Some Results on Parametric Reduction of Port-Hamiltonian Systems

within the DFG-ANR-Project INFIDHEM: Interconnected Infinite-Dimensional Systems for Heterogeneous Media

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INFIDHEM
(Interconnected INfinite-Dimensional systems for Heterogeneous Media)

- French-German project with Universities of Besançon (Prof. Le Gorrec), Toulouse (Prof. Matignon), Lyon (Prof. Maschke), Wuppertal (Prof. Jacob), Kiel (Prof. Meurer), Munich (Prof. Lohmann)
- Period: 2017-2020, DFG-ANR funded

- Fluid-thermo-structure interaction on heterogeneous media
- Active material (e.g. Piezo)
- Modelling as port-Hamiltonian system

Metallic foam as an example for heterogeneous media¹

¹ Tomography data from LAGEP, Université Claude Bernard Lyon 1 processed with iMorph (http://www.imorph.fr/)
Port-Hamiltonian-System:

\[ \dot{x} = (J - R) \nabla H(x) + Bu \]
\[ y = B^T \nabla H(x) \]

is passive, since \( \dot{H} \leq y^T u \) with pos. def. Energy Function \( H(x) \).

Linear PH-System (with \( H = \frac{1}{2} x^T Q x \))

a) in standard form:

\[ \dot{x} = (J - R) Q x + Bu \]
\[ y = B^T Q x \]

b) in co-energie form (by Trf. \( Q x = e \)):

\[ \dot{e} = (J - R) e + Bu \]
\[ y = B^T e \]

\( J \): Interconnection Matrix, skew symmetric
\( R \): Dissipation Matrix, pos. semidef.
Starting Point: Projective reduction of linear state-space models:

\[
\begin{align*}
E\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

\[
\begin{align*}
W^T EV \dot{x}_r &= W^T AV x_r + W^T Bu \\
y &= CV x_r
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= (J - R)Q x + Bu \\
\dot{z} &= V^{-1}(J - R)V^{-T}V^T QVz + V^{-1}Bu \\
y &= B^T QVz
\end{align*}
\]

Outline:

How to choose \( V, W \) so that:

- *Rational Interpolation / Moment Matching is achieved*,
- *PH structure is preserved*

How to perform parametric reduction by *Matrix Interpolation*

Application to a discretized model of the two-dimensional wave-equations
**Notation**

<table>
<thead>
<tr>
<th>Matrix</th>
<th>State vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(p)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$A_i = A(p_i)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$A_{r,i}$</td>
<td>$x_{r,i}$</td>
</tr>
<tr>
<td>$A^*_{r,i}$</td>
<td>$x^<em>_{r,i} = x^</em>_r$</td>
</tr>
<tr>
<td>$A^*_r(p)$</td>
<td>$x^<em>_{r,i} = x^</em>_r$</td>
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Structure preserving reduction of the co-energy form [1]

A projection matrix $V$ is calculated by some known method (Moment Matching, POD). Then, with the choice $W = V$ we find the reduced model in co-energy form (descriptor form)

$$V^T Q^{-1} \dot{e}_r = V^T (J - R) V e_r + V^T B u$$

$$y = \begin{bmatrix} B^T V e_r \\ B_r^T \end{bmatrix}$$

where $Q_r$ is pos. def., $J_r$ skew symmetric, $R_r$ pos. semidef.
Structure preserving reduction of the standard PH form [2, 3] (DesPH)

A projection matrix $V$ is calculated by some known method (Moment Matching, POD). Then, with the choice $W = QV$ we first find

$$V^T QV \dot{x}_r = V^T Q(J - R)QV x_r + V^T QBu$$
$$y = B^T QV x_r$$

Applying the state transformation $z_r = V^T QVx_r$ leads to the reduced model in standard PH form

$$\dot{z}_r = V^T Q(J - R)QV (V^T QV)^{-1} z_r + V^T QBu$$
$$y = B^T QV (V^T QV)^{-1} z_r$$

However, because of the state transformation, the vector $z_r$ has the physical meaning of a co-state vector $e_r = V^T QVx_r$. 


**Structure preserving reduction of the standard form (consistent state vector) [4]**

A projection matrix $V$ is calculated by some known method (Moment Matching, POD). Then, with the choice $W = QV$ we first find

\[
V^T QV \dot{x}_r = V^T Q (J - R) Q V x_r + V^T Q Bu
\]

\[y = B^T Q V x_r\]

By pre-multiplication with $(V^T Q V)^{-1}$ and by inserting the identity matrix $(V^T Q V)^{-1} (V^T Q V)$, this becomes a reduced model in standard form

\[
\dot{x}_r = (V^T Q V)^{-1} V^T Q (J - R) Q V (V^T Q V)^{-1} (V^T Q V) x_r + (V^T Q V)^{-1} V^T Q Bu
\]

\[y = B^T Q V (V^T Q V)^{-1} (V^T Q V) x_r\]

This model is in *standard form and physically consistent* (By a different derivation, it was first suggested in [4], another version can be found in [7]).
Structure-preserving reduction of the standard PH-form using Cholesky-factorization [7]

The linear system is reduced with so far unknown matrices $V$ and $W^T$. Additionally we perform the approximation $Q \approx WV^TQ$ and demand $V$ and $W^T$ to be biorthogonal ($W^TV = I$).

To achieve biorthogonality, we use Cholesky-factorization to generate $V$ and $W^T$ from a $\tilde{V}$ resulting from an arbitrary reduction method:

$$\tilde{V}^TQ\tilde{V} = R^TR$$

The reduction matrices are finally computed as:

$$V = \tilde{V}R^{-1} \quad W = QV$$
**Parametric Reduction of the co-energy form [1]**

The matrices $J, R, Q^{-1}, B$ depend on a parameter $p$. We now specify some values $p_i$ and reduce the corresponding models with corresponding individual projection matrices $V_i$:

$$V_i^TQ_i^{-1}V_i\dot{e}_{r,i} = V_i^T(J_i - R_i)V_i e_{r,i} + V_i^TB_iu$$
$$y_i = B_i^TV_i e_{r,i}$$

**Matrix Interpolation**: for some value $p$ of interest, we like to find the matrices $J_r(p), R_r(p), Q_r^{-1}(p), B_r(p)$ by *Interpolation* between the matrices $J_{r,i}, R_{r,i}, Q_{r,i}, B_{r,i}$. In order to make this interpolation physically meaningful, we first have to adapt/adjust the state spaces of the local models to each others. This is done by

- **State transformation** of each local model,
  $$x_{r,i}^* = T_i x_{r,i}$$
  with $T_i = U_i^TV_i$ (where $U$ is from an $SVD\{[V_1,\ldots,V_k]\}, [5]$) and

- **Pre-multiplication** of each local model by a matrix
  $$M_i = (V_i^TU)^{-1}$$ (see [5, 1]. Alternatives in [6]):
\[
\begin{align*}
    (V_i^T U)^{-1} V_i^T Q_i^{-1} V_i (U^T V_i)^{-1} \dot{e}_{r,i}^* &= (V_i^T U)^{-1} V_i^T (J_i - R_i) V_i (U^T V_i)^{-1} e_{r,i}^* + (V_i^T U)^{-1} V_i^T B_i u \\
y_i &= B_i^T V_i (U^T V_i)^{-1} e_{r,i}^*
\end{align*}
\]

Matrix Interpolation:
\[
\begin{align*}
    J_{r_i}(p) &= \sum_i \omega_i(p) J_{r_i} \\
    R_{r_i}(p) &= \sum_i \omega_i(p) R_{r_i} \\
    Q_{r_i}^{-1}(p) &= \sum_i \omega_i(p) Q_{r_i}^{-1} \\
    B_{r_i}(p) &= \sum_i \omega_i(p) B_{r_i}
\end{align*}
\]

with \( \omega_i \geq 0, \quad \sum_i \omega_i = 1 \)

Reduced parametric model in co-energy form:
\[
\begin{align*}
    Q_{r_i}^{-1}(p) \dot{e}_r &= (J_{r_i}(p) - R_{p_i}(p)) e_r + B_{r_i}(p) u \\
y &= B_{r_i}^T(p) e_r
\end{align*}
\]
Parametric Reduction of the standard PH form (new)

a) Interpolate descriptor form, then convert to state-space form (Int/PH)

The locally reduced models (*) in descriptor form are

\[ V_i^T Q_i V_i \dot{x}_{r,i} = V_i^T Q_i (J_{r,i} - R_{r,i}) Q_i V_i x_{r,i} + V_i^T Q_i B_i u \]
\[ y_i = B_i^T Q_i V_i x_{r,i} \]

Preparation of Interpolation by state transformation \( T_i = U^T V_i \) and pre-multiplier \( M_i = (V_i^T U)^{-1} \):

\[ \frac{Q_{r,i}}{(V_i^T U)^{-1} V_i^T Q_i V_i (U^T V_i)^{-1}} \dot{x}_{r,i}^* = \frac{J_{r,i} - R_{r,i}}{(V_i^T U)^{-1} V_i^T Q_i (J_{r,i} - R_{r,i}) Q_i V_i (U^T V_i)^{-1}} x_{r,i}^* + \frac{B_{r,i}}{(V_i^T U)^{-1} V_i^T Q_i B_i} u \]
\[ y_i = B_i^T Q_i V_i (U^T V_i)^{-1} x_{r,i}^* \]

Matrix interpolation leads to:

\[ \tilde{J}_r(p) = \sum_i \omega_i(p) J_{r,i} , \quad \tilde{R}_r(p) = \sum_i \omega_i(p) R_{r,i} , \]
\[ \tilde{Q}_r(p) = \sum_i \omega_i(p) Q_{r,i} , \quad \tilde{B}_r(p) = \sum_i \omega_i(p) B_{r,i} \]
and the reduced model in descriptor form is

\[
Q_r(p)\dot{x}_r^* = (\tilde{J}_r(p) - \tilde{R}_r(p))x_r^* + \tilde{B}_r(p)u
\]

\[
y = \tilde{B}_r^T(p)x_r^*
\]

By pre-multiplication with \(Q_r^{-1}\) and by inserting the identity matrix \(Q_r^{-1}Q_r\) we get the parametric model in standard form:

\[
\dot{x}_r^* = Q_r^{-1}(p)(\tilde{J}_r(p) - \tilde{R}_r(p)Q_r^{-1}(p)Q_r(p)x_r^* + Q_r^{-1}(p)\tilde{B}_r(p)u
\]

\[
y = \tilde{B}_r^T(p)Q_r^{-1}(p)Q_r(p)x_r^*
\]
Parametric reduction of the standard PH form (new)

b) Convert to state-space form, then interpolate (PH/Int)

The locally reduced models (** in standard form are

\[
\dot{x}_{r,i} = (J_{r,i} - R_{r,i})Q_{r,i}x_{r,i} + B_{r,i}u
\]
\[
y_{r,i} = B_{r,i}^TQ_{r,i}x_{r,i}
\]

Preparation of the interpolation by state transformation \( T_i = U^TV_i \) and by inserting an identity matrix:

\[
\dot{x}^*_r = (U^TV_i)(J_{r,i} - R_{r,i})(U^TV_i)^T(U^TV_i)^{-T}Q_i(U^TV_i)^{-1}x^*_r + (U^TV_i)B_{r,i}^*u
\]
\[
y_i = B_{r,i}^T(U^TV_i)^T(U^TV_i)^{-T}Q_i(U^TV_i)^{-1}x^*_r
\]

Matrix Interpolation determines the matrices of the reduced model in standard form:

\[
J_r^*(p) = \sum_i \omega_i(p)J_{r,i}^* , \quad R_r^*(p) = \sum_i \omega_i(p)R_{r,i}^* ,
\]
\[
Q_r^*(p) = \sum_i \omega_i(p)Q_{r,i}^* , \quad B_r^*(p) = \sum_i \omega_i(p)B_{r,i}^*
\]
Example: discretized linear wave equation [8]

\[
\frac{\partial u}{\partial t} = -\text{div} \, \mathbf{v} \\
\frac{\partial \mathbf{v}}{\partial t} = -\text{grad} \, u - c \mathbf{v}
\]

- Discretizing PDE on dual complexes leads to a Port-Hamiltonian system.
- Physical values are connected to their geometrical domain.
- The discretized equations are an exact representation of the PDE (in each geometric element)
Reduction

Full Order: 682

Reduced Order: 50

Parameter: $l_x$

$l_{x_1} = 50$

$l_{x_2} = 60$

$w_1 = w_2 = 0.5$
Interpolation
Comparison

Bode Diagram

- DesPH
- Int/PH
- PH/Int
- Comparison ROM

Magnitude (dB)

Phase (deg)

Frequency (rad/s)
Outlook: Reduction of non-linear PH-systems [7]

The PH system with nonlinear energy gradient

\[
\begin{align*}
\dot{x} &= (J - R)\nabla H(x) + Bu \\
y &= B^T \nabla H(x)
\end{align*}
\]

Is reduced with \(V\) and \(W^T\). Additional approximation:

\[
\frac{\partial}{\partial x} H(Vx_r) \approx W \frac{\partial}{\partial x_r} H_r(x_r)
\]

with \(H_r(x_r) = H(Vx_r)\) and \(W^T V = I\) (biorthogonal).

\[
W^T V \dot{x} = W^T (J - R) W \nabla H_r(x_r) + W^T Bu \\
y = B^T W \nabla H_r(x_r)
\]

The reduction matrices are generated by Snapshots and orthogonalized:

\[
V = snap[x_1 \ldots x_k] \\
W = snap[\nabla H(x_1) \ldots \nabla H(x_k)]
\]
Literature:


