Technische Universität München

Lehrstuhl für Wirtschaftsinformatik und Entscheidungstheorie

Should I Stay or Should I Go?

The No-Show Paradox in Voting and Assignment

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzender:		Prof. Dr. Rudi Zagst
Prüfer der Dissertation:	1.	Prof. Dr. Felix Brandt
	2.	Prof. Dr. Vincent Merlin Université de Caen Normandie

Die Dissertation wurde am 23.10.2018 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 08.01.2019 angenommen.

SHOULD I STAY OR SHOULD I GO?

THE NO-SHOW PARADOX IN VOTING AND ASSIGNMENT

JOHANNES HOFBAUER

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This thesis was typeset using a theme due to Hans Georg Seedig which was slightly modified and is based on LaTeX and the ClassicThesis style by André Miede, combined with the ArsClassica package by Lorenzo Pantieri. The text is set in Palatino with math in Euler, both due to Hermann Zapf. Headlines are set in Iwona by Janusz M. Nowacki, the monospace font is `Bera Mono` designed by Bitstream, Inc. Most of the graphics were created using TikZ by Till Tantau.

ABSTRACT

Voting theory revolves around the problem to aggregate possibly conflicting individual preferences of a group of voters to a satisfying collective choice of alternatives. Central objects of study are voting rules defined for this very task and their axiomatic properties. These characteristics include *Condorcet consistency*—an alternative preferred over any other by a majority of voters should be chosen uniquely—and immunity to the *no-show paradox (NSP)*: not casting one's ballot must not result in a more preferred result. In a seminal paper, Moulin (1988) shows that when we additionally require a voting rule to always select a single alternative, every Condorcet consistent rule is prone to the NSP. We continue along this iconic impossibility and study closely related questions in varying settings.

To begin with, we provide an overview of results subsequently obtained and categorize them with respect to variants of Condorcet consistency, different definitions of the NSP and ways out of single-valuedness.

Following this, we first focus on Moulin's original theorem. While susceptibility to the NSP is known for all single-valued Condorcet consistent rules, it is not well understood to which extent this is of practical relevance. For six well-known voting rules we therefore analytically and experimentally analyze how often a manipulation by strategic abstention is possible. Our findings show that, depending on different assumptions about how the electorate's preferences are structured, this strongly varies, but the likelihood is of a magnitude too high to discard the NSP as merely a theoretical problem.

We then relax the crucial assumption of single-valuedness and first allow voting rules to select sets of alternatives. In order to enable voters to compare sets we introduce so-called preference extensions and obtain corresponding set-valued versions of the NSP. Building on Fishburn's extension, we establish an incompatibility together with Pareto optimality for rules taking into account majority comparisons only. For the coarser Kelly's extension, we find the situation to be more positive and immunity to the NSP is implied by an existing variant of monotonicity.

A different framework that allows for a relaxation of single-valuedness are probabilistic voting rules, that select a probability distribution over alternatives. Due to infinitely many possible outcomes, this setting calls for the introduction of new versions of immunity to the NSP that are stronger in the sense that they even encourage participation. We study to which extent these notions are compatible with different degrees of efficiency and possible depen-

dancy on (weighted) majority comparisons only. This part is completed by a brief study of maximal lotteries, which are Condorcet consistent, efficient, and immune to the NSP up to a certain level.

Finally, we consider assignment problems where alternatives are no longer chosen collectively, but distributed to the participants individually. Probabilistic allocations prepare the ground for us to employ our previously defined new notions of immunity to the NSP and so study four well-established assignment rules. We find that all rules incentivize participation for single voters and groups alike, often in a very strong manner.

PUBLICATIONS

This thesis is based on the following publications.

[1] Incentives for participation and abstention in probabilistic social choice. In *Proceedings of the 14^{th} International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 1411–1419. IFAAMAS, 2015 (with F. Brandl and F. Brandt).

[2] Strategic abstention based on preference extensions: Positive results and computer-generated impossibilities. In *Proceedings of the 24^{th} International Joint Conference on Artificial Intelligence (IJCAI)*, 18–24. AAAI Press, 2015 (with F. Brandl, F. Brandt, and C. Geist).

[3] Random assignment with optional participation. In *Proceedings of the 16^{th} International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 326–334. IFAAMAS, 2017 (with F. Brandl and F. Brandt).

[4] Welfare maximization entices participation. *Games and Economic Behavior*, 2019 (with F. Brandl and F. Brandt), Forthcoming.

[5] Exploring the no-show paradox for Condorcet extensions using Ehrhart theory and computer simulations. In *Proceedings of the 18^{th} International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 2019 (with F. Brandt and M. Strobel), Forthcoming.[1]

1 Also presented at the 7^{th} *International Workshop on Computational Social Choice (COMSOC)*, 2018, the 14^{th} *Meeting of the Society for Social Choice and Welfare*, 2018, and the *AAMAS-IJCAI Workshop on Agents and Incentives in Artificial Intelligence*, 2018.

ACKNOWLEDGMENTS

My deepest gratitude goes out to a number of people who have accompanied me those last years that culminated in this thesis being crafted. I highly enjoyed my time in academia, not least due to them. In particular, special thanks go to. . .

- first and foremost: Felix Brandt, the officially best PhD supervisor at the mathematics department of TUM.[2] The following pages would be less filled without his constant advice, support, and encouragement. I could not have wished for a better tutor and I count myself fortunate for having had the chance to work with him.

- my further co-authors besides Felix: Florian Brandl, Christian Geist, Martin Strobel, and Martin Suderland without whom everything would have been so much harder, if possible at all.

- all other former and present colleagues from the *Algorithmic Game Theory and Computational Social Choice* research group plus our part-time external members, who made going to work feel like being among friends and who were always there for questions and discussions: Markus Brill, Martin Bullinger, Dominik Peters, Christian Saile, Hans Georg Seedig, Christian Stricker, and Warut Suksompong.

- the proofreaders of this thesis, who put so much time and effort into their comments. The devil is in the details.

- the students I had the pleasure to supervise for their Bachelor's or Master's thesis projects: Ludwig Dierks, Michael Engeßer, Antonia Schmalstieg, Dominik Spies, and Martin Suderland.

- all students I worked with during five years of teaching. You as well as I did rise to ever new challenges.

- Vincent Merlin, who helped make me feel part of the community on my first ever conference abroad and later agreed to be my second referee.

- Rudi Zagst for chairing my board of examiners.

- all further friends and colleagues in the community, who made all those conferences and meetings such an enjoyable time: Haris

2 TUM International School of Applied Mathematics Supervisory Award 2018 for excellence in doctoral research supervision (1st prize)

Aziz, Cristina Cornelio, Olivier Cailloux, Tobias Dittrich, Edith Elkind, Piotr Faliszewski, Rupert Freeman, Umberto Grandi, Ronald de Haan, Paul Harrenstein, Nick Mattei, Timo Mennle, Michael Müller, Nhan-Tam Nguyen, Clemens Puppe, Jana Rollmann, Sebastian Schneckenburger, and so many more to whom I apologize for having forgotten them here.

- Francisco Sánchez Sánchez from CIMAT whose cooperative game theory lecture started me on game theory and later social choice.
- my parents and my brother for all their support and for always being there. Furthermore, all my friends for the good times and the recreation essential to keep the overall concentration up.
- Jonas 'el Rubio' Wunderlich, for even though our academic ways separated a while back, my university career would have gone a different—harder and most surely less fun—way without him.
- my very own small family that every single day reminds me of what really matters.

Thank you. You rock.

Parts of my work were supported financially by Deutsche Forschungsgemeinschaft under grant BR 2312/10-1. Regarding travel support, I am also indebted to the TUM Graduate School, the COST Action IC1205, AAMAS 2015 and AAMAS 2017 student scholarships funded by IFAAMAS/AIJ as well as the student volunteering programs at IJCAI 2015 and IJCAI 2018.

CONTENTS

1 INTRODUCTION

> "The no-show paradox would deserve to be called the paradox of the act of voting [. . .]. The no-show paradox undermines the very rationale of voting; if by refraining from voting the end result could be strictly better, why expect people to vote?"
>
> Hannu Nurmi, 1999

Voting theory revolves around the problem to aggregate possibly conflicting individual preferences of a group of voters to a satisfying collective choice of alternatives. Central objects of study are voting rules defined for this very task and their axiomatic properties. The origins of voting theory date back as far as the 18$^{\text{th}}$ century to Borda (1784) and Condorcet (1785).[3] Over two hundred years ago, these French scientists laid the foundations for two important yet opposed families of voting rules: *scoring rules* that choose based on the alternatives' positioning within individual preferences and *Condorcet extensions* that rely on the Condorcet criterion; a notion based on pairwise majority comparisons. In particular the second group of rules will play a significant role in what is to follow.

Throughout the years, researchers have found a great many of incompatibilities of different desirable properties. These characteristics include basic fairness concepts, varying degrees of efficiency or immunity to various types of manipulation. Here, one of the presumably most important results is due to Gibbard (1973) and Satterthwaite (1975) who show that every reasonably fair voting rule that always selects a single alternative is prone to strategic manipulation, i.e., voters potentially have the chance to obtain a more preferred result by misrepresenting their preferences. In this thesis we focus on a slightly different kind of manipulation: manipulation by strategic abstention from the election process. Given this is possible for a voting rule and certain combination of individual preferences and alternatives to choose from, the rule is said to suffer from the *no-show paradox*.

The remainder of this chapter is structured as follows: we first give an informal overview of the general setting and terms in Section 1.1. This provides the basis for Section 1.2, which discusses related work about the no-show paradox: its origins, variations, and results for

3 We use the terms voting theory and social choice theory interchangeably.

modified frameworks. Section 1.3 summarizes the most important contributions presented in this thesis.

1.1 GENERAL VOTING SETTING

In the most general voting problem, we are given a set of *voters* (or *agents*) $N = \{1, \dots, n\}$ and a set of *alternatives* (or *candidates*) $A = \{a, b, c, \dots\}$, $|A| = m$. All voters i are assumed to have *preferences* $\succsim_i$ over the alternatives such that they are able to order them from best to worst, i.e., from most preferred to least preferred. If the preference ranking does not contain any ties we name it *strict* in contrast to *weak preferences* that allow for indifferences. The collection of all individual preferences is a *preference profile*, $\succsim = (\succsim_1, \dots, \succsim_n)$.

VOTING RULE. A *voting rule* f is a function mapping a preference profile to a nonempty subset of alternatives, the collective choice. If the voting rule always selects a single alternative we call it *single-valued*. In the simple case with two alternatives and two voters with opposed preferences, single-valuedness however demands that a voting rule favors either one alternative or one voter.[4] This problem can be overcome by introducing *set-valued voting rules* that may select nonempty sets of alternatives or *probabilistic voting rules* that choose a probability distribution (or *lottery*) over the alternatives. Alternatives selected by a voting rule are sometimes also called *winning alternatives*.

The *majority margin* of alternative x over alternative y is the number of voters preferring x over y minus the number of voters preferring y over x, $g_{xy} = |\{i \in N \colon x \succsim_i y\}| - |\{i \in N \colon y \succsim_i x\}|$. Voting rules that do not rely on the preference profile but on the majority margins between alternatives only are named *pairwise* while rules that only take the sign of g_{xy} into account are *majoritarian*.

CONDORCET CRITERION. Alternative x is a *Condorcet winner* if it beats every other alternative in pairwise majority comparisons, i.e., $g_{xy} > 0$ for all $y \neq x$ (Condorcet, 1785). If $g_{xy} \geqslant 0$ for all $y \neq x$, x is a *weak Condorcet winner*. A voting rule that always selects a Condorcet winner whenever one exists is called *Condorcet extension* or also said to satisfy *Condorcet consistency* or the *Condorcet criterion*. There is good reason to select a Condorcet winner, but, as already noticed by Condorcet (1785), Condorcet winners may fail to exist. The possible nonexistence of Condorcet winners is often named *Condorcet paradox*;

4 While it is commonly accepted that any voting rule should eventually select one alternative only, single-valuedness does not allow for a tie-breaking that relies solely on the individual preferences. This criticism is also expressed by, e.g., Gärdenfors (1976), Kelly (1977), Barberà (1977), Feldman (1979b), Duggan and Schwartz (2000), and Ching and Zhou (2002).

see also Example 1.1. Two important representatives of the class of Condorcet extensions are *MaxiMin* that selects the alternatives where the minimal majority margins are maximal, and the *top cycle (TC)* that selects the smallest set of alternatives such that every alternative within this set has a positive majority margin versus every outside alternative.[5]

EXAMPLE 1.1

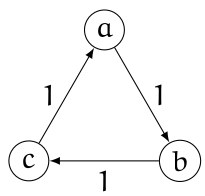
Consider the preference profiles $\succsim$ and $\succsim'$ as depicted below. When giving preference profiles, we depict individual preference rankings as columns with the most preferred alternative on top. The numbers above each column indicate how many voters share this preference relation.[6]

1	1	1
a	b	c
b	c	b
c	a	a

$\succsim$

1	1	1
a	b	c
b	c	a
c	a	b

$\succsim'$



Note that in profile $\succsim$, alternative b is a Condorcet winner while no Condorcet winner exists in $\succsim'$. The majority margins of $\succsim'$ are also depicted on the right where an arrow between two alternatives represents a positive majority margin. This preference profile and the corresponding majority graph are well-known as *Condorcet cycle*. In case the exact numbers are not relevant, e.g., when considering majoritarian voting rules, they are omitted. Both *TC* and MaxiMin choose the Condorcet winner b in $\succsim$ and $\{a, b, c\}$ in $\succsim'$.

SCORING RULES. *Scoring rules* focus on the alternatives' positioning within the preference rankings. They assign scores based on predefined *score vectors* and choose the alternatives with maximal score. Known examples include *plurality*, that selects the alternatives most often ranked first, and *Borda's rule*, where for every preference ranking an alternative is given one point per alternative ranked below (Borda, 1784). So, while plurality chooses $\{a, b, c\}$ in preference profile $\succsim$ of Example 1.1, b wins under Borda's rule.

EXTENDING PREFERENCES. Recall that every voter has preferences over the alternatives only. Hence, when allowing voting rules to

5 MaxiMin is also known as the *Simpson-Kramer method* (Black, 1958; Simpson, 1969; Kramer, 1977) while *TC* is sometimes also called *weak closure maximality*, *GETCHA*, or *Smith set* (Good, 1971; Smith, 1973; Schwartz, 1986). MaxiMin is pairwise and *TC* a majoritarian voting rule.

6 We choose this notation for the introduction only due to a sometimes large amount of voters. In all following chapters, the numbers above each column indicate which individual voters share this preference relation.

choose sets of alternatives or lotteries thereover, we face the problem that we do not know how voters compare different choices.[7] *Preference extensions* constitute concepts to lift individual preferences to preferences over sets or probability distributions.

For sets of alternatives X and Y, *Kelly's extension* prescribes that X is preferred to Y if every alternative in X is preferred to every alternative in Y (Kelly, 1977). According to *Fishburn's extension*, X is preferred to Y if all alternatives in $X \setminus Y$ are preferred to all alternatives in $X \cap Y$ are preferred to all alternatives in $Y \setminus X$ (Fishburn, 1972a). Two very intuitive extensions are the optimist and pessimist extensions. X is preferred to Y under the *optimist extension* if the most preferred alternative in X is preferred to the most preferred alternative in Y; for the *pessimist extension* the corresponding least preferred alternatives are what tips the scales.

When it comes to probabilistic voting rules, we say that lottery p is preferred to q under the *stochastic dominance extension (SD)* if the probability that p yields an alternative at least as good as x is greater or equal than the probability of q yielding an alternative at least as good as x for all choices of x (see, e.g., Gibbard, 1977; Postlewaite and Schmeidler, 1986; Bogomolnaia and Moulin, 2001). The *downward lexicographic extension (DL)*, which is a refinement of *SD*, prescribes that p is preferred to q if either p gives more probability to the most preferred alternative than q, or p gives more probability to some alternative x and just as much as q for all alternatives preferred to x (Cho, 2016).

PREFERENCE MODELS. When quantitatively studying voting rules, it is a common approach to sample preference profiles based on underlying *preference models*. A model frequently used is *impartial culture (*IC*)* where a ranking is drawn independently for each individual voter uniformly at random. Sampling under *impartial anonymous culture (*IAC*)* on the other hand gives equal probability to each anonymous preference profile, i.e., the voters' names are of no relevance.

RANDOM ASSIGNMENT. *Random assignment* is a special case of probabilistic voting theory. In accordance with the setting described above, voters (here usually named *agents*) have preferences over a set of alternatives (usually called *objects* or *houses*). Contrary to voting rules, *assignment rules* do not return a collective choice but an individual assignment for every agent. For convenience reasons, it is therefore often assumed that there is an identical number of agents and objects. Note that presuming agents only care about their own allocation, the *SD* and *DL* extensions defined before also allow to lift individual preferences to preferences over random assignments.

7 Theoretically, one could demand the individual preferences not to be over single alternatives but over sets or lotteries. This is, however, generally considered impracticable due to the exponential number of subsets and infinitely many lotteries.

In recent years, the increasing interest of computer scientists in classical voting problems together with immense technical advances has led to the evolvement of computational social choice. Focus here lies for example on insights gained with the help of computer simulations, the complexity to determine outcomes or misrepresentations, or even whole proofs found by a computer. The exact borderline between classical voting or social choice theory and computational social choice, however, is hard to define and many works hover in this grey zone. We thus count this thesis to be in good company. For a more comprehensive summary and related literature we refer to Arrow et al. (2002) and Arrow et al. (2011) for general voting theory and to Brandt et al. (2016b) as well as Shoham and Leyton-Brown (2009), Rothe (2015), and Endriss (2017) for the field of computational social choice.

1.2 THE NO-SHOW PARADOX IN THE COURSE OF TIME

The story of formal study of the voting paradox this thesis revolves around begins in 1973 with an extensive analysis of *point run-off systems*, i.e., multi-round voting rules based on score vectors.[8] Here, Smith (1973) finds that if the electorate is enlarged by an additional voter having the current winning alternative as first preference, this cannot cause said alternative to lose the election.

This concept is picked up by Fishburn and Brams (1983), who illustratively point out that a voting rule called *single-transferrable vote (STV)* suffers from four different paradoxes: the *thwarted-majorities paradox*, *multiple-districts paradox*, *more-is-less paradox*, and *no-show paradox (NSP)*. The NSP is characterized as "The addition of identical ballots with candidate x ranked last may change the winner from another candidate to x", i.e., by joining an electorate, voters with identical preferences can make their least preferred alternative the winner. Reversing situations, we see that this is equivalent to these voters changing the winning alternative from their least preferred to a more favored one by not showing up for the election.

Example 1.2

In order to capture the reassuring patina of nostalgia, we here provide the original example of the NSP due to Fishburn and Brams (1983) with only the alternatives' names simplified.

8 According to Fishburn and Brams (1983), first mentions go back to *Report of the Royal Commission appointed to enquire into electoral systems : with appendices* (1910) and Meredith (1912).

Therefore, assume a total of 1 610 voters, three alternatives, $A = \{a, b, c\}$, and preferences as given below.[9]

419	82	143	357	285	324
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

STV proceeds in rounds. It chooses the alternative ranked first by a strict majority of voters if such an alternative exists, and if not eliminates the alternative ranked first least often before starting all over again with an updated preference profile. Hence, for the above profile, we have an initial count of 501 votes for a, 500 votes for b, and 609 votes for c. Since no strict majority exists, b is eliminated and in the updated profile a is ranked first 644 times compared to 966 votes for c, which is eventually selected.

If two voters with preferences $a \succsim b \succsim c$ abstain, a is eliminated in the first round leaving a run-off between b and c. For this we count 917 votes for b and 691 votes for c. Consequently, by not participating in the election process, the two abstaining voters can sway the result in their favor.

Ray (1986) studies the likelihood of similar situations, i.e., how often the NSP occurs for *STV* and three alternatives. He finds the probability to be very high especially for cases where the outcome of plurality and *STV* differ—a situtation of special importance in the light of the discussion to exchange plurality for *STV* (see Ray, 1986).

In his seminal paper, Moulin (1988) generalizes the paradox to its form still widely accepted today: a voting rule suffers from the NSP if there exists a situation where some voter can achieve a more preferred outcome by abstaining from the election process. Rules not prone to the NSP are said to satisfy *participation*. Moulin already notes that multiple well-known rules including Borda's rule and plurality are immune to the NSP.[10] For Condorcet extensions, however, he derives an iconic impossibility theorem that serves as inspiration for a multitude of later papers: if $m \geqslant 4$ and $n \geqslant 25$, there is no single-valued voting rule satisfying Condorcet consistency and participation. If $m \leqslant 3$, both properties can be satisfied simultaneously.[11]

The minimal value of n is improved by Kardel (2014), who shows that Moulin's impossibility already holds for $n \geqslant 21$. In a very recent paper, Brandt et al. (2017a) employ SAT solving to tighten the bound; they prove that there do exist single-valued voting rules satisfying

9 Note that this example obviously is not minimal in the number of voters, but rather meant to illustrate a realistic voting situation.
10 In general, all monotonic scoring rules are immune to the NSP.
11 For instance by MaxiMin with alphabetic tie-breaking.

Condorcet consistency and participation up to $n = 11$, $m = 4$, while the two properties are incompatible for $n \geqslant 12$.

With respect to the proof for Moulin's theorem, Nurmi (1999) argues that each occurrence of the NSP is based on an underlying appearance of the Condorcet paradox. He also provides an intuitive proof sketch using this connection.

Revisiting the impossibility, we find that it actually relies on three different properties: Condorcet consistency, participation and *single-valuedness*. Virtually all subsequent work on the NSP tries to either overcome the incompatibility or find a closely related one by tackling one of those. We first focus on attempts to modify Condorcet consistency in Section 1.2.1, examine different variants of participation in Section 1.2.2, and lastly consider prior work regarding set-valued voting rules in Section 1.2.3 and probabilistic rules in Section 1.2.4. Papers combining more than one approach are placed where they fit best. Participation is related to similar properties like strategyproofness and monotonicity in Section 1.2.5, where we also conclude our overview of the NSP with some remarks.

1.2.1 Varying Condorcet Consistency

Recall from Section 1.1 that alternative x is a Condorcet winner if $g_{xy} > 0$ for all $y \neq x$. If voters' preferences are strict, this is equivalent to demanding that there does not exist an alternative y such that $|\{i \in N \colon y \succ_i x\}| \geqslant {}^1\!/_2\, n$. Hence, it is possible to generalize Condorcet winners to q-winners by naming x q-*winner* if there is no y for which $|\{i \in N \colon y \succ_i x\}| \geqslant q \cdot n$, ${}^1\!/_2 \leqslant q \leqslant 1$. A voting rule is said to be q-*consistent* if it always selects a q-winner whenever the set of q-winners is nonempty.

Holzman (1988) shows that there exist single-valued voting rules satisfying both participation and q-consistency if and only if $m \leqslant 3$ or $q \geqslant {}^{m-1}\!/_m$, i.e., there are few alternatives or the threshold q is large which corresponds to a potentially larger—but still possibly empty—set of q-winners. With one later exception (Pérez et al. (2015), see Section 1.2.3), Condorcet consistency is otherwise widely accepted as most basic of the three properties and thus left unchanged.

1.2.2 Varying Participation

We have already seen that the first two works dealing with the NSP use definitions thereof that differ from the one in Moulin's theorem. In contrast to a general comparison between winning alternatives, they only focus on the additional voter's either most (Smith, 1973) or least preferred alternative (Fishburn and Brams, 1983). It is those two ideas that most later papers pick up in order to propose variants of the NSP.

Pérez (1995) considers a setting where multiple alternatives may be chosen and defines two new properties reminiscent of participation: *monotonicity in the face of new voters* and *choice participation*. The former prescribes that when a voter with most preferred alternative x joins the electorate and x was among the winning alternatives before, it has to remain so. According to the latter, if the additional voter prefers x to y and x was chosen without him, then if y is chosen if he participates, so has to be x. Pérez develops that monotonicity in the face of new voters is incompatible with Condorcet consistency under some additional assumptions similar to duality as defined by Fishburn (1973). Choice participation cannot be satisfied simultaneously with Condorcet consistency irrespective of further properties.

Two stronger variants of the NSP are defined by Pérez (2001). A voting rule suffers from the *positive strong* NSP if there exists a situation where some voter's most preferred alternative is chosen if he abstains but not chosen if he participates. Conversely, it is prone to the *negative strong* NSP if there exists a situation where some voter's least preferred alternative is chosen if he participates but not chosen if he abstains.[12] It is shown that both strong paradoxes arise for all Condorcet extensions satisfying different dominance-inspired criteria relying on (weighted) majority comparisons. More precisely, this affects most known majoritarian and pairwise Condorcet extensions with the exception of Young's rule for the negative strong NSP and MaxiMin for both.

This result is built upon by Kasper et al. (2017) who define a Condorcet consistent set-valued voting rule that is immune to the positive strong NSP and negative strong NSP. At the same time this rule is maximal with respect to the set of alternatives chosen in the sense that all other Condorcet consistent voting rules immune to both paradoxes must choose a subset thereof.

If individual preferences are allowed to contain indifferences, the before-mentioned positive result breaks down: Duddy (2014) proves that under weak preferences, Condorcet consistency is incompatible with either strong NSP if there are at least four alternatives.

The positive strong NSP and negative strong NSP are also studied in the context of specific voting rules. Most notably, Nurmi (2004), Felsenthal and Tideman (2013), Felsenthal and Nurmi (2016), and Felsenthal and Nurmi (2018) analyze a grand variety of well-known rules and give exemplary preference profiles that illustrate violations.

Lepelley and Merlin (2000) consider scoring run-off rules in elections with three alternatives and six different versions of the NSP that distinguish between voters joining or leaving the electorate and focus on most or least preferred alternatives. Using statistical techniques, Lepelley and Merlin are able to obtain estimates for the likelihood of

12 Both strong paradoxes are defined as violations of *positive involvement* or *negative involvement* as already used by Smith (1973) (see, also, Richelson, 1978; Richelson, 1980; Saari, 1995).

the NSP to occur under the IC and IAC preference models. They find that chances for the various paradoxes to occur are lower to middle two-digit percentages for small n, and generally higher for IC than for IAC. Also, probabilities tend to decrease as n grows.

In a recent paper, this setting is revisited by Kamwa et al. (2018) who focus on *single-peaked* preferences.[13] Under this assumption, they find that multiple scoring run-off rules do not suffer from any variant of the NSP anymore while for others, e.g., the plurality run-off, the probabilities of a paradox to occur are significantly lower compared to the unrestricted domain.

Sanver and Zwicker (2009) undertake a completely different approach by defining *one-way monotonicity* as well as *half-way monotonicity* as weakening thereof. One-way monotonicity requires that for every manipulation by misrepresentation, reverse misrepresentation is not a valid manipulation. A voting rule satisfies half-way monotonicity if no voter can manipulate by completely reversing his preference ranking. Originally thought of as weaker versions of strategyproofness, Sanver and Zwicker show that half-way monotonicity is implied by participation. The converse statement is proven to hold additionally assuming homogeneity and reversal cancellation.[14] Though connections exist for three and four alternatives, one-way monotonicity is logically independent from participation. However, Sanver and Zwicker are able to develop an impossibility similar to Moulin's: no single-valued voting rule can satisfy Condorcet consistency and one-way monotonicity simultaneously.

This result is improved by Peters (2017) who establishes that for single-valued voting rules, Condorcet consistency is incompatible with half-way monotonicity. Peters' theorem is shown to hold for at least four alternatives and either 15 or 24 voters depending on the parity of n. Surprisingly, neither the statement nor the bounds change when instead considering set-valued voting rules with the optimist or pessimist extension.

Different ways to extend participation, one-way, and half-way monotonicity to set-valued voting rules are also studied by Sanver and Zwicker (2012).

Twins welcome is a weakening of participation already discussed by Moulin (1988). Intuitively, it prescribes that if a voter with preferences identical to the ones of a voter already present, i.e., his twin, joins the electorate, this must not result in a worse outcome for the two.

13 Intuitively, single-peakedness prescribes that alternatives can be ordered on a left-right axis and voters' preferences are determined by proximity to their political view. Single-peaked preferences go back to Black (1948), we refer to Elkind et al. (2017) for an overview of structured preferences.

14 *Homogeneity* prescribes that using multiple copies of the electorate does not change the outcome while *reversal cancellation* means that adding two voters with opposed preferences does not have any effect, either.

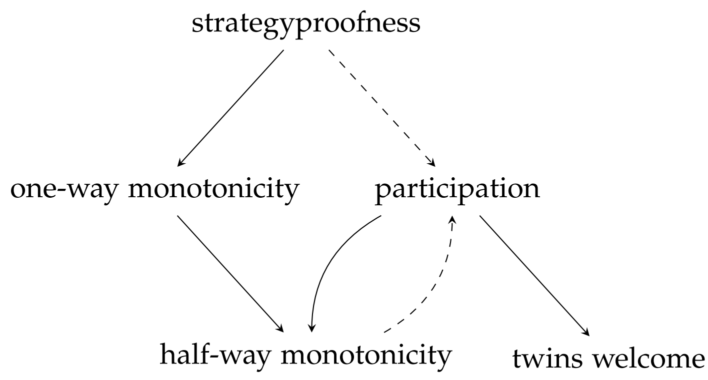


Figure 1.1: Implications between different properties related to participation; a solid arrow from one characteristic to another signifies the former implies the latter; strategyproofness implies participation if individual preferences may contain indifferences and the presence of completely indifferent voters does not change the outcome; half-way monotonicity implies participation if the voting rule satisfies homogeneity and reversal cancellation.

Moulin shows that, similar to the NSP, no single-valued Condorcet extension satisfies twins welcome.

Figure 1.1 depicts the relationships between participation and one-way monotonicity, half-way monotonicity, and twins welcome. Strategyproofness, which can also be seen as two-way monotonicity (Sanver and Zwicker, 2009), is added for the sake of completeness.

1.2.3 Varying Single-valuedness: Set-valued Voting Rules

The assumption that voting rules always have to select single alternatives is crucial for Moulin's theorem. Once we allow rules to choose multiple alternatives—most importantly when no Condorcet winner exists—there are ways to forgo the incompatibility of Condorcet consistency and participation. Note that permitting voting rules to select either a set of alternatives or a probability distribution thereover also influences the idea of participation. Different concepts of how to extend preferences over alternatives to preferences over sets or lotteries result in different notions of participation and thus varying results.

Jimeno et al. (2009) find that, assuming strict preferences and the optimist or pessimist extension, all set-valued and Condorcet consistent voting rules suffer from the NSP.[15] They also remark that if preferences are instead extended according to Kelly's or Fishburn's extension, there exist Condorcet extensions satisfying participation, e.g., a

15 In fact, Brandt et al. (2017a) note that the statement for optimistic voters is already implied by Moulin (1988).

voting rule selecting the set of weak Condorcet winners if nonempty and *TC* otherwise.

The former result is revisited by Brandt et al. (2017a) who establish tight bounds on n. More precisely, there exist Condorcet consistent voting rules satisfying participation with respect to the optimist extension for up to 16 voters while the same holds for the pessimist extension and twelve voters. For larger electorates, the two properties cannot be satisfied simultaneously.

Given that the concept of Condorcet winners is altered to q-winners, Pérez et al. (2015) show that participation with respect to the optimist extension is compatible with q-consistency.[16] In particular, they define a voting rule $f(q)$ that satisfies participation if and only if $q \geqslant 1/\varphi$ with φ being the golden ratio.

Different well-known majoritarian and pairwise Condorcet extensions are studied by Hofbauer (2014) for Kelly's, Fishburn's and Gärdenfors' extensions. Hofbauer provides numerous examples for violations of compatibility with participation.

Pérez et al. (2010) choose to also transfer the Condorcet criterion to a set-valued notion by defining a similar concept for voting rules that always select a set of fixed size $k \leqslant m$. They derive various incompatibilities for these so called *Condorcet k-correspondences* together with participation, if individual preferences are extended according to either the optimist, pessimist, or lexicographic extensions.

1.2.4 Varying Single-valuedness: Probabilistic Voting Rules

Studies of the NSP in a probabilistic framework are only conducted by a couple of very recent papers. Hofbauer (2014) provides some initial ideas for concepts that are further developed and thoroughly discussed later in this thesis. In particular, this includes the definition of different degrees of participation that prescribe that an additional voter is never worse off (participation), always at least as good off (*strong* participation), or strictly better off (*very strong* participation).

In this context, Brandt et al. (2017a) show that Condorcet consistency is incompatible with strong participation with respect to the *SD* extension.

Aziz (2016) studies connections between the *SD* and *DL* extensions for different degrees of participation and proves various implications.[17] Moreover, the so-called *maximal recursive rule* is shown to satisfy very strong participation for both *SD* and *DL*.

Subsequent work follows a similar direction. Gross et al. (2017) define a voting rule named *2-agree* that is axiomatically analyzed and proven to satisfy very strong participation for the *SD* extension. The

16 In contrast to Holzman (1988), a q-winner here is defined as an alternative that is preferred to every other alternative by at least $q \cdot n$ voters.

17 Some of these results can already be found in Hofbauer (2014).

same holds for *rank maximal equal contribution* as shown by Aziz et al. (2018b) who also introduce this new voting rule. For the special case of dichotomous preferences, i.e., preferences with only two indifference classes, Aziz et al. (2017) consider the *egalitarian rule*, *conditional utilitarian rule*, and *Nash max product* and study them also with respect to the before-mentioned variants of participation. In particular, they introduce variants of strategyproofness and existing fairness properties to convert known impossibility results to possibilities and also show that all three considered rules satisfy very strong participation.

1.2.5 Related Properties and Concluding Remarks

Participation bears analogies to multiple other well-known properties of which many have been around before the first mention of the NSP by Fishburn and Brams (1983). While the resemblance is mostly one of similar underlying ideas for some, there even exist direct implications for others. In addition to what was already discussed in Section 1.2.2, we here present the three most important concepts strategyproofness, monotonicity, and reinforcement, and discuss their relationship to participation.[18]

STRATEGYPROOFNESS. While a voting rule satisfies participation if it is immune to manipulation by strategic abstention, *strategyproofness* is defined as immunity to strategic misrepresentation of preferences.[19] Though both properties are logically independent, strategyproofness implies participation in restricted settings. For instance, Brandt (2015) notes that under mild assumptions that make it indistinguishable for a voting rule whether a voter is completely indifferent or absent, participation follows from strategyproofness.[20] Note, however, that this of course requires that individual preferences are allowed to contain indifferences.

From a real-world perspective, there are major differences of how to interpret violations of participation or strategyproofness. While being manipulable by strategic misrepresentation of preferences is undoubtedly a severe flaw of any voting rule in theory, it is open to question whether this is always of practical concern. If determining a valid manipulation is computationally hard, it may be argued that a failure of strategyproofness can be disregarded in realistic settings. Intuitively, a potentially existing manipulation becomes irrelevant if voters are unable to find it in reasonable time. This idea to use computational hardness as barrier for manipulation is introduced by

18 Other properties related to participation include for instance *manipulation by sincere truncation of preferences* (Fishburn and Brams, 1984).

19 Strategyproofness is dealt with in different contexts by, e.g., Dummett and Farquharson (1961), Gibbard (1973), Satterthwaite (1975), Gibbard (1977), Bogomolnaia and Moulin (2001), Aziz et al. (2018a), and Brandl et al. (2018) to name only a few.

20 This holds for, e.g., majoritarian or pairwise voting rules.

Bartholdi, III et al. (1989) and further discussed by, e.g., Conitzer et al. (2007), Faliszewski et al. (2010), Faliszewski and Procaccia (2010), Brandt et al. (2013), and Conitzer and Walsh (2016). When it comes to violations of participation, we find that this argument is rendered inapplicable. Since the only possible way to alter the election outcome is to abstain, determining whether this is a successful manipulation is never more complicated than computing the outcome itself. Thus, a voting rule's failure to satisfy participation can be seen as more severe compared to a violation of strategyproofness.

We are able to reach an identical conclusion using a different line of reasoning. In case of the voters knowing each other and the election process being public or information about the ballots published afterwards, it may be possible to determine whether a participant voted strategically.[21] Depending on the specific setting, such behavior might be considered immoral or even despicable, effectively forcing a voter to vote truthfully. Abstention, on the other hand, might be for the mere sake of convenience and is therefore often regarded as less objectionable. Hence, using moral as barrier against manipulation is presumably more effective for strategic misrepresentation than for strategic abstention. This again may lead to the assessment of the NSP being the more severe flaw.

MONOTONICITY. In its standard form, *monotonicity* is usually defined as follows: given an alternative is chosen by a voting rule, then if it is strengthened in the preference ranking of one voter while everything else remains unchanged, it still has to be among the chosen alternatives (see, e.g., Fishburn, 1982a). At first glance this bears some resemblances to a weaker form of participation, where giving more support to a favored alternative by joining the electorate should not cause said alternative to lose the election. Based on this observation, Nurmi (1999) poses the question whether a failure of monotonicity implies the NSP. Campbell and Kelly (2002) show that this is not the case by suggesting a non-monotonic voting rule satisfying participation.

Recently, Núñez and Sanver (2017) revisit the connection between monotonicity and participation and find some interesting implications for restricted cases or slight modifications. In particular, for only two alternatives, participation implies monotonicity and monotonicity together with homogeneity imply participation. For three or more alternatives, participation implies a weaker version of monotonicity while the converse does not hold, even together with homogeneity. Núñez and Sanver also define a lower contour set property λ that is

21 This situation is not particularly unlikely, consider for instance voting in a committee where the members know each other.

closely related to participation but logically independent from monotonicity.[22]

REINFORCEMENT. *Reinforcement* prescribes that if an alternative is chosen in two disjoint electorates independently, then it should also be chosen in the union thereof.[23] Interpreting one of these two electorates as single voter once more hints at a possible connection to participation. However, any such hope is moot as Moulin (1988) already gives two examples of voting rules satisfying participation but not reinforcement and *vice versa*. Nevertheless, Saari (1995) shows that positive involvement implies weak consistency, thus supplying a connection between variants of participation and reinforcement.

Up to now, we have always discussed the NSP for situations where the electorate changes, i.e., a voter leaves the electorate leading to a more preferred outcome, or equivalently joins and receives a less preferred result. There are, of course, settings where neither joining nor leaving is possible and the closest idea is to misrepresent as complete indifference.[24] Still, this notion is hardly discussed in the existing literature and most papers are in line with the variable electorate as originally proposed by Fishburn and Brams (1983) and Moulin (1988). Following Núñez and Sanver (2017), the difference is one of cosmetics mostly, anyway. Núñez and Sanver show that for all *regular* voting rules, participation with respect to either interpretation is equivalent.[25] We focus on variable electorates whenever applicable and consider complete indifference only when necessary, i.e., in Chapter 6.

Concepts similar to participation are also considered in slightly different contexts, e.g., by computer scientists working on voting equilibria and campaigning (Desmedt and Elkind, 2010; Baumeister et al., 2012) or in the field of judgment aggregation (Balinski and Laraki, 2011). A decision-theoretic model of participating in elections originates in works by Downs (1957), Tullock (1967), and Riker and Ordeshook (1968). These early works deal with the observation that quite often, a single voter cannot sway the outcome, i.e., does not

22 Participation implies λ while λ together with homogeneity and reversal cancellation imply participation.

23 Reinforcement is also known as *consistency* or *population consistency* and regularly studied in the literature (see, e.g., Young, 1974a; Fishburn, 1978; Myerson, 1995; Congar and Merlin, 2012). Young (1974b) shows that all Condorcet extensions violate reinforcement. Scoring rules, on the other hand, satisfy reinforcement and even are characterized by it under some additional mild assumptions (Smith, 1973; Young, 1975). Hence, they constitute a class of rules satisfying both reinforcement and participation.

24 In fact, we will model abstention as full indifference in Chapter 6 about random assignment.

25 Regularity prescribes that a voting rule's choice does not depend on completely indifferent voters. This property is satisfied by most well-known rules and identical to *independence of indifferent voters* as defined in Section 4.1.1.

gain anything by casting his ballot. The idea to model the influence of a pivotal voter as a *participation game* can be traced back to Palfrey and Rosenthal (1983), Ledyard (1984), and Palfrey and Rosenthal (1985). Recently, Levine and Palfrey (2007), Duffy and Tavits (2008), and Grillo (2017) conducted lab studies to witness individual behavior in such participation games.

For a general overview of various voting paradoxes and an extensive study of which voting rule is prone to which paradox we refer to Felsenthal (2011). When it comes to scoring rules in particular, Saari (1989) argues that if Borda's rule suffers from a paradox, so do all other scoring rules which are therefore strictly more manipulable.

1.3 CONTRIBUTIONS

This thesis includes a variety of results that complement the presented existing work in different areas. We here give an overview of our most important contributions that is in line with the ordering of later chapters and loosely so with the ordering of related work. Connections and dependencies are pointed out whenever suitable.

1.3.1 Analyzing the Likelihood of the No-Show Paradox

Recall that following Moulin (1988), every single-valued Condorcet consistent voting rule is prone to the NSP. To which extent this is the case is, however, not well understood even though Fishburn and Brams (1983) already propose "to assess the likelihood of the paradox [. . .] as an interesting possibility for investigation". This idea is pursued by Ray (1986), Lepelley and Merlin (2000), and Kamwa et al. (2018), but all three papers focus on specific definitions of the NSP and consider three alternatives only.

Our work is in line with Moulin's more general definition of participation and studies how often the NSP occurs for different single-valued Condorcet extensions. First, we focus on Copeland's rule, Black's rule, and MaxiMin with alphabetic tie-breaking and analytically compute the fraction of profiles admitting a manipulation by strategic abstention for three, three, and four alternatives, respectively, under IAC. This is done using Ehrhart theory (Ehrhart, 1962) and only possible through recent advances in computer algebra.

CONTRIBUTION 1

> We model the set of all anonymous preference profiles for n voters that are prone to the NSP as polytopes in the six-dimensional or 24-dimensional Euclidean space. Making use of Ehrhart theory and the computer program NORMALIZ we determine the exact fraction of profiles that allow for a manipulation depending

on n. We find that our analytical results are in almost perfect unison with experimental results obtained via computer simulations.

Limited to the current state of the art, it is impossible to analytically obtain exact numbers for more than four alternatives. Therefore, we rely on extensive computer simulations to get an insight into the behavior for larger values of m. In addition to the rules mentioned above, we also analyze Baldwin's, Nanson's, and Tideman's rule with alphabetic tie-breaking under the preference assumptions IC, IAC, as well as Mallows' ϕ, the urn model, and the spatial model. As far as we know, this is the only study of Condorcet extensions from a quantitative angle apart from the earlier Plassmann and Tideman (2014), Brandt et al. (2016d), and Bruns et al. (2017).

CONTRIBUTION 2
We experimentally study the manipulability of well-known single-valued Condorcet extensions. Here, we first derive results for up to 50 voters and 30 alternatives under IAC. We then fix the number of alternatives to either 4 or 30 and compare fractions for up to 1 000 or 200 voters according to five different preference models. Wherever possible we provide explanations for characteristics observed.

1.3.2 The No-Show Paradox for Set-Valued Voting Rules

We first focus on majoritarian set-valued voting rules and Fishburn's extension to lift preferences over single alternatives to preferences over sets of alternatives. To allow for the efficient use of a computer, we introduce a new participation-like condition that is based on majority graphs instead of preference profiles. This enables us to prove the incompatibility of Pareto optimality and Fishburn-participation for majoritarian voting rules via a SAT solver. Manually simplified and self-contained human-readable versions of the two corresponding computer-found proofs are included.

CONTRIBUTION 3
We show that no majoritarian and Pareto optimal set-valued voting rule satisfies Fishburn-participation whenever $m \geqslant 4$. If individual preferences are required to be strict, an identical statement holds for $m \geqslant 5$.

Next, we direct attention to Kelly's extension and find that results are a lot more positive here, even without restricting rules to be majoritarian. Most notably we find that there are attractive Condorcet extensions satisfying Kelly-participation, which stands in contrast to previous negative results for set-valued voting rules (see, e.g., Jimeno et al., 2009; Pérez et al., 2010; Brandt et al., 2017a).

CONTRIBUTION 4

We prove that set-monotonicity together with independence of indifferent voters implies Kelly-participation. For majoritarian voting rules, set-monotonicity alone suffices for the same implication, even when preferences have to be strict.

1.3.3 The No-Show Paradox for Probabilistic Voting Rules

To the best of our knowledge, participation has only been considered for probabilistic voting rules by Hofbauer (2014) before [1] which most parts of Chapter 5 are based on. In order to compare two lotteries, we rely on the SD extension mostly. We first discuss two new strengthenings of participation that are either very challenging or virtually impossible to be satisfied by single-valued or set-valued voting rules. These novel variants of participation are picked up and worked with in subsequent papers (see, e.g., Aziz, 2016; Brandt et al., 2017a; Gross et al., 2017; Aziz et al., 2017; Aziz et al., 2018b).

CONTRIBUTION 5

In addition to regular participation, we introduce two stronger versions: *strong participation* that prescribes that a voter always weakly prefers the outcome obtained when he participates compared to the outcome when he abstains. *Very strong participation* requires that a voter always strictly prefers the outcome obtained when he participates compared to the outcome when he abstains whenever this is possible, and weakly prefers the former to the latter otherwise.

In the following, we begin by studying very strong participation and obtain a number of results for different classes of probabilistic voting rules and different degrees of efficiency including *ex post* efficiency and unanimity.

CONTRIBUTION 6

We show that very strong participation is prohibitive for majoritarian voting rules, pairwise and unanimous voting rules, or Condorcet extensions. It can, however, be satisfied by *ex post* efficient voting rules.

For the weaker notions of strong participation and participation we also obtain mixed results.

CONTRIBUTION 7

We show that for majoritarian voting rules strong participation is incompatible with unanimity while participation and *ex post* efficiency are mutually exclusive. Strong participation is compatible with a strong variant of efficiency based on SD even when voting rules are required to be pairwise.

Lastly, we study a probabilistic Condorcet consistent voting rule called *maximal lotteries* and show that it satisfies a notion of participation that is stronger than regular participation but weaker than strong participation. We also characterize maximal lotteries using this version of participation. Note that strong participation alone is incompatible with Condorcet consistency (Moulin, 1988; Brandt et al., 2017a).

CONTRIBUTION 8
We show that maximal lotteries satisfy participation but fail to satisfy strong participation. In addition, we provide a unique characterization of maximal lotteries using participation with respect to the *pairwise comparison extension*.[26]

1.3.4 The No-Show Paradox for Random Assignment Rules

It is not possible to directly transfer the concept of strategic abstention to the assignment setting as we usually assume an identical number of agents and objects. This problem is tackled via two approaches that turn out to be equivalent for cases where both are applicable.

CONTRIBUTION 9
We discuss different ways to model abstention in the random assignment setting and define various degrees of participation based on results presented in Chapter 5.

Next, we apply this framework for participation to three well-known assignment rules, *random serial dictatorship*, the *(extended) probabilistic serial* rule, and the *Boston mechanism* as well as the Condorcet consistent class of *popular random assignments*. For some rules we find our results to be in sharp contrast to their susceptibility to manipulation by misrepresentation of preferences.

CONTRIBUTION 10
We show that all studied rules incentivize participation for single voters and groups of voters alike, often even in a very strong way.

UNDERLYING PUBLICATIONS

This thesis is based on various papers published over the last years and also presented at different international conferences and workshops. A full list is given on page vii. With respect to the remainder

26 A lottery p is preferred to a lottery q according to the pairwise comparison extension if the chance that p yields an alternative preferred to what q returns is higher than the probability that q returns an alternative preferred to what is given by p.

of this thesis, Chapter 3 is based on [5] while Chapter 4 is based on [2], an earlier version of which can also be found as part of Geist (2016). Chapter 5 is based on [1] and [4], parts of the latter are included in Brandl (2018), too. Lastly, Chapter 6 is based on [3].

EXCLUDED WORK

Apart from papers specified on page vii, my work also contributed to publications that do not fit the exact topic of this thesis. Though omitted, the corresponding papers are listed below for the sake of completeness:

- d-dimensional stable matching with cyclic preferences. *Mathematical Social Sciences*, 82:72–76, 2016.[27]
- Majority graphs of assignment problems and properties of popular random assignments. In *Proceedings of the 16th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 335–343. IFAAMAS, 2017 (with F. Brandt and M. Suderland).[28]

27 Also presented at the 5th *World Congress of the Game Theory Society (GAMES)*, 2016.
28 An earlier version was also presented at the 6th *International Workshop on Computational Social Choice (COMSOC)*, 2016.

2 PRELIMINARIES

In this chapter, we provide the general basis for what is to follow. We here limit ourselves to voting concepts used throughout this thesis and postpone definitions of specific notions to the corresponding chapters. In particular, the random assignment framework, which Chapter 6 is built upon exclusively, is introduced in Section 6.1. Note that some parts of what is presented here were already informally mentioned in Section 1.1.

2.1 FUNDAMENTALS

alternative
voter

Let A be a finite set of m *alternatives* and $\mathbb{N}$ be an infinite set of *voters*. By $\mathcal{F}(\mathbb{N})$ we denote the set of all finite and nonempty subsets of $\mathbb{N}$ where an element of $\mathcal{F}(\mathbb{N})$ is denoted by N, $|N| = n$. N is often also named an *electorate*. For some N, $i \in N$, we write N_{-i} for $N \setminus \{i\}$, i.e., the electorate without voter i.

electorate

(weak) preference relation

Each voter i is assumed to be endowed with a *preference relation* or *ranking* $\succsim_i$ over the alternatives. Formally, a *(weak)* preference relation is a complete, reflexive, and transitive binary relation, $\succsim_i \subseteq A \times A$.[29] The set of all preference relations over A shall be denoted by $\succsim(A)$. Whenever $x \succsim_i y$, we say that i *(weakly) prefers* x to y. By $\succ_i$ and $\sim_i$ we denote the *strict preference* as well as the *indifference* part of $\succsim_i$, respectively. Formally, $x \succ_i y$ if $x \succsim_i y$ and $y \not\succsim_i x$, and $x \sim_i y$ if $x \succsim_i y$ and $y \succsim_i x$. We denote voter i's most preferred alternatives in a set $X \subseteq A$ by $\max_{\succsim_i}(X)$,

$$\max_{\succsim_i}(X) = \{x \in X \colon x \succsim_i y \text{ for all } y \in X\}.$$

strict preference relation

In case $\succsim_i$ additionally is antisymmetric, i.e., does not contain any indifferences, we say that $\succsim_i$ is *strict*.[30] To compactly represent preferences we use comma-separated lists with all alternatives among which a voter is indifferent placed in a set; for instance $\succsim_i \colon x, \{y, z\}$ stands for $x \succ_i y \sim_i z$.

preference profile

A collection of one preference relation per voter is named *preference profile*. More precisely, we define a preference profile as a function from a set of voters N to the set of preference relations $\succsim(A)$. The set

29 *Complete* prescribes that for all $x, y \in A$, if $x \not\succsim_i y$, then $y \succsim_i x$ while *reflexive* means that for all $x \in A$, $x \succsim_i x$. Furthermore, a relation $\succsim_i$ is *transitive* if for all $x, y, z \in A$ such that $x \succsim_i y$ and $y \succsim_i z$ we have that $x \succsim_i z$.

30 Formally, a relation is *antisymmetric* if for all $x, y \in A$, $x \succsim_i y$ and $y \succsim_i x$ implies $x = y$.

of all possible preference profiles on A is denoted by $\succsim(A)^{\mathcal{F}(\mathbb{N})}$ with $\succsim(A)^N$ being the equivalent for a fixed $N \in \mathcal{F}(\mathbb{N})$. In order for voters to be able to join or leave the electorate, we define for a preference profile $\succsim \in \succsim(A)^N$ and $i \in N, j \in \mathbb{N}, S \subseteq N, T \subseteq \mathbb{N}$:

$$\succsim_{-i} = \succsim \setminus \{(i, \succsim_i)\} \qquad \succsim_{+j} = \succsim \cup \{(j, \succsim_j)\}$$
$$\succsim_{-S} = \succsim \setminus \bigcup_{k \in S} \{(k, \succsim_k)\} \qquad \succsim_{+T} = \succsim \cup \bigcup_{k \in T} \{(k, \succsim_k)\}$$

Whenever $|S|, |T| = 2$ we also write $\succsim_{-i,j}$ or $\succsim_{+i,j}$ and omit the set braces for better readability. Slightly abusing notation we regularly identify preference profiles with the collection of individual preferences only, $\succsim = (\succsim_1, \dots, \succsim_n)$. Preference profiles are often depicted in tabular form with individual voters' names in the top row and the corresponding preferences as column below with more preferred alternatives given above less preferred ones (see Example 2.1).

majority margin

Given a preference profile $\succsim \in \succsim(A)^N$, we define the *majority margin* $g_{xy}(\succsim)$, $x, y \in A$ of alternative x over alternative y as the number of voters that prefer x over y minus the number of voters who prefer y over x,

$$g_{xy}(\succsim) = |\{i \in N \colon x \succsim_i y\}| - |\{i \in N \colon y \succsim_i x\}|.$$

majority relation

Whenever $\succsim$ is clear from the context we just write g_{xy}. Note that trivially $g_{xy} = -g_{yx}$. The *majority relation* of $\succsim$ is denoted as $\succsim^M$, $x \succsim^M y$ if $g_{xy}(\succsim) \geqslant 0$, with $\succ^M$ being its strict part and $\sim^M$ the indifference part. Majority margins as well as the majority relation are of special interest as they allow for the succinct representation of preference profiles. In this respect, $(g_{xy})_{x,y \in A}$ can be written as a matrix $M \in \mathbb{Z}^{A \times A}$ while the same holds for $\succsim^M$ which can conveniently be represented as matrix $M \in \{-1, 0, 1\}^{A \times A}$. Alternatively, it is common to employ directed graphs with one vertex per alternative and either a weighted edge from x to y with weight g_{xy}, or an unweighted edge from x to y if $x \succsim^M y$. In both cases, we only depict edges with positive weight, i.e., where $g_{xy} > 0$, for reasons of clarity. These directed graphs are named *weighted majority graphs* or *majority graphs*, respectively.[31]

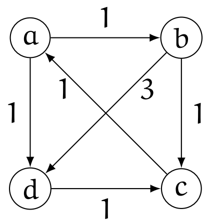
(weighted) majority graph

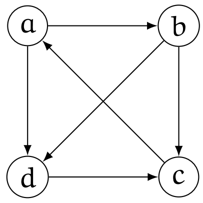
Example 2.1

Consider $N = \{1, 2, 3\}$ and $A = \{a, b, c, d\}$. A possible preference profile $\succsim \in \succsim(A)^N$ is depicted below on the left. In the middle we see the corresponding weighted majority graph with edge labels g_{xy} while the (unweighted) majority graph representing $\succsim^M$ is shown on the right.

31 Weighted majority graphs and majority graphs are also known as weighted (weak) tournaments or (weak) tournaments, respectively (see, e.g., Fischer et al., 2016; Brandt et al., 2016a).

1	2	3
a	b	c
b	d	a
d	c	b
c	a	d





For a set A of alternatives we denote by $\Delta(A)$ the set of all probability distributions (or *lotteries*) over A, i.e.,

lottery

$$\Delta(A) = \left\{ \sum_{x \in A} p(x) \cdot x \colon \sum_{x \in A} p(x) = 1 \text{ and } p(x) \geqslant 0 \text{ for all } x \in A \right\}.$$

Here, $p(x)$ is the probability lottery $p \in \Delta(A)$ awards to alternative $x \in A$. In addition, we define $p(X) = \sum_{x \in X} p(x)$ to be the total probability for alternatives $x \in X \subseteq A$. We write lotteries as convex combination of alternatives, i.e., $1/3\, a + 1/3\, b + 1/3\, c$ prescribes the uniform lottery over $\{a, b, c\}$. The *support* of a lottery $p \in \Delta(A)$ consists of all alternatives that are awarded positive probability, i.e.,

support

$$\text{supp}(p) = \{x \in A \colon p(x) > 0\}.$$

2.2 VOTING RULES

Our central objects of study are voting rules, i.e., rules that map a preference profile to a collective choice of alternatives. Based on which form this collective choice is to take, we distinguish between *single-valued*, *set-valued*, and *probabilistic voting rules*.[32] Formally, a single-valued voting rule is a function $f \colon \succsim(A)^{\mathcal{F}(\mathbb{N})} \to A$, for a set-valued voting rule we have $f \colon \succsim(A)^{\mathcal{F}(\mathbb{N})} \to 2^A \setminus \emptyset$, and a probabilistic voting rule is defined as $f \colon \succsim(A)^{\mathcal{F}(\mathbb{N})} \to \Delta(A)$. For the sake of readability, we sometimes slightly abuse notation and write $f(\succsim_i)$ instead of $f((i, \succsim_i))$ when only a single voter's preferences are of relevance. Whenever it is either clear from the context which specific type of rule we are concerned with, or we define characteristics relevant for all variants, we regularly just speak of voting rules.

Two standard properties of voting rules are anonymity and neutrality. *Anonymity* prescribes that a voting rule is invariant under the renaming of voters, i.e., $f(\succsim) = f(\succsim')$ for $N, N' \in \mathcal{F}(\mathbb{N})$ and $\succsim \in \succsim(A)^N$, $\succsim' \in \succsim(A)^{N'}$ if there exists a bijection $\pi \colon N \to N'$ and $\succsim'_i = \succsim_{\pi(i)}$ for all $i \in N$. For a permutation $\pi \colon A \to A$ and a preference relation $\succsim_i$, we define the relation $\succsim_i^\pi$ as $\pi(x) \succsim_i^\pi \pi(y)$ if $x \succsim_i y$. Now, a voting rule f is *neutral* if for all permutations $\pi \colon A \to A$ and all $\succsim \in \succsim(A)^{\mathcal{F}(\mathbb{N})}$ we have $f(\succsim^\pi) = \pi(f(\succsim))$, i.e., if all voters consistently

anonymity

neutrality

32 Note that throughout the literature they are often also referred to as resolute (or single-valued) social choice functions, irresolute (or set-valued) social choice functions, and social decision schemes or probabilistic social choice functions.

rename the alternatives in their corresponding preferences, then an identical renaming applies to $f(\succsim)$.[33] Anonymity and neutrality are often named symmetry with respect to voters and alternatives, respectively. They can be seen as basic fairness conditions as they guarantee a rule is unbiased.

pairwiseness

majoritarianness

Many well-known voting rules do not rely on the exact preference profiles but choose based on (weighted) majority comparisons only. In this sense, we say a voting rule f is *pairwise* if it is neutral and focuses on the majority margins only, i.e., $f(\succsim) = f(\succsim')$ whenever $g_{xy}(\succsim) = g_{xy}(\succsim')$ for all $x, y \in A$ where $\succsim, \succsim' \in \succsim(A)^{\mathcal{F}(\mathbb{N})}$ (see, e.g., Young, 1974b; Zwicker, 1991). A voting rule f is called *majoritarian* if it is neutral and chooses based on the majority relation only, i.e., $f(\succsim) = f(\succsim')$ whenever $\succsim^M = \succsim'^M$.[34] For majoritarian rules f we sometimes also write $f(\succsim^M)$ instead of $f(\succsim)$ for reasons of simplicity. When not reasoning about specific profiles, we generalize this to $f(G)$ which is equivalent to $f(\succsim)$ for all $\succsim \in \succsim(A)^{\mathcal{F}(\mathbb{N})}$ such that $\succsim^M = G$. Note that every majoritarian voting rule is also pairwise and every pairwise rule satisfies anonymity and neutrality.

2.3 CONDORCET CRITERION

Condorcet winner

Condorcet paradox

An alternative x is named *Condorcet winner* if it beats every other alternative in a pairwise majority comparison, i.e., x is Condorcet winner if $x \succ^M y$ for all $y \in A \setminus \{x\}$ (Condorcet, 1785). Condorcet winners are unique in the sense that whenever one exists, it is the only one.[35] As already noted by Fishburn (1977), there is good reason for a voting rule to choose a Condorcet winner as "the principle embodies the democratic precept of rule by majority will. Moreover, the majority candidate or alternative, when it exists, constitutes a stable equilibrium in that it cannot be beaten or displaced by a challenger in a direct majority vote between the two".[36] Note, however, that Condorcet winners may fail to exist which is often termed the *Condorcet paradox* (see Example 2.1).

Condorcet extension

Condorcet consistency

A voting rule that always uniquely selects the Condorcet winner whenever one exists is called a *Condorcet extension* or said to satisfy *Condorcet consistency* or the *Condorcet criterion*. For probabilistic voting rules, this transfers to giving probability one to the Condorcet winner.

33 For probabilistic voting rules this translates to $f(\succsim)(x) = f(\succsim^\pi)(\pi(x))$ for all $x \in A$.

34 Majoritarian voting rules are also known as *tournament solutions* while pairwise rules are often named *weighted tournament solutions* (see, e.g., Brandt et al., 2016a; Fischer et al., 2016). Though not strictly identical, both concepts can also be identified as C1 and C2 functions (Fishburn, 1977).

35 This is not the case for *weak Condorcet winners* x defined by $x \succsim^M y$ for all $y \in A$. Theoretically, all alternatives can be weak Condorcet winners at the same time.

36 For similar arguments see also, e.g., Dodgson (1876), Black (1958), Young (1988), and Dutta (1988).

The class of Condorcet extensions encompasses a grand variety of well-known and well-studied voting rules for all frameworks. This includes the *top cycle* (Good, 1971; Smith, 1973; Schwartz, 1986), the *uncovered set* (Fishburn, 1977; Miller, 1980), the *minimal covering set* (Dutta, 1988), and *Copeland's rule* (Copeland, 1951) which are all majoritarian set-valued voting rules. Important representatives for pairwise Condorcet consistent set-valued voting rules include *Kemeny's rule* (Kemeny, 1959), *MaxiMin* (Black, 1958), *ranked pairs* (Tideman, 1987), *Black's rule* (Black, 1958), and *Tideman's rule* (Tideman, 1987). Equipped with a suitable tie-breaking mechanism, all these rules directly define single-valued variants. When it comes to probabilistic voting rules or random assignment rules, *maximal lotteries* (Fishburn, 1984b) and their equivalent in the assignment setting, *popular random assignments* (Kavitha et al., 2011), are most established.

2.4 PARTICIPATION

We now turn to the matter of strategic manipulation, most importantly manipulation by abstention. By manipulation of a voting rule we mean that either a single voter or a group of voters can achieve a more preferred result by not submitting their corresponding preference relation $\succsim_i$ or $\succsim_S$. While the definition of a more preferred result is obvious for single-valued voting rules, it is intuitively unclear for set-valued or probabilistic rules as the individual preferences $\succsim_i$ only allow for the comparison of single alternatives. We can easily argue that a voter i with preferences $\succsim_i\colon a, b, c$ should prefer $\{a, b\}$ to $\{c\}$, but whether $\{a, c\}$ or $\{b\}$ is better for him is hard to tell.

Since it is generally considered impracticable to demand for preferences over all subsets of alternatives or lotteries thereover, we thus require an instrument to extend the preference relation $\succsim_i$. Functions fulfilling this need we name *preference extensions*. More formally, a preference extension Σ maps preferences over alternatives $\succsim_i$ to possibly incomplete preferences $\succsim_i^\Sigma$ over either sets of alternatives or lotteries.[37] A multitude of preference extensions has been established in the past decades, we refer to Barberà et al. (2004) as well as Cho (2016) and Brandt (2017) for an overview of possible ways to lift preferences to sets and lotteries, respectively. Specific extensions relevant to this thesis are discussed in Section 4.1.2 and Section 5.1.1.

preference extension

Intuitively, the no-show paradox prescribes that by abstaining from the election process, a voter receives a more preferred result. Being prone to the no-show paradox is without question a severe flaw of any voting rule, and indeed Nurmi (1999) argues that "[v]ulnerability to the no-show paradox is a serious drawback in a voting system. After

37 The terms *set extension* and *lottery extension* are also common throughout the literature.

all, any reasonable voter would expect that by voting he is contributing to the possibility that his favorite wins. The realization that the very act of communicating his true preferences by voting makes the outcome worse from his point of view than it would have been had he decided not to vote at all, may be demoralizing. It certainly undermines the very rationale of going to the polls."

no-show paradox

More formally, given a suitable preference extension Σ, we say a voting rule f is prone to the *no-show paradox (NSP)* or equivalently *manipulable by strategic abstention* if there exist A, $N \in \mathcal{F}(\mathbb{N})$ and $\succsim \in \succsim(A)^N$ such that $f(\succsim_{-i}) \succ_i^{\Sigma} f(\succsim)$ for some $i \in N$. If f is immune to the NSP we say that f satisfies Σ-*participation*.[38]

participation

group-participation

For groups of voters we additionally define a group-based notion of participation in a similar way. A voting rule f is said to satisfy Σ-*group-participation* if there are no A, $N \in \mathcal{F}(\mathbb{N})$, $\succsim \in \succsim(A)^N$, and $S \subseteq N$ such that $f(\succsim_{-S}) \succ_i^{\Sigma} f(\succsim)$ for all $i \in S$. Group-participation obviously implies participation.

For varying definitions of participation we refer to Section 1.2, which also provides further motivation and deals with related properties.

strategyproofness

group-strategyproofness

In addition to participation, we also define strategyproofness that captures immunity against manipulation by misrepresentation of preferences. Formally, for some preference extension Σ, we say a voting rule f is *manipulable by strategic misrepresentation* if there exist A, $N \in \mathcal{F}(\mathbb{N})$, $i \in N$, and $\succsim, \succsim' \in \succsim(A)^N$ with $\succsim_j = \succsim'_j$ for all $j \in N_{-i}$ such that $f(\succsim') \succ_i^{\Sigma} f(\succsim)$. If f is not manipulable by strategic misrepresentation according to extension Σ, we name it Σ-*strategyproof*. In analogy to before we also define group-strategyproofness. A voting rule f satisfies Σ-*group-strategyproofness* if there are no A, $N \in \mathcal{F}(\mathbb{N})$, $S \subseteq N$, and $\succsim, \succsim' \in \succsim(A)^N$ with $\succsim_j = \succsim'_j$ for all $j \in N \setminus S$ such that $f(\succsim') \succ_i^{\Sigma} f(\succsim)$ for all $i \in S$. Group-strategyproofness obviously is stronger than strategyproofness and thus the former implies the latter. We refer to Section 1.2.5 for a discussion of the relationship between participation and strategyproofness.

2.5 EFFICIENCY

Demanding for immunity against manipulation either by strategic abstention or misrepresentation by single voters and groups alike is undoubtedly important, but does not make a good voting rule on its own. A voting rule that always chooses an identical outcome is not manipulable but generally not ideal, either. At this point, (Pareto) efficiency becomes relevant. Intuitively, a voting rule is said to be

38 If f is single-valued or the extension is clear from the context we forgo the Σ and speak of *participation*.

efficient if it never chooses an outcome for which another one exists that is preferred by all voters.

unanimity

On its most basic level, this can be translated to *unanimity*, which prescribes that if all voters report an identical unique top alternative, this alternative has to be chosen uniquely (see, e.g., Sen, 2011; Picot and Sen, 2012; Chatterji et al., 2014). A probabilistic voting rule satisfies unanimity if this top alternative is given probability one.

Pareto dominance

Pareto optimality

ex post *efficiency*

The concept of Pareto optimality is stronger as it also restricts possible choices when voters do not share a common most preferred alternative. Given a preference profile $\succsim \in \succsim(A)^N$, alternative x *Pareto dominates* y if $x \succsim_i y$ for all $i \in N$ and there exists $j \in N$ such that $x \succ_j y$. If x is not Pareto dominated by any $y \in A$, x is called *Pareto optimal*. In the same sense, we name a single-valued or set-valued voting rule *Pareto optimal* if it never chooses Pareto dominated alternatives. For probabilistic voting rules we use the term ex post *efficiency* to prescribe that they never put positive probability on Pareto dominated alternatives (see, e.g., Gibbard, 1977; Bogomolnaia et al., 2005; Dutta et al., 2007). Pareto optimality implies unanimity, which is also satisfied by every Condorcet extension.

In Example 2.1, b Pareto dominates d. Alternatives a, b, and c are Pareto optimal just as any Pareto optimal voting rule has to choose a subset of $\{a, b, c\}$. For an *ex post* efficient probabilistic voting rule the chosen lottery p has to satisfy $p(d) = 0$.

Σ-dominance

Σ-efficiency

In particular for probabilistic voting rules, it is also common to define degrees of efficiency based on different preference extensions. For an extension Σ, we say a lottery p *Σ-dominates* lottery q in profile $\succsim \in \succsim(A)^N$ if $p \succsim_i^\Sigma q$ for all $i \in N$ and there exists $j \in N$ such that $p \succ_j^\Sigma q$. Lottery p is called *Σ-efficient* if there is no q that Σ-dominates it and similarly a probabilistic voting rule is called *Σ-efficient* if it never chooses a Σ-dominated lottery (see, e.g., Bogomolnaia and Moulin, 2001; Aziz et al., 2015; Aziz et al., 2018a).

Combining individual and social goals, i.e., immunity to different concepts of manipulation and satisfying varying degrees of efficiency, turns out to often be impossible, irrespective of the exact situation (see, e.g., Gibbard, 1973; Satterthwaite, 1975; Katta and Sethuraman, 2006; Aziz et al., 2018a; Brandl et al., 2018).

3 ANALYZING THE LIKELIHOOD OF THE NO-SHOW PARADOX

We begin this chapter by recalling the iconic impossibility theorem by Moulin (1988): every single-valued Condorcet extension is prone to the NSP. Moulin's theorem has sparked a whole line of research, see Section 1.2 for an overview. However, as previously discussed, most of the works following Moulin (1988) focus on the axioms used and the authors vary either single-valuedness, Condorcet consistency, the definition of the NSP, or multiple criteria at the same time.

A different question, posed first by Fishburn and Brams (1983), receives considerably less attention: "[i]s it indeed true that serious flaws in preferential voting such as the No-Show Paradox [...] are sufficiently rare as to cause no practical concern?" Put differently, Fishburn and Brams propose "to assess the likelihood of the paradox [...] as an interesting problem for investigation". Even though a long while has passed since then, we are aware of only three papers having tackled this issue.[39] However, all of them focus on specific definitions of the NSP and are limited to three alternatives.

We here stick to Moulin's definition of the NSP and study popular majoritarian and pairwise Condorcet extensions, among them Copeland's rule, MaxiMin, Black's, and Tideman's rule. In particular for MaxiMin, we push the boundaries of previous research and obtain exact analytical results for four alternatives and the *impartial anonymous culture* model. These numbers are obtained via modeling occurrences of the NSP by describing susceptible preference profiles using linear inequalities. Ehrhart theory (Ehrhart, 1962) allows for a count of profiles, and since the total number of profiles is known, we obtain an exact fraction.

Furthermore, we complement our analytical results by elaborate computer simulations that are perfectly aligned for up to four alternatives and allow for insights way beyond. Also included in our experiments are comparisons of different preference models that once more allow for interesting observations.

The remainder of this chapter is structured as follows: we first specify the setting considered here in Section 3.1 together with definitions of relevant voting rules and preference models. In Section 3.2, we discuss two ways of quantifying the NSP, exact and experimental analysis. Obtained results are presented and discussed in Section 3.3, which is followed by some concluding remarks in Section 3.4.

39 These being Ray (1986), Lepelley and Merlin (2000), and Kamwa et al. (2018), see Section 1.2 for some further details.

3.1 PRELIMINARIES

For this chapter, we assume all individual preferences to be strict, i.e., for any two alternatives $x, y \in A$, $x \neq y$, and every voter $i \in N$, we have that either $x \succ_i y$ or $y \succ_i x$. We therefore denote preference profiles by $\succ$ instead of $\succsim$ here in order to be clear.

3.1.1 Voting Rules

We study six different voting rules in terms of how likely it is that a voter can manipulate by abstaining strategically. The selection of rules is mainly based on three criteria:

- *Discriminability.* We want to minimize the influence of tie-breaking, which we have to make use of to obtain single-valued voting rules.
- *Simplicity.* We have to be able to model the choice sets using linear inequalities for the analytical Ehrhart theory approach and in addition voters generally prefer 'simpler' rules.
- *Efficient computability.* This is a basic requirement to enable rigorous and comprehensive simulations.[40]

In the following, we briefly define all considered voting rules.

Borda score

BLACK'S RULE. For a preference profile $\succ \in \succsim(A)^N$ and an alternative $x \in A$, we define the *Borda score* $s_x(\succ)$ of x as

$$s_x(\succ) = \sum_{i \in N} |\{y \in A \colon x \succ_i y\}|.$$

Borda winner

Black's rule

The alternatives with maximal Borda score among all alternatives are named *Borda winners* (Borda, 1784). For our modeling by linear inequalities and to be in line with other rules, we equivalently say that an alternative x is Borda winner if it maximizes $\sum_{y \in A \setminus \{x\}} g_{xy}$.[41] Now, *Black's rule* (Black, 1958) chooses the Condorcet winner whenever it exists and otherwise returns a winner according to Borda's rule:

$$f_{\text{Black}}(\succ) \in \begin{cases} x & \text{if } x \text{ is a Condorcet winner in } \succ \\ \arg\max_{x \in A} \sum_{y \in A \setminus \{x\}} g_{xy} & \text{otherwise.} \end{cases}$$

Baldwin's rule

BALDWIN'S RULE. *Baldwin's rule* (Baldwin, 1926) proceeds in multiple rounds. In each round, we drop the alternative with lowest Borda score and then continue with the preference profile reduced by one

40 Note that other discriminating Condorcet extensions such as Kemeny's rule, Dodgson's rule, and Young's rule are NP-hard to compute (see, e.g., Brandt et al., 2016b).

41 To be precise, what we define here is also known as the *asymmetric* and *symmetric Borda scores* and known to be affinely equivalent (see, e.g., Zwicker, 2016, and the proof of Theorem 5.10).

alternative, which is used to calculate updated scores. If multiple—but not all—alternatives are tied last, we delete all of them. Baldwin's rule chooses one of the alternatives remaining when no more alternative can be removed.

NANSON'S RULE. *Nanson's rule* (Nanson, 1883; Niou, 1987) is similar to Baldwin's rule in so far as it also focuses on the Borda scores and gradually eliminates alternatives. However, in contrast to before, we now remove all alternatives with average or below-average Borda score in every round. Nanson's rule returns an alternative out of those remaining when all alternatives have identical score. *Nanson's rule*

MAXIMIN. *MaxiMin* (Black, 1958), which is also known as the Simpson-Kramer method (Simpson, 1969; Kramer, 1977), looks at the worst pairwise majority comparison for each alternative. It then returns an alternative with maximal such score, formally *MaxiMin*

$$f_{\text{MaxiMin}}(\succ) \in \arg\max_{x \in A} \min_{y \in A \setminus \{x\}} g_{xy}.$$

TIDEMAN'S RULE. In contrast to MaxiMin, *Tideman's rule* (Tideman, 1987) focuses on the sum of all pairwise majority defeats. It yields an alternative where this sum is closest to zero in terms of absolute value, i.e., *Tideman's rule*

$$f_{\text{Tideman}}(\succ) \in \arg\max_{x \in A} \sum_{y \in A \setminus \{x\}} \min(0, g_{xy}).$$ [42]

COPELAND'S RULE. *Copeland's rule* (Copeland, 1951) only relies on the majority relation and not the exact majority margins. It chooses an alternative where the number of majority wins plus half the number of majority draws is maximal: *Copeland's rule*

$$f_{\text{Copeland}}(\succ) \in \arg\max_{x \in A} |\{y \in A \colon x \succ^M y\}| + 1/2\, |\{y \in A \setminus \{x\} \colon x \sim^M y\}|$$

In order to obtain well-defined single-valued voting rules, we employ alphabetic tie-breaking for all rules defined above. Note that the alphabetic ordering does not influence our results as long as we assume that there is some underlying ordering. This changes if we allow for tie-breaking based on the preference profile or choice set, something we however want our tie-breaking to be independent of. All presented voting rules can be computed in polynomial time and do not rely on the exact preference profile $\succ$, but only on the majority

42 Tideman's rule is arguably the least well-known voting rule presented here. It was proposed to efficiently approximate Dodgson's rule and is not to be confused with ranked pairs, which is sometimes also called Tideman's rule. Also note that the 'dual' rule returning alternatives for which the sum of weighted pairwise majority *wins* is maximal is not a Condorcet extension.

margins that can conveniently be represented by a skew-symmetric matrix or a weighted majority graph.

Example 3.1

Consider the preference profile $\succ$ with seven voters and four alternatives depicted below together with the matrix of pairwise majority margins.

1,2,3	4,5,6	7
a	d	b
c	c	d
b	b	a
d	a	c

$\succ$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -5 & 1 \\ -1 & 5 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \end{array}$$

In the absence of a Condorcet winner, Black's rule relies on the Borda scores which can be computed to be $s(\succ) = (10, 9, 12, 11)$ or, affinely equivalent, $(-1, -3, 3, 1)$ when determining them based on the majority margins only. Hence, $f_{\text{Black}}(\succ) = c$.

Having the lowest Borda score, b consequently is the first alternative to be eliminated when applying Baldwin's rule. After dropping c next, we have a strict majority in favor of d against a and thus $f_{\text{Baldwin}}(\succ) = d$.

In the first round of Nanson's rule, we eliminate a and b since both alternatives have a Borda score which is below average. Thereafter, we obtain a strict majority for d against c, meaning d has higher Borda score and it follows $f_{\text{Nanson}}(\succ) = d$.

For MaxiMin, we analyze all alternatives' worst pairwise majority comparison and see that a, c, and d are tied with -1. Due to alphabetic tie-breaking we have $f_{\text{MaxiMin}}(\succ) = a$.

Tideman's rule counts the sum of all pairwise majority defeats, which we find to be 2, 5, 2, and 1 for a, b, c, and d, respectively. The alternative with minimal sum is chosen, hence, $f_{\text{Tideman}}(\succ) = d$.

Lastly, Copeland's rule selects an alternative based on the number of pairwise majority wins and here breaks the tie between b and d alphabetically leading to $f_{\text{Copeland}}(\succ) = b$.

For the sake of completeness, we remark that all rules defined here are Condorcet extensions and thus suffer from the NSP (Moulin, 1988). Occurrences of the NSP for Black's, Baldwin's, and Copeland's rule require three alternatives while four alternatives are needed for MaxiMin as well as Nanson's and Tideman's rule.

It is interesting to note that whenever a Condorcet winner exists, no *weak Condorcet extension* allows for manipulation by strategic abstention by a single voter.[43] To see this, assume alternative x is the Condorcet winner, i.e., wins in a pairwise majority comparison against all other alternatives. While some of these strict majority preferences might turn to indifferences if voter i abstains from the election procedure, this can only happen for comparisons to alternatives less preferred than x according to $\succ_i$. Hence, every alternative strictly more preferred than x still loses at least the pairwise majority comparison against x, which remains a weak Condorcet winner. We deduce that irrespective of other possible weak Condorcet winners and the underlying tie-breaking, no alternative preferred to x can be chosen. Of the rules defined above, MaxiMin and Tideman's rule are weak Condorcet extensions.[44]

3.1.2 Preference Models

When analyzing properties of voting rules, it is a common approach to sample preferences according to some underlying model. A high amount of sampled preference profiles together with a—usually automated—evaluation of predefined criteria can yield interesting insights into, e.g., the expected cardinality of chosen sets for set-valued voting rules, or, as in this chapter, the frequency of voting paradoxes like the NSP. Various concepts to model preferences have been introduced over the years; we here informally and very briefly present the ones employed later on and refer to, e.g., Critchlow et al. (1991) and Marden (1995) for a detailed discussion. We focus on three parameter-free models: impartial culture, impartial anonymous culture, and the spatial model. Furthermore, we consider the urn model and Mallows' ϕ.

IMPARTIAL CULTURE. *Impartial culture* (IC) is the most basic model where each of the n voters is assigned one out of $m!$ possible preference relations uniformly at random. Hence, every possible preference profile is given identical probability. IC

IMPARTIAL ANONYMOUS CULTURE. Though using a similar idea, *impartial anonymous culture* (IAC) (Gehrlein and Fishburn, 1976) follows a slightly different approach selecting each anonymous preference profile with the same probability. More details on anonymous profiles can also be found in Section 3.2.1. Both IC and IAC are often IAC

43 A weak Condorcet extension chooses a weak Condorcet winner whenever one exists. Since every Condorcet winner automatically is a weak Condorcet winner, we obtain that every weak Condorcet extension is Condorcet consistent.

44 For both MaxiMin and Tideman's rule this holds by the observation that a weak Condorcet winner does not lose any pairwise majority comparison. Black's rule fails to be a weak Condorcet extension by definition; a counterexample for Baldwin's, Nanson's, and Copeland's rule is given by Fishburn (1977).

criticized for not being very realistic (see, e.g., Tsetlin et al., 2003; Regenwetter et al., 2006).

spatial model

SPATIAL MODEL. For the (two-dimensional) *spatial model* we sample points in the unit square uniformly at random, one for every voter and alternative each. The proximity to the alternatives then determines the voters' preferences; the closer the more preferred.

urn model

URN MODEL. The Pólya-Eggenberger *urn model* (Berg, 1985) bears similarities to IC, only individual preferences are not all sampled uniformly at random. Intuitively, we draw a preference ranking for the first voter just as for IC but than add another 10 copies of this very ranking to the 'urn' we draw from. We repeat this procedure for every ranking assigned to one of the voters, thus making already existing rankings ever more likely to be selected again.

Mallow's ϕ

MALLOWS' ϕ. Following Mallows' model (Mallows, 1957) we first sample a 'true' ranking $\succ_0$. Based on this $\succ_0$, we can compute the Kendall-tau distance (Kendall, 1938) of all other preference rankings. We now sample a ranking for every voter where a ranking is more likely to be drawn the closer it is to $\succ_0$ (in terms of Kendall-tau distance). $\phi \in (0, 1]$ is a dispersion parameter determining the influence the distance has: small values of ϕ put almost all probability on rankings close to $\succ_0$ while for $\phi = 1$, Mallows' model coincides with IC. We set $\phi = 0.8$ to obtain a reasonable variety of preferences and still be sufficiently different from IC.

The preference models we consider (such as IC, IAC, and Mallows' model) have also found widespread acceptance for the experimental analysis of voting rules within the multiagent systems and artificial intelligence community (see, e.g., Aziz et al., 2013d; Brandt and Seedig, 2014; Goldsmith et al., 2014; Oren et al., 2015; Brandt et al., 2016d).

3.2 QUANTIFYING THE NO-SHOW PARADOX

The goal in this chapter is to quantify the frequency of the NSP, i.e., to investigate for how many preference profiles a voter is incentivized to abstain from an election. In order to achieve this, we employ exact analysis via Ehrhart theory and experimental analysis via sampled preference profiles.

3.2.1 Exact Analysis via Ehrhart Theory

The imminent strength of exact analysis is that it gives reliable theoretical results. On the downside, precise computation is only feasible for very simple preference models and even then only for small values of m. We focus on IAC and make use of Ehrhart theory.

The general idea to quantify voting paradoxes via IAC has been around since the formal introduction of this preference model by Gehrlein and Fishburn (1976).[45] Still, a good thirty years were to pass until the connection to Ehrhart theory (Ehrhart, 1962) was established by Lepelley et al. (2008). We refer to Gehrlein and Lepelley (2011) for a more profound explanation of all details and an overview of results subsequently achieved.[46] The step from three to four alternatives in comparison to prior work, which equals a step from six to 24 dimensions, is only possible due to recent advances in computer algebra systems by De Loera et al. (2012) and Bruns and Söger (2015).

We follow an approach also described by Brandt et al. (2016d), who study the frequency of two single-profile paradoxes for a series of Condorcet extensions (Condorcet loser paradox and agenda contraction paradox). Brandt et al. are, to the best of our knowledge, the first to obtain analytical results for four alternatives. In a recent paper, Bruns et al. (2017) also make use of the possibility to analyze situations with $m = 4$ and look at the Condorcet efficiency of plurality and plurality with run-off as well as the structure of majority graphs and varying Borda paradoxes.

First, note that an anonymous preference profile is completely specified by the number of voters sharing each of the $m!$ possible preference rankings on m alternatives. Hence, we can sort all these rankings lexicographically, number them consecutively and so uniquely represent any anonymous profile by an integer point x in a space of $m!$ dimensions. We interpret x_i as the number of voters of type $\succ_i$, i.e., sharing preference ranking $\succ_i$. For a given number n of voters, we can easily describe the set of possible anonymous profiles as all integer points $x \in \mathbb{R}^{m!}$ that satisfy

$$\begin{aligned} x_i &\geqslant 0 \text{ for all } 1 \leqslant i \leqslant m! \text{ and} \\ \sum_{1 \leqslant i \leqslant m!} x_i &= n. \end{aligned}$$

This number is known to be $\binom{m!+n-1}{m!-1}$ (see, e.g., Feller, 1966).

Next, we want to count the number of profiles for a fixed m where a manipulation by abstention is possible for at least one voter. We do so by describing the set of all such profiles using linear (in)equalities, i.e., as a polytope P_n.[47, 48] Making use of Ehrhart theory (Ehrhart, 1962), we can determine the number of integer points in P_n and directly

45 See, e.g., Lepelley et al. (1996), Le Breton et al. (2016), and Lepelley et al. (2018).

46 See also, e.g., Wilson and Pritchard (2007), Schürmann (2013), and Le Breton et al. (2016) for some more recent papers using Ehrhart theory in the context of voting.

47 More precisely, P_n is a *dilated polytope* depending on n, $P_n = nP = \{n\vec{x} \colon \vec{x} \in P\}$.

48 We will see that a single polytope seldom suffices and we need to define a vast number thereof.

obtain the likelihood of an occurence of the NSP as fraction of the total number of anonymous profiles.

Ehrhart polynomial

period

Ehrhart shows that the number of points in P_n can be found by so-called *Ehrhart-* or *quasi-polynomials* f. Each f is not a single polynomial but rather a collection of q polynomials f_i of degree d such that $f(n) = f_i(n)$ if $n \equiv i \mod q$. q is also named the *period* of f and we intuitively have that a larger q signifies that f consists of a greater number of different polynomials. Also, a larger q generally means that f is harder to find and imposes less structure upon $f(n)$. Obtaining f is possible via computer programs like LattE (De Loera et al., 2004) or Normaliz (Bruns et al.). Brandt et al. (2016d) give a more detailed description of the general methodology.

We continue with a detailed explanation of how to model manipulation instances via polytopes for MaxiMin and $m = 4$ in Section 3.2.2. Additional modelings used later on are for Black's and Copeland's rule for $m = 3$; the respective polytopes can be found in Section 3.2.3 and Section 3.2.4.

3.2.2 Case Study: MaxiMin

For the modeling we need to give linear constraints in terms of voter types—or equivalently majority margins—that describe polytopes containing all profiles prone to the NSP. Recall the definition of MaxiMin from Section 3.1.1,

$$f_{\text{MaxiMin}}(\succ) = \arg\max_{x \in A} \min_{y \in A \setminus \{x\}} g_{xy},$$

and assume $f_{\text{MaxiMin}}(\succ) = x \in A$. For the NSP to occur, two intrinsic conditions have to be fulfilled:

1. There is a voter $i \in N$ such that $f_{\text{MaxiMin}}(\succ_{-i}) = y \neq x$.

2. For voter i, we have that $y \succ_i x$.

While for a specific instance of $A = \{a, b, c, d\}$ and a manipulation from a to b, constraint 2 only reduces the possible types of manipulators to twelve,[49] constraint 1 is more restrictive in that sense. We find that somewhat surprisingly only two out of those twelve types of voters are capable of actually making alternative b the winner when abstaining the election process, namely

$$\succ_i: c, b, a, d \quad \text{and} \quad \succ_j: d, b, a, c.$$

It can be shown that no instance exists, in which both voter types can influence the outcome in their favor. For the sake of this case study, let us focus on $\succ_i$ for the moment.

49 In twelve out of 24 permutations of $\{a, b, c, d\}$ we have b ranked above a.

A first analysis shows that for the desired manipulation to be achievable by i, it is necessary that the highest defeat of a—the winning alternative under MaxiMin—has to be versus d. Similarly, we obtain that b's highest defeat must be against c with $g_{ad} = g_{bc}$,[50] and any other defeat of b has to be lower by at least two, this being the only way i can make b the winner instead of a. These considerations give rise to a first set of basic conditions that has to hold for any manipulation instance from a to b by i, no matter a further case distinction.[51]

$$g_{ad} = g_{bc}, \qquad g_{ad} \leqslant 0, \qquad g_{ab} \geqslant g_{ad}, \qquad g_{ba} \geqslant g_{ad} + 2, \qquad x_i \geqslant 1 \tag{basis}$$

We distinguish between $g_{cd} = 0$, $g_{cd} \leqslant -1$, and $g_{cd} \geqslant 1$.

Case $g_{cd} = 0$. To achieve that $f_{\text{MaxiMin}}(\succ) = a$ and simultaneously get a valid manipulation, we have to ensure that c's and d's highest defeat is at least as large as a's:

$$g_{cd} = 0, \qquad g_{ca} \leqslant g_{ad}, \qquad g_{db} \leqslant g_{ad} \tag{A}$$

Case $g_{cd} \leqslant -1$. Regarding c and d, we deduce that just as before, their corresponding highest defeat has to be at least as large as a's. For d that means a defeat versus b, for c it could be either against a, d, or both. This is represented in the following set of constraints:

$$g_{cd} \leqslant -1, \qquad g_{db} \leqslant g_{ab} \tag{B}$$

$$g_{cd} \leqslant g_{ad}, \qquad g_{ca} \leqslant g_{ad} \tag{B1}$$

$$g_{cd} \leqslant g_{ad}, \qquad g_{ca} \geqslant g_{ad} + 1, \qquad g_{ac} \geqslant g_{ad} \tag{B2}$$

$$g_{cd} \geqslant g_{ad} + 1, \qquad g_{ca} \leqslant g_{ad} \tag{B3}$$

Case $g_{cd} \geqslant 1$. This case is almost symmetric to the previous one with reversed arguments for c and d.

$$g_{cd} \geqslant 1, \qquad g_{ca} \leqslant g_{ab} \tag{C}$$

$$g_{dc} \leqslant g_{ad}, \qquad g_{db} \leqslant g_{ad} \tag{C1}$$

$$g_{dc} \leqslant g_{ad}, \qquad g_{db} \geqslant g_{ad} + 1, \qquad g_{bd} \geqslant g_{ad} + 2 \tag{C2}$$

$$g_{dc} \geqslant g_{ad} + 1, \qquad g_{db} \leqslant g_{ad} \tag{C3}$$

Finally, the total set of profiles admitting a manipulation from a to b by i can be described by seven polytopes making use of the constraints developed above. We obtain

50 Theoretically, we only require $g_{ad} - 1 \leqslant g_{bc} \leqslant g_{ad}$. As either all g_{xy} are even or all g_{xy} are odd, this collapses to $g_{ad} = g_{bc}$.

51 Some inequalities are omitted to remove redundancies when taken together with later constraints.

- $P_1 = \text{(basis)} + \text{(A)}$,

- $P_2 = \text{(basis)} + \text{(B)} + (\text{B}_1)$, $P_3 = \text{(basis)} + \text{(B)} + (\text{B}_2)$, $P_4 = \text{(basis)} + \text{(B)} + (\text{B}_3)$,

- $P_5 = \text{(basis)} + \text{(C)} + (\text{C}_1)$, $P_6 = \text{(basis)} + \text{(C)} + (\text{C}_2)$, and $P_7 = \text{(basis)} + \text{(C)} + (\text{C}_3)$.[52]

Since we are interested in not only voter type $\succ_i$ but also $\succ_j$ and equivalently not only manipulations from a to b but also all different combinations, we need to undergo a similar reasoning 24 times. This amounts to a total of 168 disjoint polytopes to encompass all profiles prone to the NSP. We remark that even though manipulation instances are roughly in line for all 24 types of voters, there are no exact symmetries that allow for reducing the number of polytopes. This is mostly due to alphabetic—i.e., non-symmetric—tie-breaking and the required presence of a certain voter type in the electorate. Both effects diminish as n grows, but discrepancies between different types of manipulators are significant up to lower three-digit n.[53]

Even though no exact symmetries exist, it is possible to give all constraints in a general form varying based on the manipulating voter's type. This allows a computer to easily create all 168 needed polytopes.

Assume therefore that the manipulator i has preferences $\succ_i: w, x, y, z$ with (w, x, y, z) being some permutation of the alternatives $A = \{a, b, c, d\}$. Let furthermore $u > v$ denote that u precedes v in the alphabetical tie-breaking order, $u, v \in \{w, x, y, z\}$. We have that voter i can manipulate from y to x only, and the necessary (in)equalities to describe profiles where such a manipulation is possible look as follows:

$$g_{yz} = \begin{cases} g_{xw} & \text{if } y > x, \\ g_{xw} + 2 & \text{if } x > y, \end{cases} \qquad g_{yz} \leqslant 0,$$
$$g_{yx} \geqslant g_{yz}, \qquad g_{xy} \geqslant g_{xw} + 2, \qquad \text{(basis)}$$
$$x_i \geqslant 1$$

52 We choose this informal notation for the sake of readability. It is to be understood in a way that P_1 is the polytope described by (in)equalities labelled (basis) as well as (A). We additionally assume for all polytopes that the sum of voters per type adds up to n and each type consists of a nonnegative amount of voters.

53 The chance to observe ties decreases as n grows and approaches zero as n goes to infinity. Hence, were we interested in the limit only, we could ignore the tie-breaking and save a lot of work. Note, however, that the limit does not offer instructive insights for the NSP, as we will see in Section 3.3.1.

$$g_{wz} = 0, \tag{A}$$
$$g_{wy} \leqslant \begin{cases} g_{yz} & \text{if } y > w, \\ g_{yz} - 1 & \text{if } w > y, \end{cases} \qquad g_{zx} \leqslant \begin{cases} g_{xw} & \text{if } x > z, \\ g_{xw} - 1 & \text{if } z > x \end{cases}$$

$$g_{wz} \leqslant -1, \qquad g_{zx} \leqslant \begin{cases} g_{xw} & \text{if } x > z, \\ g_{xw} - 1 & \text{if } z > x \end{cases} \tag{B}$$

$$g_{wy} \leqslant \begin{cases} g_{yz} & \text{if } y > w, \\ g_{yz} - 1 & \text{if } w > y, \end{cases} \quad g_{wz} \leqslant \begin{cases} g_{yz} & \text{if } y > w, \\ g_{yz} - 1 & \text{if } w > y \end{cases} \tag{B1}$$

$$g_{wy} \leqslant \begin{cases} g_{yz} & \text{if } y > w, \\ g_{yz} - 1 & \text{if } w > y, \end{cases} \quad g_{wz} \geqslant \begin{cases} g_{yz} + 1 & \text{if } y > w, \\ g_{yz} & \text{if } w > y \end{cases} \tag{B2}$$

$$g_{wy} \geqslant \begin{cases} g_{yz} + 1 & \text{if } y > w, \\ g_{yz} & \text{if } w > y, \end{cases} \quad g_{wz} \leqslant \begin{cases} g_{yz} & \text{if } y > w, \\ g_{yz} - 1 & \text{if } w > y, \end{cases} \tag{B3}$$
$$g_{yw} \geqslant g_{yz}$$

$$g_{wz} \geqslant 1, \qquad g_{wy} \leqslant \begin{cases} g_{yz} & \text{if } y > w, \\ g_{yz} - 1 & \text{if } w > y \end{cases} \tag{C}$$

$$g_{zx} \leqslant \begin{cases} g_{xw} & \text{if } x > z, \\ g_{xw} - 1 & \text{if } z > x \end{cases} \quad g_{zw} \leqslant \begin{cases} g_{xw} & \text{if } x > z, \\ g_{xw} - 1 & \text{if } z > x \end{cases} \tag{C1}$$

$$g_{zx} \leqslant \begin{cases} g_{xw} & \text{if } x > z, \\ g_{xw} - 1 & \text{if } z > x \end{cases} \quad g_{zw} \geqslant \begin{cases} g_{xw} + 1 & \text{if } x > z, \\ g_{xw} & \text{if } z > x \end{cases} \tag{C2}$$

$$g_{zx} \geqslant \begin{cases} g_{xw} + 1 & \text{if } x > z, \\ g_{xw} & \text{if } z > x, \end{cases} \quad g_{zw} \leqslant \begin{cases} g_{xw} & \text{if } x > z, \\ g_{xw} - 1 & \text{if } z > x, \end{cases} \tag{C3}$$
$$g_{xz} \geqslant g_{xw} + 1$$

Reusing previous notation we can easily write down the respective polytopes using these linear constraints. We obtain

- $P_1 = (\text{basis}) + (A)$,

- $P_2 = (\text{basis}) + (B) + (B_1)$, $P_3 = (\text{basis}) + (B) + (B_2)$, $P_4 = (\text{basis}) + (B) + (B_3)$,
- $P_5 = (\text{basis}) + (C) + (C_1)$, $P_6 = (\text{basis}) + (C) + (C_2)$, and $P_7 = (\text{basis}) + (C) + (C_3)$.

Doing so for all 24 types of possible manipulators yields the total set of polytopes that model all profiles prone to the NSP.[54]

This approach is substantially more involved than using Ehrhart theory for other paradoxes, e.g., the Condorcet loser paradox (Brandt et al., 2016d), mainly because of three reasons.

1. While the Condorcet loser paradox relies on the majority margins only, an occurence of the NSP demands for a certain type of voter to be present in the electorate.
2. There are instances where different types of voters present may each manipulate the outcome by abstaining. Thus, cases where only a single or multiple voters are able to manipulate have to be considered separately for obtaining the total number of profiles that are subject to the NSP. This results in a significant increase in the number of polytopes required.[55]
3. Occurrence of the NSP does not only rely on a certain alternative winning, but also on *how* that alternative wins, i.e., the corresponding majority margins matter. By our modeling, we have to ensure that though a given alternative wins beforehand, all margins are exactly such that the manipulator does sway them in the proper extent to make a more preferred alternative the winner. Hence, more than a mere ordinal comparison between raw majority margins is required.

3.2.3 Polytopes for Black's Rule

For the sake of completeness, we also include the polytopes underlying the Ehrhart polynomials to compute the exact fraction of profiles admitting a manipulation by abstention for Black's rule and $m = 3$.

Note that for three alternatives, the lexicographically alphabetic ordering of voter types is as given on the right.

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$	$\succsim_6$
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

A first analysis shows that when n is even, manipulation is only possible from a Borda winner towards a Condorcet winner. Conversely, when n is odd, any possible manipulation necessarily is from

54 Similar to before, we here again additionally assume that the total number of voters sums up to n and we have a nonnegative amount of voters per voter type.

55 As a matter of fact, this cannot occur for MaxiMin. It is, however, relevant for, e.g., Black's rule.

a Condorcet winner to a Borda winner. We start with n being even where we have one polytope per voter type. Under the usual additional assumption that all x_i are nonnegative and sum up to n, these polytopes describe manipulations from a to b, a to c, b to a, b to c, c to a, and c to b, in this order for even and odd n both.

$$\begin{aligned} &g_{ab} + g_{ac} \geqslant g_{ba} + g_{bc}, && g_{ba} \geqslant 1, && x_6 \geqslant 1, \\ &g_{ab} + g_{ac} \geqslant g_{ca} + g_{cb}, && g_{bc} = 0 && \end{aligned} \tag{P1}$$

$$\begin{aligned} &g_{ab} + g_{ac} \geqslant g_{ba} + g_{bc}, && g_{ca} \geqslant 1, && x_4 \geqslant 1, \\ &g_{ab} + g_{ac} \geqslant g_{ca} + g_{cb}, && g_{cb} = 0 && \end{aligned} \tag{P2}$$

$$\begin{aligned} &g_{ba} + g_{bc} \geqslant g_{ab} + g_{ac} + 1, && g_{ab} \geqslant 1, && x_5 \geqslant 1, \\ &g_{ba} + g_{bc} \geqslant g_{ca} + g_{cb}, && g_{ac} = 0 && \end{aligned} \tag{P3}$$

$$\begin{aligned} &g_{ba} + g_{bc} \geqslant g_{ab} + g_{ac} + 1, && g_{cb} \geqslant 1, && x_2 \geqslant 1, \\ &g_{ba} + g_{bc} \geqslant g_{ca} + g_{cb}, && g_{ca} = 0 && \end{aligned} \tag{P4}$$

$$\begin{aligned} &g_{ca} + g_{cb} \geqslant g_{ab} + g_{ac} + 1, && g_{ac} \geqslant 1, && x_3 \geqslant 1, \\ &g_{ca} + g_{cb} \geqslant g_{ba} + g_{bc} + 1, && g_{ab} = 0 && \end{aligned} \tag{P5}$$

$$\begin{aligned} &g_{ca} + g_{cb} \geqslant g_{ab} + g_{ac} + 1, && g_{bc} \geqslant 1, && x_1 \geqslant 1, \\ &g_{ca} + g_{cb} \geqslant g_{ba} + g_{bc} + 1, && g_{ba} = 0 && \end{aligned} \tag{P6}$$

For these polytopes the inequalities in the left column model that the required alternative currently is the Borda winner. The (in)equalities in the second column guarantee that a manipulator can make the desired alternative Condorcet winner by abstaining as well as that with him being present, there is no Condorcet winner. The last column only demands presence of the voter type being able to manipulate.

For odd n, this only changes in so far as the first column now models that the desired alternative can be made the Borda winner by abstaining. The second column, on the other hand, describes that whichever alternative is manipulated away from currently is Condorcet winner, while it will not be so anymore after the manipulation.

$$\begin{aligned} &g_{ba} + g_{bc} \geqslant g_{ab} + g_{ac} + 1, && g_{ab} \geqslant 1, && x_3 \geqslant 1, \\ &g_{ba} + g_{bc} \geqslant g_{ca} + g_{cb} + 1, && g_{ac} = 1 && \end{aligned} \tag{P7}$$

$$\begin{aligned} g_{ca} + g_{cb} &\geqslant g_{ab} + g_{ac} + 1, & g_{ac} &\geqslant 1, & x_5 &\geqslant 1, \\ g_{ca} + g_{cb} &\geqslant g_{ba} + g_{bc} + 3, & g_{ab} &= 1 \end{aligned} \tag{P$_8$}$$

$$\begin{aligned} g_{ab} + g_{ac} &\geqslant g_{ba} + g_{bc} + 1, & g_{ba} &\geqslant 1, & x_1 &\geqslant 1, \\ g_{ab} + g_{ac} &\geqslant g_{ca} + g_{cb} + 1, & g_{bc} &= 1 \end{aligned} \tag{P$_9$}$$

$$\begin{aligned} g_{ca} + g_{cb} &\geqslant g_{ab} + g_{ac} + 3, & g_{bc} &\geqslant 1, & x_6 &\geqslant 1, \\ g_{ca} + g_{cb} &\geqslant g_{ba} + g_{bc} + 1, & g_{ba} &= 1 \end{aligned} \tag{P$_{10}$}$$

$$\begin{aligned} g_{ab} + g_{ac} &\geqslant g_{ba} + g_{bc} + 1, & g_{ca} &\geqslant 1, & x_2 &\geqslant 1, \\ g_{ab} + g_{ac} &\geqslant g_{ca} + g_{cb} + 1, & g_{cb} &= 1 \end{aligned} \tag{P$_{11}$}$$

$$\begin{aligned} g_{ba} + g_{bc} &\geqslant g_{ab} + g_{ac} + 3, & g_{cb} &\geqslant 1, & x_4 &\geqslant 1, \\ g_{ba} + g_{bc} &\geqslant g_{ca} + g_{cb} + 1, & g_{ca} &= 1 \end{aligned} \tag{P$_{12}$}$$

The total number of profiles prone to the NSP equals the sum of integer points contained in P_1 to P_{12}.

3.2.4 Polytopes for Copeland's Rule

For Copeland's rule and $m = 3$, all profiles prone to the NSP can be modeled using four polytopes only.

$$g_{ba} \geqslant 2, \quad g_{ac} \geqslant 1, \quad g_{cb} = 1, \quad x_6 \geqslant 1 \tag{P$_1$}$$

$$g_{ca} \geqslant 2, \quad g_{ab} \geqslant 1, \quad g_{bc} = 1, \quad x_4 \geqslant 1 \tag{P$_2$}$$

$$g_{ac} \geqslant 2, \quad g_{ba} \geqslant 1, \quad g_{bc} = 0, \quad x_1 \geqslant 1 \tag{P$_3$}$$

$$g_{ab} \geqslant 1, \quad g_{ca} \geqslant 1, \quad g_{bc} = 0, \quad x_2 \geqslant 1 \tag{P$_4$}$$

Note that once again, we implicitly assume a nonnegative amount of voters per voter type as well as a total number of n voters.

3.2.5 Experimental Analysis

In contrast to exact analysis, the experimental approach relies on simulations to grasp the development of different phenomena under varying conditions. On the upside, this usually allows for results for more complex problems or a larger scale of parameters, both of which might be prohibitive for exact calculations. At the same time, we however face the problem that we need a huge number of simulations per setting to get sound estimates, which in turn often requires a high-performance computer and a lot of time. Also, there remains the risk that even a vast amount of simulations fails to capture one specific, possibly crucial, effect.

Regarding the pivotal question of this section, the frequency of the NSP for various voting rules, we sample preference profiles for different combinations of n and m using the modeling assumptions explained in Section 3.1.2.

3.3 RESULTS AND DISCUSSION

We here present our results obtained by both exact analysis and computer simulations.

3.3.1 Results Under IAC

First focus on Copeland's rule and three alternatives. Our modeling in Section 3.2.4 allows for an exact analysis of the number of profiles prone to the NSP. In particular, we compute the following Ehrhart-polynomial $f(n)$ with period $q = 2$:

$$f_0(n) = {}^1\!/_{192}\, n^4 - {}^1\!/_{48}\, n^3 - {}^1\!/_{48}\, n^2 + {}^1\!/_{12}\, n$$
$$f_1(n) = {}^1\!/_{192}\, n^4 - {}^5\!/_{96}\, n^2 + {}^3\!/_{64}$$

Recall that $f(n) = f_i(n)$ if $n \equiv i \mod q$. Consequently, the fraction of profiles that admit a manipulation by strategic abstention is given by

$$\frac{f_0(n)}{\binom{n+5}{5}}$$

if n is even and

$$\frac{f_1(n)}{\binom{n+5}{5}}$$

if n is odd. This frequency of the NSP for Copeland's rule and $m = 3$ is plotted in Figure 3.1, together with results obtained by computer simulations.

With respect to Black's rule and $m = 3$, we obtain an Ehrhart-polynomial with slightly larger period $q = 6$. Once more, we can explicitly give $f(n)$ which looks as follows:

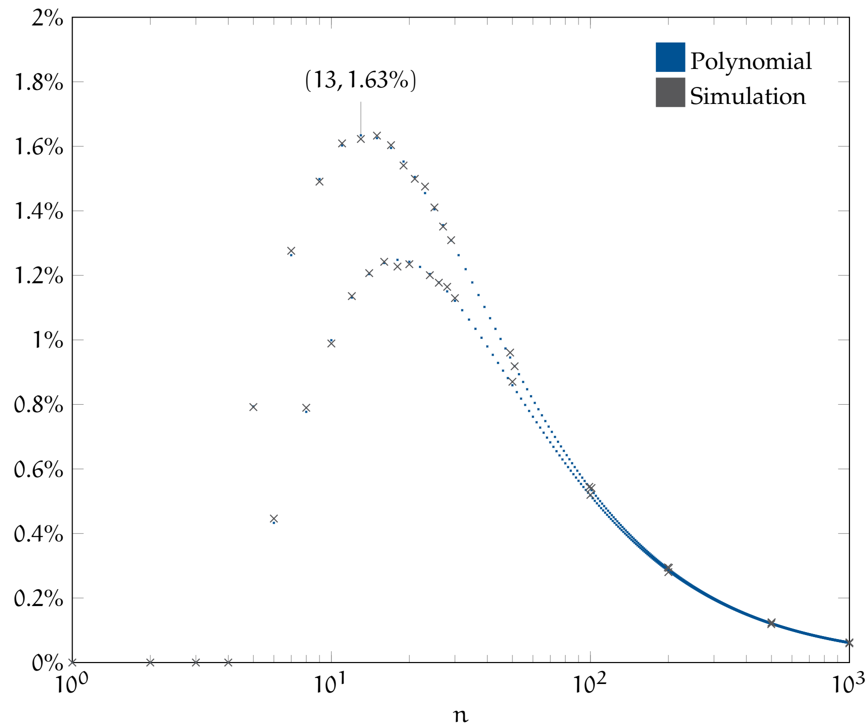


Figure 3.1: Fraction of profiles prone to the NSP for Copeland's rule and $m = 3$.

$$
\begin{aligned}
f_0(n) &= 1/192\, n^4 - 5/48\, n^2 \\
f_1(n) &= 1/192\, n^4 - 1/48\, n^3 - 7/96\, n^2 + 3/16\, n - 19/192 \\
f_2(n) &= 1/192\, n^4 - 5/48\, n^2 + 1/3 \\
f_3(n) &= 1/192\, n^4 - 1/48\, n^3 - 7/96\, n^2 + 3/16\, n + 15/64 \\
f_4(n) &= 1/192\, n^4 - 5/48\, n^2 + 1/3 \\
f_5(n) &= 1/192\, n^4 - 1/48\, n^3 - 7/96\, n^2 + 3/16\, n + 15/64
\end{aligned}
$$

The fraction of profiles prone to the NSP for Black's rule and $m = 3$ is visualized in Figure 3.2.

Similar connections between analytical and experimental results can be observed in Figure 3.3. Note that while we are able to explicitly give the Ehrhart-polynomials for Copeland's and Black's rule and $m = 3$ here, this is not possible for MaxiMin and $m = 4$ due to the polynomial's size. The corresponding polynomial $f(n)$ has a period of $q = 55\,440$, i.e., consists of 55 440 different polynomials. We deduce that no two points in Figure 3.3 are computed via the same polynomial, which makes the regularity of the curve even more remarkable.

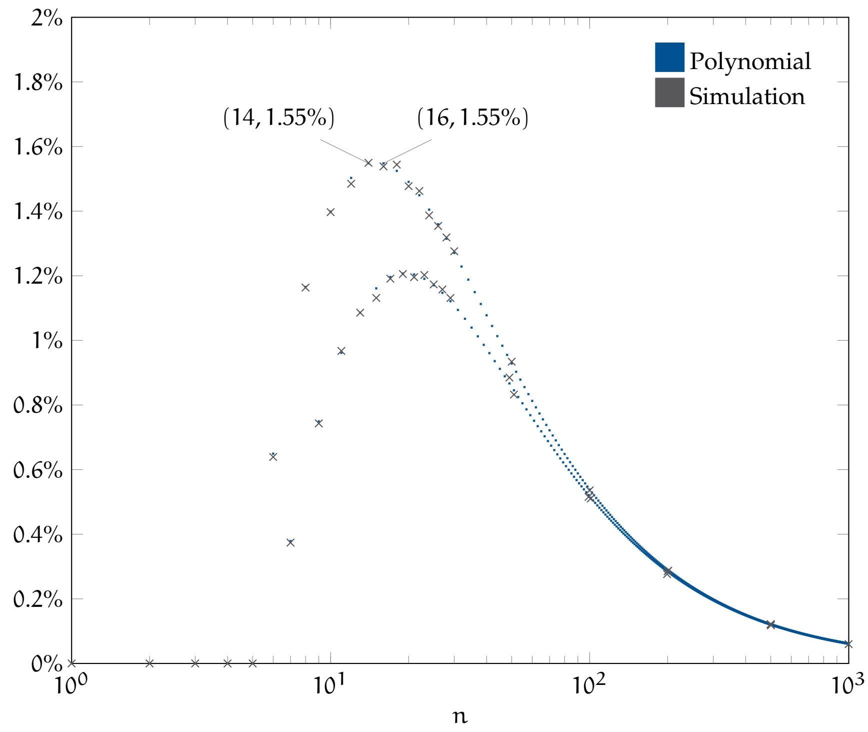


Figure 3.2: Fraction of profiles prone to the NSP for Black's rule and $m = 3$.

A couple of points come to mind when closely studying these graphs. First, we note that the results obtained by simulation almost perfectly match the exact calculations, which can be seen as strong evidence for the correctness of both. On the one hand, it confirms our modeling via polytopes, and at the same time highlights that we are running a sufficiently large amount of simulations. It additionally stands to reason that this accordance with the exact numbers also holds for larger m or even different rules, which is most useful for cases where determining the corresponding Ehrhart-polynomials or even the modeling via polytopes is infeasible.

We see that for Black's rule the maximum is attained at 14 and 16 voters with 1.55% of all profiles suffering from the NSP. For Copeland's rule the maximum is at 13 voters and 1.63% of all profiles, while for MaxiMin and $m = 4$ it is at 14 voters and a fraction of 0.55% of profiles. Hence, we can argue that for elections with few alternatives, the NSP seems to hardly cause a problem, independent of the number of voters or the voting rule considered. We deem it striking that the maxima occur at roughly the same number of voters, while this very number of voters varies between being even or odd. Also observe that Black's and Copeland's rule are more sensitive to the parity of n than MaxiMin.

Furthermore, we note that the probability for the NSP to occur converges to zero as n goes to ∞; a behavior that holds true for all voting

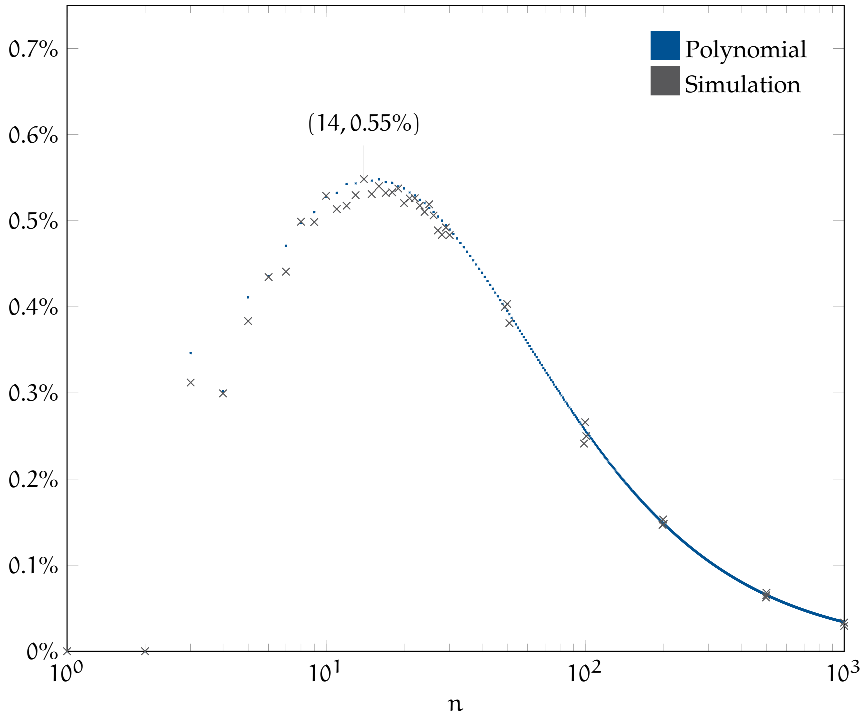


Figure 3.3: Fraction of profiles prone to the NSP for MaxiMin and $m = 4$.

rules considered and all fixed m. Intuitively, this is to be expected as for larger electorates, a single voter's power to sway the result diminishes. This first idea can be confirmed by considering the respective modeling via polytopes. Each modeling will contain at least one equality constraint, e.g., in (basis) of f_{MaxiMin} in Section 3.2.2. Consequently, the polytopes describing profiles for which a manipulation is possible are of dimension at most $m! - 1$. By Ehrhart (1962), this means that the number of those profiles can be described by a polynomial of n of degree at most $m! - 1$. The total number of profiles, on the other hand, can equivalently be determined via a polynomial of degree $m!$. Hence, the fraction of profiles prone to the NSP is upper-bounded by $O(1/n)$. Following the intuitive argument, similar behavior is to be expected for all reasonable preference models and all 'continuous' voting rules.[56]

For $m = 4$, determining the Ehrhart polynomials for both Black's as well as Tideman's rule proved to be infeasible, even when using a custom-tailored version of NORMALIZ and employing a high-performance cluster.[57] Copeland's rule unfortunately causes problems even earlier: for four alternatives the modeling via linear

56 While we are not able to give a formal definition, we intuitively mean that a voting rule is *continuous* when it does not behave entirely different for varying, e.g., odd and even, n.

57 For Black's rule, we find that the polynomial would be of period $q \approx 2.7 \cdot 10^7$ corresponding to a mid two-digit GB file size.

(in)equalities quickly becomes infeasible due to the rule only caring about unweighted majority comparisons. For all rules, $m \geqslant 5$ appears to be out of scope for years to come.

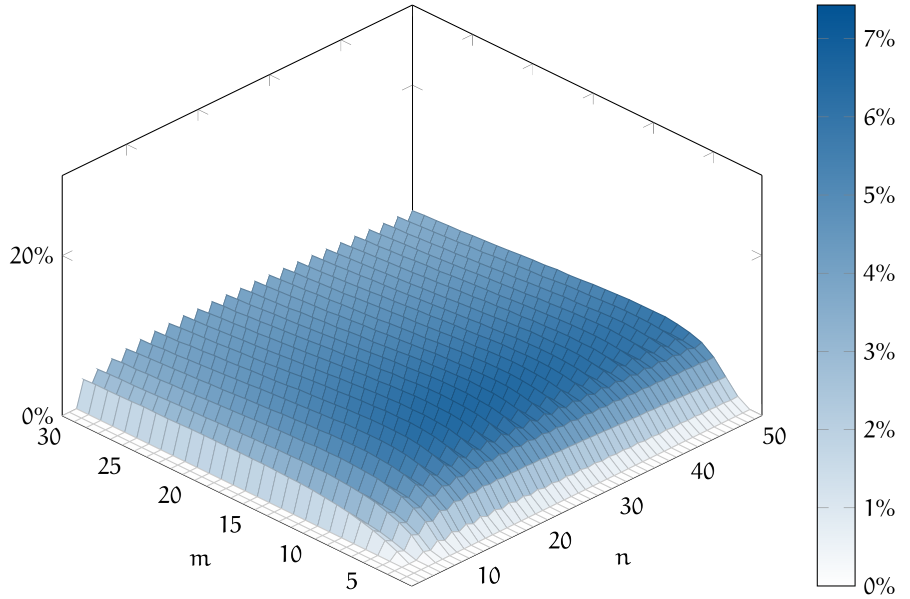


Figure 3.4: Fraction of profiles prone to the NSP for Black's rule.

We therefore rely on simulations to grasp how often the NSP can occur for different combinations of n and m up to 50 voters and 30 alternatives. Our results can be found in Figures 3.4 to 3.9. The following observations can be made.

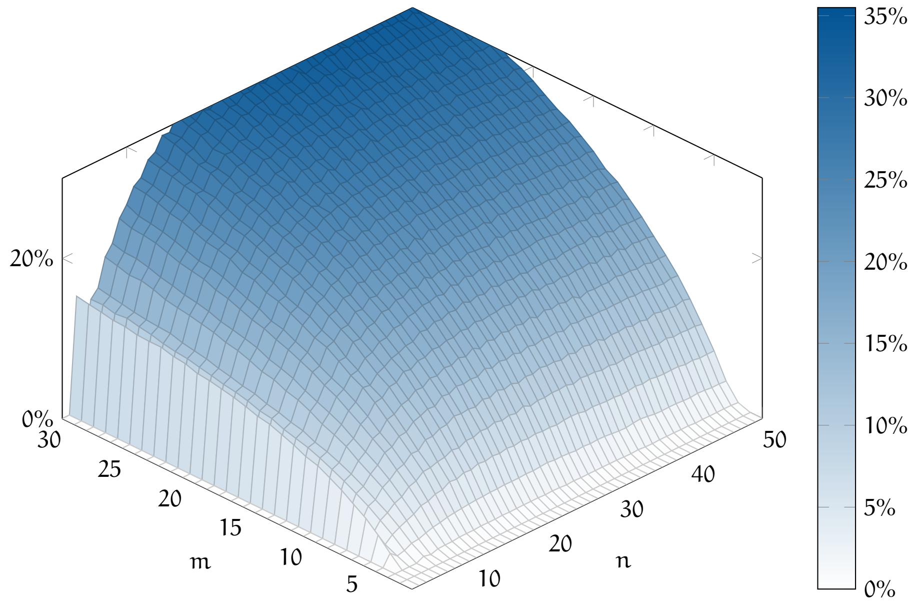


Figure 3.5: Fraction of profiles prone to the NSP for Baldwin's rule.

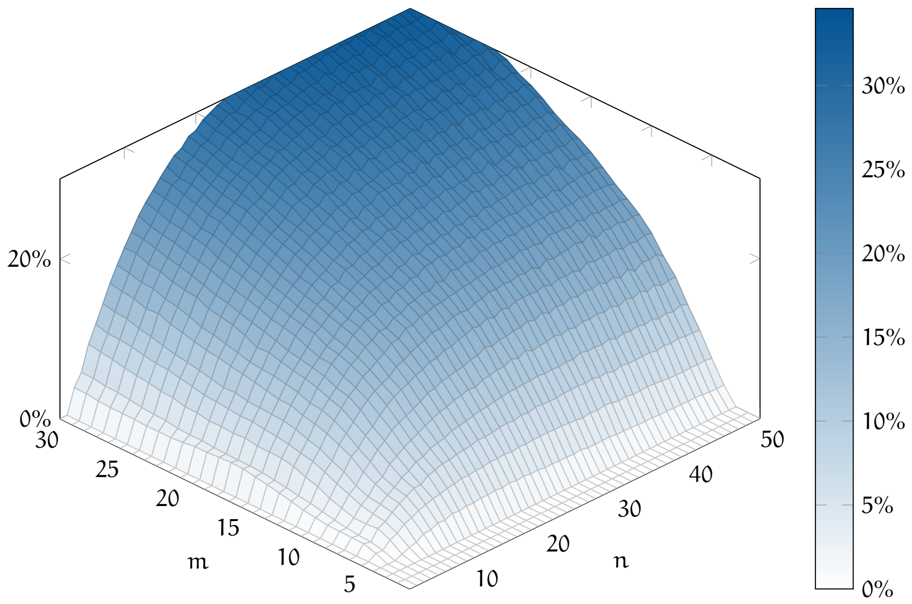


Figure 3.6: Fraction of profiles prone to the NSP for Nanson's rule.

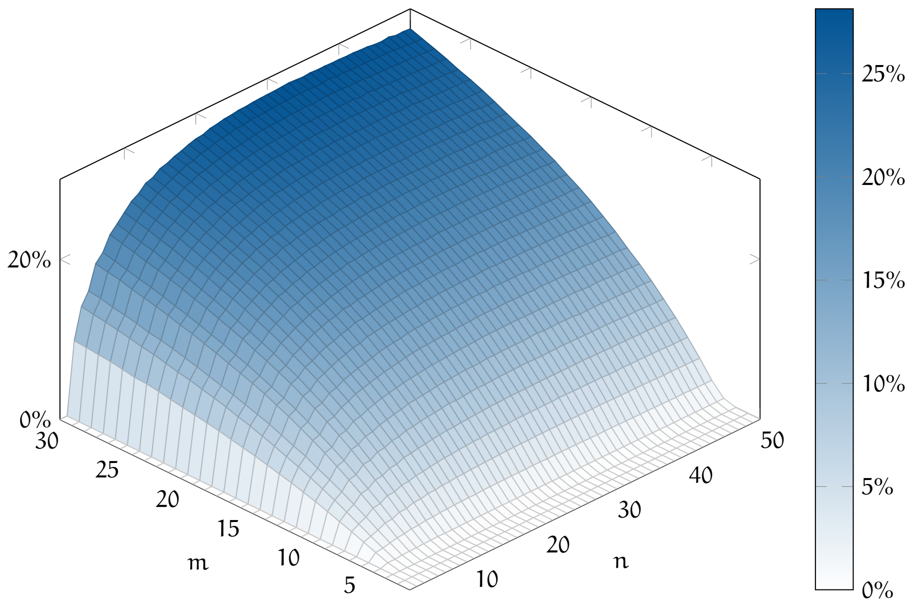


Figure 3.7: Fraction of profiles prone to the NSP for MaxiMin.

To begin with, the relatively low fraction of profiles prone to the NSP for Copeland's rule, Black's rule, and MaxiMin with a small number of alternatives increases as m grows. This increase is quite dramatic for Copeland's rule and MaxiMin. In particular, for only 20 alternatives, a rough quarter of all profiles admit a manipulation by abstention for a medium count of voters for both rules—a number too large to discard the NSP as merely a theoretical problem. Black's

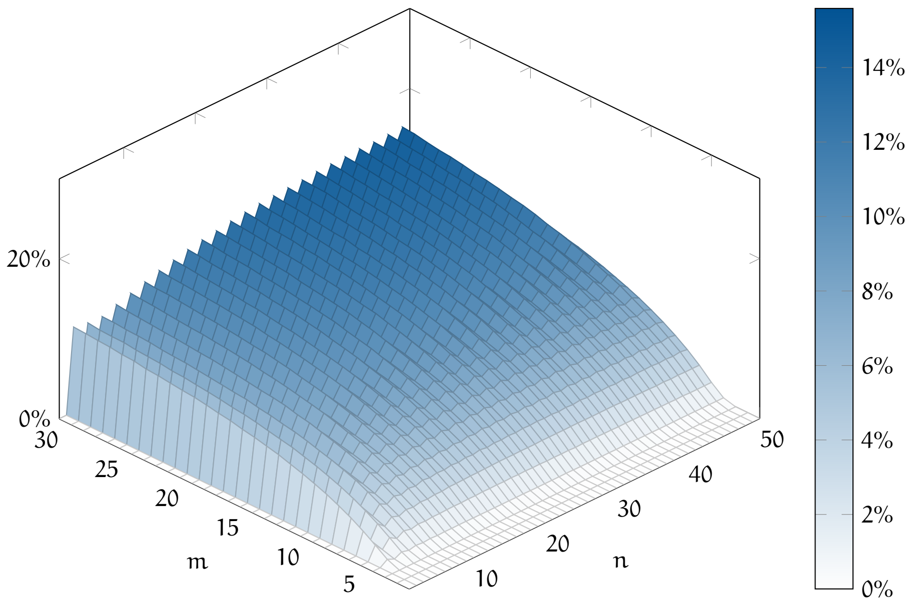


Figure 3.8: Fraction of profiles prone to the NSP for Tideman's rule.

rule, on the other hand, remains stable on a comparatively moderate level.

Especially when considering Black's, Tideman's, and Copeland's rule, we see that the parity of n crucially influences the results. However, the parity of n does not affect the fractions in a consistent way: higher fractions occur for Black's and Copeland's rule when n is even, in contrast to Tideman's rule where this happens when n is odd. For Black's rule, this is most probably due to the fact that there are more suitable profiles close to having a Condorcet winner ($g_{xy} = 0$) than profiles close to not having one ($g_{xy} = 1$).[58] Considering Copeland's rule and an odd number of voters, the score for each alternative is an integer compared to a finer scale allowing for half points when n is even. Hence, differences between alternatives are potentially more distinct for an odd number of voters which we assume makes manipulations harder to achieve. For Tideman's rule, we currently lack a convincing explanation for the observed behavior, mostly because it is hard to intuitively grasp when exactly a preference profile is manipulable.

Regarding Baldwin's and Nanson's rule as well as MaxiMin, the parity of n seems to have little effect on the numbers. More detailed analysis shows that at least for MaxiMin this appearance is deceptive: when manipulating towards an alphabetically preferred alternative, fractions are higher for even n, while the contrary holds for manip-

58 For Black's rule, manipulation is only possible either towards or away from a Condorcet winner since Borda's rule is immune to strategic abstention and manipulation is impossible from Condorcet winner to Condorcet winner.

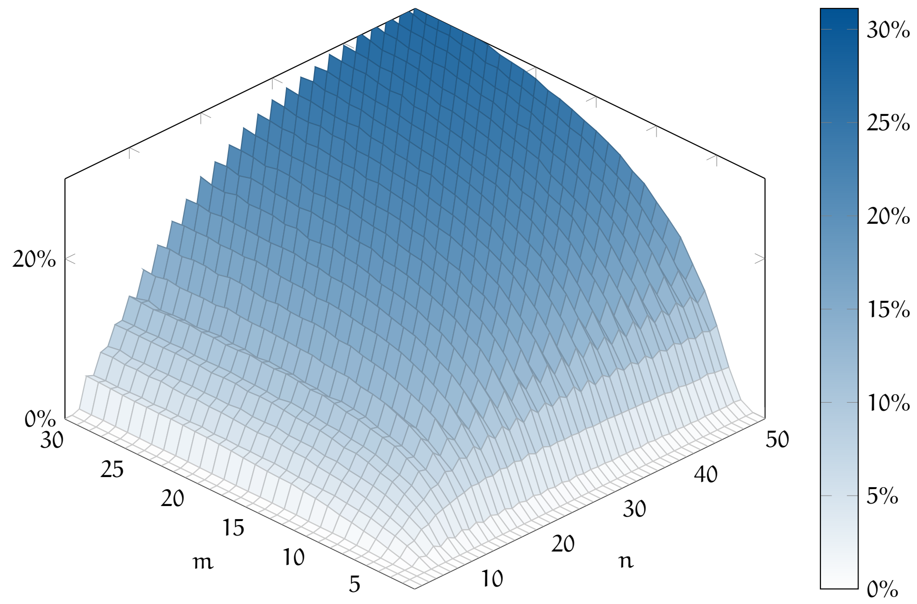


Figure 3.9: Fraction of profiles prone to the NSP for Copeland's rule.

ulations towards an alphabetically less preferred alternative. In sum, these two effects approximately cancel each other out.

Felsenthal and Nurmi (2018) argue in favor of Nanson's rule as it is—in contrast to the related Baldwin's rule—not prone to the NSP for three alternatives. We show that this difference between the two rules becomes moot for larger numbers of alternatives: the fractions of profiles allowing for a manipulation are on a roughly identical, severely high level.[59] This emphasizes that even though it seems reasonable at first glance to surmise that a voting rule based on Borda scores fares better with respect to the NSP, this is not necessarily true.

When examining Figure 3.5, the ridge at $n = 3$ immediately catches the observer's eye.

We conjecture this unique behavior of Baldwin's rule is due to preference profiles similar in structure to the one depicted on the right. In case voter 3 places sufficiently many alternatives over x, x is going to be eliminated on the way causing y to eventually be chosen. Then again, if voter 3 abstains, x is always going to be selected as long as it beats y in the tie-breaking order. Note that x and y can be chosen almost freely, all other alternatives placed virtually arbitrarily, and many profiles only similar in structure also work.

1	2	3
x	y	⋮
y		x
⋮	⋮	⋮
		y
	x	⋮

The flawless smoothness and regularity of Figures 3.4 to 3.9 are due to 10^6 runs per data point. This large number allows for all 95%

59 Felsenthal and Nurmi (2018) also show that none of the two rules fares strictly better than the other. Indeed, there are profiles where a manipulation is possible according to Baldwin's rule but not using Nanson's rule and *vice versa*.

confidence intervals to be smaller than 0.2%. Our simulations were conducted on a XeonE5-2697 v3 with 2 GB memory per job and took 35 to 48 hours for *each* data point. Since there are 1 500 data points per plot, the total runtime for all six figures easily accumulates to forty years on a single-processor machine.

3.3.2 Comparing Different Preference Models

In order to get an impression of the frequency of the NSP under different preference models, we fix the number of alternatives to be $m = 4$ or $m = 30$ and sample 10^6 profiles for increasing n up to 1 000 or 200, respectively.[60] Figure 3.10 gives the fraction of profiles prone to the NSP using either Black's, Baldwin's, or Nanson's rule. Values for MaxiMin, Tideman's, and Copeland's rule are displayed in Figure 3.11.

A close inspection of these graphs allows for multiple conclusions. First, we see that in particular Black's rule shows a severe dependency on the parity of n. For better illustration, we depict two lines per preference model to highlight this effect; which line stands for odd and which for even n is easiest checked using their corresponding point of intersection with the x-axis, which is either 1, 2, or 3 throughout. Apart from explanations given earlier, it is not completely clear why differences are more prominent for some voting rules, why we sometimes see higher percentages for odd n and other times for even n, or why for some instances there is a large discrepancy for one preference model but hardly any for another.

IC and IAC are often criticized for being unrealistic and only giving worst-case estimates (see, e.g., Tsetlin et al., 2003; Regenwetter et al., 2006). This criticism is generally confirmed by our experiments, which show that the highest fractions of profiles is prone to the NSP when the sampling is done according to IC or IAC. A notable exception is Black's rule for 30 alternatives, where a different effect prevails: for many alternatives and comparably few voters, situations in which a Condorcet winner (almost) exists appear less frequently under IC or IAC than under the other preference models. In absence thereof, Black's rule collapses to Borda's rule, which is immune to the NSP. Note that were we to conduct a dual experiment with fixed n and increasing m, the fraction of profiles prone to the NSP using Black's rule and IC or IAC would converge to zero for similar reasons.

We moreover see that IC, IAC, and the urn model exhibit identical behavior for $m = 30$. The right-hand side of Figure 3.10 and Figure 3.11 therefore seems to only feature three preference models, even though all five are depicted. This may be surprising at first but is to

60 For increasing m the computations quickly become very demanding. The values for $m = 30$ and $n \geqslant 99$ are determined with 50 000 runs each only. The size of all 95% confidence intervals is, however, still within 0.5%.

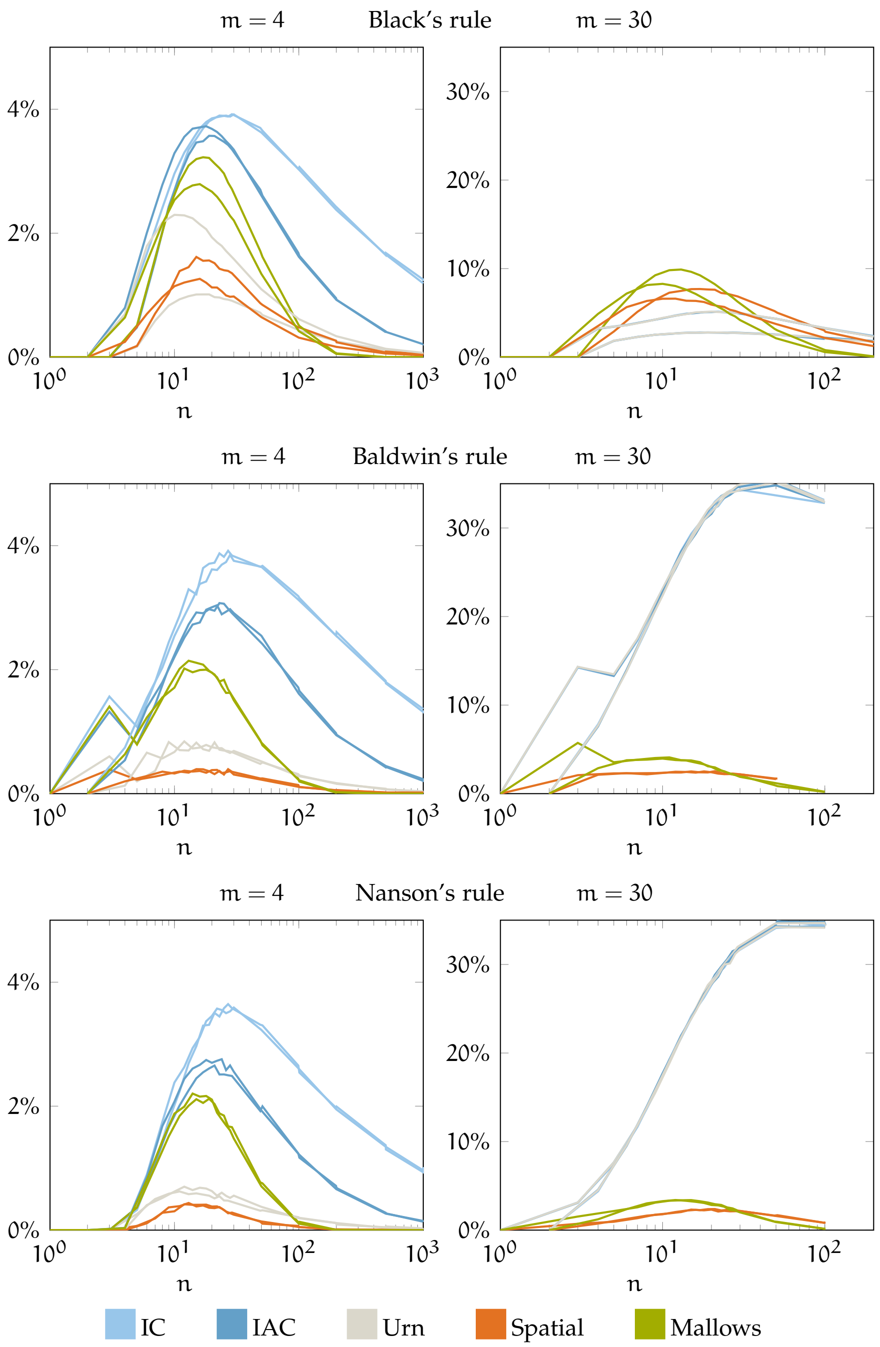


Figure 3.10: Profiles prone to the NSP for Black's, Baldwin's, and Nanson's rule, fixed m, and increasing n; two lines per preference model depending on the parity of n; IC, IAC and the urn model almost collapse for $m = 30$, resulting in a bluish grey line.

be expected since IC and IAC can equivalently be seen as urn models with parameters 0 and 1, respectively. For $30! \approx 2.7 \cdot 10^{32}$ voter types and a comparatively small n the difference between parameters 0, 1, and 10 is simply too small for a visible difference.

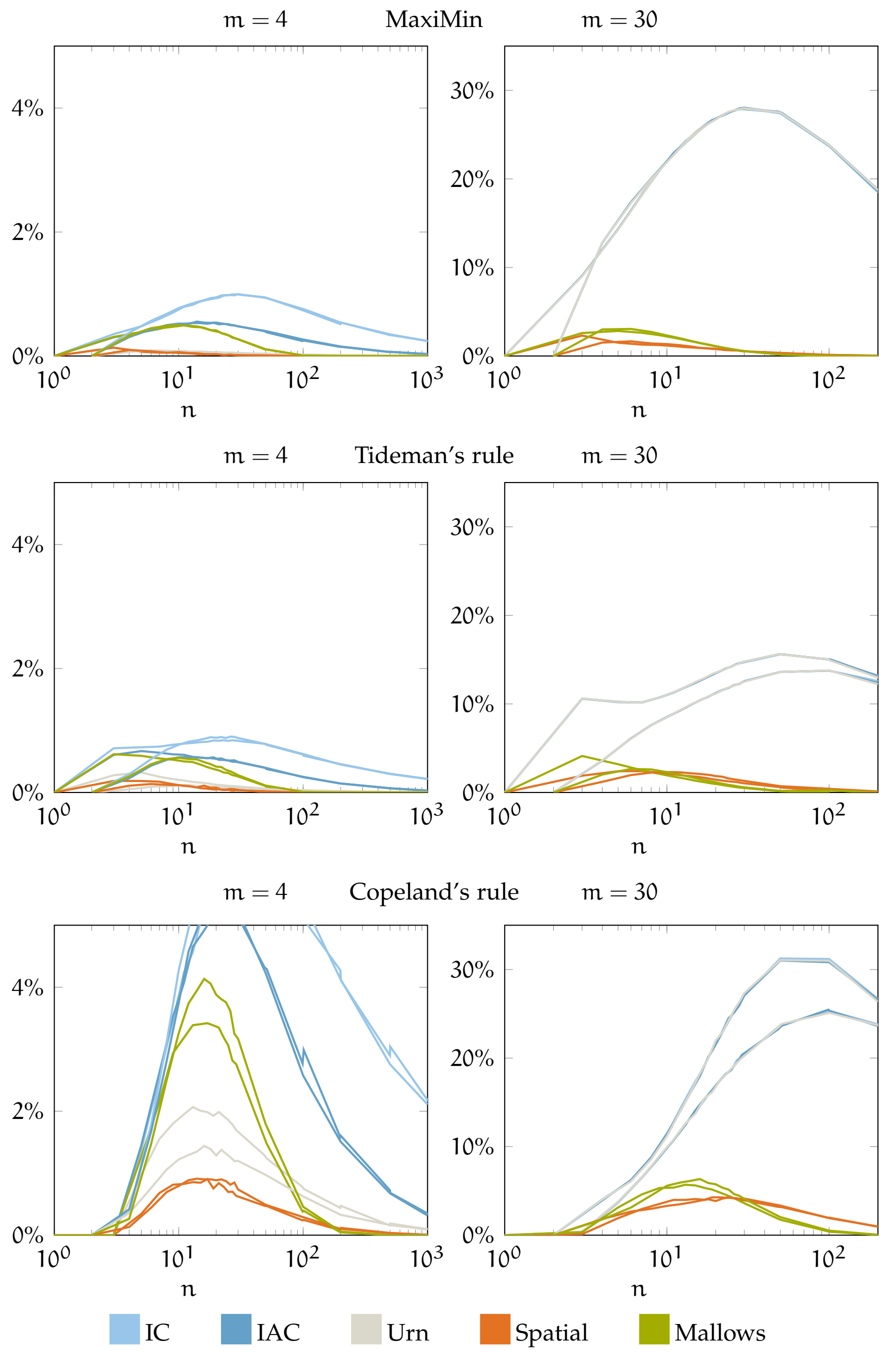


Figure 3.11: Profiles prone to the NSP for MaxiMin, Tideman's, and Copeland's rule, fixed m, and increasing n; two lines per preference model depending on the parity of n; IC, IAC and the urn model almost collapse for $m = 30$, resulting in a bluish grey line.

The large conceptual similarities between Baldwin's and Nanson's rule are also reflected in the corresponding charts. Apart from the peak at $n = 3$ for Baldwin's rule, both look almost identical for all preference models with the small difference being that Nanson's rule

	m	IC	IAC	spatial	urn	Mallows
Black	4	$3.92^{(29)}$	$3.73^{(18)}$	$1.62^{(15)}$	$2.30^{(10)}$	$3.23^{(17)}$
	30	$5.12^{(22)}$	$5.12^{(22)}$	$7.70^{(17)}$	$5.14^{(20)}$	$9.90^{(13)}$
Baldwin	4	$3.92^{(27)}$	$3.07^{(23)}$	$0.40^{(15)}$	$0.84^{(12)}$	$2.14^{(13)}$
	30	$35.4^{(49)}$	$35.4^{(51)}$	$2.54^{(21)}$	$35.7^{(49)}$	$5.73^{(3)}$
Nanson	4	$3.64^{(27)}$	$2.76^{(24)}$	$0.44^{(13)}$	$0.68^{(16)}$	$2.20^{(14)}$
	30	$34.9^{(51)}$	$34.8^{(51)}$	$2.38^{(21)}$	$34.7^{(99)}$	$3.40^{(12)}$
MaxiMin	4	$1.00^{(30)}$	$0.56^{(14)}$	$0.14^{(3)}$	$0.13^{(3)}$	$0.50^{(10)}$
	30	$28.0^{(30)}$	$28.0^{(30)}$	$2.31^{(3)}$	$28.0^{(30)}$	$3.01^{(6)}$
Tideman	4	$0.80^{(26)}$	$0.67^{(5)}$	$0.19^{(5)}$	$0.32^{(5)}$	$0.62^{(3)}$
	30	$15.6^{(51)}$	$15.6^{(49)}$	$2.42^{(7)}$	$15.6^{(49)}$	$4.12^{(3)}$
Copeland	4	$6.96^{(29)}$	$5.54^{(20)}$	$0.91^{(14)}$	$2.07^{(13)}$	$4.13^{(16)}$
	30	$31.2^{(50)}$	$31.0^{(50)}$	$4.28^{(21)}$	$31.1^{(50)}$	$6.33^{(16)}$

Table 3.1: Maximal percentage of total profiles prone to the NSP for different combinations of voting rules and preference models with $m = 4$ or $m = 30$; the number of voters n for which the maximum occurs attached in parentheses; IC, IAC, and the urn model yield almost identical numbers for $m = 30$.

appears to feature a slightly lower manipulability. Fewer rounds for winner determination thus do not seem to come at a cost with respect to the NSP.

Finally, Copeland's, Baldwin's, and Nanson's rule as well as MaxiMin to a lesser extent appear to fare exceptionally bad with respect to the NSP and IC, IAC, and the urn model. At the same time, none of these rules exhibits overly conspicuous behavior for the spatial and Mallows' model. This suggests that the risk of a possible manipulation is reduced by structural similarities in the individual preferences compared to a greater likelihood for very diverse rankings. Though generally in line with expectations, we currently do not have a profound explanation for the magnitude of this effect. For Copeland's rule, it is plausible to assume that its particularly bad performance results from the rule using less information, i.e., Copeland's rule is the only majoritarian rule considered here.

The maximal fraction of total profiles prone to the NSP for $m = 4$, $m = 30$, different voting rules, preference models, and varying values of n is given in Table 3.1. Among other things, we for instance note that the maxima constantly occur for a higher number of voters for IC (26 to 51 voters) than for Mallows' model (3 to 17 voters), a fact probably due to an increasing (expected) structure under Mallows' model and larger n. More observations are also given in Section 3.4.

3.3.3 Empirical Analysis

To get a complete picture, we also analyze the NSP for empirical data obtained from real-world elections. Unfortunately, such data is generally relatively rare and imprecise and often only fragmentarily available. A check of all 315 strict profiles contained in the PrefLib library (Mattei and Walsh, 2013) for occurrences of the NSP shows that two profiles admit a manipulation by abstention when Black's rule is used, one profile for each Copeland's, Baldwin's, and Nanson's rule, and that no manipulation is possible for MaxiMin as well as Tideman's rule.[61] While this suggests a low susceptibility to the NSP in real-world elections, much more data would be required to allow for meaningful conclusions.

3.4 CONCLUSION

While both the Condorcet criterion and immunity to the NSP are desirable, there is no single-valued voting rule satisfying both conditions (Moulin, 1988). This chapter aims at a better understanding of this incompatibility: we do know that profiles prone to the NSP exist, but how likely is it that a randomly chosen preference profile allows for a manipulation by strategic abstention? For six popular Condorcet extensions, we pursue this approach using three methods, namely analytical, experimental, and empirical analysis.

Though exact analysis using Ehrhart theory quickly comes up against a brick wall due to technical constraints, it underscores the correctness of our simulations and provides a justification why the probability for the NSP to occur converges to zero as n grows. Analysis of preference models other than IAC and larger numbers of alternatives is possible via computer experiments only. We find that for a moderate count of alternatives, a higher m comes at the price of a higher vulnerability to the NSP. Moreover, of the preference models considered, the spatial model and Mallows' ϕ generally perform best with the notable exception of Black's rule. Briefly summarized, our main results are as follows.

- When there are few alternatives, the probability of the NSP is almost negligible (less than 1% for $m = 4$, MaxiMin and Tideman's rule, and all considered preference models; less than 4% for Black's, Baldwin's, and Nanson's rule; less than 7% for Copeland's rule).
- When there are 30 alternatives and preferences are modeled using IC, IAC, and the urn model, Black's rule is least susceptible

61 For instance the profile allowing for a manipulation under Copeland's rule is immune to the NSP for all other rules. It features 10 alternatives and 30 voters. Baldwin's and Nanson's rule exhibit the NSP for the same profile.

to the NSP (< 6%), followed by Tideman's rule (< 16%), MaxiMin (< 29%), Copeland's rule (< 32%) Nanson's rule (< 35%) , and Baldwin's rule (< 36%).

- For 30 alternatives and the spatial and Mallows' model, this ordering is roughly reversed. MaxiMin and Nanson's rule are least susceptible (< 4%), followed by Tideman's rule (< 5%), Baldwin's rule (< 6%), Copeland's rule (< 7%), and Black's rule (< 10%).
- The parity of the number of voters significantly influences the manipulability of Black's, Tideman's, and Copeland's rule. Black's and Copeland's rule are more manipulable for an even number of voters whereas MaxiMin is more manipulable for an odd number of voters (under the IAC assumption).
- Whenever analysis via Ehrhart theory is feasible, the results are perfectly aligned with our simulation results, highlighting the accuracy of the experimental setup.
- Only four (out of 315) strict preference profiles in the PrefLib database are manipulable by strategic abstention (manipulations only occur for Black's, Baldwin's, Nanson's, and Copeland's rule, but not for MaxiMin and Tideman's rule).

4 THE NO-SHOW PARADOX FOR SET-VALUED VOTING RULES

After having studied the likelihood of the NSP to occur in Chapter 3, we turn to a different approach. Recall that following Moulin (1988), every *single-valued* Condorcet extension is prone to the NSP. However, this assumption of single-valuedness is often refused for being too restrictive and not realistic for many cases, among other things because it requires some tie-breaking by the rule itself even if all technically relevant criteria are identical.[62] In this spirit, Barberà et al. (2004) argue that single-valuedness is "questioned because it is formulated in terms that do not allow for multiple choices of alternatives, not even in cases where considerations of symmetry make such choices quite compelling."

We therefore turn to set-valued voting rules in this section, i.e., rules that do not necessarily select single alternatives but possibly sets of alternatives. As explained in Section 2.4, we first need to define preference extensions to obtain a variant of participation to work with; we here rely on two common extensions due to Kelly (1977) and Fishburn (1972a). Both extensions are motivated and formally defined in Section 4.1.2.

This chapter is structured as follows: we first specify notation and concepts used later on in Section 4.1. Next, Section 4.2 briefly introduces the computer-aided approach that led to some of the results presented and discussed in the following Section 4.3. Lastly, concluding remarks are given in Section 4.4.

4.1 PRELIMINARIES

For this section, we assume voting rules to be set-valued, i.e., to choose sets of alternatives, and build on notation and concepts introduced in Chapter 2. In particular, we repeatedly deal with majoritarian voting rules. Recall that a voting rule is majoritarian if the choice set only depends on the majority relation, that is, a majoritarian rule f selects the same set of alternatives for any two preference profiles $\succsim$ and $\succsim'$ as long as $\succsim^M = \succsim'^M$. Whenever we want to talk

62 Single-valuedness is criticized mostly in the context of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) by, e.g., Gärdenfors (1976), Kelly (1977), Barberà (1977), Feldman (1979b), Bandyopadhyay (1983b), Bandyopadhyay (1983a), Duggan and Schwartz (2000), Nehring (2000), Ching and Zhou (2002), and Brandt (2015).

about majority relations without linking them to a specific preference profile, we instead focus on majority graphs G. Apart of the lack of a specified profile, G is treated the same way and subject to the same operations as $\succsim^M$; we let G^{-1} be the inverse of G and write $\underline{G}$ for the strict part of G just as $\succ^M$ is the strict part of $\succsim^M$.

We proceed by defining two properties as well as a set-valued voting rule unique to this section before turning towards the two preference extensions we obtain results for.

4.1.1 Additional Properties and Notation

independence of indifferent voters

First, we introduce a very weak variable electorate condition which requires that a completely indifferent voter does not change the outcome. A voting rule f satisfies *independence of indifferent voters* (IIV) if $f(\succsim) = f(\succsim_{+i})$ for all $\succsim \in \succsim(A)^{\mathcal{F}(\mathbb{N})}$, where i is a voter who is indifferent between all alternatives, i.e., $x \sim_i y$ for all $x, y \in A$. It is easy to see that every majoritarian rule f satisfies IIV.

f-improvement

set-monotonicity

We additionally make use of set-monotonicity as originally given by Brandt (2015), who connects it to Kelly-strategyproofness. Therefore, for $\succsim, \succsim' \in \succsim(A)^N$, let $\succsim'$ be an f-*improvement* over $\succsim$ if alternatives that are chosen by f in $\succsim$ are not weakened from $\succsim$ to $\succsim'$, i.e., for all $x \in f(\succsim)$, $y \in A$, and $i \in N$, $x \succsim_i y$ implies $x \succsim'_i y$, and $y \succsim'_i x$ implies $y \succsim_i x$. A voting rule f satisfies *set-monotonicity* if $f(\succsim) = f(\succsim')$ holds whenever $\succsim'$ is an f-improvement over $\succsim$.

Pareto rule

In the following, we often relate participation and strategyproofness to Pareto optimality (see Section 2.5). Recall that an alternative is Pareto optimal if there is no other alternative that Pareto dominates it. We define the *Pareto rule (PO)* to be the rule that chooses all Pareto optimal alternatives.

4.1.2 Preference Extensions

To be able to formally reason about participation, we first have to define how voters compare sets of alternatives. As mentioned in Section 2.4, there is no obvious way to extend preferences over single alternatives to preferences over sets thereof, especially if those sets overlap. Assuming that out of this choice set one alternative is eventually selected, different possible knowledge of the voters about this *tie-breaking mechanism* results in different preference extensions (see, e.g., Gärdenfors, 1979; Ching and Zhou, 2002; Sanver and Zwicker, 2012; Brandt and Brill, 2011; Brandt, 2015; Brandt et al., 2018).

We begin by letting voters have no information whatsoever about how the final single alternative is selected out of the choice set. Under these circumstances, one set will be preferred to another only if whichever alternative the tie-breaking mechanism eventually chooses out of the former is better than everything in the latter. This leads to a very natural way of comparing two sets originally due to Kelly (1977): a voter prefers a set of alternatives X to a second set Y if he prefers every alternative in X to every alternative in Y. Formally, for all $X, Y \subseteq A$, $\succsim_i \in \succsim(A)$,

Kelly's extension

$$X \succsim_i^K Y \text{ if } x \succsim_i y \text{ for all } x \in X, y \in Y.$$

If, on the other hand, voters know that an alternative is eventually chosen according to some unknown fixed order of alternatives, e.g., the preferences of a chairman, a different extension arises. In particular, such a fixed tie-breaking mechanism guarantees that if multiple alternatives are contained in both sets, it is impossible that the most preferred one of those is chosen in one set and the least preferred one in the second. This allows a voter to also compare possibly intersecting sets without having to be indifferent in between all alternatives contained in the intersection. More precisely, a set X is preferred to another set Y if every alternative in X but not in Y is preferred to every alternative in the intersection, while every alternative in the intersection is preferred to every alternative in Y but not in X (Fishburn, 1972a). Formally, for all $X, Y \subseteq A$, $\succsim_i \in \succsim(A)$,

Fishburn's extension

$$X \succsim_i^F Y \text{ if } X \setminus Y \succsim_i^K Y \text{ and } X \succsim_i^K Y \setminus X.$$

We also say that X is Kelly-preferred or Fishburn-preferred to Y (by i) whenever $X \succsim_i^K Y$ or $X \succsim_i^F Y$ holds, respectively. The strict part of both relations is denoted by $\succ_i^K$ and $\succ_i^F$. Note that by the definitions, Fishburn's extension is a refinement of Kelly's extension. Hence, $\succsim_i^K \subseteq \succsim_i^F$ for every $\succsim_i \in \succsim(A)$, i.e., whenever one set is Kelly-preferred over another one it is also Fishburn-preferred. For a more general discussion of how to extend preferences over single alternatives to preferences over sets, we refer to Barberà et al. (2004).[63]

Example 4.1

For the sake of illustration, consider $\succsim_i$: a, b, c, d and $X = \{a, b\}$, $Y = \{a, b, c\}$, and $Z = \{b, d\}$. Then, $X \succ_i^F Y$ and $X \succ_i^K Z$. The sets Y and Z are not comparable with respect to either one of the two extensions.

Having introduced Kelly's and Fishburn's extension, we directly obtain the two notions of Kelly-participation and Fishburn-participation as well as the respective counterparts of strategyproofness. Intuitively, a voting rule f satisfies Kelly-participation or Fishburn-participation if there is no situation, where by abstaining from the election process, a voter can obtain a strictly Kelly-preferred or Fishburn-preferred result, respectively. Both variants are illustrated by the following example.

Kelly-participation

Fishburn-participation

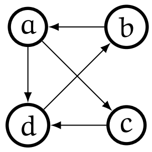
63 See also, e.g., Gärdenfors (1976), Barberà (1977), Kelly (1977), Duggan and Schwartz (2000), Barberà et al. (2001), Benoît (2002), Ching and Zhou (2002), and Özyurt and Sanver (2009).

Example 4.2

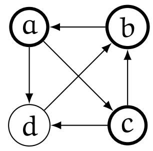
Consider the preference profile $\succsim$ with six voters and four alternatives depicted below.

1	2	3,4	5,6
c	d	a	b
d	b	c	a
b	a	d	c
a	c	b	d

$\succsim$



$\succsim^M$



$\succsim^M_{-6}$

The profile $\succsim$ induces the majority relation $\succsim^M$. A well-studied majoritarian voting rule is the *bipartisan set* (Laffond et al., 1993; Dutta and Laslier, 1999). The bipartisan set of $\succsim$ is $\{a, b, c, d\}$. If voter 6 leaves the electorate, we obtain the profile $\succsim_{-6}$, which induces the majority relation $\succsim^M_{-6}$, whose bipartisan set is $\{a, b, c\}$. Observe that $\{a, b, c\} \succ^F_6 \{a, b, c, d\}$, i.e., voter 6 can obtain a preferred outcome according to Fishburn's extension by abstaining from the election. Hence, the bipartisan set does not satisfy Fishburn-participation. However, $\{a, b, c\} \succ^K_6 \{a, b, c, d\}$ does not hold and, thus, voter 6 cannot manipulate by abstaining according to Kelly's extension. In general, the bipartisan set satisfies Kelly-participation because it satisfies set-monotonicity and IIV (see Theorem 4.9).

4.2 COMPUTER-AIDED THEOREM PROVING

For some of our results, we are going to make use of the computer-aided proving methodology described by Brandt and Geist (2016).[64] In general, this idea goes back to Tang and Lin (2009) and later Geist and Endriss (2011), who use the computer-aided method in an automated search for impossibility theorems. Following Brandt and Geist (2016)—whose approach we heavily rely on—multiple recent papers use similar methods to obtain results in different subfields: Brandt et al. (2017a) tighten the bound on the number of voters required for the central impossibility of Moulin (1988), Brandl et al. (2018) solve a challenging open problem in probabilistic voting, and Brandt et al. (2018) revisit strategyproofness for set-valued voting rules and find new incompatibilities.

The main idea of the computer-aided approach is to prove statements by encoding a finite instance as a satisfiability problem.[65] This problem can be solved by a computer using a SAT solver and, provided a (simple) reduction argument, extends to arbitrary domain sizes. We expand the framework of Brandt and Geist (2016) to also

64 See also Geist (2016) and Geist and Peters (2017) for further details.
65 See, e.g., Biere et al. (2009) for a profound discussion of satisfiability.

m	Brandt and Geist (2016)	here
3	49	823 543
4	50 625	$\sim 2.5 \cdot 10^{49}$
5	$\sim 7.9 \cdot 10^{17}$	$\sim 9.4 \cdot 10^{867}$
6	$\sim 5.8 \cdot 10^{100}$	$\sim 6.8 \cdot 10^{38\,649}$

Table 4.1: Number of different majoritarian set-valued voting rules; while Brandt and Geist (2016) could assume an odd number of voters with strict preferences, participation requires us to deal with variable electorates, and therefore weak majority relations.

cater for indifferences in the majority relations, which is an important requirement for being able to deal with the notion of participation: if a voter with at least one strict preference abstains from the election, the corresponding majority relation might already contain indifferences.

Note that the introduction of majority ties significantly increases the size of the search space (see Table 4.1), which makes any type of exhaustive search even less feasible. Apart from being able to treat such large search spaces, another major advantage of the computer-aided approach is that many similar conjectures and hypotheses (here, e.g., statements about other preference extensions) can be checked quickly using the same framework.

In the coming subsections, we are going to explain our extension and some core features of the computer-aided method; for details of the original approach, however, we refer to Brandt and Geist (2016).

4.2.1 Encoding Participation

At the core of the computer-aided approach lies an encoding of the problem to be solved as a SAT instance. For this, all axioms involved need to be stated in propositional logic. We take over the formalization of the optimized encoding by Brandt and Geist (2016), which contains the following relevant axioms: *functionality of the choice function*, the *orbit condition*, and *Pareto optimality*. Pareto optimality is encoded as being a refinement of the uncovered set.[66] What remains is to encode the notion of participation. While this encoding turns out to be similar to the one of strategyproofness defined by Brandt and Geist (2016), it is more complex and not straightforward. In particular, it requires a novel condition that is equivalent to participation for majoritarian voting rules, which we are going to call *majority-participation*.

We are going to identify preference profiles with their corresponding majority relations, i.e., for majoritarian voting rules f, we often also write $f(G)$ instead of $f(\succsim)$ whenever $G = \succsim^M$.

66 Brandt et al. (2016c) show that for every alternative not contained in the uncovered set, there exists a preference profile where this alternative is Pareto dominated.

A majoritarian voting rule f is *Fishburn-majority-manipulable by strategic abstention* if there exist majority graphs G, G' on A and a preference relation $\succsim_i \in \succsim(A)$ such that $f(G') \succ_i^F f(G)$, with

$$\underline{G} \cap \underline{G'}^{-1} = \emptyset, \tag{1}$$
$$(G \setminus G') \cup (G'^{-1} \setminus G^{-1}) \subseteq \succ_i, \text{ and} \tag{2}$$
$$(G \setminus \underline{G}) \cap (G' \setminus \underline{G'}) \subseteq \sim_i. \tag{3}$$

If the voters' preferences are required to be strict, it additionally has to hold that either G or G' is antisymmetric. A majoritarian voting rule f satisfies *Fishburn-majority-participation* if it is not Fishburn-majority-manipulable by strategic abstention.

Fishburn-majority-participation

Conditions (1) to (3) can intuitively be phrased as follows: (1) prescribes that no strict relationship may be reversed between G and G'. (2) requires that $\succsim_i$ is in line with the changes from G to G', and finally (3) means that majority ties that occur in both majority graphs must be reflected by an indifference in $\succsim_i$.

In the following lemma, we show that for majoritarian voting rules, the condition of Fishburn-majority-manipulability corresponds to an abstaining voter with preferences $\succsim_i$ who thereby obtains a preferred outcome.[67]

LEMMA 4.3

A majoritarian voting rule satisfies Fishburn-participation if and only if it satisfies Fishburn-majority-participation.

Proof. We first provide a short outline of the proof. To begin with, we show that for every preference profile $\succsim$ that allows for a Fishburn-manipulation by abstention by voter i, the two majority relations $\succsim^M$ and $\succsim^M_{-i}$ together with $\succsim_i$ satisfy all required conditions. In return, whenever we have two majority graphs G, G' and a preference relation $\succsim_i$, all with the properties stated in the definition of Fishburn-majority-participation, we can assign integer majority margins to all pairs of alternatives and, by Debord (1987), use these to determine a preference profile $\succsim'$ that induces the majority graph G'. Together with $\succsim = \succsim'_{+i}$, we obtain $\succsim'^M = G'$, $\succsim^M = G$ and thus $f(\succsim') \succ_i^F f(\succsim)$ for majoritarian voting rules f.

In more detail, we begin by showing that if a majoritarian voting rule violates Fishburn-participation, then it also violates Fishburn-majority-participation. Therefore, let f be a majoritarian voting rule such that there exists some set of voters N, $n \geqslant 2$, set of alternatives A, preference profile $\succsim \in \succsim(A)^N$, and voter $i \in N$ for whom it holds that $f(\succsim_{-i}) \succ_i^F f(\succsim)$, i.e., there is an instance where i can strategically manipulate by abstaining. Let furthermore $G = \succsim^M$ and $G' = \succsim^M_{-i}$.

67 Note that both the definition of majority-participation and Lemma 4.3 are independent of a specific preference extension, and thus also applicable to, e.g., Kelly's extension.

First note that we directly have that $f(G') \succ_i^F f(G)$. Now, continue with conditions (1) to (3) as given in the definition of Fishburn-majority-participation. Since only one voter leaves the electorate when going from $\succsim$ to $\succsim_{-i}$ and everything else remains unchanged, we cannot have a directed edge from x to y in $\underline{G}$ and a directed edge from y to x in $\underline{G'}$ for any $x, y \in A$, i.e., no edges in the strict part of the majority graphs can be reversed. Hence, $\underline{G} \cap \underline{G'}^{-1} = \emptyset$.

Next, we have that if an alternative is preferred over another alternative by a weak majority in $\succsim$ and not preferred anymore in $\succsim_{-i}$, than i must prefer the former to the latter. Conversely, given one alternative is preferred over another alternative by a majority in $\succsim_{-i}$ but not in $\succsim$, i has to prefer the latter to the former. We together obtain $(G \setminus G') \cup (G'^{-1} \setminus G^{-1}) \subseteq \succ_i$.

Additionally, for every majority comparison that was balanced at an indifference in both $\succsim$ and $\succsim_{-i}$, we necessarily also have an indifference in i's preferences: $(G \setminus \underline{G}) \cap (G' \setminus \underline{G'}) \subseteq \sim_i$.

Finally, note that either n or $n-1$ is odd, so, assuming strict individual preferences, we have that either G or G' does not contain majority indifferences, i.e., is antisymmetric.

We continue with the reverse direction and show that if f violates Fishburn-majority-participation, then it also violates Fishburn-participation. To this end, let f be a majoritarian voting rule, G and G' be two majority graphs (on some set of alternatives A), $\succsim_i \in \succsim(A)$ an individual preference ranking such that $f(G') \succ_i^F f(G)$, and let (1) to (3) as given in the definition of Fishburn-majority-participation hold. Concerning individual preferences, we start with the general case.

We begin with G' and assign margins to all majority comparisons $(x, y) \in \underline{G'}$ such that

$$w(x,y) = \begin{cases} 1 & \text{if } (x,y) \notin \underline{G}, \text{ and} \\ 3 & \text{if } (x,y) \in \underline{G}. \end{cases}$$

If $(x, y), (y, x) \in G'$ we set $w(x, y) = w(y, x) = 0$. Employing a result by Debord (1987), it is possible to construct a preference profile $\succsim'$ (for some electorate N') in a way so that $\succsim'^M = G'$ and the majority margins are according to w. Define $\succsim = \succsim'_{+i}$ (and consequently $N = N' \cup \{i\}$). We now show that $\succsim^M = G$, or, equivalently, that all changes when going from G' to G are solely due to i joining the electorate while at the same time nothing else is altered.

Therefore note that for every pair of alternatives $x, y \in A$, one of the following cases—or its neutral equivalent—applies:

1. $(x, y) \in \underline{G} \cap \underline{G'}$: In this case $w(x, y) = 3$ and i's preferences do not affect the majority comparison.

2. $(x, y) \in \underline{G}, (y, x) \in \underline{G'}$: This case is not possible due to (1) in the definition of Fishburn-majority-participation.

3. $(x, y) \in G \setminus \underline{G}, (x, y) \in \underline{G'}$: We here have $w(x, y) = 1$ and by (2) above we know that $y \succ_i x$, i.e., including i indeed causes the majority comparison to change in the required way.

4. $(x, y) \in \underline{G}, (x, y) \in G' \setminus \underline{G'}$: By (2) above we know that $x \succ_i y$ and hence adding i causes the majority indifference in G' to sway in favor of x.

5. $(x, y) \in G \setminus \underline{G}, (x, y) \in G' \setminus \underline{G'}$: By (3) above this means that $x \sim_i y$, thus i does not affect the majority comparison and it remains identical.

Consequently, $\succsim^M = G$ and we directly deduce that $f(\succsim_{-i}) \succ_i^F f(\succsim)$, i.e., based on a Fishburn-majority-manipulation, we have constructed a Fishburn-manipulation and f violates Fishburn-participation.

In case voters' preferences are required to be strict, minor details of the proof change. First note that by the definition of Fishburn-majority-participation, either G or G' has to be antisymmetric in this case. If G is antisymmetric, G' cannot be so and we have to slightly modify our margin function for G' and instead use $w(x, y) = 2$ for all $(x, y) \in \underline{G'}$. This assures that the parity of all majority comparisons is even and a corresponding preference profile consisting of strict preferences only exists. The above case distinction still applies in so far as only the first, second, and fourth case are still relevant, the third and fifth are ruled out by antisymmetry.

Next, if G' is antisymmetric, G cannot be. Using the original weighting function guarantees odd parity for all majority comparisons and hence that we can find a preference profile consisting of strict preferences. Once more, the case distinction applies.

All in all, we have that every Fishburn-manipulation constitutes a Fishburn-majority-manipulation with the required properties regarding the majority relations and given a Fishburn-majority-manipulation exists, it is possible to construct a Fishburn-manipulation. This finishes the proof. $\square$

Fishburn-majority-participation can then be encoded in propositional logic (with variables $f_{\succsim^M, X}$ representing $f(\succsim^M) = X$) as the following simple transformation shows:

$$\neg \left(f(\succsim'^M) \succ_i^F f(\succsim^M) \right) \equiv \bigwedge_{Y \succ_i^F X} (\neg f_{\succsim^M, X} \vee \neg f_{\succsim^M, Y})$$

for all majority relations $\succsim^M$, $\succsim'^M$ and preference relations $\succsim_i$ satisfying conditions (1) to (3) in the definition of Fishburn-majority-participation.

Apart from the additional axioms, extending the framework of Brandt and Geist (2016) to participation and weak individual preferences also causes some additional (partially technical) challenges, such as:

- *Search space.* The size of the search space grows significantly with the introduction of weak majority relations (see Table 4.1). This larger search space can only be managed by an optimized encoding based on identifying isomorphic graphs (we do this via *canonical* representations).[68]
- *More complex data structures.* This is more of a technical challenge to extend all data structures such that they can handle weak majority relations in addition to strict ones.

4.2.2 Proof Extraction

A very interesting feature of the approach by Brandt and Geist (2016) is the possibility to extract human-readable proofs from an unsatisfiability result by the SAT solver. This is done by computing a minimal unsatisfiable set, an inclusion-minimal set of clauses that is still unsatisfiable.[69] This minimal unsatisfiable set can then, assisted by our encoder/decoder program, be read and transformed into a standard human-readable proof. Different proofs can be found by varying the extractor for the minimal unsatisfiable set or by encoding the problem for different subdomains, such as neighborhoods of a set of profiles or randomly sampled subdomains, respectively. We refer to Brandt and Geist (2016) and Geist (2016) for a more detailed description of the technique of proof extraction.

4.3 RESULTS AND DISCUSSION

In general, participation and strategyproofness are not logically related. However, extending an observation by Brandt (2015), it can be shown that strategyproofness implies participation under certain conditions.

LEMMA 4.4
> Consider an arbitrary preference extension. Every voting rule that satisfies IIV and strategyproofness satisfies participation. When preferences are strict, every majoritarian voting rule that satisfies strategyproofness satisfies participation.

Proof. We show both statements for Kelly's extension. The same argument works for any other preference extension, including Fishburn's.

For the first statement, let f be a set-valued voting rule that satisfies IIV and Kelly-strategyproofness. Assume for contradiction that f is Kelly-manipulable by strategic abstention, i.e., there is a preference profile $\succsim$ and a voter i such that $f(\succsim_{-i}) \succ_i^K f(\succsim)$. Let $\succsim'$ be a

68 Without this optimization, only domains of up to four alternatives can be solved within a reasonable time frame.

69 We used PicoMUS, which is part of the PicoSAT distribution (Biere, 2008).

preference profile such that $\succsim'_j = \succsim_j$ for all $j \neq i$, and $\succsim'_i$ the preference relation that expresses indifference between all alternatives, i.e., $x \sim'_i y$ for all $x, y \in A$. Then, since f satisfies IIV,

$$f(\succsim') = f(\succsim_{-i}) \succ_i^K f(\succsim),$$

i.e., i can manipulate in $\succsim$ by reporting $\succsim'_i$ instead of $\succsim_i$, which contradicts Kelly-strategyproofness of f.

For the second statement, let f be a majoritarian voting rule that satisfies Kelly-strategyproofness. Assume for contradiction that f is Kelly-manipulable by strategic abstention, i.e., there is a preference profile $\succsim$ and a voter i such that $f(\succsim_{-i}) \succ_i^K f(\succsim)$. Let $2\succsim$ be a preference profile that consists of two copies of each preference relation in $\succsim$, i.e., for every $(i, \succsim_i) \in \succsim$, there are $(i_1, \succsim_{i_1}), (i_2, \succsim_{i_2}) \in 2\succsim$ such that $\succsim_i = \succsim_{i_1} = \succsim_{i_2}$. Note that $\succsim$ and $2\succsim$ have the same majority relation. $2\succsim_{-i}$ is defined analogously.

We define a preference profile $2\succsim'$ such that $2\succsim'_j = 2\succsim_j$ for all $j \neq i_1$ and $2\succsim'_{i_1} = \succsim_{i_1}^{-1}$, where $\succsim_{i_1}^{-1}$ denotes the inverse of $\succsim_{i_1}$, i.e., for all $x, y \in A$, $x \succsim_{i_1}^{-1} y$ if and only if $y \succsim_{i_1} x$. Intuitively, $2\succsim'$ is thus the same as $2\succsim$ but for the preferences of voter i_1, which are reversed. In terms of the majority relation of $2\succsim'$, we thus have that i_1 and i_2 cancel each other out, i.e., do not have an effect on the majority relation. Consequently, the majority relations of $2\succsim'$ and $2\succsim_{-i}$ are identical. Then, since f is majoritarian,

$$f(2\succsim') = f(2\succsim_{-i}) = f(\succsim_{-i}) \succ_i^K f(\succsim) = f(2\succsim),$$

i.e., i_1 can manipulate in $2\succsim$ by reporting $\succsim_{i_1}^{-1}$. This contradicts Kelly-strategyproofness of f. Note that the proof of the second statement does not require indifferences to be possible within individual preferences. □

As a consequence of Lemma 4.4, some positive results for Kelly-strategyproofness and Fishburn-strategyproofness carry over to participation.[70] We will complement these results by impossibility theorems for Fishburn-participation and a positive result for Kelly-participation, which specifically does not hold for Kelly-strategyproofness.

4.3.1 Fishburn-Participation

It turns out that Pareto optimality is incompatible with Fishburn-participation in majoritarian voting settings. The corresponding Theorems 4.7 and 4.8 and their proofs were obtained using the computer-aided method laid out in Section 4.2.[71] In order to simplify the original proofs, which were found by the computer, we first state a lemma

70 See Table 4.2 in Section 4.4 for more details.

71 A proof for Theorem 4.7 was first obtained manually. A shorter and more elegant variant is due to the computer-aided method, though.

that offers further insights into the possible choices of majoritarian voting rules that satisfy Fishburn-participation and Pareto optimality.

To state Lemma 4.5, we introduce some additional notation: an alternative x (McKelvey) covers an alternative y if x is at least as good as y compared to every other alternative (McKelvey, 1986). Formally, given a majority relation $\succsim^M$, x *covers* y if $x \succ^M y$ and, for all $z \in A$, both $y \succsim^M z$ implies $x \succsim^M z$, and $z \succsim^M x$ implies $z \succsim^M y$. The *uncovered set* of $\succsim^M$, denoted $UC(\succsim^M)$, is the set of all alternatives that are not covered by any other alternative. We inductively define $UC^k(\succsim^M)$ as the repeated application of UC, i.e., $UC^1(\succsim^M) = UC(\succsim^M)$ and

covering relation

uncovered set

$$UC^k(\succsim^M) = UC(\succsim^M|_{UC^{k-1}(\succsim^M)})$$

for $k > 1$ where $\succsim^M|_X$ means the majority relation restricted to alternatives contained in X only. Then, the *iterated uncovered set* is defined by

iterated uncovered set

$$UC^\infty(\succsim^M) = \bigcap_{k \geqslant 1} UC^k(\succsim^M).$$

Both UC and UC^∞ are illustrated in Example 4.6 after the following Lemma 4.5. By definition, UC and UC^∞ are majoritarian voting rules.

Brandt et al. (2016c) have proven that every majoritarian and Pareto optimal voting rule selects a subset of the (McKelvey) uncovered set. We show that a voting rule that additionally satisfies Fishburn-participation furthermore only depends on the majority relation between alternatives in the iterated uncovered set and only selects alternatives within the iterated uncovered set.[72]

Lemma 4.5

Let f be a majoritarian and Pareto optimal set-valued voting rule that satisfies Fishburn-participation. Let $\succsim, \succsim' \in \succsim(A)^{\mathcal{F}(\mathbb{N})}$ be preferences profiles such that $\succsim^M|_{UC^\infty(\succsim^M)} = \succsim'^M|_{UC^\infty(\succsim'^M)}$. Then

$$f(\succsim) \subseteq UC^\infty(\succsim^M)$$

and if $\succsim^M$ and $\succsim'^M$ additionally are antisymmetric, we have that

$$f(\succsim) = f(\succsim').$$

Proof. We begin the proof by showing that $f(\succsim) \subseteq UC^\infty(\succsim^M)$. Let f be a majoritarian and Pareto optimal voting rule that satisfies Fishburn-participation, $N \in \mathcal{F}(\mathbb{N})$ a set of voters, $\succsim \in \succsim(A)^N$ a preference profile and $\succsim^M$ the majority relation of $\succsim$. We prove inductively that $f(\succsim) \subseteq UC^k(\succsim^M)$ for all $k \in \mathbb{N}$.

72 Lemma 4.5 can be strengthened in various respects and also holds for all other preference extensions satisfying some mild conditions as well as probabilisitic voting rules (see Lemma 5.12).

First, let $k = 1$. Brandt et al. (2016c) have shown that if an alternative x is not in the McKelvey uncovered set $UC(\succsim^M)$, it is potentially Pareto dominated, i.e., there is a preference profile $\succsim'$ such that $\succsim^M = \succsim'^M$ and x is Pareto dominated in $\succsim'$. Hence, $x \notin f(\succsim')$ and as well $x \notin f(\succsim)$, since f is majoritarian and Pareto optimal.

Now let $k \geqslant 2$. By induction, $f(\succsim) \subseteq UC^{k-1}(\succsim^M)$. If we have that $UC^{k-1}(\succsim^M) = UC^k(\succsim^M)$, there is nothing left to show. Hence, we consider the remaining case, i.e., $UC^k(\succsim^M) \subsetneq UC^{k-1}(\succsim^M)$. By Debord (1987), we can find a preference profile $\succsim' \in \succsim(A)^{N'}$ such that $\succsim'^M = \succsim^M$ and

$$|g_{xy}(\succsim')| \begin{cases} \leqslant 1 & \text{for all } x \in UC^k(\succsim^M), y \in A \setminus UC^{k-1}(\succsim^M), \text{ and} \\ \geqslant 3 & \text{otherwise.} \end{cases}$$

This intuitively means that one or two joining voters can only affect majority comparisons in between alternatives x and y where $x \in UC^k(\succsim^M)$ and $y \in A \setminus UC^{k-1}(\succsim^M)$. Specially note that if two voters j, j' with $x \succ_j y$ and $x \succ_{j'} y$ for all such x, y join the electorate, we have that every alternative in $UC^k(\succsim^M)$ is majority-preferred over every alternative in $A \setminus UC^{k-1}(\succsim^M)$.

Let $\succsim_i$ be a preference relation such that $x \succsim_i y$ for all alternatives $x \in UC^k(\succsim'^M)$ and $y \in A \setminus UC^{k-1}(\succsim'^M)$. Clearly,

$$UC^{k-1}(\succsim'^M_{+i}) \subseteq UC^{k-1}(\succsim'^M)$$

as nothing covered before can become uncovered.

If $f(\succsim') \neq f(\succsim'_{+i})$, we can find some $\succsim_j$ such that $x \succ_j y$ for all $x \in UC^k(\succsim'^M)$ and $y \in A \setminus UC^{k-1}(\succsim'^M)$ and $f(\succsim') \succ_j^F f(\succsim'_{+j})$, which contradicts Fishburn-participation. Thus, $f(\succsim') = f(\succsim'_{+i})$. With the same reasoning, we can find $\succsim_{j'}$ such that $x \succ_{j'} y$ for all alternatives $x \in UC^k(\succsim'^M)$ and $y \in A \setminus UC^{k-1}(\succsim'^M)$ and

$$f(\succsim'_{+j,j'}) = f(\succsim'_{+j}) = f(\succsim').$$

Using a result by Brandt et al. (2016c) again, Pareto optimality of f directly implies that $f(\succsim'_{+j,j'}) \subseteq UC(\succsim^M_{+j,j'})$. By definition of $\succsim'$, $\succsim_j$ and $\succsim_{j'}$, we have that $UC(\succsim'^M_{+j,j'}) = UC^k(\succsim'^M)$. Hence,

$$f(\succsim) = f(\succsim') = f(\succsim'_{+j,j'}) \subseteq UC(\succsim'^M_{+j,j'}) = UC^k(\succsim'^M) = UC^k(\succsim^M),$$

which completes the first part of the proof.

Next, let f be as above and $\succsim$ and $\succsim'$ two preference profiles on some agenda A such that

$$\succsim^M|_{UC^\infty(\succsim^M)} = \succsim'^M|_{UC^\infty(\succsim'^M)}.$$

Suppose for contradiction that $f(\succsim) \neq f(\succsim')$. Since f is majoritarian, define without loss of generality for $\succsim^M$ majority margins $|g_{xy}|$ suitably small if $(x, y) \in \succ^M$ and $(x, y) \notin \succ'^M$, and $|g_{xy}|$ sufficiently large otherwise.[73]

73 Since $\succsim^M$ and $\succsim'^M$ are antisymmetric we obviously have $g_{xy} \neq 0$ for all $x, y \in A$.

We successively include k voters $i_1, \dots, i_k$ into the corresponding electorate, all with individual preferences such that all alternatives in $UC^\infty(\succsim'^M)$ are adjacent in $\succsim_{i_j}$. The placement of all further alternatives is to be of the way that $\succsim^M_{\{+i_1,\dots,i_k\}} = \succsim'^M$, yielding that $f(\succsim_{\{+i_1,\dots,i_k\}}) \neq f(\succsim)$. This is possible as we assumed the weights of all edges that have to be changed to be suitably small. Since the ordering of alternatives within $UC^\infty(\succsim'^M)$ does not affect the majority relation due to sufficiently large edge weights, we are able to arrange them on the way such that $x \succ_{i_j} y \succ_{i_j} z$ for all

$$x \in f(\succsim_{\{+i_1,\dots,i_{j-1}\}}) \setminus f(\succsim_{\{+i_1,\dots,i_j\}}),$$
$$y \in f(\succsim_{\{+i_1,\dots,i_{j-1}\}}) \cap f(\succsim_{\{+i_1,\dots,i_j\}}), \text{ and}$$
$$z \in UC^\infty(\succsim^M) \setminus f(\succsim_{\{+i_1,\dots,i_{j-1}\}}),$$

$1 \leqslant j \leqslant k$. Intuitively, this means that everything that was in the choice set before i_j joined the electorate but is not contained anymore, is preferred to everything that still is in the choice set, is preferred to everything that is possibly chosen now, but was not chosen before. Thus, every i_j prefers what was chosen without him to what is chosen including him.

Hence, for at least one voter i_j, $1 \leqslant j \leqslant k$, it has to hold that

$$f(\succsim_{\{+i_1,\dots,i_{j-1}\}}) \neq f(\succsim_{\{+i_1,\dots,i_j\}})$$

and by definition of this voter's preferences

$$f(\succsim_{\{+i_1,\dots,i_{j-1}\}}) \succ^F_{i_j} f(\succsim_{\{+i_1,\dots,i_j\}}).$$

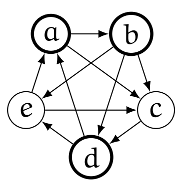
This contradicts Fishburn-participation and finishes the proof. □

The following brief example illustrates an application of Lemma 4.5.

Example 4.6

Let $\succsim$ be a preference profile on $A = \{a, b, c, d, e\}$ such that $\succsim^M$ is as depicted below. We have that $UC(\succsim^M) = \{a, b, d, e\}$ and $UC^2(\succsim^M) = UC^\infty(\succsim^M) = \{a, b, d\}$. For all majoritarian and Pareto optimal voting rules f satisfying Fishburn-participation, Lemma 4.5 gives that $f(\succsim) \subseteq \{a, b, d\}$.

Also, the choice set has to be identical to the one of any other profile with identically structured iterated uncovered set, e.g., a profile $\succsim'$ where the iterated uncovered set is a completely symmetric three-cycle on $\{a, b, d\}$. Thus, we even have $f(\succsim) = \{a, b, d\}$ due to neutrality. Similar considerations are used repeatedly in the proofs of Theorem 4.7 and Theorem 4.8.



$\succsim^M$

Now, let us turn to our impossibility theorems. The computer found these impossibilities even without using Lemma 4.5. However, the formalization of the lemma allowed the SAT solver to find smaller proofs and makes the human-readable proofs more intuitive.

THEOREM 4.7

There is no majoritarian and Pareto optimal set-valued voting rule that satisfies Fishburn-participation if $m \geqslant 4$ and $n \geqslant 6$.

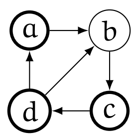
Proof. For this proof, we use an anonymous representation of preference profiles for reasons of simplicity and thus only denote how many voters share a preference ranking instead of individual identities.

Let f be a majoritarian and Pareto optimal voting rule satisfying Fishburn-participation. Note that majoritarianism implies anonymity and neutrality. We first prove the statement for $A = \{a, b, c, d\}$ and reason about the outcome of f for some specific majority graphs. Throughout this proof, we make extensive use of Lemma 4.3, which allows us to apply Fishburn-majority-participation instead of regular Fishburn-participation. To this end, we also slightly abuse notation and write G_{+i} meaning $\succsim^M_{+i}$ if $G = \succsim^M$.

Intuitively, the proof strategy is to alter the majority graphs G, G', and G'' as depicted below by letting varying voters *join* some underlying electorate. This excludes certain choices of f as a voting rule (by an application of Fishburn-majority-participation), until we reach a contradiction. For each step, i.e., each time an additional voter alters one of the majority graphs, we provide a suitable electorate inducing G, G', and G'', respectively. These electorates vary based on the joining voter's preferences and on which majority comparisons either have to or must not be changed.

Consider for instance G, that can be altered to G_{+1} by a voter with preferences $\succsim_1 : \{a, b, c\}, d$. For this to be possible, we need an underlying electorate where d has to be majority-preferred over b by a margin of exactly one, while it has to be preferred over a by a margin of at least two. These constraints make the construction of suitable electorates more demanding with respect to the number of voters needed compared to the case where only the simple majority relation is of relevance. In the figures depicting the strict part of the majority graphs, we highlight alternatives that have to be chosen by f with a thick border.

First consider G as depicted on the right. In the following, arguments using the additional voter's preferences as well as which effects him joining has on the set of possibly chosen alternatives are given on the left, while suitable electorates inducing G together with the changed majority graph are given on the right.

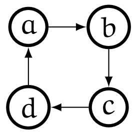


G

Whenever an electorate works for multiple cases, they are grouped in a single paragraph with the preference profile next to it.

1	1	1	1
a	b	c	d
b, c, d	c	d	a
	d	a	b
	a	b	c

Adding a voter with preferences $\succsim_1 : \{a, b, c\}, d$ yields G_{+1} where, due to symmetry, $f(G_{+1}) = \{a, b, c, d\}$. As f satisfies Fishburn-participation, nothing that is strictly preferred to $\{a, b, c, d\}$ according to $\succsim_1^F$ can be chosen in G. Thus, $d \in f(G)$. Adding another voter with $\succsim_{1'} : b, \{a, c, d\}$ also leads to majority graph G_{+1}. Hence, $f(G) \neq \{b, d\}, \{a, b, d\}, \{b, c, d\}$.

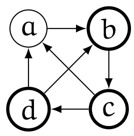


G_{+1}

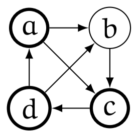
If G is altered to G_{+2} by adding a voter with $\succsim_2 : \{b, d\}, c, a$, we get $f(G_{+2}) = \{b, c, d\}$, by the fact that a is covered by d together with neutrality. Therefore, $f(G) \neq \{d\}, \{c, d\}$. Note that here $n = 5$ is the minimal number of other voters needed to construct a suitable preference profile.

1	1	1	1	1
a	a, c	b	c	d
b	d	c	d	a
c	b	d	a	b
d		a	b	c

Adding a voter with $\succsim_3 : \{a, b, d\}, c$ leads to G_{+3}, for which we have $f(G_{+3}) = \{a, c, d\}$ by the fact that b is covered by a together with neutrality. We correspondingly deduce $f(G) \neq \{a, d\}, \{a, b, c, d\}$. Hence, we conclude that $f(G) = \{a, c, d\}$.

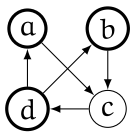


G_{+2}



G_{+3}

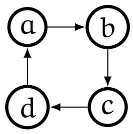
Now consider G' as shown on the right. Neutrality implies that $f(G')$ contains either neither or both of a and b.



G'

1	2	1	1
a	b	d	d
c	c	a	a, b
d	d	b	c
b	a	c	

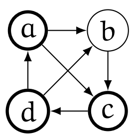
Adding a voter with preferences $\succsim'_1 : c, a, b, d$ changes G' to G'_{+1}, where $\{a, b, c, d\}$ is selected. Thus, $f(G') \neq \{c\}, \{a, b, c\}$.



G'_{+1}

1	1	1	1	1
a	a, b	b	d	d
c	c	c	a	b
d	d	d	b	a
b		a	c	c

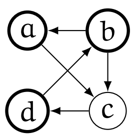
Additionally, a voter with $\succsim'_2 : \{c, d\}, a, b$ alters G' to G'_{+2}. Hence, $f(G') \neq \{d\}, \{c, d\}$.



G'_{+2}

1	1	1	2
a	a	c	d
b	c	d	b
c	d	a, b	a
d	b		c

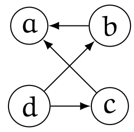
Adding a voter with $\succsim'_3 : \{b, c\}, a, d$ changes G' to G'_{+3}. G'_{+3} is isomorphic to G, which implies $f(G'_{+3}) = \{a, b, d\}$. Thus, $f(G') \neq \{a, b\}, \{a, b, c, d\}$.



G'_{+3}

Putting everything together, we deduce that $f(G') = \{a, b, d\}$.

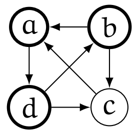
Finally, consider G'' as depicted on the right. Neutrality implies that $f(G'')$ contains either neither or both of b and c.



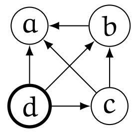
G''

Adding a voter with $\succsim''_1$: $a, \{b, d\}, c$ changes G'' to G''_{+1}. Analogously to G_{+2}, $f(G''_{+1}) = \{a, b, d\}$. Therefore, $f(G'') \neq \{a, d\}$.
Adding a voter with preferences $\succsim''_2$: $c, \{b, d\}, a$ alters G'' to G''_{+2}. Alternative d is the Condorcet winner in G''_{+2}, which implies that $f(G''_{+2}) = \{d\}$ because d covers a, b, and c. Hence, $f(G'') \neq \{b, c\}, \{b, c, d\}$.
Both a voter with $\succsim''_3$: $a, d, \{b, c\}$ as well as with $\succsim''_{3'}$: $\{a, b, c\}, d$ can alter G'' to G''_{+3}, which is isomorphic to G'. Thus, $f(G''_{+3}) = \{b, c, d\}$. This implies that $f(G'') \neq \{d\}, \{a, b, c, d\}$, otherwise a voter with $\succsim''_3$ or $\succsim''_{3'}$, respectively, could manipulate.

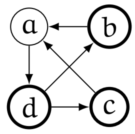
1	1	1	1
b	c	d	d
a	a	b	c
d	d	c	b
c	b	a	a



G''_{+1}



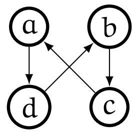
G''_{+2}



G''_{+3}

A voter with $\succsim''_4$: a, b, c, d changes G'' to G''_{+4}. Neutrality implies that $f(G''_{+4}) = \{a, b, c, d\}$. This gives that $f(G'') \neq \{a\}, \{a, b, c\}$.

1	1	1	1
a, b	c	d	d
c, d	a	b	d
	d	c	b
	b	a	a



G''_{+4}

Consequently, if Fishburn-participation is to be respected, f cannot choose anything from G'', a contradiction to f being a proper voting rule.

Now let $m \geqslant 5$. It follows from Lemma 4.5 that the choice of f does not depend on covered alternatives. Hence, the statement follows by extending the majority graphs depicted above to A such that all alternatives but a, b, c, and d are covered (for instance, by adding Condorcet losers only). □

We could verify with our computer-aided approach that this impossibility still holds for strict preferences when there are at least five alternatives and seven voters.

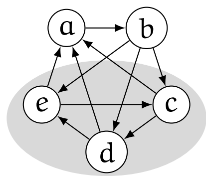
THEOREM 4.8

> There is no majoritarian and Pareto optimal set-valued voting rule that satisfies Fishburn-participation if $m \geqslant 5$ and $n \geqslant 7$, even when preferences are strict.

Proof. For this proof, we use an anonymous representation of preference profiles for reasons of simplicity and thus only denote how many voters share a preference ranking instead of individual identities.

Let f be a majoritarian and Pareto optimal voting rule satisfying Fishburn-participation and note, just as before, that majoritarianness implies anonymity and neutrality. We first prove the statement for $A = \{a, b, c, d, e\}$ and reason about the outcome of f for some specific majority graphs. Throughout this proof, we again make extensive use of Lemma 4.3, which allows us to apply Fishburn-majority-participation instead of regular Fishburn-participation. Intuitively, the basic proof strategy is similar to the proof of Theorem 4.7, due to interdependencies however more intricate.

Briefly summarized, we focus on G^1 as given on the right—essentially a three-cycle as component of another three-cycle. By neutrality we know that f has to select either all of c, d, e or none of them, i.e., there are only seven possible subsets of A that f can choose in G^1: $\{a\}, \{b\}, \{c, d, e\}, \{a, b\}, \{a, c, d, e\}, \{b, c, d, e\}$ or A.

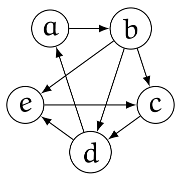


G^1

Using several auxiliary majority graphs, we successively show that if f selects $\{b\}$ or $\{a, b\}$, nothing can be chosen in G^2, if f selects $\{a, c, d, e\}$ or A, nothing can be chosen in G^3, if f selects $\{a\}$, nothing can be chosen in G^4 and finally if f chooses $\{c, d, e\}$ or $\{b, c, d, e\}$, nothing can be chosen in G^5. Thus, whatever f selects in G^1, one of the other four majority graphs is left without a possible choice set, a contradiction.

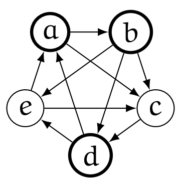
To begin with, consider the majority graph G^2 as depicted on the right together with a preference profile inducing G^2. Similar to before, we now let different voters join this electorate and examine which conclusions they allow with respect to $f(G^2)$.

1	1	1	2	1
b	b	c	d	e
c	e	a	a	a
d	c	b	b	b
e	d	d	e	c
a	a	e	a	d



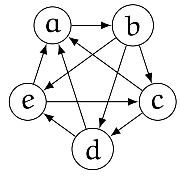
G^2

First, assume a voter with preferences $\succsim^2_1$: e, b, a, d, c joins the electorate and consequently alters G^2 to G^2_{+1}. Here, we have that $f(G^2_{+1}) = \{a, b, d\}$ according to Lemma 4.5. As nothing preferred to $\{a, b, d\}$ can be selected by f in G^2, we obtain $f(G^2) \neq \{b\}, \{a, b\}$.



G^2_{+1}

If, on the other hand, either a voter with preferences $\succsim_2^2$: d, c, e, b, a or a voter with preferences $\succsim_{2'}^2$: d, c, e, a, b joins the electorate, G^2 is changed to G^2_{+2}, which is equivalent to G^1. Suppose for the following arguments that f chooses $\{b\}$ or $\{a, b\}$ in G^1. With respect to $\succsim_2^2$, we thus obtain that $f(G^2) \neq \{c\}$, $\{d\}$, $\{e\}$, $\{b, c\}$, $\{b, d\}$, $\{b, e\}$, $\{c, d\}$, $\{c, e\}$, $\{d, e\}$, $\{b, c, d\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{c, d, e\}$, $\{b, c, d, e\}$.

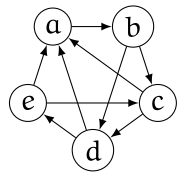


G^2_{+2}

Similarly, when considering $\succsim_{2'}^2$, we can deduce that $f(G^2) \neq \{a\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{a, b, c, d\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$, $\{a, c, d, e\}$, A. Put differently, we thus have that if $f(G^2_{+2}) = f(G^1)$ is either $\{b\}$ or $\{a, b\}$, then we obtain $f(G^2) = \emptyset$. Consequently, $f(G^1) \neq \{b\},\{a, b\}$.

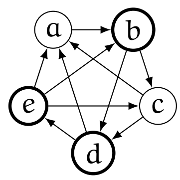
Next, consider the majority relation G^3 as given on the right together with a preference profile inducing G^3. We proceed with our well-known procedure of voters joining this electorate.

1	1	1	1	1	1
a	b	b	c	d	d
e	c	e	a	e	e
b	d	c	b	a	c
c	e	d	d	b	a
d	a	a	e	c	b



G^3

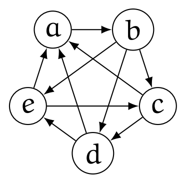
If a voter with preferences $\succsim_1^3$: c, a, e, d, b joins the electorate, he changes G^3 to G^3_{+1}, where we have that $f(G^3_{+1}) = \{b, d, e\}$. Consequently, nothing preferred to $\{b, d, e\}$ could have been selected by f in G^3 and we obtain $f(G^3) \neq \{a\}$, $\{c\}$, $\{e\}$, $\{a, c\}$, $\{a, e\}$, $\{c, e\}$, $\{d, e\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{c, d, e\}$, $\{a, c, d, e\}$, $\{b, c, d, e\}$, A.



G^3_{+1}

The same modification of G^3 to G^3_{+1} can also be achieved by voters with preferences $\succsim_2^3$: c, a, d, e, b and $\succsim_3^3$: c, a, e, b, d. This additionally gives $f(G^3) \neq \{d\}$, $\{a, d\}$, $\{c, d\}$, $\{a, c, d\}$, $\{a, b, d, e\}$ due to $\succsim_2^3$ and $f(G^3) \neq \{b, e\}$, $\{a, b, e\}$, $\{b, c, e\}$, $\{a, b, c, e\}$ because of $\succsim_3^3$.

We now let five different voters join the electorate, each of them will alter G^3 to G^3_{+4}, which is identical to G^1. For this step of the proof, suppose that $f(G^1)$ is either $\{a, c, d, e\}$ or A. Adding a voter with preferences $\succsim_4^3$: b, e, d, c, a then gives that f cannot select $\{b\}$ or $\{b, e, d\}$ in G^3. In a similar fashion, $\succsim_5^3$: b, a, e, d, c excludes $\{a, b\}$ from the set of possible choices.



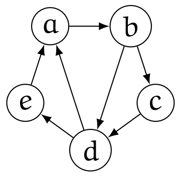
G^3_{+4}

Next, if a voter with $\succsim_6^3$: c, b, a, e, d joins the electorate, we obtain $f(G^3) \neq \{b, c\},\{a, b, c\}$. A voter with $\succsim_7^3$: d, b, a, e, c causes an identical change in the majority relation and gives that $f(G^3)$ cannot be either $\{b, d\}$ or $\{a, b, d\}$. Finally, if a voter with preferences $\succsim_8^3$: d, c, b, a, e joins the electorate, G^3 is once more altered to G^3_{+4} and we deduce $f(G^3) \neq \{b, c, d\},\{a, b, c, d\}$.

To summarize this step, we have that if $f(G^3_{+4}) = f(G^1)$ is either $\{a, c, d, e\}$ or A, then $f(G^3) = \emptyset$. Hence, $f(G^1) \neq \{a, c, d, e\}, A$.

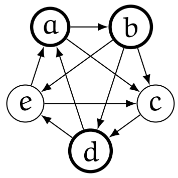
In the next step, we focus on G^4 as depicted on the right. Just as before, we provide a preference profile inducing G^4.

1	1	1	1	1	1
a	b	b	d	e	e
b	c	c	e	a	d
c	d	d	a	b	c
d	a	e	c	c	a
e	e	a	b	d	b



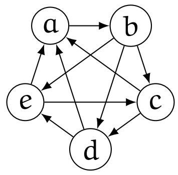
G^4

First, G^4 can be altered to G^4_{+1} if a voter with preferences $\succsim^4_1$: a, b, d, e, c joins the electorate. As we have $f(G^4_{+1}) = \{a, b, d\}$, we obtain $f(G^4) \neq \{a\}$ because otherwise joining would result in a strictly worse outcome for the corresponding voter.



G^4_{+1}

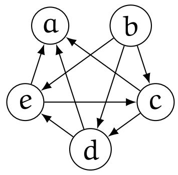
A second voter with preferences $\succsim^4_2$: b, d, e, c, a changes G^4 to G^4_{+2}, once again equal to G^1. We now assume $f(G^1) = \{a\}$. Given this presumption and that nothing preferred to $\{a\}$ may have been chosen in G^4, we deduce that $f(G^4) \neq X$ for all $X \in 2^A \setminus \{\emptyset, \{a\}\}$.
We hence have that if $f(G^4_{+2}) = f(G^1) = \{a\}$, then $f(G^4) = \emptyset$. Thus, $f(G^1) \neq \{a\}$.



G^4_{+2}

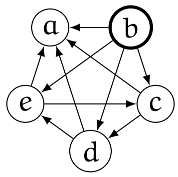
Finally, consider G^5 as given on the right together with a preference profile inducing it. Note that in G^5, c, d, and e are 'symmetric', i.e., as f satisfies neutrality, f has to either select all of c, d, e or none of them.

1	1	1	1	1	1
c	c	d	d	e	e
a	b	a	b	a	b
b	d	b	e	b	c
d	e	e	c	c	d
e	a	c	a	d	a



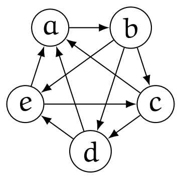
G^5

If a voter with preferences $\succsim^5_1$: d, c, e, b, a joins the electorate, he changes G^5 to G^5_{+1}, where b is the Condorcet winner and thus $f(G^5_{+1}) = \{b\}$. Consequently, $f(G^5) \neq \{c, d, e\}, \{b, c, d, e\}$.



G^5_{+1}

If, on the other hand, a voter with preferences $\succsim^5_2$: a, b, d, c, e joins, G^5 becomes G^5_{+2}, which is equal to G^1. Lastly, supposing that $f(G^1)$ is either $\{c, d, e\}$ or $\{b, c, d, e\}$, we can consequently exclude the remaining five choice possibilities from G^5: $f(G^5) \neq \{a\}, \{b\}, \{a, e\}, \{a, c, d, e\}, A$.[74]
This step thus gives that if $f(G^5_{+2}) = f(G^1)$ equals $\{c, d, e\}$ or $\{b, c, d, e\}$, then $f(G^5) = \emptyset$. We therefore have that $f(G^1) \neq \{c, d, e\}, \{b, c, d, e\}$.



G^5_{+2}

74 In case that $f(G^1) = \{b, c, d, e\}$, a voter with preferences $\succsim^5_3$: a, d, c, e, b has to join the electorate in order for us to be able to exclude $\{a, c, d, e\}$.

To put it all together, we have that for G^1, a majoritarian and Pareto optimal voting rule f satisfying Fishburn-participation has to select either $\{a\}$, $\{b\}$, $\{a, b\}$, $\{c, d, e\}$, $\{a, c, d, e\}$, $\{b, c, d, e\}$, or A due to neutrality. Yet, assuming f selects $\{b\}$ or $\{a, b\}$, we have that $f(G^2) = \emptyset$. If f selects $\{a, c, d, e\}$ or A, we deduce that $f(G^3) = \emptyset$. $f(G^1) = \{a\}$ implies that $f(G^4) = \emptyset$. Finally, if f selects $\{c, d, e\}$ or $\{b, c, d, e\}$, we obtain $f(G^5) = \emptyset$. We hence have that $f(G^1) = \emptyset$, a contradiction. Thus, such a voting rule f cannot exist and the impossibility is proven.

The extension from $m = 5$ to $m \geqslant 5$ works analogously to the extension in the proof of Theorem 4.7.

With respect to the number of voters, observe that all relevant majority graphs—G^2, G^3, G^4, and G^5—require an electorate of six voters to induce them. Together with the joining voter, we obtain the threshold of $n \geqslant 7$ voters for this impossibility theorem to hold.[75] □

Theorems 4.7 and 4.8 are both tight in the sense that, whenever there are less than four or five alternatives, respectively, there exists a voting rule that satisfies the desired properties.

An interesting question is whether these impossibilities also extend to other preference extensions. Given the computer-aided approach, this can be easily checked by simply replacing the preference extension in the SAT encoder. For instance, it turns out that the impossibility of Theorem 4.7 still holds if we consider a coarsening of Fishburn's extension that can only compare sets that are contained in each other. Kelly's extension, on the other hand, does not lead to an impossibility for $m \leqslant 5$, which will be confirmed more generally in the next section.

4.3.2 Kelly-Participation

Theorems 4.7 and 4.8 are sweeping impossibilities within the domain of majoritarian voting rules. For Kelly's extension, we obtain a much more positive result that covers attractive majoritarian and non-majoritarian rules. Brandt (2015) has shown that set-monotonicity implies Kelly-strategyproofness for strict preferences, and that no Condorcet extension is Kelly-strategyproof when preferences are weak. We prove that set-monotonicity (and the very mild IIV axiom) imply Kelly-participation even for weak preferences. We have thus found

75 Theoretically, it is also possible to not have one voter join an electorate of size six, but have a voter with reversed preferences leave the electorate in order to achieve the same change in majority relations as well as consequences on the choice set. However, exhaustive search shows that for at least one of the majority graphs it is not possible to construct an electorate of size six, such that one of the voters has exactly those reversed preferences.

natural examples of voting rules that violate Kelly-strategyproofness but satisfy Kelly-participation.[76]

Theorem 4.9

Let f be a set-valued voting rule that satisfies IIV and set-monotonicity. Then f satisfies Kelly-participation.

Proof. Let f be a set-valued voting rule that satisfies IIV and set-monotonicity. Assume for contradiction that f does not satisfy Kelly-participation. Hence, there exist a preference profile $\succsim \in \succsim(A)^N$ and a voter $i \in N$ such that $f(\succsim_{-i}) \succ_i^K f(\succsim)$. Let $X = f(\succsim)$, $Y = f(\succsim_{-i})$, and $Z = A \setminus (X \cup Y)$. By definition of $Y \succ_i^K X$, we have that $x \sim_i y$ for all $x, y \in X \cap Y$.

We define a new preference relation $\succsim_i'$ in which all alternatives in Y are tied for the first place, followed by all alternatives in $X \setminus Y$ as they are ordered in $\succsim_i$, and all remaining alternatives in one indifference class at the bottom of the ranking. Formally,

$$\succsim_{i'} = (Y \times A) \cup \succsim_i|_{X\setminus Y} \cup (A \times Z).$$

Let i'' be a voter who is indifferent between all alternatives, i.e., $x \sim_{i''} y$ for all $x, y \in A$. Since f satisfies IIV, we can deduce that $f(\succsim_{-i+i''}) = f(\succsim_{-i})$.

By definition, $\succsim_{-i+i'}$ is an f-improvement over both $\succsim$ and $\succsim_{-i+i''}$. Hence, set-monotonicity implies that $f(\succsim_{-i+i'}) = f(\succsim)$ and $f(\succsim_{-i+i'}) = f(\succsim_{-i+i''})$. In summary, we obtain

$$f(\succsim_{-i+i'}) = f(\succsim_{-i+i''}) = f(\succsim_{-i}) \succ_i^K f(\succsim) = f(\succsim_{-i+i'}),$$

which is a contradiction. □

Two rather undiscriminating voting rules that satisfy both IIV and set-monotonicity are the *Pareto rule* and the *omninomination rule*, which returns all alternatives that are ranked first by at least one voter. Majoritarian voting rules satisfy IIV by definition and there are several appealing majoritarian rules that satisfy set-monotonicity, among those for instance the *top cycle*, also known as *weak closure maximality*, *GETCHA*, or the *Smith set* (Good, 1971; Smith, 1973; Bordes, 1976; Sen, 1977; Schwartz, 1986), the *minimal covering set* (Dutta, 1990), the *bipartisan set* (see, e.g., Laffond et al., 1993; Brandt, 2015; Brandt et al., 2016a), and variations thereof (see Laslier, 1997; Dutta and Laslier, 1999; Laslier, 2000; Brandt, 2011). These majoritarian voting rules are sometimes criticized for not being discriminating enough.[77]

The computer-aided approach described in this paper can be used to find more discriminating voting rules that still satisfy Kelly-participation. We thereby found a refinement of the bipartisan set that,

76 It is easily seen that the proof of Theorem 4.9 straightforwardly extends to *group-participation*, i.e., no group of voters can obtain a unanimously more preferred outcome by abstaining.

77 Indeed, Scott and Fey (2012) show that the minimal covering set selects all alternatives in almost all large tournaments.

for $m = 5$, selects only 1.43 alternatives on average, and satisfies Kelly-participation. For comparison, the bipartisan set, which is the smallest previously known majoritarian voting rule satisfying Kelly-participation, yields 2.68 alternatives on average.

4.4 CONCLUSION

Previous results indicate a conflict between strategic non-manipulability and Condorcet consistency (Moulin, 1988; Pérez, 2001; Jimeno et al., 2009; Brandt, 2015). For example, Moulin (1988) shows that no single-valued Condorcet extension satisfies participation and Brandt (2015) proves that no set-valued Condorcet extension satisfies Kelly-strategyproofness. Theorem 4.9 addresses an intermediate question and finds that—perhaps surprisingly—Moulin's impossibility does not carry over to set-valued voting rules under the assumption that voters are unaware of how a single alternative is eventually selected out of the choice set. In this case, there exist attractive efficient Condorcet extensions that satisfy Kelly-participation, even when preferences are weak.

The situation looks slightly less promising if voters act on the assumption of the existence of a tie-breaking order. Here, we have presented elaborate computer-generated impossibilities (Theorems 4.7 and 4.8), which show that the encouraging results for Kelly-participation break down once preferences are extended by the more refined extension due to Fishburn. Note, however, that this breakdown is mostly one in terms of efficiency: The voting rule selecting a Condorcet winner whenever it exists and all alternatives otherwise (*COND*) satisfies Fishburn-participation (Gärdenfors, 1976; Brandt and Brill, 2011). For the sake of completeness, we also mention that, since Condorcet consistency implies unanimity, efficiency is not completely ruled out and we are left with at least a very weak notion thereof.

These findings improve our understanding of which behavioral assumptions allow for aggregation functions that are immune to strategic abstention. An overview of the main results of this chapter and connections to other related results is given in Table 4.2.

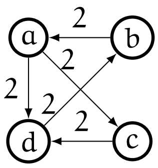
For Fishburn's extension, it would be interesting to closer examine how the picture changes if majoritarianness is no longer required. In particular, the question is whether there is a pairwise voting rule satisfying Condorcet consistency, Pareto optimality and Fishburn-participation. For any combination of three out of these four properties, we can give an affirmative answer.

- The *uncovered set* is majoritarian, i.e., also pairwise, Condorcet consistent and Pareto optimal (Fishburn, 1977). It does, however, violate Fishburn-participation (Theorem 4.7 and Theorem 4.8).

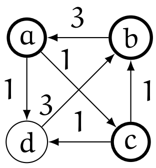
- *COND* is majoritarian, i.e., also pairwise, Condorcet consistent and satisfies Fishburn-participation (Gärdenfors, 1976; Brandt and Brill, 2011). In the absence of a Condorcet winner, it also selects potentially Pareto dominated alternatives, though.
- Borda's rule is pairwise, Pareto optimal, and satisfies Fishburn-participation (Theorem 5.10 and Moulin, 1988). On the other hand, it is well-known that it may fail to choose the Condorcet winner.
- $COND \cap PO$ satisfies Condorcet consistency, Pareto optimality and Fishburn-participation (Brandt and Brill, 2011). Still, *PO* is not pairwise.

If we however require all four properties, the situation is unclear to the best of our knowledge. While the *essential set (ES)* (Dutta and Laslier, 1999) appears to be a promising candidate at first glance because of Theorem 5.14, this unfortunately is false hope.[78]

To see this, consider the weighted majority graph G as depicted on the right.[79] Here, the essential set is $\{a, b, c, d\}$ due to a unique maximal lottery $1/5\, a + 2/5\, b + 1/5\, c + 1/5\, d$.
If a voter i with preferences $\succsim_i$: a, b, c, d leaves an electorate inducing G, we obtain G_{-i} as given on the right. This changes the essential set to $\{a, b, c\}$ as the new maximal lottery is $1/5\, a + 1/5\, b + 3/5\, c$. Since $ES(G_{-i}) \succ_i^F ES(G)$, we have a successful manipulation by strategic abstention and Fishburn-participation is violated.



G



G_{-i}

78 The essential set basically is the set of alternatives given positive probability by maximal lotteries, which are pairwise, Condorcet consistent and satisfy strong notions of efficiency and participation for probabilistic voting rules. See also Section 5.1.2 for a definition of maximal lotteries and Section 5.3.4 for a discussion of maximal lotteries and participation.

79 Even though the edge weights do not match exactly, this weighted majority graph is similar in spirit to the majority graph in Example 4.2.

		strict preferences	weak preferences
Kelly-	participation	✓ all set-monotonic voting rules that satisfy IIV (Thm. 4.9)	
	strategy-proofness	✓ all set-monotonic voting rules[a]	– no Condorcet extension[a], no pairwise & Pareto optimal voting rule ($m, n \geqslant 3$)[d]
Fishburn-	participation	– no majoritarian & Pareto optimal voting rule ($m \geqslant 5$, $n \geqslant 7$) (Thm. 4.8) ✓ few undiscriminating voting rules, e.g., *COND*[b], *TC*[b,g], and *PO*[e] (Lem. 4.4), and all scoring rules[f]	– no majoritarian & Pareto optimal voting rule ($m \geqslant 4$, $n \geqslant 6$) (Thm. 4.7)
	strategy-proofness	– no majoritarian & Pareto optimal voting rule ($m \geqslant 5$, $n \geqslant 7$)[c] ✓ few undiscriminating voting rules, e.g., *COND*[b], *TC*[b,g], and *PO*[e]	– no anonymous & Pareto optimal voting rule ($m, n \geqslant 3$)[d]

Table 4.2: Overview of results for participation and strategyproofness with respect to Kelly's and Fishburn's extension and strict/weak preferences; the symbol ✓ marks sufficient conditions or suitable rules while – marks impossibility results. [a]: Brandt (2015), [b]: Brandt and Brill (2011), [c]: Brandt and Geist (2016), [d]: Brandt et al. (2018), [e]: Feldman (1979a), [f]: Moulin (1988), [g]: Sanver and Zwicker (2012).

5 THE NO-SHOW PARADOX FOR PROBABILISTIC VOTING RULES

Probabilistic voting rules have a surprisingly long tradition that reaches back as far as ancient Greece (Headlam, 1933). Their formal analysis, however, has only begun in the relatively recent past with Zeckhauser (1969), Fishburn (1972b), and Intriligator (1973). Of late, probabilistic rules have gained increasing attention by political scientists (see, e.g., Dowlen, 2009; Stone, 2011) as well as researchers working in the field of voting theory (see, e.g., Bogomolnaia et al., 2005; Chatterji et al., 2014). In computer science, randomization is often used as a successful technique in algorithm design and the transfer to voting coincides with the rise of computational social choice (see, e.g., Conitzer and Sandholm, 2006; Procaccia, 2010; Birrell and Pass, 2011; Walsh and Xia, 2012; Service and Adams, 2012; Aziz, 2013; Aziz et al., 2014; Aziz et al., 2015; Aziz et al., 2018a). We refer to Brandt (2017) and Brandt (2018) for a more profound overview of probabilistic voting rules.

One key advantage of the probabilistic setting is higher flexibility. Where a single-valued rule only has m different choices and the number of choice sets is limited to $2^m - 1$ for set-valued rules, there is a practically endless pool of probability distributions over alternatives as long as there are at least two of them. Therefore, we are not limited to examine the classical NSP, i.e., whether a voter can benefit from abstaining from the election process. The continuity of possible choices allows us to study if participating is actually better or even strictly better than not casting one's ballot. We link these new degrees of participation to varying notions of efficiency, mostly in the context of majoritarian, pairwise, or anonymous and neutral probabilistic voting rules.

This chapter is structured as follows: we first define relevant notation unique to the probabilistic setting in Section 5.1. Next, Section 5.2 introduces two stronger notions of participation. Results are presented and discussed in Section 5.3, followed by some concluding remarks in Section 5.4.

5.1 PRELIMINARIES

We here complement Section 2 with concepts unique to the probabilistic setting. From Section 2.5 particularly recall that a voting rule satisfies *ex post* efficiency if it never puts positive probability on Pareto dominated alternatives. Σ-efficiency of a voting rule requires that it never chooses a Σ-dominated lottery.

In the following, two preference extensions for lotteries are defined in Section 5.1.1, while Section 5.1.2 explains probabilistic voting rules used in this chapter.

5.1.1 Preference Extensions

Preference extensions as introduced in Section 2.4 can be used to lift individual preferences $\succsim_i$ over single alternatives to preferences $\succsim_i^\Sigma$ over lotteries. We here define the stochastic dominance and pairwise comparison extensions used later on and point to Cho (2016) and Brandt (2017) for a more detailed discussion of further extensions.

It is a common assumption that voters are equipped with von Neumann-Morgenstern (vNM) utility functions, i.e., functions that assign a cardinal value to every alternative (von Neumann and Morgenstern, 1947). However, these utility functions are in principle unknown to a central planner, just as they usually are unclear to the individual voters themselves. In contrast, voters are typically assumed to be able to identify an ordinal ranking of alternatives, i.e., their preferences $\succsim_i$. The first extension we define tries to overcome the uncertainty regarding cardinal utility values by declaring one lottery better than another if it would be better no matter the specific vNM function.

stochastic dominance

This given, a lottery p is said to be preferred to q by voter i according to *stochastic dominance (SD)*, written $p \succsim_i^{SD} q$, if the expected utility for p is at least as large as the expected utility for q for every possible vNM utility function consistent with $\succsim_i$. Formally, we have that $p \succsim_i^{SD} q$ if for all $x \in A$, the probability that p yields an alternative at least as good as x is greater or equal than the probability that q yields an alternative at least as good as x, i.e.,

$$\sum_{y \colon y \succsim_i x} p(y) \geqslant \sum_{y \colon y \succsim_i x} q(y) \text{ for all } x \in A.$$

Note that for two lotteries p and q it might be the case that neither $p \succsim_i^{SD} q$ nor $q \succsim_i^{SD} p$, i.e., p and q may be incomparable under *SD* (see Example 5.1). The *SD* extension has been widely studied and can be considered the most well-known preference extension for lotteries (see, e.g., Gibbard, 1977; Postlewaite and Schmeidler, 1986; Bogomolnaia and Moulin, 2001).

Next, we can define a different extension without even reasoning about underlying utilities. Assuming we are unaware of the existence of any such utility function and do not know about intensities of preferences, it is still reasonable to assume that one lottery is preferred to another if it has a better chance to yield a more preferred alternative.[80]

The *pairwise comparison (PC)* extension was introduced recently by Aziz et al. (2015) and prescribes that lottery p is preferred to lottery q if the probability that p yields an alternative preferred to what q gives is greater or equal than the probability that q yields an alternative preferred to what is returned by p. Formally, $p \succsim_i^{PC} q$ if

pairwise comparison

$$\sum_{x,y\,:\,x \succsim_i y} p(x)q(y) \geqslant \sum_{x,y\,:\,x \succsim_i y} p(y)q(x).$$

With $\phi_i \in \{-1, 0, 1\}^{A \times A}$,

$$(\phi_i)_{xy} = \begin{cases} 1 & \text{if } x \succ_i y, \\ 0 & \text{if } x \sim_i y, \text{ and} \\ -1 & \text{if } y \succ_i x, \end{cases}$$

this is equivalent to defining $p \succsim_i^{PC} q$ if $p^T \phi_i\, q \geqslant 0$.[81] *PC* is a strengthening of *SD* in the sense that $p \succsim_i^{SD} q$ implies $p \succsim_i^{PC}$, i.e., whenever a lottery p is *SD*-preferred over q, p is also *PC*-preferred over q (Fishburn, 1984a). In contrast to *SD*, the *PC* extension is complete. It may, however, return intransitive preferences over lotteries, even when preferences over alternatives are transitive (see also Blavatskyy, 2006; Aziz et al., 2015; Aziz et al., 2018a; Brandl and Brandt, 2018).

Example 5.1

Consider $\succsim_i$: a, b, c and three lotteries $p = 1/2\, a + 1/2\, c$, $q = b$, and $r = 1/2\, b + 1/2\, c$. We have that $p \succ_i^{SD} r$ as well as $q \succ_i^{SD} r$ but p and q are *SD*-incomparable. For the *PC* extension, we easily see that $p \sim_i^{PC} q$.

We will use *SD* and *PC* to obtain varying degrees of efficiency. Since *PC* is a strengthening of *SD*, we have that *PC*-efficiency is stronger than *SD*-efficiency, which is stronger than *ex post* efficiency. The relationship between different concepts of efficiency is also depicted in Figure 5.2 on page 100, which summarizes many results presented later on.

5.1.2 Probabilistic Voting Rules

Multiple probabilistic voting rules are used in Section 5.3, mostly to show the compatibility of different degrees of efficiency and participation. In order to present our results more smoothly, formal definitions accompanied by a short example are put together here.

Random serial dictatorship (RSD) is the canonical generalization of

random serial dictatorship

80 While this definition theoretically allows for the underlying individual preferences to be cyclic, we do assume transitive preferences throughout.

81 It is clear from this formulation that the *PC* extension is a special case of more general *skew-symmetric bilinear (SSB)* utility functions that also allow for intensities of preferences (Fishburn, 1982b; Fishburn, 1984c; Fishburn, 1988).

random dictatorship (Gibbard, 1977) to weak preferences. Intuitively, *RSD* chooses an ordering of voters uniformly at random and allows each voter to further narrow down the set of alternatives chosen by his predecessors based on his preferences. For a later proof, it will turn out advantageous to use a rather uncommon recursive definition of *RSD*:

$$RSD(\succsim, X) = \begin{cases} \sum_{x \in X} 1/|X|\, x & \text{if } \succsim = \emptyset \text{, and} \\ \sum_{i=1}^{|\succsim|} 1/|\succsim|\, RSD(\succsim_{-i}, \max_{\succsim_i}(X)) & \text{otherwise.} \end{cases}$$

$RSD(\succsim)$ is defined as $RSD(\succsim, A)$. For an equivalent definition employing permutations, we refer to, e.g., Aziz et al. (2018a) or Section 6.2.1 in the context of random assignment. *RSD* has attracted considerable attention from researchers working on probabilistic voting, who have argued that *RSD* fares well with respect to immunity against manipulation by misrepresentation, but only satisfies comparably weak notions of efficiency (Aziz et al., 2018a).[82] Computing *RSD* has been shown to be #P-complete (Aziz et al., 2013b).

maximal lotteries

Next, we define *maximal lotteries (ML)* as first considered by Kreweras (1965) and rediscovered and thoroughly studied by Fishburn (1984b).[83] Maximal lotteries can be computed as mixed maximin strategies, or equivalently Nash equilibria, of the two-player zero-sum game given by the majority margins $g_{xy}(\succsim)$.[84, 85]

Note that *ML* does not necessarily return a unique lottery but possibly infinitely many. However, Brandl et al. (2016) show that the set of profiles admitting a unique maximal lottery is open and dense. The set of profiles admitting multiple maximal lotteries is therefore nowhere dense and thus negligible. Out of all probabilistic voting rules defined here, *ML* is the only Condorcet consistent one, i.e., it uniquely chooses a possible Condorcet winner with probability one.

BOR

Lastly, *BOR* is an artificial voting rule used later mostly for the sake of visualization. The *probabilistic Borda's rule (BOR)* chooses the

82 In fact, Brandl et al. (2018) show that no probabilistic voting rule can satisfy reasonably strong versions of strategyproofness and efficiency simultaneously.

83 Over the years, maximal lotteries or variants thereof were independently rediscovered by researchers originating from different fields including economics, mathematics, political science, and computer science (see, e.g., Laffond et al., 1993; Felsenthal and Machover, 1992; Fisher and Ryan, 1995; Rivest and Shen, 2010). Maximal lotteries were recently characterized axiomatically by Brandl et al. (2016) and are also considered in private good settings such as probabilistic matching markets and random assignment (see, e.g., Kavitha et al., 2011; Aziz et al., 2013c; Brandt et al., 2017b, and Section 6.2.4).

84 Maximin strategies and zero-sum games go back to von Neumann (1928). Though conceptually simple, zero-sum games have received considerable attention in the literature (see, e.g., von Neumann and Morgenstern, 1944; Kuhn and Tucker, 1950).

85 We interpret the alternatives as both players' actions and set the row player's payoff when playing x against y to be g_{xy} with the column player's payoff $-g_{xy} = g_{yx}$. With $M \in \mathbb{Z}^{A \times A}$, $M_{xy} = g_{xy}$, p is a maximin strategy if $p^T M\, q \geqslant 0$ for all $q \in \Delta(A)$. Mixed maximin strategies can be computed efficiently and having a zero-sum game they equal Nash equilibria (Nash, 1950; Nash, 1951).

uniform lottery over all Borda winners,

$$BOR(\succsim) = \frac{1}{|\arg\max_{y\in A} s_y(\succsim)|} \sum_{x\in\arg\max_{y\in A} s_y(\succsim)} x.$$

Note that slightly different from the definition in Section 3.1.1, Borda scores are commonly defined as

$$s_x(\succsim) = \sum_{i\in N} |\{y \in A \colon x \succ_i y\}| + 1/2\, |\{y \in A \setminus \{x\} \colon x \sim_i y\}|$$

for possible indifferences.

Example 5.2

Consider the preference profile $\succsim$ with voters $N = \{1,\dots,5\}$ and alternatives $A = \{a, b, c, d\}$ as given below.

We see that $RSD(\succsim) = 2/5\, a + 2/5\, b + 1/5\, c$. The unique maximal lottery can be computed to be $ML(\succsim) = 3/5\, a + 1/5\, b + 1/5\, c$, and we moreover determine $BOR(\succsim) = b$ as alternative b is the Borda winner.

1,2	3,4	5
a	b	c
b	d	a
d	c	b
c	a	d

5.2 STRONGER NOTIONS OF PARTICIPATION

Recall from Section 2.4 that a voting rule f is manipulable by strategic abstention for a preference extension Σ if there are $A, N \in \mathcal{F}(\mathbb{N})$, and $\succsim \in \succsim(A)^N$ such that $f(\succsim_{-i}) \succ_i^\Sigma f(\succsim)$ for some $i \in N$. f satisfies Σ-participation if it is not manipulable by strategic abstention. In game-theoretic terms, we can thus interpret participating as strictly undominated (by not voting).

Note, however, that for incomplete extensions Σ, Σ-participation often relies on the fact that two lotteries p and q are Σ-incomparable. For instance *SD*-participation only prescribes that manipulation never is possible for *all* vNM utility functions in line with the voters' preferences. It does not necessarily contradict a potential manipulation for *some* specific vNM function. This problem is addressed by *strong participation*.

strong participation

A voting rule f satisfies *strong Σ-participation* if $f(\succsim) \succsim_i^\Sigma f(\succsim_{-i})$ for all $A, N \in \mathcal{F}(\mathbb{N})$, $\succsim \in \succsim(A)^N$, and $i \in N$. Hence, strong participation is satisfied if the outcome obtained by voting always is weakly preferred to what would have resulted of abstaining. In game-theoretic terms, we correspondingly have that voting is a very weakly dominant strategy.

strong group-participation

We additionally define a group-based notion. A voting rule f is said to satisfy *strong Σ-group-participation* if for all $A, N \in \mathcal{F}(\mathbb{N})$, $\succsim \in \succsim(A)^N$, and $S \subseteq N$, $f(\succsim) \succsim_i^\Sigma f(\succsim_{-S})$ for all $i \in S$. By definition, strong group-participation implies strong participation.

The framework of probabilistic voting rules allows for the definition of a third, even stronger notion of participation.[86] It demands that a voter should always strictly prefer participating over abstaining, i.e., voting is a strictly dominant strategy, whenever an improvement is possible. This is of special interest for settings where voting is associated with a (possibly small) cost.[87] Formally, a voting rule f satisfies *very strong* Σ*-participation* if for all $A, N \in \mathcal{F}(\mathbb{N})$, $\succsim \in \succsim(A)^N$, and $i \in N$

very strong participation

$$\begin{cases} f(\succsim) \succ_i^\Sigma f(\succsim_{-i}) & \text{if } \operatorname{supp}(f(\succsim_{-i})) \not\subseteq \max_{\succsim_i}(A) \text{, and} \\ f(\succsim) \succsim_i^\Sigma f(\succsim_{-i}) & \text{otherwise.} \end{cases}$$

An equivalent formulation requires

$$\begin{cases} f(\succsim) \succ_i^\Sigma f(\succsim_{-i}) & \text{if } \exists q \in \Delta(A) \text{ such that } q \succ_i^\Sigma f(\succsim_{-i}) \text{, and} \\ f(\succsim) \succsim_i^\Sigma f(\succsim_{-i}) & \text{otherwise.} \end{cases}$$

While it is theoretically possible to also define very strong group-participation, we remark that this notion would be so strong it cannot be satisfied by any voting rule f. To see this, consider a group of two not completely indifferent abstaining voters i, j with reversed preferences. As an improvement has to be possible for at least one voter, say i, we require $f(\succsim) \succ_i^\Sigma f(\succsim_{-i,j})$, which in turn gives $f(\succsim_{-i,j}) \succ_j^\Sigma f(\succsim)$, a violation of very strong group-participation.

All three notions of participation form a hierarchy: very strong participation implies strong participation and strong participation implies participation. These relationships are also depicted in Figure 5.1. With respect to *PC* and *SD*, we have that *PC*-participation and strong *PC*-participation coincide as *PC* is complete. Moreover, we directly obtain that strong *SD*-participation implies *PC*-participation, which in turn implies *SD*-participation. These implications for different preference extensions are also visualized in Figure 5.2 on page 100.

Just as it is the case for participation, the probabilistic setting also allows for a meaningful definition of stronger notions of strategyproofness. Given an extension Σ, we say a voting rule f satisfies *strong* Σ*-strategyproofness* if $f(\succsim) \succsim_i^\Sigma f(\succsim')$ for all $A, N \in \mathcal{F}(\mathbb{N})$, $i \in N$, and $\succsim, \succsim' \in \succsim(A)^N$ with $\succsim_j = \succsim_j'$ for all $j \in N_{-i}$.[88] Very strong strategyproofness would demand that for every preference profile any voter strictly prefers telling his true preferences over submitting a misrepresentation whenever strict preference is possible. Since misrepresentation also includes, e.g., a swap of the two least preferred alternatives, no reasonable voting rule satisfies very strong strategyproofness and we forgo a formal definition thereof.

strong strategyproofness

86 This strong version of participation can of course also be defined for single-valued and set-valued rules. However, it is of questionable usefulness as it generally is prohibitive on its own in these settings.

87 Note that this cost is not necessarily monetary but may be interpreted as the effort required to determine one's preferences.

88 In the literature, the terms *weak strategyproofness* and *strategyproofness* are sometimes also used for what we call strategyproofness and strong strategyproofness.

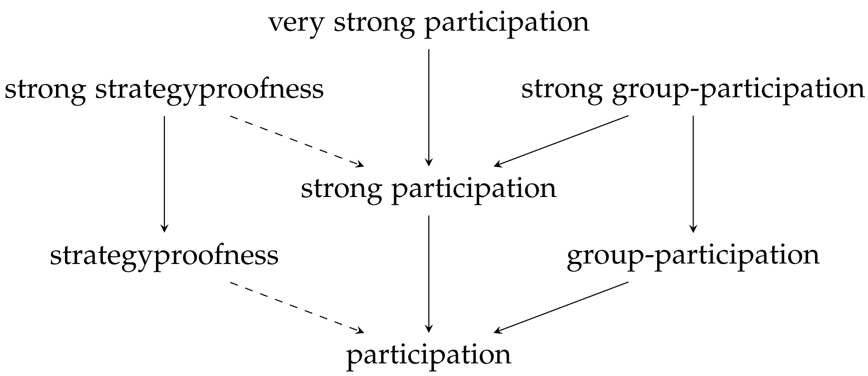


Figure 5.1: Implications between the different notions of participation and two variants of strategyproofness; a solid arrow from one notion to another signifies the former implies the latter; strategyproofness implies participation if individual preferences may contain indifferences and the presence of completely indifferent voters does not change the outcome.

For groups of voters, we define a voting rule f to satisfy *strong Σ-group-strategyproofness* if $f(\succsim) \succsim_i^\Sigma f(\succsim')$ for all $A, N \in \mathcal{F}(\mathbb{N})$, $S \subseteq N$, $i \in S$, and $\succsim, \succsim' \in \succsim(A)^N$ with $\succsim_j = \succsim'_j$ for all $j \in N \setminus S$.

strong group-strategyproofness

5.3 RESULTS AND DISCUSSION

We first direct focus to very strong *SD*-participation in Section 5.3.1. Next, we analyze strong *SD*-group-participation and strong *SD*-participation in Section 5.3.2, followed by *SD*-group-participation and *SD*-participation in Section 5.3.3. In each of these sections, we start with majoritarian voting rules and continue on to pairwise as well as anonymous and neutral ones. Lastly, we study the NSP for maximal lotteries and show that they satisfy *PC*-participation but fail to satisfy strong *SD*-participation in Section 5.3.4.

5.3.1 Very Strong *SD*-Participation

Very strong *SD*-participation is the strongest variant of participation we consider. Since every potential voter not perfectly content anyway has to be able to influence the outcome in his favor by joining the electorate, it is easy to see that very strong participation is impossible to be satisfied by a majoritarian voting rule.

THEOREM 5.3
There is no majoritarian probabilistic voting rule satisfying very strong *SD*-participation even when preferences are strict.

Proof. Let $N = \{1,2,3\}$, $A = \{a,b\}$, and $\succsim = (\succsim_1, \succsim_2, \succsim_3)$ such that $\succsim_1, \succsim_2 \colon a, b$, $\succsim_3 \colon b, a$. Note that $\succsim^M = \succsim^M_{-3} = \succsim^M_1$. Therefore, for every majoritarian probabilistic voting rule f, it holds that

$$f(\succsim) = f(\succsim_{-3}) = f(\succsim_1).$$

Assume for contradiction that f satisfies very strong *SD*-participation, i.e., we have

$$f(\succsim_{-3}) = f(\succsim) = b$$

since this is the only way to satisfy $f(\succsim) \succsim^{SD}_3 f(\succsim_{-3})$. This however contradicts $f(\succsim_{-3}) \succ^{SD}_2 f(\succsim_1)$, which would be required due to b not being voter 2's first preference. □

Whether this impossibility carries over to pairwise voting rules or there is a pairwise voting rule satisfying very strong *SD*-participation is currently open.[89] However, it is possible to show that if such a rule exists, it has to always give positive probability to all alternatives and therefore does not satisfy even the weakest notions of efficiency. This is no coincidence, as the following theorem shows that if only unanimity were required, there is no pairwise rule satisfying very strong *SD*-participation.

Theorem 5.4

There is no pairwise and unanimous probabilistic voting rule satisfying very strong *SD*-participation even when preferences are strict.

Proof. Let $N = \{1,2,3\}$, $A = \{a,b\}$, and $\succsim = (\succsim_1, \succsim_2, \succsim_3)$ be a preference profile such that $\succsim_1, \succsim_2 \colon a, b$, $\succsim_3 \colon b, a$. Note that any pairwise and unanimous probabilistic voting rule f has to choose

$$f(\succsim_{-3}) = f(\succsim_1) = a.$$

By $g_{ab}(\succsim_1) = g_{ab}(\succsim)$ we deduce $f(\succsim_1) = f(\succsim)$. Put together, it can never be the case that $f(\succsim) \succ^{SD}_3 f(\succsim_{-3})$. □

When trying to link very strong *SD*-participation to Condorcet consistency, it is easy to see that no Condorcet extension satisfies the strongest notion of participation. This directly follows from the observation that as long as the Condorcet winner beats all other alternatives by a margin of at least two, no single voter is able to influence the outcome, even when the Condorcet winner is not his most preferred alternative.

Within the unrestricted domain of probabilistic voting rules very strong *SD*-participation can be satisfied together with certain notions of efficiency. While compatibility with *SD*-efficiency is still open, for instance *RSD* at least satisfies very strong *SD*-participation and *ex post* efficiency.

89 The voting rule *EXP* previously claimed to simultaneously satisfy both properties in [1] does so only for up to three alternatives. For four or more alternatives it even violates strong *SD*-participation.

THEOREM 5.5

RSD satisfies anonymity, neutrality, *ex post* efficiency and very strong *SD*-participation.

Proof. We only prove very strong *SD*-participation here and refer to Aziz et al. (2018a) for *ex post* efficiency. Anonymity and neutrality are clear from the definition of *RSD*. Let $N \in \mathcal{F}(\mathbb{N})$, $\succsim \in \succsim(A)^N$, and $i \in N$. A first step for showing very strong *SD*-participation is to prove that $RSD(\succsim) \succsim_i^{SD} RSD(\succsim_{-i})$. It is already known that *RSD* satisfies *strong SD-strategyproofness*, i.e., $RSD(\succsim) \succsim_i^{SD} RSD(\succsim')$ for every preference profile $\succsim' \in \succsim(A)^N$ where $\succsim_j' = \succsim_j$ for all $j \neq i$ (see, e.g., Aziz et al., 2018a). If voter i is completely indifferent between all alternatives in A, it holds that $RSD(\succsim) = RSD(\succsim_{-i})$. We obtain as a direct consequence that *RSD* satisfies strong *SD*-participation.

In order to see that the even stronger notion applies, assume that $\succsim_{-i}$ allows for a strict improvement for i, i.e., there is $p \in \Delta(A)$ such that $p \succ_i^{SD} RSD(\succsim_{-i})$. Thus, at least some probability is given to alternatives not ranked first by voter i when he abstains,

$$RSD(\succsim_{-i})(\max_{\succsim_i}(A)) < 1.$$

We have

$$\begin{aligned}
&RSD(\succsim)(\max_{\succsim_i}(A)) \\
&= RSD(\succsim, A)(\max_{\succsim_i}(A)) \\
&= \sum_{j \in N} {}^1\!/_n\, RSD(\succsim_{-j}, \max_{\succsim_j}(A))(\max_{\succsim_i}(A)) \\
&= {}^1\!/_n + {}^1\!/_n \sum_{j \in N_{-i}} \underbrace{RSD(\succsim_{-j}, \max_{\succsim_j}(A))(\max_{\succsim_i}(A))}_{\geqslant RSD(\succsim_{-\{i,j\}}, \max_{\succsim_j}(A))(\max_{\succsim_i}(A))} \\
&\geqslant {}^1\!/_n + {}^{n-1}\!/_n\, \underbrace{RSD(\succsim_{-i}, A)(\max_{\succsim_i}(A))}_{<1 \text{ by assumption}} \\
&> RSD(\succsim_{-i}, A)(\max_{\succsim_i}(A)) \\
&= RSD(\succsim_{-i})(\max_{\succsim_i}(A)).
\end{aligned}$$

This shows that the total probability given to alternatives ranked first by voter i strictly increases if he chooses to participate, compared to a possible abstention. We conclude that $RSD(\succsim) \succ_i^{SD} RSD(\succsim_{-i})$ for all $i \in N$, which means *RSD* satisfies very strong *SD*-participation. □

It is noteworthy that *RSD* is by far not the only voting rule satisfying very strong *SD*-participation and *ex post* efficiency. Further voting rules can be created at will using the convex combination of *RSD* and other rules. As long as those additional voting rules satisfy strong *SD*-participation, a mixture of both inherits the very strong notion.

THEOREM 5.6

Let f_1, f_2 be two *ex post* efficient probabilistic voting rules such that f_1 satisfies very strong *SD*-participation and f_2 satisfies strong *SD*-participation. Moreover, let $\lambda \in (0,1)$. Then, a probabilistic voting rule f defined as $f = \lambda f_1 + (1-\lambda) f_2$ satisfies *ex post* efficiency and very strong *SD*-participation.

Proof. Let f_1, f_2, f, and λ be as above. First, note that if both f_1, f_2 put probability zero on all Pareto dominated alternatives $x \in A$, so does f. In addition, we have for all $N \in \mathcal{F}(\mathbb{N})$, $\succsim \in \succsim(A)^N$, $y \in A$, $i \in N$, and $k \in \{1,2\}$,

$$\sum_{x \in A : x \succsim_i y} f_k(\succsim)(x) \geqslant \sum_{x \in A : x \succsim_i y} f_k(\succsim_{-i})(x)$$

and it additionally holds for all $i \in N$ for whom an improvement is possible that there is some $y' \in A$ such that

$$\sum_{x \in A : x \succsim_i y'} f_1(\succsim)(x) > \sum_{x \in A : x \succsim_i y'} f_1(\succsim_{-i})(x).$$

We directly deduce that for all $y \in A$, $i \in N$,

$$\sum_{x \in A : x \succsim_i y} f(\succsim)(x) \geqslant \sum_{x \in A : x \succsim_i y} f(\succsim_{-i})(x)$$

and for all voters i, for whom an improvement is possible, and the corresponding y' also

$$\sum_{x \in A : x \succsim_i y'} f(\succsim)(x) > \sum_{x \in A : x \succsim_i y'} f(\succsim_{-i})(x).$$

Therefore, $f(\succsim) \succ_i^{SD} f(\succsim_{-i})$ or $f(\succsim) \succsim_i^{SD} f(\succsim_{-i})$ for all $i \in N$, depending on whether a strict improvement is possible or not. □

As a consequence, every proper convex combination of *RSD* and *BOR* satisfies very strong *SD*-participation and *ex post* efficiency. Since *BOR* will be shown to be *SD*-efficient in Theorem 5.10, shifting more and more weight away from *RSD* results in voting rules still satisfying very strong *SD*-participation and arbitrarily small violations of *SD*-efficiency. Note, however, that the number of violations is not affected by this shifting of weights and smaller violations of efficiency come at the price of smaller incentives to participate.

Recall from Section 5.2 that very strong *SD*-group-participation is a notion too strong to be satisfied by any probabilistic voting rule. Sticking to *RSD* for the moment, we conclude this section by providing an example that shows that *RSD* can be manipulated by abstaining groups of voters, i.e., does not even satisfy *SD*-group-participation.

Example 5.7

Consider $N = \{1,2,3,4\}$, $A = \{a,b,c,d\}$, and $\succsim$ as given below. Here,

$$RSD(\succsim) = 1/3\, a + 1/3\, b + 1/6\, c + 1/6\, d$$

and if voters 1 and 2 abstain, we obtain

$$RSD(\succsim_{-1,2}) = {}^1\!/_2\, a + {}^1\!/_2\, b.$$

For both voters $i \in \{1,2\}$, it holds that $RSD(\succsim_{-1,2}) \succ_i^{SD} RSD(\succsim)$, i.e., for both voters abstaining results in a strictly preferred lottery. This example furthermore illustrates a violation of *SD*-efficiency.

1	2	3	4
a, d	b, c	a, c	b, d
b	a	b	a
c	d	d	c

5.3.2 Strong *SD*-Participation

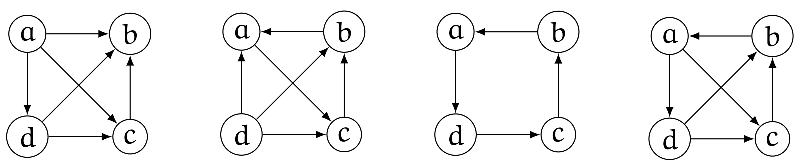
Since very strong participation implies strong participation, i.e., voting rules satisfying very strong *SD*-participation also satisfy strong *SD*-participation, positive results from the previous section carry over. Similar to Section 5.3.1, we begin our study by considering majoritarian voting rules. First, note that while there is no majoritarian probabilistic voting rule satisfying very strong *SD*-participation (Theorem 5.3), there exist majoritarian rules satisfying strong *SD*-participation. The arguably simplest example is a constant rule that always chooses the uniform distribution over all alternatives. However, this rule does, of course, not satisfy any degree of efficiency. We find this to be unsurprising as indeed only requiring unanimity leads to an impossibility.

Theorem 5.8

When $m \geqslant 4$, there is no unanimous majoritarian probabilistic voting rule satisfying strong *SD*-participation, even when preferences are required to be strict.

Proof. Let $A = \{a, b, c, d\}$. For contradiction, suppose f is a probabilistic voting rule satisfying majoritarianness, unanimity, and strong *SD*-participation. Throughout this proof, we often argue about majority graphs G without necessarily connecting them to a specific underlying preference profile. Hence, we often also write $f(G)$ instead of $f(\succsim) = f(\succsim^M)$ if $\succsim^M = G$.

First, look at majority graph G_α as depicted below and note that G_α is transitive and could thus be induced by only one voter with preferences $\succsim_i$: a, d, c, b. By unanimity, we obtain $f(G_\alpha) = a$.



G_α G_β G_γ G

Now, let two new voters α, α' with identical preference relation $\succsim_\alpha, \succsim_{\alpha'}$: b, a, c, d join an electorate with preference profile $\succsim$ inducing G_α. For the following argument, assume $\succsim$ to be of the form that $|g_{xy}(\succsim)| \geqslant 3$ for all $(x,y) \in A \times A \setminus \{(a,b),(b,a)\}$, $x \neq y$, and $g_{ab}(\succsim) = 1$. The additional voters alter the majority graph in a way such that $\succsim^M_{+\alpha,\alpha'} = G$. By strong *SD*-participation we have that $f(G) \succsim^{SD}_{\alpha} f(G_\alpha)$. Consequently, neither c nor d may receive positive probability in G,

$$f(G)(c) = f(G)(d) = 0.$$

Next, consider majority graph G_β that is induced by, e.g., $(\succsim_{j_1}, \succsim_{j_2}, \succsim_{j_3})$,

$$\succsim_{j_1}: d,a,c,b, \qquad \succsim_{j_2}: d,b,a,c, \qquad \succsim_{j_3}: d,c,b,a.$$

Using unanimity we obtain $f(G_\beta) = d$. Analogously to before, let two voters β, β' with preferences $\succsim_\beta, \succsim_{\beta'}: a,d,b,c$ join a different electorate with preference profile $\succsim'$ inducing G_β. We suppose $\succsim'$ to be of the form that $|g_{xy}(\succsim')| \geqslant 3$ for all $(x,y) \in A \times A \setminus \{(a,d),(d,a)\}$, $x \neq y$, and $g_{da}(\succsim') = 1$. This changes the majority graph of $\succsim'$ such that $\succsim'^M_{+\beta,\beta'} = G$. Once more, voter β has to weakly prefer $f(G)$ to $f(G_\beta)$ due to strong *SD*-participation, $f(G) \succsim^{SD}_{\beta} f(G_\beta)$. Consequently,

$$f(G)(b) = f(G)(c) = 0$$

and together with $f(G)(d) = 0$ we deduce $f(G) = a$.

Due to neutrality, we know that

$$f(G_\gamma) = 1/4\, a + 1/4\, b + 1/4\, c + 1/4\, d.$$

We now add a single voter γ, $\succsim_\gamma: d,b,a,c$, to an electorate with preference profile $\succsim''$ inducing G_γ, where

$$g_{ad}(\succsim'') = g_{dc}(\succsim'') = g_{cb}(\succsim'') = g_{ba}(\succsim'') \geqslant 2.$$

The majority graph is thus altered such that $\succsim''^M_{+\gamma} = G$. Note that for γ, $f(G)$ and $f(G_\gamma)$ are incomparable according to the *SD*-extension and in particular $f(G) \not\succsim^{SD}_{\gamma} f(G_\gamma)$. This contradicts that f satisfies strong *SD*-participation and concludes the proof. □

Note that Condorcet consistency implies unanimity. Hence, Theorem 5.8 also answers whether there is a majoritarian Condorcet extension satisfying strong *SD*-participation.

Corollary 5.9

When $m \geqslant 4$, there is no majoritarian probabilistic voting rule satisfying Condorcet consistency and strong *SD*-participation, even when preferences are required to be strict.

We now turn to pairwise voting rules. In contrast to very strong *SD*-participation which is mutually exclusive with even the weak notion of unanimity (Theorem 5.4), there exist pairwise voting rules satisfying strong *SD*-participation and *SD*-efficiency. A possible representative is *BOR* that randomizes over all Borda winners.

Theorem 5.10
BOR is pairwise and satisfies *SD*-efficiency as well as strong *SD*-participation.

Proof. We begin by showing that *BOR* is pairwise. Therefore, observe that

$$|\{i \in N\colon x \succ_i y\}| + 1/2\, |\{i \in N\colon x \sim_i y\}| = 1/2\, n + 1/2\, g_{xy}.$$

We can thus rewrite the definition of $s_x(\succsim)$ as

$$\begin{aligned} s_x(\succsim) &= \sum_{y \in A \setminus \{x\}} |\{i \in N\colon x \succ_i y\}| + 1/2\, |\{i \in N\colon x \sim_i y\}| \\ &= \sum_{y \in A \setminus \{x\}} 1/2\, n + 1/2\, g_{xy} \\ &= 1/2\, n(m-1) + 1/2 \sum_{y \in A \setminus \{x\}} g_{xy}. \end{aligned}$$

Hence, the order of the $s_x(\succsim)$ and thus also the outcome of *BOR* only depends on g_{xy} and consequently *BOR* is pairwise.

In order to see that *BOR* satisfies strong *SD*-participation, consider the following: if by joining an electorate N_{-i}, some voter i can force an alternative a into the set of Borda winners without dropping any other, then a has to be ranked above the other Borda winners in $\succsim_i$, i.e., $BOR(\succsim) \succsim_i^{SD} BOR(\succsim_{-i})$. On the other hand, if i can force an alternative b out of the set of Borda winners by participating, b has to be ranked below the other Borda winners in $\succsim_i$ giving once more $BOR(\succsim) \succsim_i^{SD} BOR(\succsim_{-i})$. A combination of both arguments yields that even for some alternatives being added to the set of Borda winners as well as others being left out when i participates in the election, we always get $BOR(\succsim) \succsim_i^{SD} BOR(\succsim_{-i})$.

Finally, suppose *BOR* does not satisfy *SD*-efficiency, i.e., there exists some electorate $N \in \mathcal{F}(\mathbb{N})$, preference profile $\succsim \in \succsim(A)^N$, and lottery $p \in \Delta(A)$ such that $p \succsim_i^{SD} BOR(\succsim)$ for all $i \in N$. First, note that therefore

$$\sum_{x \in \text{supp}(p)} p(x) s_x(\succsim) \geqslant \sum_{x \in \text{supp}(BOR(\succsim))} BOR(\succsim)(x) s_x(\succsim).$$

Stated differently and taking into account that one voter has to strictly *SD*-prefer p, the (weighted) average Borda score of the alternatives in $\text{supp}(p)$ would have to be greater than the one of the alternatives in $\text{supp}(BOR(\succsim))$. This contradicts the fact that *BOR* chooses the alternatives with maximal Borda score and concludes the proof of Theorem 5.10. □

Note that following very similar arguments, it is easy to show that randomizing uniformly over the winners of any scoring rule with strictly monotonically decreasing scoring vector satisfies strong *SD*-participation and *SD*-efficiency. However, such rules in general do not have to be pairwise.

We now turn to groups of voters. While *BOR* appears to be a good choice for a first glance, we find that it only satisfies *SD*-group-participation. This can be seen by using arguments resembling those employed in the proof of Theorem 5.10. If an abstaining group of voters S causes some alternative x to drop out of the set of Borda winners, x has to receive a higher Borda score and thus be more preferred on average by members of S when compared to any alternative remaining or becoming a Borda winner. In particular, there exists at least one voter $i \in S$ for whom $f(\succsim_{-S}) \not\succsim_i^{SD} f(\succsim)$. Equivalently, if by abstaining S can force an alternative x into the set of Borda winners, x has to be less preferred on average compared to any alternative becoming or remaining a Borda winner. Consequently, there is at least one $i \in S$ for whom $f(\succsim_{-S}) \not\succsim_i^{SD} f(\succsim)$. In total, this contradicts a successful manipulation by $S \subseteq N$. The fact that *BOR* does not satisfy strong *SD*-group-participation follows from Theorem 5.11 below.

In absence of requirements for efficiency, strong *SD*-group-participation can easily be satisfied by, e.g., a voting rule always selecting the uniform distribution over all alternatives. If some notion of efficiency would be demanded, though, no anonymous and neutral voting rule can meet these conditions as the following theorem shows.

THEOREM 5.11

> There is no anonymous and neutral probabilistic voting rule satisfying unanimity and strong *SD*-group-participation.

Proof. Let $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$ with $\succsim_1: a, b, c$, $\succsim_2: b, c, a$, and $\succsim_3: c, a, b$. Any anonymous and neutral voting rule f has to choose $f(\succsim) = 1/3\, a + 1/3\, b + 1/3\, c$. Unanimity of f additionally yields $f(\succsim_1) = f(\succsim_{-2,3}) = a$. However, $f(\succsim) \not\succsim_3^{SD} f(\succsim_{-2,3})$ and thus f does not satisfy strong *SD*-group-participation. □

5.3.3 *SD*-Participation

In contrast to before, only requiring *SD*-participation allows for majoritarian and mildly efficient probabilistic voting rules. The voting rule that returns the Condorcet winner whenever one exists and the uniform lottery over all alternatives otherwise is majoritarian and unanimous. Also, neither a single nor multiple voters are able to either make their most preferred alternative Condorcet winner nor to prevent their least preferred alternative from being Condorcet winner by leaving the electorate. Thus, both *SD*-participation and *SD*-group-participation are satisfied.

We find that it is not possible to further strengthen the degree of efficiency to *ex post* efficiency and at the same time preserve *SD*-

participation and majoritarianness. For the following proof of Theorem 5.13, recall Lemma 4.5 linking Pareto optimality and the uncovered set for set-valued voting rules satisfying Fishburn-particpation. A slightly adapted (and weaker) version for probabilistic rules helps to simplify the proof of this section's main theorem. Note that we also employ notation defined in the context of Lemma 4.5, in particular the covering relation and the uncovered set.

LEMMA 5.12

Let f be a majoritarian and *ex post* efficient probabilistic voting rule that satisfies *SD*-participation. Let $\succsim$ and $\succsim'$ be preference profiles on A such that $\succsim^M|_{UC(\succsim^M)} = \succsim'^M|_{UC(\succsim'^M)}$. Then

$$f(\succsim) \subseteq UC(\succsim^M)$$

and if $\succsim^M$ and $\succsim'^M$ additionally are antisymmetric, we have that

$$f(\succsim) = f(\succsim').$$

With Lemma 5.12 at hand, we now show the incompatibility of *ex post* efficiency and *SD*-participation for majoritarian rules.

THEOREM 5.13

When $m \geqslant 4$, there is no majoritarian probabilistic voting rule satisfying *ex post* efficiency and *SD*-participation.

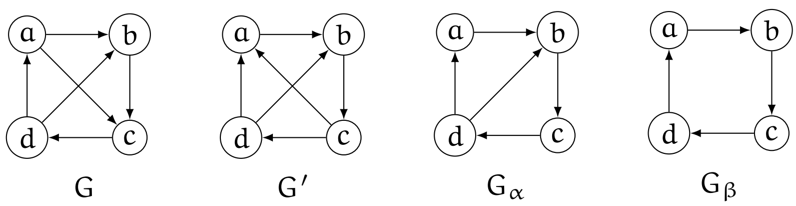
Proof. The proof follows ideas reminiscent of the proof of Theorem 5.8 in the sense that we assume for contradiction a probabilistic voting rule f with the desired properties exists. Arguing about multiple majority graphs, we are able to gradually eliminate possible choices of f for a certain majority graph which culminates in f not being able to select any alternative. Similar to before, we regularly write $f(G)$ instead of $f(\succsim^M)$ if the majority graph of $\succsim$ equals G, $\succsim^M = G$.

Start the proof by letting $A = \{a, b, c, d\}$ and f a majoritarian probabilistic voting rule satisfying *ex post* efficiency and *SD*-participation. First, note that by Lemma 5.12, alternatives not in $UC(\succsim^M)$ receive probability zero in $f(\succsim)$. In addition, we have that only majority comparisons within $UC(\succsim)$ effect the outcome of f.

We begin by examining majority graphs of the structure G as depicted below. Since alternative b is covered by a, we deduce $f(G)(b) = 0$. Furthermore, due to neutrality, we get

$$f(G) = 1/3\, a + 1/3\, c + 1/3\, d.$$

Now direct attention to G_α.



Let $\succsim$ be a preference profile inducing G_α and additionally assume $|g_{xy}(\succsim)| \geqslant 2$ for all $(x, y) \in A \times A \setminus \{(a, c), (c, a)\}$, $x \neq y$. The uncovered set consists of all alternatives and there do not exist any symmetries, so we cannot make immediate statements concerning $f(G_\alpha)$. Yet, using two auxiliary majority graphs, we show that $f(G_\alpha)$ can be determined exactly.

If a voter α with preferences $c \succ_\alpha a$ leaves the electorate, the majority graph of $\succsim_{-\alpha}$ equals G, regardless of how b and d are ranked. Since f satisfies *SD*-participation, it must hold that $f(\succsim_{-\alpha}) \not\succ^{SD}_\alpha f(\succsim)$. Assume for contradiction $f(G_\alpha)(c) < 1/3$. In this case, the flexibility of b and d allows to fix individual preferences $\succsim_\alpha$ in a way such that $f(\succsim_{-\alpha}) \succ^{SD}_\alpha f(\succsim)$. Consequently, $f(G_\alpha)(c) \geqslant 1/3$.

Conversely, if another voter α' equipped with preferences $a \succ_{\alpha'} c$ leaves the electorate, the majority graph of $\succsim$ changes to G'. Note that G' is isomorphic to G, i.e., equivalent to G modulo a permutation of the alternatives. In contrast to G, alternative a is covered by d in G' and we obtain

$$f(\succsim_{-\alpha'}) = f(G') = 1/3\, b + 1/3\, c + 1/3\, d.$$

Similar to before, it has to hold that $f(\succsim_{-\alpha'}) \not\succ^{SD}_{\alpha'} f(\succsim)$. In order to prevent manipulation by a voter with preferences $\succsim_{\alpha'}\colon a, c, \{b, d\}$, we first deduce

$$f(G_\alpha)(a) + f(G_\alpha)(c) \leqslant 1/3.$$

Together with $f(G_\alpha)(c) \geqslant 1/3$ known from above, we can conclude $f(G_\alpha)(c) = 1/3$ and $f(G_\alpha)(a) = 0$. Now, if $f(G_\alpha)(b) \neq f(G_\alpha)(d)$, this would allow for a manipulation by abstention by either $\succsim_{\alpha'}\colon a, c, b, d$ or $\succsim_{\alpha'}\colon a, c, d, b$, depending on whether we have $f(G_\alpha)(d) > f(G_\alpha)(b)$ or $f(G_\alpha)(b) > f(G_\alpha)(d)$. We deduce $f(G_\alpha)(b) = f(G_\alpha)(d)$ and putting everything together

$$f(G_\alpha) = 1/3\, b + 1/3\, c + 1/3\, d.$$

On the other hand, any preference profile $\succsim' \in \succsim(A)^{\mathcal{F}(\mathbb{N})}$ inducing majority graph G_β necessarily results in the lottery

$$f(G_\beta) = 1/4\, a + 1/4\, b + 1/4\, c + 1/4\, d$$

because of neutrality. Finally, note that a voter β with preferences $\succsim_\beta : \{a, c\}, d, b$ joining a suitable electorate with preference profile $\succsim'$ changes $\succsim'^M$ such that it equals G_α afterwards. From above, we know that

$$f(G_\alpha) = 1/3\, b + 1/3\, c + 1/3\, d.$$

Since $f(G_\beta) \succ^{SD}_\beta f(G_\alpha)$, voter β has the possibility of *SD*-manipulation by strategic abstention contradicting the initial assumption that f satisfies *SD*-participation. This concludes the proof. □

5.3.4 Participation for Maximal Lotteries

We now forsake participation with respect to stochastic dominance and turn to maximal lotteries and *PC*-participation. Some additional notation is required to conveniently present our main theorem. Recall from Section 5.1.1 that $p \succsim_i^{PC} q$ if $p^T \phi_i q \geqslant 0$ for ϕ_i as defined before. For an electorate $N \in \mathcal{F}(\mathbb{N})$, $S \subseteq N$, and preference profile $\succsim \in \succsim(A)^N$, let $\phi_S = \sum_{i \in S} \phi_i$. Using this terminology, we have that $p \in ML(\succsim)$ if $p^T \phi_N q \geqslant 0$ for all $q \in \Delta(A)$.

THEOREM 5.14
Every probabilistic voting rule returning maximal lotteries satisfies *PC*-group-participation but violates strong *SD*-participation.

Proof. Let $N \in \mathcal{F}(\mathbb{N})$, $S \subsetneq N$, and $\succsim \in \succsim(A)^N$. For $p \in ML(\succsim)$ and $p' \in ML(\succsim_{-S})$, we then have

$$\begin{aligned} p^T \phi_N q &\geqslant 0 \text{ for all } q \in \Delta(A), \text{ and} \\ p'^T \phi_{N \setminus S} q &\geqslant 0 \text{ for all } q \in \Delta(A) \end{aligned}$$

as p and p' are maximal lotteries. Note that by skew-symmetry of all ϕ_i, we have that

$$\begin{aligned} & p'^T \phi_{N \setminus S} q \geqslant 0 \text{ for all } q \in \Delta(A) \\ \Leftrightarrow\ & q^T \phi_{N \setminus S} p' \leqslant 0 \text{ for all } q \in \Delta(A). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} p^T \phi_S p' &= p^T (\phi_N - \phi_{N \setminus S}) p' \\ &= \underbrace{p^T \phi_N p'}_{\geqslant 0} - \underbrace{p^T \phi_{N \setminus S} p'}_{\leqslant 0} \\ &\geqslant 0. \end{aligned}$$

Hence, there has to exist at least one $i \in S$ for which we have that $p^T \phi_i p' \geqslant 0$, i.e., $p \succsim_i^{PC} p'$.

To see that strong *SD*-participation is violated, consider $A = \{a, b, c\}$ and the preference profile $\succsim = (\succsim_1, \dots, \succsim_6)$ as given on the right. It is easily seen that

1,2	3,4	5,6
a	b	c
b	c	a
c	a	b

$$ML(\succsim) = 1/3\, a + 1/3\, b + 1/3\, c.$$

Now, if an additional voter with preferences $\succsim_7$: a, b, c joins the electorate, we compute $ML(\succsim_{+7}) = 3/7\, a + 1/7\, b + 3/7\, c$. Obviously $ML(\succsim_{+7}) \not\succsim_7^{SD} ML(\succsim)$, a violation of strong *SD*-participation. This concludes the proof. □

Theorem 5.14 is of special interest as *ML* satisfies a particularly strong notion of efficiency, namely *PC*-efficiency (Aziz et al., 2018a).

		SD-efficient	ex post efficient	unanimous	unrestricted
very strong SD-part.	majoritarian	–	–	–	–
	pairwise	–	–	–	?
	anonymous and neutral	?	✓	✓	✓
strong SD-part.	majoritarian	–	–	–	✓✓
	pairwise	✓	✓	✓	✓✓
	anonymous and neutral	✓	✓	✓	✓✓
SD-part.	majoritarian	–	–	✓✓	✓✓
	pairwise	✓✓	✓✓	✓✓	✓✓
	anonymous and neutral	✓✓	✓✓	✓✓	✓✓

Table 5.1: Existence of probabilistic voting rules combining certain notions of efficiency and participation; ✓ and ✓✓ indicate the existence of probabilistic voting rules satisfying single-voter participation and group-participation, respectively.

While we have that *ML* fares worse than, e.g., *RSD* or *BOR* in terms of participation, it is strictly more efficient than both. Also note that *ML* is pairwise and recall from Section 5.1.2 that it additionally satisfies Condorcet consistency.

5.4 CONCLUSION

In this chapter, we defined and analyzed different degrees of participation for probabilistic voting rules and studied their compatibility with varying notions of efficiency. Results obtained in Section 5.3.1 to Section 5.3.3 are summarized in Table 5.1 for abstention by single voters and groups alike. Positive results carry over from group-participation to participation, from stronger to weaker notions of efficiency and participation and from majoritarianness to pairwiseness to anonymity and neutrality. Implications are opposed when considering impossibilities.

To recap the most important points, we have seen that the voting rule *BOR* satisfies desirable properties. Apart of the strong notion of *SD*-efficiency, *BOR* also fares well in terms of resistance against manipulation by strategic abstention both for single voters as well as groups thereof. This spotlights the difference between strong *SD*-participation and the related strong *SD*-strategyproofness, which is incompatible with anonymity, neutrality, and *SD*-efficiency (Bogomolnaia and Moulin, 2001).[90]

For *RSD*, we have seen that any voter not already perfectly content with the current outcome can strictly improve his expected utility by joining an electorate. It is worth noting that *RSD*, variants thereof, or convex combinations of different voting rules including *RSD* are not the only probabilistic voting rules to satisfy very strong *SD*-participation (see also Section 1.2.4). This seems to suggest that characterizing *RSD* using very strong *SD*-participation might well require additional technical and non-obvious properties. Also note that in contrast to *BOR*, *RSD* can be *SD*-manipulated by abstaining groups of voters.

Following Theorem 5.13, when attention is restricted to majoritarian probabilistic voting rules, *ex post* efficiency is incompatible with *SD*-participation. This impossibility also holds for the complete downward and upward lexicographic extensions (see, e.g., Cho, 2016) as well as the pairwise comparison extension. It is unknown whether this result still holds when stronger preference extensions, e.g., bilinear dominance (see, e.g., Aziz et al., 2018a), are considered.

Theorem 5.14 is of special interest when seen together with the central result of Moulin (1988). While Moulin shows that every Condorcet consistent single-valued voting rule is prone to the NSP, Theorem 5.14 establishes that there do exist attractive probabilistic voting rules satisfying Condorcet consistency and a reasonable notion of (group-)participation.[91] In this sense, we can say that Moulin's seminal impossibility theorem for single-valued voting rules does not extend to the probabilistic framework.

Note however, that when demanding a stronger notion of participation instead, the original incompatibility remains intact: There is no Condorcet consistent probabilistic voting rule satisfying strong *SD*-participation (Brandt et al., 2017a).

All of these important points are also captured in Figure 5.2. When considering the solid lines, this diagram hints at a tradeoff between efficiency and participation similar to the tradeoff between efficiency and strategyproofness established in recent works by Brandl et al. (2018) and Aziz et al. (2018a). Based on this observation, we deem it interesting to study whether there exists a probabilistic voting rule satisfying *SD*-efficiency and very strong *SD*-participation, a question open so far. In addition, we also leave the question whether very

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90 Interestingly, *BOR* is known to be particularly vulnerable to strategic manipulation by misrepresentation as it is *single-winner manipulable* (Taylor, 2005). This means *BOR* not only violates strong *SD*-strategyproofness, but strategyproofness with respect to *any* preference extension.

91 For a more thorough discussion of further desirable properties of maximal lotteries, we here refer to Brandl et al. (2016).

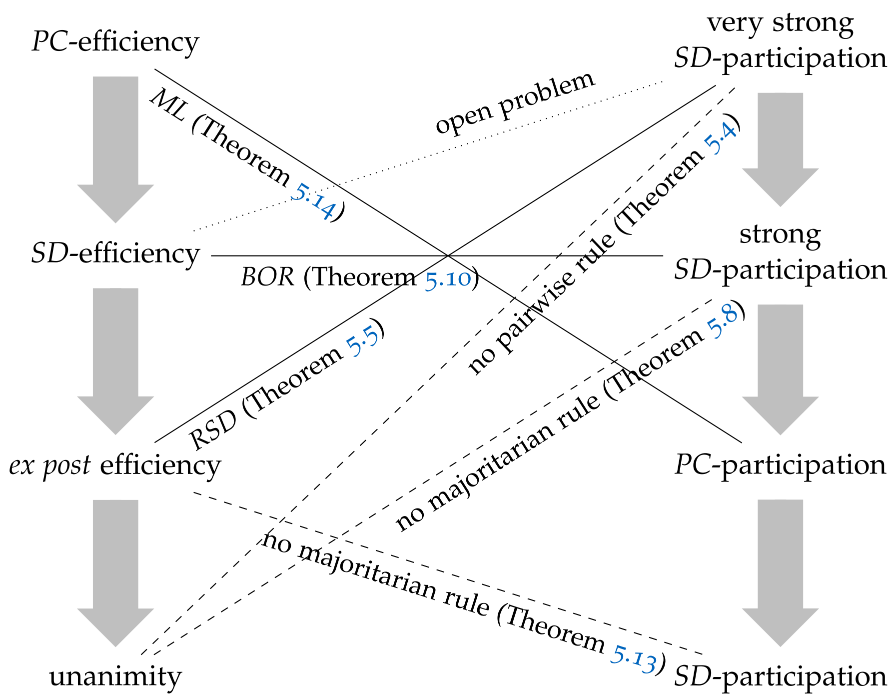

Figure 5.2: Relationships between efficiency and participation concepts; an arrow from one notion of efficiency or participation to another denotes that the former implies the latter; a solid line indicates that there exist probabilistic voting rules with the given properties; a dashed line indicates that no probabilistic voting rule with the given properties exists; the dotted line marks an open problem.

strong *SD*-participation can be satisfied by pairwise rules for future research.

We conclude this chapter with some remarks concerning results presented here or extensions thereof.

Remark 5.15
For probabilistic voting rules, it is also possible to define a notion of participation even stronger than very strong participation. This notion demands that a voter should always strictly prefer the outcome with him participating over what would result had he abstained, i.e., we omit the restriction to situations where he is not completely content when abstaining. Satisfaction thereof, however, requires a voting rule to always select lotteries that give positive probability to every alternative. Consequently, every meaningful degree of efficiency is violated.

Voting rules satisfying this strongest notion of participation nevertheless exist, an easy example is a rule f that mixes *RSD* and a uniform lottery over all alternatives based on the number of voters present. Formally,

$$f(\succsim) = {}^{1}/_{n} \left(\sum_{x \in A} {}^{1}/_{m}\, x \right) + {}^{n-1}/_{n}\, RSD(\succsim).$$

We remark that there are situations where a (moderate) violation of efficiency might even be desired. This particularly applies to repeated decisions, where even though voters have preferences over the alternatives, they also appreciate grander variety to some extent, e.g., restaurants for daily lunch breaks or artists played on the radio.

REMARK 5.16

A lemma weaker than Lemma 5.12 in various aspects suffices to show Theorem 5.13. In particular, the statement that for any majoritarian and *ex post* efficient probabilistic voting rule f, we have that $\text{supp}(f(\succsim)) \subseteq UC(\succsim^M)$ for all $\succsim \in \succsim(A)^{\mathcal{F}(\mathbb{N})}$ alone allows for an almost identical—and only slightly more complicated—proof.

REMARK 5.17

It is possible to show that the converse of Theorem 5.14 also holds under rather mild technical assumptions. More precisely, we can show that every homogenous and Condorcet consistent probabilistic voting rule satisfying a notion of participation focusing on accumulated welfare with respect to the *PC* extension has to return maximal lotteries.[92]

In addition, Theorem 5.14 as well as the converse implication even hold in the more general domain of *SSB* utility theory that allows for intensities of preferences.

REMARK 5.18

A proof similar to the one of Theorem 5.14 shows that probabilistic voting rules returning maximal lotteries also satisfy *SD*-one-way-monotonicity. This stands in contrast to Sanver and Zwicker (2009) and Peters (2017), who show that no single-valued voting rule satisfies Condorcet consistency and half-way-monotonicity, a weakening of one-way-monotonicity and participation (see also Section 1.2.2).

REMARK 5.19

Point voting schemes are a class of probabilistic voting rules defined by Barberà (1979) on the basis of what Gibbard (1977) calls unilateral and duple rules. Giving each alternative probability based on the sum of scores it receives from all voters, they can be seen as a way to generalize scoring rules to the

92 A voting rule is homogenous if using multiple copies of the electorate does not affect the outcome.

probabilistic framework. Point voting schemes satisfy strong *SD*-participation, and in case there are at least two voters with different preference rankings, they even satisfy very strong *SD*-participation.

6 THE NO-SHOW PARADOX FOR RANDOM ASSIGNMENT RULES

We now drop one of the central modeling assumptions we have adhered to up to now. In all previous settings—be it single-valued, set-valued or probabilistic voting—voters' preferences were aggregated to a collective choice relevant to everybody. For the remainder of this thesis, we redirect focus to individual incentives, i.e., we assume every involved voter is assigned a personal alternative.

This situation is well-known as *assignment* (or *house allocation*) and regularly considered in multiagent systems and microeconomic theory (see, e.g., Chevaleyre et al., 2006; Sönmez and Ünver, 2011; Manlove, 2013; Bouveret et al., 2016). In this context, it is common to name voters *agents*, who have preferences over *objects* instead of alternatives. Rules assigning objects to agents will be called *assignment rules*.

A central problem is how to find an assignment based on the individual preferences only that satisfies varying notions of fairness. Since objects are indivisible and generally unique and two agents might be equipped with identical preferences, it is easy to see that no deterministic assignment rule can treat both identically. Just as it was the case in the domain of voting, introducing randomization constitutes an elegant way out (see, e.g., Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2001; Che and Kojima, 2010; Budish et al., 2013).

For this, we need to define *random assignment rules* that award to every agent a lottery over objects. Under the common assumption that we have an identical amount of agents and objects, a valid *random assignment* thus requires every agent to receive a total probability of one of objects and every object to be distributed with a total probability of one, too. In order to allow agents to compare lotteries, we pick up the concept of stochastic dominance discussed in Section 5.1.1.

In this framework, we now study incentives for participation for three well-known random assignment rules—random serial dictatorship, the probabilistic serial rule, and the Boston mechanism—and the class of popular random assignments. Our results are largely positive for single agents and groups thereof alike: we show that participation is strictly incentivized for single agents by all three rules, while manipulation might be possible for some but never all vNM utility functions when popular random assignments are considered. In addition, all rules as well as the class of popular random assignments satisfy varying notions of *SD*-group-participation. An overview of our results is provided in Table 6.1 on page 121.

This chapter is structured as follows: we formally define the random assignment setting in Section 6.1. In Section 6.2, we look at all three random assignment rules and popular random assignments one after another and study them with respect to single-agent and group-participation. A conclusion and final remarks are given in Section 6.3.

6.1 RANDOM ASSIGNMENT SETTING

The framework required to study assignment problems and random assignments is closely related to the (probabilisitic) voting setting. Hence, while we formally introduce concepts that are either new or variations of previously defined ones, we merely explain others informally and point to the section containing the corresponding definition.

Let $N = \{1, \dots, n\}$ be a set of n agents and O a set of n objects. Together with a preference profile $\succsim \in \succsim(O)^N$, the triple $(N, O, \succsim)$ constitutes an *assignment problem*. For all $i \in N$, we denote by k_i the number of indifference classes in $\succsim_i$ and by O_i^k the union of the upper k indifference classes for $k \in [k_i] = \{1, \dots, k_i\}$.

assignment problem

A *deterministic assignment* (or *pure matching*) is a one-to-one map from N to O. We identify a deterministic assignment M with a permutation matrix in $\mathbb{R}^{N \times O}$, where $M_{io} = 1$ if agent i is assigned object o and $M_{io} = 0$ otherwise. The set of all deterministic assignments is denoted by $\mathcal{M}$.

deterministic assignment

In line with notation previously used, a *random assignment* p is a probability distribution (or lottery) over deterministic assignments, i.e., $p \in \Delta(\mathcal{M})$. We represent a random assignment p as bistochastic matrix in $\mathbb{R}^{N \times O}$ where $p(i, o)$ is the probability that agent i is assigned object o.[93] Note that by the Birkhoff-von Neumann Theorem, every bistochastic matrix can be written as probability distribution over deterministic assignments (see, e.g., Kavitha et al., 2011). If we have that $p = f(\succsim)$ for some random assignment rule f, we also write $p_{-i} = f(\succsim_{-i})$ and $p_{-S} = f(\succsim_{-S})$ with slight abuse of notation.[94]

random assignment

For a set of agents $S \subseteq N$ and a set of objects $O' \subseteq O$, we denote by $p(S, O')$ the sum of probabilities of agents in S for objects in O' in the random assignment p. Formally, $p(S, O') = \sum_{(i,o) \in S \times O'} p(i, o)$. In case either S or O' is a singleton, we write $p(i, O')$ or $p(S, o)$ for convenience, respectively. To simplify notation, for $p \in \Delta(\mathcal{M})$ and $i \in N$, we write $p(i)$ for the i^{th} row of p, i.e., the lottery over O assigned to agent i.

93 A matrix M is bistochastic if all entries are nonnegative and every row and every column sums up to one, i.e., $M_{ij} \geqslant 0$ and $\sum_i M_{ij} = \sum_j M_{ij} = 1$ for all i, j.

94 See below for a brief discussion of how to interpret $\succsim_{-i}$ and $\succsim_{-S}$ in the random assignment setting that demands for an identical number of agents and objects.

EXAMPLE 6.1

As an example consider the random assignment p depicted on the right where $N = \{1, 2, 3\}$ and $O = \{a, b, c\}$. Here, $p(1, a) = 1/3$ and $p(3) = 1/4\, b + 3/4\, c$.

$$p = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 1/12 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

To determine the agents' preferences over random assignments, we rely on two assumptions. First, we assume an agent only cares about the lottery he is assigned, not about what other agents receive.[95] Moreover, we suppose agents compare lotteries using stochastic dominance (see Section 5.1.1). Picking up Example 6.1 and assuming agent 1 has preferences $\succsim_1 : a, b, c$, we thus have that $p(1) \succ_1^{SD} p(3)$ and $p(2) \succ_1^{SD} p(3)$. $p(1)$ and $p(2)$ are not comparable for agent 1 according to *SD*.

For this setting of random assignment, we want to study whether agents may face an incentive to abstain from the allocation procedure. First, note that by definition, we require the number of agents and objects to be equal in any assignment problem. We therefore define abstention by letting an agent declare complete indifference. This leads to a natural notion of participation in settings where agents always receive an object, regardless of whether they participate in the mechanism or not. Using a different interpretation of participation, for a group of agents $S \subseteq N$, we can assume that objects are first allocated to agents in $N \setminus S$ and whichever objects remain are distributed uniformly among agents in S.

For random serial dictatorship and the probabilistic serial rule, both concepts of abstention coincide and we will use whichever suits our needs best throughout the proofs to come. The Boston mechanism is defined for strict preferences only and it is intuitively unclear how to extend it to possible indifferences. Hence, we stick to the interpretation of abstainers being rewarded what is left after a first round in Section 6.2.3. Popular random assignments, on the other hand, do not allow for multiple rounds of allocation and we thus go with the idea of complete indifference in Section 6.2.4.

In order to study possible incentives for participating or abstaining, we use the degrees of single-agent and group-participation defined in Section 5.2:

- *Very strong participation* requires that participating always yields a strictly preferred allocation whenever this is possible.
- *Strong participation* prescribes that participating always yields a weakly preferred allocation.
- *Participation* forbids that for some assignment problem, abstaining yields a more preferred result.

95 Such preferences are also known as *nonexogenous preferences* and commonly assumed in the context of assignment or fair division (see, e.g., Bouveret et al., 2016).

Since we restrict ourselves to stochastic dominance in this chapter, we will henceforth drop the *SD* for all notions of participation.

Note that every assignment problem can be seen as a voting problem, where the set of voters equals the set of agents and the set of alternatives is the set of deterministic assignments (Aziz et al., 2013c). Under the assumption that a voter is indifferent in between any two assignments awarding him an identical allocation, we can easily lift preferences over objects to preferences over assignments. This transformation, however, causes a large blow-up as the original n objects induce $n!$ deterministic assignments, i.e., alternatives.

6.2 RESULTS AND DISCUSSION

We now introduce three popular random assignment rules together with a class of random assignments and study to which extent they allow for manipulation by strategic abstention, or, rather, incentivize participation. As motivational example, consider a company that assigns office space to workers using the probabilistic serial (*PS*) rule. The default preference preassigned to every worker is complete indifference and it is up to him to update his preferences before a given deadline or not. We prove that a worker is always *strictly* better off (whenever an improvement is possible at all) by updating his preferences and thus participating in the mechanism, no matter what his underlying vNM utility function is. By contrast, it is well-known that *PS* fails to satisfy strategyproofness.[96]

In the following, we point out similar differences with respect to the degrees of participation and strategyproofness satisfied whenever suitable. As a basic principle, we find that all rules fare a lot better in terms of participation than in terms of strategyproofness.

6.2.1 Random Serial Dictatorship

The characteristic feature of *random serial dictatorship (RSD)*, also known as *random priority*, is its resistance to strategic manipulation by a single agent, i.e., *RSD* satisfies strong strategyproofness (see, e.g., Barberà et al., 1998; Bogomolnaia and Moulin, 2001). This directly implies that *RSD* also satisfies strong participation. However, *RSD* violates group-strategyproofness. By contrast and perhaps surprisingly, we will show that *RSD* satisfies strong group-participation.

Typically, *RSD* is defined for the special case where all agents have strict preferences over objects. Our definition extends *RSD* to the

96 While *PS* satisfies strategyproofness for strict preferences, it fails to do so for general preferences (Bogomolnaia and Moulin, 2001; Katta and Sethuraman, 2006). Note once more that what is named strategyproofness here is sometimes referred to as *weak strategyproofness* in the literature. *PS* fails to satisfy the stronger notion of strategyproofness here called strong strategyproofness even for strict preferences.

full preference domain (see also Bogomolnaia et al., 2005; Aziz et al., 2013a). For better exposition, we start by defining *RSD* for agents with strict preferences: first, a permutation of agents is drawn uniformly at random, then the agents successively choose their most preferred object among the remaining objects according to the order given by the permutation. For general preferences, this process is not well-defined as agents may have multiple most preferred objects. In this case, an agent narrows down the set of assignments to assignments in which he is allocated one of them.

Note that *RSD* as we define it for the assignment setting is equivalent to *RSD* in the domain of voting (see Section 5.1.2). In particular, the random assignment returned by *RSD* corresponds exactly to the lottery over deterministic assignments that would have been returned by *RSD* if the assignment problem was transferred to a voting problem first. We here provide an alternative definition relying on permutations that will turn out to be more convenient.

Formally, let Π_N be the set of all permutations of N. For a preference relation $\succsim_i$ and a set of deterministic assignments $\mathcal{M}' \subseteq \mathcal{M}$, recall that

$$\max_{\succsim_i}(\mathcal{M}') = \{M \in \mathcal{M}' \colon M \succsim_i M' \text{ for all } M' \in \mathcal{M}'\}$$

is the set of most preferred assignments according to $\succsim_i$ in $\mathcal{M}'$.[97] For $\succsim \in \succsim(O)^N$, $\pi \in \Pi_N$, and $k \in \{1, \dots, n\}$, we define inductively

$$\sigma^k(\succsim, \pi) = \begin{cases} \max_{\succsim_{\pi(1)}}(\mathcal{M}) & \text{if } k = 1 \text{, and} \\ \max_{\succsim_{\pi(k)}}(\sigma^{k-1}(\succsim, \pi)) & \text{if } k \in \{2, \dots, n\}. \end{cases}$$

Then, $\sigma^n(\succsim, \pi)$ is the outcome of *serial dictatorship* according to the permutation π. Note that this set may contain more than one deterministic assignment. We resolve this ambiguity by randomizing uniformly over these assignments and define $\sigma(\succsim, \pi)$ as the uniform distribution over $\sigma^n(\succsim, \pi)$. *RSD* is defined by randomizing uniformly over all permutations of agents, i.e.,

serial dictatorship

random serial dictatorship

$$RSD(\succsim) = {}^1/_{n!} \sum_{\pi \in \Pi_N} \sigma(\succsim, \pi).$$

EXAMPLE 6.2

Consider the assignment problem $(N, O, \succsim)$ with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, and $\succsim$ as given below.[98]

$$\succsim = \begin{array}{ll} 1\colon & \{a, b\}, c \\ 2\colon & a, b, c \\ 3\colon & b, a, c \end{array} \qquad RSD(\succsim) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 0 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

97 If agent i is assigned object o in M and o' in M', then $M \succsim_i M'$ holds if $o \succsim_i o'$.

98 In the assignment setting it is common to denote preference profiles horizontally instead of vertically, i.e., agents' preferences are depicted as rows instead of columns with more preferred objects to the left.

To illustrate the definition, we explain the computation for the permutation $\pi = (1,2,3)$. Agent 1 narrows down the set of assignments to all assignments where he is assigned either object a or object b. Out of these, agent 2 prefers the assignments where he is assigned object a (and hence agent 1 is assigned object b). In the only remaining assignment, agent 3 is assigned object c.

Our first result states that *RSD* satisfies very strong participation.

Corollary 6.3
RSD satisfies very strong participation.

Proof. The statement follows directly from Theorem 5.5 and the observation that assignment is a special case of voting (see, e.g., Aziz et al., 2013c). □

We proceed by showing that, in contrast to Theorem 6.4, *RSD* violates group-strategyproofness.[99] To this end, consider the assignment problem $(N, O, \succsim)$ with $N = \{1,2,3,4\}$, $O = \{a,b,c,d\}$ and $\succsim$ as follows.

$$\succsim \; = \begin{array}{ll} 1: & a,b,c,d \\ 2: & a,b,c,d \\ 3: & b,a,d,c \\ 4: & b,a,d,c \end{array} \qquad RSD(\succsim) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}$$

Then, the group consisting of all four agents can manipulate by reporting the preferences $\succsim'$.

$$\succsim' \; = \begin{array}{ll} 1: & a,c,b,d \\ 2: & a,c,b,d \\ 3: & b,d,a,c \\ 4: & b,d,a,c \end{array} \qquad RSD(\succsim') = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

Next, we consider group-participation.

Theorem 6.4
RSD satisfies strong group-participation.

Proof. The proof relies on considering the outcome of serial dictatorship for all possible permutations of agents in N. We make use of the fact that if S abstains, it is irrelevant whether we consider permutations of $N \setminus S$ and distribute the remaining probabilities uniformly among agents in S, or instead consider all permutations of N with agents in S being completely indifferent. Hence, for every permutation of agents in $N \setminus S$ where S receives some probability $\alpha > 0$ of $o \in O$, we have $\binom{|N|}{|S|}|S|!$ permutations of agents in N where S receives

99 This statement is included to illustrate the contrast with group-participation. It follows from *RSD*'s well-documented lack of *SD*-efficiency (see also Bade, 2016).

the very same probability α.[100] Note that the fraction of profiles in which agent $i \in S$ precedes all other agents in S is exactly $1/|S|$. Assuming object o is i's first choice, he would thus have received o with probability at least α in all permutations where he is first out of S, had he participated. Going back to all $\binom{|N|}{|S|}|S|!$ permutations of agents in N, we hence have that when S participates, i receives o with probability at least $1/|S|\,\alpha$ while i's probability for o is exactly $1/|S|\,\alpha$ when S abstains.

Formally, we show that for all assignment problems $(N, O, \succsim)$ and agents $i \in N$, we have that $p(i, O_i^k) \geqslant p_{-S}(i, O_i^k)$ for all $S \subseteq N$, $i \in S$, and $k \in [k_i]$. Here, we use $p = \mathit{RSD}(\succsim)$. To this end, let $(N, O, \succsim)$ be an assignment problem and choose $S \subseteq N$, $i \in S$, and $k \in [k_i]$ arbitrarily. We begin with the case where agents in S abstain, i.e., they are completely indifferent. Recall that under this circumstance, for *RSD* it is irrelevant whether we include agents in S in the sequence of agents or only focus on $N \setminus S$ and distribute the remaining probabilities uniformly.

Consequently, we first consider permutations of $N \setminus S$ only. By $\Pi_{N \setminus S}$ denote the set of all permutations of $N \setminus S$. Let $\pi \in \Pi_{N \setminus S}$ and let α_π be the corresponding probability share of O_i^k given to S.

Note that for each π, there exist $\binom{|N|}{|S|}|S|!$ permutations of N using which serial dictatorship yields the same probability share α_π of O_i^k for S. A fraction of exactly $1/|S|$ of these sequences list i as j^{th} agent out of S. Thus, i precedes all agents in S in $1/|S|$ of sequences and had S participated, he would have received a probability share of at least $\min\{1, \alpha_\pi\}$ of O_i^k in these cases. In another $1/|S|$ of the sequences, i comes second and had S participated, he would have received a probability share of at least $\max\{0, \min\{1, \alpha_\pi - 1\}\}$ of O_i^k. For the general case of i being at the l^{th} position of agents in S, he would have received a probability share of at least

$$\max\{0, \min\{1, \alpha_\pi - (l-1)\}\}$$

of O_i^k had S participated.

Summing up all possible positions with respect to agents in S, we obtain that had S participated, i would have received a probability share of at least

$$\lfloor\alpha_\pi\rfloor/|S| + 1/|S|\,(\alpha_\pi - \lfloor\alpha_\pi\rfloor)$$

of O_i^k. Here, the first summand corresponds to positions where i would receive a full probability share of one, while the second summand models situations in which i would receive a probability share of only $\alpha_\pi - \lfloor\alpha_\pi\rfloor < 1$. Note that since

$$\lfloor\alpha_\pi\rfloor/|S| + 1/|S|\,(\alpha_\pi - \lfloor\alpha_\pi\rfloor) = 1/|S|\,\alpha_\pi,$$

100 We consider a single object $o \in O$ and not a set of objects $O' \subseteq O$ as would be required in order to show strong group-participation for reasons of exposition. See the formal proof for arguments employing sets.

we have that had S participated, i would have received at least the same probability share of objects in O_i^k for all orderings $\pi \in \Pi_{N\setminus S}$. We consequently have that $p_{-S}(i, O_i^k) \leqslant p(i, O_i^k)$.

For the sake of clarity, we put all (in)equalities together and obtain

$$\begin{aligned} p_{-S}(i, O_i^k) &= {}^1/_{|S|}\, p_{-S}(S, O_i^k) \\ &= {}^1/_{(|N\setminus S|)!} \sum_{\pi \in \Pi_{N\setminus S}} {}^1/_{|S|}\, \alpha_\pi \\ &= {}^1/_{(|N\setminus S|)!} \sum_{\pi \in \Pi_{N\setminus S}} {}^{\lfloor \alpha_\pi \rfloor}/_{|S|} + {}^1/_{|S|}\, (\alpha_\pi - \lfloor \alpha_\pi \rfloor) \\ &\leqslant p(i, O_i^k), \end{aligned}$$

which completes the proof. □

6.2.2 Probabilistic Serial

In contrast to *RSD*, the *probabilistic serial (PS)* rule is a relatively new assignment rule proposed by Bogomolnaia and Moulin (2001). They also study *PS* axiomatically and find that in the domain of strict preferences—for which *PS* is originally defined only—it satisfies *SD*-efficiency, envy-freeness and strategyproofness but violates strong strategyproofness.

probabilistic serial Intuitively, *PS* works as follows: We assume all objects to be edible and of equal size one. At time $t = 0$, all agents simultaneously begin to eat their respective favorite object at uniform speed. If an object is completely consumed, all agents involved at this point switch to their most preferred remaining object, which they then start to eat. Reaching time $t = 1$, no more objects are available and each agent has consumed a total amount of one. Finally, the fraction an agent has eaten of a certain object corresponds directly to the probability with which it is awarded to him.

Example 6.5

Consider the assignment problem $(N, O, \succsim)$ with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, and $\succsim$ as given below.

$$\succsim = \begin{array}{ll} 1: & a, b, c \\ 2: & a, c, b \\ 3: & b, a, c \end{array} \qquad PS(\succsim) = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

In the beginning, agents 1 and 2 are eating a while agent 3 is eating b. At time $t = 1/2$, a is completely consumed while half of b has been eaten by agent 3. 1 and 2 continue on to b and c, their respective second most preferred objects. At time $t = 3/4$, b is completely consumed as well and all agents simultaneously finish c. Put together, the random assignment returned by *PS* is as given above.

The extension of *PS* to the full preference domain that is most widely accepted is due to Katta and Sethuraman (2006). They name their algorithm, which is heavily based on maximal flows in networks, *extended probabilistic serial*. Katta and Sethuraman prove that it still satisfies *SD*-efficiency and envy-freeness but is no longer strategyproof.[101] Whenever we refer to *PS* in the sequel, we mean the generalization by Katta and Sethuraman (2006).

EXAMPLE 6.6

To get a first taste, consider the assignment problem $(N, O, \succsim)$ with $N = \{1,2,3,4\}$, $O = \{a, b, c, d\}$, and $\succsim$ as given below. In contrast to a naive generalization, agent 1 is not 'eating' a and b simultaneously but instead he is reserving some probability of $\{a, b\}$ with identical uniform speed. At $t = 2/3$, we arrive at a point where three agents have each reserved a share of $2/3$ of $\{a, b\}$—or at least one object out of it—meaning that the set $\{a, b\}$ is completely consumed. We therefore say that $\{a, b\}$ is a *bottleneck* and *PS* proceeds by gradually identifying subsequent bottlenecks and distributing the included objects to the competing agents in a fair way. For the preference profile $\succsim$ below, *PS* thus finds the bottlenecks $\{a, b\}$, $\{c\}$, $\{d\}$ in chronological order.

$$\succsim = \begin{array}{ll} 1: & \{a,b\}, c, d \\ 2: & a, c, b, d \\ 3: & b, a, d, c \\ 4: & c, d, a, b \end{array} \qquad PS(\succsim) = \begin{pmatrix} 1/3 & 1/3 & 1/9 & 2/9 \\ 2/3 & 0 & 1/9 & 2/9 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 7/9 & 2/9 \end{pmatrix}$$

As stated before, the implementation makes use of flows on networks that are redesigned after each bottleneck. We omit a more detailed and formal explanation of *PS* for general preferences in the interest of space and refer to Katta and Sethuraman (2006).

Recently, *PS* was generalized to the more general domain of voting by Aziz and Stursberg (2014), who call the resulting rule the *egalitarian simultaneous reservation* rule. Interestingly, this rule violates strong participation (Aziz, 2016).

We show that within the domain of assignment, *PS* fares not only better but indeed optimal with respect to manipulation by strategic abstention, i.e., *PS* satisfies the strongest notions of participation considered here: very strong participation and strong group-participation. This stands in contrast to the voting domain as well as to the related concept of strategyproofness.

THEOREM 6.7

PS satisfies very strong participation.

Proof. It follows from Theorem 6.8 below that *PS* satisfies strong participation. We now prove that even the stronger notion of *very* strong

101 More so, Katta and Sethuraman (2006) show that *SD*-efficiency and envy-freeness are incompatible with strategyproofness on the full preference domain.

participation holds. Therefore, focus on the first level of preferences and let p be the random assignment returned by *PS*. First, note that

$$p_{-i}(i, O_i^1) \leqslant p(i, O_i^1) \leqslant 1$$

is implied by strong participation. Furthermore, if $p(i, O_i^1) = 1$, very strong participation is satisfied. Thus, the only case remaining for examination is

$$p_{-i}(i, O_i^1) \leqslant p(i, O_i^1) < 1.$$

We will show that it always holds that $p_{-i}(i, O_i^1) < p(i, O_i^1)$, which implies very strong participation.

Note that by the algorithm used for *PS* (see Katta and Sethuraman, 2006), we have that $0 < p(i, O_i^1)$. In addition, $p(i, O_i^1) < 1$ by the above assumption, thus, O_i^1 has to be part of some bottleneck $B \subseteq O$ that occurs at time t_B. Let the set of agents who cause said bottleneck to occur be S_B. We additionally use the notation Γ taken from Katta and Sethuraman (2006) and slightly modify it to better fit our needs: $\Gamma_t(S)$ denotes the union of objects that agents S are eating (or reserving) at time t.

We distinguish whether B is the first bottleneck or not:

1. O_i^1 is part of the first bottleneck. We consequently have that $\Gamma_{t_B}(S_B) = \Gamma_{t_B}(S_B \setminus \{i\})$ because otherwise a different bottleneck would have occurred earlier for agents $S_B \setminus \{i\}$. To see this, assume for contradiction $\Gamma_{t_B}(S_B) \neq \Gamma_{t_B}(S_B \setminus \{i\})$. It follows that

   $$|\Gamma_{t_B}(S_B \setminus \{i\})| \leqslant |\Gamma_{t_B}(S_B)| - 1$$

   and thus

   $$\frac{|\Gamma_{t_B}(S_B \setminus \{i\})|}{|S_B \setminus \{i\}|} < \frac{|\Gamma_{t_B}(S_B)|}{|S_B|}.$$

   This contradicts the first bottleneck appearing for S_B—a different one would have appeared earlier for $S_B \setminus \{i\}$. Hence, $\Gamma_{t_B}(S_B) = \Gamma_{t_B}(S_B \setminus \{i\})$.

   Since $|S_B| - |S_B \setminus \{i\}| = 1$, we conclude that

   $$\frac{|\Gamma_{t_B}(S_B \setminus \{i\})|}{|S_B \setminus \{i\}|} \leqslant 1.$$

   Hence, given that i abstains, we still have a bottleneck that includes agents $S_B \setminus \{i\}$ (not necessarily the first) and it holds that $0 = p_{-i}(i, O_i^1) < p(i, O_i^1)$.[102]

2. O_i^1 is not part of the first bottleneck. For the bottleneck including O_i^1, we have that

   $$p(S_B, B) = |\Gamma_{t_B}(S_B)| = |B|$$

102 Theoretically, agents $S_B \setminus \{i\}$ do not necessarily belong to the same bottleneck. However, they will each be part of some bottleneck before the algorithm terminates. We omit details for the sake of readability.

and $p(S_B, B) < |S_B|$. Consequently $p(i', B) < 1$ for all $i' \in S_B$ and for similar arguments as above we have that

$$\Gamma_{t_B}(S_B) = \Gamma_{t_B}(S_B \setminus \{i\}).$$

Hence,

$$p(S_B \setminus \{i\}, B) < |\Gamma_{t_B}(S_B \setminus \{i\})|,$$

which means that the bottleneck B' including O_i^1 will occur strictly later at time $t = t_{B'} \leqslant 1$ for a possibly different group of agents $S_{B'} \supseteq S_B \setminus \{i\}$.[103] At this point, we have

$$p_{-i}(S_{B'}, B') = |\Gamma_{t_{B'}}(S_{B'})|.$$

Since $t_{B'} > t_B$ and thus $p_{-i}(i', B') > p(i', B')$ for all $i' \in S_{B'}$ as well as $p_{-i}(i', B') = p(i', B)$ for all $i' \in S_B$, it holds that

$$p(S_B, B) < p_{-i}(S_B, B).$$

Putting everything together, we obtain

$$\begin{aligned} p_{-i}(i, O_i^1) &\leqslant |\Gamma_{t_{B'}}(S_{B'})| - p_{-i}(S_{B'} \setminus \{i\}, B') \\ &< |\Gamma_{t_B}(S_B)| - p(S_B \setminus \{i\}, B) \\ &= p(i, B) \\ &= p(i, O_i^1). \end{aligned}$$

Thus, $p_{-i}(i, O_i^1) < p(i, O_i^1)$ for both cases and very strong participation is satisfied. □

THEOREM 6.8
PS satisfies strong group-participation.

Proof. Let $(N, O, \succsim)$ be an assignment problem with $O = \{o_1, \dots, o_n\}$, $S \subseteq N$ the group of agents that abstains, and $p = PS(\succsim)$. In the case where S participates, we call the bottlenecks that appear when executing the algorithm in order to determine *PS* $B_1, B_2, B_3, \dots \subseteq O$, where the naming is done in chronological order with arbitrary tie-breaking. Denote by $\beta(O_i^k)$ the minimal $l \in \mathbb{N}$ such that

$$O_i^k \subseteq \bigcup_{j \in [l]} B_j.$$

We want to show that $p \succsim_i^{SD} p_{-S}$ for all $i \in S$, which is equivalent to

$$p(i, O_i^k) \geqslant p_{-S}(i, O_i^k)$$

for all $i \in S$, $k \in [k_i]$.

First, we claim that $p_{-S} \succsim_i^{SD} p$ for all $i \in N \setminus S$. This holds true as all original bottlenecks either remain unchanged when S abstains or

103 As before, $S_B \setminus \{i\}$ do not necessarily contribute to the same bottleneck, they may be part of different ones. However, all of them occur at some point between t_B and the hypothetical $t_{B'}$. We once more omit details for the sake of readability.

occur later (in a possibly changed version). In particular, they cannot appear earlier, as less agents compete for the objects. Hence,

$$p_{-S}(i, O_i^k) \geqslant p(i, O_i^k)$$

for all $i \in N \setminus S$, $k \in [k_i]$, which proves the claim.

Now consider any agent $i \in S$, $k \in [k_i]$ and define

$$B = \bigcup_{j \in [\beta(O_i^k)]} B_j,$$

the set of all objects that are part of some bottleneck up to $B_{\beta(O_i^k)}$. We have that $p(i, O_i^k) = p(i, B)$ since for all B_j, $j \in [\beta(O_i^k)]$, such that $B_j \cap O_i^k = \emptyset$, i is not awarded any probability. However, they are completely consumed by other agents until all objects in B are consumed, hence, $p(i, B_j) = 0$ holds for them.

Note that i is awarded some probability in bottleneck $B_{\beta(O_i^k)}$, which means that at this point, no other agent can have received more total probability of B than i. In particular, this holds for all agents in S. We thus conclude that

$$p(i, B) \geqslant {}^{1}/_{|S|}\, p(S, B).$$

Concerning the total probability awarded up to the moment of bottleneck $B_{\beta(O_i^k)}$, we have that

$$p(S, B) + p(N \setminus S, B) = |B|$$

and consequently

$$p(S, B) = |B| - p(N \setminus S, B).$$

We now make use of our initial claim about agents not in S preferring p_{-S} to p, and conclude

$$|B| - p(N \setminus S, B) \geqslant |B| - p_{-S}(N \setminus S, B).$$

A variant of the sum formula of $|B|$ which we used before yields

$$|B| - p_{-S}(N \setminus S, B) = p_{-S}(S, B).$$

Recall that by our definition of abstention, a group S that does not participate is given the 'remaining' probability of all objects which is then distributed evenly among agents in S. Thus,

$${}^{1}/_{|S|}\, p_{-S}(S, B) = p_{-S}(i, B)$$

and since $O_i^k \subseteq B$, we have that

$$p_{-S}(i, B) \geqslant p_{-S}(i, O_i^k).$$

Putting everything together, we obtain a chain of (in)equalities summarizing our proof:

$$\begin{aligned} p(i, O_i^k) &= p(i, B) \\ &\geqslant {}^1/_{|S|}\, p(S, B) \\ &= {}^1/_{|S|}\, [|B| - p(N \setminus S, B)] \\ &\geqslant {}^1/_{|S|}\, [|B| - p_{-S}(N \setminus S, B)] \\ &= {}^1/_{|S|}\, p_{-S}(S, B) \\ &= p_{-S}(i, B) \\ &\geqslant p_{-S}(i, O_i^k) \end{aligned}$$

This shows that strong group-participation is satisfied by *PS*. □

6.2.3 Boston Mechanism

The *Boston mechanism (BM)* originates from the practical problem of school choice (see, e.g., Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005; Abdulkadiroğlu et al., 2009). Given students have varying preferences over different schools in the surrounding area, but each school can only accept a limited amount of students, how should vacant seats be distributed? In this context, *BM* is arguably one of the simplest rules: Consider only top-ranked schools in the first round and assign a seat at the top-ranked school to every student as long as there are enough available seats; if not break ties uniformly at random. Now, remove all students who have been assigned a seat and their respective seats, and consider the students' second most-preferred schools in the next round. Again, seats are assigned to students and ties are broken uniformly at random. This procedure continues until no students are left.[104]

Boston mechanism

EXAMPLE 6.9
Consider the assignment problem $(N, O, \succsim)$ with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, and $\succsim$ as given below.

$$\succsim = \begin{array}{ll} 1: & a, b, c \\ 2: & a, c, b \\ 3: & b, a, c \end{array} \qquad BM(\succsim) = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

In the beginning, agents 1 and 2 are applying to a while agent 3 applies to b. Since 3 is the only one focusing on b at this moment, b is awarded to him. a, on the other hand, goes to agents 1 and 2 with probability 1/2 each. *BM* now branches out.

104 Note that what we describe here is sometimes also called *naive* Boston mechanism in contrast to the *adaptive* Boston mechanism, where students apply to their most-preferred school that still has free seats (see, e.g., Alcalde, 1996; Mennle and Seuken, 2014). We here consider the naive *BM* for simplicity. Our results, however, also hold for the adaptive *BM*.

First, assume a was given to 1. Having been assigned an object, agents 1 and 3 as well as objects a and b leave the market after round one and in round two, agent 2 applies to his second most preferred object c. Being the only candidate, 2 receives c and the mechanism terminates.

Next, assume a was given to 2. Similar to above, agents 2 and 3 and objects a and b leave the market after round one. In the second round, 1 would want to apply to b which is, however, not available anymore. Hence, we directly continue with round three where 1 applies to and receives his third most preferred object c.

Since both cases happen with identical probability $1/2$ each, we obtain the random assignment $BM(\succsim)$ given above.

In our framework, we assume there is an equal number of schools and students with only one seat per school. In addition, we require that individual preferences are strict—it is unclear how to define *BM* for general preferences.

Unfortunately, the relative straightforwardness of *BM* comes at a price: Among other shortcomings, *BM* may yield unstable assignments and is easily manipulable by a large number of agents (Abdulkadiroğlu and Sönmez, 2003; Chen and Sönmez, 2006; Ergin and Sönmez, 2006).[105] These findings reduced *BM*'s popularity among both researchers and practitioners, indeed it has often been replaced by the famous *deferred acceptance mechanism* due to Gale and Shapley (1962). Nevertheless, *BM* is still considered an important assignment rule, which is also reflected by a recent axiomatic characterization due to Kojima and Ünver (2014).

With respect to participation, it turns out that, when only single agents abstain, it fares equally well as *RSD* and *PS*, i.e., it satisfies very strong participation. When considering abstention by groups of agents, results are mixed. While *BM* satisfies group-participation, it violates strong group-participation, which is satisfied by both *RSD* and *PS*.

Theorem 6.10

BM satisfies very strong participation.

Proof. Let $(N, O, \succsim)$ be an assignment problem with $N = \{1, \ldots, n\}$, $O = \{o_1, \ldots, o_n\}$, $i \in N$, and $p = BM(\succsim)$. Without loss of generality, we may assume that i has preferences $\succ_i\colon o_1, \ldots, o_n$. First, assume that i is the only agent who ranks o_1 at the top. In this case, we have that $p(i, o_1) = 1$ and very strong participation is satisfied as i gets the best possible result when participating.

105 Following these results some recent papers analyze manipulations of school choice mechanisms in general and *BM* in particular based on real-world data (see, e.g., Burgess et al., 2015; Dur et al., 2018).

Now, suppose there exists another agent who also lists o_1 as top choice. We have that $p_{-i}(i, o_1) = 0$ and $p(i, o_1) > 0$. In addition, for all agents $i' \in N_{-i}$ and $k \in [k_{i'}]$, we have that

$$p_{-i}(i', O^k_{i'}) \geqslant p(i', O^k_{i'}).$$

This holds true as reduced competition cannot 'harm' the remaining agents.

Going back to the abstaining agent i, we compare $p(i, o_j)$ to $p_{-i}(i, o_j)$ for $2 \leqslant j \leqslant k_i$. We have that if $p_{-i}(i, o_j) > p(i, o_j)$ for some j, then

$$p_{-i}(N_{-i}, o_j) < p(N_{-i}, o_j).$$

By the observation above, it however holds that

$$p_{-i}(i', O^k_{i'}) \geqslant p(i', O^k_{i'})$$

for all $i' \neq i$ and $k \in [k_{i'}]$, which means that

$$p(N_{-i}, O^{j-1}_i) < p_{-i}(N_{-i}, O^{j-1}_i),$$

where

$$p_{-i}(i, o_j) - p(i, o_j) \leqslant p_{-i}(N_{-i}, O^{j-1}_i) - p(N_{-i}, O^{j-1}_i).$$

Hence,

$$p_{-i}(i, O^k_i) \leqslant p(i, O^k_i)$$

for all $k \in [k_i]$ and

$$p_{-i}(i, o_1) = 0 < p(i, o_1)$$

by the initial assumption.

Put less formally, even though i's probability for some object o_j can increase, his maximum gain in probability is capped by the sum of probabilities he has lost for objects $\{o_1, \dots, o_{j-1}\}$. Together with the fact that by abstaining, i loses all probability for o_1, this shows very strong participation. □

THEOREM 6.11
BM does not satisfy strong group-participation.

Proof. Consider the following assignment problem $(N, O, \succsim)$ with agents $N = \{1, 2, 3, 4\}$, objects $O = \{a, b, c, d\}$, and $\succsim$ as given below and the corresponding random assignment $p = BM(\succsim)$.

$$\succsim = \begin{array}{ll} 1: & a, b, c, d \\ 2: & a, b, c, d \\ 3: & b, a, c, d \\ 4: & d, a, b, c \end{array} \qquad p = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If agents 1, 2, and 3 abstain, i.e., $S = \{1,2,3\}$, each of them is assigned the uniform lottery over objects a, b, and c, i.e.,

$$p_{-S} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

But $p \not\succsim_i^{SD} p_{-S}$ for $i \in \{1,2\}$. Hence, *BM* violates strong group-participation. □

THEOREM 6.12
BM satisfies group-participation.

Proof. In order to prove group-participation of *BM*, we have to show that for no assignment problem $(N, O, \succsim)$ and $S \subseteq N$ it holds that $p_{-S} \succ_i^{SD} p$ for all $i \in S$ where $p = BM(\succsim)$.

We first consider the case where at least two agents i, i' out of S have disjoint most preferred objects. For reasons of readability, assume i's favorite object is o_1. In this instance, we have that S' total probability for objects top-ranked among S cannot increase when S abstains, i.e., $p_{-S}(S, o) \leqslant p(S, o)$ for all $o \in O_j^1$, $j \in S$. Since it also holds that $p(i, o_1) \geqslant p(j, o_1)$ for all $j \in S$ and

$$p(i, o_1) > p(i', o_1) = 0,$$

we obtain in total

$$\begin{aligned} p_{-S}(i, o_1) &= 1/|S|\, p_{-S}(S, o_1) \\ &\leqslant 1/|S|\, p(S, o_1) \\ &< p(i, o_1). \end{aligned}$$

Consequently, $p_{-S} \not\succ_i^{SD} p$.

Now assume all agents in S have identical first k levels of preferences. If $p_{-S}(S, O_i^k) = |S|$, $i \in S$, then also $p(S, O_i^k) = |S|$ and $p \succsim_i p_{-S}$ for all $i \in S$. If on the other hand $p_{-S}(S, O_i^k) < |S|$, $i \in S$, then either $p(S, O_i^k) = |S|$ and consequently $p \succsim_i^{SD} p_{-S}$ for all $i \in S$ or

$$p(i, O_i^{k+1}) > p_{-S}(i, O_i^{k+1})$$

for similar reasons as above, and thus $p_{-S} \not\succ_i^{SD} p$. This completes the proof. □

6.2.4 Popular Random Assignments

We finally consider a class of random assignment rules that is based on the notion of popularity. Popularity is first considered in the context of deterministic assignments by Gärdenfors (1975).[106] An assign-

106 An increasing amount of attention by researchers led to a variety of papers dealing with popularity in the past decade (see, e.g., Abraham et al., 2007; Kavitha and Nasre, 2009; Biró et al., 2010; McDermid and Irving, 2011; Cseh et al., 2015; Brandl and Kavitha, 2018), see also Cseh (2017) for a detailed overview and more explanations.

ment is *popular* if there exists no other assignment that is preferred by a majority of those agents that are not indifferent in between both assignments. Popular assignments correspond to weak Condorcet winners in social choice theory and unfortunately do not have to exist. *popular assignment*

The problem of potential nonexistence is addressed by Kavitha et al. (2011), who introduce popular *random* assignments. A random assignment is *popular* if there does not exist another random assignment that is preferred by an *expected* majority of agents. Formally, we first define a function ϕ_i for every agent $i \in N$, $\phi_i\colon O \times O \to \mathbb{R}$, *popular random assignment*

$$\phi_i(o, o') = \begin{cases} 1 & \text{if } o \succ_i o', \\ -1 & \text{if } o' \succ_i o, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Making use of this notation and assuming that $O = \{o_1, o_2, \dots\}$, a random assignment p is popular if

$$\sum_{i \in [n]} \sum_{j,j' \in [n]} p(i, o_j)\, p'(i, o_{j'})\, \phi_i(o_j, o_{j'}) \geqslant 0 \text{ for all } p' \in \Delta(A).$$

A *popular random assignment rule* is a rule that always returns some popular random assignment. *popular random assignment rule*

Example 6.13

Consider the assignment problem $(N, O, \succsim)$ with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, and $\succsim$ as given below.

$$\succsim \; = \begin{array}{ll} 1\colon & a, b, c \\ 2\colon & a, c, b \\ 3\colon & b, a, c \end{array} \qquad p_\lambda = \begin{pmatrix} \lambda & 0 & 1-\lambda \\ 1-\lambda & 0 & \lambda \\ 0 & 1 & 0 \end{pmatrix}$$

This assignment problem allows for two popular assignments, p_0 and p_1, i.e., p_λ as given above with $\lambda \in \{0, 1\}$. In addition, we have infinitely many popular random assignments, namely p_λ, $0 \leqslant \lambda \leqslant 1$.

In contrast to *RSD*, whose outcome is #P-complete to compute (Aziz et al., 2013b), popular random assignments can be found efficiently via linear programming (Kavitha et al., 2011). However, even though finding popular random assignment is computationally easy, there unfortunately is no intuitive procedure that makes finding them simple for humans. Since popular random assignments need not be unique, this is especially true if one is tasked to find *all* of them.[107]

The axiomatic study of popular random assignment rules was initiated by Aziz et al. (2013c), who show that all popular random assignment rules satisfy *SD*-efficiency and there always exists at least one

107 Brandt et al. (2017b) study the uniqueness of popular random assignments and give some conditions under which either a unique or infinitely many popular random assignments exist.

popular random assignment satisfying equal treatment of equals.[108] On the other hand, popularity is incompatible with envy-freeness and strong *SD*-strategyproofness if $n \geqslant 3$—impossibilities that are strengthened to weak envy-freeness and *SD*-strategyproofness for $n \geqslant 5$ and $n \geqslant 7$, respectively, by Brandt et al. (2017b).

Aziz et al. (2013c) also point out that popular random assignment rules are a special case of probabilistic voting rules returning *maximal lotteries* (see Section 5.1.2 and Section 5.3.4). To this effect, recall Theorem 5.14, which states that every probabilistic voting rule returning maximal lotteries satisfies *PC*-group-participation. We therefore directly obtain the following statement.

Corollary 6.14

All popular random assignment rules satisfy group-participation.

Proof. The statement follows directly from Theorem 5.14 and the observation that assignment is a special case of voting (see, e.g., Aziz et al., 2013c). □

Again, this result stands in contrast to results about strategyproofness because popular random assignment rules are manipulable (Aziz et al., 2013c; Brandt et al., 2017b).

For the remainder of this section, we make the reasonable assumption that popular random assignment rules assign the same lottery to all abstaining agents, i.e., to all agents that are indifferent between all objects. It turns out that the strongest notions of participation and group-participation we consider are not satisfied by popular random assignments.

Theorem 6.15

All popular random assignment rules violate very strong participation and strong group-participation.

Proof. We start with very strong participation. To this end, let $(N, O, \succsim)$ be an assignment problem with $N = \{1,2,3\}$, $O = \{a,b,c\}$, and $\succsim$ as depicted below.

$$\succsim = \begin{array}{ll} 1: & a, c, b \\ 2: & a, b, c \\ 3: & a, b, c \end{array} \qquad p = \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 1-\lambda & 0 \\ 1-\lambda & \lambda & 0 \end{pmatrix}$$

For this assignment problem, all popular random assignments are of the form p with $0 \leqslant \lambda \leqslant 1$. Hence, $p(1,c) = p_{-1}(1,c) = 1$ even though $c \notin O_1^1$, which violates very strong participation.

108 Popular random assignments even satisfy the stronger notion of *PC*-efficiency.

	very strong participation		strong group-participation		group-participation	
RSD	✓	(Cor. 6.3)	✓	(Thm. 6.4)	✓	
PS	✓	(Thm. 6.7)	✓	(Thm. 6.8)	✓	
BM	✓	(Thm. 6.10)	–	(Thm. 6.11)	✓	(Thm. 6.12)
PRA	–	(Thm. 6.15)	–	(Thm. 6.15)	✓	(Cor. 6.14)

Table 6.1: Overview of results; by definition, strong group-participation implies group-participation, i.e., a checkmark in the second column implies a checkmark in the third column; *PRA* stands for rules returning popular random assignments.

For strong group-participation, consider the assignment problem $(N', O', \succsim')$ with $N' = \{1, 2, 3, 4\}$, $O' = \{a, b, c, d\}$ and $\succsim'$ as depicted below.

$$\succsim' = \begin{array}{ll} 1: & a, b, c, d \\ 2: & a, b, c, d \\ 3: & b, a, c, d \\ 4: & d, a, b, c \end{array}$$

All popular random assignments for this assignment problem are of the form p' with $0 \leqslant \lambda \leqslant 1$. Now, if $S = \{1, 2, 3\}$ abstains, only agent 4 remains and p'_{-S}, as given below, is the unique popular random assignment.

$$p' = \begin{pmatrix} \lambda & 0 & 1-\lambda & 0 \\ 1-\lambda & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad p'_{-S} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For at least one agent $i \in \{1, 2\}$, we have that

$$p'(i, O_i^2) \leqslant 1/2 < 2/3 = p'_{-S}(i, O_i^2)$$

contradicting strong group-participation. □

6.3 CONCLUSION

We studied well-known random assignment rules under the assumption that participation is optional. In the assignment setting, our main concern are not agents who deliberately abstain to improve their assignment, mostly because this requires the agents to be very well-informed about the others' preferences. Rather, we think of situations where participation is associated with a small effort or cost, e.g., for figuring out one's own preferences. Our positive results show that participation is encouraged because it can only lead to more utility (sometimes even strictly). Participation is also desirable from the planner's perspective because it is required to identify efficient assignments of the objects.

Our results, which are summarized in Table 6.1, show that all considered rules satisfy a weak notion of participation (even for groups of agents). Perhaps surprisingly, *RSD* and *PS* even satisfy a strong notion of group-participation that is prohibitive in the more general voting domain (Theorem 5.11).[109] Moreover, all considered rules except popular random assignment rules even provide strict incentives to participate. Whether popular random assignments satisfy strong participation remains an interesting, but presumably challenging, open problem.

Remark 6.16

It is possible to define a more general property, the satisfaction of which by a random assignment rule is sufficient to imply very strong participation. To this end, say that a random assignment rule satisfies this monotonicity-like property if, when an indifference class in $\succsim_i$ is split into two with everything else remaining unchanged, the total probability awarded to i for the new lower indifference class as well as probabilities for all less preferred indifference classes may not increase. This alone already implies strong participation. If we additionally demand that—whenever possible—strictly more probability is given to the top indifference class of $\succsim_i$ compared to when the upper two indifference classes of $\succsim_i$ were merged, we obtain very strong participation.

The proof showing the link to strong participation works via first assuming i abstains, i.e., is completely indifferent, and then building $\succsim_i$ bottom-up. Thus, we let the least preferred objects form a new bottom indifference class and proceed by repeating this step until we end up with $\succsim_i$. By transitivity of the *SD*-extension, we directly obtain strong participation; very strong participation follows from the above-mentioned requirement.

First, note that *RSD* and *PS* can be shown to satisfy this monotonicity-like property as well as the additional demand for the top indifference class, i.e., we could also have shown very strong participation this way. Additionally, we see that both conditions seem quite natural and in particular the first one has the alluring characteristic that it incentivizes agents to specify their preferences in more detail. We would expect all reasonable random assignment rules that focus on individual preferences instead of (weighted) majority comparisons and assign objects in a 'continuous' way to satisfy both conditions. This highlights that in the context of random assignment, pro-

109 More precisely, strong group-participation is prohibitive when we additionally require a very weak natural notion of efficiency. On its own, strong group-participation is satisfied by, e.g., any constant rule.

viding a strict incentive for participating is way more common than in the domain of (probabilistic) voting. In a reverse conclusion, we might deduce that a failure of (very) strong participation is more severe for random assignment rules compared to probabilistic voting rules—a possible argument against the practical application of any such rule.

For the sake of completeness, we remark that our monotonicity-like property informally introduced above heavily relies on agents' preferences to be allowed to contain indifferences. Hence, while we could have used it in order to show very strong participation of *RSD* and *PS*, it does not help when arguing about *BM* or similar rules that require strict preferences.

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