot g)(H,\boldsymbol{P})A\Omega = \mathbb{1}_{H_m}(H,\boldsymbol{P})g(H,\boldsymbol{P})A\Omega \\ &= E(H_m)g(H,\boldsymbol{P})A\Omega, \end{aligned} \tag{3.1.8}$$

so that $g(H,\boldsymbol{P})A\Omega \in E(H_m)\mathscr{H}$.

Secondly we obtain from Theorem 3.1.3 (ii) similarly that $g_\Delta(H,\boldsymbol{P}) \longrightarrow E(H_m)$ strongly for $\Delta \nearrow H_m$, so that for sufficiently large $\Delta \subset H_m$ we have $\|g_\Delta(H,\boldsymbol{P})A\Omega - E(H_m)A\Omega\| \leq \epsilon/2$. Finally we can estimate

$$\begin{aligned} \left\|\Psi_1' - g_\Delta(H,\boldsymbol{P})A\Omega\right\| &= \|E(H_m)(\Psi_1 - g_\Delta(H,\boldsymbol{P})A\Omega)\| \\ &\leq \|E(H_m)(\Psi_1 - A\Omega)\| + \|E(H_m)(A\Omega - g_\Delta(H,\boldsymbol{P})A\Omega)\| \leq \epsilon, \end{aligned} \tag{3.1.9}$$

establishing density of these one-particle vectors in $E(H_m)\mathscr{H}$. □

Here we have used only the cyclicity (HK5) and the spectral gap condition (HK6$^\sharp$), but otherwise the argument is identical to the strictly local case. Let us note that the axiomatic operator-algebraic approach itself does not provide much further structural information on the state space $\mathscr{H}$. Identity (3.1.7) is the first step for making the connection from the spectral analysis of the Wigner particle content of a wedge-local QFT model to the localization structure of observables $A \in \mathfrak{A}(\mathcal{W})$. Together with the translation isomorphisms $\alpha_x(A) := U(x)AU(x)^*$ these are the central mathematical objects of the wedge-local operator-algebraic framework. But in Lemma 3.1.4 the relation of the constructed vectors Ψ_1 in terms of the space-time structure provided by $\mathfrak{A}(\mathcal{W})$ is not yet clear. Heuristically the choice of vectors $A\Omega$, $A \in \mathfrak{A}(\mathcal{W})$, and the fact that the construction (3.1.7) can be accomplished with $g \in C_c^\infty(\mathbb{R}^{s+1})$ having a smooth continuation outside the mass shell H_m implies that the wedge-localization information is not completely destroyed when passing from $A\Omega$ to $g(H,\boldsymbol{P})A\Omega$.

Lemma 3.1.5. *Let $f \in L^1(\mathbb{R}^{s+1})$ and $A \in \mathfrak{A}(\mathcal{W})$.*

(i) The weak integral

$$A(f) := \alpha_f(A) = \int \mathrm{d}^{s+1}x\, f(x)\alpha_x(A) \tag{3.1.10}$$

defines an element of the inductive limit $\overline{\cup_{x\in\mathbb{R}^{s+1}}\mathfrak{A}(\mathcal{W}+x)}^{\|\cdot\|} \subset \mathrm{B}(\mathscr{H})$.

(ii) For rapidly decreasing f, $A(f)$ is almost-wedge-local with respect to $\mathcal{W}$ in the following sense: denoting by $\mathcal{W}^r \in \mathbf{Reg}_W$ a wedge containing $\mathcal{W} + \mathscr{C}_r$ there exists a family of operators $A_r \in \mathfrak{A}(\mathcal{W}^r)$ $(r > 0)$ converging rapidly to $A(f)$. That is, for any $N \in \mathbb{N}$ there exists a constant C_N with

$$\|A_r - A(f)\| \leq \frac{C_N}{1 + r^N}. \tag{3.1.11}$$

Here $\mathscr{C}_r := \{(t,\mathbf{x}) \in \mathbb{R}^{s+1} : |t| + |\mathbf{x}| < r\}$ denotes the standard double cone of radius $r > 0$.

(iii) Due to translation-invariance of Ω *we have*

$$A(f)\Omega = (2\pi)^{(s+1)/2} \hat{f}(H, \boldsymbol{P}) A\Omega. \tag{3.1.12}$$

Proof. (standard) (i) follows by writing $A_r := A(f_r)$, where $f_r(x) := f(x)\mathbb{1}_{\mathscr{C}_r}(x)$. Here $A_r \in \mathrm{B}(\mathscr{H})$ is well defined due to continuity of $\alpha_x(A) = U(x)AU(x)^*$ with respect to the weak operator topology. Further $A_r \in \mathfrak{A}(\mathcal{W}^r)$ because $\alpha_x(A) \in \mathfrak{A}(\mathcal{W}^r)$ for all $x \in \operatorname{supp} f_r \subset \mathscr{C}_r$ by covariance (HK2), isotony (HK1), and weak integration preserves this inclusion as $\mathfrak{A}(\mathcal{W}^r)$ is weakly closed. The norm limit as $r \to \infty$ exists due to $\|A(f) - A(f_r)\| \le \|A\| \cdot \left\|\mathbb{1}_{\mathbb{R}^{s+1}\setminus\mathscr{C}_r} \cdot f\right\|_{L^1(\mathbb{R}^{s+1})} \longrightarrow 0$ as $r \to \infty$ as a consequence of $f \in L^1(\mathbb{R}^{s+1})$, so that it is in the quasi-wedge-local algebra associated to the centering of the localization wedge of A. For (ii) we note that for rapidly decreasing f we have the stronger estimate $\left\|\mathbb{1}_{\mathbb{R}^{s+1}\setminus\mathscr{C}_r} \cdot f\right\|_{L^1(\mathbb{R}^{s+1})} \le C_N/(1+r^N)$.

Finally, (iii) follows from the translation invariance of the vacuum by direct computation,

$$\begin{aligned}
A(f)\Omega &= \int \mathrm{d}^{s+1}x f(x)\alpha_x(A)\Omega = \int \mathrm{d}^{s+1}x f(x)U(x)AU(x)^*\Omega \\
&= \int \mathrm{d}^{s+1}x f(x)U(x)A\Omega = \int \mathrm{d}^{s+1}x f(x) \int \mathrm{d}E(k)\mathrm{e}^{\mathrm{i}k\cdot x}A\Omega \\
&= (2\pi)^{(s+1)/2} \int \mathrm{d}E(k)A\Omega \int \frac{\mathrm{d}^{s+1}x}{(2\pi)^{(s+1)/2}} f(x)\mathrm{e}^{\mathrm{i}k\cdot x} \\
&= (2\pi)^{(s+1)/2} \int \mathrm{d}E(k)A\Omega\, \hat{f}(k) = (2\pi)^{(s+1)/2}\hat{f}(H,\boldsymbol{P})A\Omega,
\end{aligned} \tag{3.1.13}$$

where (3.1.6) has been used twice and the integrations can be exchanged as a consequence of the integrability of $f(x)\mathrm{e}^{\mathrm{i}k\cdot x}$ with respect to the product measure (due to integrability of f and finiteness of the spectral measure). □

Let us conclude this section by noting that (ii) and (iii) provide the theoretical background for the construction of almost-wedge-local creation operators $B = A(f)$. Namely we can take $f = (2\pi)^{(s+1)/2}\check{g}$ with $g \in C_c^\infty(\mathbb{R}^{s+1})$ and $A \in \mathfrak{A}(\mathcal{W})$ as in Lemma 3.1.4: smoothness of g implies that the inverse Fourier transform $\check{g}$ is rapidly decreasing so that Lemma 3.1.5 (ii) applies. An interesting point of the two constructions is the fact that the operators $g(H,\boldsymbol{P})$ in Lemma 3.1.4 do not depend on $g(k)$ for $k \notin H_m$ due to the restrictions on $\operatorname{supp} g$ and $\operatorname{supp} E_{(H,\boldsymbol{P})}$ (and thereby also $A(\check{g})\Omega$ due to (iii)), whereas the smeared operators $A(\check{g})$ in general do depend on $g(k)$ also for $k \notin H_m$.

3.2 Spectral theory of automorphism groups

In Lemma 3.1.5 (iii) we have seen that the smearing operation (3.1.10) can be used to control the energy-momentum spectrum of $A(f)\Omega$. In fact, there holds a corresponding spectral restriction also for the action of $A(f)$ on the general spectral subspaces $E(\Delta)\mathscr{H}$, $\Delta \subset \mathbb{R}^{s+1}$. This can be established by means of the Arveson spectral theory for automorphism groups of C*-algebras [Arv80] [BR1, Sec. 3.2.3]. Here we will briefly introduce the Arveson spectrum and describe its application to Haag-Ruelle scattering theory.

Let $\mathfrak{A}$ be a C*-algebra, and let $\mathbb{R}^n \ni x \longmapsto \alpha_x : \mathfrak{A} \longrightarrow \mathfrak{A}$ be an additive automorphism group of $\mathfrak{A}$ such that the smeared operators $\alpha_f(A)$ as in (3.1.10) are well defined for all $f \in \mathscr{S}(\mathbb{R}^n)$, $A \in \mathfrak{A}$. In the present context this can be justified using the quantum field theory structure as sketched in the proof of Lemma 3.1.5 (i), but in a general C*-algebraic setting an additional continuity condition[1] on the action of α_x is required. Due to the bound $\|\alpha_f(A)\| \le \|A\| \, \|f\|_{L^1(\mathbb{R}^n)}$, the mapping $\alpha(A) : \mathscr{S}(\mathbb{R}^n) \ni f \longmapsto \alpha_f(A)$ defines for each $A \in \mathfrak{A}$ an operator-valued distribution, whose Fourier transform can be defined by duality as the operator-valued distribution

$$\hat{\alpha}(A) : g \longmapsto \hat{\alpha}_g(A) := \alpha_{\check{g}}(A), \tag{3.2.1}$$

where $\check{g}$ denotes the inverse Fourier transform of $g \in \mathscr{S}(\mathbb{R}^n)$.

Definition 3.2.1 (Arveson spectrum)**.** *The Arveson spectrum* $\mathrm{Sp}_A\, \alpha$ *of an operator* $A \in \mathfrak{A}$ *with respect to the automorphism group* $\mathbb{R}^n \ni x \longmapsto \alpha_x$ *is defined as the support of the operator-valued distribution* $\mathscr{S}(\mathbb{R}^n) \ni g \longmapsto \hat{\alpha}_g(A)$*.*

Proposition 3.2.2 (elementary properties of the Arveson spectrum)**.** *For* $A, B \in \mathfrak{A}$ *we have*

$$\begin{aligned} \mathrm{Sp}_{A+\lambda B}\, \alpha &\subset \mathrm{Sp}_A\, \alpha \cup \mathrm{Sp}_B\, \alpha, \;\; (\lambda \in \mathbb{C}), && (3.2.2)\\ \mathrm{Sp}_{A^*}\, \alpha &= -\mathrm{Sp}_A\, \alpha, && (3.2.3)\\ \mathrm{Sp}_{\alpha_x(A)}\, \alpha &= \mathrm{Sp}_A\, \alpha, \;\; (x \in \mathbb{R}^n), && (3.2.4)\\ \mathrm{Sp}_{\alpha_f(A)}\, \alpha &\subset \mathrm{Sp}_A\, \alpha \cap \mathrm{supp}\, \hat{f}, \;\; (f \in \mathscr{S}(\mathbb{R}^n)). && (3.2.5) \end{aligned}$$

Proof. The above relations on the supports of $\hat{\alpha}(A)$ are consequences of corresponding elementary relations for the Fourier transform (3.2.1), see e.g. [BR1, Lem. 3.2.38, 3.2.42]. □

For automorphism groups which are implemented by a strongly continuous unitary group $x \longmapsto U(x)$, i.e., $\alpha_x(A) = U(x)AU(x)^*$, the spectral transfer relation establishes a useful connection between the Arveson spectrum of $A \in \mathfrak{A}$ and the action of A on the spectral subspaces $E_G(\Delta)\mathscr{H}$ associated to the generators $G = (G_1, \ldots, G_n)$ of $U(x) = \exp(\mathrm{i}x \cdot G)$.

Lemma 3.2.3 (Spectral transfer relation)**.** *Let* $A \in \mathrm{B}(\mathscr{H})$*. Then for any Borel set* $\Delta \subset \mathbb{R}^n$*,*

$$AE_G(\Delta)\mathscr{H} \subset E_G(\overline{\Delta + \mathrm{Sp}_A\, \alpha})\mathscr{H}. \tag{3.2.6}$$

Proof. Follows from [Arv80, Thm. 3.5], or see [BDN15, App. B]. □

Let us return to the setting of local or wedge-local quantum field theory and the analysis of the action of $A(f)$ with respect to energy-momentum spectral measure $E(\Delta) := E_{(H,\boldsymbol{P})}(\Delta)$. From (3.2.5) and (3.2.6) we can infer that

$$A(f)E(\Delta)\mathscr{H} \subset E(\overline{\Delta + \mathrm{supp}\, \hat{f}})\mathscr{H}, \tag{3.2.7}$$

or equivalently

$$A(f)E(\Delta) = E(\overline{\Delta + \mathrm{supp}\, \hat{f}})A(f)E(\Delta). \tag{3.2.8}$$

When choosing $\Delta = \{0\}$ this is clearly consistent with the computation of $A(f)\Omega$ carried out in Lemma 3.1.5 (iii). Further we obtain analogously from (3.2.3) that

$$A(f)^*E(\Delta) = E(\overline{\Delta - \mathrm{supp}\, \hat{f}})A(f)^*E(\Delta). \tag{3.2.9}$$

In the following we will also have compact $\Delta \subset \mathbb{R}^n$ and compactly supported $\hat{f}$ so that the closures inside the spectral projections can be dropped.

For quantum field theory the first conclusion is that the smearing operation can be used to construct creation-operator approximants whose action and iterated action preserve the

[1] See e.g. [Arv80, p. 213 f.] or [BR1, Secs. 3.1.2 and 2.5.3].

(non-closed) subspace of finite energy states. This is a prerequisite for the applicability of energy bounds, which can be proven in local quantum field theories using the method of Buchholz as described in Section 3.3.

As a second application which is important for Haag-Ruelle theory one may use the annihilation relation (3.2.9) to obtain the clustering properties for massive creation-operator approximants in models with lower mass gap using an argument found in [DG14; Dy14], cf. also [BF82, Sec. 3] and [BDN15]. Here a refinement of the choice of the smearing function taken in Lemma 3.1.4 is required.

Lemma 3.2.4. *Consider a local or wedge-local quantum field theory with an isolated mass shell $H_m \subset \operatorname{supp} E_{(H,\boldsymbol{P})}$. Let $K \subset K'$ be compact subsets of H_m, such that K and $H_m \setminus K'$ are separated by a positive distance, and $A \in \mathfrak{A}$. Then there exists a Haag-Ruelle auxiliary function $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ such that*

(i) $A(\chi)\Omega \in E(K')\mathscr{H}$.

(ii) $E(K)A(\chi)\Omega = E(K)A\Omega$.

(iii) $A(\chi)^*\Psi = \Omega \cdot \langle \Omega, A(\chi)^*\Psi \rangle$ *for all* $\Psi \in E(K')\mathscr{H}$, *and in particular for* $\Psi = A(\chi)\Omega$.

Proof. (i) and (ii) follow from Lemma 3.1.5 (iii) as one can construct $\hat{\chi} \in C_c^\infty(\mathbb{R}^{s+1})$ satisfying $\hat{\chi}(k) = 1$ for $k \in K$, and which has support in an arbitrarily small neighbourhood $N \supset K$. In particular we can arrange due to the mass gap that $N \cap (\operatorname{supp} E \setminus H_m) = \emptyset$ and similarly that $N \cap (\operatorname{supp} E \setminus K') = \emptyset$.

To obtain (iii) we see from (3.2.9) that

$$\begin{aligned} A(\chi)^*\Psi = A(\chi)^*E(K')\Psi &= E(K' - \operatorname{supp}\hat{\chi})A(\chi)^*E(K')\Psi \\ &= E((K' - \operatorname{supp}\hat{\chi}) \cap \bar{V}^+)A(\chi)^*E(K')\Psi, \end{aligned} \tag{3.2.10}$$

where we used the spectral condition (HK6). Hence it only remains to show that there exists a sufficiently small neighbourhood $N \supset K$ within which to construct $\hat{\chi}$ as above, such that $(K' - N) \cap H_\mu = \emptyset$ for all $\mu \geq m$, where m denotes the lower mass gap of the theory as in (HK6$^\sharp$). For this we can make the ansatz $N := K + N_0$, where $N_0 \subset \mathbb{R}^{s+1}$ is some compact neighbourhood of zero energy-momentum. Now $(K - K') \setminus \{0\}$ contains only space-like vectors due to the geometry of the relativistic mass shell. Therefore $K - K'$ is disjoint from $\cup_{\mu \geq m} H_\mu \supset \operatorname{supp} E \setminus \{0\}$ and by compactness of $K - K'$ these two sets are separated by a positive distance $\delta > 0$. Hence we have for sufficiently small neighbourhoods N_0 of zero and $N = K + N_0$ that $(K' - N) \cap \operatorname{supp} E = \{0\}$ and if required we can further shrink N_0 to assure that the restrictions from (i) are satisfied. Hence for $\hat{\chi} \in C_c^\infty(\mathbb{R}^{s+1})$ satisfying $\hat{\chi}(k) = 1$, $\operatorname{supp}\hat{\chi} \subset N$ the final claim (iii) follows from (3.2.10), the Haag-Ruelle mass gap condition (HK6$^\sharp$) and the uniqueness of the vacuum (HK4). □

The clustering relations $\mathcal{B}_{1\tau}(f_1)^*\mathcal{B}_{2\tau}(f_2)\Omega = \Omega \cdot \langle \Omega, \mathcal{B}_{1\tau}(f_1)^*\mathcal{B}_{2\tau}(f_2)\Omega \rangle$ for creation-operator approximants constructed in terms of $B_k = A_k(\chi)$ with χ as above follow from Lemma 3.2.4 (iii) by a direct computation.

3.3 Spectral analysis of local operators

Energy bounds provide a general strategy for obtaining control over unbounded operators, which have the special property of becoming norm bounded when restricted to subspaces of finite energy-momentum $E(\Delta)\mathscr{H}$, ($\Delta \subset \mathbb{R}^{s+1}$ bounded). With the method of Buchholz [Bu90a], such energy bounds can be established in the context of local Haag-Kastler nets for energy decreasing almost-local operators as a consequence of locality and the C*-property $\|A^*A\| = \|A\|^2$. The analysis of Buchholz is based on two versatile central lemmas, which are an important tool in many modern treatments of scattering theory in local QFT. In the present section we will use the standard abbreviations $A(t, \mathbf{x}) := \alpha_{(t,\mathbf{x})}(A)$ and $A(\mathbf{x}) := \alpha_{(0,\mathbf{x})}(A)$ for $(t, \mathbf{x}) \in \mathbb{R}^{s+1}$ and $A \in \mathrm{B}(\mathscr{H})$.

Lemma 3.3.1 ([Bu90a], Lemma 2.1)**.** *Let $B \in \mathrm{B}(\mathscr{H})$, $n \in \mathbb{N}$, and denote by P_n the orthogonal projection onto the kernel of B^n. Then*

$$\|BP_n\|^2 = \|P_n B^* B P_n\| \leq (n-1)\,\|[B, B^*]\|\,. \tag{3.3.1}$$

Proof. By definition we have $\|P_1 B^* B P_1\| = \|BP_1\|^2 = 0$, and the result follows by induction

$$\begin{aligned}\|P_n B^* B P_n\| &= \|BP_n\|^2 = \|P_{n-1} B P_n\|^2 \leq \|P_{n-1}B\|^2 = \|P_{n-1} B B^* P_{n-1}\| \\ &\leq \|P_{n-1} B^* B P_{n-1}\| + \|[B, B^*]\| \leq (n-2)\,\|[B, B^*]\| + \|[B, B^*]\|\,. \qquad \square\end{aligned}$$

Lemma 3.3.2 ([Bu90a], Lemma 2.2)**.** *Let $K \subset \mathbb{R}^{s+1}$ compact, $B \in \mathrm{B}(\mathscr{H})$ and denote by P_n the orthogonal projection onto the intersection of the kernels of the n-fold products of space-like translated operators $B_1(\mathbf{x}_1)\ldots B_n(\mathbf{x}_n)$ for any configuration of $\mathbf{x}_1, \ldots \mathbf{x}_n \in \mathbb{R}^s$. Then*

$$\left\| P_n \int_K \mathrm{d}^s x\, (B^*B)(\mathbf{x})\, P_n \right\| \leq (n-1) \int\limits_{K-K} \mathrm{d}^s x\, \|[B^*, B(\mathbf{x})]\|\,. \tag{3.3.2}$$

These bounds also enable a refined analysis of the spectral structure of the automorphisms of space-like translations [Bu90a; Dy10; Hrd14]. For scattering theory the method has been successfully applied in various contexts. Some examples are the massless scattering theory [DH14; AD17; Du17], Haag-Ruelle theory for embedded mass shells [Dy05; Hrd13; Du17], the analysis of Araki-Haag detectors [Bu90a; DyG13; Du13], and applications in the theory of particle weights [BPS91].

For the Reeh-Schlieder-based construction the strengthened bound (3.3.2) has been used [Du17], but the proof of this bound is more elaborate and will not be reviewed here. Let us now briefly discuss the basic strategy of the derivation of τ-uniform energy bounds for the standard creation-operator approximants

$$\mathcal{B}_\tau(f) := \int \mathrm{d}^s x f(\tau, \mathbf{x}) B(\tau, \mathbf{x}), \tag{3.3.3}$$

where f is a regular Klein-Gordon solution for $m > 0$, $B = A(\chi)$ with $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ whose Fourier transform $\hat{\chi}$ is compactly supported inside the interior of $\bar{V}^+$, and $A \in \mathfrak{A}(\mathcal{O})$ for some bounded region $\mathcal{O} \subset \mathbb{R}^{s+1}$. We recall that the direct estimate (cf. also Section 3.5)

$$\|\mathcal{B}_\tau(f)\| \leq \|A\|\,\|\chi\|_{L^1(\mathbb{R}^{s+1})}\,\|f_\tau\|_{L(\mathbb{R}^s)} \leq C(1 + |\tau|^{s/2}) \tag{3.3.4}$$

becomes trivial in the scattering theoretic limits $\tau \to \pm\infty$.

Proposition 3.3.3. *For any compact $\Delta \subset \mathbb{R}^{s+1}$ we have $\|\mathcal{B}_\tau(f)E(\Delta)\| \leq C_\Delta$ with C_Δ independent of $\tau \in \mathbb{R}$.*

Proof. First we use the energy-momentum transfer relation to write

$$\begin{aligned}\|\mathcal{B}_\tau(f)E(\Delta)\| &= \|E(\Delta + \operatorname{supp}\hat{\chi})\mathcal{B}_\tau(f)E(\Delta)\| \\ &\leq \|E(\Delta + \operatorname{supp}\hat{\chi})\mathcal{B}_\tau(f)\| = \|\mathcal{B}_\tau(f)^* E(\Delta + \operatorname{supp}\hat{\chi})\|\,. \end{aligned} \tag{3.3.5}$$

Setting $\Delta' := \Delta + \operatorname{supp}\hat{\chi}$ we apply Lemma 3.3.1 to the energy-decreasing operator $B := \mathcal{B}_\tau(f)^*$ and obtain

$$\|BP_n\|^2 \leq (n-1)\,\|[B, B^*]\| = (n-1)\,\|[\mathcal{B}_\tau(f)^*, \mathcal{B}_\tau(f)]\|\,. \tag{3.3.6}$$

To establish the energy bound it remains to show that $E(\Delta') \leq P_n$ for sufficiently large n (depending on Δ and $\operatorname{supp}\hat{\chi}$), and that the commutator norm can be uniformly bounded in τ. For the former we note that as a consequence of the energy-momentum transfer relation and the spectral condition (HK6) we have

$$B^n E(\Delta') = E(\Delta' - \sum_n \operatorname{supp}\hat{\chi})B^n E(\Delta') = E(\bar{V}^+ \cap (\Delta' - \sum_n \operatorname{supp}\hat{\chi}))B^n E(\Delta'), \tag{3.3.7}$$

which vanishes for sufficiently large n by compactness of Δ' and using the fact that $\hat{\chi}$ is by assumption compactly supported at strictly positive energies. Hence we have for sufficiently large n that $E(\Delta')\mathscr{H} \subset \ker B^n = P_n\mathscr{H}$ and

$$\|\mathcal{B}_\tau(f)^* E(\Delta + \operatorname{supp}\hat{\chi})\| = \|\mathcal{B}_\tau(f)^* P_n E(\Delta + \operatorname{supp}\hat{\chi})\| \leq \|\mathcal{B}_\tau(f)^* P_n\|\,. \tag{3.3.8}$$

Finally, to estimate the commutator we write

$$\begin{aligned}\|[\mathcal{B}_\tau(f), \mathcal{B}_\tau(f)^*]\| &\leq \int \mathrm{d}^s x\, \mathrm{d}^s y\, |f(\tau, \mathbf{x})|\, |f(\tau, \mathbf{y})^*|\, \|[B(\tau, \mathbf{x}), B(\tau, \mathbf{y})^*]\| \\ &\leq \|f_\tau\|_\infty \int \mathrm{d}^s x\, \mathrm{d}^s y\, |f(\tau, \mathbf{x})|\, \|[B, B^*(0, \mathbf{y} - \mathbf{x})^*]\| \\ &\leq \|f_\tau\|_\infty \int \mathrm{d}^s x\, \mathrm{d}^s y'\, |f(\tau, \mathbf{x})|\, \left\|[B, B^*(0, \mathbf{y}')^*]\right\| \\ &= \|f_\tau\|_\infty\, \|f_\tau\|_{L^1(\mathbb{R}^s)} \int \mathrm{d}^s y'\, \left\|[B, B^*(0, \mathbf{y}')^*]\right\| \\ &\leq C_{f,\chi}(1 + |\tau|)^{-s/2} \cdot (1 + |\tau|^{s/2})\, \|B\|^2 \leq C'\, \|B\|^2\,, \end{aligned}$$

where we used the standard estimates for regular Klein-Gordon solutions and the integrability of $\|[B, B^*(0, \mathbf{y}')^*]\| \leq C_N^{\hat{\chi}}\, \|B\|^2\, (1 + |\mathbf{x}'|)^{-N}$ for sufficiently large N. □

Remark 3.3.4. The energy bounds of Proposition 3.3.3 can also be derived as in [Bu90a] by making use of deeper results from [Bu90a] on the Fourier transform of the space-like translates $\alpha_{\mathbf{x}}(A)$ of local operators $A \in \mathfrak{A}(\mathcal{O})$.

3.4 Tomita-Takesaki modular theory

The Tomita-Takesaki modular theory is an essential tool in the theory of von Neumann algebras. Let $\mathfrak{M} \subset \mathrm{B}(\mathscr{H})$ be a von Neumann algebra and $\Omega \in \mathscr{H}$ and be a vector which is cyclic and separating for $\mathfrak{M}$. In this general context the modular theory provides an important emergent symmetry structure.

Definition 3.4.1 (modular objects). *Using the adjoint $*$-operation on $\mathfrak{M}$ one obtains via*

$$S : \begin{cases} \mathfrak{M}\Omega \longrightarrow \mathfrak{M}\Omega \\ A\Omega \longmapsto A^*\Omega \end{cases} \tag{3.4.1}$$

a closable unbounded operator. The polar decomposition $S = J\Delta^{1/2}$ yields the antiunitary modular conjugation J and the positive modular operator Δ.

Details can be found e.g. in [BR1, Sec. 2.5] or [KR2, Sec. 9.2]. For our purposes we will only require the fundamental Tomita-Takesaki theorem, which justifies the importance of the above construction.

Theorem 3.4.2 (Tomita-Takesaki).

$$J\mathfrak{M}J = \mathfrak{M}', \qquad \Delta^{\mathrm{i}\tau}\mathfrak{M}\Delta^{-\mathrm{i}\tau} = \mathfrak{M} \quad (\forall\tau \in \mathbb{R}). \tag{3.4.2}$$

Proof. See [BR1, Sec. 2.5]. □

Tomita-Takesaki theory has found various fruitful applications to QFT in the operator algebraic approach, see e.g. [BW75; BDL90; Bor95; Bor00; BGL02; BL04]. In the wedge-local multi-particle scattering theory, modular theory provides a method to obtain a dense subspace of vectors $\mathscr{H}_\mathcal{W} \subset \mathscr{H}$ which satisfy the wedge swapping symmetry for any given wedge $\mathcal{W} \in \mathbf{Reg}_W$. More precisely, for $\Psi \in \mathscr{H}_\mathcal{W}$ there exist $A \in \mathfrak{A}(\mathcal{W})$ and $A^\perp \in \mathfrak{A}(\mathcal{W}')$ such that

$$\Psi = A\Omega = A^\perp\Omega. \tag{3.4.3}$$

This symmetry is a central ingredient for justifying the construction of ordered multi-particle scattering states and for proving their Fock structure in the general wedge-local setting.

Let us briefly describe the argument establishing the existence of swappable Ψ in general wedge-local models. For $A = A^* \in \mathfrak{M} := \mathfrak{A}(\mathcal{W})$ we have formally[2]

$$A\Omega = A^*\Omega = SA\Omega = J\Delta^{1/2}A\Omega = J\Delta^{1/2}A\Delta^{-1/2}J\Omega, \tag{3.4.4}$$

and the linear combinations of the vectors $\Psi = A\Omega$ are dense by cyclicity of Ω for $\mathfrak{M}$. For a suitable weakly dense set of $A \in \mathfrak{M}$ one can proceed to show that $A_1 := \Delta^{1/2}A\Delta^{-1/2}$ defines an element of $\mathfrak{M}$ (here one has to take into account that Δ is unbounded and that (3.4.2) requires $\tau \in \mathbb{R}$). Then we obtain from the Tomita-Takesaki theorem $A^\perp := JA_1J \in \mathfrak{M}'$, and the wedge duality condition (HK2$^\sharp$) yields $A^\perp \in \mathfrak{A}(\mathcal{W})' = \mathfrak{A}(\mathcal{W}')$ as required to establish the swapping symmetry.

3.5 Stationary phase analysis

The original Haag-Ruelle estimates for the construction of two- and multi-particle scattering states have been significantly strengthened by Hepp [Hep65; Hep66], by taking into account the propagation geometry of Klein-Gordon wave packets. Geometrical propagation estimates for regular Klein-Gordon solutions can be established using standard stationary- and non-stationary phase techniques, and have been first worked out by Ruelle [Ru62, Sec. 3].

[2] I am grateful to Detlev Buchholz for pointing out this argument.

The stationary phase analysis is a method to establish bounds on the asymptotics of integrals of the form
$$\int \mathrm{d}^s k\, \mathrm{e}^{\mathrm{i}\xi\omega(\mathbf{k})} u(\mathbf{k}) \tag{3.5.1}$$
as a function of $\xi > 0$ in the oscillatory regime of large ξ. Typical assumptions are $u \in C_c^\infty(\mathbb{R}^s)$ and $\omega : \mathbb{R}^s \longrightarrow \mathbb{R}$ being arbitrarily often differentiable on the support of u. In many practical applications the analysis of (3.5.1) can be reduced to two basic estimates concerning the special cases that the *phase function* ω has no critical point or a single critical point $\mathbf{k}_0 \in \mathbb{R}^s$ with invertible second derivative on the support of u.

Lemma 3.5.1 (non-stationary phase estimate, [RS3], Thm. XI.14)**.** *Let the phase function $\omega : \mathbb{R}^s \longrightarrow \mathbb{R}$ be arbitrarily often differentiable, $u \in C_c^\infty(\mathbb{R}^s)$, and $\nabla\omega(\mathbf{k}) \neq 0$ for all $\mathbf{k} \in \operatorname{supp} u$. Then for any $\xi \in \mathbb{R}$*
$$\left| \int \mathrm{d}^s k\, \mathrm{e}^{\mathrm{i}\xi\omega(\mathbf{k})} u(\mathbf{k}) \right| \leq C_N (1 + |\xi|)^{-N} \|u\|_{C^N(\mathbb{R}^s)}\,, \tag{3.5.2}$$
where the constants C_N can be chosen uniformly for all functions $u \in C_c^\infty(\mathbb{R}^s)$ with support in a fixed compact set $K \subset \mathbb{R}^s$, and $\|u\|_{C^N(\mathbb{R}^s)} := \sup_{\mathbf{x}\in\mathbb{R}^s} \sum_{\alpha\in\mathbb{N}_0^s, |\alpha|\leq N} |\partial^\alpha u(\mathbf{x})|$.

Lemma 3.5.2 (stationary-phase estimate, [RS3], Thm. XI.15)**.** *Let the phase function $\omega : \mathbb{R}^s \longrightarrow \mathbb{R}$ be arbitrarily often differentiable, $u \in C_c^\infty(\mathbb{R}^s)$. Assume that $\nabla\omega(\mathbf{k}) \neq 0$ for all $\mathbf{k} \in \operatorname{supp} u \setminus \{\mathbf{k}_0\}$ with the exception of one stationary point $\mathbf{k}_0 \in \operatorname{supp} u$, $\nabla\omega(\mathbf{k}_0) = 0$, with invertible second derivative $D^2\omega(\mathbf{k}_0)$. Then*
$$\left| \int \mathrm{d}^s k\, \mathrm{e}^{\mathrm{i}\xi\omega(\mathbf{k})} u(\mathbf{k}) \right| \leq C (1 + |\xi|)^{-s/2} \|u\|_{C^N(\mathbb{R}^s)}\,, \tag{3.5.3}$$
for some $N > s/2$ and with C depending on ω.

It is instructive to begin by visualizing the geometrical meaning of the different regimes of the Klein-Gordon stationary-phase estimates by a simple example using the classical dynamics of free classical point particles.

Remark 3.5.3 (propagation of classical point particles)*.* The classical trajectory of a point particle passing through a point $\mathbf{x}_0 \in \mathbb{R}^s$ at time $t = 0$ with non-relativistic velocity $\mathbf{v} \in \mathbb{R}^s$ is given by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$, $t \in \mathbb{R}$. If only the initial positions and velocities are known to be constrained within some given sets $\mathbf{X}_0 \subset \mathbb{R}^s$ and $\mathbf{V} \subset \mathbb{R}^s$, respectively, we get simply
$$\mathbf{x}(t) \in \mathbf{X}_0 + t\mathbf{V}. \tag{3.5.4}$$
Finally for scattering theory we are interested in large $|t|$. In this regime there is a simple trick which allows to assume $\mathbf{X}_0 = \{0\}$. Using that velocities are at most equal to the speed of light, $\mathbf{V} \subset \overline{\mathrm{B}}_1(0)$, and $\mathbf{X}_0 \subset \mathrm{B}_R(0)$ for some $R > 0$, we simply pass to a slightly larger neighbourhood ($\delta > 0$)
$$\mathbf{V}^\delta := \{\mathbf{w} \in \mathbb{R}^s : \exists \mathbf{v} \in \mathbf{V} : |\mathbf{w} - \mathbf{v}| < \delta\} = \mathbf{V} + \mathrm{B}_\delta(0). \tag{3.5.5}$$
Then we have for sufficiently large $t \geq T_0 := T_0(\mathbf{X}_0, \delta) := R/\delta$ that $\mathbf{x}(t) \in t\mathbf{V}^\delta$. This provides a heuristic explanation for the enlargement of the essential support regions in the following Lemma 3.5.4.

With this example in mind let us return to regular positive-energy Klein-Gordon solutions
$$f(t, \mathbf{x}) := \int \mathrm{d}^s k\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_m(\mathbf{k})t} \tilde{f}(\mathbf{k}), \quad \tilde{f} \in C_c^\infty(\mathbb{R}^s), \quad \omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}. \tag{3.5.6}$$

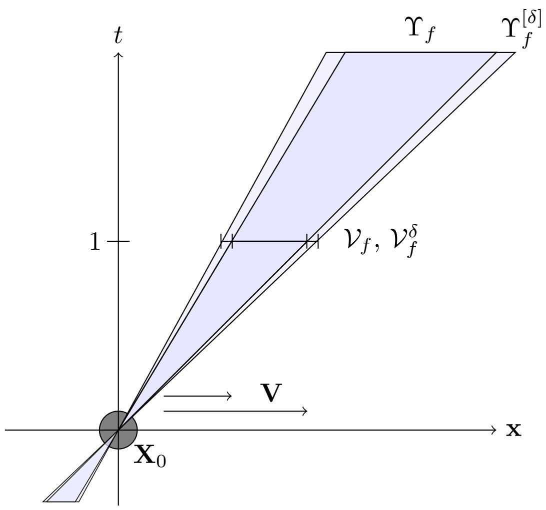

Figure 3.1: Velocity supports and classical propagation region.

Setting $\xi := t$ inspection of the phase function $\omega_{\mathbf{x}/t}(\mathbf{k}) := \mathbf{k}\cdot\mathbf{x}/t - \omega_m(\mathbf{k})$ shows that it is free of critical points on the support of the integrand when evaluated for parameters $(t, \mathbf{x}) \in \mathbb{R}^{s+1}$ satisfying

$$\frac{\mathbf{x}}{t} \notin \left\{ \frac{\mathbf{k}}{\omega_m(\mathbf{k})} \in \mathbb{R}^s : \mathbf{k} \in \operatorname{supp} \tilde{f} \right\} =: \mathbf{V}_f. \tag{3.5.7}$$

Comparison to Remark 3.5.3 shows that $\mathbf{V}_f$ can be interpreted as a set of velocities and from this we may expect $|f(t, \mathbf{x})|$ to decay rapidly outside the propagation region

$$\Upsilon_f := \{\xi \cdot (\omega_m(\mathbf{k}), \mathbf{k}) \in \mathbb{R}^{s+1} : \mathbf{k} \in \operatorname{supp} \tilde{f}, \xi \in \mathbb{R}\}. \tag{3.5.8}$$

For uniform decay estimates it is helpful to introduce the uniformly enlarged neighbourhoods

$$\Upsilon_f^{[\delta]} := \{\xi \cdot (1, \mathbf{v}') \in \mathbb{R}^{s+1} : \mathbf{v}' \in \mathbf{V}_f^{\delta},\ \xi \in \mathbb{R}\}, \quad \text{where } \mathbf{V}_f^{\delta} := \mathbf{V}_f + \mathrm{B}_\delta(0), \quad (\delta > 0), \tag{3.5.9}$$

and $\mathrm{B}_\delta(0) \subset \mathbb{R}^s$ denotes the centered open ball of radius δ.

Lemma 3.5.4 (Klein-Gordon estimates)**.** *Let f be a regular positive-energy solution of the Klein-Gordon equation for mass $m > 0$ of the form* (3.5.6) *and let $\delta > 0$.*

(i) $|f(t, \mathbf{x})| \leq C_{N,\delta}(1 + |t| + |\mathbf{x}|)^{-N}$ *for all* $(t, \mathbf{x}) \in \mathbb{R}^{s+1} \setminus \Upsilon_f^{[\delta]}$ *and any* $N \in \mathbb{N}$.

(ii) $|f(t, \mathbf{x})| \leq C_f(1 + |t|)^{-s/2}$ *for all* $(t, \mathbf{x}) \in \mathbb{R}^{s+1}$.

Proof. Estimate (ii) is established in [RS3, Thm. XI.17]. For (i) see e.g. [A, Sec. 5.3]. □

The geometric bound (ii) is of particular significance for Haag-Ruelle theory. It can be used to prove commutator estimates for creation-operator approximants with rapid asymptotic decay via the Hepp strategy, if the corresponding velocity supports are disjoint or suitably ordered.

4 Literature and discussion

In this section we will briefly review related and relevant works, which provide the wider scientific context and potential applications for our results. We will begin with a short summary of motivations for the axiomatic and wedge-local perspectives, and comment on the important role of the wedge-local view-point in the more recent operator-algebraic approach to constructive quantum field theory [Le15]. Within this wedge-local constructive programme the existence of a large classes of integrable QFT models in two dimensions has been established, and also constructions for wedge-local theories in higher dimensions are available in the literature.

4.1 Emergence of the wedge-local perspective

Let us begin with a brief description of influential general results which highlight the importance of wedge algebras in QFT. The first appearance of operator algebras $\mathfrak{A}(\mathcal{W})$ associated to wedge-regions $\mathcal{W}$ in axiomatic quantum field theory is found in the work of Bisognano and Wichmann [BW75]. They compute the modular objects J and Δ as defined in Section 3.4 for wedge algebras $\mathfrak{A}(\mathcal{W})$ which are generated by a Wightman field $\phi(f)$. In this case the action of the modular objects is geometric, and for $\mathcal{W} = \mathcal{W}_{\mathrm{r}}$ explicitly given by the *Bisognano-Wichmann property*

$$\begin{aligned} \Delta^{\mathrm{i}\tau} &= U(\Lambda^{2\pi\tau}), \\ J &= \Theta U(R_1(\pi)), \end{aligned} \tag{BW}$$

where U denotes the representation of the Lorentz group of the Wightman QFT, Λ^{β} are the standard Lorentz boosts (1.3.9) with rapidity $\beta \in \mathbb{R}$, Θ denotes the PCT operator of the Wightman theory and $R_1(\pi)$ is the rotation by π about the x_1-axis. The main objective of [BW75] was the verification of Haag duality $\mathfrak{A}(\mathcal{O})' = \mathfrak{A}(\mathcal{O}')$. For regions $\mathcal{O} \subset \mathbb{R}^{s+1}$ which are intersections of wedges, duality of the corresponding intersection algebras is inferred via wedge-duality (HK2$^\sharp$) and this is the point where (BW) enters. One should mention that the modular objects for bounded space-time regions $\mathcal{O} \subset \mathbb{R}^{s+1}$ are in general unknown. There are some notable exceptions such as the computation of the modular objects for the massless free field by Hislop and Longo [HL82], cf. [Bor00] and references therein. For wedge regions the situation is much better: the fact that the arguments of Bisognano and Wichmann apply for interacting Wightman theories suggest that (BW) is an important general structural property of quantum field theory which holds in all reasonable and physically relevant models.

Moving ahead to the general two-dimensional operator-algebraic formulation, Borchers [Bor92] has established a far-reaching generalization of (BW). He investigated the setting of the modular objects of an abstract von Neumann algebra $\mathfrak{M}$ together with a unitary group $\xi \longmapsto U(\xi)$ which satisfies $U(\xi)\mathfrak{M} \subset \mathfrak{M}$ for $\xi > 0$ and has a positive generator. In this context he established the commutation relations

$$\Delta^{\mathrm{i}\tau} U(\xi) \Delta^{-\mathrm{i}\tau} = U(\mathrm{e}^{2\pi\tau}\xi), \qquad JU(a)J = U(-a), \tag{4.1.1}$$

on general grounds. Here, the requirements imposed on U can be seen as an abstraction of the properties of light-like translations $\xi \mapsto U(\xi, \xi)$ acting on a wedge algebra $\mathfrak{M} = \mathfrak{A}(\mathcal{W}_\mathrm{r})$. This explains the connection to (BW), as in the latter setting the relations (4.1.1) are consequences of (BW). These results of Borchers also mark the beginning of the investigation of wedge-local nets in their own right, going beyond the study of wedge algebras in the context of the Wightman or local Haag-Kastler frameworks.

Yet the relations (4.1.1) are in essence weaker properties compared to (BW), as they are independent of the Lorentz covariance and the geometric action of the modular group, see [Bor00, Sec. 3] and references otherein. The latter modular covariance property has important physical implications in the operator-algebraic formulation, such as the spin-statistics connection [GL95].

In the scattering-theoretic context perhaps the most interesting and important general insight motivating the study of wedge-local observables and the corresponding operator algebras is the discovery of wedge-local polarization-free generators in interacting models. In integrable interacting quantum field theories they were identified by Schroer [Sch99; SW00], and a subsequent general existence result was obtained with Borchers and Buchholz [BBS01].

Definition 4.1.1 (Polarization-free generator)**.** *Consider a local or wedge-local QFT with one-particle space $\mathscr{H}_1$. A* polarization-free generator *(PFG) is a closable operator G such that*

(1) $G\Omega \in \mathscr{H}_1 \setminus \{0\}$,

(2) G is affiliated to $\mathfrak{A}(\mathcal{O})$ for some (possibly unbounded) region $\mathcal{O} \subset \mathbb{R}^d$.

In other words, G creates one-particle states from the vacuum and is non-trivially localizable. A basic example of a PFG is a smeared scalar *free* Wightman field $\phi_0(f)$ evaluated for a compactly supported test function $f \in C_c^\infty(\mathbb{R}^d)$. On the other hand, in interacting theories this simple example with bounded localization appears to be in conflict with a classical theorem of Jost and Schroer [SW, Thm. 4.15], cf. [Mun12]. Remarkably it can be shown that PFG which are wedge-local always exist in any local or wedge-local quantum field theory, as long as the Bisognano-Wichmann condition (BW) is satisfied [BBS01]. At the same time the authors caution the reader that PFG will in general have rather complicated domains. The PFG should further be expected to be mathematically less regular than free quantum fields, and for the clarification of this issue a tentative notion of well-behaved PFGs has been investigated in [BBS01].

Definition 4.1.2 (temperate PFG)**.** *A PFG G is* temperate *iff its domain $\mathrm{D}(G)$ is invariant under the action of space-time translations $U(x)$, and for any $\Psi \in \mathrm{D}(G)$ the norm of the vector-valued function $x \longmapsto GU(x)\Psi$, $(x \in \mathbb{R}^d)$, is bounded by some polynomial in $|x|_c := |x_0| + |\mathbf{x}|$.*

Non-temperateness is a serious obstruction for the scattering-theoretic analysis in terms of the PFG. Yet only non-temperate polarization free generators are compatible with interaction in higher dimensional local QFT, with the exception of two-dimensional interacting models without particle production [BBS01]. And even in theories without particle production, non-temperateness may appear, posing a difficult challenge for the construction of wedge algebras [CT15]. Let us conclude by noting that our presently developed wedge-local scattering theory is formulated in terms of bounded wedge-local observables, so that our results apply also in the non-temperate situation.

4.2 Constructive wedge-local QFT

It appears evident that the class of wedge-local theories is strictly larger than the class of wedge-local theories which can be obtained from an underlying local Haag-Kastler net, cf. [LTU17]. The recent literature points to the fact that the wedge-local setting is more flexible in comparison to the local framework of quantum field theory, and that wedge-local models are more accessible for direct and rigorous constructions. The most notable family of models, which appears to be naturally accommodated by the wedge local setting are the two-dimensional integrable quantum field theories with factorizing S-matrices, which have been intensely studied in the physics literature, see [AAR] and references therein. To illustrate the domain of applicability of the wedge-local scattering theory, we will describe a few examples of such constructions. The detailed reviews of Lechner and Summers [Le15; Sum12] contain most relevant references from the literature. Hence we will only describe some important concepts and ideas.

4.2.1 Direct constructions of wedge-local nets

One of the first direct constructions of wedge-local theories appeared in the context of the theory of standard subspaces of Brunetti, Guido and Longo [BGL02]. Here a wedge-local second quantized net is constructed in a canonical way for any given positive-energy representation of the Poincaré group. In this case there are also general arguments to show that intersections of wedge algebras corresponding to space-like cones are also non-trivial and act cyclically on the Fock vacuum.[1]

In particular in space-time dimension two, important examples of interacting wedge-local nets can be directly constructed in a very explicit way in terms of polarization-free generators [Sch99; Le03]. In this approach the input is the two-particle S-matrix. The classification and construction of such two-particle S-matrices with the required properties has been a central topic in the form-factor programme.[2] Wedge algebras can be generated similarly as for free quantum fields,

$$\mathfrak{A}(\mathcal{W}_\ell) := \{\mathrm{e}^{\mathrm{i}\phi(f)},\ f \in \mathscr{S}(\mathbb{R}^2),\ \operatorname{supp} f \subset \mathcal{W}_\ell\}'', \tag{4.2.1}$$

and similarly for translates $\mathcal{W}_\ell + x$, $(x \in \mathbb{R}^2)$ of the left wedge $\mathcal{W}_\ell = \mathcal{W}_{\mathrm{r}}{}' = -\mathcal{W}_{\mathrm{r}}$. Here the explicit ansatz for polarization-free generators recognized by Schroer is given by

$$\phi(x) = \int \mathrm{d}\theta\ \left(\mathrm{e}^{\mathrm{i}p_\mu(\theta)x^\mu} z^*(\theta) + \mathrm{e}^{-\mathrm{i}p_\mu(\theta)x^\mu} z(\theta)\right), \tag{4.2.2}$$

where $p_\mu x^\mu := p^0x^0 - p^1x^1$, and $p^\mu(\theta) := m(\cosh\theta, \sinh\theta)$. The single difference in comparison to the ordinary scalar free field is that the bosonic creation-annihilation operator-distributions have been replaced by the Zamolodchikov-Faddeev generators $z(\theta)$ defined by the relations $(\theta, \theta' \in \mathbb{R})$,

$$\begin{aligned} z(\theta)z(\theta') &= S_2(\theta-\theta')z(\theta')z(\theta), \\ z^*(\theta)z^*(\theta') &= S_2(\theta-\theta')z^*(\theta')z^*(\theta), \\ z(\theta)z^*(\theta') &= S_2(\theta'-\theta)z^*(\theta')z(\theta) + \delta(\theta-\theta'). \end{aligned} \tag{4.2.3}$$

[1] In some cases the localizability can be further improved to cyclicity of the vacuum for the intersection algebras of bounded regions, but for infinite-spin representations satisfying the Bisognano-Wichmann property it has been shown that one cannot go beyond localization in space-like cones [LMR16].

[2] Many examples of such S-matrices have been constructed, see [AAR] and references therein. In the scalar case there even exists a complete classification, see [Le06], Sec. 3 and App. A.

The wedge-local theory is then constructed on the representation within the unsymmetrized Fock space. This representation contains the Fock vacuum Ω and can be used to construct the exponentiated generators. Thus the weak closure in (4.2.1) yields a von Neumann algebra for which Ω is cyclic and separating, if the S_2-function has a holomorphic extension to the physical strip $\mathbb{R} + \mathrm{i}(0, \pi)$ [Le03].

Precisely at the transition to the operator-algebraic description the wedge-local perspective enters and has to be justified. By explicit computation the field operators (4.2.2) appear to be completely delocalized unless $S_2(\theta) = 1$ for all $\theta \in \mathbb{R}$. Still, Schroer realized that $\phi(x)$ can be interpreted as localized in $\mathcal{W}_\ell + x$, for the reason that in the vacuum Hilbert space there also exists a corresponding reflected field

$$\phi'(x) = J\phi(-x)J \tag{4.2.4}$$

satisfying

$$[\phi(x), \phi'(y)] = 0 \text{ for any } y - x \in \mathcal{W}_\mathrm{r}. \tag{4.2.5}$$

Here the antilinear involution J is defined via its action on improper states

$$Jz^*(\theta_1) \dots z^*(\theta_n)\Omega := z^*(\theta_n) \dots z^*(\theta_1)\Omega. \tag{4.2.6}$$

As consequence of the existence of the reflected field one obtains that Ω is separating for $\mathfrak{A}(\mathcal{W}_\ell)$. In other words, ϕ' generates corresponding von Neumann algebras $\mathfrak{A}(\mathcal{W}_\mathrm{r} + x)$ which commute with $\mathfrak{A}(\mathcal{W}_\ell + x)$ [Le03; Le06].

The analysis of these models can be carried much further. Locality is established by showing that Ω is also cyclic for the intersection algebras $\mathfrak{A}(\mathcal{W}_\ell \cap (\mathcal{W}_\mathrm{r} + x)) := \mathfrak{A}(\mathcal{W}_\ell) \cap \mathfrak{A}(\mathcal{W}_\mathrm{r} + x)$, $(x \in \mathcal{W}_\ell)$. This is accomplished using advanced operator-algebraic techniques [BDL90; BL04; Le06]. A notable point in these constructions is that the local observables enter the wedge algebras with the weak closure operation (4.2.1). This is seen e.g. in wedge-local operator expansions [BC13]. Explicit examples of local operators have been constructed and can be characterized in terms of their expansion with respect to the Zamolodchikov generators [BC18]. For the known tractable examples the algebraic structure of wedge-local operators appears to be mathematically simpler. In this regard it is important to understand which physical features of QFT models are contained in the general wedge-local description.

Constructions of integrable models with more general particle spectra and non-scalar S-matrices have been initiated out by Lechner, Schützenhofer, and Alazzawi [LS14; Ala14; AL17]. Their approach also requires analyticity of the two-particle S-matrix in the physical strip. This class of scattering data is claimed to contain many physically interesting theories such as the Sinh-Gordon [Le06] or $O(n)$-symmetric nonlinear sigma models studied in [Ala14]. As the two-particle S-matrix is the defining input, one also obtains many models for which no corresponding Lagrangian is known. The correspondence between models with a Lagrangian description and the suitable factorizing S-matrices does not seem to be established with full mathematical rigour. The proof of locality in such theories has so far been completed for scalar models with $S(0) = -1$ [Le08] (cf. [Ala14]), and for certain "diagonal" tensor models with analogous antisymmetry at zero rapidity [AL17], establishing non-triviality of $\mathfrak{A}(\mathscr{C}_R)$ up to double cone regions $\mathscr{C}_R$ with certain minimal radius $R > R_0 > 0$. Locality up to $R_0 = 0$ has been shown for a class of generalizations of the Federbush model [Tan14].

Many examples of the discovered S-matrices from the form factor programme also contain isolated poles in the physical strip, which are physically interpreted as the presence of bound states. For such models the ansatz (4.2.2) will no longer work. Cadamuro and Tanimoto constructed a compensating bound-state operator $\chi(f)$ which restores wedge locality of $\tilde{\phi}(f) :=$

$\phi(f) + \chi(f)$ at the level of weak commutation on a suitable domain. These new wedge-local fields are more singular compared to the Zamolodchikov generators used by Lechner. Together with their reflections the $\tilde{\phi}(f)$ are no longer temperate and many technical challenges arise in the construction of wedge-algebras, e.g. bounded functions such as $\exp(\mathrm{i}\tilde{\phi}(f))$ are difficult to define [CT15]. More precisely, the problems of constructing self-adjoint extensions and establishing their strong commutativity have to be addressed [Tan16]. These constructions are particularly interesting as non-temperate polarization-free generators would also arise in non-integrable models with particle production or in higher dimensional theories [BBS01]. Hence progress in this direction can be expected to provide more experience and relevant technical tools for novel operator-algebraic constructions, and can be seen as important step towards non-integrable and higher-dimensional theories.

In the literature wedge-local and other partially localizable fields have also been investigated on higher-dimensional Minkowski space-time. Buchholz and Summers studied a scalar fermionic model and explicitly construct observables localized in wedge-regions as well as with localization in wedge-intersections $\mathcal{W} \cap (\mathcal{W}' + x)$, with $x \in \mathcal{W}$ [BS07]. Another interesting class of wedge-local models has been constructed and studied by Grosse and Lechner [GL07], see Section 1.3.2. They explicitly computed the two-particle S-matrix using Haag-Ruelle theory and discovered that the matrix elements carry a non-trivial phase factor, which shows that these models are interacting quantum field theories. There are further explicit expressions for higher correlation functions [GL07], but it was not yet clear whether these correlations can be interpreted in terms of scattering reactions.

There are various arguments which support that the Grosse-Lechner models in higher dimensions are genuinely wedge-local. That is, intersections of algebras over families of wedges resulting in compact regions are expected to be small or trivial [BLS11]. The problem of concretely determining the intersection algebras

$$\mathfrak{A}(\mathcal{W}_{\mathrm{r}}) \cap \mathfrak{A}(\mathcal{W}_{\mathrm{r}}{}' + x), \ (x \in \mathcal{W}_{\mathrm{r}}) \tag{4.2.7}$$

is still open even in two-dimensional Grosse-Lechner models. Buchholz and Summers [BS07] have shown that in the case of the Ising model the size and structure of intersections of two opposite wedge algebras depends on the dimension of space-time. The insights of [GL07] also inspired a larger body of research on deformation constructions in wedge-local quantum field theory, which is briefly discussed below.

4.2.2 Construction of wedge-local QFT via deformations

Buchholz and Summers have shown that the model of Grosse and Lechner can be re-expressed as a deformation of the scalar free field theory on the operator algebraic level. The remarkable aspect of the deformation procedure is that it can be formulated entirely in terms of the general wedge-local structure [BS08; BLS11]. In this form it can be applied to any given wedge-local net $(\mathfrak{A}_0(\mathcal{W}), \alpha, \Omega)$ to define a continuous family of new inequivalent wedge-local theories $(\mathfrak{A}_Q(\mathcal{W}), \alpha, \Omega)$ for any admissible Q (see (1.3.21)). An essential ingredient of the construction is the warped convolution

$$A_Q := \int \mathrm{d}E_{(H,\boldsymbol{P})}(p)\ \alpha_{Qp}(A), \tag{4.2.8}$$

formally written here as an oscillatory operator-valued integral [BS08], where $A \in \mathfrak{A}_0(\mathcal{W})$. Due to the operator valued integrand, the integration with respect to the energy-momentum spectral measure $E_{(H,\boldsymbol{P})}$ goes beyond standard spectral calculus. A rigorous definition of (4.2.8) yielding

$A_Q \in \mathrm{B}(\mathscr{H})$ for suitably regular $A \in \mathfrak{A}_{\mathrm{reg}}(\mathcal{W}) \subset \mathfrak{A}(\mathcal{W})$ can be given using oscillatory integral techniques [BLS11], and it can be shown that the von Neumann algebras

$$\mathfrak{A}_Q(\mathcal{W}) := \{A_{Q_\mathcal{W}},\ A \in \mathfrak{A}_{\mathrm{reg}}(\mathcal{W})\}'', \tag{4.2.9}$$

with $Q_\mathcal{W}$ as in Section 1.3.2, satisfy the required commutation relations to define a new wedge-local net. Again at the level of the two-particle scattering matrix computed using Haag-Ruelle theory, the inequivalence can be seen as the deformed S-matrix acquires an additional phase factor relative to the undeformed two-particle S-matrix.

Subsequently more general deformations of (free) Wightman fields leading to various modifications of the scattering matrix have been found [Le12]. Another operator-algebraic approach based on Longo-Witten endomorphisms [LW11] has been used to construct interesting two-dimensional massless theories, some of which even display features in the scattering data resembling particle creation [BT13]. The problem of constructing wedge-local massive models featuring particle production in scattering reactions is a difficult but important open question.

Finally the deformation approaches also convey a very valuable pedagogical message about quantum field theory: in these constructions the time evolution and the Hamiltonian are taken over unchanged from the undeformed theory, and in the simplest cases the latter is non-interacting. Hence one may perceive it as puzzling that the new theories exhibit non-trivial scattering without requiring modification of the free time evolution. This strongly reinforces the old wisdom that the physically relevant data are contained in the algebraic structure of the observables of a theory and the relevance of their localization properties [Ha, Chap. III].

4.2.3 Construction of wedge-local product theories and asymptotic completeness

The free product construction for von Neumann algebras has been applied to construct novel products of wedge-local models in the work of Longo, Tanimoto and Ueda [LTU17]. These authors show that local algebras constructed from free product algebras with infinitely many identical factors must be trivial. It appears to be unknown whether finite free product models can contain non-trivial strictly local subtheories in the wedge-local setting, as discussed in [LTU17, Sec. 5].

The free product construction also illustrates an interesting aspect of wedge-local scattering theory: in the free product theory of two massive free fields, the incoming and outgoing velocity-ordered two particle scattering states with one-particle vectors from different factors are orthogonal. Due to the simple geometric characterization of the outgoing and incoming two-particle scattering states, one may conclude a failure of asymptotic completeness of velocity-ordered two-particle states in these models [LTU17]. The insights obtained from these examples provided significant motivation to carefully define the wedge-local wave operators on velocity ordered Fock spaces, as explained in Section 2.1.3.

4.3 Scattering for particles with embedded mass shells

The scattering theory for quantum fields as originally developed by Haag, Ruelle and Hepp [Ha58; Ru62; Hep65] provided the first mathematically rigorous construction of scattering states in axiomatic quantum field theory. One notable motivation in this construction came from the fact that an interacting quantum field theory can give rise to bound states of particles, and in this case there are no corresponding “elementary fields” describing these composite particles. This suggests that the collision theory should be constructed and analyzed in an axiomatic setting.

In the Haag-Kastler framework the basic Haag-Ruelle construction makes use of *almost-local* $B \in \mathfrak{A} = \overline{\bigcup_{\mathcal{O}\in\mathbf{Reg}_b} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$ which create one-particle states from the vacuum, that is $B\Omega \in E(H_m)\mathscr{H}$. The construction of such B via space-time smearing and spectral calculus for particles with isolated mass shells is well known, see Section 3.1, or [Hep66, § 5.] for a discussion within the Wightman framework.

The basic smearing argument can also be successful for theories with more general particle spectra. In this case a natural approach is to make use of more refined information about the algebraic structure of the observables of the model. Doplicher, Haag and Roberts [DHR69, Sec. VII] construct collision states for one-particle states having mass shells isolated within the corresponding localizable superselection sector. It was later recognized by Buchholz and Fredenhagen that the localizability assumption is not required. They established localizability in space-like cones for any charged representation of the quasilocal algebra whose energy-momentum spectrum is bounded from below by an isolated mass shell [BF82].

The first construction of scattering states for particles with embedded mass shells was given by Herbst [Hrb71, Sec. IV. C], assuming an isolated vacuum state and certain regularity of the background spectrum.

Definition 4.3.1 (Herbst spectral condition)**.** *A one-particle state $\Psi_1 \in E_{(H,\boldsymbol{P})}(H_m)\mathscr{H}$ is regular in the sense of Herbst with exponent $\epsilon > 0$ if there exists a local operator A s.t.*

(i) $\Psi_1 = E_{(H,\boldsymbol{P})}(H_m)A\Omega$,

(ii) $\left\|E_{(H,\boldsymbol{P})}(H_m^\delta \setminus H_m)A\Omega\right\| \leq C\delta^\epsilon$, *for some* $C > 0$ *and all* $\delta > 0$,

where $H_m^\delta := \{(\omega_\mu(\mathbf{k}), \mathbf{k}) \in \mathbb{R}^{s+1} : \mathbf{k} \in \mathbb{R}^s, |\mu - m| < \delta\}$ *denote covariant δ-neighbourhoods of the mass shell H_m.*

The construction of scattering states of Herbst-regular one-particle vectors without the lower mass gap requirement was achieved by Dybalski [Dy05], and also LSZ-reduction formulas can be established in this setting [BS05]. On the other hand, it had in the mean time become clear that the Herbst particle picture of Definition 4.3.1, relying on the Wigner definition, does not apply to electrically charged particles. This has been shown by a general argument based on the Gauss law [Bu86]. A more general particle concept which is deemed suitable for the model-independent analysis of electrically charged particles is the theory of particle weights developed by Buchholz, Porrmann, and Stein [BPS91]. The challenges of constructing inclusive collision cross sections in a setting of particle weights have so far not been addressed in a satisfactory manner and are discussed e.g. in [Dy12]. An alternative approach which is conjectured to provide better control for scattering situation is based on infravacuum representations of Kraus-Polley-Reents type, cf. [AD17; CD18] and also the discussions in [Hrd13; DH14].

In this context the Herbst particle concept is relevant, in particular for electrically neutral particles. Some concrete physical examples are the analysis of S-matrix elements of configurations of Hydrogen atoms or other stable electrically neutral bound states. In a more recent contribution Herdegen observed that the spectral background condition (ii) of Herbst can be further relaxed [Hrd13]. It would be interesting to understand whether the Reeh-Schlieder effect [Du17] can contribute to a further relaxation of the required assumptions, cf. Chapter 5.

The necessity to impose the Herbst condition does not arise in the construction of scattering states for massless Wigner particles [Bu75a; Bu75b; Bu77], as one can rely on Huygens' principle to establish convergence of asymptotic fields. Improvements and simplifications of Buchholz' argument have been discussed in [Bu90a] and more recently in [DH14; AD17].

Let us conclude this section with a brief discussion of models which exhibit embedded mass shells and are accessible for rigorous construction. The simplest examples are obtained as tensor products of free fields with suitable mass spectrum or within the class of generalized free fields, see e.g. [BLOT, Sec. 8.4 D]. These models obviously exhibit only trivial scattering and do not require further analysis regarding their particle structure. The generalized free field models with continuous Lehmann measure show further unphysical behaviour, namely concerning their phase space structure, the failure of asymptotic completeness, and the time-slice property, cf. the survey in [Hor, Sec. 3.3]. Nevertheless they provide relevant examples for the context of embedded mass shells, as they demonstrate that the violation of Herbst-type spectral conditions by local observables is compatible with the Wightman and Haag-Kastler axioms, cf. [Dy05, Sec. 4].

Within rigorously constructed interacting theories only few models with embedded mass shells are known. Physically it has to be taken into account that in two-component models with $m_2 > 2m_1$ the presence of interactions can cause the heavier particle to become unstable, as suggested by perturbation theory [Wei95, Sec. 3.8]. Still, there are two examples for interacting models with embedded mass shells which are outside the perturbative regime due to the formation of bound states. In $P(\phi)_2$-models there can be two-particle bound states, but these have masses below the scattering continuum. It has been shown that also models with three-particle bound states exist [Nev81]. Another example for particles with embedded mass shells which may soon be within reach of the operator-algebraic constructive programme are certain integrable quantum field theories with bound states. For example, relevant mass spectra with stable embedded massive particles appear in the Sine-Gordon model for sufficiently small values of the coupling constant $\nu < 1/3$, cf. [CT16, Sec. 2.1].

Returning briefly to the wedge-local setting, we expect that the constructions of Sections 2.1.1 and 2.1.3 can be generalized along similar lines as in [Hrb71; Dy05] or [Du17] to the case of massive particles with embedded mass shells. In the general wedge-local analysis a Herbst- or Reeh-Schlieder-type condition for the particle with embedded mass shell is required, for the same technical reasons that have been encountered in local QFT. In particular the class of wedge-local models with lower mass gap appears to be within reach due to the availability of general clustering estimates, see e.g. [Fre85] or the energy-momentum transfer method described in Section 3.2. In massless wedge-local models the status of such clustering estimates has not yet been clarified. A scattering theory of waves has been constructed in the two-dimensional wedge-local massless case in the work of Dybalski and Tanimoto [DT11]. Here clustering of asymptotic fields can be established on kinematical grounds, cf. [Bu75b]. The general higher-dimensional massless wedge-local case appears to be challenging and requires more advanced techniques, perhaps similar to the ergodic theory argument used in [AD17].

5 Outlook, future directions, and open problems

In this thesis, N-particle Haag-Ruelle scattering theory for massive wedge-local quantum field theory has been developed in the general operator algebraic setting. In local QFT we studied the situation of massive particles with embedded mass shells, and established a novel approach for the construction of corresponding scattering states using non-local vacuum correlation effects. Our work may be regarded as contributions to what we may call the *central conjecture* of the Haag-Ruelle approach to scattering theory.

Conjecture. *Any quantum theory with vacuum and one-particle states, and sufficiently strong localization of observables must also describe corresponding N-particle states and (possibly trivial) scattering processes.*

Previously it appeared that geometrical properties of wedge-local theories require restriction of the scattering theoretic analysis to $N = 2$. In our approach this is no longer necessary, and the convergence and Fock structure of velocity-ordered scattering states with arbitrarily many particles has been proven. The required wedge-swapping symmetry of one-particle vectors is trivially realized in local quantum field theories, and it has been established in general wedge-local models using Haag duality for wedges (HK2$^\sharp$) and Tomita-Takeski theory. In local QFT our construction is consistent with conventional scattering theory, cf. [Du18, Lem. 20]. Let us remark, that in comparison to standard Haag-Ruelle theory our Fock structure argument is technically slightly simpler, as we do not require double commutator estimates due to the velocity ordering. In the wedge-local context our results imply that the multi-particle structure in general wedge-local models with isolated mass shells must be as rich as in local quantum field theories.

5.1 Excluding QFT models with pathological Wigner particles

So far the existence of scattering states cannot be claimed for massive Wigner particles in general if the mass shells are embedded, not even in local QFT. More precisely, the currently available results still permit the existence of local quantum field theories $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$, which exhibit a non-trivial one-particle subspace

$$E(H_m)\mathscr{H} \neq \{0\} \tag{5.1.1}$$

for some $m > 0$, while at the same time the corresponding scattering state approximants

$$\Psi_\tau := \mathcal{B}_{1\tau}(f_1)\mathcal{B}_{2\tau}(f_2)\Omega, \ (\tau \in \mathbb{R}), \tag{5.1.2}$$

behave pathologically at large τ, e.g. $\Psi_\tau \rightharpoonup 0$ as $\tau \to \pm\infty$. Here we wrote $\mathcal{B}_{k\tau}(f_k)$ for generic creation-operator approximants which are suitable for embedded mass shells, e.g. using the conventional τ-dependent smearing approach [Hrb71; Dy05; Hrd13], or Reeh-Schlieder-type operator families [Du17].

In the following we will refer to such hypothetical theories as *models with pathological Wigner particles*, and we note that the results of Buchholz [Bu75a; Bu75b; Bu77] exclude the existence of massless pathological Wigner particles. We believe that it is an interesting question whether massive pathological Wigner particles can also be ruled out on general grounds.

To this end, we note that the results [Hrb71; Dy05; Hrd13] and [Du17] constrain the structure of models with pathological Wigner particles. Namely the local algebras must be such, that all $A \in \mathfrak{A}(\mathcal{O})$ with $E(H_m)A\Omega \neq 0$ violate the spectral condition introduced by Herbst (see Definition 4.3.1), i.e. the spectral background satisfies the lower bound

$$\left\| E(H_m^\delta \setminus H_m)A\Omega \right\| \geq C\delta^\epsilon \tag{5.1.3}$$

for any $C > 0$ and $\epsilon > 0$ as soon as the δ-neighbourhood $H_m^\delta \supset H_m$ becomes sufficiently small (depending on C and ϵ). As remarked in [Dy05], such violation of the Herbst bound is indeed compatible with the Wightman and Haag-Kastler axioms, and examples are provided by generalized free fields with suitable Lehmann measure. It is clear that the Haag-Ruelle approximants Ψ_τ are convergent in these models, albeit rather slowly with the rate of convergence solely governed by the convergence of the one-particle problem.

In addition we now know from our results in [Du17], that any counterexample must also violate the Reeh-Schlieder condition[1]

$$\begin{aligned} \|A_\beta\Omega - \Psi_1\| &\leq C\beta, \\ \|A_\beta\| &\leq C\beta^{-\gamma}. \end{aligned} \tag{5.1.4}$$

One strategy to rule out the scenario of pathological Wigner particles could consist in a proof that (5.1.4) and (5.1.3) are sufficient properties to conclude that the model must be equivalent to a generalized free field. In fact there exist characterization theorems of similar kind, e.g. the criterion of Baumann in terms of the decay of the two-point function at large energies [Bau86]. We believe that the failure of conditions (5.1.3) combined with the violation of (5.1.4) are strong constraints. Yet the mathematical analysis of their status is not obvious, as we will further discuss in Section 5.2.

From the viewpoint of physics, pathological Wigner particles might perhaps be difficult to distinguish from regular Wigner particles. On the level of the Haag-Ruelle approximants Ψ_τ we have that even if the Ψ_τ are not convergent, their variation is roughly proportional to $\|\partial_\tau\Psi_\tau\| \sim 1/\tau$ and thus perhaps so slow to be non-detectable on typical experimental time scales. Additionally the states Ψ_τ can be constructed with an approximate Fock structure at each fixed τ, which improves as $\tau \to \pm\infty$, and from this perspective the mathematical interpretation of Ψ_τ for large enough τ as approximate scattering states could be justified also for pathological Wigner particles.

Yet, one may question the correctness of a phenomenological interpretation of the Ψ_τ in this context. A more fundamental description of collision processes in QFT is the formulation of Araki and Haag [AH67]. In this approach the collision cross sections may in principle be well-defined and could be constructed in terms of coincidence arrangements of detectors along the lines of [BPS91], even if the "approximate S-matrix elements" $\langle\Psi_\tau, \Psi'_{-\tau'}\rangle$ fluctuate in the limits of large $\tau, \tau' > 0$. However, so far the convergence of coincidence arrangements of detectors can be established only for non-pathological Wigner particles, namely on domains of scattering

[1] Here (5.1.4) is written with an additional constant $C > 0$ which will be convenient for the discussion of the status this condition. For the purposes of scattering theory such constants can be absorbed by rescaling, slight enlargement of the degree γ and restricting to sufficiently small $\beta > 0$.

states [AH67; Du13], and for particles with isolated mass shells also on arbitrary finite energy states [DyG13] with the additional restriction to certain coincidence arrangements.

5.2 Status of the Reeh-Schlieder condition

For free fields and integrable models with temperate polarization free generators it is not difficult to construct local or wedge-local Reeh-Schlieder families, respectively, as we pointed out in [Du17]. Namely making use of self-adjointness of $\phi(f)$ for real-valued test functions $f \in C_c^\infty(\mathbb{R}^{s+1})$ we can simply set for any $\gamma > 0$

$$A_\beta := \phi(f) \exp(-\beta \, |\phi(f)|^{1/\gamma}), \tag{5.2.1}$$

which will be a family of local, respectively wedge-local, operators satisfying the Reeh-Schlieder conditions for degree γ. The ansatz (5.2.1) also works for polarization free generators G satisfying $\Omega \in \mathrm{D}(|G|^{1+\epsilon})$, and this domain condition appears to be independent of the temperateness of G. However, the Reeh-Schlieder families obtained from polarization free generators will in general only be wedge-local, which is acceptable for the scattering-theoretic construction in presence of a lower mass gap. Otherwise wedge-locality of A_β may not be sufficient due to the unknown status of clustering, as already discussed in Section 4.3.

A further construction of Reeh-Schlieder families can be given starting from the spectral regularity condition of Herbst. The resulting operator families A_β are local, but we have only weaker control over $\|A_\beta\|$.

Proposition 5.2.1 (Construction of Reeh-Schlieder families of exponential type)**.** *Assume that $\Psi_1 \in E(H_m)\mathscr{H}$ is Herbst-regular. Then there exists for any $\epsilon' > 0$ a vector $\Psi_1' \in E(H_m)\mathscr{H}$ with $\|\Psi_1' - \Psi_1\| < \epsilon'$ which satisfies the following Reeh-Schlieder condition: there exists a family of local operators $A_\beta \in \mathfrak{A}(\mathcal{O})$, $(\beta > 0)$, s.t. for any compact set $\Delta \subset \mathbb{R}^{s+1}$*

$$\|E(\Delta)(A_\beta\Omega - \Psi_1)\| \leq C_\Delta \beta, \tag{5.2.2}$$

$$\ln \|A_\beta\| \leq C\beta^{-\gamma}, \tag{5.2.3}$$

for suitable constants C, C_Δ, with γ proportional to the inverse ϵ^{-1} of the Herbst exponent.

Proof (sketch). Such A_β can be constructed even without making use of the Reeh-Schlieder effect, simply by taking suitable smearing functions. For this let $\chi \in C_c^\infty(\mathbb{R}^{s+1})$ be a compactly supported function with almost exponentially decaying Fourier transform $|\hat{\chi}(p)| \leq \exp(-|p|^\nu)$, with respect to the Euclidean norm and $\nu < 1$, e.g. as constructed in [Hör90, Thm. 1.3.5]. Then we set $B = A(\chi)$ where $A \in \mathfrak{A}(\mathcal{O})$, s.t. $\Psi_1 = E(H_m)A\Omega$, is obtained from the Herbst condition. To construct A_β we apply the differential operators $P_{N,\alpha} := (\mathbb{1} - \alpha(\Box + m^2)^2)^N$, $\alpha > 0$, $N \in \mathbb{N}$, with the d'Alembertian $\Box = \partial_{x^0}^2 - \nabla_{\mathbf{x}}^2$ to define

$$A_\beta := P_{N(\beta),\alpha(\beta)} B(x)\big|_{x=0},$$

which have the same compact localization region as B. For suitable choices of $N(\beta)$ and $\alpha(\beta)$ it can be shown that these operators fulfill the two bounds (5.2.2) and (5.2.3), by using Herbst-regularity of $A\Omega$ and the almost-exponential decay of $\hat{\chi}$, respectively. $\square$

By a careful analysis of the Haag-Ruelle argument one can see, that Reeh-Schlieder families of exponential type (5.2.2), (5.2.3) are also sufficient for the construction of scattering states, if the QFT model under consideration has a lower mass gap. Let us note that the strong exponential norm divergence of the bound (5.2.3) does not give evidence of the failure of the strengthened Reeh-Schlieder condition (5.1.4), as the Reeh-Schlieder effect has not even been used in the rather naive construction given in Proposition 5.2.1.

As a next step the status of the Reeh-Schlieder condition should be investigated in generalized free field models. In this case the construction of Reeh-Schlieder families reduces to the one-particle problem, as we discussed with some partial results in [Du17, App. C]. It is well known that the Reeh-Schlieder property for free theories is equivalent to anti-locality properties of the fractional Klein-Gordon operators [SG65], and the Reeh-Schlieder property is also not specific to flat space-times [Ver93; SVW02]. Havin and Jöricke [HJ94, Sec. 5] discuss various anti-local operators which appear in mathematical physics. The difficulty to quantify anti-locality effects, as would be required to establish the strengthened Reeh-Schlieder condition, seems to be due to the fact that such aspects of anti-local operators are subtle and related to the study of *ill-posed* problems in analysis [HJ94, p. 474].

Beyond the free models, one may conjecture that the strength of the Reeh-Schlieder effect is stronger in interacting theories. E.g. in $P(\phi)_2$-models one can in principle use all higher Wick powers to subtract vacuum polarization present in $\phi(f)\Omega$. This provides additional freedom beyond the choice of the test function, but also leads to more singular large-energy behaviour. In this regard, it appears be interesting to investigate the quantitative structure of the vacuum polarization of more tractable interacting models. A large class of local observables which could be analyzed has recently been constructed and described in fairly explicit form for the integrable Ising model [BC18].

5.3 Asymptotic completeness in wedge-local quantum field theory

Finally, let us briefly discuss the question of completeness of the particle interpretation of quantum field theories. The first full proof of asymptotic completeness in interacting relativistic QFT is due to Lechner, and appeared in the context of the operator-algebraic construction of integrable quantum field theories in two dimensions [Le08]. Here strict locality of the wedge-local models described in Section 4.2 is established first, and then density of the standard Haag-Ruelle scattering states in the S-symmetrized Fock space is proven [Le08]. The arguments of Lechner could not be used to establish interacting asymptotically complete models for $d > 1+1$. The reason is the unknown and questionable status of locality in the higher-dimensional interacting wedge-local models constructed so far. Locality seemed an indispensable requirement to construct N-particle scattering states with $N \geq 3$. The necessity of such states for asymptotic completeness was suggested by the structure of all interacting wedge-local models known so far.

Due to our existence result for N-particle states, it is now clear that the previously constructed two-particle states can never be dense in the vacuum Hilbert space $\mathscr{H}$ of massive wedge-local theories. Let us note that the necessity for N-particle states does not arise in the purely massless two-dimensional case, where asymptotically complete interacting models have been established by Dybalski and Tanimoto [DT11].

In the massive wedge-local case we may ask whether the ordered scattering states

$$\Psi^{\pm} = \mathbb{W}^{\pm}_{\mathcal{W}} \Psi_n, \quad \Psi_n = \Psi_1^1 \otimes \Psi_1^2 \otimes \ldots \otimes \Psi_1^n \in \Gamma^{\succ_{\mathcal{W}} / \prec_{\mathcal{W}}}(\mathscr{H}_1) \tag{5.3.1}$$

span dense subspaces of $\mathscr{H}$. In local QFT models the ordering condition is not a problematic restriction. Namely using the conventional bosonic or fermionic statistics we can rearrange

$$\Psi^{+} = \lim_{\tau \to \infty} \mathcal{B}_{1\tau}(f_1) \mathcal{B}_{2\tau}(f_2) \ldots \mathcal{B}_{n\tau}(f_n) \Omega \tag{5.3.2}$$

into any desired order specified by a permutation $\pi \in \mathfrak{S}_n$,

$$\Psi^+ = \pm \lim_{\tau\to\infty} \mathcal{B}_{\pi(1)\tau}(f_{\pi(1)})\mathcal{B}_{\pi(2)\tau}(f_{\pi(2)}) \dots \mathcal{B}_{\pi(n)\tau}(f_{\pi(n)})\Omega. \tag{5.3.3}$$

This can be used to obtain the required velocity-ordering of creation operators for non-exceptional velocity configuration. In the covariant formulation of Section 2.1.2, the non-orderable velocity configurations are sets of Lebesgue measure zero in $\mathbb{R}^{ns}$ for any $\mathcal{W} \in \mathbf{Reg}_W$. Hence the ordered asymptotic completeness of wave operators

$$\mathbb{W}^{\pm}_{\mathcal{W}}\Gamma^{\succ_{\mathcal{W}}/\prec_{\mathcal{W}}}(\mathscr{H}_1) = \mathscr{H} \tag{5.3.4}$$

can hold in wedge-local models, if the latter comply with our usual experiences about particle statistics from local QFT. On one hand we are cautioned by the examples for the failure of ordered two-particle asymptotic completeness in the wedge-local models from [LTU17] that such intuitions from local QFT can be false in the general wedge-local setting. On the other hand it is very reasonable to expect that constructions like the deformation method based on warped convolutions as in [BLS11] do not create such unintuitive features.

As a matter of fact it is possible to compare the scattering states constructed in a wedge-local QFT $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$ with the corresponding scattering states obtained in the deformed model $(\mathfrak{A}_Q, \alpha, \mathscr{H}, \Omega)$ defined via warped convolution, cf. the discussion of the massless case in [DT11]. More explicitly, we consider an ordered outgoing scattering state

$$\Psi^+ := \lim_{\tau\to\infty} \mathcal{B}_{1\tau}(f_1)\mathcal{B}_{2\tau}(f_2) \dots \mathcal{B}_{n\tau}(f_n)\Omega, \tag{5.3.5}$$

and we would like to understand its relation to the corresponding scattering state obtained in the deformed theory

$$\Psi^+_Q := \lim_{\tau\to\infty} \mathcal{B}_{Q1\tau}(f_1)\mathcal{B}_{Q2\tau}(f_2) \dots \mathcal{B}_{Qn\tau}(f_n)\Omega. \tag{5.3.6}$$

Using the definition of warped convolutions we can write the deformed scattering state approximants $\Psi_{Q\tau}$ formally in terms of the spectral calculus of the energy-momentum operators and the translates of the original Haag-Ruelle approximants as

$$\begin{aligned}\Psi_{Q\tau} := \int \mathrm{d}E(q_1)\alpha_{Qq_1}(\mathcal{B}_{1\tau}(f_1)) \int \mathrm{d}E(q_2)\alpha_{Qq_2}(\mathcal{B}_{2\tau}(f_2)) \int \mathrm{d}E(q_3) \dots \\ \dots \int \mathrm{d}E(q_n)\alpha_{Qq_n}(\mathcal{B}_{n\tau}(f_n))\Omega.\end{aligned} \tag{5.3.7}$$

Here we know from Theorem 2.1.3 that the asymptotic limits

$$\lim_{\tau\to\infty} \alpha_{Qq_1}(\mathcal{B}_{1\tau}(f_1))\alpha_{Qq_2}(\mathcal{B}_{2\tau}(f_2)) \dots \alpha_{Qq_n}(\mathcal{B}_{n\tau}(f_n))\Omega \tag{5.3.8}$$

define scattering states in the range of the undeformed wave operator $\mathbb{W}^+_{\mathcal{W}}\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1)$. This comparison can be made rigorous using the oscillatory integral techniques from [BLS11], and it can be inferred from expression (5.3.8) that the deformation produces phase factors depending on the particle momenta. Such comparison arguments suggest that the spaces of deformed and undeformed scattering states

$$\mathbb{W}^+_{Q\mathcal{W}}\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1) = \mathbb{W}^+_{\mathcal{W}}\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1) \tag{5.3.9}$$

coincide in the Hilbert space of the interacting theory, and from this one obtains that the property of ordered asymptotic completeness is stable under warped convolutions. As the ordered configurations $\Gamma^{\succ w/\prec w}(\mathscr{H}_1)$ are dense in the Fock space of the free theory, we also obtain asymptotic completeness of the Grosse-Lechner models with respect to our construction of scattering states. Thereby the models constructed by Grosse and Lechner are established as the first asymptotically complete interacting quantum field theories in space-time dimension $d > 1{+}1$. The detailed arguments will be provided in a forthcoming publication [Du18b].

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List of included refereed publications

The publications are listed in anti-chronological order. All publications below are core publications for this thesis.

1. Duell M., N-Particle Scattering in Wedge-local Quantum Field Theory, Commun. Math. Phys. 364, 203–232 (2018)
2. Duell M., Strengthened Reeh-Schlieder Property and Scattering in Quantum Field Theories Without Mass Gaps, Commun. Math. Phys. 352, 935–966 (2017)

Publication 1. N-Particle Scattering in Relativistic Wedge-Local Quantum Field Theory

In this publication we study the multi-particle scattering problem in the general operator-algebraic setting of wedge-local quantum field theory. Wedge-local models emerged from various recent constructive results, such as [Le06; BLS11], but a general scattering-theoretic analysis for this class of theories was only known at the two-particle level.

1. Construction of Scattering States

The first main result of this publication is the convergence of N-particle Haag-Ruelle scattering states, based solely on a wedge-swapping symmetry and ordering of wave-packets.

Theorem 1 (Wedge-local N-particle Haag-Ruelle theorem). *Let $\Lambda \in \mathcal{L}_+^\uparrow$ and $\Psi_1^j = E_m A_j \Omega = E_m A_j^\perp \Omega$ with $A_j \in \mathfrak{A}(\mathcal{W})$, $A_j^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$ and let $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ be an admissible auxiliary function (supported in a sufficiently small neighbourhood of the isolated mass shell).*

(i) For regular positive-energy Klein-Gordon solutions f_j satisfying $\mathcal{V}_{f_n}^\Lambda \prec_\mathcal{W} \mathcal{V}_{f_{n-1}}^\Lambda \prec_\mathcal{W} \ldots \prec_\mathcal{W} \mathcal{V}_{f_1}^\Lambda$, the scattering state approximants $\Psi_n^\Lambda(\tau) := B_{1\tau}^\Lambda(f_1) B_{2\tau}^\Lambda(f_2) \ldots B_{n\tau}^\Lambda(f_n)\Omega$ converge in norm for $\tau \to \infty$.

(ii) For $\Lambda \in \mathcal{L}^*(\mathcal{W})$ scalar products of any two outgoing $\Psi_n^{+,\Lambda} := \lim_{\tau\to\infty} B_{1\tau}^\Lambda(f_1)\ldots B_{n\tau}^\Lambda(f_n)\Omega$, $\Psi_{n'}^{\prime +,\Lambda} := \lim_{\tau\to\infty} B_{1\tau}^{\prime\Lambda}(f_1')\ldots B_{n'\tau}^{\prime\Lambda}(f_{n'}')\Omega$ constructed w.r.t. the same wedge $\mathcal{W}$ satisfy the Fock structure relation $\left\langle \Psi_n^{+,\Lambda}, \Psi_{n'}^{\prime +,\Lambda} \right\rangle = \delta_{nn'} \prod_{j=1}^n \left\langle B_{j\tau}^\Lambda(f_j)\Omega, B_{j\tau}^{\prime\Lambda}(f_j')\Omega \right\rangle$.

This shows that the multi-particle structure of wedge-local quantum field theories with isolated mass shells must be as rich as in the case of local QFT. To our knowledge there is no previous general construction and existence result for $n \geq 3$ in the literature. The result is also given in a covariant formulation admitting general localization wedges, thereby the study of Lorentz-covariance properties and their possible asymptotic breaking by wedge-locality becomes possible [BLS11].

2. Well-definedness of Scattering Data and Poincaré Covariance

We further find that wave-operators and scattering data are well defined, but may depend on the localization wedge $\mathcal{W}$ used for their preparation.

Theorem 2. *Assuming wedge-duality (HK2$^\sharp$), the wedge-local Haag-Ruelle construction induces well-defined wave operators $\mathbb{W}_\mathcal{W}^\pm : \Gamma^{\succ_\mathcal{W}/\prec_\mathcal{W}}(\mathscr{H}_1) \longrightarrow \mathscr{H}$ defined on velocity ordered Fock spaces. For $\lambda = (a, \Lambda) \in \mathcal{P}_+^\uparrow$ we obtain covariance $\mathbb{W}_{\mathcal{W}+a}^\pm = \mathbb{W}_\mathcal{W}^\pm$ and $U(\lambda)\mathbb{W}_\mathcal{W}^\pm = \mathbb{W}_{\Lambda\mathcal{W}}^\pm U_0(\lambda)$.*

Theorem 3. *S-matrices $S_{\mathrm{f\,i}}^{\mathcal{W}_\mathrm{f},\mathcal{W}_\mathrm{i}} := (\mathbb{W}_{\mathcal{W}_\mathrm{f}}^+)^* \mathbb{W}_{\mathcal{W}_\mathrm{i}}^-$, and wedge transition maps $S_{\mathrm{f\,f}}^{\mathcal{W}',\mathcal{W}} := (\mathbb{W}_{\mathcal{W}'}^+)^* \mathbb{W}_\mathcal{W}^+$, $S_{\mathrm{i\,i}}^{\mathcal{W}',\mathcal{W}} := (\mathbb{W}_{\mathcal{W}'}^-)^* \mathbb{W}_\mathcal{W}^-$. satisfy Poincaré-covariance identities $U_0(\lambda) S_{\mathrm{f\,i}}^{\mathcal{W}_\mathrm{f},\mathcal{W}_\mathrm{i}} U_0(\lambda)^* = S_{\mathrm{f\,i}}^{\Lambda\mathcal{W}_\mathrm{f},\Lambda\mathcal{W}_\mathrm{i}}$, $U_0(\lambda) S_{\mathrm{f\,f}}^{\mathcal{W},\mathcal{W}'} U_0(\lambda)^* = S_{\mathrm{f\,f}}^{\Lambda\mathcal{W},\Lambda\mathcal{W}'}$, $U_0(\lambda) S_{\mathrm{i\,i}}^{\mathcal{W},\mathcal{W}'} U_0(\lambda)^* = S_{\mathrm{i\,i}}^{\Lambda\mathcal{W},\Lambda\mathcal{W}'}$. If the wave operators are asymptotically complete we have further $S_{\mathrm{f\,i}}^{\mathcal{W}_\mathrm{f},\mathcal{W}_\mathrm{i}} = S_{\mathrm{f\,f}}^{\mathcal{W}_\mathrm{f},\mathcal{W}_\mathrm{f}'} S_{\mathrm{f\,i}}^{\mathcal{W}_\mathrm{f}',\mathcal{W}_\mathrm{i}'} S_{\mathrm{i\,i}}^{\mathcal{W}_\mathrm{i}',\mathcal{W}_\mathrm{i}}$.*

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I am the sole author of this publication. The swapping argument was found by myself during research on scattering theory for embedded mass shells [Du17]. The problem of Lorentz covariance in the wedge-local setting was suggested by W. Dybalski.

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Communications in
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N-Particle Scattering in Relativistic Wedge-Local Quantum Field Theory

Maximilian Duell

Zentrum Mathematik, Technische Universität München, 85747 Garching, Germany.
E-mail: duell@ma.tum.de

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Abstract: Multi-particle scattering states are constructed for massive Wigner particles in the general operator-algebraic setting of wedge-local quantum field theory. The apparent geometrical restriction of the conventional wedge-local Haag–Ruelle argument to two-particle scattering states is overcome with a swapping symmetry argument based on wedge duality.

1. Introduction

Wedge locality has become an increasingly prominent concept in mathematical physics ever since wedge duality was established in the Wightman framework by Bisognano and Wichmann [BiW75]. In particular, while interacting local quantum field theories (QFT) in four dimensions are still missing, non-trivial wedge-local QFT have emerged in recent years [GL07,BLS11]. This provides strong motivation to develop N-particle scattering theory in the wedge-local setting, which is the goal of the present paper.

The classical Wigner particle concept can still be consistently formulated in wedge-local theories as it does not depend on any notion of localization in configuration space. Accordingly, we may define massive single particle states $\Psi_1 \in \mathscr{H}$ as eigenvectors corresponding to positive eigenvalues of the relativistic mass operator[1] $M := \sqrt{H^2 - \boldsymbol{P}^2}$. Two-particle scattering states were then constructed in [GL07,BS08] along the lines of Haag–Ruelle, using that two particles can be separated by two wedge regions [BBS01]. Scattering states with a larger number of particles however appeared inaccessible or even unnatural in the wedge-local setting as a result of a simple geometric consideration: it is impossible to write down three or more wedge-local operators whose localization regions are space-like separated.[2]

[1] Here $(H, \boldsymbol{P})$ denote the energy-momentum operators of a given wedge-local QFT model, and $M \geq 0$ due to the relativistic spectral condition. The precise framework is presented in Sect. 2.

[2] This is best visualized by noting that the standard wedge $\mathcal{W}_{\mathrm{r}} = \{|x^0| < x^1\}$ restricted to the time-zero hyperplane yields the half space $\{x_1 > 0\}$—more than two half-spaces cannot be disjoint.

In this paper we give a construction of scattering states for an arbitrary number of massive Wigner particles in the general wedge-local setting. Underlying our arguments is a simple *swapping* symmetry, which follows from wedge duality and augments cyclicity of the vacuum Ω for wedge algebras. It states that for a given wedge-local bounded operator $A \in \mathfrak{A}(\mathcal{W}) \subset \mathrm{B}(\mathscr{H})$ localized in a wedge $\mathcal{W} \subset \mathbb{R}^d$ there exists[3] $A^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$ such that

$$A\Omega = A^\perp \Omega, \tag{1}$$

where $A^\perp$ is localized in a translate $\mathcal{W}^\perp := \mathcal{W}' + x, x \in \mathbb{R}^d$, of the causal complement $\mathcal{W}'$ in Minkowski space of dimension $d = s + 1$. The symmetry (1) itself has been known for some time in the context of integrable models,[4] but its utility for the construction of scattering states seems to have so far escaped the attention of the experts. In fact, its application in scattering theory appears very natural from the perspective of the causal geometry of wedge regions.

Let us now explain the role of the swapping relation (1) for scattering theory by sketching the convergence argument as an example. Recalling standard definitions of Haag–Ruelle theory [Ha58,Ru62], we select $A_k \in \mathfrak{A}(\mathcal{W})$ $(1 \leq k \leq n)$ with non-vanishing projection $\Psi_1^k = E_{\{M=m\}} A_k \Omega$ onto one-particle space of mass $m > 0$ and smear their space-time translates $\alpha_x(A_k) := U(x) A_k U(x)^*$ first with an auxiliary Schwartz function $\chi \in \mathscr{S}(\mathbb{R}^d)$ and afterwards with a positive-energy Klein–Gordon solution f_k (also for mass m) to obtain *creation-operator approximants*

$$B_k := A_k(\chi) := \int \mathrm{d}^d x \;\chi(x) \alpha_x(A_k), \tag{2}$$

$$B_{k\tau}(f_k) := \int \mathrm{d}^s x \;f_k(\tau, \mathbf{x}) \alpha_{(\tau,\mathbf{x})}(B_k), \quad (\tau \in \mathbb{R}). \tag{3}$$

The smearing operation (2) suitably restricts the energy-momentum transfer, while (3) may be understood as a comparison dynamics in the sense of scattering theory. More precisely due to mass gaps we may arrange $B_k \Omega \in E_{\{M=m\}} \mathscr{H}$ for suitable χ (supported in a sufficiently small neighbourhood of the mass shell) and then $B_{k\tau}(f_k)\Omega = \tilde{f}_k(\boldsymbol{P}) B_k \Omega$ is a one-particle state created from the vacuum, which is independent of the parameter $\tau \in \mathbb{R}$. The n-particle scattering states are now to be constructed as

$$\Psi_n^\pm := \lim_{\tau \to \pm\infty} \Psi_n(\tau), \quad \Psi_n(\tau) := B_{1\tau}(f_1) B_{2\tau}(f_2) \ldots B_{n\tau}(f_n)\Omega, \tag{4}$$

where existence of the limits can be reduced to the one-particle convergence if the norm of pairwise commutators is sufficiently decaying with $\tau \to \pm\infty$. However, even if the Klein–Gordon solutions f_k describe wave packets which separate for large enough τ, we should not expect such mutual commutation of the $B_{k\tau}(f_k)$ in a general wedge-local model.

Here the swapping relation (1) enters and yields a second family of creation operators defined analogously in terms of $A_k^\perp$ which satisfy

$$B_{k\tau}^\perp(f_k)\Omega = B_{k\tau}(f_k)\Omega. \tag{5}$$

[3] Up to technical points to be discussed in Sect. 3.1.

[4] Swapping relations are mentioned e.g. in [BS08] above Thm. 3.2 for bounded operators, in [Le03] below (3.13) for wedge-local fields, and indirectly in even earlier works of Schroer. The general connection to wedge-duality has been investigated in depth by Borchers [Bor95], Rem. 1.1 and subsequent comments.

Let us specialize to the exemplary outgoing case $\tau \to \infty$. Then for suitably propagating wave packets f_k we obtain an asymptotic commutator decay across the two operator families

$$\left\| \left[B_{j\tau}(f_j), B_{k\tau}^{\perp}(f_k) \right] \right\| \leq C_N (1+\tau)^{-N} \quad \text{for} \quad 1 \leq j < k \leq n, \quad \tau > 0, \tag{6}$$

where the emergence of an ordering constraint $j < k$ conditionally on the outgoing regime $\tau > 0$ is a typical trait of the causal geometry of wedges. To establish convergence of (4) we estimate via Cook's method ($0 < \tau_1 < \tau_2$)

$$\|\Psi_n(\tau_2) - \Psi_n(\tau_1)\| = \left\| \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \; \partial_\tau \Psi_n(\tau) \right\| \leq \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \; \|\partial_\tau \Psi_n(\tau)\| \,, \tag{7}$$

where the integrand on the right hand side is expanded using the product rule. To estimate the resulting terms we make use of (5) to write

$$\begin{aligned} &B_{1\tau}(f_1) \ldots (\partial_\tau B_{k\tau}(f_k)) \ldots B_{n\tau}(f_n)\Omega \\ &\quad = B_{1\tau}(f_1) \ldots (\partial_\tau B_{k\tau}(f_k)) \ldots B_{n-1\,\tau}(f_{n-1}) B_{n\tau}^{\perp}(f_n)\Omega \\ &\quad = B_{n\tau}^{\perp}(f_n) B_{1\tau}(f_1) \ldots (\partial_\tau B_{k\tau}(f_k)) \ldots B_{n-1\,\tau}(f_{n-1})\Omega \\ &\qquad + \text{(commutators)}, \end{aligned}$$

where commutator terms vanish rapidly as $\tau \to \infty$ by (6), $\left\| B_{j\tau}(f_j) \right\| \leq C(1+|\tau|^{s/2})$ and $\|B_{j\tau}^{\perp}(f_j)\| \leq C(1+|\tau|^{s/2})$. Iterating a total of $n-k$ times, the derivative term will act directly on the vacuum so that we can make use of $\partial_\tau B_{k\tau}(f_k)\Omega = 0$ as in standard Haag–Ruelle theory. Altogether (6) and polynomial norm growth of $B_{j\tau}(f_j)$, $B_{j\tau}^{\perp}(f_j)$ yield for $\tau > 0$ the rapid decay

$$\|B_{1\tau}(f_1) \ldots (\partial_\tau B_{k\tau}(f_k)) \ldots B_{n\tau}(f_n)\Omega\| \leq C'_N (1+\tau)^{-N}.$$

Summing up these terms, we obtain convergence of outgoing scattering states Ψ_n^+ from Cook's method (7). A similar swapping argument yields the Fock structure of these scattering states for any number of particles $n \geq 0$. For $n \leq 2$ swapping is strictly speaking not necessary, as scattering states can be directly constructed via $\lim_{\tau\to\infty} B_\tau(f) B_\tau^{\perp}(f^{\perp})\Omega$ as in [BBS01, GL07]. Lastly it is important to point out that beyond swapping, it is also necessary that all operators A_k entering in (4) are localizable in a common wedge $\mathcal{W}$. Further, the propagation velocities of f_k must be suitably restricted to match the wedge geometry and be in correspondence with the fixed ordering of creation-operator approximants in (4), as will be made precise in Sects. 3 and 4.

Our construction applies in particular to the model of Grosse and Lechner [GL07]. This model originated from a proposed quantum field theory on a non-commutative space-time, which may be motivated from gravitational considerations [DFR95]. Only later a reinterpretation as wedge-local quantum field theory on ordinary Minkowski space-time was discovered and it was shown that this model exhibits non-trivial 2-particle scattering [GL07]. The curious message of [GL07] was that the model itself is Poincaré-covariant, while Lorentz symmetry is broken at the level of scattering states. To clarify this effect, which is impossible in local quantum field theories, we carry out a general analysis of Poincaré covariance of the scattering states in Sect. 5. We intend to apply our results to extend the pioneering analysis of Grosse and Lechner to the multi-particle scattering data in a subsequent publication.

This paper is structured as follows. In Sect. 2 we introduce the wedge-local variant of the Haag–Kastler framework providing the standing assumptions of our construction. The wedge-local Haag–Ruelle theorem is established in Sect. 3 under certain geometrical restrictions allowing for a streamlined proof. These restrictions are lifted in Sect. 4, where we also obtain residual Lorentz covariance properties and pave the ground for a general discussion of wave operators and S-matrices in Sect. 5.

2. Wedge-Local Quantum Field Theories

Our results are valid for Quantum Field Theory models defined on general Minkowski space-time $\mathbb{R}^d$, whose metric we take in the mainly-minus convention and we denote the spatial dimension by $s := d - 1$. The family of wedge regions is defined as the orbit $\mathcal{P}\mathcal{W}_{\mathrm{r}} := \{\lambda\mathcal{W}_{\mathrm{r}} = \Lambda\mathcal{W}_{\mathrm{r}} + x : \lambda = (x, \Lambda) \in \mathcal{P}\}$ of the conventional *Rindler wedge*[5] $\mathcal{W}_{\mathrm{r}} := \{(t, \mathbf{x}) \in \mathbb{R}^d : |t| < x^1\}$ under the action of the Poincaré group $\mathcal{P}$ [BiW75].

A wedge-local quantum field theory model in operator-algebraic formulation is specified by mathematical objects $(\mathfrak{A}, \alpha, \mathscr{H}, \Omega)$, where $\mathscr{H}$ is the Hilbert space of pure states containing the *vacuum* as a distinguished unit vector $\Omega \in \mathscr{H}$. The *wedge-local net* $\mathfrak{A}$ is a mapping from the family wedge regions $\mathcal{P}\mathcal{W}_{\mathrm{r}} \ni \mathcal{W}$ to von Neumann algebras $\mathfrak{A}(\mathcal{W}) \subset \mathrm{B}(\mathscr{H})$, which serves to describe Einstein causality at the quantum mechanical level. Poincaré symmetry acts on the wedge-local net $\mathfrak{A}$ by a given group of isomorphisms[6] α_λ and we denote by $\lambda = (x, \Lambda) \in \mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow$ the elements of the proper orthochronous Poincaré group.

Guided by physical intuition we ask that these objects satisfy wedge-local variants of the Haag–Kastler postulates, which are concerned with the algebraic and representation-theoretic properties of $\mathfrak{A}$. Firstly, for any choice of wedge regions $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2$ we have

$$\begin{aligned}
&\textbf{Isotony} && \mathfrak{A}(\mathcal{W}_1) \subset \mathfrak{A}(\mathcal{W}_2) \text{ for } \mathcal{W}_1 \subset \mathcal{W}_2, && \text{(HK1)}\\
&\textbf{Locality} && \mathfrak{A}(\mathcal{W}_1) \subset \mathfrak{A}(\mathcal{W}_2)' \text{ for } \mathcal{W}_1 \subset \mathcal{W}_2', && \text{(HK2)}\\
&\textbf{Wedge-Duality} && \mathfrak{A}(\mathcal{W}') = \mathfrak{A}(\mathcal{W})', && (\text{HK2}^\sharp)\\
&\textbf{Translation-Covariance} && \alpha_x(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}(\mathcal{W} + x), \quad x \in \mathbb{R}^d, && \text{(HK3)}\\
&\textbf{Poincaré-Covariance} && \alpha_\lambda(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}(\lambda\mathcal{W}), \quad \lambda \in \mathcal{P}_+^\uparrow. && (\text{HK3}^\sharp)
\end{aligned}$$

Here the Minkowski causal complement $\mathcal{W}' = -\Lambda\mathcal{W}_{\mathrm{r}} + x$ of $\mathcal{W} = \Lambda\mathcal{W}_{\mathrm{r}} + x$ is also a wedge region and $\mathfrak{A}(\mathcal{W})'$ denotes the commutant of $\mathfrak{A}(\mathcal{W})$ relative to $\mathrm{B}(\mathscr{H})$.

On the representation-theoretic side we further assume that translations are unitarily implemented on the vacuum Hilbert space $\mathscr{H}$ by a strongly continuous $s+1$-parameter group, $\alpha_x(A) = U(x)AU(x)^*$. The representing unitaries are generated by the *energy-momentum operators* via $U(x) = U(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$, whose joint spectral resolution in terms of projection-operator-valued measures will be denoted by $\Delta \longmapsto E(\Delta) := E_{(H,\boldsymbol{P})}(\Delta)$. Focusing also in particular on the analysis of scattering theory it will be

[5] In the literature, $\mathcal{W}_{\mathrm{r}}$ is sometimes simply called the *standard wedge* or *right wedge*.

[6] The formulation of our main results requires only space-time translations. With some abuse of notation we denote translation automorphisms by the same letter α, or α_x, where $x \in \mathbb{R}^d$ is identified with $\lambda_x = (x, \mathbb{1}) \in \mathcal{P}_+^\uparrow$. In particular the basic version of the framework given by (HK1)–(HK6) suffices for multi-particle scattering provided a suitable swapping assumption holds, and we will state explicitly when the strengthened variants ($\text{HK2}^\sharp$) or ($\text{HK3}^\sharp$) are required.

convenient to further impose the following standard assumptions concerned with the vacuum representation and its one-particle spectrum,

$$\textbf{Uniqueness of } \boldsymbol{\Omega} \quad E(\{0\})\mathscr{H} = \mathbb{C}\Omega, \tag{HK4}$$

$$\textbf{Cyclicity of } \boldsymbol{\Omega} \quad \overline{\mathfrak{A}(\mathcal{W})\Omega} = \mathscr{H}, \tag{HK5}$$

$$\textbf{Mass Gap} \quad H_m \subset \operatorname{supp} E \subset \{0\} \cup H_m \cup \bar{H}_M \subset \bar{V}^+, \tag{HK6}$$

for some $M > m > 0$, where $H_m := \{(\omega_m(\mathbf{p}), \mathbf{p}) : \mathbf{p} \in \mathbb{R}^s\}$, $\omega_m(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m^2}$, is the (positive) hyperboloid of mass $m > 0$ and $\bar{H}_M := \{(\omega, \mathbf{p}) : \mathbf{p} \in \mathbb{R}^s, \omega \geq \omega_M(\mathbf{p})\}$ denotes the convex hull of H_M. Note that (HK6) implies in particular that the one-particle subspace $\mathscr{H}_1$ and the associated orthogonal projector $E_m := E(H_m)$ are non-trivial. We may extend any given wedge-local net also to regions obtained as sum of a given wedge and any open bounded region $\mathcal{O} \subset \mathbb{R}^{s+1}$ by setting $\mathfrak{A}(\mathcal{O} + \mathcal{W}) := (\cup_{x\in\mathcal{O}}\mathfrak{A}(\mathcal{W} + x))''$.

For later convenience we will also introduce some refined terminology for wedge regions concerning their geometry in the case of more than two dimensions. Recalling that any wedge region can be written as $\mathcal{W} = \Lambda\mathcal{W}_\mathrm{r} + x$, we may define the corresponding centered wedge as $\mathcal{W}_\mathrm{c} := \Lambda\mathcal{W}_\mathrm{r}$. $\mathcal{W}_\mathrm{c}$ is uniquely characterized by the coordinate origin being contained in its edge, and we will call such wedges *centered*. This concept may be motivated heuristically by noting that scattering situations are concerned with phenomena at very large distances, making finite translation by $x \in \mathbb{R}^d$ in a sense negligible. Centered wedges $\mathcal{W}$ are convex cones in the sense that $\mathcal{W} + \mathcal{W} \subset \mathcal{W}$. This assures that the causal ordering given via the *precursor* relation [BBS01]

$$\mathcal{O}_1 \prec_\mathcal{W} \mathcal{O}_2 :\Longleftrightarrow \mathcal{O}_2 - \mathcal{O}_1 \subset \mathcal{W}_\mathrm{c} \tag{8}$$

for non-empty regions $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^d$ is transitive, anti-symmetric, and irreflexive (hence asymmetric). Thus the precursor relation is a (strict) partial order, which is in fact Poincaré covariant. Namely, for any $\lambda = (x, \Lambda) \in \mathcal{P}$, any wedge $\mathcal{W}$ and any sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^{s+1}$ we have

$$\mathcal{O}_2 \prec_\mathcal{W} \mathcal{O}_1 \Longleftrightarrow \lambda\mathcal{O}_2 \prec_{\Lambda\mathcal{W}} \lambda\mathcal{O}_1, \tag{9}$$

as follows from the definition (8) and $\mathcal{O}_1 - \mathcal{O}_2 \subset \mathcal{W}_\mathrm{c} \Longleftrightarrow \lambda\mathcal{O}_1 - \lambda\mathcal{O}_2 \subset \Lambda\mathcal{W}_\mathrm{c}$. Individual points $x_1, x_2 \in \mathbb{R}^d$ can also be ordered, writing simply $x_1 \prec_\mathcal{W} x_2 :\Longleftrightarrow \{x_1\} \prec_\mathcal{W} \{x_2\}$.

We recall that the causal complement $\mathcal{W}'$ of any wedge region $\mathcal{W}$ is also a wedge-region, and it is clear that $(\mathcal{W}_\mathrm{c})' = (\mathcal{W}')_\mathrm{c}$. We say that $\mathcal{W}'$ is the *complementary* wedge to $\mathcal{W}$. More generally we will say that a wedge $\mathcal{W}^\perp$ is *opposite* to a given wedge $\mathcal{W}$ if $\mathcal{W}^\perp$ can be translated into the complement of $\mathcal{W}$, i.e. if for some $x \in \mathbb{R}^d$ we have $\mathcal{W}^\perp + x \subset \mathcal{W}'$. Lastly we will see that the construction of scattering states is most convenient for the geometrical situation of a given wedge whose edge is parallel to the time-zero hyperplane. This is equivalent to $\mathcal{W} = R\mathcal{W}_\mathrm{r} + x$ for $x \in \mathbb{R}^d$ and some spatial rotation $R \in \mathrm{SO}(s) \subset \mathcal{L}_+^\uparrow$, and we will call such wedges $\mathcal{W}$ *upright* or *non-tilted*. This is relevant as for upright $\mathcal{W}$ the restriction of $\prec_\mathcal{W}$ to certain hyperplanes behaves almost like a total relation, which will be helpful for establishing the Fock structure of scattering states in Sect. 3.2.

Lemma 1 ("quasi-totality" of $\prec_\mathcal{W}$ for velocity supports). *Let $\mathcal{W}$ be an upright wedge and let $\mathcal{V}_k, \mathcal{V}_k' \subset \mathbb{R}^{s+1}$, $(k = 1, 2)$, be sets of the form ("velocity supports")*

$$\mathcal{V}_k = \{1\} \times \mathbf{V}_k, \quad \mathbf{V}_k \subset \mathbb{R}^s, \quad (\textit{similarly for } \mathcal{V}_k') \tag{10}$$

satisfying

$$\mathcal{V}_2 \prec_{\mathcal{W}} \mathcal{V}_1, \quad \textit{and} \quad \mathcal{V}_2' \prec_{\mathcal{W}} \mathcal{V}_1'. \tag{11}$$

Then necessarily at least one of the two relations

$$\mathcal{V}_2' \prec_{\mathcal{W}} \mathcal{V}_1, \quad \textit{or} \quad \mathcal{V}_2 \prec_{\mathcal{W}} \mathcal{V}_1' \tag{12}$$

must be satisfied as well.

Proof. Let Λ be s.t. $\Lambda\mathcal{W} = \mathcal{W}_\mathrm{r}$ and note that as $\mathcal{W}$ is upright we can choose Λ as a spatial rotation. We obtain by (9) that

$$\Lambda\mathcal{V}_2 \prec_{\mathcal{W}_\mathrm{r}} \Lambda\mathcal{V}_1, \text{ and } \Lambda\mathcal{V}_2' \prec_{\mathcal{W}_\mathrm{r}} \Lambda\mathcal{V}_1'.$$

Due to the choice as spatial rotation, the sets $\bar{\mathcal{V}}_k := \Lambda\mathcal{V}_k$ are still of the form (10), and analogously for $\bar{\mathcal{V}}_k'$. Dropping bars, the two assumptions (11) for $\mathcal{W} = \mathcal{W}_\mathrm{r}$ translate to inequalities

$$\mathbf{e}_1 \cdot (\mathbf{g}_1 - \mathbf{g}_2) > 0 \text{ and } \mathbf{e}_1 \cdot (\mathbf{g}_1' - \mathbf{g}_2') > 0 \ \forall\, \mathbf{g}_k \in \mathbf{V}_k, \quad \mathbf{g}_k' \in \mathbf{V}_k', \quad (k = 1, 2), \tag{13}$$

where $\mathbf{e}_1 \in \mathbb{R}^s$ denotes the spatial unit-vector in 1-direction. Assuming that $\mathcal{V}_2' \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_1$ is false, there must be $g_2'^* = (1, \mathbf{g}_2'^*) \in \mathcal{V}_2'$, $g_1^* = (1, \mathbf{g}_1^*) \in \mathcal{V}_1$ forming an ordering "obstruction". Namely,

$$\begin{aligned} \neg(\mathcal{V}_2' \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_1) &\iff \neg(\forall\, \mathbf{g}_1 \in \mathbf{V}_1 \ \forall\, \mathbf{g}_2' \in \mathbf{V}_2' : \ \mathbf{e}_1 \cdot (\mathbf{g}_1 - \mathbf{g}_2') > 0) \\ &\iff \exists\, \mathbf{g}_1^* \in \mathbf{V}_1 \ \exists\, \mathbf{g}_2'^* \in \mathbf{V}_2' : \ \mathbf{e}_1 \cdot (\mathbf{g}_1^* - \mathbf{g}_2'^*) \le 0. \end{aligned} \tag{14}$$

For any given $\mathbf{g}_2 \in \mathbf{V}_2$ and $\mathbf{g}_1' \in \mathbf{V}_1'$ we can now estimate by transitivity

$$\mathbf{e}_1 \cdot (\mathbf{g}_1' - \mathbf{g}_2) = \mathbf{e}_1 \cdot (\mathbf{g}_1' - \mathbf{g}_2'^*) + \mathbf{e}_1 \cdot (\mathbf{g}_2'^* - \mathbf{g}_1^*) + \mathbf{e}_1 \cdot (\mathbf{g}_1^* - \mathbf{g}_2) > 0,$$

where we used that the first and last term on the right are strictly positive for any $\mathbf{g}_2 \in \mathbf{V}_2$ and $\mathbf{g}_1' \in \mathbf{V}_1'$ as particular instances of (13) and the middle term is non-negative due to (14). By definition this implies $\mathcal{V}_2 \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_1'$. □

Finally, let us remark that given any Haag–Kastler net of von Neumann algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ defined for open bounded regions $\mathcal{O} \subset \mathbb{R}^d$, there exists a canonical associated wedge-local net. On the other hand starting from a wedge-local net the question of existence or non-existence of local observables can be highly non-trivial, as explained in the recent review of Lechner [Le15]. While previously the existence of suitable localized operators was always regarded as essential for going beyond two-particle scattering states, cf. [BBS01] or [Le06] Section 6, we will see in the following that scattering theory in most wedge-local models can be studied in reasonable generality without any reference to local observables.

3. Construction of Scattering States

3.1. Swapping relations for opposite wedge algebras. At the core of our subsequent arguments to establish convergence and Fock structure of scattering states will be certain swapping identities, such as (1). Due to the mass gap assumption and with our desired application to the construction of scattering states it will be in fact sufficient if the swapping relation holds only after projection to the one-particle subspace.

Definition 2 (*swapping symmetry of single-particle states*). We say that a single-particle vector $\Psi_1 \in E_m\mathscr{H}$ of mass $m > 0$ is *swappable* with respect to a given wedge $\mathcal{W}$ if there exist operators $A \in \mathfrak{A}(\mathcal{W})$, $A^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, localized in $\mathcal{W}$ and an opposite wedge $\mathcal{W}^\perp = \mathcal{W}' + x$, $x \in \mathbb{R}^d$, respectively, such that

$$\Psi_1 = E_m A\Omega = E_m A^\perp\Omega. \tag{15}$$

As a matter of fact, the swapping relations (1), (15) can be obtained as a consequence of wedge duality (HK2$^\sharp$), which is a basic and well-established structural property in quantum field theory [BiW75,Mo18].

Lemma 3 (D. Buchholz, private communications (2017)). *In a wedge-local QFT satisfying duality* (HK2$^\sharp$) *there exist $A \in \mathfrak{A}(\mathcal{W})$, $A^\perp \in \mathfrak{A}(\mathcal{W}')$ such that*

$$\Psi^{\mathcal{W}} := A\Omega = A^\perp\Omega. \tag{16}$$

Moreover the subspace $\mathscr{H}^{\mathcal{W}} \subset \mathscr{H}$ of swappable vectors $\Psi^{\mathcal{W}}$ associated to any fixed wedge $\mathcal{W}$ is dense.

The proof of Buchholz relies on standard Tomita–Takesaki theory, but is somewhat detached from our otherwise purely scattering-theoretic analysis. To make the present paper self-contained we will provide a version of his argument in Appendix B. Alternatively it is possible to build the entire theory of Tomita and Takesaki upon swapping relations $A\Omega = A'\Omega$, see e.g. [KR2, Ch. 9.2, p. 625 ff.]. In this context the space of all swappable vectors sometimes appears the operator-algebraic literature as so-called *modular Hilbert algebra* or *Tomita algebra*. In spite of the abstract general method, the argument is constructive and hence may turn out useful for future studies of concrete models.

Corollary 4. *Assuming* (HK2$^\sharp$)*, single-particle vectors satisfying the swapping relation* (15) *w.r.t. any given wedge $\mathcal{W}$ are dense in the single-particle space $\mathscr{H}_1 := E_m\mathscr{H}$.*

The swapping relation (16) is established in Appendix B exactly for the case of "touching" wedges $\mathcal{W}^\perp = \mathcal{W}'$. Then $\mathcal{W}^\perp \cap \mathcal{W}$ is empty, so that (16) becomes nontrivial also for local theories. For space-time dimension $d \geq 2+1$ the dense sets of swappable vectors constructed in Appendix B have a non-trivial dependence on $\mathcal{W}$. For the purposes of scattering theory on the other hand it is sufficient if the wedges are merely opposite, admitting overlaps as in Definition 2, so that swapping becomes trivially satisfiable in local QFT. Interestingly certain wedge-local models also admit a dense subspace of vectors which are swappable in the sense of Definition 2 for all wedges simultaneously. An example for such behaviour is found in the class of deformed local theories constructed in [BLS11], see loc. cit., eq. (2.7).

One simple and immediate consequence of the swapping relation is the consistency of our definition of scattering states (4) with previous discussions of two-particle scattering in wedge-local models [GL07,BS08], where the physically more intuitive opposite-localization prescription $\Psi_2^+ := \lim_{\tau\to\infty} B_\tau(f)B_\tau^\perp(f^\perp)\Omega$ has been used. With the swapping relation as main technical tool at hand, we may write

$$\Psi_2^+ = \lim_{\tau\to\infty} B_\tau(f)B_\tau^\perp(f^\perp)\Omega = \lim_{\tau\to\infty} B_\tau(f)\bar{B}_\tau(f^\perp)\Omega, \tag{17}$$

where $\bar{B}_\tau(f^\perp)$ is defined in terms of $\bar{A} \in \mathfrak{A}(\mathcal{W})$ with $\bar{A}\Omega = A^\perp\Omega$. The new prescription from (17) with all operators localized in the same wedge $\mathcal{W}$ now generalizes to N-particle scattering theory, as will be seen in the next section.

3.2. Wedge-local Haag–Ruelle theorem. As comparison dynamics for the construction of scattering states we may restrict to *regular positive-energy* Klein–Gordon solutions f_k, which are of the form

$$f_k(t,\mathbf{x}) = \int \frac{\mathrm{d}^s k}{(2\pi)^s} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}-\mathrm{i}\omega_m(\mathbf{k})t}\ \tilde{f}_k(\mathbf{k}), \tag{18}$$

$$\omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2+m^2},\ \ \tilde{f}_k \in C_c^\infty(\mathbb{R}^s). \tag{19}$$

Definition 5 (*Haag–Ruelle creation operator approximants*). For $A \in \mathfrak{A}(\mathcal{W})$, $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$, and f a regular positive-energy Klein–Gordon solution we set for $\tau \in \mathbb{R}$

$$B := A(\chi) = \int \mathrm{d}^{s+1}x\ \chi(x)\alpha_x(A), \tag{20}$$

$$B_\tau(f) := \int \mathrm{d}^s x\ f(\tau,\mathbf{x})\alpha_{(\tau,\mathbf{x})}(B). \tag{21}$$

Equations (18)–(21) provide the standing notation for the rest of the paper and in particular for our main result Theorem 6. Further any operators $B \in \mathrm{B}(\mathscr{H})$ in the following are obtained via $B := A(\chi)$ from a corresponding $A \in \mathfrak{A}(\mathcal{W})$, with χ chosen as in Lemma 7, in accordance with the mass gap (HK6).

The restrictions on propagation of wave packets mentioned in the introduction are made precise using the precursor relation (8), to constrain the *velocity supports*

$$\mathcal{V}_{f_k} := \{(1,\mathbf{k}/\omega_m(\mathbf{k})) : \mathbf{k} \in \operatorname{supp} \tilde{f}_k\}. \tag{22}$$

Basic intuition for handling localizations of creation-operator approximants comes from the fact that regular f_k are rapidly decreasing outside the cone $\Upsilon_{f_k}^\delta := \mathbb{R}\mathcal{V}_{f_k}^\delta$ generated by any δ-neighbourhood $\mathcal{V}_{f_k}^\delta \supset \mathcal{V}_{f_k}$, as seen from standard non-stationary phase estimates.[7]

Theorem 6. *Fix a wedge $\mathcal{W}$ and let $\Psi_1^k \in \mathscr{H}_1$ $(1 \le k \le n)$ be single-particle vectors isolated from the remaining energy-momentum spectrum which satisfy the swapping relation $\Psi_1^k = E_m A_k\Omega = E_m A_k^\perp\Omega$, $A_k \in \mathfrak{A}(\mathcal{W})$, $A_k^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, and define $B_k = A_k(\chi)$ (with χ as in Lemma 7 below).*

[7] The velocity support estimates for regular Klein–Gordon solutions are due to Ruelle [Ru62], see also [RS3, Sec. XI.3, App. 1] or [A, Thm. 5.3]. Via such estimates, disjointness $\mathcal{V}_k \cap \mathcal{V}_j = \emptyset$, $(k \neq j)$ is sufficient for local QFT to control equal-time commutators [Hep65], and to some limited extent also non-equal time-commutators [Du17].

(i) For any family of regular positive-energy Klein–Gordon solutions f_k satisfying

$$\mathcal{V}_{f_n} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}} \prec_{\mathcal{W}} \dots \prec_{\mathcal{W}} \mathcal{V}_{f_1}, \tag{23}$$
$$\Psi_n(\tau) := B_{1\tau}(f_1)B_{2\tau}(f_2)\dots B_{n\tau}(f_n)\Omega, \quad (\tau \in \mathbb{R}), \tag{24}$$

converges in norm for $\tau \to \infty$.

(ii) Let $\Psi_n^+ := \lim_{\tau\to\infty}\Psi_n(\tau)$, $\Psi'^+_{n'} := \lim_{\tau\to\infty}\Psi'_{n'}(\tau)$ be scattering states as in (6), constructed from operators localizable with respect to the same wedge $\mathcal{W}$. Then for upright $\mathcal{W}$ their scalar products can be computed using the Fock prescription

$$\left\langle \Psi_n^+, \Psi'^+_{n'} \right\rangle = \delta_{nn'} \prod_{k=1}^{n} \left\langle B_{k\tau}(f_k)\Omega, B'_{k\tau}(f'_k)\Omega \right\rangle, \tag{25}$$

where the right-hand side is independent of τ.

Analogous statements hold for convergence and Fock structure of any two incoming scattering states defined as the limit of $\Psi_n(\tau)$ for $\tau \to -\infty$, assuming the reversed ordering of wave packets

$$\mathcal{V}_{f_n} \succ_{\mathcal{W}} \mathcal{V}_{f_{n-1}} \succ_{\mathcal{W}} \dots \succ_{\mathcal{W}} \mathcal{V}_{f_1} \tag{26}$$

(*while keeping the operator ordering of* (24)).

We should point out that the ordering prescription (23) is not new. Such relations are well known from the form-factor programme and related constructive QFT literature, see e.g. [Smi, p. 8], [Le06, esp. Sec. 6], and references therein. The construction of Theorem 6 is in fact consistent with these debut appearances of the ordering conditions (23), (26),[8] and enables the generalization to arbitrary Poincaré frames as needed for higher dimensional space-times, to be established in Sect. 4. Even for the two-dimensional case we note that in contrast to the results from [BBS01,Le06] our arguments require neither the existence of local observables, nor temperateness of suitable polarization-free generators.

Lemma 7 (Haag–Ruelle Lemma, wedge-local version). *Let $A \in \mathfrak{A}(\mathcal{W})$ and $K \subset K' \subset H_m$ be compact subsets of the mass shell, such that K can be separated from $H_m \setminus K'$ by a smooth function. Then there exists a suitable $\chi \in \mathscr{S}(\mathbb{R}^{s+1})$ (with $\hat{\chi}$ supported in a sufficiently small neighbourhood of the mass shell as dictated by the mass gaps* (HK6)) *such that $B := A(\chi)$ satisfies*

(i) $B\Omega \in E(K')\mathscr{H} \subset E(H_m)\mathscr{H}$,
(ii) $E(K)B\Omega = E(K)A\Omega$,
(iii) $B^*\Omega = 0$,

[8] To readers familiar with [Smi], [Le06] the ordering prescriptions (23), (26) may at first sight appear to be in conflict with the established conventions that for rapidities $\beta_1 < \beta_2 < \dots < \beta_n$,

$$z^*(\beta_1)z^*(\beta_2)\dots z^*(\beta_n)|\Omega\rangle = |\beta_1,\dots,\beta_n\rangle^{\text{out}},$$
$$z^*(\beta_n)z^*(\beta_{n-1})\dots z^*(\beta_1)|\Omega\rangle = |\beta_1,\dots,\beta_n\rangle^{\text{in}},$$

as e.g. in [Le06, Thm. 6.1.2]. Clearly, rewriting in terms of velocities $v_k := (1, \tanh\beta_k)$ gives $\beta_1 < \beta_2 < \dots < \beta_n \iff v_1 \prec_{\mathcal{W}_\text{r}} v_2 \prec_{\mathcal{W}_\text{r}} \dots \prec_{\mathcal{W}_\text{r}} v_n$. Consistency with (23), (26) can be seen by noting that in the conventions of [Le06] the combination $\phi(f) = z^*(\hat{f}^+) + z(\hat{f}^-)$ is actually affiliated to the algebra of the left wedge $\mathcal{W}_\ell = -\mathcal{W}_\text{r}$, for which we have $v_1 \prec_{\mathcal{W}_\text{r}} v_2 \prec_{\mathcal{W}_\text{r}} \dots \prec_{\mathcal{W}_\text{r}} v_n \iff v_1 \succ_{\mathcal{W}_\ell} v_2 \succ_{\mathcal{W}_\ell} \dots \succ_{\mathcal{W}_\ell} v_n$.

(iv) $B^*\Psi_1 = E_\Omega B^*\Psi_1$ *for all* $\Psi_1 \in E(K' \cap H_m)\mathscr{H}$*, where* $E_\Omega := |\Omega\rangle\langle\Omega|$.

(v) B *is* almost wedge-local *(w.r.t.* $\mathcal{W}$*), i.e. for any* $r > 0$ *there exists* $B_r \in \mathfrak{A}(\mathcal{W} + \mathscr{C}_r)$ *so that for any* $N \in \mathbb{N}$ *we have for a suitable* $C_N > 0$ *that*

$$\|B - B_r\| \leq \frac{C_N}{1 + r^N}. \tag{27}$$

Here $\mathscr{C}_r := \{x \in \mathbb{R}^{s+1} : |x^0| + |\mathbf{x}| < r\}$ *denotes the double cone of radius* r.

The main spectral statements above, (i), (iii), and (iv), may be understood by noting that the smearing operation $B := A(\chi)$ restricts the Arveson spectrum[9] $\mathrm{Sp}_B\alpha \subset \mathrm{supp}\,\hat{\chi}$. A significant modification in comparison to the standard results from the local case [Ha58,Ru62] appears in (v), where the statement of the lemma needs to be adapted for the wedge-local case. Leaving aside the localization for a moment, we directly obtain that the creation-operator approximants $B_\tau(f)$ from Definition 5 satisfy most of the standard properties required for the Haag–Ruelle construction.

Proposition 8 (elementary properties of B and B_τ).

(i) $B_\tau(f)\Omega = \tilde{f}(\boldsymbol{P})B\Omega$ *for all* $\tau \in \mathbb{R}$.

(ii) If $A\Omega = A^\perp\Omega$*, the corresponding Haag–Ruelle operators satisfy* $B_\tau(f)\Omega = B_\tau^\perp(f)\Omega$.

(iii) $\partial_\tau B_\tau(f)\Omega = 0$.

(iv) $\|B_\tau(f)\| \leq C(1 + |\tau|^{s/2})$.

(v) $\partial_\tau B_\tau(f)$ *exists in norm and* $\|\partial_\tau B_\tau(f)\| \leq C'(1 + |\tau|^{s/2})$.

(vi) $B_{1\tau}(f_1)^* B_{2\tau}(f_2)\Omega = E_\Omega B_{1\tau}(f_1)^* B_{2\tau}(f_2)\Omega$*, independently of velocity supports and operators possibly associated to different wedges* $\mathcal{W}_1$*,* $\mathcal{W}_2$*, where* $E_\Omega := |\Omega\rangle\langle\Omega|$.

The proofs of Lemma 7 and Proposition 8 carry over literally from standard Haag–Ruelle theory up to aspects pertaining to the weakened localization and hence shall be skipped here. For further details we refer to [A, Sec. 5] or [Du17]. The most serious consequence of wedge-locality is expressed by the following localization and commutator estimates, which provide the technical background responsible for the break-down of the standard Haag–Ruelle arguments beyond the two-particle case.

Lemma 9. *Let* $A \in \mathfrak{A}(\mathcal{W})$*. For any* $\tau \in \mathbb{R}$ *and* $\delta > 0$ *the corresponding* $B_\tau := B_\tau(f)$ *can be approximated by* $B_\tau^{(\delta)} \in \mathfrak{A}(\tau\mathcal{V}_f + \mathscr{C}_{\delta|\tau|} + \mathcal{W})$*,* $(\delta > 0)$*, such that for any* $N \in \mathbb{N}$

$$\left\| B_\tau^{(\delta)} - B_\tau \right\| \leq \frac{C_N^\delta}{1 + |\tau|^N}, \tag{28}$$

where the constants C_N^δ *depend on* f*,* A *and* χ*, but are independent of* τ.

For later use in Sect. 4 we note that analogous approximants $\bar{B}_\tau^{(\delta)}$ *exist if* f *is replaced by the pointwise product* $\bar{f} := fh$ *with a polynomially bounded measurable function* $h : \mathbb{R}^d \to \mathbb{C}$.

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[9] See e.g. [Arv82] or [BDN15], Sec. 3.

Corollary 10 (commutators with ordered velocity support). *Let B, $B^{\perp}$ be as in Lemma 7 for a pair of opposite wedges $\mathcal{W}$, $\mathcal{W}^{\perp}$, respectively, and let f, $f^{\perp}$ be ordered by $\mathcal{V}_{f^{\perp}} \prec_{\mathcal{W}} \mathcal{V}_f$. Then for any $\tau > 0$,*

$$\left\| \left[B_{\tau}^{\perp}(f^{\perp}), B_{\tau}(f) \right] \right\| \leq \frac{C_N}{1 + |\tau|^N}, \tag{29}$$

where C_N depend on operators and smearing functions as in Lemma 9. For $\tau < 0$ estimate (29) holds under the reversed ordering assumption $\mathcal{V}_f \prec_{\mathcal{W}} \mathcal{V}_{f^{\perp}}$. The commutator estimate extends to the cases that one or both of the operators in (29) are replaced by their adjoints or τ-derivatives.

For the convenience of the reader, the technical proofs of Lemma 9 and Corollary 10 are provided in Appendix A.

Proof of Theorem 6. **Ad (i)**. Setting $\Psi_n(\tau) := B_{1\tau}(f_1)B_{2\tau}(f_2)\dots B_{n\tau}(f_n)\Omega$ we want to establish convergence for $\tau \to \infty$. Due to Proposition 8 (v) and (iv), Cook's method is applicable and we can write for $0 < \tau_1 < \tau_2$

$$\|\Psi_n(\tau_2) - \Psi_n(\tau_1)\| = \left\| \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \; \partial_\tau \Psi_n(\tau) \right\| \leq \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \; \|\partial_\tau \Psi_n(\tau)\| \, . \tag{30}$$

Convergence will follow from the rapid decay estimate $\|\partial_\tau \Psi_n(\tau)\| \leq C_N \tau^{-N}$ for $\tau > 0$.

The latter is obtained by induction with respect to the number of particles n, with starting case $n = 1$ given by $\partial_\tau \Psi_n(\tau) = 0$ as seen in Proposition 8 (iii). For the induction step we write

$$\begin{aligned} \partial_\tau \Psi_n(\tau) &= \partial_\tau (B_{1\tau}(f_1)B_{2\tau}(f_2)\dots B_{n-1\,\tau}(f_{n-1}))\; B_{n\tau}(f_n)\Omega \\ &\quad + B_{1\tau}(f_1)\dots B_{n-1\,\tau}(f_{n-1})\; \partial_\tau B_{n\tau}(f_n)\Omega \\ &= \partial_\tau (B_{1\tau}(f_1)B_{2\tau}(f_2)\dots B_{n-1\,\tau}(f_{n-1}))\; B_{n\tau}^{\perp}(f_n)\Omega, \end{aligned} \tag{31}$$

where we first used Proposition 8 (iii) to drop the term with derivative operator acting directly on the vacuum and used that the swapping relation (15) implies $B_{n\tau}(f_n)\Omega = B_{n\tau}^{\perp}(f_n)\Omega$. Now there are oppositely wedge-localized pairs of HR-operators whose commutators can be controlled using Corollary 10, and we may estimate for $\tau > 0$

$$\begin{aligned} \|\partial_\tau \Psi_n(\tau)\| \leq & \left\| B_{n\tau}^{\perp}(f_n) \right\| \left\| \partial_\tau \Psi_{n-1}(\tau) \right\| \\ & + \left\| \left[\partial_\tau B_{1\tau}(f_1)\dots B_{n-1\,\tau}(f_{n-1}), B_{n\tau}^{\perp}(f_n) \right] \right\| \|\Omega\| \, . \end{aligned} \tag{32}$$

Here the first summand is rapidly decreasing for $\tau \to \infty$ by the induction assumption and Proposition 8 (iv). The second summand can be generously bounded from above by expanding the derivative and commutator as

$$\sum_{k=1}^{n-1}\sum_{j=1}^{n-1} B_{1\tau}(f_1)\dots(\partial_\tau B_{k\tau}(f_k))\dots\left[B_{j\tau}(f_j), B_{n\tau}^{\perp}(f_n) \right]\dots B_{n-1\,\tau}(f_{n-1}). \tag{33}$$

Estimating the corresponding operator norm in (32) by expanding in terms of $\|B_{k\tau}(f_k)\| \leq C_k(1+|\tau|^{s/2})$, $\|\partial_\tau B_{k\tau}(f_k)\| \leq C_k'(1+|\tau|^{s/2})$, $\left\|\left[B_{j\tau}(f_j), B_{n\tau}^{\perp}(f_n)\right]\right\| \leq C_N(1+\tau)^{-N}$, and $\left\|\left[\partial_\tau B_{k\tau}(f_k), B_{n\tau}^{\perp}(f_n)\right]\right\| \leq C_N(1+\tau)^{-N}$ yields an overall rapid decay. Here we used that Corollary 10 applies due to transitivity of the precursor ordering. Together we obtain that (32) decays faster than any polynomial, and thus convergence of outgoing scattering states follows from (30). The existence of incoming states follows analogously for opposite operator ordering.

Ad (ii). As before let $\Psi_n^+ := \lim_{\tau\to\infty} B_{1\tau}(f_1)\ldots B_{n\tau}(f_n)\Omega$. Further we take a second scattering state $\Psi_{n'}^{\prime +} := \lim_{\tau\to\infty} B'_{1\tau}(f'_1)\ldots B'_{n'\tau}(f'_{n'})\Omega$ defined with respect to the same wedge $\mathcal{W}$ and denote the minimum number of particles by $N := \min(n, n')$. We will assume instead of upright $\mathcal{W}$ only the following weaker technical ordering condition: adjacent pairs of velocity supports are precursor-comparable from the rear also across the two families, in the sense that

$$\forall\, 0 \le j < N : \mathcal{V}_{f_{n-j}} \prec_{\mathcal{W}} \mathcal{V}_{f'_{n'-j-1}} \text{ or } \mathcal{V}_{f'_{n'-j}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-j-1}}. \tag{34}$$

For upright wedges (34) follows from Lemma 1, but the argument based on (34) can be applied also for non-upright $\mathcal{W}$, e.g. to compute $\left\|\Psi_n^+\right\|^2 = \langle \Psi_n^+, \Psi_n^+\rangle$.

The proof of the Fock relation (25) is now by induction on the minimum number of particles N. By continuity of the scalar product we may write

$$\left\langle \Psi_n^+, \Psi_{n'}^{\prime +}\right\rangle = \lim_{\tau\to\infty} \left\langle \Omega, B_{n\tau}(f_n)^* \ldots B_{1\tau}(f_1)^*\, B'_{1\tau}(f'_1)\ldots B'_{n'\tau}(f'_{n'})\Omega\right\rangle. \tag{35}$$

For $N = 0$ the Fock identity (25) follows from $\|\Omega\| = 1$ or Lemma 7 (iii), in the respective cases vacuum-vacuum or for a non-zero number of creation operators. Assuming (25) holds up to the minimum number of $N-1$ particles, we now distinguish the two cases $\mathcal{V}_{f_n} \prec_{\mathcal{W}} \mathcal{V}_{f'_{n'-1}}$ or $\mathcal{V}_{f'_{n'}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}}$ obtained from (34), as they determine the side of (35) on which the swapping should be performed. Let us proceed for the case $\mathcal{V}_{f'_{n'}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}}$, by swapping

$$\begin{aligned}
\left\langle \Psi_n(\tau), \Psi'_{n'}(\tau)\right\rangle &= \left\langle \Omega, B_{n\tau}(f_n)^* \ldots B_{1\tau}(f_1)^*\, B'_{1\tau}(f'_1)\ldots B'_{n'\tau}(f'_{n'})\Omega\right\rangle \\
&= \left\langle \Omega, B_{n\tau}(f_n)^* \ldots B_{1\tau}(f_1)^*\, B'_{1\tau}(f'_1)\ldots B'^{\perp}_{n'\tau}(f'_{n'})\Omega\right\rangle \\
&= \left\langle \Omega, B^*_{n\tau} B'^{\perp}_{n'\tau} B^*_{n-1\,\tau}\ldots B^*_{1\tau}\, B'_{1\tau}\ldots B'_{n'-1\,\tau}\Omega\right\rangle \\
&\quad + \left\langle \Omega, B^*_{n\tau}\left[B^*_{n-1\,\tau}\ldots B^*_{1\tau}\, B'_{1\tau}\ldots B'_{n'-1\,\tau}, B'^{\perp}_{n'\tau}\right]\Omega\right\rangle,
\end{aligned}$$

where in the last step and below we suppress obvious wave packet dependences. Expanding the commutator gives

$$\begin{aligned}
&\sum_{k=1}^{n-1} B^*_{n-1\,\tau}\ldots\left[B^*_{k\tau}, B'^{\perp}_{n'\tau}\right]\ldots B^*_{1\tau} B'_{1\tau}\ldots B'_{n'-1\,\tau} \\
&\qquad + B^*_{n-1\,\tau}\ldots B^*_{1\tau} \sum_{k=1}^{n'-1} B'_{1\tau}\ldots\left[B'_{k\tau}, B'^{\perp}_{n'\tau}\right]\ldots B'_{n'-1\,\tau}.
\end{aligned}$$

Here Corollary 10 applies due to $\mathcal{V}_{f'_{n'}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}}$, the assumed orderings (23) of the velocity supports of f_k and f'_k within each family, and transitivity of the precursor ordering. This yields $\left\|\left[B_{k\tau}(f_k)^*, B'^{\perp}_{n'\tau}(f'_{n'})\right]\right\| \le C_N(1+\tau)^{-N}$ and $\left\|\left[B'_{k'\tau}(f'_{k'}), B'^{\perp}_{n'\tau}(f'_{n'})\right]\right\| \le C_N(1+\tau)^{-N}$ for $1 \le k \le n-1$ and $1 \le k' \le n'-1$, respectively, so that together with

$\|B^*_{j\tau}\| \le C_j(1+|\tau|^{s/2})$ and $\|B'_{j\tau}\| \le C'_j(1+|\tau|^{s/2})$ from Proposition 8 (iv), we can estimate for $\tau > 0$

$$\left|\left\langle \Psi_n(\tau), \Psi'_{n'}(\tau)\right\rangle - \left\langle \Omega, B^*_{n\tau} B'^{\perp}_{n'\tau} B^*_{n-1\,\tau} \dots B^*_{1\tau}\, B'_{1\tau} \dots B'_{n'-1\,\tau}\Omega\right\rangle\right| \le C_N \tau^{-N}. \quad (36)$$

As $\lim_{\tau\to\infty} \left\langle \Psi_n(\tau), \Psi_{n'}(\tau')\right\rangle$ exists by part (i) of this theorem, which was established above,

$$\begin{aligned}\lim_{\tau\to\infty} \left\langle \Psi_n(\tau), \Psi'_{n'}(\tau)\right\rangle &= \lim_{\tau\to\infty} \left\langle (B'^{\perp}_{n'\tau})^* B_{n\tau}\Omega,\, B^*_{n-1\,\tau} \dots B^*_{1\tau}\, B'_{1\tau} \dots B'_{n'-1\,\tau}\Omega\right\rangle \\ &= \lim_{\tau\to\infty} \left\langle \Omega, (B'^{\perp}_{n'\tau})^* B_{n\tau}\Omega\right\rangle \left\langle \Omega, B^*_{n-1\,\tau} \dots B^*_{1\tau}\, B'_{1\tau} \dots B'_{n'-1\,\tau}\Omega\right\rangle,\end{aligned}$$

where the right hand side was rewritten using the clustering identity from Proposition 8 (vi). The existence of the limit on the right-hand side now follows for the one-particle matrix element in the first factor from Proposition 8 (iii), and for the second factor from the induction assumption, respectively. Taken together we obtain the Fock formula (25).

For the complementary ordering $\mathcal{V}_{f'_{n'}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}}$ in (35) we swap instead on the opposite side, making use of $B_{n\tau}(f_n)\Omega = B^{\perp}_{n\tau}(f_n)\Omega$. Following otherwise the same chain of arguments we obtain the limit (25) also in this case, concluding the induction step. □

To close this section let us recall that in dimension 1 + 1 all wedges are upright in a trivial sense. In higher dimension the restriction to upright wedges is unnatural as it singles out a subfamily of localization wedges which is not fully Poincaré covariant. We will later see that the uprightness restriction is of a technical nature arising due to the a priori Lorentz-frame dependent formulation of Haag–Ruelle theory. Consequently it can be lifted by passing to a variant of the Haag–Ruelle creation operator approximants (3) adapted to the reference frame of a given (non-upright) operator localization wedge $\mathcal{W}$.

4. Localization in General Wedges

The goal of this section is to remove the assumption of localization of operators in upright wedges from Theorem 6 (ii), as will be needed for a physically satisfactory discussion of the known Poincaré-covariant wedge-local models (e.g. as in [BLS11]). We recall that these additional considerations are specific to the case of spatial dimension $s > 1$. The following simple example illustrates the causal restrictions in the non-upright case which invalidate Lemma 1 and allows to visualize how these are resolved below.

Remark 11 (canonical non-upright wedge). A non-upright wedge can be obtained by boosting the Rindler wedge $\mathcal{W}_\mathrm{r} = \{x \in \mathbb{R}^d : |x^0| < x^1\}$, $d \ge 3$, in x^2-direction with rapidity $\beta \in \mathbb{R} \setminus \{0\}$, yielding

$$\mathcal{W} := \Lambda^{(2)}_\beta \mathcal{W}_\mathrm{r} = \{x \in \mathbb{R}^d : \left|\cosh(\beta)x^0 - \sinh(\beta)x^2\right| < x^1\}. \quad (37)$$

For concreteness we may take $d = 3$. The relevant part determining the precursor ordering of velocity supports $\mathcal{V}_1 \prec_{\mathcal{W}} \mathcal{V}_2 \iff \mathcal{V}_2 - \mathcal{V}_1 \subset \mathcal{W}$ is the restriction of $\mathcal{W}$ to $\{x^0 = 0\}$. For the upright case $\beta = 0$ this restriction is a half plane, and the opposite ordering $\mathcal{V}_2 \prec_{\mathcal{W}} \mathcal{V}_1$ corresponds to inclusion in the complementary open half-plane. Exactly this special geometrical situation is necessary for the validity of Lemma 1.

Further it suggests the physical interpretation that the scattering states constructed in Theorem 6 cover the entire 2-particle velocity space up to a set of measure zero.[10]

However for the non-upright case $\beta \neq 0$ the restriction of $\mathcal{W}$ to $\{x^0 = 0\}$ yields merely a cone $C := \{\mathbf{x} \in \mathbb{R}^{d-1} : \left|\sinh(\beta)x^2\right| < x^1\}$. Hence there is a non-trivial region of the two-particle velocity space which cannot be locally decomposed into ordered configurations. For example we may take the corresponding velocity supports concentrated in sufficiently small neighbourhoods of points $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ for which

$$v_1 \not\prec_{\mathcal{W}} v_2 \text{ and } v_2 \not\prec_{\mathcal{W}} v_1 \quad\Longleftrightarrow\quad \mathbf{v}_2 - \mathbf{v}_1 \in \mathbb{R}^2 \setminus (C \cup (-C)) =: \Xi, \tag{38}$$

where a "causally forbidden" region Ξ appears, having vanishing measure only if $\beta = 0$.

4.1. Haag–Ruelle theorem with adapted Lorentz frame. Difficulties as in (38) result from the implicit Lorentz-frame dependence of the Haag–Ruelle operators $B_\tau(f)$. Nevertheless the latter have turned out to be well suited for the construction with upright $\mathcal{W}$. This motivates us to adjust the construction from Theorem 6 for general $\mathcal{W}$ by passing to a suitable reference frame.[11]

Definition 12 (Adapted Haag–Ruelle operators). For a general (possibly non-upright) wedge $\mathcal{W}$, $A \in \mathfrak{A}(\mathcal{W})$, $B = A(\chi)$ as before and regular positive-energy Klein–Gordon solutions f, we set for $\tau \in \mathbb{R}$

$$B_\tau^\Lambda(f) := \int \mathrm{d}^s x \; f(\Lambda(\tau, \mathbf{x}))\alpha_{(\Lambda(\tau,\mathbf{x}))}(B), \tag{39}$$

where $\Lambda \in \mathcal{L}^*(\mathcal{W}) := \{\Lambda \in \mathcal{L}_+^\uparrow : \Lambda\mathcal{W}_\mathrm{r} = \mathcal{W}_\mathrm{c}\}$ or more generally $\Lambda \in \mathcal{L}_+^\uparrow$.

In fact, such $B_\tau^\Lambda(f)$ appear naturally in the discussion of Lorentz covariance in standard Haag–Ruelle theory. Here we just introduce them in an ad-hoc manner, even if the wedge-local net may not be Lorentz covariant. In the following we will see that they can equally well serve as creation-operator approximants, which will turn out suitable for our cause. We should emphasize that no Lorentz transformation is applied to B—only the hyperplane used for smearing the translates $\alpha_x(B)$ is modified. Fortunately it is not necessary to repeat our arguments from Sect. 3.2. We will instead infer the existence of the limits

$$\Psi_n^{+,\Lambda} := \lim_{\tau\to\infty} B_{1\tau}^\Lambda(f_1) B_{2\tau}^\Lambda(f_2) \ldots B_{n\tau}^\Lambda(f_n)\Omega \tag{40}$$

and their Fock structure for suitably ordered wave packets from a redefinition of the wedge-local net and the results of Sect. 3.2. The basic observation is that the modification of passing from f to $f^\Lambda(x) := f(\Lambda x)$ and from translation by α_x to modified translation automorphisms $\alpha_x^\Lambda := \alpha_{\Lambda x}$ entering in (39) are both compatible with the underlying structures in a sense to be made precise now.

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[10] An important caveat here is that this simple and compelling picture contains implicitly the assumption of conventional (e.g. bosonic) particle statistics. This may be misleading in the general wedge-local setting, as illustrated in recent examples constructed by Longo, Tanimoto, and Ueda [LTU17, Sec. 5].

[11] Constructions using Lorentz-covariant creation-operator approximants (e.g. [Her13]) face similar problems as in (38) when applied in a wedge-local setting.

Lemma 13. *Let* $\Lambda \in \mathcal{L}_+^\uparrow$.

(i) $f^\Lambda(x) := f(\Lambda x)$ *defines a regular positive-energy Klein–Gordon solution iff* f *is a regular positive-energy Klein–Gordon solution.*

(ii) Setting $\alpha_x^\Lambda := \alpha_{\Lambda x}$ *and* $\mathfrak{A}^\Lambda(\mathcal{W}) := \mathfrak{A}(\Lambda\mathcal{W})$, $(\mathfrak{A}^\Lambda, \alpha^\Lambda, \Omega)$ *is a wedge-local quantum field theory satisfying* (HK1)–(HK6)*, and possibly* (HK2$^\sharp$), (HK3$^\sharp$)*, iff the corresponding assumptions hold for* $(\mathfrak{A}, \alpha, \Omega)$.

If (HK3$^\sharp$) *holds, we set further* $\alpha_\lambda^\Lambda(A) := \alpha_{(\Lambda x, \Lambda\Lambda_1\Lambda^{-1})}(A)$ *for* $\lambda = (x, \Lambda_1) \in \mathcal{P}_+^\uparrow$.

Proof. Lorentz invariance of the Klein–Gordon equation (i) is standard, so let us only comment that the restriction to orthochronous Lorentz transformation is essential for preserving the positive-energy property, and that the regularity property can be concluded via the representation (19) and standard (non-)stationary phase estimates.

Statement (ii) follows from elementary computations which we illustrate for the example of (HK3$^\sharp$). Letting $\lambda = (x, \Lambda_1) \in \mathcal{P}_+^\uparrow$ we obtain

$$\alpha_\lambda^\Lambda \mathfrak{A}^\Lambda(\mathcal{W}) = \alpha_{(\Lambda x, \Lambda\Lambda_1\Lambda^{-1})}\mathfrak{A}(\Lambda\mathcal{W}) = \mathfrak{A}(\Lambda\Lambda_1\Lambda^{-1}\Lambda\mathcal{W} + \Lambda x) = \mathfrak{A}^\Lambda(\Lambda_1\mathcal{W} + x)$$

where we used that (HK3$^\sharp$) holds for the original net $\mathfrak{A}$. □

It should be noted that Lemma 13 (ii) applies also to wedge-local nets which are not Poincaré covariant (HK3$^\sharp$). In particular the basic definitions, $\alpha_x^\Lambda := \alpha_{\Lambda x}$, and $\mathfrak{A}^\Lambda(\mathcal{W}) := \mathfrak{A}(\Lambda\mathcal{W})$, do not make use of Lorentz-transformation isomorphisms, they are only a passive redefinition on the level of the wedge-local net.

To establish the Haag–Ruelle theorem for the adapted scattering state approximants in (40) we rewrite the adapted Haag–Ruelle operators in terms of the boosted net of Lemma 13 as

$$B_\tau^\Lambda(f) = \int d^s x \; f^\Lambda(\tau, \mathbf{x})\alpha_{(\tau,\mathbf{x})}^\Lambda(B), \text{ and similarly} \tag{41}$$

$$\begin{aligned} B = A(\chi) &= \int d^d x' \, \chi(x')\alpha_{x'}(A) \\ &= \int d^d x \, \chi(\Lambda x)\alpha_{\Lambda x}(A) = \int d^d x \, \chi^\Lambda(x)\alpha_x^\Lambda(A), \end{aligned}$$

where we used Lorentz invariance of $d^d x$. Due to $\chi^\Lambda(x) := \chi(\Lambda x) \in \mathscr{S}(\mathbb{R}^d)$ we know that B is almost wedge-local also for the redefined net $\mathfrak{A}^\Lambda$. Therefore Theorem 6 may be applied to the rewritten operators (41). It remains to rephrase the statement of Theorem 6 from the boosted net $(\mathfrak{A}^\Lambda, \alpha^\Lambda, \Omega)$ to return to the terminology of the original theory $(\mathfrak{A}, \alpha, \Omega)$.

Let now $\mathcal{W}$ be any wedge, $\Psi_1^j = E_m A_j \Omega = E_m A_j^\perp \Omega$, $A_j \in \mathfrak{A}(\mathcal{W})$, $A_j^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, $(1 \leq j \leq n)$, and $\Lambda \in \mathcal{L}_+^\uparrow$. Then Ψ_1^j are obviously also swappable with respect to the boosted net and in particular $A_j \in \mathfrak{A}^\Lambda(\Lambda^{-1}\mathcal{W})$. For $\Lambda \in \mathcal{L}^*(\mathcal{W})$ we get $A_j \in \mathfrak{A}^\Lambda(\mathcal{W}_\mathrm{r} + \Lambda^{-1}x)$, for some $x \in \mathbb{R}^d$ depending on $\mathcal{W}$, where $\Lambda^{-1}\mathcal{W}_\mathrm{c} = \mathcal{W}_\mathrm{r}$ is upright. Hence assuming uprightness is redundant for the adapted Haag–Ruelle construction with $\Lambda \in \mathcal{L}^*(\mathcal{W})$. Secondly we see from (41) that applying Theorem 6 to outgoing scattering-state approximants interpreted via the boosted net now requires the ordering

$$\mathcal{V}_{f_n^\Lambda} \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_{n-1}^\Lambda} \prec_{\mathcal{W}_\mathrm{r}} \dots \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_1^\Lambda}, \tag{42}$$

with $\mathcal{V}_{f_j^\Lambda}$ as in (22), denoting the velocity support of $f_j^\Lambda(x) := f_j(\Lambda x)$. In terms of the original net, (42) is by covariance of the ordering relation (9) equivalent to

$$\Lambda\mathcal{V}_{f_n^\Lambda} \prec_{\mathcal{W}} \Lambda\mathcal{V}_{f_{n-1}^\Lambda} \prec_{\mathcal{W}} \dots \prec_{\mathcal{W}} \Lambda\mathcal{V}_{f_1^\Lambda}. \tag{43}$$

This is also consistent with a corresponding localization of the adapted Haag–Ruelle operators (41) similarly as in Lemma 9, but with respect to adapted velocity supports

$$\mathcal{V}_{f_j}^\Lambda := \Lambda\mathcal{V}_{f_j^\Lambda}. \tag{44}$$

The result of this discussion is summarized in Theorem 14.

Theorem 14. *Let $\Lambda \in \mathcal{L}_+^\uparrow$ and $\Psi_1^j = E_m A_j \Omega = E_m A_j^\perp \Omega$ with $A_j \in \mathfrak{A}(\mathcal{W})$, $A_j^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$ (as in Theorem 6).*

(i) For regular positive-energy Klein–Gordon solutions f_j satisfying

$$\mathcal{V}_{f_n}^\Lambda \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}}^\Lambda \prec_{\mathcal{W}} \dots \prec_{\mathcal{W}} \mathcal{V}_{f_1}^\Lambda, \tag{45}$$

the scattering state approximants $\Psi_n^\Lambda(\tau) := B_{1\tau}^\Lambda(f_1)B_{2\tau}^\Lambda(f_2)\dots B_{n\tau}^\Lambda(f_n)\Omega$ converge in norm for $\tau \to \infty$.

(ii) For $\Lambda \in \mathcal{L}^*(\mathcal{W})$ scalar products of $\Psi_n^{+,\Lambda} := \lim_{\tau\to\infty} B_{1\tau}^\Lambda(f_1)\dots B_{n\tau}^\Lambda(f_n)\Omega$, $\Psi_{n'}^{\prime +,\Lambda} := \lim_{\tau\to\infty} B_{1\tau}^{\prime\Lambda}(f_1')\dots B_{n'\tau}^{\prime\Lambda}(f_{n'}')\Omega$ constructed w.r.t. the same wedge $\mathcal{W}$ satisfy

$$\left\langle \Psi_n^{+,\Lambda}, \Psi_{n'}^{\prime +,\Lambda} \right\rangle = \delta_{nn'} \prod_{j=1}^{n} \left\langle B_{j\tau}^\Lambda(f_j)\Omega, B_{j\tau}^{\prime\Lambda}(f_j')\Omega \right\rangle. \tag{46}$$

Analogous statements hold for incoming scattering states assuming opposite ordering.

4.2. Lorentz-Frame Independence and Residual Covariance. For the adapted Haag–Ruelle operators $B_{j\tau}^\Lambda(f_j)$, convergence of scattering-state approximants $\Psi_n^\Lambda(\tau) := B_{1\tau}^\Lambda(f_1)\dots B_{n\tau}^\Lambda(f_n)\Omega$ has now been established for general wedges, i.e. upright or tilted. The new ordering restrictions (45) appear optimal in the context of Remark 11, and the Fock structure follows without additional assumptions for $\Lambda \in \mathcal{L}^*(\mathcal{W})$. However as in standard Haag–Ruelle theory, the choice of HR-operators $B_{j\tau}^\Lambda(f_j)$ creating a given one-particle vector $\Psi_1^j = B_{j\tau}^\Lambda(f_j)\Omega$ is not unique. Fock structure [Theorem 14 (ii)] implies only for fixed Λ, that resulting scattering states do not depend on this freedom of choosing $B_{j\tau}^\Lambda(f_j)$. In the following we will exclude also any unphysical dependence on $\Lambda \in \mathcal{L}^*(\mathcal{W})$, for which one has to handle the possibility of associated non-trivial changes in the localizations of $B_{j\tau}^\Lambda(f_j)$. We begin by considering the Λ-dependence of one-particle vectors, to be followed by a discussion of the influence on ordering conditions and finally on scattering states. It should be emphasized that the present section addresses only the Lorentz-frame independence of the construction of scattering states. Covariance of scattering states under (HK3$^\sharp$) is a related but separate issue to be considered later (see Theorem 24).

Lemma 15. *Let $\Lambda \in \mathcal{L}_+^\uparrow$ and f a regular positive-energy Klein–Gordon solution.*

(i) The wave packet of $f^\Lambda(x) := f(\Lambda x)$ as defined in (19) is given by

$$\tilde{f}^\Lambda(\mathbf{k}) = \frac{\omega_m(\Lambda_m(\mathbf{k}))}{\omega_m(\mathbf{k})}\tilde{f}(\Lambda_m(\mathbf{k})), \tag{47}$$

where $\tilde{f}$ is the wave packet of f and $\Lambda_m(\mathbf{k})$ denotes the spatial part of $\Lambda\cdot(\omega_m(\mathbf{k}),\mathbf{k})$. In particular, $\operatorname{supp}\tilde{f}^\Lambda = \Lambda_m^{-1}(\operatorname{supp}\tilde{f})$.

(ii) The Λ-dependence of one-particle vectors is

$$B_\tau^\Lambda(f)\Omega = \frac{\omega_m(\boldsymbol{P})}{\omega_m(\Lambda_m^{-1}(\boldsymbol{P}))}\tilde{f}(\boldsymbol{P})E(H_m)B\Omega. \tag{48}$$

These one-particle formulas are well-known from the discussion of Lorentz-covariance in the local case and we will only briefly sketch the computations in Appendix A. They are important for the present discussion, as (48) suggests a non-trivial dependence of $\lim_{\tau\to\infty}\Psi_n^\Lambda(\tau)$ on the auxiliary boost Λ. However the dependence can be absorbed by passing to Klein–Gordon solutions $f_j^{(\Lambda)}$ defined via modified wave packets $\tilde{f}_j^{(\Lambda)}(\mathbf{p}) := \frac{\omega_m(\Lambda_m^{-1}(\mathbf{p}))}{\omega_m(\mathbf{p})}\tilde{f}_j(\mathbf{p})$, which have identical velocity supports and give via (48) that

$$B_{j\tau}^\Lambda(f_j^{(\Lambda)})\Omega = \tilde{f}_j(\boldsymbol{P})E(H_m)B_j\Omega, \text{ for any } \Lambda \in \mathcal{L}_+^\uparrow. \tag{49}$$

While the above argument coincides with the familiar result from local QFT, the discussion of scattering-state dependence requires additional care in the wedge-local case due to additional ordering requirements. For brevity reasons we shall focus on Λ-dependence only within the preferred class of reference frames for a given localization wedge $\mathcal{W}$ defined by $\mathcal{L}^*(\mathcal{W}) := \{\Lambda \in \mathcal{L}_+^\uparrow : \Lambda\mathcal{W}_\mathrm{r} = \mathcal{W}_\mathrm{c}\}$ as in Theorem 14.[12]

Remark 16. Clearly any $\Lambda, \Lambda' \in \mathcal{L}^*(\mathcal{W})$ are related by an element $\bar{\Lambda} := \Lambda^{-1}\Lambda'$ from the stabilizer $\mathrm{Stab}_{\mathcal{L}_+^\uparrow}\mathcal{W}_\mathrm{r} := \{\Lambda \in \mathcal{L}_+^\uparrow : \Lambda\mathcal{W}_\mathrm{r} = \mathcal{W}_\mathrm{r}\} \cong \mathrm{O}(1,1)_+^\uparrow \times \mathrm{SO}(d-2)$, where the first factor is generated by boosts Λ_β in x^1-direction ($\beta \in \mathbb{R}$), and the second by rotations fixing x^1. In particular we note for later reference that $\mathrm{Stab}_{\mathcal{L}_+^\uparrow}\mathcal{W}_\mathrm{r}$ is path connected, and that we may smoothly interpolate between any $\Lambda, \Lambda' \in \mathcal{L}^*(\mathcal{W})$ via arbitrarily often differentiable maps $\Lambda^\gamma : [0,1] \to \mathcal{L}^*(\mathcal{W})$ such that $\Lambda^0 = \Lambda$, $\Lambda^1 = \Lambda'$.

Proposition 17 ($\mathcal{L}^*(\mathcal{W})$-invariance of velocity ordering). *For regular Klein–Gordon solutions f_1, f_2 and any $\Lambda, \Lambda' \in \mathcal{L}^*(\mathcal{W})$ we have*

$$\mathcal{V}_{f_1}^\Lambda \prec_\mathcal{W} \mathcal{V}_{f_2}^\Lambda \Longleftrightarrow \mathcal{V}_{f_1}^{\Lambda'} \prec_\mathcal{W} \mathcal{V}_{f_2}^{\Lambda'}. \tag{50}$$

Proof. By covariance (9) we have $\mathcal{V}_{f_1}^\Lambda \prec_\mathcal{W} \mathcal{V}_{f_2}^\Lambda \Longleftrightarrow \mathcal{V}_{f_1^\Lambda} \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_2^\Lambda}$ and similarly for Λ', allowing us to reduce (50) to the case $\mathcal{W} = \mathcal{W}_\mathrm{r}$ up to boosts acting on f_j. Thus (50) amounts to a property of the relativistic velocity transformation law. Let us assume that $\mathcal{V}_{f_1^{\Lambda'}} \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_2^{\Lambda'}}$. By Remark 16 we may write $\Lambda' = \Lambda\bar{\Lambda}$, $\bar{\Lambda} = \Lambda_\beta R_1$

[12] Preliminary computations suggest that Theorem 14 also extends to all $\Lambda \in \mathcal{L}_+^\uparrow$ as long as the ordering (45) holds for $\Lambda \in \mathcal{L}^*(\mathcal{W})$.

with a boost Λ_β in x^1-direction of rapidity $\beta \in \mathbb{R}$ and a spatial rotation R_1 preserving x^1. Hence from $f_j^{\Lambda'} = f_j^{\Lambda\Lambda_\beta R_1} = (f_j^{\Lambda\Lambda_\beta})^{R_1}$, $(j = 1, 2)$, we obtain for the spatial projection $\mathbf{V}_{f_j^{\Lambda'}}$ of $\mathcal{V}_{f_j^{\Lambda'}}$ that

$$\mathbf{V}_{f_j^{\Lambda'}} = \left\{ \frac{\mathbf{k}}{\omega_m(\mathbf{k})} : \mathbf{k} \in R_1^{-1}(\operatorname{supp} \tilde{f}_j^{\Lambda\Lambda_\beta}) \right\} = \left\{ \frac{R_1^{-1}\mathbf{k}}{\omega_m(\mathbf{k})} : \mathbf{k} \in \operatorname{supp} \tilde{f}_j^{\Lambda\Lambda_\beta} \right\} = R_1^{-1}\mathbf{V}_{f_j^{\Lambda\Lambda_\beta}}.$$

Here we used Proposition 15 (i), that R from the rotation subgroup of $\mathcal{L}_+^\uparrow$ act on H_m by $R_m(\mathbf{k}) = R\mathbf{k}$, and $\omega_m(R_1^{-1}\mathbf{k}) = \omega_m(\mathbf{k})$. By covariance (9)

$$\mathcal{V}_{f_1^{\Lambda'}} \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_2^{\Lambda'}} \iff R_1^{-1}\mathcal{V}_{f_1^{\Lambda\Lambda_\beta}} \prec_{\mathcal{W}_\mathrm{r}} R_1^{-1}\mathcal{V}_{f_2^{\Lambda\Lambda_\beta}} \iff \mathcal{V}_{f_1^{\Lambda\Lambda_\beta}} \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_2^{\Lambda\Lambda_\beta}},$$

where we used that $R_1\mathcal{W}_\mathrm{r} = \mathcal{W}_\mathrm{r}$, as R_1 is a rotation preserving x_1. The remaining x^1-boost gives

$$\begin{aligned}\mathbf{V}_{(f_j^\Lambda)^{\Lambda_\beta}} &= \left\{ \frac{\mathbf{k}}{\omega_m(\mathbf{k})} : \mathbf{k} \in (\Lambda_\beta)_m^{-1}(\operatorname{supp} \tilde{f}_j^\Lambda) \right\} = \left\{ \frac{(\Lambda_{-\beta})_m(\mathbf{k})}{\omega_m((\Lambda_{-\beta})_m(\mathbf{k}))} : \mathbf{k} \in \operatorname{supp} \tilde{f}_j^\Lambda \right\} \\ &= \left\{ \frac{((\sinh(-\beta)\omega_m(\mathbf{k}) + \cosh(-\beta)k^1), k^2, \dots, k^s)}{\cosh(-\beta)\omega_m(\mathbf{k}) + \sinh(-\beta)k^1} : \mathbf{k} \in \operatorname{supp} \tilde{f}_j^\Lambda \right\},\end{aligned}$$

where we used the group action property $(\Lambda_\beta)_m^{-1}(\mathbf{k}) = (\Lambda_\beta^{-1})_m(\mathbf{k}) = (\Lambda_{-\beta})_m(\mathbf{k})$. From this we obtain $\mathcal{V}_{f_1^{\Lambda\Lambda_\beta}} \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_2^{\Lambda\Lambda_\beta}} \iff \forall \mathbf{k}_2 \in \operatorname{supp} \tilde{f}_2^\Lambda,\ \mathbf{k}_1 \in \operatorname{supp} \tilde{f}_1^\Lambda$:

$$\frac{-\sinh(\beta)\omega_m(\mathbf{k}_2) + \cosh(\beta)k_2^1}{\cosh(\beta)\omega_m(\mathbf{k}_2) - \sinh(\beta)k_2^1} - \frac{-\sinh(\beta)\omega_m(\mathbf{k}_1) + \cosh(\beta)k_1^1}{\cosh(\beta)\omega_m(\mathbf{k}_1) - \sinh(\beta)k_1^1} > 0.$$

Passing to the common denominator and using $\cosh(\beta)^2 - \sinh(\beta)^2 = 1$, this is equivalent to $k_2^1/\omega_m(\mathbf{k}_2) - k_1^1/\omega_m(\mathbf{k}_1) > 0$. As the equivalence holds for all $\mathbf{k}_2 \in \operatorname{supp} \tilde{f}_2^\Lambda$, $\mathbf{k}_1 \in \operatorname{supp} \tilde{f}_1^\Lambda$, we have shown that $\mathcal{V}_{f_1^\Lambda} \prec_{\mathcal{W}_\mathrm{r}} \mathcal{V}_{f_2^\Lambda}$. $\square$

This establishes that all choices $\Lambda \in \mathcal{L}^*(\mathcal{W})$ are equivalent with respect to the ordering restriction. That is a prerequisite for the following commutator estimate, which extends Corollary 10 and will be required for comparing scattering states defined for distinct $\Lambda \in \mathcal{L}^*(\mathcal{W})$.

Lemma 18 (Uniform norm and commutator estimates). *Let $A \in \mathfrak{A}(\mathcal{W})$, $A^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, and let $f, f^\perp$ be regular Klein–Gordon solutions. For a continuously differentiable compact curve $\Lambda^\gamma \in \mathcal{L}^*(\mathcal{W})$, $\gamma \in [0, 1]$, we define $f^{(\Lambda^\gamma)}$, $f^{\perp(\Lambda^\gamma)}$, as in (49).*

(i) The corresponding adapted HR-operators from Definition 12 satisfy the norm bounds

$$\left\| B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)}) \right\| \leq C(1 + |\tau|^{s/2}), \quad \left\| \partial_\gamma B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)}) \right\| \leq C'(1 + |\tau|^{s/2+1}), \qquad (51)$$

and analogously for $B_\tau^{\perp\Lambda^\gamma}(f^{\perp(\Lambda^\gamma)})$. The constants C, C' are uniform in $\gamma \in [0, 1]$ and $\tau \in \mathbb{R}$ but depend on A, χ, f and on the curve Λ^γ.

(ii) For $\tau > 0$ and ordered $\mathcal{V}^\Lambda_{f^\perp} \prec_{\mathcal{W}} \mathcal{V}^\Lambda_f$ (w.r.t. any one $\Lambda \in \mathcal{L}^*(\mathcal{W})$) we have

$$\left\| [B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)}), B_\tau^{\perp\Lambda^\gamma}(f^{\perp(\Lambda^\gamma)})] \right\| \leq C_N \tau^{-N}, \tag{52}$$

$$\left\| [\partial_\gamma B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)}), B_\tau^{\perp\Lambda^\gamma}(f^{\perp(\Lambda^\gamma)})] \right\| \leq C'_N \tau^{-N}, \tag{53}$$

with C_N, C'_N independent of $\gamma \in [0, 1]$, $\tau > 0$, but otherwise depending on operators, smearing functions, and Λ^γ as in (i). Analogous estimates hold for $\tau < 0$ if the opposite ordering of wave packets is assumed.

The γ-independent estimates of Lemma 18 are established in Appendix A. Together with the previous results it can now be seen that for any scattering state $\Psi_n^{+,\Lambda}$, $\Lambda \in \mathcal{L}^*(\mathcal{W})$ given any other $\Lambda' \in \mathcal{L}^*(\mathcal{W})$, we can construct a corresponding $\Psi_n^{\prime +,\Lambda'}$ s.t. $\Psi_n^{\prime +,\Lambda'} = \Psi_n^{+,\Lambda}$. This shows also that for any given wedge $\mathcal{W}$ the range of scattering states from Theorem 14, constructed via correspondingly localized operators $A_k \in \mathfrak{A}(\mathcal{W})$ (and $A_k^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$), is independent of the reference frame specified by the auxiliary $\Lambda \in \mathcal{L}^*(\mathcal{W})$.

Theorem 19 (Λ-independence of scattering states). *Let $\Psi_1^j = E_m A_j \Omega = E_m A_j^\perp \Omega$ with $A_j \in \mathfrak{A}(\mathcal{W})$, $A_j^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, and let f_k be regular KG-solutions such that for some $\Lambda_0 \in \mathcal{L}^*(\mathcal{W})$ we have $\mathcal{V}^{\Lambda_0}_{f_n} \prec_{\mathcal{W}} \mathcal{V}^{\Lambda_0}_{f_{n-1}} \prec_{\mathcal{W}} \ldots \prec_{\mathcal{W}} \mathcal{V}^{\Lambda_0}_{f_1}$. Then for any $\Lambda' \in \mathcal{L}^*(\mathcal{W})$ the scattering states*

$$\Psi_n^{+,(\Lambda')} := \lim_{\tau \to \infty} B_{1\tau}^{\Lambda'}(f_1^{(\Lambda')}) \ldots B_{1\tau}^{\Lambda'}(f_n^{(\Lambda')})\Omega \tag{54}$$

are well-defined and the limiting outgoing state $\Psi_n^{+,(\Lambda')}$ is independent of Λ'. An analogous result holds for incoming scattering states.

Proof. Convergence follows from Proposition 17 and Theorem 14. Using the above preparations we can establish Λ-independence by generalizing the arguments familiar from the local case. Due to Remark 16 we can interpolate between the two reference frames specified by $\Lambda^0 = \Lambda_0$ and $\Lambda^1 = \Lambda'$ with a differentiable curve $\Lambda^\gamma \in \mathcal{L}^*(\mathcal{W})$, $\gamma \in [0, 1]$. Now we estimate for $\tau > 0$ inductively with respect to the particle number n that

$$\left\| \Psi_n^{(\Lambda)}(\tau) - \Psi_n^{(\Lambda')}(\tau) \right\| \leq \int_0^1 \mathrm{d}\gamma \, \left\| \partial_\gamma \Psi_n^{(\Lambda^\gamma)}(\tau) \right\| \leq C_N \tau^{-N}.$$

For $n = 1$ this follows from (49) with $C_N = 0$. The induction step is established by expanding $\|\partial_\gamma \Psi_n^{(\Lambda^\gamma)}(\tau)\| \leq \left\|(\partial_\gamma (B_{1\tau}^\gamma \ldots B_{n-1\,\tau}^\gamma)) B_{n\tau}^\gamma \Omega\right\| + \left\| B_{1\tau}^\gamma \ldots B_{n-1\,\tau}^\gamma \partial_\gamma B_{n\tau}^\gamma \Omega \right\|$ where we abbreviated $B_{j\tau}^\gamma := B_{j\tau}^{\Lambda^\gamma}(f_j^{(\Lambda^{(\gamma)})})$. Here the second term vanishes due to (49) and the first term may be estimated by swapping

$$\begin{aligned}
\left\|(\partial_\gamma (B_{1\tau}^\gamma \ldots B_{n-1\,\tau}^\gamma)) B_{n\tau}^\gamma \Omega\right\| &= \left\|(\partial_\gamma (B_{1\tau}^\gamma \ldots B_{n-1\,\tau}^\gamma)) B_{n\tau}^{\perp\gamma} \Omega\right\| \\
&\leq \left\| B_{n\tau}^{\perp\gamma} \right\| \left\|(\partial_\gamma (B_{1\tau}^\gamma \ldots B_{n-1\,\tau}^\gamma))\Omega\right\| \\
&\quad + \left\| [\partial_\gamma (B_{1\tau}^\gamma \ldots B_{n-1\,\tau}^\gamma), B_{n\tau}^{\perp\gamma}] \right\|
\end{aligned}$$

where both terms are rapidly decreasing in τ. For the first term this is obtained from the induction assumption and $\|B_{j\tau}^{\perp\gamma}\| \leq C(1+|\tau|^{s/2})$ (uniformly in $\gamma \in [0,1]$). The second term is estimated by expansion of the commutator similarly as in (33) and inserting the γ-uniform bounds from Lemma 18. □

5. Wave Operators, S-Matrix, and Wedge Transitions

We have now sufficient understanding of the construction from Sect. 4.1 to begin with a general and model-independent analysis of the multi-particle scattering data in wedge-local models. In particular we propose a formalism for wave operators and S-matrices, which emphasizes the potential physical peculiarities of multi-particle scattering in the wedge-local setting. These considerations will provide the foundation for the study of the multi-particle structure of the Grosse-Lechner model and related wedge-local theories in subsequent work.

Guided by conventional Haag–Ruelle theory we additionally need to address restrictions of our construction regarding swapping and ordering conditions. Regarding the former it will be convenient to introduce in addition to the one-particle space $\mathscr{H}_1 := E_m\mathscr{H}$ the (non-closed) subspaces

$$\begin{aligned} \mathscr{H}_1^{\mathcal{W}} &:= \{\Psi_1 \in \mathscr{H}_1 : \Psi_1 \text{ swappable w.r.t. } \mathcal{W}+x \text{ for some } x \in \mathbb{R}^d\}, \\ \mathscr{H}_{1c}^{\mathcal{W}} &:= \{\tilde{f}(\boldsymbol{P})\Psi_1 : \Psi_1 \in \mathscr{H}_1^{\mathcal{W}},\ \tilde{f} \in C_c^\infty(\mathbb{R}^s)\}. \end{aligned} \tag{55}$$

It is clear that $\mathscr{H}_1^{\mathcal{W}} = \mathscr{H}_1^{\mathcal{W}+y} = U(y)\mathscr{H}_1^{\mathcal{W}} = \mathscr{H}_1^{\mathcal{W}'+y'}$ for any $y, y' \in \mathbb{R}^d$ by symmetry of the definition, and if covariance (HK3$^\sharp$) applies $U(\Lambda)\mathscr{H}_1^{\mathcal{W}} = \mathscr{H}_1^{\Lambda\mathcal{W}}$. Lastly Lemma 3 shows that wedge-duality (HK2$^\sharp$) yields $\overline{\mathscr{H}_1^{\mathcal{W}}} = E(H_m)\mathscr{H}$ for any wedge $\mathcal{W}$. Further independent of duality $\mathscr{H}_{1c}^{\mathcal{W}} \subset \overline{\mathscr{H}_1^{\mathcal{W}}}$ is dense by spectral calculus, but one should not expect $\mathscr{H}_{1c}^{\mathcal{W}}$ to be a subspace of $\mathscr{H}_1^{\mathcal{W}}$, cf. [BBS01] Lemma 3.4. It is clear by definition that for any one particle vector $\Psi_1^k \in \mathscr{H}_{1c}^{\mathcal{W}}$ we can find associated creation operators such that $\Psi_1^k = B_{k\tau}^{\Lambda}(f_k)\Omega = B_{k\tau}^{\perp\Lambda}(f_k)\Omega$, so that we can proceed to the corresponding ordered scattering states. The basic conceptual issue to be addressed in the passage from the Haag–Ruelle construction to the wave operators and the S-matrix concerns the potential implicit dependence of scattering states on the choice of creation-operator approximants $B_{k\tau}^{\Lambda}(f_k)$.

Lemma 20. *Let $A_k, A_k' \in \mathfrak{A}(\mathcal{W})$ together with KG-solutions f_k, f_k' and auxiliary functions $\chi, \chi' \in \mathscr{S}(\mathbb{R}^d)$ (cf. Lemma 7) such that $B_{k\tau}^{\Lambda}(f_k)\Omega = B_{k\tau}^{\prime\Lambda'}(f_k')\Omega$ with $\mathcal{V}_n \prec_{\mathcal{W}} \mathcal{V}_{n-1} \prec_{\mathcal{W}} \cdots \prec_{\mathcal{W}} \mathcal{V}_1$ where $\mathcal{V}_k := \mathcal{V}_{f_k}^{\Lambda}$ and analogously for $\mathcal{V}_k' := \mathcal{V}_{f_k'}^{\Lambda'}$, $\Lambda, \Lambda' \in \mathcal{L}^*(\mathcal{W})$. Then the outgoing limits Ψ_n^+, $\Psi_{n'}^{\prime +}$ of $\Psi_n(\tau) := B_{1\tau}^{\Lambda}(f_1)\ldots B_{n\tau}^{\Lambda}(f_n)\Omega$ and $\Psi_n'(\tau) := B_{1\tau}^{\prime\Lambda'}(f_1')\ldots B_{n\tau}^{\prime\Lambda'}(f_n')\Omega$ coincide. The same holds for incoming limits with ordering assumptions replaced by $\mathcal{V}_1 \prec_{\mathcal{W}} \mathcal{V}_2 \prec_{\mathcal{W}} \cdots \prec_{\mathcal{W}} \mathcal{V}_n$.*

Proof. For $\Lambda = \Lambda'$ we find directly $\left\|\Psi_n^+ - \Psi_n^{\prime +}\right\|^2 = \left\|\Psi_n^+\right\|^2 - 2\,\mathrm{Re}\langle\Psi_n^+, \Psi_n^{\prime +}\rangle + \left\|\Psi_n^{\prime +}\right\|^2$. This vanishes, as due to Fock structure (Theorem 14 (ii)) and coinciding one-particle vectors we obtain $\langle\Psi_n^+, \Psi_n^{\prime +}\rangle = \left\|\Psi_n^+\right\|^2 = \left\|\Psi_n^{\prime +}\right\|^2$. The case of general $\Lambda, \Lambda' \in \mathcal{L}^*(\mathcal{W})$ follows from the above via Theorem 19. □

Further one can make sense of velocity supports and the corresponding ordering assumptions without reference to Klein–Gordon solutions. For a single-particle state $\Psi_1 \in \mathscr{H}_1$ the classical propagation region and the corresponding Λ-velocity support ($\Lambda \in \mathcal{L}_+^\uparrow$) are given in terms of the energy-momentum spectral measure $E_{(H,\boldsymbol{P})}(\Delta)$ ($\Delta \subset \mathbb{R}^{s+1}$ Borel) by

$$\Upsilon_{\Psi_1} := \{t \cdot (\omega, \mathbf{k}) : (\omega, \mathbf{k}) \in \mathrm{supp}(E_{(H,\boldsymbol{P})}\Psi_1),\ t \in \mathbb{R}\},$$

$$\mathcal{V}_{\Psi_1}^{\Lambda} := \Upsilon_{\Psi_1} \cap \Lambda T_1, \quad T_1 := \{(1, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^s\}. \tag{56}$$

The precursor ordering is lifted to a relation on one-particle vectors $\Psi_1, \Psi_1' \in \mathscr{H}_1$ by

$$\Psi_1' \prec_{\mathcal{W}} \Psi_1 :\Longleftrightarrow \mathcal{V}_{\Psi_1'}^{\Lambda} \prec_{\mathcal{W}} \mathcal{V}_{\Psi_1}^{\Lambda}, \tag{57}$$

with some $\Lambda \in \mathcal{L}^*(\mathcal{W})$. We recall from Sect. 4.1 that this prescription is distinguished in the non-upright case, as it leads to the largest range of constructed scattering states in terms of admissible wave-packet configurations, or equivalently to the weakest ordering restrictions possible with the present method. The definition is consistent, as the resulting relation does not depend on the choice of Λ within $\mathcal{L}^*(\mathcal{W})$ due to Proposition 17.

The multi-particle configurations accessible via our wedge-local Haag–Ruelle construction can be conveniently expressed by the following notion of ordered Fock spaces replacing the conventional definition based on bosonic or fermionic statistics.

Definition 21. The ordered tensor products over one-particle Hilbert space $\mathscr{H}_1$ with respect to a partial order $\prec$ on $\mathscr{H}_1$ are defined as closure $\otimes_{\prec}^n \mathscr{H}_1 := \overline{\hat{\otimes}_{\prec}^n \mathscr{H}_1}$ of the finite linear spans

$$\hat{\otimes}_{\prec}^n \mathscr{H}_1 := \mathrm{span}\{\Psi_1^1 \otimes \ldots \otimes \Psi_1^n : \Psi_1^k \in \mathscr{H}_1, \Psi_1^1 \prec \Psi_1^2 \prec \ldots \prec \Psi_1^n\}. \tag{58}$$

Using the conventions $\hat{\otimes}_{\prec}^0 \mathscr{H}_1 := \mathbb{C}\Omega$, $\hat{\otimes}_{\prec}^1 \mathscr{H}_1 := \mathscr{H}_1$ we obtain corresponding ordered Fock spaces $\Gamma^{\prec}(\mathscr{H}_1) := \bigoplus_{n=0}^{\infty} \otimes_{\prec}^n \mathscr{H}_1$. The subspace of finite linear combinations of ordered tensor product vectors with $\Psi_1^k \in \mathscr{H}_1' \subset \mathscr{H}_1$ will be denoted by $\Gamma_0^{\prec}(\mathscr{H}_1') := \hat{\bigoplus}_{n=0}^{\infty} \hat{\otimes}_{\prec}^n \mathscr{H}_1'$.

To proceed to the scattering data note that $\Gamma_0^{\prec_{\mathcal{W}}}(\mathscr{H}_{1c}^{\mathcal{W}}) \subset \Gamma^{\prec_{\mathcal{W}}}(\mathscr{H}_1)$ is dense and the wave operators are defined by linear extension of the isometries obtained from the wedge-local Haag–Ruelle construction of Theorem 14. Just as for ordinary bosonic- and fermionic statistics, unsymmetrized Fock space $\Gamma^{\mathrm{u}}(\mathscr{H}_1) := \bigoplus_{n=0}^{\infty} \mathscr{H}_1^{\otimes n}$ provides a common enveloping space into which ordered tensor products and Fock spaces embed naturally. The possible dependence of scattering states on a given wedge of reference $\mathcal{W}$, noted by Grosse and Lechner [GL07], extends also to multi-particle scattering states and is most consequently expressed on the level of wave operators.

Definition 22 (Wave operators). For any given centered wedge $\mathcal{W}$ we set

$$\mathbb{W}_{\mathcal{W}}^{+} : \begin{cases} \Gamma_0^{\succ_{\mathcal{W}}}(\mathscr{H}_{1c}^{\mathcal{W}}) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau \to \infty} B_{1\tau}^{\Lambda}(f_1) \ldots B_{n\tau}^{\Lambda}(f_n)\Omega, \end{cases} \tag{59}$$

$$\mathbb{W}_{\mathcal{W}}^{-} : \begin{cases} \Gamma_0^{\prec_{\mathcal{W}}}(\mathscr{H}_{1c}^{\mathcal{W}}) \longrightarrow \mathscr{H}, \\ \Psi_1^1 \otimes \ldots \otimes \Psi_1^n \longmapsto \lim\limits_{\tau \to -\infty} B_{1\tau}^{\Lambda}(f_1) \ldots B_{n\tau}^{\Lambda}(f_n)\Omega, \end{cases} \tag{60}$$

where for $\Lambda \in \mathcal{L}^*(\mathcal{W})$ suitable $B^{\Lambda}_{k\tau}(f_k)\Omega = \Psi^k_1$ with B_k swappable and almost wedge-local w.r.t. the given wedge $\mathcal{W}$ can be obtained for $\Psi^k_1 \in \mathscr{H}^{\mathcal{W}}_{1c}$ via (48). In the two-dimensional case we can take any $\Lambda \in \mathcal{L}^{\uparrow}_{+}$ also for translates of $\mathcal{W}_{\mathrm{r}}{}'$.[13]

Proposition 23. *Assuming wedge-duality (HK2$^\sharp$), the wave operators (59), (60) are well-defined and extend to bounded linear isometries $\mathbb{W}^+_{\mathcal{W}} : \Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1) \longrightarrow \mathscr{H}$, and $\mathbb{W}^-_{\mathcal{W}} : \Gamma^{\prec_{\mathcal{W}}}(\mathscr{H}_1) \longrightarrow \mathscr{H}$.*

Proof. *Well-definedness of $\mathbb{W}^+_{\mathcal{W}}$ on $\Gamma^{\succ_{\mathcal{W}}}_0(\mathscr{H}^{\mathcal{W}}_{1c})$ follows by noting that the computation from the proof of Lemma 20 extends to linear combinations of Ψ^+_n. As the Fock structure also implies isometry of $\mathbb{W}^+_{\mathcal{W}}$, the wave operators further extend to the closures $\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1) = \overline{\Gamma^{\succ_{\mathcal{W}}}_0(\mathscr{H}^{\mathcal{W}}_{1c})}$ by continuity and using that $\overline{\mathscr{H}^{\mathcal{W}}_{1c}} = \mathscr{H}_1$ (Lemma 3). The construction of $\mathbb{W}^-_{\mathcal{W}}$ is analogous on the oppositely ordered spaces.* □

Due to translation covariance it is sufficient to consider $\mathbb{W}^\pm_{\mathcal{W}}$ for centered wedges $\mathcal{W} = \Lambda\mathcal{W}_{\mathrm{r}}$. In other words we will now see that the wave operators in fact depend on the wedge $\mathcal{W}$ only modulo translations. Given (HK3$^\sharp$) this symmetry consideration in fact extends to the full Poincaré group, whose action $U_0(\lambda)$ on $\Gamma^{\mathrm{u}}(\mathscr{H}_1)$ is defined by

$$U_0(\lambda)\left(\Psi^1_1 \otimes \Psi^2_1 \otimes \cdots \otimes \Psi^n_1\right) := (U(\lambda)\Psi^1_1) \otimes (U(\lambda)\Psi^2_1) \otimes \cdots \otimes (U(\lambda)\Psi^n_1). \quad (61)$$

While $U_0(x)$ preserves velocity-ordered Fock spaces, boosts act in general non-trivially. Explicitly, (9) shows that $U_0(\Lambda)\Gamma^{\prec_{\mathcal{W}}}(\mathscr{H}_1) = \Gamma^{\prec_{\Lambda\mathcal{W}}}(\mathscr{H}_1)$, $U_0(\Lambda)\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1) = \Gamma^{\succ_{\Lambda\mathcal{W}}}(\mathscr{H}_1)$, and analogously for the subspaces $\Gamma^{\prec_{\mathcal{W}}}_0(\mathscr{H}^{\mathcal{W}}_{1c})$.

Theorem 24. *For $\lambda = (a, \Lambda) \in \mathcal{P}^{\uparrow}_{+}$ we have $\mathbb{W}^\pm_{\mathcal{W}+a} = \mathbb{W}^\pm_{\mathcal{W}}$ and $U(\lambda)\mathbb{W}^\pm_{\mathcal{W}} = \mathbb{W}^\pm_{\Lambda\mathcal{W}}U_0(\lambda)$.*

Proof. The first statement follows trivially from translation symmetry of Definition 22. For the second statement let us consider only the outgoing case, and note that it is sufficient to establish the identities for special Ψ^+_n of ordered tensor product form

$$\begin{aligned}\Psi^+_n &= \mathbb{W}^+_{\mathcal{W}}(B^{\Lambda'}_{1\tau}(f_1)\Omega \otimes \cdots \otimes B^{\Lambda'}_{n\tau}(f_n)\Omega)\\ &= \lim_{\tau\to\infty} B^{\Lambda'}_{1\tau}(f_1)\ldots B^{\Lambda'}_{n\tau}(f_n)\Omega.\end{aligned}$$

with auxiliary boost $\Lambda' \in \mathcal{L}^*(\mathcal{W})$ and velocity supports ordered correspondingly, that is by $\mathcal{V}^{\Lambda'}_{f_1} \succ_{\mathcal{W}} \mathcal{V}^{\Lambda'}_{f_2} \succ_{\mathcal{W}} \cdots \succ_{\mathcal{W}} \mathcal{V}^{\Lambda'}_{f_n}$. From continuity of $U(\lambda)$, we obtain

$$U(\lambda)\Psi^+_n = \lim_{\tau\to\infty} U(\lambda)B^{\Lambda'}_{1\tau}(f_1)U(\lambda)^*U(\lambda)\ldots U(\lambda)^*U(\lambda)B^{\Lambda'}_{n\tau}(f_n)\Omega. \quad (62)$$

Using $U(\lambda)U(x) = U(\Lambda x)U(\lambda)$, the adjoint action of $U(\lambda)$ yields due to

$$\begin{aligned}U(\lambda)B^{\Lambda'}_{j\tau}(f_j)U(\lambda)^* &= \int d^s x f_j(\Lambda'(\tau,\mathbf{x}))\, U(\lambda)\alpha_{\Lambda'(\tau,\mathbf{x})}(B_j)U(\lambda)^*\\ &= \int d^s x f'_j(\Lambda\Lambda'(\tau,\mathbf{x}))\, \alpha_{\Lambda\Lambda'(\tau,\mathbf{x})}(B'_j) = B'^{\Lambda\Lambda'}_{j\tau}(f'_j) \quad (63)\end{aligned}$$

[13] The $d = 2$ exception can be dropped by extending the formalism, and results of Sect. 4.2, to make use of the larger sets of auxiliary Lorentz transformations $\{\Lambda \in \mathcal{L}^{\uparrow}_{+} : \Lambda^{-1}\mathcal{W}$ upright$\}$ instead of $\mathcal{L}^*(\mathcal{W}_{\mathrm{r}}{}') = \emptyset$.

again a Haag–Ruelle operator with $B'_j := U(\Lambda, a) B_j U(\Lambda, a)^*$ from the class of almost-wedge local operators considered in Lemma 7 (with respect to the transformed wedge $\Lambda\mathcal{W}$) and $f'_j(x) := f_j(\Lambda^{-1}x)$. Starting from (62) covariance of $\mathbb{W}^+_{\mathcal{W}}$ is now obtained via

$$\begin{aligned} U(\lambda)\Psi_n^+ &= \lim_{\tau\to\infty} B'^{\Lambda\Lambda'}_{1\tau}(f'_1) B'^{\Lambda\Lambda'}_{2\tau}(f'_2) \ldots B'^{\Lambda\Lambda'}_{n\tau}(f'_n)\Omega \\ &= \mathbb{W}^+_{\Lambda\mathcal{W}}((B'^{\Lambda\Lambda'}_{1\tau}(f'_1)\Omega) \otimes \ldots \otimes (B'^{\Lambda\Lambda'}_{n\tau}(f'_n)\Omega)) \\ &= \mathbb{W}^+_{\Lambda\mathcal{W}}((U(\Lambda)B^{\Lambda'}_{1\tau}(f_1)\Omega) \otimes \ldots \otimes (U(\Lambda)B^{\Lambda'}_{n\tau}(f_n)\Omega)) \\ &= \mathbb{W}^+_{\Lambda\mathcal{W}} U_0(\Lambda)(B^{\Lambda'}_{1\tau}(f_1)\Omega \otimes \ldots \otimes B^{\Lambda'}_{n\tau}(f_n)\Omega). \end{aligned}$$

Here we first used (63), well-definedness of the wave-operators (Proposition 23), then again (63), and lastly (61). Finally we extend by linearity and continuity to all of $\Gamma^{\succ_{\mathcal{W}}}(\mathscr{H}_1)$, whereby we obtain the covariance identity. □

For local theories $\mathbb{W}^\pm_{\mathcal{W}}$ are equivalent to the conventional Haag–Ruelle wave operators as a consequence of Lemma 20. Therefore in local theories they must be $\mathcal{W}$-independent and Lorentz-covariant up to suitable identification of ordered Fock spaces by standard arguments. In the general wedge-local setting on the other hand, a non-trivial dependence of $\mathbb{W}^\pm_{\mathcal{W}}$ on the wedge $\mathcal{W}$ should be expected, as noticed in [GL07]. The resulting asymptotic breaking of Lorentz symmetry in higher dimensions will be strongly model dependent, so that it is beyond the scope of our present general analysis. The lesson to be learned is that there must be a residual Lorentz covariance with respect to the stabilizer of $\mathcal{W}_c$ in any wedge-local theory.

Finally let us note that also the S-matrix in wedge-local theories, as accessible via our construction with suitable ordering restrictions, will inherit the wedge-dependence of the wave operators.

Definition 25 (*S-matrix and wedge-transition maps*). The S-matrices and wedge-transition maps between final and initial states are defined as

$$S^{\mathcal{W}_f,\mathcal{W}_i}_{f\,i} := (\mathbb{W}^+_{\mathcal{W}_f})^*\mathbb{W}^-_{\mathcal{W}_i}, \quad S^{\mathcal{W}',\mathcal{W}}_{f\,f} := (\mathbb{W}^+_{\mathcal{W}'})^*\mathbb{W}^+_{\mathcal{W}}, \quad S^{\mathcal{W}',\mathcal{W}}_{i\,i} := (\mathbb{W}^-_{\mathcal{W}'})^*\mathbb{W}^-_{\mathcal{W}}. \tag{64}$$

depending on centered wedges $\mathcal{W}_f$, $\mathcal{W}_i$, $\mathcal{W}$, $\mathcal{W}'$ entering in the Haag–Ruelle construction.

Theorem 26. *S-matrices and wedge transition maps (64) are Poincaré-covariant in the sense that for $\lambda = (a, \Lambda) \in \mathcal{P}^\uparrow_+$ we have*

$$\begin{aligned} &U_0(\lambda) S^{\mathcal{W}_f,\mathcal{W}_i}_{fi} U_0(\lambda)^* = S^{\Lambda\mathcal{W}_f,\Lambda\mathcal{W}_i}_{fi}, \\ &U_0(\lambda) S^{\mathcal{W},\mathcal{W}'}_{ff} U_0(\lambda)^* = S^{\Lambda\mathcal{W},\Lambda\mathcal{W}'}_{ff}, \qquad U_0(\lambda) S^{\mathcal{W},\mathcal{W}'}_{ii} U_0(\lambda)^* = S^{\Lambda\mathcal{W},\Lambda\mathcal{W}'}_{ii}. \end{aligned}$$

If the wave operators are asymptotically complete (i.e. have dense range in $\mathscr{H}$) we have additional transition identities such as $S^{\mathcal{W}_f,\mathcal{W}_i}_{f\,i} = S^{\mathcal{W}_f,\mathcal{W}'_f}_{f\,f} S^{\mathcal{W}'_f,\mathcal{W}'_i}_{f\,i} S^{\mathcal{W}'_i,\mathcal{W}_i}_{i\,i}$.

Proof. *Covariance identities follow from Proposition 24. The wedge-transition formula is a consequence of (64) using that asymptotic completeness and isometry of $\mathbb{W}^+_{\mathcal{W}'_f}$ imply $\mathbb{W}^+_{\mathcal{W}'_f}(\mathbb{W}^+_{\mathcal{W}'_f})^* = \mathbb{1}$ and analogously for $\mathbb{W}^-_{\mathcal{W}'_i}$.* □

It is important to highlight that in our construction the localization wedge $\mathcal{W}$ must agree among all creation operators used to define a scattering state. Additionally even if there is a non-trivial overlap between two distinct ordered Fock spaces, for non-vanishing $\Psi \in \Gamma^{\succ \mathcal{W}}(\mathscr{H}_1) \cap \Gamma^{\succ \mathcal{W}'}(\mathscr{H}_1)$ one will in general have $\mathbb{W}^{+}_{\mathcal{W}}\Psi \neq \mathbb{W}^{+}_{\mathcal{W}'}\Psi$. The analysis of this localization-dependence can be carried much further in models with stronger (e.g. string-like) localization. In this case also scattering states can be constructed for mixed string-directions and the dependence on these directions can be taken into account on the level of the asymptotic Fock spaces [FGR96].

6. Concluding Remarks

We developed N-particle scattering theory for general wedge-local quantum field theories with isolated mass shells. In particular, we constructed scattering states for arbitrarily many particles, even with reduced localization information available from wedge-locality. This implies also that the asymptotic particle structure of wedge-local models with isolated mass shells must be as rich as for strictly local theories.

With multi-particle scattering states at hand, we may proceed to the problem of asymptotic completeness (AC) which, in spite of recent progress [Le06,DT11,DG14], is largely open both in the local and wedge-local setting. Using our construction of N-particle scattering states, we intend to establish AC in the wedge-local model of Grosse and Lechner [GL07]. This will give the first example of a relativistic wedge-local theory in 4-dimensional space-time, which is interacting and asymptotically complete. Furthermore, we expect that the non-trivial S-matrix of this model will be factorizing, which is an unusual feature in higher dimensions. On the other hand, interesting counterexamples to two-particle asymptotic completeness have recently been constructed in wedge-local setting [LTU17, Sec. 5]. These models also ought to be instructive at the multi-particle level within the presently developed multi-particle scattering theory.

Returning to the axiomatic viewpoint, it is not known whether the existence of an interpolating wedge-local net has any consequences on the properties of an S-matrix beyond the basic symmetry principles discussed in Sect. 5. As a first step, one may ask whether there is any meaningful generalization of the LSZ reduction formula for the wedge-local setting. The latter provides the conventional point of departure for investigating analyticity properties of the S-matrix [A, Sec. 5.6]. Phrased differently, one may ask in which generality the inverse scattering problem is solvable within the class of wedge-local models. While this appears to be an ambitious question, there are in fact related positive results for non-local models [BaW84], or for a certain class of field theories formulated on Krein spaces [AG01].

Lastly, let us point out that a general scattering theory for massless particles in the wedge-local setting curiously appears to require new ideas. In particular many of the conventional technical results may fail without mass gaps, including energy bounds and clustering estimates. These were indispensable tools in all previous constructions of scattering states in the local setting without mass gaps, see e.g. [Bu77,Dy05,Her13, AD17,Du17].

A. Some Technical Arguments

For the convenience of the reader we will briefly explain how the standard proof of commutator estimates for Haag–Ruelle operators also yields the corresponding results in the wedge-local setting. Due to the covariance arguments from Sect. 4.1 it is sufficient to consider the case of non-adapted HR-operators corresponding to $B_\tau^\Lambda(f)$ with $\Lambda = \mathbb{1}$.

Lemma 27. *Let f be a regular Klein–Gordon solution of mass $m > 0$.*

(i) $|f(t,\mathbf{x})| \le C/(1+|t|^{s/2})$ *for any* $(t,\mathbf{x}) \in \mathbb{R}^{s+1}$,
(ii) $|f(t,\mathbf{x})| \le C_{\epsilon,N}/(1+|t|^N+|\mathbf{x}|^N)$ *for* $(t,\mathbf{x}) \in \mathbb{R}^d \setminus \Upsilon_f^\epsilon$,
(iii) $\|f_t\|_{L^1(\mathbb{R}^s)} \le C(1+|t|^{s/2})$, *where* $f_t(\mathbf{x}) := f(t,\mathbf{x})$,

where $\epsilon > 0$, and $N \in \mathbb{N}$ are arbitrary, $C > 0$, $C_{\epsilon,N} > 0$ are suitable constants depending on f, and $\Upsilon_f^\epsilon := \mathbb{R}\mathcal{V}_f^\epsilon$ is the cone generated by the ϵ-enlarged velocity support $\mathcal{V}_f^\epsilon := \{(1,\mathbf{v}) \in \mathbb{R}^d : \exists (1,\mathbf{v}') \in \mathcal{V}_f,\ |\mathbf{v}-\mathbf{v}'| < \epsilon\}$.

Lemma 27 follows from (non-)stationary phase analysis, see e.g. [RS3, Sec. XI.3, App. 1] or [A, Thm. 5.3].

Proof of Lemma 9. Let $\delta > 0$ be given and $B_r \in \mathfrak{A}(\mathcal{W}+\mathscr{C}_r)$, $\|B - B_r\| \le C_N/(1+r^N)$ as in Lemma 7. Suitable wedge-local approximants may then be obtained by restricting the integration in the definition of $B_\tau(f)$ to the asymptotically dominant part $f^\uparrow(x) := f(x)\mathbb{1}_{\Upsilon_f^{\delta/2}}(x)$ (Lemma 27) and setting $r(\tau) := \delta\,|\tau|\,/2$ to obtain

$$B_\tau^{(\delta)} := (B_{r(\tau)})_\tau(f^\uparrow) = \int d^s x\ f^\uparrow(\tau,\mathbf{x})\alpha_{(\tau,\mathbf{x})}(B_{r(\tau)}) \in \mathfrak{A}(\mathcal{W}+\mathscr{C}_{\delta|\tau|/2}+\tau\mathcal{V}_f^{\delta/2}),$$

where the localization was computed for given $\tau \in \mathbb{R}$ by covariance, isotony and noting that $\Upsilon_f^{\delta/2} \cap \{x \in \mathbb{R}^d : x^0 = \tau\} = \tau\mathcal{V}_f^{\delta/2} \subset \tau\mathcal{V}_f + \mathscr{C}_{\delta|\tau|/2}$ and $\mathscr{C}_{\delta|\tau|/2}+\mathscr{C}_{\delta|\tau|/2} \subset \mathscr{C}_{\delta|\tau|}$. The approximation in norm is established by $\|B_\tau(f) - B_\tau^{(\delta)}\| = \|B_\tau(f) - (B_{r(\tau)})_\tau(f^\uparrow)\| \le \|(B - B_{r(\tau)})_\tau(f)\| + \|(B_{r(\tau)})_\tau(f - f^\uparrow)\| \le \|B - B_{r(\tau)}\|\,\|f_\tau\|_{L^1(\mathbb{R}^s)} + \|B_{r(\tau)}\|\|f_\tau - f_\tau^\uparrow\|_{L^1(\mathbb{R}^s)}$ using $\|B_r\| \le \|B\|+C_1$, $\|f_\tau - f_\tau^\uparrow\|_{L^1(\mathbb{R}^s)} \le C_N'/(1+|\tau|^N)$ due to Lemma 27 and that $\|B - B_{r(\tau)}\| \le C_N/(1+\delta^N|\tau|^N)$ is sufficient to compensate the polynomial growth in Lemma 27 (iii) and obtain overall $\|B_\tau(f) - B_\tau^{(\delta)}\| \le C_{\delta,N}/(1+|\tau|^N)$.

Proof of Corollary 10. To estimate $\|[B_\tau^\perp(f^\perp), B_\tau(f)]\|$, let $\delta > 0$ and $B_\tau^{(\delta)}$, $B_\tau^{\perp(\delta)}$ corresponding approximants as from Lemma 9, i.e. $B_\tau^{(\delta)} \in \mathfrak{A}(\tau\mathcal{V}_f + \mathscr{C}_{\delta|\tau|} + \mathcal{W})$, s.t. $\|B_\tau^{(\delta)} - B_\tau(f)\| \le C_N^\delta/(1+|\tau|^N)$, and let analogously $B_\tau^{\perp(\delta)} \in \mathfrak{A}(\tau\mathcal{V}_{f^\perp}+\mathscr{C}_{\delta|\tau|}+\mathcal{W}^\perp)$, s.t. $\|B_\tau^{\perp(\delta)} - B_\tau^\perp(f^\perp)\| \le C_N'^\delta/(1+|\tau|^N)$.

Choosing $\delta > 0$ sufficiently small the localization regions of $B_\tau^{(\delta)}$ and $B_\tau^{\perp(\delta)}$ will be space-like separated for any large enough $\tau > 0$: By assumption we have $\mathcal{V}_f - \mathcal{V}_{f^\perp} \subset \mathcal{W}_c$ with $\mathcal{V}_f - \mathcal{V}_{f^\perp}$ compact and $\mathcal{W}_c$ open. Thus there exists $\epsilon > 0$ such that $\mathcal{V}_f - \mathcal{V}_{f^\perp} + \mathscr{C}_\epsilon \subset \mathcal{W}_c$, where $\mathscr{C}_\epsilon := \{x \in \mathbb{R}^d : |x|_c = |x^0| + |\mathbf{x}| < \epsilon\}$ and as $\mathcal{W}_c$ is a convex cone this implies also $\tau(\mathcal{V}_f - \mathcal{V}_{f^\perp} + \mathscr{C}_\epsilon) \subset \mathcal{W}_c$ for any $\tau > 0$. To obtain space-like separation recall that $\mathcal{W} = \mathcal{W}_c + x_1$, $\mathcal{W}^\perp = \mathcal{W}_c' + x_2$, for $x_1, x_2 \in \mathbb{R}^d$. Thus we get for $\delta < \epsilon/3$ and $\tau > 3(|x_1|_c + |x_2|_c)/\epsilon$ and any $|x_1'|_c < \delta$, $|x_2'|_c < \delta$ that

$$\begin{aligned}
&\tau(\mathcal{V}_f - \mathcal{V}_{f^\perp} + \frac{x_1 - x_2}{\tau} + x_1' - x_2') + \mathcal{W}_c \subset \mathcal{W}_c = (\mathcal{W}_c^\perp)' \\
&\quad \Longrightarrow \tau\mathcal{V}_f + x_1 + \tau x_1' + \mathcal{W}_c \subset (\mathcal{W}_c^\perp + \tau\mathcal{V}_{f^\perp} + x_2 + \tau x_2')',
\end{aligned}$$

where we used $\mathcal{W}_c + \mathcal{W}_c = \mathcal{W}_c$ and that $\mathcal{O}_1 + \mathcal{O}_2 \subset \mathcal{O}_3' \iff \mathcal{O}_1 \subset (\mathcal{O}_3 - \mathcal{O}_2)'$ for arbitrary $\mathcal{O}_k \subset \mathbb{R}^d$. Due to $\tau\mathscr{C}_\delta = \mathscr{C}_{\delta\tau}$ this is equivalent to $\mathcal{W} + \tau\mathcal{V}_f + \mathscr{C}_{\delta\tau} \subset (\mathcal{W}^\perp + \tau\mathcal{V}_{f^\perp} + \mathscr{C}_{\delta\tau})'$ for $\delta < \epsilon/3$ and $\tau > 3(|x_1|_c + |x_2|_c)/\epsilon$, as claimed.

For such τ, δ we now obtain from locality that $[B_\tau^{\perp(\delta)}, B_\tau^{(\delta)}] = 0$, which implies the commutator estimate by expanding

$$\begin{aligned}\left\|[B_\tau^\perp(f^\perp), B_\tau(f)]\right\| &= \left\|[B_\tau^\perp(f^\perp) - B_\tau^{\perp(\delta)} + B_\tau^{\perp(\delta)}, B_\tau(f) - B_\tau^{(\delta)} + B_\tau^{(\delta)}]\right\| \\ &\leq \left\|[B_\tau^\perp(f^\perp) - B_\tau^{\perp(\delta)}, B_\tau(f) - B_\tau^{(\delta)} + B_\tau^{(\delta)}]\right\| \\ &\quad + \left\|[B_\tau^{\perp(\delta)}, B_\tau(f) - B_\tau^{(\delta)}]\right\| + \left\|[B_\tau^{\perp(\delta)}, B_\tau^{(\delta)}]\right\|, \end{aligned} \tag{65}$$

where $\|[B_\tau^\perp(f^\perp) - B_\tau^{\perp(\delta)}, B_\tau(f)]\| \leq 2\|B_\tau^\perp(f^\perp) - B_\tau^{\perp(\delta)}\| \, \|B_\tau(f)\| \leq 2C_{N'}^\delta C/(1 + |\tau|^{N'}) \cdot (1 + |\tau|)^{s/2} \leq C_N' \tau^{-N}$ due to Lemma 9 and Proposition 8 (iv) and analogously for the second non-vanishing commutator.

Proof of Proposition 15. **Ad (i)** The wave packet $\tilde{f}_\tau^\Lambda$ of f_τ^Λ can be computed via Fourier inversion theorem by noting that

$$\begin{aligned} f_\tau^\Lambda(\mathbf{x}) &= \int \frac{\mathrm{d}^s k}{(2\pi)^s} \, \mathrm{e}^{-\mathrm{i}(\omega_m(\mathbf{k}),\mathbf{k})^\mu (\Lambda(\tau,\mathbf{x}))_\mu} \tilde{f}(\mathbf{k}) \\ &= \int \frac{\mathrm{d}^s k}{(2\pi)^s \omega_m(\mathbf{k})} \, \mathrm{e}^{-\mathrm{i}(\Lambda^{-1}(\omega_m(\mathbf{k}),\mathbf{k}))^\mu (\tau,\mathbf{x})_\mu} \tilde{f}(\mathbf{k}) \, \omega_m(\mathbf{k}) \\ &= \int \frac{\mathrm{d}^s k'}{(2\pi)^s \omega_m(\mathbf{k}')} \, \mathrm{e}^{-\mathrm{i}(\omega_m(\mathbf{k}'),\mathbf{k}')^\mu (\tau,\mathbf{x})_\mu} \tilde{f}(\Lambda_m(\mathbf{k}')) \, \omega_m(\Lambda_m(\mathbf{k}')), \end{aligned}$$

where we substituted $\mathbf{k}' := \Lambda_m^{-1}(\mathbf{k})$ after rewriting with respect to the standard Lorentz-invariant measure $\mathrm{d}^s k/\omega_m(\mathbf{k})$ (more precisely Λ_m-invariant, see e.g. [RS2] Thm. IX.37) and used that $(\Lambda(\omega_m(\mathbf{k}), \mathbf{k}))^0 = \omega_m(\Lambda_m(\mathbf{k}))$ due to $(\omega_m(\mathbf{k}), \mathbf{k}) \in H_m$ and Lorentz-invariance of the mass hyperboloid H_m.

Ad (ii) We obtain

$$\begin{aligned} B_\tau^\Lambda(f)\Omega &= \int \mathrm{d}^s x \; f^\Lambda(\tau, \mathbf{x}) \, \mathrm{e}^{\mathrm{i}(\Lambda^{-1}P)^\mu(\tau,\mathbf{x})_\mu} B\Omega = \mathrm{e}^{\mathrm{i}H_\Lambda \tau} \int \mathrm{d}^s x \; f^\Lambda(\tau, \mathbf{x}) \, \mathrm{e}^{-\mathrm{i}\boldsymbol{P}_\Lambda \cdot \mathbf{x}} B\Omega \\ &= \mathrm{e}^{\mathrm{i}H_\Lambda \tau} \int \mathrm{d}^s x \, \mathrm{d}E_{\boldsymbol{P}_\Lambda}(\mathbf{p}) \; f_\tau^\Lambda(\mathbf{x}) \, \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} B\Omega = \mathrm{e}^{\mathrm{i}H_\Lambda \tau} \tilde{f}_\tau^\Lambda(\boldsymbol{P}_\Lambda) B\Omega. \end{aligned} \tag{66}$$

Here we first used translation-invariance of Ω, $P^\mu(\Lambda x)_\mu = (\Lambda^{-1}P)^\mu x_\mu$, and then we abbreviated $(H_\Lambda, \boldsymbol{P}_\Lambda) := \Lambda^{-1}(H, \boldsymbol{P})$, $f_\tau^\Lambda(\mathbf{x}) := f(\Lambda(\tau, \mathbf{x}))$. Further due to (47), $\tilde{f}_\tau^\Lambda(\mathbf{k}) = \frac{\omega_m(\Lambda_m(\mathbf{k}))}{\omega_m(\mathbf{k})} \tilde{f}(\Lambda_m(\mathbf{k})) \mathrm{e}^{-\mathrm{i}\omega_m(\mathbf{k})t}$, and therefore $\mathrm{e}^{-\mathrm{i}\omega_m(\boldsymbol{P}_\Lambda)t} B\Omega = \mathrm{e}^{-\mathrm{i}\omega_m(\boldsymbol{P}_\Lambda)t} E(H_m) B\Omega = \mathrm{e}^{-\mathrm{i}H_\Lambda t} E(H_m) B\Omega$, so that τ-dependent terms cancel in (66). Finally (48) is obtained by inserting $\boldsymbol{P}_\Lambda = \Lambda_m^{-1}(\boldsymbol{P})$. □

Lemma 28. *Let $\{f^\gamma\}_{\gamma \in I}$ be a family regular KG-solutions whose associated family of wave packets is compact with respect to the Schwartz topology. Then estimates (i)–(iii) of Lemma 27 hold also with constants which can be chosen uniformly in γ. The rapid decay from (ii) holds in particular with respect to $\Upsilon^\epsilon := \bigcup_{\gamma \in I} \Upsilon_{f^\gamma}^\epsilon$.*

Proof (sketch). As (iii) is obtained from (i) and (ii) by integration, it suffices to establish the latter two uniformly in γ. These follow from the fact that the (non-) stationary-phase analysis yields constants C, $C_{N,\epsilon}$ with explicit wave packet dependences (see [RS3, Sec. XI.3, App. 1]*) via the C^k-norms*

$$\|\tilde{f}\|_{C^k(\mathbb{R}^s)} := \sum_{|\alpha|\leq k} \sup_{\mathbf{p}\in\mathbb{R}^s} |(\partial^\alpha_{\mathbf{p}} \tilde{f})(\mathbf{p})|, \ k \in \mathbb{N}_0, \tag{67}$$

with summation over multi-indices $\alpha \in \mathbb{N}_0^s$ of order at most k. Hence $C^\gamma_{N,\epsilon}$, C^γ are continuous w.r.t. $\tilde{f}^\gamma$, from which it follows by compactness that γ-uniform constants can be chosen, as claimed. □

Proof of Lemma 18 (sketch). We begin by noting for the second estimate in (51) and for (53) that

$$\begin{aligned}\partial_\gamma B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)}) = &\int \mathrm{d}^s x\, (\partial_\mu B)(\Lambda^\gamma(\tau,\mathbf{x}))\, f^{(\Lambda^\gamma)}(\Lambda^\gamma(\tau,\mathbf{x}))\, w^\mu_{(\tau,\mathbf{x},\gamma)} \\ &+ \int \mathrm{d}^s x\, B(\Lambda^\gamma(\tau,\mathbf{x}))\partial_\gamma f^{(\Lambda^\gamma)}(\Lambda^\gamma(\tau,\mathbf{x})),\end{aligned} \tag{68}$$

with implied Minkowski summation over μ, and where $w^\mu_{(\tau,\mathbf{x},\gamma)} := (\partial_\gamma \Lambda^\gamma(\tau,\mathbf{x}))^\mu$ satisfies the γ-uniform bound $|w^\mu_{(\tau,\mathbf{x},\gamma)}| \leq C(|\tau|+|\mathbf{x}|)$ due to continuous differentiability and compactness of $\gamma \mapsto \Lambda^\gamma$.

Ad (i). To obtain the norm estimates (i) uniformly, we write

$$\|B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)})\| \leq \int \mathrm{d}^s x\, |f^{(\Lambda^\gamma)}(\Lambda^\gamma(\tau,\mathbf{x}))|\, \big\|B(\Lambda^\gamma(\tau,\mathbf{x}))\big\| = \|B\| \cdot \|(f^{(\Lambda^\gamma)})^{\Lambda^\gamma}_\tau\|_{L^1(\mathbb{R}^s)}, \tag{69}$$

where due to (47) we have for $(f^{(\Lambda^\gamma)})^{\Lambda^\gamma}(\tau,\mathbf{x}) := f^{(\Lambda^\gamma)}(\Lambda^\gamma(\tau,\mathbf{x}))$ the wave packet $\tilde{f}(\Lambda^\gamma_m(\mathbf{p}))$, and $(f^{(\Lambda^\gamma)})^{\Lambda^\gamma}_\tau(\mathbf{x}) := (f^{(\Lambda^\gamma)})^{\Lambda^\gamma}(\tau,\mathbf{x})$. For the second estimate in (51) we can analogously estimate the two summands in (68) using also $\big\|\partial_\mu B\big\| \leq \|A\|\,\big\|\partial_\mu \chi\big\|_{L^1(\mathbb{R}^{s+1})}$. The desired γ-uniform τ-estimates (51) now follow from corresponding bounds obtained via Lemma 28 on $\|(f^{(\Lambda^\gamma)})^{\Lambda^\gamma}_\tau\|_{L^1(\mathbb{R}^s)}$, $\|\partial_\gamma(f^{(\Lambda^\gamma)})^{\Lambda^\gamma}_\tau\|_{L^1(\mathbb{R}^s)}$, $\|(f^{(\Lambda^\gamma)})^{\Lambda^\gamma}_\tau w_\tau\|_{L^1(\mathbb{R}^s)}$, where $w_\tau(\mathbf{x}) := C(|\tau| + |\mathbf{x}|)$ was inserted as upper bound on $w^\mu_{(\tau,\mathbf{x},\gamma)}$. There we can argue by verifying the γ-uniform norm bounds (67) by direct computation, yielding the uniform variants of bounds (i), (ii) from Lemma 27. The estimates (51) follow by integration, and presence of w_τ yields an additional power of $|\tau|$ in the inner region $|\mathbf{x}| < (1+\epsilon)\,|\tau|$ of the integral and can be compensated by (ii) on the complement of this region.

Ad (ii). The γ-uniform commutator estimates (52), (53) are obtained by analogous arguments as given in the proofs of Lemma 9 and Corollary 10, again making use of the γ-uniform KG-estimates from Lemma 28. More precisely we obtain wedge-local approximants $(B_\tau^{\perp\gamma})^{(\delta)} := (B_\tau^{\perp\Lambda^\gamma}(f^{\perp(\Lambda^\gamma)}))^{(\delta)} \in \mathfrak{A}(\mathcal{W}^\perp + \tau\mathcal{V}^{\Lambda^\gamma}_{f^\perp} + \mathscr{C}_{\delta|\tau|})$ ($\delta > 0$) such that for any $N \in \mathbb{N}$, $\|(B_\tau^{\perp\gamma})^{(\delta)} - B_\tau^{\perp\Lambda^\gamma}(f^{\perp(\Lambda^\gamma)})\| < C_N/(1+\tau^N)$. Analogously there are wedge-local approximants $(B_\tau^\gamma)^{(\delta)} \in \mathfrak{A}(\mathcal{W} + \tau\mathcal{V}^{\Lambda^\gamma}_f + \mathscr{C}_{\delta|\tau|})$ for $B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)})$ and $(\partial_\gamma B_\tau^\gamma)^{(\delta)} \in \mathfrak{A}(\mathcal{W} + \tau\mathcal{V}^{\Lambda^\gamma}_f + \mathscr{C}_{\delta|\tau|})$ for $\partial_\gamma B_\tau^{\Lambda^\gamma}(f^{(\Lambda^\gamma)})$, which are rapidly converging in norm with γ-uniform constants. Here we also used that $\mathcal{V}^{\Lambda^\gamma}_{f(\Lambda^\gamma)} = \mathcal{V}^{\Lambda^\gamma}_{f}$ and $\mathcal{V}^{\Lambda^\gamma}_{f^\perp(\Lambda^\gamma)} = \mathcal{V}^{\Lambda^\gamma}_{f^\perp}$ for all γ. Estimates (52), (53) now follow as in (65) assuming correctly ordered wave packets, depending on the incoming or outgoing cases, $\tau < 0$ and $\tau > 0$, respectively. □

B. Swapping Relations and Modular Theory

According to Tomita–Takesaki theory one defines for a von Neumann algebra $\mathfrak{M}$ with cyclic and separating vector Ω the positive self-adjoint *modular operator* Δ and the anti-unitary *modular conjugation* J as the unique polar decomposition of $S : \mathfrak{M}\Omega \longrightarrow \mathfrak{M}\Omega$ given by

$$SA\Omega := A^*\Omega, \quad S = J\Delta^{1/2}. \tag{70}$$

The central Tomita-Takesaki theorem [BR1, Thm. 2.5.14] states that

$$J\mathfrak{M}J = \mathfrak{M}', \quad \text{and} \quad \Delta^{\mathrm{i}\tau}\mathfrak{M}\Delta^{-\mathrm{i}\tau} = \mathfrak{M}, \ (\tau \in \mathbb{R}). \tag{71}$$

In our case we take $\mathfrak{M} = \mathfrak{A}(\mathcal{W})$, so that the modular objects $S_{\mathcal{W}}$, $J_{\mathcal{W}}$ and $\Delta_{\mathcal{W}}$ will depend on the wedge $\mathcal{W}$. It is clear that $S_{\mathcal{W}}\Omega = S_{\mathcal{W}}\mathbb{1}\Omega = \Omega = S_{\mathcal{W}'}\Omega$ and one has further [BR1, Prop. 2.5.11]

$$\Delta_{\mathcal{W}}\Omega = \Omega, \ J_{\mathcal{W}}\Omega = \Omega. \tag{72}$$

The basic idea for the proof of Lemma 3 is that for given self-adjoint $A = A^* \in \mathfrak{A}(\mathcal{W})$, $S_{\mathcal{W}}$ acts trivially on $A\Omega$ so that (70)–(72) yield up to domain questions

$$A\Omega = A^*\Omega = S_{\mathcal{W}}A\Omega = J_{\mathcal{W}}\Delta_{\mathcal{W}}^{1/2}A\Omega = \underbrace{J_{\mathcal{W}}\Delta_{\mathcal{W}}^{1/2}A\Delta_{\mathcal{W}}^{-1/2}J_{\mathcal{W}}}_{=:A^\perp}\Omega. \tag{73}$$

Proof of Lemma 3. We follow the argument of Buchholz [Bu17]. As we keep $\mathcal{W}$ fixed, we drop wedge indices on the modular objects. To establish existence we consider vectors of the form $\Psi = A\Omega$ with $A^* = A \in \mathfrak{A}(\mathcal{W})$. Rigorous control over (73) is then obtained by passing to operators A_δ, $(\delta > 0)$, which are "regularized" with respect to the adjoint action of the modular group by setting

$$A_\delta := \int \frac{\mathrm{d}\tau}{\sqrt{2\pi\delta}}\, \mathrm{e}^{-\frac{\tau^2}{2\delta}}\, \Delta^{\mathrm{i}\tau}A\Delta^{-\mathrm{i}\tau}. \tag{74}$$

From the Tomita-Takesaki theorem (71) we see that the integrand is pointwise in $\mathfrak{A}(\mathcal{W})$ so that $A_\delta \in \mathfrak{A}(\mathcal{W})$ as wedge-algebras are weakly closed. Secondly we obtain from strong continuity of $\Delta^{\mathrm{i}\tau}$ that $A_\delta \rightharpoonup A$ in the weak operator topology, so that by modular invariance of Ω we have $A_\delta\Omega \to A\Omega$ in norm as $\delta \to 0$. Further due to (74) the adjoint action of the modular group on A_δ may be computed explicitly as

$$\Delta^{\mathrm{i}t}A_\delta\Delta^{-\mathrm{i}t} = \int \frac{\mathrm{d}\tau}{\sqrt{2\pi\delta}}\, \mathrm{e}^{-\frac{\tau^2}{2\delta}}\, \Delta^{\mathrm{i}\tau+\mathrm{i}t}A\Delta^{-\mathrm{i}\tau-\mathrm{i}t} = \int \frac{\mathrm{d}\tau'}{\sqrt{2\pi\delta}}\, \mathrm{e}^{-\frac{(\tau'-t)^2}{2\delta}}\, \Delta^{\mathrm{i}\tau'}A\Delta^{-\mathrm{i}\tau'}. \tag{75}$$

Returning to (73) we now define $\bar{A}_\delta := \Delta^{1/2}A_\delta\Delta^{-1/2}$ as a quadratic form on a suitable domain. It will be convenient to restrict to $\mathrm{D}_\omega(\Delta^\pm) := \{E_\Delta([k, K])\Psi : \Psi \in$

$\mathscr{H}, 0 < k < K\}$, which is dense in $\mathscr{H}$ by spectral calculus. For $\Psi_1, \Psi_2 \in D_\omega(\Delta^\pm)$ the function $t \longmapsto \langle \Psi_1, \Delta^{it} A_\delta \Delta^{-it} \Psi_2 \rangle$ is entire analytic. It further coincides for $t \in \mathbb{R}$ with the entire function defined by the right hand side of (75). By analyticity these two entire functions coincide for all $t \in \mathbb{C}$ so that

$$\langle \Psi_1, \Delta^{1/2} A_\delta \Delta^{-1/2} \Psi_2 \rangle = \int \frac{d\tau}{\sqrt{2\pi\delta}}\, e^{-\frac{\tau^2}{2\delta}} \langle \Psi_1, \Delta^{i\tau+1/2} A \Delta^{-i\tau-1/2} \Psi_2 \rangle \tag{76}$$

$$= \int \frac{d\tau'}{\sqrt{2\pi\delta}}\, e^{-\frac{(\tau'+i/2)^2}{2\delta}} \langle \Psi_1, \Delta^{i\tau'} A \Delta^{-i\tau'} \Psi_2 \rangle. \tag{77}$$

From (77) we see firstly that (76) in fact defines a bounded bilinear form, so that $\bar{A}_\delta$ extends to a bounded operator on all of $\mathscr{H}$, and secondly that $\bar{A}_\delta \in \mathfrak{A}(\mathcal{W})$ by repeating the argument below (74). Thus the swapping partner may be obtained as in (73) by setting $A_\delta^\perp := J \bar{A}_\delta J$, and noting that $A_\delta^\perp \in \mathfrak{A}(\mathcal{W}')$ due to (71) and wedge duality (HK2$^\sharp$).

To establish density of swappable vectors let $\Psi \in \mathscr{H}$ and $\epsilon > 0$. By cyclicity of Ω there exists $A \in \mathfrak{A}(\mathcal{W})$ such that $\|\Psi - A\Omega\| \leq \epsilon/2$. We may then decompose $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) =: A_1 + iA_2$ such that the above argument applies to $A_k\Omega$, $k = 1, 2$, and the swapping partner of A_δ is then given by $A_\delta^\perp := (A_1)_\delta^\perp + i(A_2)_\delta^\perp$. Choosing $\delta > 0$ sufficiently small yields $\|\Psi - A_\delta\Omega\| \leq \|\Psi - A\Omega\| + \|A_1\Omega - (A_1)_\delta\Omega\| + \|A_2\Omega - (A_2)_\delta\Omega\| \leq \epsilon$ so that we obtain density. $\square$

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Publication 2. Strengthened Reeh-Schlieder Property and Scattering in Quantum Field Theory without Mass Gaps

Let $\Psi \in \mathscr{H}$. A Reeh-Schlieder family for Ψ of degree $\gamma \geq 0$ is an operator family $(A_\beta)_{\beta>0}$ localized in some fixed bounded region $\mathcal{O} \subset \mathbb{R}^4$ satisfying for sufficiently small $\beta > 0$

$$\|A_\beta\Omega - \Psi\| \leq \beta, \qquad \|A_\beta\| \leq \beta^{-\gamma}. \tag{1}$$

In this publication we construct multi-particle scattering states using (1), and develop required non-equal-time commutation estimates, energy-bounds, and clustering estimates compatible with the $\beta \to 0$ norm singularity in (1). The status of (1) in models is also discussed. The main result is the following Haag-Ruelle theorem based on the Reeh-Schlieder condition (1).

Theorem 1. *Let $A_{1\beta}, \ldots, A_{n\beta}$ be Reeh-Schlieder families of finite degree less than some $\gamma > 0$, let $f_1, \ldots, f_n$ be regular positive-energy Klein-Gordon solutions with disjoint velocity supports, and take a scaling exponent $\mu \in (0, \frac{\kappa}{4n\gamma})$*

(i) $\Psi_\tau := \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega$ is convergent in norm as $\tau \to \pm\infty$.

(ii) The limit is independent of the choice of μ, $A_{k\beta}$ and f_k within the specified restrictions, as long as the associated operators $\mathcal{B}'_{k\tau}$ create on the vacuum the same family of single-particle states $\Psi_k^{(1)} = \lim_{\tau\to\infty} \mathcal{B}_{k\tau}\Omega$.

(iii) The scalar products of any outgoing scattering states are given by the Fock formula $\langle\Psi^+, \Psi'^+\rangle = \delta_{nn'} \sum_{\pi\in\mathfrak{S}_n} \prod_{k=1}^n \left\langle \Psi_k^{(1)}, \Psi'^{(1)}_{\pi(k)} \right\rangle$, and similarly for incoming states.

Non-equal time commutators and Haag-Ruelle construction

The following estimate collects the basic idea from [Du13], which enables the reduction of the scattering state convergence to the one-particle problem and is used in the discretized Cook's method.

Lemma 2 (local difference estimate)**.** *Let $f_1, \ldots f_n$ be regular KG-solutions with disjoint velocity supports. Then there exists $\rho > 0$ and a suitable scaling, so that N-particle state approximants can be estimated w.r.t. one-particle differences with controlled error terms*

$$\left\|\Psi_{\tau_2} - \Psi'_{\tau_1}\right\| \leq C_1 \sum_{k=1}^{n} \left\|\mathcal{B}_{k\tau_2}\Omega - \mathcal{B}'_{k\tau_1}\Omega\right\| + C_2 \left|\tau\right|^{n\gamma\mu-\kappa/4}, \text{ where } \tau_1, \tau_2 \in [1, 1+\rho] \cdot \tau. \tag{2}$$

The geometrical limitation here is important and required due to the corresponding restrictions in non-equal time commutator estimates.

Lemma 3 (non-equal-time commutator estimate)**.** *Let $A_{k\beta}$, $(k = 1, 2)$, be Reeh-Schlieder families of finite degree, take regular Klein-Gordon solutions f_k with disjoint velocity supports and assume a fixed scaling $\beta(\tau) = |\tau|^{-\mu}$, $\mu > 0$. Then there exists $\rho > 0$ and for any $N \in \mathbb{N}$ a constant $C_N > 0$, such that for arbitrary $\tau \in \mathbb{R}$ and all τ_1, τ_2 from the corresponding interval spanned by τ and $\tau + \rho\tau$,*

$$\|[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}]\| \leq C_N(1 + |\tau|)^{-N}.$$

Together with novel multi-operator clustering estimates and energy-bounds developed in the paper, the local difference estimate (2) is obtained. Cook-type summation of the left-hand side differences in (2) over a telescopic expansion of a geometric series $\tau_k := (1 + \rho)^k\tau_0$ yields the convergence of scattering states. The Fock stucture is established by adapting the strategy from [Dy05].

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I am the sole author of this publication. The idea to make use of non-local QFT correlation effects in scattering theory has been suggested by K. Fredenhagen. Energy bounds are proven via a strategy proposed by W. Dybalski.

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Strengthened Reeh–Schlieder Property and Scattering in Quantum Field Theories Without Mass Gaps

Maximilian Duell

Technische Universität München, 85747 Garching, Germany. E-mail: duell@ma.tum.de

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Abstract: We develop Haag–Ruelle scattering theory for Wigner particles in local relativistic Quantum Field Theory without assuming mass gaps or any other restrictions on the spectrum of the mass operator near the particle masses. Our approach is based on the Reeh–Schlieder property of the vacuum state. It is shown that a strengthened variant of this property, concerning the relative approximation error for single-particle states, implies the existence of scattering states.

1. Introduction

The infrared problem in Quantum Electrodynamics (QED) has attracted a lot of attention in the mathematical physics literature of the last decade. Consistent scattering theory has been developed for various physical processes involving charged particles (‘electrons’), neutral massive particles (‘atoms’) and massless particles (‘photons’). Some of these results were obtained in non-relativistic models of QED [CFP10,DyP13,MS14], others in the general setting of algebraic QFT [BR14,AD15,Dy05,Hrd13,DH14]. In spite of all these efforts, even the seemingly simple case of scattering of several atoms is still not fully under control.

This may be explained by the fact that atoms in QED constitute a prototypical example of an *embedded* particle. In other words, single-atom states correspond to eigenvalues of the mass operator which are not isolated, but embedded in a continuous mass spectrum, arising e.g. from states consisting of multiple lighter particles (photons). For the construction of scattering states, such background particles need to be separated from the desired single-atom states. In the framework of Haag–Ruelle theory, this separation could so far only be achieved with the help of technical assumptions[1] on the spectral measure of the mass operator near the particle masses. Such spectral conditions were first proposed by Herbst [Hrb71] and we might consider them to be a remnant of the original Haag–Ruelle mass-gap assumption [Ha58,Ru62,Hep65].

[1] See e.g. [Hrb71,Dy05,Hrd13,DH14].

As the physical meaning of these assumptions has remained obscure, the existence of scattering states of atoms still lacks a conceptually clear explanation. Aiming at such an explanation, we develop Haag–Ruelle scattering theory for atoms, relying on certain non-local correlations of the vacuum state. The required condition is only slightly stronger than the well-established Reeh–Schlieder property and it permits to the best of our knowledge the first proof of existence of scattering states of massive embedded Wigner particles without a priori requiring a spectral condition of Herbst type.

The Reeh–Schlieder property states that the vacuum Ω is cyclic for any algebra $\mathfrak{A}(\mathcal{O})$ of observables[2] localized in a bounded space-time region $\mathcal{O}$. That is, given any vector $\Psi \in \mathscr{H}$ (for example describing an atom essentially localized far from the region $\mathcal{O}$), there exists a family of observables $(A_\beta)_{\beta>0}$ from $\mathfrak{A}(\mathcal{O})$ such that

$$\lim_{\beta \to 0} \|A_\beta \Omega - \Psi\| = 0. \tag{1}$$

While $\|A_\beta \Omega\|$ clearly remains bounded, we note that the operator norms $\|A_\beta\|$ may tend to infinity as $\beta \to 0$. As it will be important for our investigation to quantify this growth, we will say that Ψ is a vector of finite *Reeh–Schlieder degree* if there exists a family of operators $(A_\beta)_{\beta>0}$ localized in some fixed bounded space-time region $\mathcal{O}$, such that for some $\gamma > 0$ we have

$$\|A_\beta\| \leq \beta^{-\gamma} \quad \text{and} \quad \|A_\beta \Omega - \Psi\| \leq \beta. \tag{2}$$

In this paper we will construct scattering states of configurations of atoms whose single-particle states are generated by such families with finite Reeh–Schlieder degree γ. Condition (2) is readily verified for free scalar fields,[3] but it seems that not much progress has been made in understanding such relations since the seminal work of Haag and Swieca [HS65]. In theories where Herbst's spectral condition is satisfied, one can construct an operator family $(A_\beta)_{\beta>0}$ satisfying a weakened variant ($\mathrm{RS}^\flat$) of (2) (see concluding discussion), but the status of (2) in interacting theories is currently not clear and constitutes a difficult technical problem outside the scope of this work.

Let us now describe in non-technical terms the relevance of (2) for Haag–Ruelle scattering theory. Take a single-atom state Ψ of finite Reeh–Schlieder degree and let $(A_\beta)_{\beta>0}$ be a corresponding *Reeh–Schlieder family* from formula (2). Since $(A_\beta)_{\beta>0}$ play a role of creation operators, it is technically convenient to smear them with the Fourier transform of a function $\hat{\chi} \in C_c^\infty(\mathbb{R}^4 \setminus \bar{V}^-)$ yielding a family of almost-local operators

$$B_\beta := \int \mathrm{d}^4x \; \chi(x) A_\beta(x), \quad (\beta > 0), \tag{3}$$

where $A_\beta(x)$ denotes the translate of A_β in space-time by x. Following the standard prescription we pick a regular positive-energy solution f of the Klein–Gordon equation with the mass of the atom and set

$$\mathcal{B}_\tau := \int \mathrm{d}^3x \; f(\tau, \mathbf{x}) B_{\beta(\tau)}(\tau, \mathbf{x}), \quad \text{with} \;\; \beta(\tau) := \tau^{-\mu}, \;\; \mu > 0 \text{ fixed}. \tag{4}$$

[2] In the case of QED these algebras should be generated by bounded functions of suitably smeared electromagnetic fields and the electric current, cf. [Bu86].

[3] A free scalar field $\phi(f)$ is self-adjoint for real-valued f and $\phi(f)\Omega$, $f \in C_c^\infty(\mathbb{R}^4)$, yield a dense subset of single-particle states. If $\operatorname{supp} f \subset \mathcal{O}$ we can simply set $A_\beta := \phi(f) \exp(-\beta|\phi(f)|^{1/\gamma}) \in \mathfrak{A}(\mathcal{O})$ to obtain Reeh–Schlieder families of arbitrarily small degrees $\gamma > 0$. For further examples see Appendix C.

We will call $\mathcal{B}_\tau$ an (approximating) *creation operator* of Ψ since it has the property

$$\lim_{\tau\to\infty} \mathcal{B}_\tau \Omega = (2\pi)^2 \hat{\chi}(H, \boldsymbol{P}) \tilde{f}(\mathbf{P}) \left(\lim_{\tau\to\infty} A_{\beta(\tau)}\Omega \right) = (2\pi)^2 \hat{\chi}(H, \boldsymbol{P}) \tilde{f}(\boldsymbol{P})\Psi \tag{5}$$

(see Proposition 3). That is, it asymptotically creates Ψ from the vacuum up to an inessential function of the energy-momentum operators $(H, \boldsymbol{P})$ (which can be arranged to be equal to one if Ψ has bounded energy). Since we inserted a Reeh–Schlieder family in (4), we obtain convergence in (5) without the ergodic averaging used in earlier works [Dy05, Bu77]. We also note that (5) holds even for Ψ of infinite Reeh–Schlieder degree. The need to assume finiteness of the Reeh–Schlieder degree of Ψ arises only at the level of n-atom scattering states, $n \geq 2$—the case to which we now proceed.

Let Ψ_1, Ψ_2 be two single-atom states with disjoint velocity supports and finite Reeh–Schlieder degree. Let $\mathcal{B}_{1\tau}, \mathcal{B}_{2\tau}$ be the corresponding creation operators constructed as above. The scattering state describing these two atoms is given by the limit as $\tau \to \infty$ of the family[4]

$$\Psi_\tau := \mathcal{B}_{1\tau}\mathcal{B}_{2\tau}\Omega.$$

The conventional Cook-argument to establish convergence does not apply here due to the additional τ-dependence via the Reeh–Schlieder family in (4). Therefore, we base our proof on a discretized analog of Cook's argument involving summability of the telescopic expansion

$$\left\| \Psi_{\tau_N} - \Psi_{\tau_0} \right\| \leq \sum_{k=0}^{N-1} \left\| \Psi_{\tau_{k+1}} - \Psi_{\tau_k} \right\| \tag{6}$$

in the limit $N \to \infty$ (here $\tau_k := (1+\rho)^k \tau_0$, $\tau_0 > 0$, and $\rho > 0$ is sufficiently small). The first term in this sum has the form

$$\Psi_{\tau_1} - \Psi_{\tau_0} = \mathcal{B}_{1\tau_1}(\mathcal{B}_{2\tau_1} - \mathcal{B}_{2\tau_0})\Omega + (\mathcal{B}_{1\tau_1} - \mathcal{B}_{1\tau_0})\mathcal{B}_{2\tau_0}\Omega. \tag{7}$$

Exploiting locality and the fact that $|\tau_1 - \tau_0|$ is small, we obtain that $[(\mathcal{B}_{1\tau_1} - \mathcal{B}_{1\tau_0}), \mathcal{B}_{2\tau_0}]$ is rapidly decreasing with τ_0 and thus it suffices to study the expressions

$$\mathcal{B}_{1\tau_1}(\mathcal{B}_{2\tau_1} - \mathcal{B}_{2\tau_0})\Omega, \qquad \mathcal{B}_{2\tau_0}(\mathcal{B}_{1\tau_1} - \mathcal{B}_{1\tau_0})\Omega. \tag{8}$$

Let us concentrate on the first term above: Thanks to the smearing operation (3) which restricts the energy-momentum transfers of the creation operators, we can write

$$\|\mathcal{B}_{1\tau_1}(\mathcal{B}_{2\tau_1} - \mathcal{B}_{2\tau_0})\Omega\| \leq \|\mathcal{B}_{1\tau_1} E(\Delta)\| \|(\mathcal{B}_{2\tau_1} - \mathcal{B}_{2\tau_0})\Omega\|, \tag{9}$$

where $E(\Delta)$ is a projection onto a compact subset Δ of the energy-momentum spectrum. Now exploiting formula (5) and results from [Bu90a], which give $\|\mathcal{B}_{1\tau_1} E(\Delta)\| \leq C\|A_{1\beta(\tau_1)}\|$, we can estimate (9) by

$$\|A_{1\beta(\tau_1)}\| \|A_{2\beta(\tau_1)}\Omega - A_{2\beta(\tau_0)}\Omega\| \leq \|A_{1\beta(\tau_1)}\|(\|A_{2\beta(\tau_1)}\Omega - \Psi_2\| + \|A_{2\beta(\tau_0)}\Omega - \Psi_2\|) \tag{10}$$

up to an overall constant, and the analysis of the second term in (8) gives an analogous bound. By substituting such estimates into (6), it is easy to obtain convergence of Ψ_τ, provided Ψ_1, Ψ_2 are of Reeh–Schlieder degree $\gamma < 1$ [cf. relations (2), (4)]. A similar

[4] For clarity reasons we consider here only outgoing states. The incoming case $\tau \to -\infty$ is analogous.

discussion of n-atom scattering states could suggest that single-atom states of arbitrarily small Reeh–Schlieder degree are needed. It turns out that this is not the case: by careful geometrical analysis and application of corresponding novel multi-operator clustering estimates (cf. Lemmas 8 and 16, respectively) we develop complete Haag–Ruelle scattering theory for single-atom states of arbitrarily large Reeh–Schlieder degree. Although atoms are our prime example, the construction works equally well for photons,[5] which demonstrates the robustness of our approach. We hope that this investigation will pave the way to a definite unifying solution of the problem of scattering of Wigner particles in algebraic QFT.

This paper is structured as follows: In Sect. 2 we state the basic assumptions underlying this work and introduce the Reeh–Schlieder degree of Hilbert-space vectors. Section 3 gives an exposition of our variant of Haag–Ruelle creation operators and establishes some of their basic properties. Section 4 provides the fundamental technical tool of the discretized Cook's method: we derive rapid norm decay of non-equal time commutators of creation operators. In Sects. 5 and 6 we establish clustering estimates and study their consequences relevant for refined handling of the norm growth of the creation operator approximants. All these results are then combined in Sect. 7 to prove convergence of scattering states and to establish their Fock structure in Sect. 8.

2. Framework and Assumptions

As the basis for our considerations we take a Haag–Kastler theory in the vacuum representation, i.e. a net $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O}) \subset \mathrm{B}(\mathscr{H})$ of von Neumann algebras associated to bounded open regions $\mathcal{O} \subset \mathbb{R}^4$ in Minkowski space-time.[6] Space-time translations by vectors $x = (t, \mathbf{x}) \in \mathbb{R}^4$ are represented on the Hilbert space $\mathscr{H}$ by a strongly-continuous group of unitary operators $U(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$, generated by the strongly-commuting family of the self-adjoint *energy-momentum* operators $(H, \boldsymbol{P})$. Their joint spectral measure is denoted by $E(\Delta) := E_{(H,\boldsymbol{P})}(\Delta)$ for any Borel set $\Delta \subset \mathbb{R}^4$. The *vacuum* is a normalized translation-invariant vector $\Omega \in \mathscr{H}$. Finally, translations of operators $A \in \mathrm{B}(\mathscr{H})$ are induced by U according to $A(x) := \alpha_x(A) := U(x)AU(x)^*$. We will use the following version of the Haag–Kastler postulates,

$$\textbf{Isotony} \quad \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2) \text{ for } \mathcal{O}_1 \subset \mathcal{O}_2 \tag{HK1}$$

$$\textbf{Locality} \quad \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)' \text{ for } \mathcal{O}_1 \subset \mathcal{O}_2' \tag{HK2}$$

$$\textbf{Covariance} \quad \alpha_x(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O} + x) \tag{HK3}$$

$$\textbf{Uniqueness of } \boldsymbol{\Omega} \quad E(\{0\})\mathscr{H} = \mathbb{C}\Omega \tag{HK4}$$

$$\textbf{Spectrum Condition} \quad \operatorname{supp} E_{(H,\boldsymbol{P})} \subset \bar{V}^+ \tag{HK5}$$

$$\textbf{Reeh–Schlieder Property} \quad \overline{\mathfrak{A}(\mathcal{O})\Omega} = \mathscr{H} \tag{HK6}$$

for any non-empty open bounded regions $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^4$ and any $x \in \mathbb{R}^4$. Here, $\mathfrak{A}(\mathcal{O})'$ is the commutant of $\mathfrak{A}(\mathcal{O})$ in $\mathrm{B}(\mathscr{H})$ and $\mathcal{O}' := \{y \in \mathbb{R}^4 : (y-x)^2 < 0 \ \forall x \in \mathcal{O}\}$ defines the causal complement of $\mathcal{O}$. Further, $\bar{V}^\pm := \{x \in \mathbb{R}^4 : x^2 \geq 0, \pm x^0 \geq 0\}$ is the future or past light cone, respectively. For future reference we denote by $\mathfrak{A}$ the C*-inductive limit of the local net and by $H_m := \{p \in \mathbb{R}^4 : p^0 = \sqrt{\mathbf{p}^2 + m^2}\}$ the mass hyperboloid of a particle of mass $m \geq 0$.

[5] In contrast to atoms, scattering theory of photons is well understood since [Bu77].

[6] We take the space-time metric with signature $(+, -, -, -)$.

Next, we define the *Reeh–Schlieder degree* $\gamma_{\mathrm{RS}} \geq 0$ of a vector $\Psi \in \mathscr{H}$ as the infimum over all $\gamma \geq 0$ for which there exists an open bounded region $\mathcal{O}$ and a family of observables $(A_\beta)_{\beta>0}$ from $\mathfrak{A}(\mathcal{O})$ such that for all sufficiently small $\beta > 0$ we have

$$\left\| A_\beta \Omega - \Psi \right\| \leq \beta, \qquad \left\| A_\beta \right\| \leq \beta^{-\gamma}. \tag{11}$$

We will call $(A_\beta)_{\beta>0}$ a *Reeh–Schlieder family* (of degree γ). If no such family exists, we will say that Ψ is a vector of infinite Reeh–Schlieder degree. But we note that, by the standard Reeh–Schlieder property (HK6),[7] at least the first inequality of (11) can always be satisfied for non-empty regions $\mathcal{O} \subset \mathbb{R}^4$.

We amend the Haag–Kastler postulates by the following more specific assumptions, which can be seen in combination as a sharpened Wigner concept of a particle:

(HK5′) In addition to (HK5), the relativistic mass operator $M := \sqrt{H^2 - \boldsymbol{P}^2}$ has an eigenvalue $m \geq 0$. In other words $E_m := E(H_m) \neq 0$.

(HK6′) The single-particle subspace $\mathscr{H}_m := E_m \mathscr{H}$ contains a dense subset of vectors of finite Reeh–Schlieder degree.

Under the above assumptions, (HK1)–(HK4), (HK5′), and (HK6′), our results from Sects. 7 and 8 below allow to construct wave-operators and the S-matrix in the usual manner (see e.g. [Dy09] App. A).

3. Creation Operators and Their Basic Properties

Given a single-atom state $\Psi_1 \in E(H_m)\mathscr{H}$ of mass $m \geq 0$ we now want to find a corresponding family of creation operators $\mathcal{B}_\tau$, which is suitable for the construction of scattering states. By the Reeh–Schlieder property (HK6) we can always fix some non-empty bounded open region $\mathcal{O} \subset \mathbb{R}^4$ and pick a corresponding family of local operators $(A_\beta)_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$ as in formula (1).

The Klein–Gordon equation will provide a free reference dynamics for comparison to the large-τ asymptotics of the translated operator family $A_\beta(\tau, \mathbf{x}) := U(\tau, \mathbf{x}) A_\beta U(\tau, \mathbf{x})^*$, $x = (\tau, \mathbf{x}) \in \mathbb{R}^4$, $\beta > 0$, when taking the simultaneous limit $\beta \to 0$. We will say that $f : \mathbb{R}^4 \longrightarrow \mathbb{C}$ is a *regular positive-energy Klein–Gordon solution* (of *mass* $m \geq 0$) if it can be written as

$$f(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_m(\mathbf{k})t}\, \tilde{f}(\mathbf{k}), \quad \omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}, \tag{12}$$

where the *wave-packet* $\tilde{f}$ has to be smooth and compactly supported. For the case $m = 0$ we will also add the standard requirement $\mathbf{0} \notin \operatorname{supp} \tilde{f}$, as it leads to improved decay in the interior of the light cone which will be technically convenient in Sect. 4.

Taking a Reeh–Schlieder family A_β for a given single-particle state $\Psi \in E(H_m)\mathscr{H}$ of mass $m \geq 0$ and a regular positive-energy Klein–Gordon solution f of the same mass, we may modify the standard prescription for creation-operator approximants by admitting the following additional time-dependence of the smeared operators,

$$\mathcal{A}_\tau := \int \mathrm{d}^3 x\, f(\tau, \mathbf{x}) A_{\beta(\tau)}(\tau, \mathbf{x}). \tag{13}$$

[7] If the Haag–Kastler net under consideration is obtained from a suitable Wightman theory (e.g. satisfying certain energy bounds [Bu90b]), property (HK6) holds as a consequence of the original results of Reeh and Schlieder [RS61]. Alternatively, (HK6) follows from assuming *additivity* of the Haag–Kastler net, see e.g. [A], Thm. 4.14.

For now it will suffice to demand that the *scaling function* β satisfies $\beta(\tau) \longrightarrow 0$ for $\tau \to \pm\infty$.[8] The operator family $\mathcal{A}_\tau$ then already satisfies some properties which are characteristic for creation operators, as might be expected from the close similarity to standard Haag–Ruelle theory.[9] Before proceeding we would like to perform some further standard modifications needed for the multi-particle case, which will lead to improved differentiability and impose restrictions on energy-momentum transfers [see Proposition 3 (iii)].

Remark 1 (uniform differentiability of A_β). By a standard smearing argument, restricting A_β (for fixed β) to the $*$-algebra of smooth operators $\mathfrak{A}_0(\mathcal{O})$, for which $(t, \mathbf{x}) \longmapsto A_\beta(t, \mathbf{x})$ is arbitrarily often differentiable in norm, results in no loss of generality. It is important for our purposes that this smearing argument directly generalizes to yield *uniformly differentiable* families, i.e.

$$\left\| \partial_\alpha A_\beta \right\| \le C_\alpha \left\| A_\beta \right\| \tag{14}$$

for all multi-indices $\alpha \in \mathbb{N}_0^4$ and some β-independent constants C_α. In the following we will therefore assume that all appearing Reeh–Schlieder families A_β are smooth and uniformly differentiable.

Further it will be convenient to have at hand a related operator family with common compact energy-momentum transfers disjoint from a neighbourhood of the origin. To achieve this we have to give up strict localization and smear the family A_β with the Fourier transform of a function $\hat{\chi} \in C_c^\infty(\mathbb{R}^4 \setminus \bar{V}^-)$. We will denote the resulting family of almost-local[10] operators by

$$B_\beta := A_\beta(\chi).$$

With these preparations we can introduce our family of creation operator approximants.

Definition 2 (*Creation operator approximant*). Let $A_\beta \in \mathfrak{A}(\mathcal{O})$ be a uniformly differentiable Reeh–Schlieder family for $\Psi_1 \in E(H_m)\mathscr{H}$, $m \ge 0$. Fixing $\hat{\chi} \in C_c^\infty(\mathbb{R}^4 \setminus \bar{V}^-)$ we set $B_\beta := A_\beta(\chi)$ and for $\tau \in \mathbb{R}$ and a regular positive-energy Klein–Gordon solution f of the same mass m we define *creation-operator approximants* as

$$\mathcal{B}_\tau := \int \mathrm{d}^3x \; f(\tau, \mathbf{x}) B_{\beta(\tau)}(\tau, \mathbf{x}). \tag{15}$$

We will often make use of the fact that $\mathcal{B}_\tau$ are related to the simpler operator family $\mathcal{A}_\tau$ by convolution algebra. Let us collect the most important properties of these families of operators.

Proposition 3 (Basic properties of creation operators). *For an arbitrary operator family $A_\beta \in \mathrm{B}(\mathscr{H})$ define B_β, $\mathcal{A}_\tau$ and $\mathcal{B}_\tau$ as before. Then*

(i) $\mathcal{B}_\tau = \mathcal{A}_\tau(\chi)$.
(ii) $\|\mathcal{B}_\tau\| \le C\,\|\mathcal{A}_\tau\| \le C'(1 + |\tau|^N)\left\| A_{\beta(\tau)} \right\|$ *with suitable constants* $C, C', N > 0$.

[8] For concreteness the reader may take $\beta(\tau) := |\tau|^{-\mu}$, with $\mu > 0$ fixed. We will later see that this is a suitable choice in the context of Reeh–Schlieder families of finite degree.

[9] See e.g. [Ha58,Ru62], [Dy05], or [A] Ch. 5.

[10] See Appendix B.

(iii) *For any closed* $\Delta \subset \mathbb{R}^4$*, we have the* energy-momentum transfer relations

$$B_\beta E(\Delta)\mathscr{H} \subset E(\Delta + \mathrm{supp}\,\hat{\chi})\mathscr{H},$$
$$B_\beta^* E(\Delta)\mathscr{H} \subset E(\Delta - \mathrm{supp}\,\hat{\chi})\mathscr{H}.$$

(iv) *There exists a neighbourhood of zero* $\mathcal{U} \subset \mathbb{R}^4$ *such that* $B_\beta^* E(\mathcal{U}) = 0$.

(v) $B_\beta^* \Omega = 0$.

(vi) *If* $A_\beta \Omega \to \Psi_1 \in E(H_m)\mathscr{H}$ *where* $m \geq 0$ *denotes the mass of* f*, then*

$$\lim_{\tau\to\pm\infty} \mathcal{A}_\tau \Omega = \tilde{f}(\boldsymbol{P})\Psi_1, \quad \textit{and similarly} \quad \lim_{\tau\to\pm\infty} \mathcal{B}_\tau \Omega = \tilde{f}(\boldsymbol{P})\Psi_1', \tag{16}$$

with $\Psi_1' := \lim_{\beta\to 0} B_\beta \Omega = (2\pi)^2 \hat{\chi}(H, \boldsymbol{P})\Psi_1$.

Properties (*iii*)–(*v*) *also hold with* $\mathcal{B}_\tau$ *in place of* B_β *without further modifications.*

Proof. (i) is equivalent to $(\alpha_\tau(A_{\beta(\tau)}(\chi)))(f_\tau) = ((\alpha_\tau(A_{\beta(\tau)}))(f_\tau))(\chi)$, where $f_\tau(\mathbf{x}) := f(\tau, \mathbf{x})$, and this follows from convolution algebra. Property (ii) is a consequence of Hölder's inequality $\|A(f)\| \leq \|A\| \cdot \|f\|_1$ and the standard polynomial bounds for spatial L^1-norms of Klein–Gordon solutions [RS3, Appendix 1 to XI.3]. For the proof of relation (iii) we refer to the literature of Arveson spectral theory—e.g. [Arv80]. To establish (iv), we note that by assumption $-\,\mathrm{supp}\,\hat{\chi}$ is compact and disjoint from the closed set $\bar{V}^+$, so that for a sufficiently small neighbourhood $\mathcal{U}$ of the origin there holds $(\mathcal{U} - \mathrm{supp}\,\hat{\chi}) \cap \bar{V}^+ = \emptyset$. By (iii) and the spectrum condition (HK5) it follows that $B_\beta^* E(\mathcal{U})\mathscr{H} \in E(\mathcal{U} - \mathrm{supp}\,\hat{\chi})\mathscr{H} = \{0\}$. Identity (v) is a direct consequence of (iv), as $\Omega \in E(\mathcal{U})\mathscr{H}$ for any neighbourhood of zero $\mathcal{U}$. The relations for $\mathcal{B}_\tau$ follow by similar argument after using identity (i).

It remains to verify that $\mathcal{A}_\tau$ and $\mathcal{B}_\tau$ provide solutions for the single-particle problem (vi). By spectral calculus we obtain

$$\mathcal{A}_\tau \Omega = \tilde{f}_\tau(\boldsymbol{P})U(\tau)A_{\beta(\tau)}\Omega = \tilde{f}(\boldsymbol{P})\mathrm{e}^{\mathrm{i}(H-\omega_m(\boldsymbol{P}))\tau} A_{\beta(\tau)}\Omega.$$

As Ψ_1 is invariant under the unitaries $V(\tau) := \mathrm{e}^{\mathrm{i}(H-\omega_m(\boldsymbol{P}))\tau}$ we may directly estimate

$$\|\mathcal{A}_\tau \Omega - \tilde{f}(\boldsymbol{P})\Psi_1\| = \|\mathcal{A}_\tau \Omega - \tilde{f}(\boldsymbol{P})V(\tau)\Psi_1\| \leq \|\tilde{f}\|_\infty \left\|A_{\beta(\tau)}\Omega - \Psi_1\right\|.$$

The convergence of $\mathcal{B}_\tau \Omega$ follows then from (i) by writing $\mathcal{B}_\tau \Omega = (2\pi)^2 \hat{\chi}(H, \boldsymbol{P})\mathcal{A}_\tau \Omega$. □

An important consequence of the energy-momentum transfer relation (iii) is the following energy bound. The key point is that the estimate can be made uniform in τ relative to the norm of the underlying Reeh–Schlieder families, as long as we consider the restriction of creation operators to a subspace of bounded energy. Our analysis was somewhat inspired by Herdegen's work [Hrd13], but we rely on different aspects of Buchholz' results [Bu90a] given in Lemma 4.

Lemma 4 ([Bu90a], Lemma 2.2). *Let* $K \subset \mathbb{R}^3$ *compact,* $B \in \mathrm{B}(\mathscr{H})$ *and denote by* P_n *the orthogonal projection onto the intersection of the kernels of the n-fold products of translated operators* $B(\mathbf{x}_1)\ldots B(\mathbf{x}_n)$ *for any configuration of* $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^3$*. Then*

$$\left\| P_n \int_K \mathrm{d}^3x\ (B^*B)(\mathbf{x}) P_n \right\| \leq (n-1) \int_{\Delta K} \mathrm{d}^3x \left\|[B^*, B(\mathbf{x})]\right\|,$$

where integration on the right is over all element-wise differences $\Delta K := K - K$.

Proposition 5 (Energy bounds). *Without further restrictions on the families of operators A_β, $A_{k\beta} \in \mathfrak{A}(\mathcal{O})$, we have for any compact $\Delta \subset \mathbb{R}^4$,*

$$\|\mathcal{B}_\tau E(\Delta)\| \leq C \left\| A_{\beta(\tau)} \right\|, \tag{17}$$

$$\left\| \mathcal{B}_{1\tau_1} \ldots \mathcal{B}_{n\tau_n} E(\Delta) \right\| \leq C \prod_{k=1}^{n} \left\| A_{k\beta(\tau_k)} \right\|, \tag{18}$$

where the constant C depends on Δ, $\mathcal{O}$, $\operatorname{supp} \hat{\chi}$, the number of operators n, and the corresponding wave packets $\tilde{f}$, $\tilde{f}_k$, but it is independent of τ.

Proof. To establish (17), let $\Delta \subset \mathbb{R}^4$ be a given compact set. By a partition argument, we can assume that $\operatorname{supp} \hat{\chi}$ is contained in a compact, convex set disjoint from $\bar{V}^-$. The compact common energy-momentum transfer (cf. Proposition 3 (iii)) of $\mathcal{B}_\tau$ then allows us to write

$$\|\mathcal{B}_\tau E(\Delta)\| = \left\| E(\Delta + \operatorname{supp} \hat{\chi}) \mathcal{B}_\tau E(\Delta) \right\| \leq \left\| E(\Delta') \mathcal{B}_\tau \right\| = \left\| \mathcal{B}_\tau^* E(\Delta') \right\|,$$

where $\Delta' := \Delta + \operatorname{supp} \hat{\chi}$ is compact as well.

To make the connection with Lemma 4, we note that by iterated application of Proposition 3 (iii) and translation-invariance of finite-energy subspaces, we obtain

$$B_\beta^*(\mathbf{x}_1) \ldots B_\beta^*(\mathbf{x}_n) E(\Delta) \mathscr{H} \subset E(\Delta - \Sigma_n \operatorname{supp} \hat{\chi}) \mathscr{H},$$

where $\Sigma_n \operatorname{supp} \hat{\chi} := \{y_1 + \cdots + y_n : y_k \in \operatorname{supp} \hat{\chi}\} = n \operatorname{supp} \hat{\chi}$ due to convexity. By the Hyperplane Separation Theorem, we obtain $(\Delta' - \Sigma_n \operatorname{supp} \hat{\chi}) \cap \bar{V}^+ = \emptyset$ for sufficiently large $n \in \mathbb{N}$. This implies via the spectrum condition (HK5) that for such n, the projections P_n appearing in Lemma 4 may be estimated from below by $E(\Delta')\mathscr{H} \subset P_n \mathscr{H}$. With these preparations we can estimate

$$\begin{aligned}
\left\| \mathcal{B}_\tau^* E(\Delta') \right\| \leq \left\| \mathcal{B}_\tau^* P_n \right\| &\leq \sup_{\substack{\Psi \in \mathscr{H} \\ \|\Psi\|=1}} \int d^3x \, |f(\tau, \mathbf{x})| \left\| B_{\beta(\tau)}^*(\tau, \mathbf{x}) P_n \Psi \right\| \\
&\leq \left(\int d^3x \, |f(\tau, \mathbf{x})|^2 \right)^{1/2} \left(\sup_{\substack{\Psi \in \mathscr{H} \\ \|\Psi\|=1}} \int d^3x \left\| B_{\beta(\tau)}^*(\tau, \mathbf{x}) P_n \Psi \right\|^2 \right)^{1/2}.
\end{aligned}$$

The first factor is constant by the Plancherel identity (cf. Prop. 12 (iv)). For estimating the second factor we choose an arbitrarily large compact region $K \subset \mathbb{R}^3$ and obtain from Lemma 4 that

$$\begin{aligned}
\sup_{\substack{\Psi \in \mathscr{H} \\ \|\Psi\|=1}} \int_K d^3x \left\| B_{\beta(\tau)}^*(\tau, \mathbf{x}) P_n \Psi \right\|^2 &= \sup_{\substack{\Psi \in \mathscr{H} \\ \|\Psi\|=1}} \left\langle \Psi, P_n \int_K d^3x \, (B_{\beta(\tau)} B_{\beta(\tau)}^*)(\tau, \mathbf{x}) P_n \Psi \right\rangle \\
&= \left\| P_n \int_K d^3x \, (B_{\beta(\tau)} B_{\beta(\tau)}^*)(\tau, \mathbf{x}) P_n \right\| \\
&\leq (n-1) \int_{\Delta K} d^3x \left\| \left[B_{\beta(\tau)}, B_{\beta(\tau)}^*(\mathbf{x}) \right] \right\|.
\end{aligned}$$

The family B_β and its adjoint are uniformly almost-local (as defined in Appendix B), so that the remaining integral can be estimated by $2C_\chi \|A_{\beta(\tau)}\|^2 \cdot d^3$, where d depends only on the size of the localization region of A_β. This yields a bound which is uniform in ΔK and by taking $K \nearrow \mathbb{R}^3$ we obtain the energy bound for a single operator.

Then the bound (18) on multiple creation operators follows directly by induction: the compact common energy-momentum transfer of the family $\mathcal{B}_{k\tau}$ yields

$$\begin{aligned}\left\|\mathcal{B}_{1\tau_1}\ldots\mathcal{B}_{n\tau_n}E(\Delta)\right\| &= \left\|\mathcal{B}_{1\tau_1}\ldots\mathcal{B}_{n-1\,\tau_{n-1}}E(\Delta+\text{supp}\,\hat{\chi})\mathcal{B}_{n\tau_n}E(\Delta)\right\| \\ &\leq \left\|\mathcal{B}_{1\tau_1}\ldots\mathcal{B}_{n-1\,\tau_{n-1}}E(\Delta+\text{supp}\,\hat{\chi})\right\|\cdot\left\|\mathcal{B}_{n\tau_n}E(\Delta)\right\| \\ &\leq C^{(n-1)}_{\Delta+\text{supp}\,\hat{\chi}}\left(\prod_{k=1}^{n-1}\left\|A_{k\beta(\tau_k)}\right\|\right)\cdot C_\Delta\left\|A_{n\beta(\tau_n)}\right\|.\end{aligned}$$

□

4. Geometry of Non-equal Time Commutators

The goal of this section is to study the decay behaviour of commutators $\left[\mathcal{B}_{1\tau_1},\mathcal{B}_{2\tau_2}\right]$ for distinct asymptotic parameters $\tau_1 \neq \tau_2$. The strongest known decay estimates for equal times $\tau_1 = \tau_2$ have been established for the case, where the defining Klein–Gordon solutions f_1, f_2 have disjoint support in momentum space [Hep65]. This corresponds to the physically reasonable assumption that the two particles will separate at large times. We will restrict our analysis to this setting and begin by reviewing required results on regular Klein–Gordon solutions $f : \mathbb{R}^4 \longrightarrow \mathbb{C}$ with mass $m \geq 0$, as defined in (12).

The geometry of the asymptotic behaviour of f can be intuitively understood in terms of the set of velocities corresponding to the momenta $\mathbf{k} \in \text{supp}\,\tilde{f}$. Accordingly we define the *velocity support* of f by $\boldsymbol{\Gamma}_{\tilde{f}} := \{\mathbf{k}/\omega_m(\mathbf{k}) \in \mathbb{R}^3 : \ \mathbf{k} \in \text{supp}\,\tilde{f}\}$. Let us recall how this definition allows for a compact formulation of the classical result of Ruelle [Ru62] on the decay of Klein–Gordon solutions outside the velocity-support cone. We provide a unified treatment of the massive and massless case.

Lemma 6 (Velocity-support estimate). *Let f be a regular solution of the Klein–Gordon equation with mass $m \geq 0$. The following estimate holds for any $N \in \mathbb{N}$ with suitable constants $C_N > 0$ and any $(t, \mathbf{x}) \in \mathbb{R}^4$ satisfying $\mathbf{x}/t \notin \boldsymbol{\Gamma}_{\tilde{f}}$,*

$$|f(t,\mathbf{x})| \leq \frac{C_N}{\delta^N\,|t|^N},$$

where δ denotes the distance of $\mathbf{x}/t$ from the set $\boldsymbol{\Gamma}_{\tilde{f}}$.

For regular massive Klein–Gordon solutions, geometrical propagation properties such as the above can be found in various textbooks, e.g. [A] Thm. 5.3. We will skip the standard proof, which makes use of the non-stationary phase method (see e.g. [RS3], Appendix 1 to XI.3). Lemma 6 is applicable in particular in the case of $\mathbf{x}/t$ approaching the velocity support $\boldsymbol{\Gamma}_{\tilde{f}}$. This will be needed later in Proposition 12 to establish certain norm estimates in the massless case.

For the purpose of rapid decay of commutators, it is actually sufficient to make use of Lemma 6 in some fixed neighbourhood $\mathbf{U} \supset \boldsymbol{\Gamma}_{\tilde{f}}$. One obtains the following simple rapid-decay estimate with respect to time *and* space outside a corresponding enlarged neighbourhood of the cone generated by the velocity support.

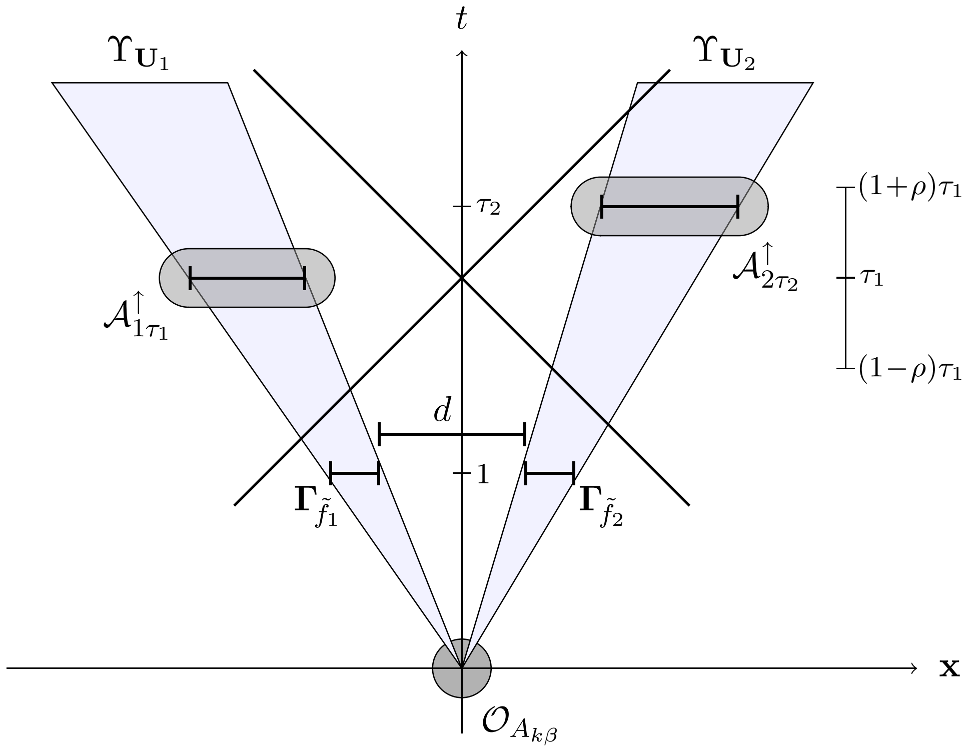

Fig. 1. Localization regions of asymptotically dominant parts $\mathcal{A}^{\uparrow}_{k\tau_k}$ with disjoint velocity supports and $\tau_1 \neq \tau_2$ (schematically; a separating pair of wedges is indicated, restricting $|\tau_2 - \tau_1|$)

Corollary 7. *Let f be a regular solution of the Klein–Gordon equation with mass $m \geq 0$ and let $\mathbf{U} \supset \mathbf{\Gamma}_{\tilde{f}}$ be any (slightly larger) neighbourhood of the velocity support. Then the restriction of f to the complement of the cone*

$$\Upsilon_{\mathbf{U}} := \{(t, t\mathbf{v}) \in \mathbb{R}^4, \mathbf{v} \in \mathbf{U}, t \in \mathbb{R}\}$$

is rapidly decreasing, i.e. for any $N \in \mathbb{N}$ we have

$$|f(t, \mathbf{x})| \leq C_N(1 + |t| + |\mathbf{x}|)^{-N} \qquad \forall\,(t, \mathbf{x}) \in \mathbb{R}^4 \setminus \Upsilon_{\mathbf{U}},$$

with suitable $C_N > 0$ depending on N, $\tilde{f}$, and the distance between $\mathbb{R}^3 \setminus \mathbf{U}$ and $\mathbf{\Gamma}_{\tilde{f}}$.

While our construction of collision states will make use of the creation operators $\mathcal{B}_{k\tau}$, it is clear that additional technical difficulties arise due to the loss of strict locality when passing from localized Reeh–Schlieder families $A_{k\beta} \in \mathfrak{A}(\mathcal{O})$ (with $\mathcal{O}$ independent of β) to the almost-local operators $B_{k\beta} := A_{k\beta}(\chi)$. We recall that the thus obtained compact energy-momentum transfers of $B_{k\beta}$ were essential for establishing energy bounds in Proposition 5.

One strategy to resolve these complications, which makes arguments based on locality particularly transparent, is to first establish corresponding results for the operators $\mathcal{A}_{k\tau}$, as these have better localization properties. Statements which are sufficiently stable under smearing can then be carried over to $\mathcal{B}_{k\tau} = \mathcal{A}_{k\tau}(\chi)$ [see Proposition 3 (i)]. For this reason we want to additionally allow space-time translates $\alpha_x(\mathcal{B}_\tau)$ with $x \in \mathbb{R}^4$ restricted to suitable bounded regions in space-time. We note for clarification that $\alpha_x(\mathcal{A}_{k\,\tau+t})$ differs from $\alpha_{x+(t,0)}(\mathcal{A}_{k\tau})$ due to the time evolution of the Klein–Gordon solution and the underlying time-dependent Reeh–Schlieder family.

The geometrical content of Lemma 8 is illustrated in Fig. 1. Regarding the depicted situation it is clear that in order to obtain rapid decay the allowed translation vectors $x = (x^0, \mathbf{x}) \in \mathbb{R}^4$ will have to be subjected to a similar restriction as the time differences $|\tau_2 - \tau_1|$. In the context of causal distance estimates, it will be convenient to specify this restriction by introducing the norm $|x|_c := |x^0| + |\mathbf{x}|$, where $|\mathbf{x}| := \sqrt{\mathbf{x}^2}$ denotes the Euclidean length of $\mathbf{x} \in \mathbb{R}^3$. The centered open balls generated by this norm are the familiar double cones $\mathscr{C}_R = \{x \in \mathbb{R}^4 : |x|_c < R\}$ with radius $R > 0$.

Lemma 8. *There exists a constant $C > 0$, such that for any f_1, f_2 with velocity supports separated by a positive distance $d > 0$, the following estimate holds for any $N \in \mathbb{N}$, $x \in \mathbb{R}^4$ and $\tau_1, \tau_2 \in \mathbb{R}$ satisfying $|x|_c + |\tau_2 - \tau_1| \leq C d^2 \cdot \tau_{\min}$,*

$$\left\| [\mathcal{A}_{1\tau_1}, \alpha_x(\mathcal{A}_{2\tau_2})] \right\| \leq C_N \left\| A_{1\beta(\tau_1)} \right\| \left\| A_{2\beta(\tau_2)} \right\| \cdot (1 + \tau_{\min})^{-N}. \tag{19}$$

Here, $\tau_{\min} := \min(|\tau_1|, |\tau_2|)$ and the constants C_N depend only on N, f_k and the size of the localization regions of $A_{k\beta}$.

Proof. We can assume without restriction that $\tau_{\min} = |\tau_1|$. Further it is enough to establish (19) for $|\tau_1|$ sufficiently large,[11] and for this case we will make use of a suitable common asymptotic decomposition of the Klein–Gordon solutions f_k. By definition, the corresponding velocity supports $\mathbf{\Gamma}_{\tilde{f}_1}$ and $\mathbf{\Gamma}_{\tilde{f}_2}$ are closed subsets of the closed unit ball. Aiming at the application of Corollary 7, it is clear that we can find neighbourhoods $\mathbf{U}_1$ and $\mathbf{U}_2$ of the velocity supports $\mathbf{\Gamma}_{\tilde{f}_1}$ and $\mathbf{\Gamma}_{\tilde{f}_2}$, which are separated by a distance of at least $d/2$ and which are contained in some fixed larger ball. For concreteness we may assume without loss of generality that $\mathbf{v} \in \mathbf{U}_{1/2}$ always satisfy[12] $|\mathbf{v}| \leq 2$.

Denoting by $\mathbb{1}_{\Upsilon_{\mathbf{U}_k}}$ the characteristic function of the cone $\Upsilon_{\mathbf{U}_k}$ (as defined in Corollary 7) we introduce the following decompositions into *asymptotically dominant* and *negligible parts*,

$$f_k = f_k^{\uparrow} + f_k^{\downarrow}, \quad f_k^{\uparrow}(x) := f_k(x) \cdot \mathbb{1}_{\Upsilon_{\mathbf{U}_k}}(x),$$

and similarly $\mathcal{A}_{k\tau} = \mathcal{A}_{k\tau}^{\uparrow} + \mathcal{A}_{k\tau}^{\downarrow}$, $(k = 1, 2)$, denote the induced decompositions of creation operators. By Corollary 7, we obtain

$$\left\| \mathcal{A}_{k\tau_k} - \mathcal{A}_{k\tau_k}^{\uparrow} \right\| = \left\| \mathcal{A}_{k\tau_k}^{\downarrow} \right\| \leq C'_N \left\| A_{k\beta(\tau_k)} \right\| \cdot (1 + |\tau_k|)^{-N}.$$

This implies that it is sufficient to analyse the commutator of the dominant parts as can be seen from the following estimate, which holds uniformly in $x \in \mathbb{R}^4$,

$$\left\| [\mathcal{A}_{1\tau_1}, \mathcal{A}_{2\tau_2}(x)] \right\| \leq \left\| [\mathcal{A}_{1\tau_1}^{\uparrow}, \mathcal{A}_{2\tau_2}^{\uparrow}(x)] \right\| + C''_N \left\| A_{1\beta(\tau_1)} \right\| \left\| A_{2\beta(\tau_2)} \right\| \cdot (1 + \tau_{\min})^{-N}.$$

We will now verify that the commutator of the dominant parts vanishes for sufficiently large τ_1 in the claimed region of x and τ_k. As a standard consequence of the Haag–Kastler axioms we obtain

$$\mathcal{A}_{k\tau_k}^{\uparrow} \in \mathfrak{A}(\mathcal{O}_{k,\tau_k}), \qquad \text{with } \mathcal{O}_{k,\tau_k} := \mathscr{C}_R + \tau_k \cdot (\{1\} \times \mathbf{U}_k),$$

[11] On any bounded interval $|\tau_k| \leq \tau_{\max}$ ($\tau_{\max}$ fixed), we may use Proposition 3 (ii) to obtain $\left\| [\mathcal{A}_{1\tau_1}, \alpha_x(\mathcal{A}_{2\tau_2})] \right\| \leq C_{\tau_{\max}} \left\| A_{1\beta(\tau_1)} \right\| \left\| A_{2\beta(\tau_2)} \right\|$, which is compatible with (19) for sufficiently large C_N.

[12] Such a bound will be important later in the proof. The concrete choice of the constant has no physical significance, but it will influence the magnitude of the proportionality constant C controlling time-differences in the statement of the lemma.

where we picked a sufficiently large radius $R > 0$ such that the double cone $\mathscr{C}_R$ provides a common bounded localization region of the families $A_{k\beta}$. Then we have by covariance $\mathcal{A}^{\uparrow}_{2\tau_2}(x) \in \mathfrak{A}(\mathcal{O}_{2,\tau_2}+x)$. To estimate the causal distance of any two points $y_1 \in \mathcal{O}_{1,\tau_1}$ and $y_2 \in \mathcal{O}_{2,\tau_2}+x$ from the respective support regions, we write them as $y_1 = o_1+\tau_1\cdot(1, \mathbf{v}_1)$, $y_2 = o_2 + \tau_2 \cdot (1, \mathbf{v}_2) + x$, with $o_1, o_2 \in \mathscr{C}_R$ and $\mathbf{v}_k \in \mathbf{U}_k$. We can then see that

$$y_2 - y_1 = [(\tau_2, \tau_2\mathbf{v}_2) - (\tau_1, \tau_1\mathbf{v}_1)] + o_2 + x - o_1.$$

In the end we will impose a suitable restriction on $u := o_2 + x - o_1$ and therefore the space-like separation of y_1 and y_2 needs to be derived from the difference term inside the brackets, which we denote by $w := (\tau_2, \tau_2\mathbf{v}_2) - (\tau_1, \tau_1\mathbf{v}_1)$. We compute

$$\begin{aligned} w^2 &= (\tau_2 - \tau_1)^2 - (\tau_2\mathbf{v}_2 - \tau_1\mathbf{v}_1)^2, \\ |\tau_2\mathbf{v}_2 - \tau_1\mathbf{v}_1| &= |\tau_2\mathbf{v}_2 - \tau_1\mathbf{v}_2 + \tau_1(\mathbf{v}_2 - \mathbf{v}_1)| \\ &\geq -|\tau_1 - \tau_2|\,|\mathbf{v}_2| + |\tau_1|\,|\mathbf{v}_2 - \mathbf{v}_1|\,, \end{aligned} \tag{20}$$

and thus

$$w^2 \leq -\tau_1^2(\mathbf{v}_2 - \mathbf{v}_1)^2 + 2\,|\tau_1 - \tau_2|\,|\tau_1|\,|\mathbf{v}_2 - \mathbf{v}_1|\,|\mathbf{v}_2| + (\tau_1 - \tau_2)^2.$$

We note that by the non-vanishing negative coefficient of the quadratic term, w will become space-like for large enough $|\tau_1|$ if sufficient restrictions are placed on $|\tau_2 - \tau_1|$. By a similar argument also the perturbation of adding u can be controlled, as can be seen from

$$(y_2 - y_1)^2 = w^2 + 2w \cdot u + u^2 \leq w^2 + 2\,|w|_c\,|u|_c + |u|_c^2\,, \tag{21}$$

where we used the Cauchy-Schwarz inequality. Now assume that $|\tau_2 - \tau_1|+|x|_c \leq \bar{\rho}\,|\tau_1|$ for some constant $\bar{\rho} > 0$ (to be determined). Using that our choice of $\mathbf{U}_k$ implies $|\mathbf{v}_k| \leq 2$, $0 < d \leq |\mathbf{v}_2 - \mathbf{v}_1| \leq 4$, we can then further estimate

$$\begin{aligned} w^2 &\leq -d^2\tau_1^2 + (16\bar{\rho} + \bar{\rho}^2)\tau_1^2, \\ |w|_c &\leq 3\,|\tau_2 - \tau_1| + 4\,|\tau_1| \leq (4 + 3\bar{\rho})\,|\tau_1|\,, \\ |u|_c &\leq |o_1|_c + |o_2|_c + |x|_c \leq 2R + \bar{\rho}\,|\tau_1|\,. \end{aligned}$$

To simplify the resulting bound on $(y_2 - y_1)^2$, let us choose firstly $\bar{\rho} \leq 1$ and then subsequently $|\tau_1| \geq 2R/\bar{\rho}$. This allows us to eliminate unimportant scales by writing $|w|_c \leq 7\,|\tau_1|$, $|u|_c \leq 2\bar{\rho}\,|\tau_1|$ and $\bar{\rho}^2 \leq \bar{\rho}$. Then we obtain from (21) that with a suitable numerical constant $C > 0$,

$$(y_2 - y_1)^2 \leq -d^2\tau_1^2 + C^{-1}\bar{\rho}\tau_1^2.$$

This proves that any choice $0 < \bar{\rho} < Cd^2$ (< 1) leads to space-like localization regions of the dominant parts, and so by locality $[\mathcal{A}^{\uparrow}_{1\tau_1}, \mathcal{A}^{\uparrow}_{2\tau_2}(x)] = 0$ for $|\tau_1| > 2R/\bar{\rho}$ under the assumed restriction on τ_2 and x. □

With this technical preparation we can now establish asymptotic commutation of the creation operators $\mathcal{B}_{k\tau}$ with disjoint velocity supports at non-equal times. We can also appreciate now how the power-law scaling $\beta(\tau) = |\tau|^{-\mu}$ (for large enough $|\tau|$), $\mu > 0$, plays a distinguished role: for this choice the norm terms $\|A_{k\beta(\tau)}\|$ can be absorbed due to the rapid decay in Lemma 8. While these commutator estimates may still be improved in a suitably adapted setting, already the results of the next section will impose sharp restrictions on the scaling parameter μ.

Theorem 9 (non-equal-time commutator estimate). *Let $A_{k\beta}$, $(k = 1, 2)$, be Reeh–Schlieder families of finite degree,*[13] *take regular Klein–Gordon solutions f_k with disjoint velocity supports and assume a fixed scaling $\beta(\tau) = |\tau|^{-\mu}$, $\mu > 0$. Setting $\rho := Cd^2/2 \in (0, 1)$ with C, d as in Lemma 8, there exists for any $N \in \mathbb{N}$ a constant $C_N > 0$, such that for arbitrary $\tau \in \mathbb{R}$ and all τ_1, τ_2 from the corresponding interval spanned by τ and $\tau + \rho\tau$,*

$$\left\| [\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}] \right\| \leq C_N (1 + |\tau|)^{-N}.$$

Proof. We have $\mathcal{B}_{k\tau_k} = \mathcal{A}_{k\tau_k}(\chi)$, with $\chi \in \mathscr{S}(\mathbb{R}^4)$ and so we obtain

$$\left\| \left[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2} \right] \right\| \leq \int \mathrm{d}^4x \, \mathrm{d}^4y \;\; |\chi(x)| \, |\chi(y)| \, \left\| \left[\mathcal{A}_{1\tau_1}, \mathcal{A}_{2\tau_2}(y - x) \right] \right\| . \tag{22}$$

We decompose the integral into the region $|x|_c \leq \rho\,|\tau|\,/2$ and its complement, and similarly for the y-integration. As a consequence of our assumptions we have a polynomial bound $\left\| A_{k\beta(\tau)} \right\| \leq |\tau|^{\mu\gamma}$ and restricting to τ_1, τ_2 from the claimed interval we obtain for fixed $x \in \mathbb{R}^4$ that

$$\begin{aligned} \int \mathrm{d}^4y \;\; |\chi(y)| \, \left\| \left[\mathcal{A}_{1\tau_1}, \mathcal{A}_{2\tau_2}(x - y) \right] \right\| &\leq 2 \, \|\chi\|_1 \, \left\| f_{1\tau_1} \right\|_1 \, \left\| f_{2\tau_2} \right\|_1 \, \left\| A_{1\beta(\tau_1)} \right\| \, \left\| A_{2\beta(\tau_2)} \right\| \\ &\leq C \, |\tau|^M , \end{aligned}$$

for some large enough $M > 0$ and the estimate holds uniformly in x. This now implies that the integral of (22) restricted to the outside region $|x|_c \geq \rho\,|\tau|\,/2$ is rapidly decreasing: we can estimate it by a product of the above polynomially bounded function with the rapidly decreasing function obtained by integrating $|\chi(x)|$ over the retracting regions given by $|x|_c \geq \rho\,|\tau|\,/2$. By a similar argument we can assume that also $|y|_c \leq \rho\,|\tau|\,/2$ and so we can write with suitable constants C'_N,

$$\left\| \left[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2} \right] \right\| \leq \frac{C'_N}{1 + |\tau|^N} \;\; + \int\limits_{|x|_c, |y|_c \leq \rho|\tau|/2} \mathrm{d}^4x \, \mathrm{d}^4y \;\; |\chi(x)| \, |\chi(y)| \, \left\| \left[\mathcal{A}_{1\tau_1}, \mathcal{A}_{2\tau_2}(x - y) \right] \right\| .$$

Assuming the given restriction $|\tau_1 - \tau_2| \leq \rho\,|\tau| \;\; (\leq \rho\tau_{\min})$ we obtain $|\tau_2 - \tau_1| + |x - y|_c \leq 2\rho\,|\tau| \leq Cd^2\tau_{\min}$. Therefore Lemma 8 is applicable, which yields

$$\int\limits_{|x|_c, |y|_c \leq \rho|\tau|/2} \mathrm{d}^4x \, \mathrm{d}^4y \;\; |\chi(x)| \, |\chi(y)| \, \left\| \left[\mathcal{A}_{1\tau_1}, \mathcal{A}_{2\tau_2}(x - y) \right] \right\| \leq \frac{C''_{N'} \left\| A_{1\beta(\tau_1)} \right\| \left\| A_{2\beta(\tau_2)} \right\|}{(1 + \tau_{\min})^{N'}} .$$

As $\tau_{\min} \geq |\tau|$ we can proceed similarly as before and choose N' large enough (depending on the desired decay order N, the scaling μ, and $\left\| A_{k\beta(\tau_k)} \right\|$) to compensate for the polynomial growth of $\left\| A_{k\beta(\tau_k)} \right\|$. $\square$

It is clear that the same reasoning applies, if we replace one or more creation operator approximants by their adjoints. For later use in Sect. 8, we also mention the following equal-time result regarding double commutators with one additional creation operator which may have arbitrary velocity support. This follows from Theorem 9 by a well-known decomposition argument.

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[13] For Lemma 9, it is sufficient if the operator families $A_{k\beta}$ are uniformly localized ($A_{k\beta} \in \mathfrak{A}(\mathcal{O})$, with bounded $\mathcal{O}$ independent of β) and have at most polynomial norm growth $\left\| A_{k\beta} \right\| \leq \beta^{-\gamma}$, $(\gamma \geq 0)$.

Corollary 10 (Decay of double commutators). *In the setting of Theorem 9 let $\mathcal{B}_\tau$ be an additional creation operator approximant defined in terms of a regular Klein–Gordon solution f (without restrictions on its velocity support) and an additional Reeh–Schlieder family A_β of finite degree. Then,*

$$\|[[\mathcal{B}_\tau, \mathcal{B}_{1\tau}], \mathcal{B}_{2\tau}]\| \leq C_N(1+|\tau|)^{-N}$$

The same estimate holds if we replace one or more operators by their adjoints.

Proof. By a smooth decomposition of the wave packet $\tilde{f} = \tilde{f}_{1^c} + \tilde{f}_{2^c}$, such that the resulting commutators $[\mathcal{B}_\tau^{k^c}, \mathcal{B}_{k\tau}]$ are both rapidly decreasing in norm, the result follows directly from Theorem 9 and the Jacobi identity. □

The results of this section seem to be somewhat similar in spirit to Theorem 2 (ii) of [Hrd13], although their role in our verification of convergence of scattering states by discretized time sequences is quite different. A similar result can be found in [Du13].

5. Large Space-Like Translations and Clustering

In this section we prove the following clustering property for the operator families $\mathcal{A}_{k\tau}$,

$$\lim_{\tau\to\infty} E_\Omega^\perp[\mathcal{A}_{1\tau}^*, \mathcal{A}_{2\tau}]\Omega = 0, \tag{23}$$

with $E_\Omega := |\Omega\rangle\langle\Omega|$, $E_\Omega^\perp := \mathbb{1} - E_\Omega$, and where in contrast to Sect. 4 no restrictions are imposed on velocity supports. We will require that the scaling $\mu > 0$ has been chosen sufficiently small (depending on the Reeh–Schlieder degrees). Combined with the single-particle convergence established in Proposition 3 (vi), relation (23) implies that also the limit of $[\mathcal{A}_{1\tau}^*, \mathcal{A}_{2\tau}]\Omega$ exists and is proportional to the vacuum. Similarly as in Sect. 4 we will admit some relative translations of the two operators in (23), so that the results can be carried over to the corresponding expressions involving the operators $\mathcal{B}_{k\tau}$ in Sect. 6. These estimates will play a key role for our proof of convergence of scattering states.

Our treatment is chiefly inspired by Section 3 of [Dy05] and corresponding earlier results of Buchholz [Bu77]. We rely similarly on space-like decay of matrix elements of local operators, as established by the well-known Araki–Hepp–Ruelle Theorem. For smooth operators $B \in \mathfrak{A}_0(\mathcal{O})$ a variant of this decay estimate may be conveniently expressed in terms of the norm $\|B\|_{\mathrm{AHR}} := \|B\| + \|\partial_0 B\|$.

Theorem 11 (Araki–Hepp–Ruelle [AHR62]). *Let $A_k \in \mathfrak{A}_0(\mathscr{C}_{R_k})$, $k = 1, 2$. Then for any $|\mathbf{x}| \geq 2(R_1 + R_2)$, we have*

$$\left|\left\langle \Omega, A_1 U(\mathbf{x}) E_\Omega^\perp A_2 \Omega\right\rangle\right| \leq \frac{C_{\mathrm{AHR}}(R_1+R_2)^3}{|\mathbf{x}|^2} \|A_1\|_{\mathrm{AHR}} \|A_2\|_{\mathrm{AHR}}. \tag{24}$$

The constant C_{AHR} is universal, but we note that estimate (24) with its quadratic decay is specific to theories on physical Minkowski space-time $\mathbb{R}^4$.

To establish the clustering estimate (23) we will have to assume that $A_\beta \in \mathfrak{A}_0(\mathcal{O})$ for small enough $\beta > 0$ and that $\|A_\beta\|_{\mathrm{AHR}}$ is not growing too fast. Both assumptions follow from the uniform differentiability property discussed in Remark 1. Further we will make use of the velocity support estimate of Lemma 6 supplemented by well-known globally valid norm estimates for Klein–Gordon solutions, which we collect in Proposition 12.

Proposition 12. *Let f be a regular solution of the Klein–Gordon equation with mass $m \geq 0$ and set $f_\tau(\mathbf{x}) := f(\tau, \mathbf{x})$. Then for any $p \geq 1$ and $0 < \epsilon < 1$ the following estimates hold.*

(i) $|f(t, \mathbf{x})| \leq C_N \epsilon^{-N}(1 + |t| + |\mathbf{x}|)^{-N}$ *for* $|\mathbf{x}| \geq (1 + \epsilon)\,|t|$ *and any* $N \in \mathbb{N}$.
(i_0) If $m = 0$, *then (i) holds also for any* $|\mathbf{x}| \leq (1 - \epsilon)\,|t|$.
(ii) For $m > 0$, *we have* $\|f_\tau\|_\infty \leq C(1 + |\tau|)^{-3/2}$ *everywhere.*
(ii_0) $\|f_\tau\|_\infty \leq C(1 + |\tau|)^{-1}$ *everywhere.*
(iii) For $m > 0$, $\|f_\tau\|_p^p \leq C_p(1 + |\tau|^{\frac{3}{2}\cdot(2-p)})$.
(iii_0) If $m = 0$, *then* $\|f_\tau\|_p^p \leq C_{\epsilon,p}(1 + |\tau|^{2-p+\epsilon})$ *for any* $\epsilon > 0$.
(iv) $\|f_\tau\|_2 = (2\pi)^{-3/2}\,\|\tilde{f}\|_2$ *is constant (even if* $m = 0$*).*

All appearing constants depend on the wave packet of f and norms are taken in $L^p(\mathbb{R}^3)$.

Proof. (i) and (i_0) can be established as consequences of the velocity support estimate of Corollary 7. Note that for $m = 0$ we assumed $\mathbf{0} \notin \operatorname{supp} \tilde{f}$. The global estimates (ii) and (ii_0) are proven e.g. in [RS3], Theorems XI.17 and XI.18. (iii) and (iii_0) with $\epsilon = 1$ follow by decomposing the integration according to the regions of validity of the respective versions of (i), (ii), i.e. for $m = 0$ we may take $I_\tau := \{||\mathbf{x}| - |\tau|| \leq d\,|\tau|\}$ and its complement. The present result for (iii_0) with $0 < \epsilon < 1$ follows by setting $d = d(\tau) = |\tau|^{-\nu}$ for any $0 < \nu < 1$ with $\nu := 1 - \epsilon$ and by making use of Lemma 6. Finally, (iv) is a consequence of the Plancherel identity. □

Lemma 13. *Let the creation-operator approximants $\mathcal{A}_{k\tau}$ be defined in terms of operator families $A_{1\beta}$ and $A_{2\beta}$ which are localized in the standard double cone $\mathscr{C}_R$ ($R > 0$). For any $x_1, x_2 \in \mathbb{R}^4$, we have*

$$\left\| E_\Omega^\perp[\mathcal{A}_{1\tau}(x_1)^*, \mathcal{A}_{2\tau}(x_2)]\Omega \right\|^2 \leq \frac{C(R + |x_2 - x_1|_c)^9}{|\tau|^\kappa} \cdot \left\| A_{1\beta(\tau)} \right\|_{\mathrm{AHR}}^2 \left\| A_{2\beta(\tau)} \right\|_{\mathrm{AHR}}^2, \tag{25}$$

where $|x|_c := |x^0| + |\mathbf{x}|$. Here $\kappa = 3/2$ in the case of $m > 0$ and for $m = 0$ we can choose $\kappa = 1 - \epsilon$ for any $\epsilon > 0$ with C depending on ϵ and the wave packets $\tilde{f}_k$.

Proof. By translation invariance, it is sufficient to establish the estimate for the relative translation by $x := x_2 - x_1$. We may express the norm square as a vacuum expectation value by writing

$$\begin{aligned}
\left\| E_\Omega^\perp[\mathcal{A}_{1\tau}^*, \mathcal{A}_{2\tau}(x)]\Omega \right\|^2 &= \left| \left\langle \Omega, [\mathcal{A}_{2\tau}(x)^*, \mathcal{A}_{1\tau}] E_\Omega^\perp[\mathcal{A}_{1\tau}^*, \mathcal{A}_{2\tau}(x)]\Omega \right\rangle \right| \\
&= \left| \int \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_4\ f_{2\tau}^*(\mathbf{x}_1) f_{1\tau}(\mathbf{x}_2) f_{1\tau}^*(\mathbf{x}_3) f_{2\tau}(\mathbf{x}_4) K(\tau, x, \mathbf{x}_1, \ldots, \mathbf{x}_4) \right| \\
&\leq \int \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_4\, |f_{2\tau}(\mathbf{x}_1)|\,|f_{1\tau}(\mathbf{x}_2)|\,|f_{1\tau}(\mathbf{x}_3)|\,|f_{2\tau}(\mathbf{x}_4)| \cdot |K(\tau, x, \mathbf{x}_1, \ldots, \mathbf{x}_4)|,
\end{aligned} \tag{26}$$

where due to time-translation invariance the matrix element K can be written as

$$K := \left\langle \Omega, [A_{2\beta,x}(\mathbf{x}_1)^*, A_{1\beta}(\mathbf{x}_2)] E_\Omega^\perp[A_{1\beta}(\mathbf{x}_3)^*, A_{2\beta,x}(\mathbf{x}_4)]\Omega \right\rangle.$$

For compact notation, we introduced the abbreviation $A_{2\beta,x} := \alpha_x(A_{2\beta})$ and we suppressed the τ-dependence of $\beta = \beta(\tau)$.

Now we can estimate K by combining its support properties resulting from locality (HK2) with the space-like decay estimates from Theorem 11 in a manner which seems to be originally due to Buchholz [Bu77]. More precisely, by covariance, isotony and the geometry of double cones, the standard double cone $\mathscr{C}_{R_2+|x|_c}$ provides a localization region for the translated operator family $A_{2\beta,x}$. Therefore K can only be non-zero if

$$\begin{aligned} |\mathbf{x}_1 - \mathbf{x}_2| &\leq R_1 + R_2 + |x|_c \\ |\mathbf{x}_3 - \mathbf{x}_4| &\leq R_1 + R_2 + |x|_c \end{aligned} \tag{27}$$

are both satisfied. This (for fixed x) finite restriction on the relative differences $\mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{x}_3 - \mathbf{x}_4$ now allows for successfully estimating the integrand of (26) for large enough relative distance $\mathbf{x}_2 - \mathbf{x}_3$ "across" $E_\Omega^\perp$ by means of Theorem 11.

Restricting the integral (26) to the region subject to the constraints (27), which we shall denote by $M \subset (\mathbb{R}^3)^4$, we find that for points $(\mathbf{x}_1, \dots, \mathbf{x}_4) \in M$, the two appearing commutators can be localized in suitably translated double cones, whose radii can be simultaneously bounded from above by $R' := 2(R_1 + R_2) + |x|_c$, i.e.

$$\begin{aligned} C_1 &:= [A_{2\beta,x}(\mathbf{x}_1)^*, A_{1\beta}(\mathbf{x}_2)] \in \mathfrak{A}(\mathcal{O}_{\mathbf{x}_2}), \quad \mathcal{O}_{\mathbf{x}_2} := \mathscr{C}_{R'} + (0, \mathbf{x}_2), \\ C_2 &:= [A_{1\beta}(\mathbf{x}_3)^*, A_{2\beta,x}(\mathbf{x}_4)] \in \mathfrak{A}(\mathcal{O}_{\mathbf{x}_3}), \quad \mathcal{O}_{\mathbf{x}_3} := \mathscr{C}_{R'} + (0, \mathbf{x}_3). \end{aligned}$$

Note that C_1 and C_2 are both differentiable by the product rule, as a consequence of the assumed differentiability of the families $A_{k\beta}$. To apply Theorem 11 we subdivide M into the region $M_1 := \{(\mathbf{x}_1, \dots, \mathbf{x}_4) \in M : |\mathbf{x}_2 - \mathbf{x}_3| > 2R'\}$ and its complement $M_2 := M \setminus M_1$ and write

$$\left\| E_\Omega^\perp [\mathcal{A}_{1\tau}^*, \mathcal{A}_{2\tau}(x)]\Omega \right\|^2 \leq I_{M_1} + I_{M_2},$$

where I_{M_k} denotes the integration part of (26) over the subregion M_k. On M_1 we have by Theorem 11,

$$|K| \leq \frac{C_{\mathrm{AHR}}(2R')^3}{|\mathbf{x}_2 - \mathbf{x}_3|^2} C_A, \quad C_A := \|C_1\|_{\mathrm{AHR}} \|C_2\|_{\mathrm{AHR}} \leq 4 \left\| A_{1\beta} \right\|_{\mathrm{AHR}}^2 \left\| A_{2\beta} \right\|_{\mathrm{AHR}}^2 .$$

Also note that trivially $|K| \leq C_A$ holds everywhere. Here we made use of

$$\begin{aligned} \|C_2\|_{\mathrm{AHR}} &\leq \left\| [A_1^*, A_2] \right\| + \left\| [\partial_0 A_1^*, A_2] \right\| + \left\| [A_1^*, \partial_0 A_2] \right\| \\ &\leq 2(\left\| A_1^* \right\| \|A_2\| + \left\| \partial_0 A_1^* \right\| \|A_2\| + \left\| A_1^* \right\| \|\partial_0 A_2\|) \\ &\leq 2 \left\| A_1^* \right\|_{\mathrm{AHR}} \|A_2\|_{\mathrm{AHR}} = 2 \|A_1\|_{\mathrm{AHR}} \|A_2\|_{\mathrm{AHR}} , \end{aligned}$$

and similarly for $\|C_1\|_{\mathrm{AHR}}$, where we suppressed dependencies on β, x and $\mathbf{x}_k$. This allows us to estimate

$$
\begin{aligned}
I_{M_1} &= \int_{M_1} \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_4\, |f_{2\tau}(\mathbf{x}_1)|\, |f_{1\tau}(\mathbf{x}_2)|\, |f_{1\tau}(\mathbf{x}_3)|\, |f_{2\tau}(\mathbf{x}_4)| \cdot |K(\tau, x, \mathbf{x}_1, \ldots, \mathbf{x}_4)| \\
&\leq \int_{M_1} \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_4\, |f_{2\tau}(\mathbf{x}_1)|\, |f_{1\tau}(\mathbf{x}_2)|\, |f_{1\tau}(\mathbf{x}_3)|\, |f_{2\tau}(\mathbf{x}_4)|\, \frac{C_{\mathrm{AHR}}(2R')^3}{|\mathbf{x}_2 - \mathbf{x}_3|^2} C_A \\
&= C_{\mathrm{AHR}} C_A (2R')^3 \int_{|\mathbf{x}_2 - \mathbf{x}_3| > 2R'} \mathrm{d}^3x_2 \mathrm{d}^3x_3 \frac{|f_{1\tau}(\mathbf{x}_2)|\, |f_{1\tau}(\mathbf{x}_3)|}{|\mathbf{x}_2 - \mathbf{x}_3|^2} \int_{|\mathbf{x}_1 - \mathbf{x}_2| < R'} \mathrm{d}^3x_1\, |f_{2\tau}(\mathbf{x}_1)| \int_{|\mathbf{x}_3 - \mathbf{x}_4| < R'} \mathrm{d}^3x_4\, |f_{2\tau}(\mathbf{x}_4)| \\
&\leq (2R')^9\, \|f_{2\tau}\|_\infty^2 \cdot C_{\mathrm{AHR}} C_A \cdot \int_{|\mathbf{x}_2 - \mathbf{x}_3| > 2R'} \mathrm{d}^3x_2 \mathrm{d}^3x_3\ \ |f_{1\tau}(\mathbf{x}_2)|\, |f_{1\tau}(\mathbf{x}_3)|\, \frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|^2}.
\end{aligned} \tag{28}
$$

Here and below, all appearing p-norms ($1 \leq p \leq \infty$) are on $L^p(\mathbb{R}^3)$-spaces associated to spatial smearing functions. We proceed by first estimating the d^3x_3 subintegral for fixed $\mathbf{x}_2$ using Cauchy-Schwarz (all integrals below over $\{\mathbf{x}_3 \in \mathbb{R}^3 : |\mathbf{x}_2 - \mathbf{x}_3| > 2R'\}$)

$$
\int \mathrm{d}^3x_3 \frac{|f_{1\tau}(\mathbf{x}_3)|}{|\mathbf{x}_2 - \mathbf{x}_3|^2} \leq \|f_{1\tau}\|_2 \cdot \left(\int \frac{\mathrm{d}^3x_3}{|\mathbf{x}_2 - \mathbf{x}_3|^4} \right)^{1/2} \leq C_{R^{-1}} \|f_{1\tau}\|_2.
$$

Here both terms are uniformly bounded in τ, by the Plancherel identity or explicit computation,[14] respectively.

Plugging this into the remaining d^3x_2-integration in (28), we have now shown that

$$
I_{M_1} \leq C_{\mathrm{AHR}} C_A C_{R^{-1}} (2R')^9\, \|f_{2\tau}\|_\infty^2\, \|f_{1\tau}\|_2\, \|f_{1\tau}\|_1\,.
$$

On M_2 we estimate similarly using $|K| \leq C_A$,

$$
\begin{aligned}
I_{M_2} &\leq \int_{M_2} \mathrm{d}^3x_1 \ldots \mathrm{d}^3x_4\, |f_{2\tau}(\mathbf{x}_1)|\, |f_{1\tau}(\mathbf{x}_2)|\, |f_{1\tau}(\mathbf{x}_3)|\, |f_{2\tau}(\mathbf{x}_4)|\, C_A \\
&\leq C_A (2R')^9\, \|f_{2\tau}\|_\infty^2\, \|f_{1\tau}\|_\infty\, \|f_{1\tau}\|_1\,.
\end{aligned}
$$

The result now follows from Proposition 12. □

6. Consequences of the Clustering Estimate

With the clustering estimate for the operators $\mathcal{A}_{k\tau}$ from Lemma 13 at hand, it is straightforward to prove clustering of the creation operators $\mathcal{B}_{k\tau}$ for Reeh–Schlieder families of finite degree.

Proposition 14. *For uniformly differentiable Reeh–Schlieder families $A_{1\beta}$, $A_{2\beta}$, and regular Klein–Gordon solutions f_1, f_2 of mass $m \geq 0$, we have*

$$
\left\| E_\Omega^\perp \mathcal{B}_{1\tau}^* \mathcal{B}_{2\tau} \Omega \right\| \leq \frac{C}{|\tau|^{\kappa/2}} \left\| A_{1\beta(\tau)} \right\|_{\mathrm{AHR}} \left\| A_{2\beta(\tau)} \right\|_{\mathrm{AHR}}.
$$

Here, C depends on χ, localization regions of $A_{k\beta}$, wave packets, and κ (see Lemma 13).

[14] Performing the second integral in spherical coordinates around $\mathbf{x}_2$ leads to the radial integration beginning at $2R' > R > 0$, which can be estimated uniformly in $|x|_c$ by a finite constant $C_{R^{-1}}$.

Proof. As $\mathcal{B}_{1\tau}^*\Omega = 0$, we can replace the product $\mathcal{B}_{1\tau}^*\mathcal{B}_{2\tau}$ acting on the vacuum by the commutator $[\mathcal{B}_{1\tau}^*, \mathcal{B}_{2\tau}]$. Making use of $\mathcal{B}_{k\tau} = \mathcal{A}_{k\tau}(\chi)$, $\chi \in \mathscr{S}(\mathbb{R}^4)$, and Lemma 13, we obtain

$$\begin{aligned}\left\| E_\Omega^\perp[\mathcal{B}_{1\tau}^*, \mathcal{B}_{2\tau}]\Omega \right\| &\leq \int d^4x_1 d^4x_2 \;\; |\chi(x_1)|\,|\chi(x_2)| \cdot \left\| E_\Omega^\perp[\mathcal{A}_{1\tau}^*(x_1), \mathcal{A}_{2\tau}(x_2)]\Omega \right\| \\ &\leq \int d^4x_1 d^4x_2 \;\; |\chi(x_1)|\,|\chi(x_2)|\, \frac{C'(R+|x_1-x_2|_c)^{9/2}}{|\tau|^{\kappa/2}} \left\| A_{1\beta(\tau)} \right\|_{\mathrm{AHR}} \left\| A_{2\beta(\tau)} \right\|_{\mathrm{AHR}} \\ &= \frac{C}{|\tau|^{\kappa/2}} \left\| A_{1\beta(\tau)} \right\|_{\mathrm{AHR}} \left\| A_{2\beta(\tau)} \right\|_{\mathrm{AHR}} .\end{aligned}$$

□

For Reeh–Schlieder families of finite degree Proposition 14 simplifies further, yielding a constraint for admissible choices of scaling. In the following γ always denotes the (finite) largest appearing degree, i.e. $\|A_{k\beta}\|_{\mathrm{AHR}} \leq \beta^{-\gamma}$ for small enough $\beta > 0$ and all $k = 1, \ldots, n$. From now on we will also adopt the canonical scaling $\beta(\tau) := |\tau|^{-\mu}$, $\mu > 0$.

Corollary 15. *Let the Reeh–Schlieder families $A_{1\beta}$, $A_{2\beta}$ have finite degrees. Under the assumptions of Lemma 14, there exists a $C > 0$ such that for large enough τ we have*

$$\left\| E_\Omega^\perp \mathcal{B}_{1\tau}^* \mathcal{B}_{2\tau}\Omega \right\| \leq C\,|\tau|^{2\gamma\mu-\kappa/2},$$

Consequently for any $0 < \mu < \frac{\kappa}{4\gamma}$ we obtain

$$\lim_{\tau\to\infty} E_\Omega^\perp \mathcal{B}_{1\tau}^* \mathcal{B}_{2\tau}\Omega = 0.$$

Proof. Follows immediately from inserting $\|A_{k\beta(\tau)}\|_{\mathrm{AHR}} \leq C'\beta(\tau)^{-\gamma} = C'\,|\tau|^{\gamma\mu}$ into the estimate of Proposition 14. □

While Corollary 15 will be sufficient to establish the Fock structure of scattering states in Sect. 8, our proof of convergence relies on an extension of this result, which is concerned with the case of multiple creation operators. The resulting Lemma 16 combines energy bounds and clustering estimates in a novel way. It may be considered our main technical result.

Lemma 16 (multi-operator clustering). *For $\tau_1, \ldots, \tau_n \in \mathbb{R}$ denote by $|\tau_{\min}| > 0$ and $|\tau_{\max}|$ the minimum and maximum of absolute values $|\tau_k|$, ($1 \leq k \leq n$), respectively. Then for large enough $\tau_{\min}$,*

$$\left\| E_\Omega^\perp \left(\prod_{k=1}^{n} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) \Omega \right\| \leq C\,|\tau_{\max}|^{2n\gamma\mu} \cdot |\tau_{\min}|^{-\kappa/2}. \tag{29}$$

The constant C is independent of the τ_k, but depends on the number of pairs n, wave packets, Reeh–Schlieder families, and the smearing function χ.

Proof. We will show by induction that

$$\left\| E_\Omega^\perp \left(\prod_{k=1}^{n} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) \Omega \right\| \leq C \sum_{j=1}^{n} \left(\prod_{k=1}^{j-1} \| E(\Delta) \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} E(\Delta) \| \right) \left\| E_\Omega^\perp \mathcal{B}_{j\tau_j}^* \mathcal{B}_{j\tau_j} \Omega \right\| , \tag{30}$$

where $\Delta \subset \mathbb{R}^4$ is a large enough compact set depending on supp $\hat{\chi}$ and the number of pairs $n \in \mathbb{N}$. From this we obtain by applying 2-operator clustering (Corollary 15), the energy bound of Lemma 5, and the finite-degree Reeh–Schlieder estimates that for large enough $|\tau_{\min}|$, (29) holds as claimed. For $n = 1$, statement (30) has been established in Corollary 15. Assuming that (30) holds for $n - 1$ pairs, we write

$$\begin{aligned} \left\| E_\Omega^\perp \left(\prod_{k=1}^{n} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) \Omega \right\| &= \left\| E_\Omega^\perp \left(\prod_{k=1}^{n-1} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) (E_\Omega + E_\Omega^\perp) \mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n} \Omega \right\| \\ &\leq \left\| E_\Omega^\perp \left(\prod_{k=1}^{n-1} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) E_\Omega \mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n} \Omega \right\| \\ &\quad + \left\| E_\Omega^\perp \left(\prod_{k=1}^{n-1} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) E_\Omega^\perp \mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n} \Omega \right\| . \end{aligned} \tag{31}$$

Now we would like to estimate the second term by 2-operator clustering (Corollary 15). Regarding the applicability of energy bounds from Proposition 5, we note that by Proposition 3 (iii), $\mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n}$ has compact energy-momentum transfer $\Delta' :=$ supp $\hat{\chi}$ − supp $\hat{\chi}$. Therefore we can insert an energy-momentum projection onto Δ' (which commutes with $E_\Omega^\perp$) and estimate

$$\begin{aligned} &\left\| E_\Omega^\perp \left(\prod_{k=1}^{n-1} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) E(\Delta') E_\Omega^\perp \mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n} \Omega \right\| \\ &\qquad \leq \left\| E_\Omega^\perp \left(\prod_{k=1}^{n-1} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) E(\Delta') \right\| \cdot \left\| E_\Omega^\perp \mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n} \Omega \right\| \\ &\qquad \leq \left(\prod_{k=1}^{n-1} \| E(\Delta) \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} E(\Delta) \| \right) \cdot \left\| E_\Omega^\perp \mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n} \Omega \right\| , \end{aligned}$$

where we have chosen the compact set $\Delta \subset \mathbb{R}^4$ large enough (depending on n) to contain the sum of the energy-momentum transfers differences Δ' of all creation-annihilation operator pairs. Similarly we estimate the first term in (31) by making use of the one-dimensional nature of the projection E_Ω, and the induction assumption,

$$\begin{aligned} &\left\| E_\Omega^\perp \left(\prod_{k=1}^{n-1} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) E_\Omega \mathcal{B}_{n\tau_n}^* \mathcal{B}_{n\tau_n} \Omega \right\| = \left\| \mathcal{B}_{n\tau_n} \Omega \right\|^2 \cdot \left\| E_\Omega^\perp \left(\prod_{k=1}^{n-1} \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} \right) \Omega \right\| \\ &\qquad \leq C \cdot \sum_{j=1}^{n-1} \left\| E_\Omega^\perp \mathcal{B}_{j\tau_j}^* \mathcal{B}_{j\tau_j} \Omega \right\| \cdot \left(\prod_{k=1}^{j-1} \| E(\Delta) \mathcal{B}_{k\tau_k}^* \mathcal{B}_{k\tau_k} E(\Delta) \| \right) . \end{aligned}$$

Here we also made use of the fact that $\left\|\mathcal{B}_{n\tau_n}\Omega\right\| \leq C$ is uniformly bounded in τ_n by convergence to the corresponding single-particle state [see Proposition 3 (vi)]. Taken together, these two estimates complete the induction step. □

A useful consequence of multi-operator clustering, which will be important for us later, is the boundedness of scattering-state approximants, i.e. vectors resulting from iterated application of creation-operators to the vacuum. In fact, a similar result was used by Buchholz for the collision theory of massless bosons [Bu77]. While the proofs of Buchholz' results can be simplified[15] using methods from harmonic analysis [Bu90a], our construction is based on operator families A_β with diverging norms in the limit $\beta \to 0$. This norm growth will be inherited by energy bounds for creation operators, if they are derived by means of Proposition 5. In the vacuum sector of a local theory however, we can establish uniform estimates on scattering-state approximants by relying on the good behaviour of $A_\beta\Omega$ via the previously established clustering properties, similarly as in [Bu77].

Corollary 17. *Assume disjoint velocity supports. For any scaling* $0 < \mu < \frac{\kappa}{4\gamma(n-1)}$, *there exists a* $C > 0$, *such that for all sufficiently large* $\tau \in \mathbb{R}$ *and all* τ_k *from the corresponding interval spanned by* τ *and* $\tau + \rho\tau$,

$$\left\|\mathcal{B}_{1\tau_1} \dots \mathcal{B}_{n\tau_n}\Omega\right\| \leq C,$$

with ρ *as in Theorem 9 (for* $n = 1$, *any* $\mu \in (0, \infty)$ *is admissible).*

Proof. The proof is by induction on the number of particles n. For $n = 1$, the claim follows by convergence to the corresponding single-particle state as proven in Proposition 3 (vi). For the general case it will be sufficient to establish the claim for large enough $|\tau|$, as can be seen from the simple polynomial estimate of Proposition 3. Let us now assume that the statement holds for n particles. For simplicity we set $\mathcal{B}_k := \mathcal{B}_{k\tau_k}$ and write

$$\begin{aligned}
\|\mathcal{B}_1 \dots \mathcal{B}_{n+1}\Omega\|^2 &= \left\langle \Omega, \mathcal{B}_{n+1}^* \dots \mathcal{B}_1^*\mathcal{B}_1 \dots \mathcal{B}_{n+1}\Omega\right\rangle \\
&= \left\langle \Omega, \mathcal{B}_{n+1}^*\mathcal{B}_{n+1}\mathcal{B}_n^* \dots \mathcal{B}_1^*\mathcal{B}_1 \dots \mathcal{B}_n\Omega\right\rangle \\
&\quad + \left\langle \Omega, \mathcal{B}_{n+1}^*\left[\mathcal{B}_n^* \dots \mathcal{B}_1^*\mathcal{B}_1 \dots \mathcal{B}_n, \mathcal{B}_{n+1}\right]\Omega\right\rangle,
\end{aligned}$$

where the absolute value of the second term is bounded, as it vanishes for $|\tau| \to \infty$ for any choice of scaling by the rapid decay of commutators (Theorems 9). This decay can compensate the norm growth of the creation-operator approximants, which is at most polynomial—even when using the naive estimate of Proposition 3.

Therefore it is sufficient to establish boundedness of the matrix element

$$\begin{aligned}
&\left\langle \Omega, \mathcal{B}_{n+1}^*\mathcal{B}_{n+1}\mathcal{B}_n^* \dots \mathcal{B}_1^*\mathcal{B}_1 \dots \mathcal{B}_n\Omega\right\rangle \\
&= \left\langle \Omega, \mathcal{B}_{n+1}^*\mathcal{B}_{n+1}(E_\Omega + E_\Omega^\perp)\mathcal{B}_n^* \dots \mathcal{B}_1^*\mathcal{B}_1 \dots \mathcal{B}_n\Omega\right\rangle \\
&= \|\mathcal{B}_{n+1}\Omega\|^2\,\|\mathcal{B}_1 \dots \mathcal{B}_n\Omega\|^2 + \left\langle \Omega, \mathcal{B}_{n+1}^*\mathcal{B}_{n+1}E_\Omega^\perp\mathcal{B}_n^* \dots \mathcal{B}_1^*\mathcal{B}_1 \dots \mathcal{B}_n\Omega\right\rangle. \quad (32)
\end{aligned}$$

The first term of (32) provides the dominant contribution in the limit $|\tau| \to \infty$: its two factors are bounded by the induction assumption and the one-particle case. The second term can be written as the sum of $\left\langle \Omega, \mathcal{B}_{n+1}^*\mathcal{B}_{n+1}E_\Omega^\perp\mathcal{B}_n^*\mathcal{B}_n \dots \mathcal{B}_1^*\mathcal{B}_1\Omega\right\rangle$ and further matrix

[15] See e.g. [AD15].

elements involving at least one commutator of operators involving disjoint velocity supports. As before, the latter are rapidly decreasing by Theorem 9. We can conclude the proof by applying the Cauchy-Schwarz inequality to the remaining term

$$\left|\left\langle \Omega, \mathcal{B}_{n+1}^* \mathcal{B}_{n+1} E_\Omega^\perp \mathcal{B}_n^* \mathcal{B}_n \ldots \mathcal{B}_1^* \mathcal{B}_1 \Omega \right\rangle\right| \leq \left\| E_\Omega^\perp \mathcal{B}_{n+1}^* \mathcal{B}_{n+1} \Omega \right\| \cdot \left\| E_\Omega^\perp \mathcal{B}_n^* \mathcal{B}_n \ldots \mathcal{B}_1^* \mathcal{B}_1 \Omega \right\|$$

where both factors vanish in the limit $|\tau| \to \infty$ for any sufficiently small choice of scaling μ by Lemma 16. □

7. Convergence of Scattering State Approximants

For this section we adopt the standing assumptions that $A_{1\beta}, \ldots, A_{n\beta}$ are Reeh–Schlieder families of finite degree and we take $f_1, \ldots, f_n$ to be regular positive-energy Klein–Gordon solutions of the corresponding mass with pairwise disjoint velocity supports.

Theorem 18. *Let the Reeh–Schlieder families $A_{1\beta}, \ldots, A_{n\beta}$ have degrees less than some finite value $\gamma > 0$ and take a scaling exponent $\mu \in (0, \frac{\kappa}{4(n-1)\gamma})$ (κ as in Lemma 13).*

(i) The family $\Psi_\tau := \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega$ is convergent in norm as $\tau \to \pm\infty$.
(ii) The limit is independent of the choice of μ, $A_{k\beta}$ and f_k within the specified restrictions, as long as the associated operators $\mathcal{B}'_{k\tau}$ create on the vacuum the same family of single-particle states $\Psi_k^{(1)} = \lim_{\tau\to\infty} \mathcal{B}_{k\tau}\Omega$.

Avoiding differentiability assumptions on $A_{k\beta}$ with respect to the parameter β, we will proceed by a discrete variant of Cook's method, thereby reducing the convergence of the scattering state approximants Ψ_τ to the convergence of single-particle state approximants $\mathcal{B}_{k\tau}\Omega$. Recall that for Reeh–Schlieder families $A_{k\beta}$, we have quantitative control over the convergence of the single-particle problem by Proposition 3 (vi).

The restrictions on the time differences to obtain rapid decay of commutators in Theorem 9 suggests to consider the restrictions of Ψ_τ to sequences

$$\tau_k = (1+\rho)^k \tau_0, \quad \tau_0 \neq 0 \text{ arbitrary}, \tag{33}$$

and $\rho > 0$ depending on the separation of velocity supports as explained in Theorem 9.

As preparation for proving Theorem 18 we will first show that we can relate the norm of differences $\Psi_{\tau_2} - \Psi_{\tau_1}$ to corresponding single-particle expressions $\left\|\mathcal{B}_{k\tau_2}\Omega - \mathcal{B}_{k\tau_1}\Omega\right\|$ at least *"locally"*, i.e. if we place sufficient restrictions on the differences $|\tau_2 - \tau_1|$. We will give a unified account for proving both parts of Theorem 18 by comparing the scattering state approximants associated to two possibly distinct families of creation operators with comparable velocity supports. Thereto let $A_{k\beta}, A'_{k\beta} \in \mathfrak{A}(\mathcal{O})$ be uniformly differentiable Reeh–Schlieder families of finite degree, and choose regular positive-energy Klein–Gordon solutions $f_1, \ldots, f_n$ and $f'_1, \ldots f'_n$ of mass $m \geq 0$ such that all pairs with $j \neq k$ (including mixed pairs f_j, f'_k) have disjoint velocity supports. We denote the corresponding creation operators by $\mathcal{B}_{k\tau}, \mathcal{B}'_{k\tau}$ and set

$$\Psi_\tau := \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega, \quad \Psi'_\tau := \mathcal{B}'_{1\tau} \ldots \mathcal{B}'_{n\tau}\Omega.$$

Remark 19 (change of scaling). Anticipating also the proof of Theorem 18 (ii), we may also allow the creation operator families $\mathcal{B}_{k\tau}$ and $\mathcal{B}'_{k\tau}$ to be defined using distinct choices of scaling $\beta_k(\tau) := |\tau|^{-\mu_k}$, $\beta'_k(\tau) := |\tau|^{-\mu'_k}$. On the first reading, this detail can safely be ignored, but it is easily seen that the statement and proof of Lemma 20 can even be kept invariant under this generalization if we simply denote the smallest appearing scaling exponent by $\mu := \min\{\mu_k, \mu'_k\ (1 \le k \le n)\} > 0$. The required extensions of Theorem 9, Lemma 16, and Corollary 17 follow directly by similar considerations.

Lemma 20. *Take $\rho > 0$ as given in Theorem 9 (using the smallest value suitable for all disjoint pairs of velocity supports), and choose sufficiently small scaling $\mu > 0$ (cf. Corollary 17). Then there exist constants $C_1, C_2 > 0$, such that for sufficiently large $|\tau| > 0$ and any subsequent choice of τ_1, τ_2 from the interval spanned by τ and $(1+\rho)\tau$, we have*

$$\left\| \Psi_{\tau_2} - \Psi'_{\tau_1} \right\| \le C_1 \sum_{k=1}^{n} \left\| \mathcal{B}_{k\tau_2}\Omega - \mathcal{B}'_{k\tau_1}\Omega \right\| + C_2\, |\tau|^{n\gamma\mu-\kappa/4} .$$

Proof. For $n = 1$ the statement is trivial. For $n \ge 2$ we can estimate telescopically

$$\left\| \Psi_{\tau_2} - \Psi'_{\tau_1} \right\| \le \sum_{k=1}^{n} \left\| \mathcal{B}_{1\tau_2} \dots \mathcal{B}_{k-1\,\tau_2} (\mathcal{B}_{k\tau_2} - \mathcal{B}'_{k\tau_1}) \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'_{n\tau_1}\Omega \right\| .$$

The claim is obtained if the following estimate can be established for each $1 \le k \le n$,

$$\begin{aligned} &\left\| \mathcal{B}_{1\tau_2} \dots \mathcal{B}_{k-1\,\tau_2} (\mathcal{B}_{k\tau_2} - \mathcal{B}'_{k\tau_1}) \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'_{n\tau_1}\Omega \right\|^2 \\ &\quad \le C_1 \left\| \mathcal{B}_{k\tau_2}\Omega - \mathcal{B}'_{k\tau_1}\Omega \right\|^2 + C_2\, |\tau|^{2\gamma n\mu-\kappa/2} . \end{aligned} \tag{34}$$

We will prove this inequality by making use of the rapid decay of restricted non-equal time commutators together with the energy bound and clustering. Introducing the abbreviation $\Delta_\tau \mathcal{B}_k := \mathcal{B}_{k\tau_2} - \mathcal{B}'_{k\tau_1}$, we can write

$$\begin{aligned} &\left\| \mathcal{B}_{1\tau_2} \dots \mathcal{B}_{k-1\,\tau_2} (\Delta_\tau \mathcal{B}_k) \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'_{n\tau_1}\Omega \right\|^2 \\ &\quad = \left\langle \Omega, \mathcal{B}'^*_{n\tau_1} \dots \mathcal{B}'^*_{k+1\,\tau_1} (\Delta_\tau \mathcal{B}_k)^* \mathcal{B}^*_{k-1\,\tau_2} \dots \mathcal{B}^*_{1\tau_2} \mathcal{B}_{1\tau_2} \dots \mathcal{B}_{k-1\,\tau_2} (\Delta_\tau \mathcal{B}_k) \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'_{n\tau_1}\Omega \right\rangle \\ &\quad \le \left| \left\langle \Omega, \mathcal{B}^*_{1\tau_2} \mathcal{B}_{1\tau_2} \dots \mathcal{B}^*_{k-1\,\tau_2} \mathcal{B}_{k-1\,\tau_2} \cdot \mathcal{B}'^*_{k+1\,\tau_1} \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'^*_{n\tau_1} \mathcal{B}'_{n\tau_1} (\Delta_\tau \mathcal{B}_k)^* (\Delta_\tau \mathcal{B}_k)\Omega \right\rangle \right| \\ &\qquad + C_M\, |\tau|^{-M} , \end{aligned} \tag{35}$$

and the rapidly decreasing error can be subsumed into the C_2-term of (34). To obtain Eq. (35), we made multiple use of the non-equal-time commutator estimate[16] of Lemma 9, which is sufficiently strong for overcompensating to any desired inverse polynomial order the asymptotic growth of the elementary estimate $\|\mathcal{B}_{k\tau}\| \le C_k(1 + |\tau|^{N+\gamma\mu})$ and similar estimates for adjoints and primed operators (see Proposition 3).

The remaining term in (35) still contains the asymptotically dominant contribution, which we will now extract using the clustering estimate. Inserting an identity operator $(E_\Omega + E_\Omega^\perp)$ after $(\Delta_\tau \mathcal{B}_k^*)(\Delta_\tau \mathcal{B}_k)\Omega$ and making use of subadditivity and decay of commutators yields

$$\begin{aligned} &\left| \left\langle \Omega, \mathcal{B}^*_{1\tau_2} \mathcal{B}_{1\tau_2} \dots \mathcal{B}^*_{k-1\,\tau_2} \mathcal{B}_{k-1\,\tau_2} \cdot \mathcal{B}'^*_{k+1\,\tau_1} \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'^*_{n\tau_1} \mathcal{B}'_{n\tau_1} (\Delta_\tau \mathcal{B}_k^*) (\Delta_\tau \mathcal{B}_k)\Omega \right\rangle \right| \\ &\quad \le \| E_\Omega^\perp \mathcal{B}'^*_{n\tau_1} \mathcal{B}'_{n\tau_1} \dots \mathcal{B}'^*_{k+1\,\tau_1} \mathcal{B}'_{k+1\,\tau_1} \cdot \mathcal{B}^*_{k-1\,\tau_2} \mathcal{B}_{k-1\,\tau_2} \dots \mathcal{B}^*_{1\tau_2} \mathcal{B}_{1\tau_2}\Omega \| \\ &\qquad \cdot \| (\Delta_\tau \mathcal{B}_k^*) (\Delta_\tau \mathcal{B}_k)\Omega \| \\ &\qquad\quad + \| \mathcal{B}_{1\tau_2} \dots \mathcal{B}_{k-1\,\tau_2} \cdot \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'_{n\tau_1}\Omega \|^2 \cdot \| (\Delta_\tau \mathcal{B}_k)\Omega \|^2 + C_M\, |\tau|^{-M} . \end{aligned}$$

[16] For the status of Theorem 9 in the context of non-equal scaling, cf. Remark 19 and Footnote 13.

Both terms depend on the convergence speed of the single-particle problem, although—anticipating the results of Sect. 8—we expect the second summand to be dominant for large τ: By boundedness of scattering state approximants (Corollary 17)

$$\left\| \mathcal{B}_{1\tau_2} \dots \mathcal{B}_{k-1\,\tau_2} \cdot \mathcal{B}'_{k+1\,\tau_1} \dots \mathcal{B}'_{n\tau_1} \Omega \right\|^2 \leq C_1$$

for suitable $C_1 > 0$. It remains to be shown that the first summand has the same asymptotics as the C_2-term of (34). By the clustering result with multiple pairs of creation- and annihilation-operator approximants of Lemma 16, we obtain that

$$\left\| E_\Omega^\perp \mathcal{B}'^*_{n\tau_1} \mathcal{B}'_{n\tau_1} \dots \mathcal{B}'^*_{k+1\,\tau_1} \mathcal{B}'_{k+1\,\tau_1} \cdot \mathcal{B}^*_{k-1\,\tau_2} \mathcal{B}_{k-1\,\tau_2} \dots \mathcal{B}^*_{1\tau_2} \mathcal{B}_{1\tau_2} \Omega \right\| \leq C_2 \, |\tau|^{2(n-1)\gamma\mu - \kappa/2} ,$$

which also made use of the time restriction yielding $|\tau| \leq |\tau_k| \leq (1+\rho)\,|\tau|$. The second factor is estimated making use of the energy bound,

$$\begin{aligned} \left\| (\Delta_\tau \mathcal{B}_k^*)(\Delta_\tau \mathcal{B}_k)\Omega \right\| &= \left\| (\Delta_\tau \mathcal{B}_k^*) E(\Delta)(\Delta_\tau \mathcal{B}_k)\Omega \right\| \\ &\leq \left\| (\Delta_\tau \mathcal{B}_k^*) E(\Delta) \right\| \cdot \| (\Delta_\tau \mathcal{B}_k)\Omega \| \\ &\leq C_3 \, |\tau_2|^{\gamma\mu} \leq C_3 (1+\rho)^{\gamma\mu} \, |\tau|^{\gamma\mu} =: C_3' \, |\tau|^{\gamma\mu} , \end{aligned}$$

where the energy-momentum projection onto the compact set $\Delta := \operatorname{supp} \hat{\chi}$ can be inserted due to $\Delta_\tau \mathcal{B}_k \Omega \in E(\Delta)\mathscr{H}$. Altogether we obtain (34), completing the proof. □

The convergence of scattering state approximants Ψ_τ is now easily established by iterated application of Lemma 20.

Proof of Theorem 18. Ad (i). We estimate by writing a telescopic sum and making use of subadditivity of the norm,

$$\left\| \Psi_{\tau_L} - \Psi_{\tau_0} \right\| \leq \sum_{k=1}^{L} \left\| \Psi_{\tau_k} - \Psi_{\tau_{k-1}} \right\| .$$

We have by construction that τ_k, τ_{k-1} are contained in the interval spanned by τ_{k-1} and $(1+\rho)\tau_{k-1}$. Thus Lemma 20 is applicable with $\mathcal{B}_{k\tau} = \mathcal{B}'_{k\tau}$. Fixing the scaling parameter $\mu > 0$ such that $\delta := \kappa/4 - n\gamma\mu > 0$, all assumptions of Lemma 20 are satisfied and we obtain

$$\left\| \Psi_{\tau_L} - \Psi_{\tau_0} \right\| \leq \sum_{k=1}^{L} \left(C_1 \sum_{j=1}^{n} \left\| \mathcal{B}_{j\tau_k} \Omega - \mathcal{B}_{j\tau_{k-1}} \Omega \right\| + C_2 \, |\tau_{k-1}|^{-\delta} \right) . \tag{36}$$

Now, the single-particle convergence property of the Reeh–Schlieder families implies

$$\left\| \mathcal{B}_{j\tau_k} \Omega - \mathcal{B}_{j\tau_{k-1}} \Omega \right\| \leq \left\| \mathcal{B}_{j\tau_k} \Omega - \Psi_j^{(1)} \right\| + \left\| \Psi_j^{(1)} - \mathcal{B}_{j\tau_{k-1}} \Omega \right\| \leq C \, |\tau_{k-1}|^{-\mu} ,$$

where $\Psi_j^{(1)} = \lim_{\tau \to \pm\infty} \mathcal{B}_{j\tau} \Omega$. Applying this estimate and inserting $\tau_k = (1+\rho)^k \tau_0$, we can take care of both terms in (36) by writing

$$\left\| \Psi_{\tau_L} - \Psi_{\tau_0} \right\| \leq C' \sum_{k=1}^{L} |\tau_{k-1}|^{-\mu'} = C' \, |\tau_0|^{-\mu'} \cdot \sum_{k=1}^{L} (1+\rho)^{-\mu'(k-1)}, \tag{37}$$

with $\mu' := \min(\mu, \delta)$. Clearly, the geometric series is convergent for $L \to \infty$. Independence of the limit from the choice of the sequence τ_k, i.e. convergence of Ψ_τ as a function of the continuous parameter τ, may be inferred from a second invocation of Lemma 20 or directly from (37).

Ad (ii). This is another direct consequence of Lemma 20, which implies for equal times but distinct creation operators, with possibly distinct choices of scaling in the allowed region, that

$$\left\| \Psi_\tau - \Psi'_\tau \right\| \leq C_1 \sum_{j=1}^{n} \left\| \mathcal{B}_{j\tau}\Omega - \mathcal{B}'_{j\tau}\Omega \right\| + C_2 \, |\tau|^{-\delta} ,$$

where as before $\delta := \kappa/4 - n\gamma\mu > 0$. If $\lim_\tau \mathcal{B}_{j\tau}\Omega = \lim_\tau \mathcal{B}'_{j\tau}\Omega$, we obtain that the limits of Ψ_τ and Ψ'_τ coincide and that they are invariant under changes of scaling as claimed. $\square$

8. Fock Structure of Scattering States

Finally, we want to establish the Fock structure of scattering states, which provides a simple formula for computing scalar products of any two scattering states in terms of their single-particle components. An important consequence is the non-vanishing of the limits defining the scattering states and it is the essential ingredient to establish the extension of wave operators to the full asymptotic Fock spaces (cf. [Dy09] App. A). With the clustering relation of creation-operators of Corollary 15 at hand, the arguments leading to the Fock structure of scattering states are well-known and we can not refrain from rephrasing them, e.g. from [Dy05]. We will use the abbreviation $[n] := \{1, 2, \ldots, n\} \subset \mathbb{N}$ for finite subsets of natural numbers and $\mathfrak{S}_n$ denotes the symmetric group of degree n in its defining representation, i.e. acting on $[n]$.

We now consider two scattering state approximants ($n, n' \in \mathbb{N}_0$)

$$\Psi_\tau := \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega, \qquad\qquad \Psi'_\tau := \mathcal{B}'_{1\tau} \ldots \mathcal{B}'_{n'\tau}\Omega,$$

such that $\mathcal{B}_{k\tau}$ and $\mathcal{B}'_{k\tau}$ have disjoint velocity supports within each family. Assuming finite Reeh–Schlieder degrees, the *outgoing* and *incoming scattering states* $\Psi^{\pm} := \lim_{\tau\to\pm\infty} \Psi_\tau$, respectively, are well-defined by Theorem 18 for sufficiently small choices of scaling $\beta(\tau) = |\tau|^{-\mu}$, $\mu > 0$, and similarly for $\Psi'^{\pm} := \lim_{\tau\to\pm\infty} \Psi'_\tau$. We denote the corresponding single-particle states by $\Psi_k^{(1)} := \lim_{\tau\to\infty} \mathcal{B}_{k\tau}\Omega$, $(1 \leq k \leq n)$ and $\Psi'^{(1)}_{k'} := \lim_{\tau\to\infty} \mathcal{B}'_{k'\tau}\Omega$, $(1 \leq k' \leq n')$.

Theorem 21 (Fock structure). *The scalar products of any two outgoing scattering states of the above form are given by*[17]

$$\left\langle \Psi^+, \Psi'^+ \right\rangle = \delta_{nn'} \sum_{\pi\in\mathfrak{S}_n} \prod_{k=1}^{n} \left\langle \Psi_k^{(1)}, \Psi'^{(1)}_{\pi(k)} \right\rangle, \tag{38}$$

and similarly for incoming states.

[17] As usual, the right-hand side of (38) is consistently interpreted for $n > n'$, yielding vanishing scalar products also in this case (as a consequence of the vanishing Kronecker delta $\delta_{nn'}$).

Proof. For simplicity we treat only the outgoing case $\tau \to +\infty$. By continuity of the scalar product, the left-hand side of (38) can be written as the limit $\tau \to \infty$ of

$$\langle \Psi_\tau, \Psi'_\tau \rangle = \langle \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega, \mathcal{B}'_{1\tau} \ldots \mathcal{B}'_{n'\tau}\Omega \rangle, \tag{39}$$

where we can assume identical scaling $\mu > 0$ for both sides by Theorem 18 (ii). Now we perform induction with respect to the number of particles n' (assuming without restriction that $n' \geq n$). For each n' and $n = 0$, statement (38) is equivalent to $\|\Omega\| = 1$ for $n = 0$ and $\langle \Psi'^+, \Omega \rangle = 0$ for $n' > 0$. The latter follows from eq. (39) and the spectral support argument of Proposition 3 (v).

Assuming now that (38) holds for $n - 1$ particles, one can show by means of Corollary 10 and Corollary 15 that, up to terms vanishing for $|\tau| \to \infty$, (39) equals

$$\sum_{k=1}^{n'} \langle \Omega, \mathcal{B}^*_{n\tau} \ldots \mathcal{B}^*_{2\tau} \mathcal{B}'_{1\tau} \ldots \mathcal{B}'_{k-1\,\tau} \mathcal{B}'_{k+1\,\tau} \ldots \mathcal{B}'_{n'\tau} E_\Omega \mathcal{B}^*_{1\tau} \mathcal{B}'_{k\tau} \Omega \rangle$$

$$\xrightarrow{\tau\to\infty} \sum_{k=1}^{n'} \left(\left(\delta_{n-1,n'-1} \sum_{\pi \in \mathfrak{S}_{n-1}(1,k)} \prod_{l=2}^{n} \left\langle \Psi_l^{(1)}, \Psi'^{(1)}_{\pi(l)} \right\rangle \right) \cdot \left\langle \Psi_1^{(1)}, \Psi'^{(1)}_k \right\rangle \right),$$

where $\mathfrak{S}_{n-1}(1, k)$ denotes the set of bijective maps π between the two sets of numbers $[n] \setminus \{1\}$ and $[n] \setminus \{k\}$ and convergence is inferred from the induction assumption. Note that while $\mathfrak{S}_{n-1}(1, k)$ is by itself not a group (its elements are maps between different sets and thus cannot be composed), it can nevertheless be identified with the subset of $\pi \in \mathfrak{S}_n$ for which $\pi(1) = k$. This implies that

$$\lim_{\tau\to\infty} \langle \Psi_\tau, \Psi'_\tau \rangle = \delta_{nn'} \sum_{k=1}^{n} \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi(1)=k}} \prod_{l=1}^{n} \left\langle \Psi_l^{(1)}, \Psi'^{(1)}_{\pi(l)} \right\rangle = \delta_{nn'} \sum_{\pi \in \mathfrak{S}_n} \prod_{l=1}^{n} \left\langle \Psi_l^{(1)}, \Psi'^{(1)}_{\pi(l)} \right\rangle.$$

□

9. Conclusions and Outlook

We have established the existence and Fock structure of scattering states corresponding to single-particle states $\Psi_1 \in E(H_m)\mathscr{H}$ with finite Reeh–Schlieder degree. This requires the existence of a family of local operators $(A_\beta)_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$ such that

$$\|A_\beta\Omega - \Psi_1\| \leq \beta, \quad \|A_\beta\| \leq \beta^{-\gamma}. \tag{RS}$$

Beyond (RS) our method has no further dependence on the concrete mechanism (e.g. additional ergodic averaging as in [Dy05]) yielding a limit of $A_\beta\Omega$ in the single-particle space. We have seen that the Haag–Ruelle construction can be adapted, so that any finite degree γ is feasible. Thus an arbitrarily strong polynomial growth of $\|A_\beta\|$ relative to the convergence of $A_\beta\Omega$ to the single-particle vector Ψ_1 can be handled.

As mentioned in the introduction, Assumption (RS) is readily verified in free field theory (cf. also Appendix C). Its status in concrete interacting models or within the general axiomatic framework is beyond the scope of the present work and poses an interesting problem for future research. We will briefly summarize our current understanding regarding the validity of conditions of strengthened Reeh–Schlieder type and also give some additional supporting arguments for our approach to the construction of scattering states. We shall refrain from going into technical details, as we intend to provide them elsewhere.

(a) Quantitative improvements in the construction of scattering states regarding the strength of condition (RS) are possible. Most notably in theories with lower mass gap one can show that already $(A_\beta)_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$,

$$\left\| E(\Delta)(A_\beta\Omega - \Psi_1) \right\| \leq C_\Delta \beta, \qquad \ln \left\| A_\beta \right\| \leq \beta^{-\gamma}, \tag{RS$^\flat$}$$

is sufficient for establishing scattering theory. Here $\Delta \subset \mathbb{R}^4$ is an arbitrary compact set, and $C_\Delta > 0$ does not depend on β. Intuitively, the stronger norm increase in (RS$^\flat$) may be compensated by the exponential space-like clustering in these models.

(b) It was already pointed out that previous constructions of scattering states of embedded (massive) particles commonly need to assume additional regularity of the spectral measure near the particle masses. Here we briefly comment on the relation of such regularity assumptions to conditions of Reeh–Schlieder type. For spectral regularity according to Herbst, one requires that there exist local operators $A \in \mathfrak{A}(\mathcal{O})$ such that in addition to a nonvanishing single-particle component $E_m A\Omega$, one has for a suitable $\epsilon > 0$ and all small enough $\delta > 0$, [Hrb71,Dy05][18]

$$\left\| E(H_m^\delta \setminus H_m)A\Omega \right\| \leq C\delta^\epsilon, \quad \text{where } H_m^\delta := \bigcup_{|\mu - m| < \delta} H_\mu, \tag{H}$$

and that the set of single particle vectors obtained from such operators is dense in the single particle space $E_m \mathscr{H}$.
Starting from an operator $A \in \mathfrak{A}(\mathcal{O})$ as in (H), one can show by a very crude but general construction using differential operators that there exists a dense set of single particle states $\Psi_1 \in E_m\mathscr{H}$, which are generated by operators satisfying (RS$^\flat$), with $\gamma > 0$ inversely proportional to the Herbst constant ϵ from (H). Here we do not even need to invoke the Reeh–Schlieder property—one may make use of the non-local nature of the energy-projection $E(\Delta)$ in condition (RS$^\flat$) to generate single-particle states (even if Δ is larger than a subset of the mass hyperboloid). Improving upon this result appears to require a more detailed quantitative understanding of the non-local correlations implied by the Reeh–Schlieder theorem, which may be model-dependent—cf. also Appendix C.

(c) We restricted our analysis to uniformly localized Reeh–Schlieder families solely for technical convenience. The present method may be refined to admit families $A_\beta \in \mathfrak{A}(\mathscr{C}_{R_\beta})$ similarly as in (RS), but localized in double cones $\mathscr{C}_{R_\beta}$ of polynomially growing radii $R_\beta := \beta^{-N}$ (for some $N > 0$).
A similar delocalization commonly enters in previous approaches via ergodic averaging prescriptions [Hrb71,Dy05,Hrd13,DH14]. Due to the geometrical limitations discussed in Sect. 4, this delocalization appears to necessitate Herbst-type spectral conditions [Hrb71] in these works. Allowing a weakened localization $A_\beta \in \mathfrak{A}(\mathscr{C}_{R_\beta})$ might help to understand the relation of such spectral conditions to the Reeh–Schlieder condition (RS).

A more concrete investigation of (RS) can be carried out using the concept of polarization-free generators [BBS01]. In this setting, we can derive a wedge-local variant of the Reeh–Schlieder condition from the domain condition $\Omega \in \mathrm{D}(T^{1+\epsilon})$ for some $\epsilon > 0$, where $T \geq 0$ denotes the self-adjoint part of the polar decomposition of a suitable polarization-free generator $G = UT$. With this input we can proceed as in free field theory and set $A_\beta := UT\mathrm{e}^{-\beta T^\epsilon}$ to obtain *wedge-local* Reeh–Schlieder

[18] Weakened variants of (H) have also been discussed recently, see e.g. [Hrd13,DH14].

families of degree $\gamma = \epsilon^{-1}$. If a correponding variant of Theorem 11 holds for oppositely localized pairs of such wedge-local operators, as it is the case in purely massive theories [Fre85], our results may be extended to yield a construction of two-particle scattering states for embedded Wigner particles.

In this setting, it is problematic to imitate the Haag–Ruelle construction by directly smearing polarization-free generators G due to the complicated structure of the domains $D(G)$. It has been shown that even ostensibly weak temperateness assumptions with respect to the action of space-time translations on $\mathrm{D}(G)$ imply triviality of scattering in massive theories on Minkowski space with spatial dimension $s > 1$ [BBS01]. Therefore it is a subtle question whether the above domain condition is compatible with non-trivial scattering.[19]

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A. Notation and Conventions

For the Minkowski space-time metric we use the convention $k \cdot x := k^0 x^0 - \mathbf{k} \cdot \mathbf{x}$ for $k, x \in \mathbb{R}^4$. Accordingly, the Fourier transform of a Schwartz functions $f \in \mathscr{S}(\mathbb{R}^4)$ is defined by

$$\hat{f}(k) := \frac{1}{(2\pi)^2} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}k\cdot x} f(x). \tag{40}$$

The wave-packet $\tilde{f}$ of a regular Klein–Gordon solution f (as defined in Section 3), is related to a corresponding partial (spatial) inverse transform of $f_t(\mathbf{x}) := f(t, \mathbf{x})$ at $t = 0$ by a factor $(2\pi)^{3/2}$.
The Fourier transform on the extended space $x = (\mathbf{x}, s)$ and space-time $\underline{x} = (x^0, \mathbf{x}, s)$ (see Appendix C) is defined for $\underline{f} \in \mathscr{S}(\mathbb{R}^5)$ and $\mathbf{f} \in \mathscr{S}(\mathbb{R}^4)$ by

$$\begin{aligned}\hat{\underline{f}}(\omega, \mathbf{k}, \mu) &:= \frac{1}{(2\pi)^{5/2}} \int \mathrm{d}^5x\, \mathrm{e}^{\mathrm{i}\omega x^0 - \mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\mu s}\, \underline{f}(x^0, \mathbf{x}, s),\\ \hat{\mathbf{f}}(\mathbf{k}, \mu) &:= \frac{1}{(2\pi)^2} \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\mu s}\, \mathbf{f}(\mathbf{x}, s).\end{aligned}$$

For $x = (t, \mathbf{x}) \in \mathbb{R}^4$ we write $A(x) := \alpha_x(A) := U(x) A U(x)^*$ and similarly for $\alpha_t(A)$ and $\alpha_{\mathbf{x}}(A)$. By weak integration, these automorphisms of the global algebra induce for given $A \in \mathfrak{A}$ (regular) operator-valued distributions

$$A(f) := \int \mathrm{d}^4x\; f(x) \alpha_x(A), \quad f \in \mathscr{S}(\mathbb{R}^4)$$

and similar distributions $A(g)$ are obtained for spatial smearing with $g \in \mathscr{S}(\mathbb{R}^3)$.

[19] Preliminary computations suggest that $\Omega \in \mathrm{D}(T^{1+\epsilon})$ could be fulfilled in certain 1+1-dimensional integrable models with non-temperate polarization free generators G [CT15] [Yoh Tanimoto, private communications]. A definite assessment requires the construction of a Borchers triple for these models, which has not yet been completed at the time of writing of this work.

B. Uniformly Almost-Local Operator Families

An operator $A \in \mathfrak{A}$ is *almost-local* if there exists for any $r > 0$ a double-cone localized operator $A_r \in \mathfrak{A}(\mathscr{C}_r)$, such that for each $N \in \mathbb{N}$ with a suitable constant C_N we have

$$\|A - A_r\| \leq \frac{C_N}{(1+r)^N}. \tag{41}$$

For certain families $(A_\beta) \subset \mathfrak{A}$ of almost-local operators, the behaviour of corresponding constants $C_{N,\beta}$ in (41) with respect to the parameter $\beta > 0$ can be quantified in a simple manner.

Proposition 22. *Let $A_\beta \in \mathfrak{A}(\mathcal{O})$ ($\beta > 0$) be an operator family localized in a fixed bounded region $\mathcal{O} \subset \mathbb{R}^3$ and let $\chi \in \mathscr{S}(\mathbb{R}^4)$. Then the family of almost-local operators $B_\beta := A_\beta(\chi)$ is* uniformly almost-local *relative to $\|A_\beta\|$ in the following sense: for each $\beta > 0$ there are $B_{\beta,r} \in \mathfrak{A}(\mathscr{C}_r)$ ($r > 0$), such that for all $N \in \mathbb{N}$*

$$\exists C_N > 0 \; \forall \beta > 0 : \left\|B_\beta - B_{\beta,r}\right\| \leq \frac{C_N \left\|A_\beta\right\|}{1+r^N}. \tag{42}$$

Notably, the constants C_N are uniform in β. This also implies

$$\int \mathrm{d}^3x \left\|\left[B_\beta, B_\beta^*(\mathbf{x})\right]\right\| \leq C_{\chi,\mathcal{O}} \left\|A_\beta\right\|^2. \tag{43}$$

Proof. Let us assume for concreteness that $A_\beta \in \mathfrak{A}(\mathscr{C}_R)$ with the double-cone radius $R > 0$ fixed. As $\chi \in \mathscr{S}(\mathbb{R}^4)$, we obtain natural candidates for approximating local operators

$$B_{\beta,r} := \int\limits_{|x|_c < r-R} \mathrm{d}^4x \; \chi(x) A_\beta(x) \in \mathfrak{A}(\mathscr{C}_r)$$

(for $r \leq R$ we simply set $B_{\beta,r} = 0$). By the rapid decay of χ, we get for $r > 2R$,

$$\left\|B_\beta - B_{\beta,r}\right\| \leq \left\|A_\beta\right\| \cdot \int\limits_{|x|_c \geq r-R} \mathrm{d}^4x \;\; |\chi(x)| \leq \frac{C_N \left\|A_\beta\right\|}{1+(r-R)^N} \leq \frac{C'_{N,R} \left\|A_\beta\right\|}{1+r^N}.$$

Together with the trivial estimate $\left\|B_\beta\right\| \leq \left\|A_\beta\right\| \|\chi\|_1$ for $r \leq 2R$, this implies (42). To obtain (43) we use an $|\mathbf{x}|$-dependent local approximation $B_{\beta,r}$ under the integral: choosing $r = r(\mathbf{x}) := |\mathbf{x}|/2$ the commutator $[B_{\beta,r(\mathbf{x})}, B^*_{\beta,r(\mathbf{x})}(\mathbf{x})]$ will vanish by locality and thereby we have reduced the integrand to terms proportional to the approximation error. More explicitly we rewrite the left-hand side as

$$\int \mathrm{d}^3x \left\|\left[(B_\beta - B_{\beta,r(\mathbf{x})}) + B_{\beta,r(\mathbf{x})}, (B^*_\beta(\mathbf{x}) - B^*_{\beta,r(\mathbf{x})}(\mathbf{x})) + B^*_{\beta,r(\mathbf{x})}(\mathbf{x})\right]\right\|.$$

After expanding the commutator (preserving the two differences in brackets) and utilizing subadditivity, $\|[B_{\beta,r(\mathbf{x})}, B^*_{\beta,r(\mathbf{x})}(\mathbf{x})]\|$ vanishes for all $\mathbf{x}$ by construction (due to locality). All remaining terms will contain at least one difference $B_\beta - B_{\beta,r(\mathbf{x})}$ or its translate. Using (42) we can now directly estimate the integral,

$$\left\|\left[B_\beta - B_{\beta,r(\mathbf{x})}, B^*_{\beta,r(\mathbf{x})}(\mathbf{x})\right]\right\| \leq 2\left\|B_\beta - B_{\beta,r(\mathbf{x})}\right\| \left\|B^*_{\beta,r(\mathbf{x})}(\mathbf{x})\right\| \leq \frac{2C_N \left\|A_\beta\right\|^2}{1+r^N}.$$

Taking N sufficiently large we obtain convergence of the integral and (43). □

C. Reeh–Schlieder Families in Generalized Free Models

Let us briefly discuss the status of condition (RS) for noninteracting theories with embedded mass shell. Generalized free theories have proven useful to study Herbst-type spectral conditions (H) ([Dy05], Sec. 4, see also [Hor90, Ch. 3.3, esp. p. 264 ff.] for a general review), and we think that the following considerations might also give some hints concerning strengthened Reeh–Schlieder properties in interacting theories.[20] The generalized free field $\phi(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$, may be interpreted as a certain superposition of ordinary free fields $\phi_\mu(f)$ of mass $\mu \geq 0$ with weight measure $\mathrm{d}\rho(\mu)$ describing the mass spectrum of the theory. For our purposes, ρ should consist of a delta measure at the desired particle mass $m \geq 0$ and some continuous background spectrum. We will take

$$\rho := \delta_m + \rho_{\mathrm{cont}}, \quad \rho_{\mathrm{cont}}(\Delta) := \int_{\Delta\cap[0,m+1]} \mathrm{d}\mu \, \frac{1}{|\mu - m|^{1-\epsilon}} + \alpha\lambda(\Delta), \tag{44}$$

for Borel sets $\Delta \subset [0, \infty)$, where λ denotes Lebesgue measure. The parameter $\epsilon > 0$ controls the regularity in the vicinity of the particle mass, i.e. regarding the Herbst condition (H). Additionally, the support properties of ρ, governed by $\alpha \in \{0, 1\}$, are of (perhaps unexpected) relevance for the Reeh–Schlieder problem.

On the bosonic Fock space $\mathscr{F}_\rho := \Gamma(\mathscr{H}_{1,\rho})$ over the single-particle space $\mathscr{H}_{1,\rho} := L^2(\mathbb{R}^3) \otimes L^2([0, \infty), \mathrm{d}\rho)$ we obtain a Wightman field in terms of the Segal operators $\Phi_S(\psi) := (a^*(\psi) + a(\psi))/\sqrt{2}$, $\psi \in \mathscr{H}_{1,\rho}$, for real-valued test functions $f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^4)$ by

$$\phi(f) = \Phi_S(\omega^{-1/2} \hat{f}_+), \tag{45}$$

where the argument contains the restriction $\hat{f}_+(\mathbf{p}, \mu) := \hat{f}(\omega_\mu(\mathbf{p}), \mathbf{p})$, $\omega_\mu(\mathbf{p}) := \sqrt{\mathbf{p}^2 + \mu^2}$, and ω denotes the corresponding (unbounded) multiplication operator on $\mathscr{H}_{1,\rho}$. The representation of translation group is generated by the second quantization of the multiplication operators $(\omega, \mathbf{p})$, and setting $W(f) := \mathrm{e}^{\mathrm{i}\phi(f)}$, we obtain a corresponding Haag–Kastler net for bounded open regions $\mathcal{O} \subset \mathbb{R}^4$ by

$$\mathfrak{A}(\mathcal{O}) := \{W(f) : \; f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^4), \; \mathrm{supp}\, f \subset \mathcal{O}\}''. \tag{46}$$

It will be convenient to adopt Landau's formulation [Lan74], as it gives a simple reinterpretation of $\mathfrak{A}(\mathcal{O})$ in terms of time-zero fields. For Schwartz test functions $\underline{f}(\underline{x})$, $\underline{x} = (x^0, \mathbf{x}, s)$, from here on assumed to be symmetric in s, where s may be interpreted as new auxiliary space-like[21] variable conjugate to the mass μ, set $\underline{\phi}(\underline{f}) := \Phi_S(\omega^{-1/2}\hat{\underline{f}}_+)$, $\hat{\underline{f}}_+(\mathbf{p}, \mu) := \hat{\underline{f}}(\omega_\mu(\mathbf{p}), \mathbf{p}, \mu)$. Analogously to (46), we obtain an extended net $\underline{\mathfrak{A}}(\underline{\mathcal{O}})$ on $\mathbb{R}^5$.

It is easily seen that extended field $\underline{\phi}(\underline{f})$ and its time derivative $\underline{\phi}_t(\underline{f}) := -\underline{\phi}(\partial_t \underline{f})$ admit well-defined restrictions to time-zero fields

$$\underline{\phi}_0(\mathbf{f}) = \Phi_S(\omega^{-1/2}\hat{\mathbf{f}}), \qquad \underline{\pi}_0(\mathbf{f}) = \Phi_S(\mathrm{i}\omega^{1/2}\hat{\mathbf{f}}) \tag{47}$$

[20] Due to vacuum polarization $\phi(f)\Omega$ cannot have sharp mass for interacting theories, i.e. there is some spectral background $E_m^\perp \phi(f)\Omega \neq 0$. Generalized free fields simulate this in a simplistic way via (44).

[21] However the field $\underline{\phi}(\underline{f})$ should not be expected to be local in the direction of s.

for test functions $\mathbf{f} \in \mathscr{S}(\mathbb{R}^4, \mathbb{R})$ defined on the extended $(\mathbf{x}, s)$-space. In terms of corresponding extended double cones $\underline{\mathscr{C}}_R := \{(t, \mathbf{x}, s) \in \mathbb{R}^{3+2} : |t| + \sqrt{\mathbf{x}^2 + s^2} < R\}$, $(R > 0)$, Landau gave the following characterization of the net (46).

Theorem 23. [Lan74]. $\mathfrak{A}(\mathscr{C}_R) = \underline{\mathfrak{A}}(\underline{\mathscr{C}}_R)$. *Furthermore, these algebras are generated by bounded functions of the time-zero fields* (47) *with test functions* $\mathbf{f} \in \mathscr{S}(\mathbb{R}^4, \mathbb{R})$ *supported in the ball* $\mathrm{B}_R = \underline{\mathscr{C}}_R\big|_{t=0}$.

Proposition 24. [Lan74]. *If the defining measure ρ of the generalized free field is exponentially decreasing, then* $\mathfrak{A}(\mathscr{C}_R) = \underline{\mathfrak{A}}(\mathscr{C}_R \times \mathbb{R})$.

For choosing $\alpha = 0$ in (44), we may conclude that the strengthened Reeh–Schlieder property holds for the net $\mathfrak{A}$: take a family of test functions $\mathbf{f}_\beta \in C_c^\infty(\mathbb{R}^4)$, such that $\{0\} \times \operatorname{supp} \mathbf{f}_\beta \subset \mathcal{O} \times \mathbb{R}$ and with Fourier transforms converging sufficiently rapidly to a smooth limit supported on the sharp-mass subset $\mathbb{R}^3 \times \{m\}$. By Proposition 24 we can make such a choice which is compatible with bounded functions of $\phi_\beta := \underline{\phi}_0(\mathbf{f}_\beta)$, such as $A_\beta := \phi_\beta \mathrm{e}^{-\beta|\phi_\beta|^N}$, being contained in the local algebra $\mathfrak{A}(\mathcal{O})$, thus confirming the validity of (RS). Regarding (RS) we may summarize:

Proposition 25. *For generalized free field models defined by* (44) *with $\alpha = 0$, there exists a dense set of sharp-mass single-particle states generated by Reeh–Schlieder families of arbitrarily small degree $\gamma > 0$ independently of the choice of ϵ in* (44).

A fortiori, a continuity argument then shows that the sharp-mass free field net $\mathfrak{A}_m(\mathcal{O})$ is a subnet of $\mathfrak{A}(\mathcal{O})$. To obtain a non-trivial example we should thus choose $\alpha = 1$. We conclude with a short consideration of this difficult case, for which the assumptions of Proposition 24 are violated.
Given a bounded double-cone region $\mathscr{C}_R$ and a single-particle vector $\Psi_1 \in \mathscr{H}_{1,\rho}$ (say $\Psi_1 = \underline{\phi}_0(\mathbf{f})\Omega$, with $\mathbf{f} \in \mathscr{S}(\mathbb{R}^4)$ supported in a very large region) we would like to find a family of smeared field operators ϕ_β localized in $\mathscr{C}_R$, such that $\left\|\phi_\beta \Omega - \Psi_1\right\| \leq \beta$. For this purpose it will be convenient to introduce the following closed single-particle subspaces ($\mathbf{f} \in \mathscr{S}(\mathbb{R}^4)$) in the setting of Theorem 23,

$$\mathscr{H}_{\underline{\phi}_0, \mathrm{B}_R} := \overline{\{\phi_0(\mathbf{f})\Omega,\ \operatorname{supp} \mathbf{f} \subset \mathrm{B}_R\}}, \quad \mathscr{H}_{\underline{\pi}_0, \mathrm{B}_R} := \overline{\{\pi_0(\mathbf{f})\Omega,\ \operatorname{supp} \mathbf{f} \subset \mathrm{B}_R\}}. \tag{48}$$

The orthogonal projections P_ϕ, P_π corresponding to (48) may be used to iteratively define approximations of Ψ_1 by vectors from (48) or equivalently, generated by $\mathscr{C}_R$-localized operators. Underlining error terms after each half-step we begin with

$$\Psi_1 = P_\phi \Psi_1 + \underline{(1 - P_\phi)\Psi_1} = P_\phi \Psi_1 + P_\pi P_\phi^\perp \Psi_1 + \underline{P_\pi^\perp P_\phi^\perp \Psi_1} = \cdots$$

Similarly, after N iterations the remaining error is given by $\|(P_\pi^\perp P_\phi^\perp)^N \Psi_1\|$. By the von Neumann alternating projection theorem [vN50, Thm. 13.7], $(P_\pi^\perp P_\phi^\perp)^N$ in fact converges strongly to the orthogonal projection onto the intersection $\mathscr{H}_{\underline{\phi}_0, \mathrm{B}_R}^\perp \cap \mathscr{H}_{\underline{\pi}_0, \mathrm{B}_R}^\perp = (\mathscr{H}_{\underline{\phi}_0, \mathrm{B}_R} + \mathscr{H}_{\underline{\pi}_0, \mathrm{B}_R})^\perp$. The latter is trivial by the Reeh–Schlieder theorem, implying convergence of our iterative procedure. An upper bound on the degree of sharp-mass Reeh–Schlieder families along the lines of (RS) or ($\mathrm{RS}^\flat$) may be inferred from the speed of convergence $\|(P_\pi^\perp P_\phi^\perp)^N \Psi_1\| \to 0$, $\Psi_1 \in E_m \mathscr{H}_{1,\rho}$ or equivalent geometrical information regarding the situation of Ψ_1 in relation to the spaces (48). This is presently still under investigation.

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