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Value of Information Analysis in Feedback Control

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Zusammenfassung

Diese Doktorarbeit beschäftigt sich mit dem Informationswert in der Regelung von cyber-physischen Systemen. Ein geregeltes cyber-physisches System ist ein Rückmeldungssystem in dem die Messwerte über eine Rückkopplung zu einem Regler kommuniziert werden, welcher das Systemverhalten in gewünschter Weise beeinflusst. Die Reglergüte in solch einem System hängt von der verfügbaren Information am Regler ab. Wie jedoch die Reglergüte von der Qualität der Information abhängt und wie die Information selbst gesammelt werden soll, wurde bisher noch nicht ausreichend und systematisch erforscht. Dies dient als Motivation in dieser Arbeit eine Informationswertanalyse in Bezug auf die Auswirkung von Unsicherheiten in einem Rückmeldungssystem durchzuführen. Unsicherheiten und limitierte Ressourcen sind zwei wichtige Faktoren in cyber-physischen Systemen. Ressourcenlimitierungen werden von verschiedenen Limitierungen in Kommunikation, Rechenleistung und Energie hervor-gerufen, während Unsicherheiten von fehlerhaften und nicht vollständigen Messwerten, Modellierungsfehlern und Störungen im System verursacht werden. Daher wird in dieser Thesis ein theoretischer Rahmen entwickelt um drei Probleme in der Regelung von cyber-physischen Systemen zu behandeln, welche auf den ersten Blick unterschiedlich sind, aber, wie gezeigt wird, starke Gemeinsamkeiten aufweisen. Für jedes Problem wird das Rahmenwerk in Form eines LQG-Reglers präsentiert und versucht die Reglergüte unter einer Limitierung in dem Informationsfluss des Systems zu maximieren. Diese Limitierung ist entweder als eine Beschränkung in der Kommunikationsrate, der verfügbaren Energie oder in dem Datenschutzniveau ausgedrückt. Jede Problemstellung wird als dynamisches Spiel mit zwei verteilten Spielern modelliert, welche versuchen das stochastische, nicht vollständig beobachtbare Regelungssystem mit einer Rückkoppelungsstrategie zu beeinflussen. In diesem Spiel nimmt der Regler die Rolle eines Spielers ein, welcher nicht über vollständige Kenntnisse und alle Informationen des Systems verfügt. Der zweite Spieler ist ein Mechanismus, der die Unsicherheiten des Reglers imitiert. Unter gewissen Annahmen kann man die optimalen Strategien jedes Spielers in diesen Problemen charakterisieren und zeigen, dass jedes Strategieprofil einem Nash-Gleichgewicht entspricht. Des Weiteren kann gezeigt werden, dass sich diese optimalen Strategien separieren lassen.

Schlüsselwörter. Kommunikation, gerichtete Information, Energie, ereignisbasierte Abtastung, fading channels, LQG-Regelung, Nash-Gleichgewicht, Optimale Strategien, nicht vollständig beobachtbare Systeme, Privatsphäre, Informationswert.

Abstract

This thesis is concerned with the value of information in control of cyber-physical systems. At its simplest, a controlled cyber-physical system is a feedback system in which sensory information is communicated via a feedback loop to a controller that seeks to modify the behavior of the system in a desirable way. Indeed, the control performance in such a system depends on the information that is available to the controller. However, how the control performance changes with respect to the quality of information and how the information itself should be collected have not yet been studied thoroughly and systematically. This motivates us in this thesis to carry out a value of information analysis by considering the impact of uncertainty on a feedback system. Two inevitable issues that arise in cyber-physical systems are resource constraints and uncertainties. Resource constraints are caused by various limitations in communication, computation, and energy; and uncertainties are due to incomplete or imperfect observations, unknown parameters in the mathematical model, and disturbances that affect the system. In this thesis, we develop a unified theoretical framework for the study of three problems, which seem to be different but as we show in fact share substantial similarities, in control of cyber-physical systems. In each problem, we present our framework in the context of linear quadratic Gaussian control, and seek to maximize the control performance subject to a constraint that restricts the information flow in the system. This constraint is expressed by either a sampling rate, an energy capacity, or a privacy level. We formulate each problem as a dynamic game with two distributed decision makers that use closed-loop policies to influence the underlying stochastic dynamical system that is partially observable. In this game, one decision maker is the controller that does not possess all possible knowledge and information related to the system, and the other one is a mechanism that controls the uncertainty of the controller. We characterize, under certain assumptions, the optimal policies of the decision makers in each of these problems such that the corresponding policy profile represents a Nash equilibrium. We prove that a separation between the optimal policies is achievable.

Keywords. communication, directed information, energy, event-triggered sampling, fading channels, linear quadratic Gaussian control, Nash equilibrium, optimal policies, partially observable systems, privacy, value of information.

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CHAPTER 1

Introduction

This thesis is concerned with the *value of information* in control of cyber-physical systems. Cyber-physical systems are complex systems that integrate communication, computation, and control in a way that enable us to efficiently observe and influence the physical world [**1**]. At its simplest, a controlled cyber-physical system is a feedback system in which sensory information is communicated via a feedback loop to a controller that seeks to modify the behavior of the system in a desirable way. Indeed, the control performance in such a system depends on the information that is available to the controller. However, how the control performance changes with respect to the quality of information and how the information itself should be collected have not yet been studied thoroughly and systematically. This motivates us in this thesis to carry out a value of information analysis by considering the impact of uncertainty on a feedback system. The concept of value of information has widely been used in multiple disciplines including information economics [**2**], risk management [**3**], and stochastic programming [**4**]. Generally speaking, value of information is defined as the value that is assigned to the reduction of uncertainty from the decision maker's perspective given a piece of information [**5**]. Besides, a value of information analysis identifies the optimal information collection strategy among a set of alternatives that maximizes the decision maker's interest [**6**]. We believe that the value of information is a building block for construction of a theory that systematically integrates information/communication theory with control theory.

Two inevitable issues that arise in cyber-physical systems are *resource constraints* and *uncertainties*. Resource constraints are caused by various limitations in communication, computation, and energy; and uncertainties are due to incomplete or imperfect observations, unknown parameters in the mathematical model, and disturbances that affect the system. Clearly, in control under resource constraints one has to make a trade-off between competing objectives, and in control under uncertainty one needs to take into account different possibilities in the system's behavior. In this thesis, we develop a unified theoretical framework for the study of three problems, which seem to be different but as we show in fact share substantial similarities, in control of cyber-physical systems. In each problem, we present our framework in the context of linear quadratic Gaussian control, and seek to maximize the control performance subject to a constraint that restricts the *information flow* in the system (i.e., in the communication channel that is placed between the process and controller). This constraint is expressed by either a sampling rate, an energy capacity, or a privacy level. We formulate each problem as a dynamic game with two distributed decision makers that use *closed-loop policies* to influence the underlying stochastic dynamical system that is *partially observable* (i.e., only imperfect information is available to the decision makers). In this game, one decision maker is the controller that does not possess all possible knowledge and information related to the system, and the other one is a mechanism that controls the uncertainty of the controller. In particular, in the first problem where there is a constraint on the sampling rate this mechanism is an *event trigger* in which the decision variable is the binary event $\delta_k \in \{0,1\}$ representing whether an observation is transmitted or not. In the second problem where there is a constraint on the energy capacity, the mechanism is a *power scheduler* in which the decision variable is the packet success rate $\mathsf{PSR}_k \in (0,1)$ representing the probability for successful transmission of an observation. Finally, in the third problem where there is a constraint on the privacy level the mechanism is a *privacy filter* in which the decision variable is the perturbation covariance $N_k \succeq 0$ representing the uncertainty for obfuscation of an observation. In the rest of this chapter, we describe the motivations, review the related work, and specify our contributions for each of these three problems.

1.1. Event Triggering Mechanisms

Ubiquitous communication networks are essential for control of cyber-physical systems [**7**]. Indeed, a major issue in any communication networks is congestion, which must be mitigated. Traditionally, in a networked control system, observations of the process are periodically sampled and transmitted to the controller because this facilitates the design of such a system [**8**]. However, it has been conceived that not every sampled observation of the process has the same effect on the performance of a networked control system, and one can employ a mechanism, i.e., event trigger, that transmits an observation only when a significant event occurs [**9**]. As a result, a reduction in the sampling rate, which is an appropriate index for packet switching networks, in event-triggered control can be expected. In the first part of this thesis, we seek to jointly design an event trigger and a controller in a networked control system by constructing a trade-off between the sampling rate and control performance.

1.1.1. Related Work. In a seminal work, Åström and Berhardsson [**9**] showed for a first-order continuous-time stochastic process under a sampling rate constraint that event-triggered sampling outperforms periodic sampling in the sense of mean error variance. This work fostered extensive research in event-triggered control. In a networked control system an event trigger can be employed at the sensor side to reduce the sampling rate in the observation channel, or at the controller side to reduce the sampling rate in the control channel. We should point out that what we are interested in here is the former in which the controller and event trigger are distributed. In the former, one primarily deals with an optimization problem with an event trigger and a controller as the decision makers. Yet, this problem for the joint design of the event trigger and controller is in general intractable (see e.g., [**10**,**11**]). The reasons are that the optimal estimator at the controller is nonlinear with no analytical solution, estimation and control are coupled due to a dual effect, and the event trigger and controller have nonclassical information patterns. Nevertheless, one can still characterize the solutions of this problem under certain conditions. Herein, to elucidate the essence of this problem, we neglect network-induced effects such as quantization, packet dropouts, and time-varying delays.

Several works have addressed optimal event-triggered estimation, and characterized the optimal triggering policies [**12**–**15**]. In particular, Xu and Hespanha [**12**] studied optimal event-triggered estimation with perfect information by discarding the *negative information* (i.e., information associated with non-transmitted observations). They searched in the space of stochastic triggering policies, and showed that the optimal triggering policy is indeed deterministic. Rabi and Baras [**13**] formulated optimal event-triggered estimation with perfect information as an optimal multiple stopping time problem by discarding the negative information, and showed that the optimal triggering policy for first-order systems is symmetric. Later, Lipsa and Martins [**14**] used majorization theory to study optimal event-triggered estimation with perfect information without discarding the negative information, and proved for first-order systems that the optimal estimator is linear and the optimal triggering policy is symmetric. Moreover, Molin and Hirche [**15**] developed an iterative algorithm for obtaining the optimal estimator and optimal triggering policy in optimal event-triggered estimation with perfect information that is applicable to systems with arbitrary noise distributions. They studied the convergence properties of the algorithm for first-order systems, and obtained a result that coincides with that in [**14**]. As explained before, in the joint design of the event trigger and controller a separation between estimation and control is not given a priori. Therefore, the results in the aforementioned studies do not apply directly to optimal event-triggered control. However, there exist a number of studies that have addressed optimal event-triggered control, and characterized the optimal control policies [**11**, **16**–**18**]. In particular, Molin and Hirche [**16**, **17**] investigated optimal event-triggered control with perfect and imperfect information, and showed that the optimal control policy is a certainty-equivalence policy while assuming that the triggering policy is a function of primitive variables. Ramesh *et al.* [**11**] studied dual effect in optimal event-triggered control with perfect information, and proved that the dual effect in general exists. In addition, they showed that the certainty-equivalence principle holds if and only if the triggering policy is independent of the control policy. Recently, Demirel *et al.* [**18**] addressed optimal event-triggered control with imperfect information by adopting a stochastic triggering policy that is independent of the control policy, and proved that the optimal control policy is a certainty-equivalence policy. Unlike these studies, in addition to scrutinizing the notion of value of information in optimal event-triggered control, we herein characterize both optimal triggering policy and optimal control policy such that the corresponding policy profile represents a Nash equilibrium. Besides, we synthesize a closed-form suboptimal triggering policy with a performance guarantee. We show that our analysis is tractable and also extensible to high-order systems with imperfect information.

A special class of event-triggered estimation and event-triggered control is sensor scheduling in which open-loop triggering policies are employed. Sensor scheduling can be traced back to the 1970s. However, recently Trimpe and D'Andrea [**19**] and Leong *et al.* [**20**] adopted sensor scheduling for networked control systems, and obtained open-loop triggering policies in terms of the estimation error covariances. It is also worth mentioning that in a rather different setup from what we consider in this study, Antunes and Heemels [**21**] considered a networked control system in which the event trigger and controller are both collocated. They proposed an approximation algorithm, and showed that a performance improvement with respect to periodic control can be guaranteed. Our approximation algorithm is inspired by this idea. Nevertheless, herein unlike the above work, the event trigger and controller are distributed.

1.2. Transmission Power Mechanisms

Wireless sensors provide a safe and flexible solution for control of cyber-physical systems [**22**]. In general, wireless sensors possess sensing, data processing, and communicating capabilities. Thanks to their unique characteristics, we can simply integrate wireless sensors into physical systems, and realize unprecedented wireless control systems in which ubiquitous sensors are connected to controllers over wireless networks. Nevertheless, wireless sensors have limited power supplies that must be utilized efficiently. A promising technique in this respect is power control [**23**], in which a mechanism, i.e., power scheduler, adapts the transmit power for transmission of observations such that the energy consumption is reduced. In the second part of this thesis, we seek to jointly design a power scheduler and a controller in a wireless control system by constructing to a trade-off between the energy capacity and control performance.

1.2.1. Related Work. Transmit power is a fundamental degree of freedom in wireless networks, and power control as a technique for adaptation of transmit powers has systematically been studied since the 1970s. A comprehensive survey of the models, algorithms, and methodologies in power control is provided in [**24**]. We should point out that in wireless communication systems the objective of power control can be management of energy, interference, or connectivity. To that end, one make a trade-off between the network performance (e.g., throughput, delay, or capacity) and energy resource, and obtains the optimal transmit power as a function of the state of the channel. Indeed, the transmit power influences the received signal-to-noise power ratio, and subsequently the probability of *packet dropouts* [**25**]. In wireless control systems, where freshness of information is vital and retransmission of dropped packets is not favorable, packet dropouts can have severe effects on the control performance or even yield instability [**26**]. Therefore, in contrast to wireless communication systems in which the transmit power is adapted only to the state of the channel, the transmit power in wireless control systems must be adapted to the state of the channel and of the dynamical system.

Previous research in control community has first recognized the severe effects of packet dropouts on stability. Early works have considered erasure channels with Bernoulli distributions, i.e., *i.i.d. erasure channels* [**26–29**]. In a seminal work, Sinopoli *et al.* [**27**] studied mean-square stability of the Kalman filter over i.i.d. erasure channels, and proved that there exists a *critical point* on the packet success rate below which the expected estimation error covariance is unbounded. Besides, Schenato *et al.* [**26**] addressed optimal control over i.i.d. erasure channels. They proved that there exists a separation between estimation and control when the receipt acknowledgment is available, and showed that below a critical point the optimal controller fails to stabilize the system. Later, serval studies have employed erasure channels modeled by a two-state Markov chain, i.e., *Gilbert-Elliott channels*, to capture the temporal correlation of wireless channels [**30–33**]. In this respect, Wu *et al.* [**30**] studied stability of the Kalman filter over Gilbert-Elliott channels, and proved that there exists a *critical region* defined by both recovery rate and failure rate outside which the expected prediction error covariance is unbounded. In addition, Mo *et al.* [**31**] investigated optimal control over Gilbert-Elliott channels. They proved that the separation principle still holds when the receipt acknowledgment is available, and showed that outside a critical region the optimal controller cannot stabilize the system. Eventually, a number of studies have employed *fading channels* in order to take into account the time variation of wireless channels [**34**–**36**]. In particular, Quevedo *et al.* [**34**] investigated stability of the Kalman filter over fading channels with correlated gains, and established a sufficient condition that ensures the exponential boundedness of the expected estimation error covariance. Moreover, Elia [**35**] addressed the stabilization problem in the robust mean-square stability sense over fading channels by modeling the fading as stochastic model uncertainty, and designed a controller with the largest stability margin.

Recently, several works have adopted power control to reduce energy consumption in wireless sensors in estimation and control tasks [**37**–**41**]. In particular, Leong *et al.* [**37**] studied power control for estimation over fading channels, and derived the optimal power policy that minimizes the probability of outage (i.e. the probability of the event that the estimation error covariance exceeds a given threshold) subject to a constraint on the average total power. Moreover, Quevedo *et al.* [**38**] investigated power control for estimation over fading channels, and characterized the optimal power policy that minimizes the average total power subject to a stability condition ensuring that the expected estimation error covariance is exponentially bounded. In fact, the studies in [**37**–**39**] derive open-loop power policies in terms of the estimation error covariances. However, there are two works that have characterized closed-loop policies [**40**, **41**]. In particular, Ren *et al.* [**40**] studied the joint design of an estimator and a power scheduler for estimation with perfect information over fading channels, and based on the common information approach proved for first-order systems that the optimal estimator is linear and the optimal power policy is deterministic and symmetric. Closely related to our study, Gatsis *et al.* [**41**] addressed the joint design of a power scheduler and a controller for control with perfect information over fading channels by discarding the negative information, and showed that the optimal power policy is deterministic and the optimal control policy is certainty-equivalence. In contrast to the work in [**41**], we herein consider imperfect information, and characterize the optimal power policy and optimal control policy such that the corresponding policy profile represents a Nash equilibrium.

1.3. Privacy Protection Mechanisms

Cloud computing offers a flexible and scalable solution for control of cyber-physical systems [**42**]. Literally, clouds possess unlimited computing and storage resources that are ubiquitously available to the users. Moreover, clouds can have global information, which may not be available locally at each user. In this respect, we define *cloud control* as the use of cloud computing services and infrastructure to perform control tasks in the systems where the users are dynamical agents. In cloud control, the required information for accomplishing a control task has to be shared by each agent with the cloud provider. However, the cloud provider can be *honest-but-curious*, who may exploit the private information of the agents. This raises a number of privacy issues, and necessitate the use of a mechanism, i.e., privacy filter, that protects the private information of each agent according to its privacy preference. In the third part of this thesis, we seek to jointly design a privacy filter and a controller in a cloud control system by constructing a trade-off between the privacy level and control performance.

1.3.1. Related Work. There is a large and growing body of work in the literature on privacy protection, where in particular database privacy from both statistical and data-mining perspectives [**43**], [**44**], location privacy [**45**], and smart-meter privacy [**46**] have received remarkable attention since the 1970s. In privacy protection, one has to make a balance between *private information* (i.e., information that should be hidden) and *public information* (i.e., information that can be revealed) while guaranteeing a satisfactory quality of service, which is basically a function of the public information. Privacy protection can be achieved by various techniques including data encryption (e.g., multi-party computation [**47**] and homomorphic encryption [**48**]) data anonymization (e.g., k-anonymity [**49**] and l-diversity [**50**]) and data perturbation (e.g., differential privacy [**51**]). Data encryption introduces additional computation and latency that is often unacceptable in practice [**52**]. Moreover, it has been shown that data anonymization is insufficient to protect privacy [**53**]. In our study herein, we concentrate on data perturbation, which is readily applicable to control tasks.

It was only recently that researchers in the control community have been attracted to privacy protection in control tasks [**54–58**]. In particular, Cortés *et al.* [**54**] provided an overview of various affine privacy filters with additive Laplace or Gaussian noise that enforce differential privacy for consensus and distributed optimization. Wang *et al.* [**55**] studied affine privacy filters that guarantee differential privacy for control tasks. They sought to minimize the output entropy for linear systems, and demonstrated that there is a lower bound on the entropy that is achieved by an affine privacy filter with additive Laplace noise. Hale *et al.* [**56**] used an affine privacy filter with additive Gaussian noise that enforces differential privacy for control of multi-agent systems. They showed that the optimal control policy is certainty-equivalence, and obtained a bound on the log-determinant of the steady-state estimation error covariance. Jia *et al.* [**57**] proposed mutual information as a privacy measure, and studied privacy protection in partially observable Markov decision processes. They developed an approximation algorithm based on Gibbs sampling for computing the privacy loss. For linear systems, they showed that the optimal control policy is certainty-equivalence if the privacy filter is affine with additive Gaussian noise. Closely related to our work, Tanaka *et al.* [**58**] investigated the use of directed information as a privacy measure in control systems following an axiomatic argument. For linear systems with perfect information, they proved that the optimal (open-loop) privacy filter is an affine policy with additive Gaussian noise whose perturbation covariance is obtained by a linear-matrix-inequality algorithm depending on the estimation error covariances and that the optimal control policy is certainty-equivalence. In contrast to the work in [**58**], we herein consider closed-loop policies, and characterize the optimal perturbation policy and optimal control policy such that the corresponding policy profile represents a Nash equilibrium.

In our work, we measure the leakage of private information by directed information. Directed information was first introduced by Marko [**59**], and then formalized by Massey [**60**] as a natural generalization of mutual information for characterizing *causality* [**61**] in feedback systems. Later, Kramer [**62**] introduced the concept of causal conditioning, and extended the use of directed information to communication networks with feedback. Directed information has been adopted in multiple disciplines for characterizing phenomena in which causality is of interest, including information theory [**63**], neuroscience [**64**], thermodynamics [**65**], portfolio theory [**66**], swarm intelligence [**67**], video processing [**68**], and gambling [**69**]. A fundamental justification of the use of directed information as a loss function for causal systems was provided by Jiao *et al.* [**70**]. They indicated that logarithmic loss functions are the only measures that satisfy the data-processing axiom, and among them directed information is the the unique measure that takes causality into account.

1.4. Thesis Outline and Contributions

The thesis comprises five chapters and one appendix. The present chapter serves the purpose of introducing the content of the thesis, the three intermediate chapters delineate the main results, and the last chapter presents concluding remarks and possible directions for future research. The intermediate chapters can be studied without any particular order. The main contributions of the thesis are highlighted as follows:

In Chapter 2, we present our framework, in its basic form, for a networked control system with imperfect information, and jointly design an event trigger and a controller by constructing a trade-off between the sampling rate and control performance. We obtain the optimal policies such that the corresponding policy profile represents a Nash equilibrium. In particular, we prove, under certain assumptions, that the optimal closed-loop triggering policy is a threshold policy that depends on estimation errors and that the optimal closed-loop control policy is a certainty-equivalence policy. Moreover, we characterize the value of information in the trade-off between the sampling rate and control performance. Finally, we synthesize a closed-form suboptimal triggering policy with a performance guarantee.

In Chapter 3, we extend our framework to a wireless control system with imperfect information by taking into account the state of a wireless fading channel, and jointly design a power scheduler and a controller by constructing a trade-off between the energy capacity and control performance. We introduce the specification of the wireless fading channel, and determine the relation between the required transmit power and packet success rate. We obtain the optimal policies such that the corresponding policy profile represents a Nash equilibrium. In particular, we prove, under certain assumptions, that the optimal closed-loop power policy is a nonlinear policy with packet success rate that depends on estimation errors and the channel gain and that the optimal closed-loop control policy is a certainty-equivalence policy. Moreover, we characterize the value of information in the trade-off between the energy capacity and control performance. Finally, we synthesize a closed-form suboptimal power policy with a performance guarantee.

In Chapter 4, we further extend our framework to a cloud control system with imperfect information by considering the causally-conditioned directed information from the private information to the public information, and jointly design a privacy filter and a controller by constructing a trade-off between the privacy level and control performance. We adopt a perturbation-based privacy filter, and calculate the causally-conditioned directed information for the considered system. We obtain the optimal policies such that the corresponding policy profile represents a Nash equilibrium. In particular, we prove, under certain assumptions, that the optimal closed-loop perturbation policy is an affine policy with additive Gaussian noise whose covariance depends on estimation errors and that the optimal closed-loop control policy is a certainty-equivalence policy. Moreover, we characterize the value of information in the trade-off between the privacy level and control performance. Finally, we synthesize a closed-form suboptimal perturbation policy with a performance guarantee.

1.5. Notations

We collect here for easy reference many of the symbols we use throughout this thesis. Vectors, matrices, and sets are represented by lower case, upper case, and Calligraphic letters like x, X, and $\mathcal{X}$ respectively. The sequence of all vectors $x_t,\ t = 0, \ldots, k$, is represented by $\mathbf{x}_k$, and the sequence of all vectors $x_t,\ t = k, \ldots, N$ for a specific N, is represented by $\mathbf{x}^k$. The indicator function of a subset $\mathcal{A}$ of a set $\mathcal{X}$ is denoted by $f(x) = \mathbb{1}_{\mathcal{A}}$ where $x \in \mathcal{X}$. The identity matrix is denoted by I. For matrices X and Y, the relations $X \succ 0$ and $Y \succeq 0$ denote that X and Y are positive definite and positive semi-definite respectively. The probability distribution of the stochastic variable x is represented by $\mathsf{P}(x)$. The expected value and covariance of x are represented by $\mathsf{E}[x]$ and $\mathsf{cov}[x]$ respectively.

CHAPTER 2

Optimal Event Trigger in Networked Control

In this chapter, we consider a networked control system, and seek to jointly design an event trigger and a controller by constructing a trade-off between the sampling rate and control performance. Based on our unified framework, we develop the optimal estimators and characterize the optimal policies such that the corresponding policy profile represents a Nash equilibrium. Moreover, we define the value of information in the trade-off between the sampling rate and control performance, and show the relation between the value of information and optimal triggering policy. This chapter is organized in the following way. We formulate the problem in Section 2.1. We provide the main results of the chapter in Section 2.2. We present a numerical example in Section 2.3. Finally, we give a summary in Section 2.4.

2.1. Problem Formulation

In this section, we describe the system model and define the indices for the sampling rate and control performance. Then, we formulate the main problem of the chapter. The material here requires basic knowledge of stochastic control theory and game theory.

2.1.1. System Model. Consider a stochastic process with dynamics generated by the following linear discrete-time time-varying state system:

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{2.1}$$

for $0 \leq k \leq N$ and with initial condition x_0 where $x_k \in \mathbb{R}^n$ is the state of the process, $A_k \in \mathbb{R}^{n\times n}$ is the state matrix, $B_k \in \mathbb{R}^{n\times m}$ is the input matrix, $u_k \in \mathbb{R}^m$ is the control input to be decided by a controller, $w_k \in \mathbb{R}^n$ is a white Gaussian noise with zero mean and covariance $W_k \succ 0$, and N is a finite terminal time. It is assumed that x_0 is a Gaussian vector with mean m_0 and covariance M_0. At each time, a noisy output of the process is measured by a sensor, which is given by

$$y_k = C_k x_k + v_k, \tag{2.2}$$

for $0 \leq k \leq N$ where $y_k \in \mathbb{R}^p$ is the output of the process, $C_k \in \mathbb{R}^{p\times n}$ is the output matrix, and $v_k \in \mathbb{R}^p$ is a white Gaussian noise with zero mean and covariance $V_k \succ 0$. It is assumed that x_0, w_k, and v_k are mutually independent for all $0 \leq k \leq N$. In addition, it is assumed that (A_k, B_k) is controllable and (A_k, C_k) is observable.

We employ an event trigger with event variable $\delta_k \in \{0, 1\}$ that determines whether an observation is transmitted or not (see Fig. 2.1). At time k, the output of the process y_k is available at the event trigger instantly, and is transmitted to

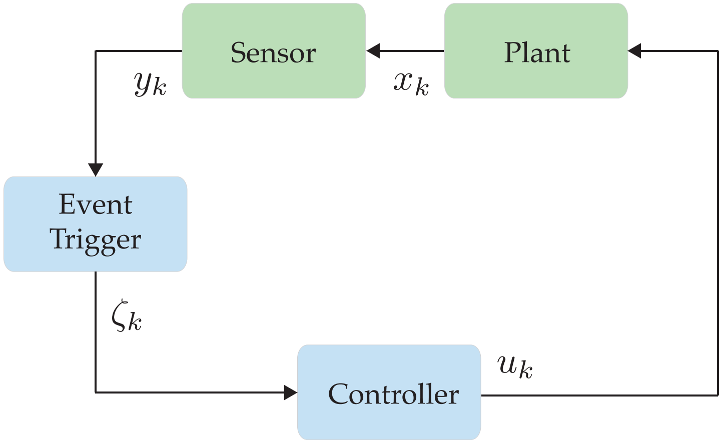

FIGURE 2.1. Schematic view of a networked control system with an event trigger. At each time, the output of the process is available at the event trigger instantly, and is transmitted to the controller sporadically with one-step delay.

the controller with one-step delay if $\delta_k = 1$. Therefore, we have

$$\zeta_{k+1} = \begin{cases} y_k, & \text{if } \delta_k = 1, \\ \varnothing, & \text{otherwise}, \end{cases} \tag{2.3}$$

where ζ_{k+1} is the transmitted output of the process subject to one-step delay.

In the sequel, we assume that the following assumption holds.

Assumption 2.1. *The information associated with non-transmitted observations, i.e., when $\delta_k = 0$, is discarded at the controller.*

Remark 2.1. *Assumption 2.1 leads to a Gaussian conditional distribution at the controller, and allows us to obtain the optimal estimator at the controller in a tractable way.*

2.1.2. Trade-off Problem. Consider a triggering policy $\pi = \{\delta_0, \ldots, \delta_N\}$ and a control policy $\mu = \{u_0, \ldots, u_N\}$. Let $\mathcal{I}_k^t$ and $\mathcal{I}_k^c$ denote the admissible information set of the event trigger and of the controller at time k respectively, and $\mathcal{P}$ and $\mathcal{M}$ denote the admissible set of triggering policies and of control policies respectively. Then, $\pi \in \mathcal{P}$ if δ_k is a measurable function of $\mathcal{I}_k^t$ for all $0 \leq k \leq N$, and $\mu \in \mathcal{M}$ if u_k is a measurable function of $\mathcal{I}_k^c$ for all $0 \leq k \leq N$.

We would like to make a trade-off between the sampling rate and control performance. We measure the sampling rate by an average number of transmissions, i.e.,

$$R(\pi, \mu) = \tfrac{1}{N+1} \mathsf{E}\left[\textstyle\sum_{k=0}^{N} \alpha_k \delta_k\right], \tag{2.4}$$

where α_k is a weighting coefficient. Moreover, we measure the control performance by an average cost function penalizing the state deviation and control effort, i.e.,

$$J(\pi, \mu) = \tfrac{1}{N+1} \mathsf{E}\left[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u_k^T R_k u_k\right], \tag{2.5}$$

where $Q_k \succeq 0$ and $R_k \succ 0$ are weighting matrices. The desired trade-off is formulated by the following optimization problem:

$$\text{minimize} \quad \lambda J(\pi, \mu) + (1 - \lambda) R(\pi, \mu), \tag{2.6}$$

for $\lambda \in (0,1)$ over $\pi \in \mathcal{P}$ and $\mu \in \mathcal{M}$. Equivalently, we can solve the following optimization problem:

$$\text{minimize} \quad \Psi(\pi,\mu), \tag{2.7}$$

where

$$\Psi(\pi,\mu) = \mathsf{E}\left[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u_k^T R_k u_k + \theta_k \delta_k\right], \tag{2.8}$$

where $\theta_k = \alpha_k(1-\lambda)/\lambda$.

Now, we can formally define the value of information.

Definition 2.1. *The value of information* VoI_k *in the trade-off between the sampling rate and control performance is the variation in the value function associated with the optimization problem in (2.7) with respect to an observation at the controller.*

In the sequel, we shall obtain π^* and μ^* such that (π^*, μ^*) represents a Nash equilibrium, and characterize the value of information VoI_k.

2.2. Main Results

In this section, our goal is to characterize the optimal policies. Let us define the admissible information set of the event trigger at time k as the set of the current and prior outputs of the process, i.e.,

$$\mathcal{I}_k^t = \left\{ y_t \;\middle|\; t \le k \right\}, \tag{2.9}$$

and the admissible information set of the controller at time k as the set of the prior transmitted outputs of the process and prior event variables, i.e.,

$$\mathcal{I}_k^c = \left\{ y_t, \delta_{t'} \;\middle|\; t, t' < k, \delta_t = 1 \right\}. \tag{2.10}$$

We see that the latter is a subset of the former, i.e., $\mathcal{I}_k^c \subseteq \mathcal{I}_k^t$. Note that the information set $\mathcal{I}_k^c$ satisfies Assumption 2.1.

2.2.1. Optimal Estimators. Given that the process is partially observable, both event trigger and controller have to estimate the state of the process. We shall derive the optimal estimators based on the Bayesian analysis. The next two propositions give the optimal estimators with respect to the information sets $\mathcal{I}_k^t$ and $\mathcal{I}_k^c$ respectively, and show that such estimators are linear.

Proposition 2.1. *The conditional expectation* $\mathsf{E}[x_k|\mathcal{I}_k^t]$ *with the following dynamics minimizes the mean-square error at the event trigger:*

$$\check{x}_k = A_{k-1}\check{x}_{k-1} + B_{k-1}u_{k-1} + H_k\big(y_k - C_k(A_{k-1}\check{x}_{k-1} + B_{k-1}u_{k-1})\big), \tag{2.11}$$

$$\Sigma_k = \big((A_{k-1}\Sigma_{k-1}A_{k-1}^T + W_{k-1})^{-1} + C_k^T V_k^{-1} C_k\big)^{-1}, \tag{2.12}$$

where

$$H_k = \Sigma_k C_k^T V_k^{-1}, \tag{2.13}$$

for $1 \le k \le N$ *with initial conditions* $\check{x}_0 = m_0 + \Sigma_0 C_0^T V_0^{-1}(y_0 - C_0 m_0)$ *and* $\Sigma_0 = (M_0^{-1} + C_0^T V_0^{-1} C_0)^{-1}$ *where* $\check{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^t]$ *and* $\Sigma_k = \mathsf{cov}[x_k|\mathcal{I}_k^t]$.

Proof. The output y_k is available at the event trigger at each time. Hence, it is clear that given the information set $\mathcal{I}_k^t$ at the event trigger, the state estimate minimizing the mean-square error is the conditional expectation $\mathsf{E}[x_k|\mathcal{I}_k^t]$, and the optimal estimator is the Kalman filter (see e.g., [**71**]). □

Proposition 2.2. *The conditional expectation* $\mathsf{E}[x_k|\mathcal{I}_k^c]$ *with the following dynamics minimizes the mean-square error at the controller:*

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + \delta_{k-1}K_{k-1}(y_{k-1} - C_{k-1}\hat{x}_{k-1}), \tag{2.14}$$

$$P_k = A_{k-1}P_{k-1}A_{k-1}^T + W_{k-1} - \delta_{k-1}K_{k-1}C_{k-1}P_{k-1}A_{k-1}^T, \tag{2.15}$$

where

$$K_{k-1} = A_{k-1}P_{k-1}C_{k-1}^T(C_{k-1}P_{k-1}C_{k-1}^T + V_{k-1})^{-1}, \tag{2.16}$$

for $1 \le k \le N$ *with initial conditions* $\hat{x}_0 = m_0$ *and* $P_0 = M_0$ *where* $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^c]$ *and* $P_k = \mathsf{cov}[x_k|\mathcal{I}_k^c]$.

Proof. Given the information set $\mathcal{I}_k^c$ at the controller, the state estimate minimizing the mean-square error is clearly the conditional expectation $\mathsf{E}[x_k|\mathcal{I}_k^c]$. From the definition, $\hat{x}_{k+1} = \mathsf{E}[x_{k+1}|\mathcal{I}_{k+1}^c]$ and $P_{k+1} = \mathsf{cov}[x_{k+1}|\mathcal{I}_{k+1}^c]$. Taking the conditional expectation of (2.1), we get

$$\hat{x}_{k+1} = A_k\,\mathsf{E}[x_k|\mathcal{I}_{k+1}^c] + B_k u_k, \tag{2.17}$$

$$P_{k+1} = A_k\,\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c]A_k^T + W_k. \tag{2.18}$$

From Assumption 2.1, the controller makes no inference when $\delta_k = 0$. However, the controller receives $\zeta_{k+1} = y_k$ at time $k+1$ and subsequently can make an inference when $\delta_k = 1$. Let us define $\xi_k = [x_k^T \quad y_k^T]^T$. We can easily show that

$$\mathsf{E}[\xi_k|\mathcal{I}_k^c] = \begin{bmatrix} \hat{x}_k \\ C_k\hat{x}_k \end{bmatrix}, \tag{2.19}$$

$$\mathsf{cov}[\xi_k|\mathcal{I}_k^c] = \begin{bmatrix} P_k & P_kC_k^T \\ C_kP_k & C_kP_kC_k^T + V_k \end{bmatrix}. \tag{2.20}$$

Now, we can use Lemma A.2 together with the conditional distribution specified by the mean and covariance in (2.19), (2.20), and get

$$\mathsf{E}[x_k|\mathcal{I}_k^c, y_k] = \hat{x}_k + K_k'(y_k - C_k\hat{x}_k),$$

$$\mathsf{cov}[x_k|\mathcal{I}_k^c, y_k] = P_k - K_k'C_kP_k,$$

where $K_k' = P_kC_k^T(C_kP_kC_k^T + V_k)^{-1}$. Therefore, from the definition of δ_k, we can write

$$\mathsf{E}[x_k|\mathcal{I}_{k+1}^c] = \hat{x}_k + \delta_k K_k'(y_k - C_k\hat{x}_k), \tag{2.21}$$

$$\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c] = P_k - \delta_k K_k'C_kP_k, \tag{2.22}$$

where we used the definition of $\mathcal{I}_{k+1}^c$ given δ_k at the controller at time $k+1$. We obtain the results by substituting (2.21), (2.22) in (2.17), (2.18) respectively. □

As it becomes evident later, in addition to the Kalman filter given in Proposition 2.1, the event trigger needs to construct a copy of the estimator given in Proposition 2.2. This is possible because $\mathcal{I}_k^c \subseteq \mathcal{I}_k^t$.

2.2.2. Optimal Policies. We shall design the optimal policies using backward induction. Let $e_k = x_k - \hat{x}_k$ be the estimation error and $\nu_k = y_k - C_k \hat{x}_k$ be the innovation both associated with the estimator at the controller. Moreover, let $\varepsilon_k = \check{x}_k - \hat{x}_k$ be the mismatch estimation error associated with the estimators at the event trigger and controller. We can obtain

$$\mathsf{E}[e_k|\mathcal{I}_k^t] = \mathsf{E}[x_k - \hat{x}_k|\mathcal{I}_k^t] = \check{x}_k - \hat{x}_k = \varepsilon_k, \tag{2.23}$$

$$\mathsf{cov}[e_k|\mathcal{I}_k^t] = \mathsf{cov}[x_k|\mathcal{I}_k^t] = \Sigma_k. \tag{2.24}$$

The next theorem characterizes the structures of the optimal triggering policy and optimal control policy, and shows that there exists a separation between the optimal designs of the event trigger and controller.

Theorem 2.1. *Let $S_k \succeq 0$ be a matrix that satisfies the condition in Lemma A.1. The optimal closed-loop triggering policy is a threshold policy given by*

$$\delta_k^* = \mathbb{1}_{\mathrm{VoI}_k \geq 0}, \tag{2.25}$$

where VoI_k is the value of information defined as

$$\mathrm{VoI}_k = \nu_k^T K_k^T \Gamma_{k+1}(2A_k\varepsilon_k - K_k\nu_k) - \theta_k + \varrho_k, \tag{2.26}$$

where ϱ_k is a variable that depends on ε_k and ν_k, and the optimal closed-loop control policy is a certainty-equivalence policy given by

$$u_k^* = -L_k\hat{x}_k, \tag{2.27}$$

where L_k is the control gain defined as

$$L_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k. \tag{2.28}$$

Proof. We need to show that (π^*, μ^*) represents a Nash equilibrium. Using the optimal control policy μ^* in the cost function $\Psi(\pi, \mu)$ given by Lemma A.1, we obtain

$$\Psi(\pi, \mu^*) = \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{\theta_k\delta_k + w_k^T S_{k+1} w_k + e_k^T L_k^T (B_k^T S_{k+1} B_k + R_k) L_k e_k\Big\}\Big],$$

where we used the definition of the estimation error e_k. Following the fact that x_0 and w_k are independent of the triggering policy, associated with $\Psi(\pi, \mu^*)$, we define the value function V_k^t as

$$V_k^t = \min_{\boldsymbol{\delta}^k} \mathsf{E}\Big[\textstyle\sum_{t=k}^N \theta_t\delta_t + e_{t+1}^T \Gamma_{t+1} e_{t+1}\Big|\mathcal{I}_k^t\Big],$$

where $\Gamma_k = L_k^T(B_k^T S_{k+1} B_k + R_k)L_k$ with the exception of $\Gamma_{N+1} = 0$. From the additivity of the value function V_k^t, we have

$$\begin{aligned} V_k^t &= \min_{\delta_k} \mathsf{E}\Big[\theta_k\delta_k + e_{k+1}^T\Gamma_{k+1}e_{k+1} \\ &\quad + \min_{\delta_{k+1}} \mathsf{E}\Big[\theta_{k+1}\delta_{k+1} + e_{k+2}^T\Gamma_{k+2}e_{k+2} + \dots \Big|\mathcal{I}_{k+1}^t\Big]\Big|\mathcal{I}_k^t\Big] \\ &= \min_{\delta_k} \mathsf{E}\Big[\theta_k\delta_k + e_{k+1}^T\Gamma_{k+1}e_{k+1} + V_{k+1}^t\Big|\mathcal{I}_k^t\Big], \end{aligned}$$

with initial condition $V_{N+1}^t = 0$. We prove by induction that the value function V_k^t is independent of the control policy. Clearly, the claim is satisfied for time $N+1$.

We assume that the claim holds at time $k+1$, and we shall prove that it also holds at time k. We can write the dynamics of the estimation error at the controller as

$$e_{k+1} = A_k e_k + w_k - \delta_k K_k \nu_k. \tag{2.29}$$

Thus, we find

$$\begin{aligned}
&\mathsf{E}[e_{k+1}^T \Gamma_{k+1} e_{k+1} | \mathcal{I}_k^t] \\
&\quad = \mathsf{E}\Big[e_k^T A_k^T \Gamma_{k+1} A_k e_k + w_k^T \Gamma_{k+1} w_k + \delta_k^2 \nu_k^T K_k^T \Gamma_{k+1} K_k \nu_k \\
&\qquad + 2 e_k^T A_k^T \Gamma_{k+1} w_k - 2\delta_k \nu_k^T K_k^T \Gamma_{k+1} w_k - 2\delta_k \nu_k^T K_k^T \Gamma_{k+1} A_k e_k \Big| \mathcal{I}_k^t \Big] \\
&\quad = \varepsilon_k^T A_k^T \Gamma_{k+1} A_k \varepsilon_k + \mathrm{tr}(A_k^T \Gamma_{k+1} A_k \Sigma_k) + \mathrm{tr}(\Gamma_{k+1} W_k) \\
&\qquad + \delta_k \nu_k^T K_k^T \Gamma_{k+1} K_k \nu_k - 2\delta_k \nu_k^T K_k^T \Gamma_{k+1} A_k \varepsilon_k,
\end{aligned}$$

where in the second equality we used the definitions of ε_k and Σ_k and the facts that ν_k is $\mathcal{I}_k^t$-measurable and that w_k is independent of e_k. Hence, we have

$$\begin{aligned}
V_k^t = \min_{\delta_k} \Big\{ &\theta_k \delta_k + \varepsilon_k^T A_k^T \Gamma_{k+1} A_k \varepsilon_k + \mathrm{tr}(A_k^T \Gamma_{k+1} A_k \Sigma_k) \\
&+ \mathrm{tr}(\Gamma_{k+1} W_k) + \delta_k \nu_k^T K_k^T \Gamma_{k+1} K_k \nu_k \\
&- 2\delta_k \nu_k^T K_k^T \Gamma_{k+1} A_k \varepsilon_k + \mathsf{E}[V_{k+1}^t | \mathcal{I}_k^t] \Big\}.
\end{aligned} \tag{2.30}$$

The minimizer in (2.30) is obtained as $\delta_k^* = \mathbb{1}_{\mathrm{VoI}_k \geq 0}$ where

$$\mathrm{VoI}_k = \nu_k^T K_k^T \Gamma_{k+1} (2 A_k \varepsilon_k - K_k \nu_k) - \theta_k + \varrho_k,$$

where $\varrho_k = \mathsf{E}[V_{k+1}^t | \mathcal{I}_k^t, \delta_k = 0] - \mathsf{E}[V_{k+1}^t | \mathcal{I}_k^t, \delta_k = 1]$. From the hypothesis assumption ϱ_k is independent of the control policy. Hence, we conclude that V_k^t is independent of the control policy. This complete the induction.

Now, using the the triggering policy π^* in the cost function $\Psi(\pi, \mu)$ given by Lemma A.1, we obtain

$$\begin{aligned}
\Psi(\pi^*, \mu) = \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{ &\theta_k \mathbb{1}_{\mathrm{VoI}_k \geq 0} + w_k^T S_{k+1} w_k \\
&+ (u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \Big\} \Big],
\end{aligned}$$

where $\Lambda_k = B_k^T S_{k+1} B_k + R_k$. Following the fact that x_0, VoI_k, and w_k are independent of the control policy, associated with $\Psi(\pi^*, \mu)$, we define the auxiliary value function V_k^c as

$$V_k^c = \min_{\mathbf{u}^k} \mathsf{E}\Big[\textstyle\sum_{t=k}^N (u_t + L_t x_t)^T \Lambda_t (u_t + L_t x_t) \Big| \mathcal{I}_k^c \Big].$$

From the additivity of the auxiliary value function V_k^c, we obtain

$$\begin{aligned} V_k^c &= \min_{u_k} \mathsf{E}\Big[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \\ &\quad + \min_{u_{k+1}} \mathsf{E}\Big[(u_{k+1} + L_{k+1} x_{k+1})^T \Lambda_{k+1} \\ &\quad \times (u_{k+1} + L_{k+1} x_{k+1}) + \dots \Big| \mathcal{I}_{k+1}^c\Big]\Big|\mathcal{I}_k^c\Big] \\ &= \min_{u_k} \mathsf{E}\Big[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) + V_{k+1}^c \Big| \mathcal{I}_k^c\Big], \end{aligned}$$

with initial condition $V_{N+1}^c = 0$. We prove by induction that the auxiliary value function V_k^c is a function of P_k. Clearly, the claim is satisfied for time $N+1$. We assume that the claim holds at time $k+1$, and we shall prove that it also holds at time k. Using the identity $x_k = \hat{x}_k + e_k$, we find

$$\begin{aligned} &\mathsf{E}\left[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \middle| \mathcal{I}_k^c\right] \\ &\quad = \mathsf{E}\Big[(u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) \\ &\qquad + e_k^T L_k^T \Lambda_k L_k e_k + 2(u_k + L_k \hat{x}_k)^T \Lambda_k L_k e_k \Big| \mathcal{I}_k^c\Big] \\ &\quad = (u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) + \mathrm{tr}(L_k^T \Lambda_k L_k P_k), \end{aligned}$$

where in the second equality we used the fact that $\hat{x}_k$ is $\mathcal{I}_k^c$-measurable and $\mathsf{E}[e_k | \mathcal{I}_k^c] = 0$. Hence, we have

$$\begin{aligned} V_k^c = \min_{u_k} \Big\{ &(u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) \\ &+ \mathrm{tr}(L_k^T \Lambda_k L_k P_k) + \mathsf{E}[V_{k+1}^c | \mathcal{I}_k^c] \Big\}. \end{aligned} \tag{2.31}$$

The minimizer in (2.31) is obtained as $u_k^* = -L_k \hat{x}_k$. Moreover, we conclude that V_k^c is a function of P_k. This completes the induction and also the proof. □

According to Theorem 2.1, the optimal triggering policy depends on ε_k and ν_k, and is independent of the control policy. Besides, the error covariance P_k in (2.15) does not depend on $\mathbf{u}_{k-1}$. Hence, the control has no dual effect. In Theorem 2.1, we expressed the value of information based on the value function V_k^t and corresponding to the successful transmission of an observation. This is emphasized in the following definition.

Definition 2.2. *The value of information* VoI_k *in the trade-off between the sampling rate and control performance is given by*

$$\mathrm{VoI}_k = \nu_k^T K_k^T \Gamma_{k+1} (2 A_k \varepsilon_k - K_k \nu_k) - \theta_k + \varrho_k.$$

Remark 2.2. *The optimal triggering policy provided above depends on the variable ϱ_k. Although ϱ_k can be computed with an arbitrary accuracy by solving recursively the optimality equation in (2.30), its computation is expensive. Next, we shall introduce a procedure for approximation of this variable.*

2.2.3. Approximation Algorithm. We here provide a rollout algorithm [**72**] for approximation of the variable ϱ_k and the value of information VoI_k, and accordingly synthesize a closed-form suboptimal triggering policy with a performance guarantee that can readily be implemented. Let $\bar{\pi} = \{\bar{\delta}_0, \dots, \bar{\delta}_N\}$ be a periodic policy with $\delta_k = 1$ for all $0 \leq k \leq N$. The following algorithm gives an approximation of the variable ϱ_k.

Algorithm 2.1. *An approximation of the variable ϱ_k associated with the policy $\bar{\pi}$ is given by*

$$\varrho_k^{\bar{\pi}} = \mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^t, \delta_k = 0] - \mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^t, \delta_k = 1], \tag{2.32}$$

where

$$\mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^t, \delta_k] = \mathsf{E}\left[\textstyle\sum_{t=k+1}^{N} \theta_t \delta_t + e_{t+1}^T \Gamma_{t+1} e_{t+1} \middle| \mathcal{I}_k^t, \delta_k \right],$$

with $\boldsymbol{\delta}^{k+1} = \bar{\boldsymbol{\delta}}^{k+1}$.

The next theorem guarantees that it is possible to synthesize a suboptimal triggering policy that outperforms $\bar{\pi}$.

Theorem 2.2. *Let π^+ be a suboptimal triggering policy obtained based on Theorem 2.1 and Algorithm 2.1 and μ be the optimal control policy. Then,*

$$\Psi(\pi^+, \mu^*) \leq \Psi(\bar{\pi}, \mu^*). \tag{2.33}$$

PROOF. We shall show that $\Psi_k(\pi^+, \mu^*) \leq \Psi_k(\bar{\pi}, \mu^*)$ for any k and all initial conditions. In order to show that, it is enough to show $V_k^{\pi^+} \leq V_k^{\bar{\pi}}$. We prove this by induction. Clearly, $V_{N+1}^{\pi^+} = V_{N+1}^{\bar{\pi}} = 0$. Assume that the claim holds for $k+1$. We have

$$\begin{aligned} V_k^{\pi^+} &= \mathsf{E}\left[\theta_k \delta_k^+ + e_{k+1}^T \Gamma_{k+1} e_{k+1} + V_{k+1}^{\pi^+} \middle| \mathcal{I}_k^t \right] \\ &\leq \mathsf{E}\left[\theta_k \delta_k^+ + e_{k+1}^T \Gamma_{k+1} e_{k+1} + V_{k+1}^{\bar{\pi}} \middle| \mathcal{I}_k^t \right] \\ &\leq \mathsf{E}\left[\theta_k \bar{\delta}_k + e_{k+1}^T \Gamma_{k+1} e_{k+1} + V_{k+1}^{\bar{\pi}} \middle| \mathcal{I}_k^t \right] = V_k^{\bar{\pi}}, \end{aligned}$$

where the first and second equalities come from backward induction, the first inequality from the induction hypothesis, and the second inequality from the definition of the suboptimal triggering policy π^+. □

In the next proposition, we synthesize a closed-form suboptimal triggering policy with a performance guarantee.

Proposition 2.3. *A suboptimal triggering policy that outperforms the periodic policy $\bar{\pi}$ is given by*

$$\delta_k^+ = \mathbb{1}_{\mathrm{VoI}_k^{\bar{\pi}} \geq 0}, \tag{2.34}$$

where

$$\begin{aligned} \mathrm{VoI}_k^{\bar{\pi}} &= \nu_k^T K_k^T \Gamma_{k+1} (2 A_k \varepsilon_k - K_k \nu_k) - \theta_k \\ &\quad + \textstyle\sum_{t=k+2}^{N} \bar{e}_t^{0T} \Gamma_t \bar{e}_t^0 + \mathrm{tr}(\Gamma_t \bar{P}_t^0) - \bar{e}_t^{1T} \Gamma_t \bar{e}_t^1 - \mathrm{tr}(\Gamma_t \bar{P}_t^1), \end{aligned} \tag{2.35}$$

and

$$\bar{e}^0_{t+1} = (A_t - K^0_t C_t)\bar{e}^0_t,$$

$$\bar{P}^0_{t+1} = (A_t - K^0_t C_t)\bar{P}^0_t (A_t - K^0_t C_t)^T + W_t + K^0_t V_t K^{0T}_t,$$

$$P^0_{t+1} = A_t P^0_t A^T_t + W_t - K^0_t C_t P^0_t A^T_t,$$

$$\bar{e}^1_{t+1} = (A_t - K^1_t C_t)\bar{e}^1_t,$$

$$\bar{P}^1_{t+1} = (A_t - K^1_t C_t)\bar{P}^1_t (A_t - K^1_t C_t)^T + W_t + K^1_t V_t K^{1T}_t,$$

$$P^1_{t+1} = A_t P^1_t A^T_t + W_t - K^1_t C_t P^1_t A^T_t,$$

where

$$K^0_t = A_t P^0_t C^T_t (C_t P^0_t C^T_t + V_t)^{-1},$$

$$K^1_t = A_t P^1_t C^T_t (C_t P^1_t C^T_t + V_t)^{-1},$$

for $t \geq k+1$ *with initial conditions* $\bar{e}^0_{k+1} = A_k \varepsilon_k$, $\bar{P}^0_{k+1} = A_k \Sigma_k A^T_k + W_k$, $P^0_{k+1} = A_k P_k A^T_k + W_k$, $\bar{e}^1_{k+1} = A_k \varepsilon_k - K_k \nu_k$, $\bar{P}^1_{k+1} = A_k \Sigma_k A^T_k + W_k$, *and* $P^1_{k+1} = A_k P_k A^T_k + W_k - K_k C_k P_k A^T_k$.

PROOF. For the proof, it is enough to derive $\varrho^{\bar{\pi}}_k$ based on the periodic policy $\bar{\pi}$. First, note that

$$\begin{aligned}\mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^e_k, \delta_k] &= \mathsf{E}\left[\textstyle\sum_{t=k+1}^N \theta_t + e^T_{t+1} \Gamma_{t+1} e_{t+1} \middle| \mathcal{I}^t_k, \delta_k \right] \\ &= \textstyle\sum_{t=k+1}^N \theta_t + \bar{e}^T_{t+1} \Gamma_{t+1} \bar{e}_{t+1} + \mathrm{tr}(\Gamma_{t+1} \bar{P}_{t+1}),\end{aligned}$$

where in the first equality we used the definition of $V^{\bar{\pi}}_{k+1}$ and the fact that $\delta_t = 1$ for all $t \geq k+1$ and in the second equality the definitions $\bar{e}_t = \mathsf{E}[e_t | \mathcal{I}^t_k, \delta_k]$ and $\bar{P}_t = \mathsf{cov}[e_t | \mathcal{I}^t_k, \delta_k]$ for all $t \geq k+1$.

From the dynamics of the estimation error in (2.29), given the fact that $\delta_t = 1$ for all $t \geq k+1$, we obtain

$$e_{t+1} = (A_t - K_t C_t) e_t + w_t - K_t v_t.$$

Accordingly, when $\delta_k = 0$, we have

$$\bar{e}^0_{t+1} = (A_t - K^0_t C_t)\bar{e}^0_t,$$

$$\bar{P}^0_{t+1} = (A_t - K^0_t C_t)\bar{P}^0_t (A_t - K^0_t C_t)^T + W_t + K^0_t V_t K^{0T}_k,$$

and

$$P^0_{t+1} = A_t P^0_t A^T_t + W_t - K^0_t C_t P^0_t A^T_t,$$

$$K^0_t = A_t P^0_t C^T_t (C_t P^0_t C^T_t + V_t)^{-1},$$

for $t \geq k+1$ with initial conditions $\bar{e}^0_{k+1} = A_k \varepsilon_k$, $\bar{P}^0_{k+1} = A_k \Sigma_k A^T_k + W_k$, and $P^0_{k+1} = A_k P_k A^T_k + W_k$. The initial conditions $\bar{e}^1_{k+1}$ and $\bar{P}^1_{k+1}$ were obtained by using (2.29). Hence, we find

$$\mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^t_k, \delta_k = 0] = \textstyle\sum_{t=k+1}^N \theta_t + \bar{e}^{0T}_{t+1} \Gamma_{t+1} \bar{e}^0_{t+1} + \mathrm{tr}(\Gamma_{t+1} \bar{P}^0_{t+1}).$$

Moreover, when $\delta_k = 1$, we have

$$\bar{e}^1_{t+1} = (A_t - K^1_t C_t)\bar{e}^1_t,$$

$$\bar{P}^1_{t+1} = (A_t - K^1_t C_t)\bar{P}^1_t (A_t - K^1_t C_t)^T + W_t + K^1_t V_t K^{1T}_k,$$

and

$$P^1_{t+1} = A_t P^1_t A^T_t + W_t - K^1_t C_t P^1_t A^T_t,$$

$$K^1_t = A_t P^1_t C^T_t (C_t P^1_t C^T_t + V_t)^{-1},$$

for $t \geq k+1$ with initial conditions $\bar{e}^1_{k+1} = A_k \varepsilon_k - K_k \nu_k$, $\bar{P}^1_{k+1} = A_k \Sigma_k A^T_k + W_k$, and $P^1_{k+1} = A_k P_k A^T_k + W_k - K_k C_k P_k A^T_k$. The initial conditions $\bar{e}^1_{k+1}$ and $\bar{P}^1_{k+1}$ were obtained by using (2.29). Hence, we find

$$\mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^t_k, \delta_k = 1] = \textstyle\sum_{t=k+1}^N \theta_t + \bar{e}^{1T}_{t+1} \Gamma_{t+1} \bar{e}^1_{t+1} + \operatorname{tr}(\Gamma_{t+1} \bar{P}^1_{t+1}).$$

Finally, following the definition of $\varrho^{\bar{\pi}}_k$, we have

$$\begin{aligned}
\varrho^{\bar{\pi}}_k &= \mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^t_k, \delta_k = 0] - \mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^t_k, \delta_k = 1] \\
&= \textstyle\sum_{t=k+1}^N \Big\{ \bar{e}^{0T}_{t+1} \Gamma_{t+1} \bar{e}^0_{t+1} + \operatorname{tr}(\Gamma_{t+1} \bar{P}^0_{t+1}) \\
&\qquad - \bar{e}^{1T}_{t+1} \Gamma_{t+1} \bar{e}^1_{t+1} - \operatorname{tr}(\Gamma_{t+1} \bar{P}^1_{t+1}) \Big\} \\
&= \textstyle\sum_{t=k+2}^N \bar{e}^{0T}_t \Gamma_t \bar{e}^0_t + \operatorname{tr}(\Gamma_t \bar{P}^0_t) - \bar{e}^{1T}_t \Gamma_t \bar{e}^1_t - \operatorname{tr}(\Gamma_t \bar{P}^1_t),
\end{aligned}$$

where the last equality comes from the fact that $\Gamma_{N+1} = 0$. Incorporating this into (2.26), we obtain the result. □

2.3. Numerical Example

In this section, we show an application of the theoretical framework we developed in this chapter. Consider an inverted pendulum on a cart observed by an internal sensor that communicates with an internal controller through a controller area network (see Fig. 2.2). The continuous-time equations of motion linearized around the unstable equilibrium are given by

$$(I + ml^2)\ddot{\phi} - mgl\phi = ml\ddot{x},$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u,$$

where ϕ is the pitch angle of the pendulum, x here is the position of the cart, u here is the force applied to the cart, I here is the moment of inertia of the pendulum, m is the mass of the pendulum, l is the length to the pendulum's center of mass, g is the gravity, M is the mass of the cart, and b is the coefficient of friction for the cart. We assume the following parameters: $I = 0.006$ kg.m^2, $m = 0.2$ kg, $l = 0.3$ m, $g = 9.81$ m/s^2, $M = 0.5$ kg, and $b = 0.1$ N/m/sec. The sensor can only measure the position and pitch angle. The discrete-time dynamics of form (2.1) obtained by a zero-hold transformation with sampling frequency of 100 Hz and the sensor

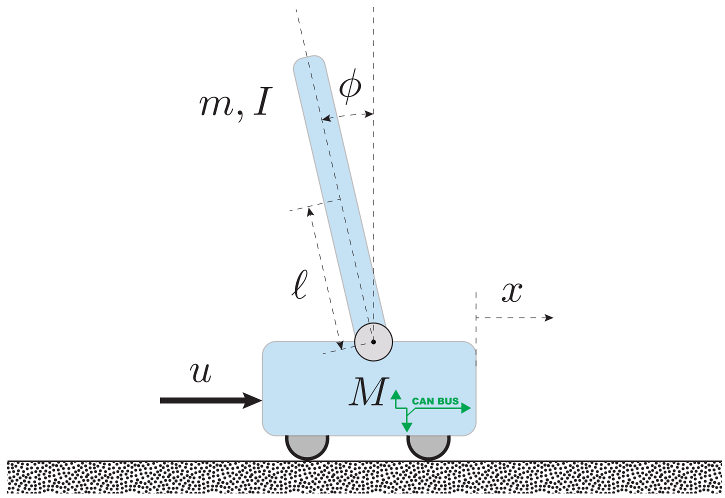

FIGURE 2.2. Model of an inverted pendulum on a cart. The sensor and controller are both insourced. The sensor is connected to the controller through a controller area network.

model of form (2.2) together with the covariance matrices are given by

$$A_k = \begin{bmatrix} 1.0000 & 0.0100 & 0.0001 & 0.0000 \\ 0.0000 & 0.9982 & 0.0267 & 0.0001 \\ 0.0000 & 0.0000 & 1.0016 & 0.0100 \\ 0.0000 & -0.0045 & 0.3122 & 1.0016 \end{bmatrix}, \; B_k = \begin{bmatrix} 0.0001 \\ 0.0182 \\ 0.0002 \\ 0.0454 \end{bmatrix},$$

$$C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, V_k = \begin{bmatrix} 0.0020 & 0.0000 \\ 0.0000 & 0.0010 \end{bmatrix},$$

$$W_k = \begin{bmatrix} 0.0006 & 0.0003 & 0.0001 & 0.0006 \\ 0.0003 & 0.0008 & 0.0003 & 0.0004 \\ 0.0001 & 0.0003 & 0.0007 & 0.0006 \\ 0.0006 & 0.0004 & 0.0006 & 0.0031 \end{bmatrix},$$

with initial conditions $m_0 = [0\ 0\ 0.2\ 0]^T$ and $M_0 = 10W_k$. For this system, we are interested in designing an event trigger that is employed at the sensor and a controller that is collocated with the actuator. The cost function of form (2.8) is specified by the weights $Q_{N+1} = \text{diag}\{1, 1, 1000, 1\}$, $Q_k = \text{diag}\{1, 1, 1000, 1\}$, $R_k = 1$, and $\theta_k = 150$ for all $0 \le k \le N$ where $N = 500$. The state estimates $\check{x}_k$ and $\hat{x}_k$ are provided by Proposition 2.1 and Proposition 2.2 respectively. From Theorem 2.1, it follows that the optimal triggering policy is $\delta_k^* = \mathbb{1}_{\text{VoI}_k \ge 0}$ and the optimal control policy is $u_k^* = -L_k \hat{x}_k$. We approximated the value of information VoI_k by using Proposition 2.3, and obtained the control gain L_k by solving the Riccati equation in Lemma A.1. We carried out a simulation experiment. For a realization of the system, Fig. 2.3 and Fig. 2.4 illustrate the trajectories of the main variables of the system. In particular, the trajectories of the position, velocity, pitch angle, and pitch rate are shown in Fig. 2.3. Moreover, the trajectories of the value of information, event, and control are shown in Fig. 2.4. In this experiment, the value of information became positive only 18 times, which lead to the transmission of the observation at each of those times. Besides, we observe that the system could still achieve a good control performance while the sampling rate was reduced by 96.4% with respect to the periodic policy.

2.4. Summary

In this chapter, we provided a theoretical framework for the analysis and design of networked control systems with event triggers. We formulated the problem as a dynamic game, and characterized the optimal triggering policy and optimal control policy such that the corresponding policy profile represents a Nash equilibrium. We proved, under certain assumptions, that the optimal closed-loop triggering policy is a threshold policy that depends particularly on the estimation innovation at the controller and mismatch estimation error; and that the optimal closed-loop control policy is a certainty-equivalence policy. Moreover, we synthesized a closed-form suboptimal triggering policy with a performance guarantee. We propose that further research should be undertaken in extension of the present framework to networks of interacting systems that share a common communication channel. In such a case, a multiple access scheme should be designed.

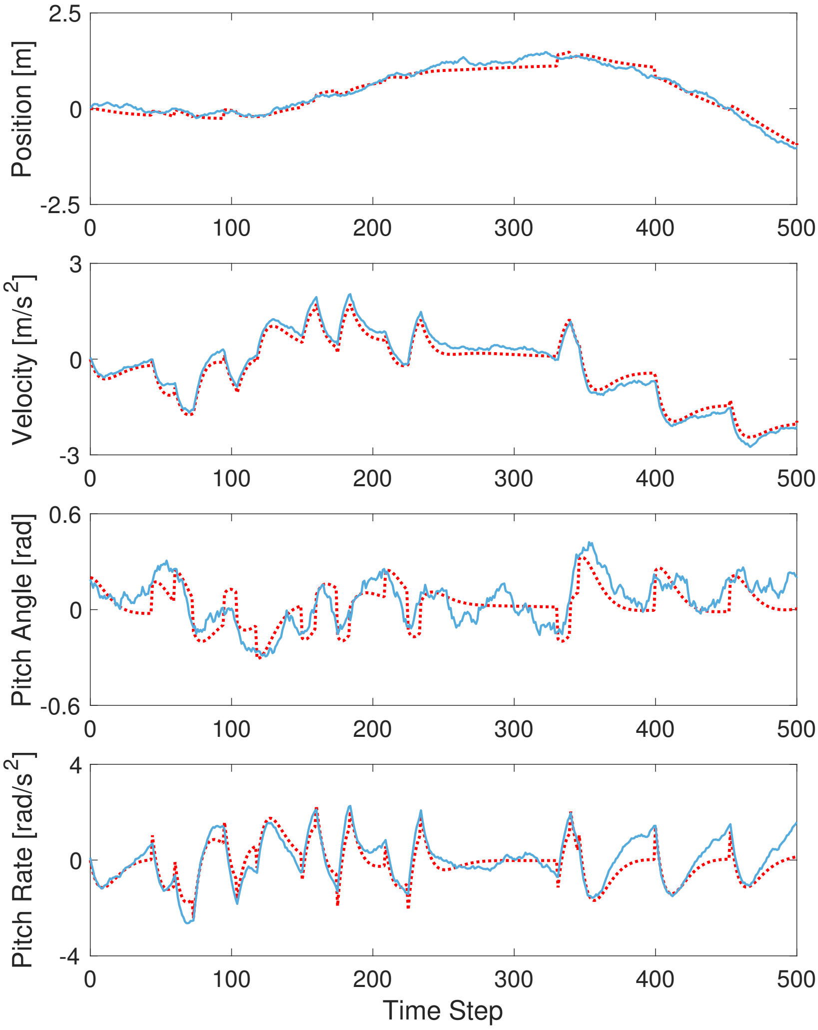

FIGURE 2.3. Trajectories of the position, velocity, pitch angle, and pitch rate. The solid lines represent the state components and the dotted lines represent the state estimate components at the controller.

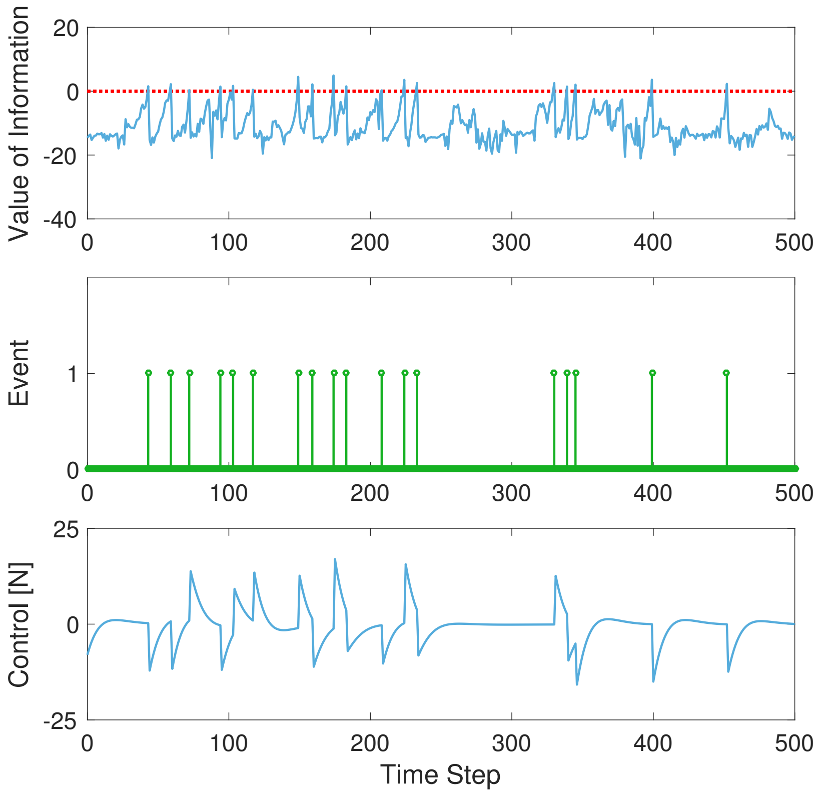

FIGURE 2.4. Trajectories of the value of information, event, and control. The dotted line in the diagram of the value of information represents the zero values. The value of information is scaled by one tenth.

CHAPTER 3

Optimal Power Scheduler in Wireless Control

In this chapter, we consider a wireless control system, and seek to jointly design a power scheduler and a controller by constructing a trade-off between the energy capacity and control performance. Based on our unified framework, we develop the optimal estimators and characterize the optimal policies such that the corresponding policy profile represents a Nash equilibrium. Moreover, we define the value of information in the trade-off between the energy capacity and control performance. This chapter is organized in the following way. We formulate the problem in Section 3.1. We provide the main results of the chapter in Section 3.2. We present a numerical example in Section 3.3. Finally, we give a summary in Section 3.4.

3.1. Problem Formulation

In this section, we describe the system model, introduce the specification of the wireless channel, and determine the relation between the required transmit power and packet success rate. Moreover, we define the indices for the energy capacity and control performance. Finally, we formulate the main problem of the chapter. The material here requires basic knowledge of communication theory, stochastic control theory, and game theory.

3.1.1. System Model. Consider a stochastic process with dynamics generated by the following linear discrete-time time-varying state system:

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{3.1}$$

for $0 \leq k \leq N$ and with initial condition x_0 where $x_k \in \mathbb{R}^n$ is the state of the process, $A_k \in \mathbb{R}^{n \times n}$ is the state matrix, $B_k \in \mathbb{R}^{n \times m}$ is the input matrix, $u_k \in \mathbb{R}^m$ is the control input to be decided by a controller, $w_k \in \mathbb{R}^n$ is a white Gaussian noise with zero mean and covariance $W_k \succ 0$, and N is a finite terminal time. It is assumed that x_0 is a Gaussian vector with mean m_0 and covariance M_0. At each time, a noisy output of the process is measured by a *wireless sensor*, which is given by

$$y_k = C_k x_k + v_k, \tag{3.2}$$

for $0 \leq k \leq N$ where $y_k \in \mathbb{R}^p$ is the output of the process, $C_k \in \mathbb{R}^{p \times n}$ is the output matrix, and $v_k \in \mathbb{R}^p$ is a white Gaussian noise with zero mean and covariance $V_k \succ 0$. It is assumed that x_0, w_k, and v_k are mutually independent for all $0 \leq k \leq N$. In addition, it is assumed that (A_k, B_k) is controllable and (A_k, C_k) is observable.

The sensor is connected to the controller through a wireless fading channel, and a power scheduler is employed at the sensor that adapts the transmit power (see Fig. 3.1). At time k, the output of the process y_k is available at the power scheduler instantly, and is received by the controller with one-step delay if the transmission is successful. In the following, we describe the specification of the wireless channel.

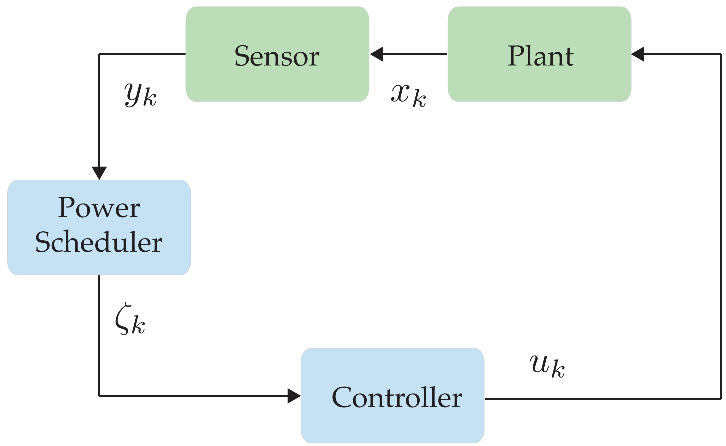

FIGURE 3.1. Schematic view of a wireless control system with a power scheduler. At each time, the output of the process is available at the power scheduler instantly, and is received by the controller sporadically with one-step delay.

3.1.2. Wireless Channel Specification. Consider a discrete-time additive white Gaussian noise (AWGN) channel with the following channel input-output relationship:

$$r_k = \sqrt{g_k} s_k + n_k, \tag{3.3}$$

where r_k is the channel output, $g_k \geq 0$ is the channel gain (also known as channel state information), s_k is the channel input, and n_k is a white Gaussian noise with zero mean and power spectral density N_0. The channel gain g_k is in general a stochastic variable representing path loss, shadowing, and multipath effects. The channel gain g_k can change at each time with or without correlation over time. We assume that the channel is block fading, i.e., the channel gain g_k remains constant during each packet transmission (i.e., block), but might change from block to block. We also assume that the channel gain g_k is known at both controller and power scheduler before transmission at time k given an ideal feedback channel.

Let the transmit power at time k be p_k^{tx}, which is constrained by $0 \leq p_k^{tx} \leq p_{max}$. The received power is obtained by $p_k^{rx} = g_k p_k^{tx}$. Moreover, the received signal-to-noise power ratio SNR_k, defined as the ratio of the received signal power to the noise power within the bandwidth of the transmitted signal [**73**], is given by

$$\mathsf{SNR}_k = \frac{g_k p_k^{tx}}{N_0 W} = \frac{E_k R}{N_0 W}, \tag{3.4}$$

where E_k is the received signal energy per bit, R here is the communication rate, and W here is the noise bandwidth.

The bit sequence corresponding to each observation is modulated by the transmitter into a carrier signal. It is then transmitted over the channel, and is eventually detected by the receiver (see Fig. 3.2). For any modulation and detection techniques, the performance of the wireless channel, specifying the bit error rate BER_k, depends only on the ratio E_k/N_0 and not on any other detailed characteristics of the signal and noise [**73**]. Hence, we can write

$$\mathsf{BER}_k = \phi_1\left(\tfrac{E_k}{N_0}\right), \tag{3.5}$$

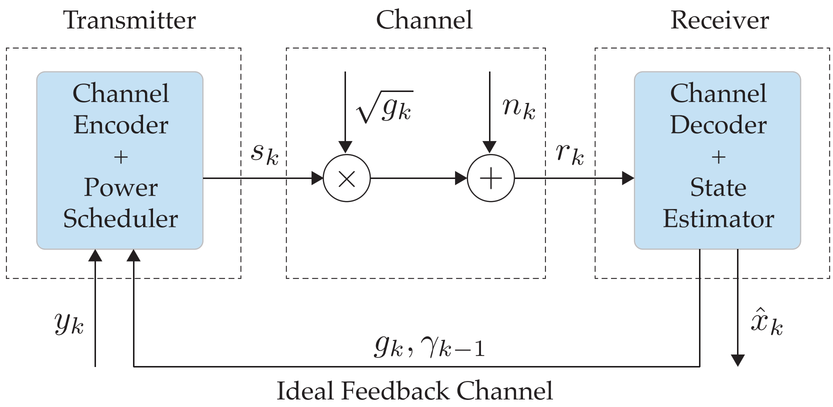

FIGURE 3.2. Wireless communication over an AWGN channel that connects the sensor to the controller.

where $\phi_1(.)$ is a function often expressed in terms of Q-function. Note that $\mathsf{BER}_k \in (0, 0.5]$ where 0.5 corresponds to the worst case.

Remark 3.1. *It is shown in [25] that the performance of AWGN channels corresponding to modulation with coherent detection and perfect recovery of the carrier frequency and phase can generally be expressed by*

$$\phi_1\left(\frac{E_k}{N_0}\right) = \rho_0 Q\left(\sqrt{\rho_1 \frac{E_k}{N_0}}\right), \tag{3.6}$$

where ρ_0 and ρ_1 are constants depending on the particular form of modulation.

Each observation is transmitted as a network packet that encapsulates the associated bit sequence. Thus, given the bit error rate BER_k, one can obtain the packet success rate PSR_k for transmission of a packet as

$$\mathsf{PSR}_k = \phi_2(\mathsf{BER}_k), \tag{3.7}$$

where $\phi_2(.)$ is a function depending on the error correcting code and the length of each packet ℓ. Note that $\mathsf{PSR}_k \in [\epsilon_0, \epsilon_1)$ where $\epsilon_0 = \phi_2(0.5)$ and $\epsilon_1 = \phi_2(0)$. The packet success rate PSR_k tends to the extreme values 0 and 1 for large ℓ and p_{max} respectively.

Remark 3.2. *If all single bit errors in a packet can be detected, the packet success rate for transmission of a packet with ℓ bits is specified by*

$$\phi_2(\mathsf{BER}_k) = (1 - \mathsf{BER}_k)^{\ell}, \tag{3.8}$$

where ℓ is the length of the packet.

Putting (3.4), (3.5), (3.7) together, we can obtain the required transmit power at time k for a given packet success rate as

$$p_k = \frac{N_0 R}{g_k} \phi^{-1}(\mathsf{PSR}_k), \tag{3.9}$$

where $\phi(.) = \phi_2(\phi_1(.))$ and p_k is used instead of p_k^{tx} for the sake of keeping the notation simple. Fig. 3.3 depicts the required transmit power as a function of the packet success rate for a specific wireless fading channel. In the sequel, we assume that $p_{max} < \infty$ leads to a successful transmission of the observation y_k at time k almost surely, i.e., $\mathsf{PSR}_k \simeq 1$. Moreover, we consider PSR_k as the decision variable

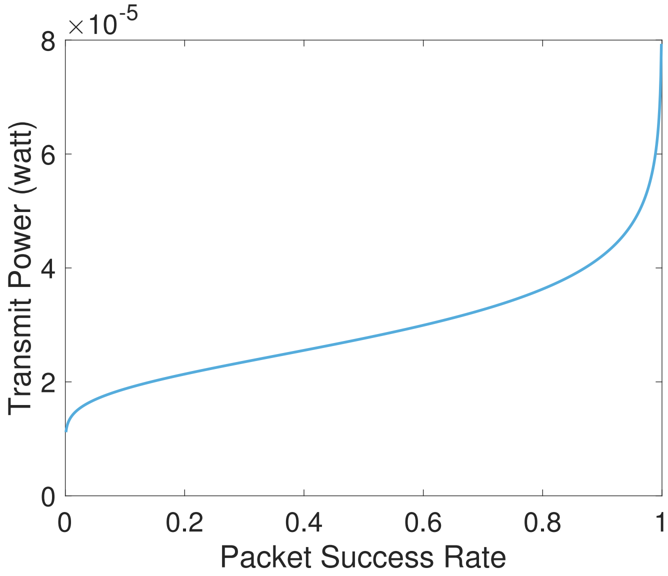

FIGURE 3.3. The required transmit power as a function of the packet success rate for a specific wireless fading channel described in Section 3.3.

of the power scheduler. Clearly, given g_k and PSR_k, the transmit power p_k at each time is obtained by (3.9).

Remark 3.3. *In practice, the wireless sensor may decide not to transmit the observation at time k if PSR_k is very small. In this case, the transmit power should be set to zero.*

We model packet dropouts according to a random arrival process γ_k with probability distribution $\mathsf{P}(\gamma_k = 1) = \mathsf{PSR}_k$ where $\gamma_k = 1$ if y_k is received successfully and $\gamma_k = 0$ otherwise. Therefore, we have

$$\zeta_{k+1} = \begin{cases} y_k, & \text{if } \gamma_k = 1, \\ \varnothing, & \text{otherwise,} \end{cases} \tag{3.10}$$

where ζ_{k+1} is the successfully received output of the process subject to one-step delay. Packets are received with one-step delay, and packets that are not received successfully are discarded without being retransmitted. Besides, the receipt acknowledgement of the packet transmitted at time k is available at the power scheduler at time $k+1$ via the ideal feedback channel.

In the sequel, we assume that the following assumptions hold.

Assumption 3.1. *The arrival variables γ_k for all $0 \leq k \leq N$ are conditionally independent given channel gains and transmit powers, i.e,*

$$\mathsf{P}(\gamma_0, \ldots, \gamma_k | \mathbf{g}_k, \mathbf{p}_k) = \Pi_{t=0}^{k} \mathsf{P}(\gamma_t | g_t, p_t), \tag{3.11}$$

for $k \geq 0$.

Assumption 3.2. *The information associated with dropped observations, i.e., when $\gamma_k = 0$, is discarded at the controller.*

Remark 3.4. *Assumption 3.1 comes from the fact that the packet success rate at each time statistically depends only on the channel gain and transmit power at* *that time. This model incorporates i.i.d. erasure channels and Gilbert-Elliot channels as special cases. Assumption 3.2 leads to a Gaussian conditional distribution at the controller, and allows us to obtain the optimal estimator at the controller in a tractable way.*

3.1.3. Trade-off Problem. Consider a power policy $\pi = \{p_0, \dots, p_N\}$ and a control policy $\mu = \{u_0, \dots, u_N\}$. Let $\mathcal{I}_k^s$ and $\mathcal{I}_k^c$ denote the admissible information set of the power scheduler and of the controller at time k respectively, and $\mathcal{P}$ and $\mathcal{M}$ denote the admissible set of power policies and of control policies respectively. Then, $\pi \in \mathcal{P}$ if p_k is a measurable function of $\mathcal{I}_k^s$ for all $0 \le k \le N$, and $\mu \in \mathcal{M}$ if u_k is a measurable function of $\mathcal{I}_k^c$ for all $0 \le k \le N$.

We would like to make a trade-off between the energy capacity and control performance. We measure the energy capacity by an average transmit power, i.e.,

$$E(\pi, \mu) = \tfrac{1}{N+1} \mathsf{E}\left[\textstyle\sum_{k=0}^{N} \alpha_k p_k \right], \tag{3.12}$$

where $\alpha_k \ge 0$ is a weighting coefficient. Moreover, we measure the control performance by an average cost function penalizing the state deviation and control effort, i.e.,

$$J(\pi, \mu) = \tfrac{1}{N+1} \mathsf{E}\left[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u_k^T R_k u_k \right], \tag{3.13}$$

where $Q_k \succeq 0$ and $R_k \succ 0$ are weighting matrices. The desired trade-off is formulated by the following optimization problem:

$$\text{minimize} \quad \lambda J(\pi, \mu) + (1 - \lambda) E(\pi, \mu), \tag{3.14}$$

for $\lambda \in (0, 1)$ over $\pi \in \mathcal{P}$ and $\mu \in \mathcal{M}$. Equivalently, we can solve the following optimization problem:

$$\text{minimize} \quad \Psi(\pi, \mu), \tag{3.15}$$

where

$$\Psi(\pi, \mu) = \mathsf{E}\left[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u_k^T R_k u_k + \theta_k p_k \right], \tag{3.16}$$

where $\theta_k = \alpha_k (1 - \lambda)/\lambda$.

Now, we can formally define the value of information.

Definition 3.1. *The value of information* VoI_k *in the trade-off between the energy capacity and control performance is the variation in the value function associated with the optimization problem in (3.15) with respect to a reliable observation at the controller.*

In the sequel, we shall obtain π^* and μ^* such that (π^*, μ^*) represents a Nash equilibrium, and characterize the value of information VoI_k.

3.2. Main Results

In this section, our goal is to characterize the optimal policies. Let us define the admissible information set of the power scheduler at time k as the set of the current and prior outputs of the process, current and prior channel gains, and prior arrival variables, i.e.,

$$\mathcal{I}_k^s = \left\{ y_t, g_t, \gamma_{t'} \;\middle|\; t \le k, t' < k \right\}, \tag{3.17}$$

and the admissible information set of the controller at time k as the set of the prior successfully received outputs of the process, current and prior channel gains, and prior arrival variables, i.e.,

$$\mathcal{I}_k^c = \Big\{ y_t, g_{t'}, \gamma_{t''} \;\Big|\; t, t'' < k, t' \leq k, \gamma_t = 1 \Big\}. \tag{3.18}$$

We see that the latter is a subset of the former, i.e., $\mathcal{I}_k^c \subseteq \mathcal{I}_k^s$. Note that the information set $\mathcal{I}_k^c$ satisfies Assumption 3.2.

3.2.1. Optimal Estimators. Given that the process is partially observable, both power scheduler and controller have to estimate the state of the process. We shall derive the optimal estimators based on the Bayesian analysis. The next two propositions give the optimal estimators with respect to the information sets $\mathcal{I}_k^s$ and $\mathcal{I}_k^c$ respectively, and show that such estimators are linear.

Proposition 3.1. *The conditional expectation* $\mathsf{E}[x_k|\mathcal{I}_k^s]$ *with the following dynamics minimizes the mean-square error at the power scheduler:*

$$\check{x}_k = A_{k-1}\check{x}_{k-1} + B_{k-1}u_{k-1} + H_k\big(y_k - C_k(A_{k-1}\check{x}_{k-1} + B_{k-1}u_{k-1})\big), \tag{3.19}$$

$$\Sigma_k = \big((A_{k-1}\Sigma_{k-1}A_{k-1}^T + W_{k-1})^{-1} + C_k^T V_k^{-1} C_k\big)^{-1}, \tag{3.20}$$

where

$$H_k = \Sigma_k C_k^T V_k^{-1}, \tag{3.21}$$

for $1 \leq k \leq N$ *with initial conditions* $\check{x}_0 = m_0 + \Sigma_0 C_0^T V_0^{-1}(y_0 - C_0 m_0)$ *and* $\Sigma_0 = (M_0^{-1} + C_0^T V_0^{-1} C_0)^{-1}$ *where* $\check{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^s]$ *and* $\Sigma_k = \mathsf{cov}[x_k|\mathcal{I}_k^s]$.

Proof. The output y_k is available at the power scheduler at each time. Hence, it is clear that given the information set $\mathcal{I}_k^s$ at the power scheduler, the state estimate minimizing the mean-square error is the conditional expectation $\mathsf{E}[x_k|\mathcal{I}_k^s]$, and the optimal estimator is the Kalman filter (see e.g., [**71**]). □

Proposition 3.2. *The conditional expectation* $\mathsf{E}[x_k|\mathcal{I}_k^c]$ *with the following dynamics minimizes the mean-square error at the controller:*

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + \gamma_{k-1}K_{k-1}(y_{k-1} - C_{k-1}\hat{x}_{k-1}), \tag{3.22}$$

$$P_k = A_{k-1}P_{k-1}A_{k-1}^T + W_{k-1} - \gamma_{k-1}K_{k-1}C_{k-1}P_{k-1}A_{k-1}^T, \tag{3.23}$$

where

$$K_{k-1} = A_{k-1}P_{k-1}C_{k-1}^T(C_{k-1}P_{k-1}C_{k-1}^T + V_{k-1})^{-1}, \tag{3.24}$$

for $1 \leq k \leq N$ *with initial conditions* $\hat{x}_0 = m_0$ *and* $P_0 = M_0$ *where* $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^c]$ *and* $P_k = \mathsf{cov}[x_k|\mathcal{I}_k^c]$.

Proof. Given the information set $\mathcal{I}_k^c$ at the controller, the state estimate minimizing the mean-square error is clearly the conditional expectation $\mathsf{E}[x_k|\mathcal{I}_k^c]$. From the definition, $\hat{x}_{k+1} = \mathsf{E}[x_{k+1}|\mathcal{I}_{k+1}^c]$ and $P_{k+1} = \mathsf{cov}[x_{k+1}|\mathcal{I}_{k+1}^c]$. Taking the conditional expectation of (3.1), we get

$$\hat{x}_{k+1} = A_k\,\mathsf{E}[x_k|\mathcal{I}_{k+1}^c] + B_k u_k, \tag{3.25}$$

$$P_{k+1} = A_k\,\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c]A_k^T + W_k. \tag{3.26}$$

From Assumption 3.2, the controller makes no inference when $\gamma_k = 0$. However, the controller receives $\zeta_{k+1} = y_k$ at time $k+1$ and subsequently can make an inference when $\gamma_k = 1$. Let us define $\xi_k = [x_k^T \;\; y_k^T]^T$. We can easily show that

$$\mathsf{E}[\xi_k | \mathcal{I}_k^c] = \begin{bmatrix} \hat{x}_k \\ C_k \hat{x}_k \end{bmatrix}, \tag{3.27}$$

$$\mathsf{cov}[\xi_k | \mathcal{I}_k^c] = \begin{bmatrix} P_k & P_k C_k^T \\ C_k P_k & C_k P_k C_k^T + V_k \end{bmatrix}. \tag{3.28}$$

Now, we can use Lemma A.2 together with the conditional distribution specified by the mean and covariance in (3.27), (3.28), and get

$$\mathsf{E}[x_k | \mathcal{I}_k^c, y_k] = \hat{x}_k + K_k'(y_k - C_k \hat{x}_k),$$

$$\mathsf{cov}[x_k | \mathcal{I}_k^c, y_k] = P_k - K_k' C_k P_k,$$

where $K_k' = P_k C_k^T (C_k P_k C_k^T + V_k)^{-1}$. Therefore, from the definition of γ_k, we can write

$$\mathsf{E}[x_k | \mathcal{I}_{k+1}^c] = \hat{x}_k + \gamma_k K_k'(y_k - C_k \hat{x}_k), \tag{3.29}$$

$$\mathsf{cov}[x_k | \mathcal{I}_{k+1}^c] = P_k - \gamma_k K_k' C_k P_k, \tag{3.30}$$

where we used the definition of $\mathcal{I}_{k+1}^c$ given g_{k+1} and γ_k at the controller at time $k+1$. We obtain the results by substituting (3.29), (3.30) in (3.25), (3.26) respectively. □

As it becomes evident later, in addition to the Kalman filter given in Proposition 3.1, the power scheduler needs to construct a copy of the estimator given in Proposition 3.2. This is possible because $\mathcal{I}_k^c \subseteq \mathcal{I}_k^s$.

3.2.2. Optimal Policies. We shall design the optimal policies using backward induction. Let $e_k = x_k - \hat{x}_k$ be the estimation error and $\nu_k = y_k - C_k \hat{x}_k$ be the innovation both associated with the estimator at the controller. Moreover, let $\varepsilon_k = \check{x}_k - \hat{x}_k$ be the mismatch estimation error associated with the estimators at the power scheduler and controller. We can obtain

$$\mathsf{E}[e_k | \mathcal{I}_k^s] = \mathsf{E}[x_k - \hat{x}_k | \mathcal{I}_k^s] = \check{x}_k - \hat{x}_k = \varepsilon_k, \tag{3.31}$$

$$\mathsf{cov}[e_k | \mathcal{I}_k^s] = \mathsf{cov}[x_k | \mathcal{I}_k^s] = \Sigma_k. \tag{3.32}$$

The next theorem characterizes the structures of the optimal power policy and optimal control policy, and shows that there exists a separation between the optimal designs of the power scheduler and controller.

Theorem 3.1. *Let $S_k \succeq 0$ be a matrix that satisfies the condition in Lemma A.1. The optimal closed-loop power policy is a nonlinear policy given by*

$$p_k^* = \tfrac{N_0 R}{g_k} \phi^{-1}(\mathsf{PSR}_k^*), \tag{3.33}$$

where

$$\begin{aligned} \mathsf{PSR}_k^* = \operatorname*{argmin}_{\mathsf{PSR}_k} \Big\{ & \tfrac{\theta_k N_0 R}{g_k} \phi^{-1}(\mathsf{PSR}_k) \\ & + \mathsf{PSR}_k \, \nu_k^T K_k^T \Gamma_{k+1} (K_k \nu_k - 2 A_k \varepsilon_k) + \hat{\varrho}_k \Big\}, \end{aligned} \tag{3.34}$$

where $\hat{\varrho}_k$ is a variable that depends on ε_k and ν_k, and the optimal closed-loop control policy is a certainty-equivalence policy given by

$$u_k^* = -L_k \hat{x}_k, \tag{3.35}$$

where L_k is the control gain defined as

$$L_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k. \tag{3.36}$$

PROOF. We need to show that (π^*, μ^*) represents a Nash equilibrium. Using the optimal control policy μ^* in the cost function $\Psi(\pi, \mu)$ given by Lemma A.1, we obtain

$$\begin{aligned}\Psi(\pi, \mu^*) = \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{\theta_k p_k + w_k^T S_{k+1} w_k \\ + e_k^T L_k^T (B_k^T S_{k+1} B_k + R_k) L_k e_k\Big\}\Big],\end{aligned}$$

where we used the definition of the estimation error e_k. Following the fact that x_0 and w_k are independent of the power policy, associated with $\Psi(\pi, \mu^*)$, we define the value function V_k^s as

$$V_k^s = \min_{\mathsf{PSR}^k} \mathsf{E}\Big[\textstyle\sum_{t=k}^N \theta_t p_t + e_{t+1}^T \Gamma_{t+1} e_{t+1} \Big| \mathcal{I}_k^s\Big],$$

where $\Gamma_k = L_k^T (B_k^T S_{k+1} B_k + R_k) L_k$ with the exception of $\Gamma_{N+1} = 0$. From the additivity of the value function V_k^s, we have

$$\begin{aligned}V_k^s &= \min_{\mathsf{PSR}_k} \mathsf{E}\Big[\theta_k p_k + e_{k+1}^T \Gamma_{k+1} e_{k+1} \\ &\quad + \min_{\mathsf{PSR}_{k+1}} \mathsf{E}\Big[\theta_{k+1} p_{k+1} + e_{k+2}^T \Gamma_{k+2} e_{k+2} + \dots \Big| \mathcal{I}_{k+1}^s\Big] \Big| \mathcal{I}_k^s\Big] \\ &= \min_{\mathsf{PSR}_k} \mathsf{E}\Big[\theta_k p_k + e_{k+1}^T \Gamma_{k+1} e_{k+1} + V_{k+1}^s \Big| \mathcal{I}_k^s\Big],\end{aligned}$$

with initial condition $V_{N+1}^s = 0$. We prove by induction that the value function V_k^s is independent of the control policy. Clearly, the claim is satisfied for time $N+1$. We assume that the claim holds at time $k+1$, and we shall prove that it also holds at time k. We can write the dynamics of the estimation error at the controller as

$$e_{k+1} = A_k e_k + w_k - \gamma_k K_k \nu_k. \tag{3.37}$$

Thus, we find

$$\begin{aligned}&\mathsf{E}[e_{k+1}^T \Gamma_{k+1} e_{k+1} | \mathcal{I}_k^s] \\ &\quad = \mathsf{E}\Big[e_k^T A_k^T \Gamma_{k+1} A_k e_k + w_k^T \Gamma_{k+1} w_k + \gamma_k^2 \nu_k^T K_k^T \Gamma_{k+1} K_k \nu_k \\ &\qquad + 2 e_k^T A_k^T \Gamma_{k+1} w_k - 2\gamma_k \nu_k^T K_k^T \Gamma_{k+1} w_k - 2\gamma_k \nu_k^T K_k^T \Gamma_{k+1} A_k e_k \Big| \mathcal{I}_k^s\Big] \\ &\quad = \varepsilon_k^T A_k^T \Gamma_{k+1} A_k \varepsilon_k + \mathrm{tr}(A_k^T \Gamma_{k+1} A_k \Sigma_k) + \mathrm{tr}(\Gamma_{k+1} W_k) \\ &\qquad + \mathsf{PSR}_k\, \nu_k^T K_k^T \Gamma_{k+1} K_k \nu_k - 2\, \mathsf{PSR}_k\, \nu_k^T K_k^T \Gamma_{k+1} A_k \varepsilon_k,\end{aligned}$$

where in the second equality we used the definitions of ε_k and Σ_k and the facts that ν_k is $\mathcal{I}_k^s$-measurable and that w_k is independent of e_k. Hence, we have

$$\begin{aligned} V_k^s = \min_{\mathsf{PSR}_k} \Big\{ & \theta_k p_k + \varepsilon_k^T A_k^T \Gamma_{k+1} A_k \varepsilon_k + \mathrm{tr}(A_k^T \Gamma_{k+1} A_k \Sigma_k) \\ & + \mathrm{tr}(\Gamma_{k+1} W_k) + \mathsf{PSR}_k\, \nu_k^T K_k^T \Gamma_{k+1} K_k \nu_k \\ & - 2\,\mathsf{PSR}_k\, \nu_k^T K_k^T \Gamma_{k+1} A_k \varepsilon_k + \mathsf{E}[V_{k+1}^s | \mathcal{I}_k^s] \Big\}. \end{aligned} \tag{3.38}$$

The minimizer in (3.38) is obtained as

$$\begin{aligned} \mathsf{PSR}_k^* = \operatorname*{argmin}_{\mathsf{PSR}_k} \Big\{ & \tfrac{\theta_k N_0 R}{g_k} \phi^{-1}(\mathsf{PSR}_k) \\ & + \mathsf{PSR}_k\, \nu_k^T K_k^T \Gamma_{k+1} (K_k \nu_k - 2 A_k \varepsilon_k) + \hat{\varrho}_k \Big\}, \end{aligned}$$

where $\hat{\varrho}_k = \mathsf{E}[V_{k+1}^s | \mathcal{I}_k^s]$. From the hypothesis assumption $\hat{\varrho}_k$ is independent of the control policy. Hence, we conclude that V_k^s is independent of the control policy. This complete the induction.

Now, using the the power policy π^* in the cost function $\Psi(\pi, \mu)$ given by Lemma A.1, we obtain

$$\begin{aligned} \Psi(\pi^*, \mu) = \mathsf{E} \Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{ & \tfrac{\theta_k N_0 R}{g_k} \phi^{-1}(\mathsf{PSR}_k^*) + w_k^T S_{k+1} w_k \\ & + (u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \Big\} \Big], \end{aligned}$$

where $\Lambda_k = B_k^T S_{k+1} B_k + R_k$. Following the fact that x_0, PSR_k^*, and w_k are independent of the control policy, associated with $\Psi(\pi^*, \mu)$, we define the auxiliary value function V_k^c as

$$V_k^c = \min_{\mathbf{u}^k} \mathsf{E} \Big[\textstyle\sum_{t=k}^N (u_t + L_t x_t)^T \Lambda_t (u_t + L_t x_t) \Big| \mathcal{I}_k^c \Big].$$

From the additivity of the auxiliary value function V_k^c, we obtain

$$\begin{aligned} V_k^c &= \min_{u_k} \mathsf{E} \Big[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \\ &\quad + \min_{u_{k+1}} \mathsf{E} \Big[(u_{k+1} + L_{k+1} x_{k+1})^T \Lambda_{k+1} \\ &\quad \times (u_{k+1} + L_{k+1} x_{k+1}) + \dots \Big| \mathcal{I}_{k+1}^c \Big] \Big| \mathcal{I}_k^c \Big] \\ &= \min_{u_k} \mathsf{E} \Big[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) + V_{k+1}^c \Big| \mathcal{I}_k^c \Big], \end{aligned}$$

with initial condition $V_{N+1}^c = 0$. We prove by induction that the auxiliary value function V_k^c is a function of P_k. Clearly, the claim is satisfied for time $N+1$. We assume that the claim holds at time $k+1$, and we shall prove that it also holds at time k. Using the identity $x_k = \hat{x}_k + e_k$, we find

$$\begin{aligned}
&\mathsf{E}\left[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \middle| \mathcal{I}_k^c\right] \\
&\quad = \mathsf{E}\Big[(u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) \\
&\qquad + e_k^T L_k^T \Lambda_k L_k e_k + 2(u_k + L_k \hat{x}_k)^T \Lambda_k L_k e_k \Big| \mathcal{I}_k^c\Big] \\
&\quad = (u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) + \mathrm{tr}(L_k^T \Lambda_k L_k P_k),
\end{aligned}$$

where in the second equality we used the fact that $\hat{x}_k$ is $\mathcal{I}_k^c$-measurable and $\mathsf{E}[e_k | \mathcal{I}_k^c] = 0$. Hence, we have

$$\begin{aligned}
V_k^c = \min_{u_k} \Big\{ &(u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) \\
&+ \mathrm{tr}(L_k^T \Lambda_k L_k P_k) + \mathsf{E}[V_{k+1}^c | \mathcal{I}_k^c] \Big\}.
\end{aligned} \tag{3.39}$$

The minimizer in (3.39) is obtained as $u_k^* = -L_k \hat{x}_k$. Moreover, we conclude that V_k^c is a function of P_k. This completes the induction and also the proof. □

According to Theorem 3.1, the optimal power policy depends on ε_k, ν_k, and g_k, and is independent of the control policy. Besides, the error covariance P_k in (3.23) does not depend on $\mathbf{u}_{k-1}$. Hence, the control has no dual effect. In Theorem 3.1, we expressed the value of information based on the value function V_k^s and corresponding to the successful transmission of an observation. This is emphasized in the following definition.

Definition 3.2. *The value of information* VoI_k *in the trade-off between the energy capacity and control performance is given by*

$$\mathrm{VoI}_k = \nu_k^T K_k^T \Gamma_{k+1} (2 A_k \varepsilon_k - K_k \nu_k) - \theta_k p_{max} + \varrho_k, \tag{3.40}$$

where $\varrho_k = \mathsf{E}[V_{k+1}^s | \mathcal{I}_k^s, \gamma_k = 0] - \mathsf{E}[V_{k+1}^s | \mathcal{I}_k^s, \gamma_k = 1]$.

Remark 3.5. *The optimal power policy provided above depends on the variable* $\hat{\varrho}_k$. *Although* $\hat{\varrho}_k$ *can be computed with an arbitrary accuracy by solving recursively the optimality equation in (3.38), its computation is expensive. Next, we shall introduce a procedure for approximation of this variable.*

3.2.3. Approximation Algorithm.

We here provide a rollout algorithm [**72**] for approximation of the variable $\hat{\varrho}_k$ and the value of information VoI_k, and accordingly synthesize a closed-form suboptimal power policy with a performance guarantee that can readily be implemented. Let $\bar{\pi} = \{\bar{p}_0, \ldots, \bar{p}_N\}$ be a power policy with packet success rates $\mathsf{PSR}_k \simeq 1$ for all $0 \leq k \leq N$. The following algorithm gives an approximation of the variable $\hat{\varrho}_k$.

Algorithm 3.1. *An approximation of the variable* $\hat{\varrho}_k$ *associated with the policy* $\bar{\pi}$ *is given by*

$$\hat{\varrho}_k^{\bar{\pi}} = \mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^s], \tag{3.41}$$

where

$$\mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^s] = \mathsf{E}\left[\textstyle\sum_{t=k+1}^{N} \theta_t p_t + e_{t+1}^T \Gamma_{t+1} e_{t+1} \middle| \mathcal{I}_k^s \right],$$

with $\mathbf{p}^{k+1} = \bar{\mathbf{p}}^{k+1}$.

Similar to Theorem 2.2, we can prove that it is possible to synthesize a suboptimal power policy that outperforms the policy $\bar{\pi}$. In the next proposition, we synthesize a closed-form suboptimal power policy with a performance guarantee.

Proposition 3.3. *A suboptimal power policy that outperforms the policy* $\bar{\pi}$ *is given by*

$$p_k^+ = \tfrac{N_0 R}{g_k}\phi^{-1}(\mathsf{PSR}_k^+), \tag{3.42}$$

where

$$\begin{aligned} \mathsf{PSR}_k^+ = \operatorname*{argmin}_{\mathsf{PSR}_k} \Big\{ & \tfrac{\theta_k N_0 R}{g_k}\phi^{-1}(\mathsf{PSR}_k) + \mathsf{PSR}_k\, \nu_k^T K_k^T \Gamma_{k+1}(K_k \nu_k - 2A_k\varepsilon_k) \\ & - \mathsf{PSR}_k \Big(\textstyle\sum_{t=k+2}^N \bar{e}_t^{0T}\Gamma_t \bar{e}_t^0 + \mathrm{tr}(\Gamma_t \bar{P}_t^0) - \bar{e}_t^{1T}\Gamma_t\bar{e}_t^1 - \mathrm{tr}(\Gamma_t \bar{P}_t^1)\Big)\Big\}, \end{aligned} \tag{3.43}$$

and

$$\bar{e}_{t+1}^0 = (A_t - K_t^0 C_t)\bar{e}_t^0,$$

$$\bar{P}_{t+1}^0 = (A_t - K_t^0 C_t)\bar{P}_t^0(A_t - K_t^0 C_t)^T + W_t + K_t^0 V_t K_t^{0T},$$

$$P_{t+1}^0 = A_t P_t^0 A_t^T + W_t - K_t^0 C_t P_t^0 A_t^T,$$

$$\bar{e}_{t+1}^1 = (A_t - K_t^1 C_t)\bar{e}_t^1,$$

$$\bar{P}_{t+1}^1 = (A_t - K_t^1 C_t)\bar{P}_t^1(A_t - K_t^1 C_t)^T + W_t + K_t^1 V_t K_t^{1T},$$

$$P_{t+1}^1 = A_t P_t^1 A_t^T + W_t - K_t^1 C_t P_t^1 A_t^T,$$

where

$$K_t^0 = A_t P_t^0 C_t^T (C_t P_t^0 C_t^T + V_t)^{-1},$$

$$K_t^1 = A_t P_t^1 C_t^T (C_t P_t^1 C_t^T + V_t)^{-1},$$

for $t \geq k+1$ *with initial conditions* $\bar{e}_{k+1}^0 = A_k \varepsilon_k$, $\bar{P}_{k+1}^0 = A_k \Sigma_k A_k^T + W_k$, $P_{k+1}^0 = A_k P_k A_k^T + W_k$, $\bar{e}_{k+1}^1 = A_k\varepsilon_k - K_k \nu_k$, $\bar{P}_{k+1}^1 = A_k\Sigma_k A_k^T + W_k$, *and* $P_{k+1}^1 = A_k P_k A_k^T + W_k - K_k C_k P_k A_k^T$.

PROOF. For the proof, it is enough to derive $\hat{\varrho}_k^{\bar{\pi}}$ based on the policy $\bar{\pi}$. First, note that

$$\begin{aligned} \mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^s, \gamma_k] &= \mathsf{E}\Big[\textstyle\sum_{t=k+1}^N \theta_t \bar{p}_t + e_{t+1}^T \Gamma_{t+1} e_{t+1} \Big| \mathcal{I}_k^s, \gamma_k\Big] \\ &= \textstyle\sum_{t=k+1}^N \theta_t \bar{p}_t + \bar{e}_{t+1}^T \Gamma_{t+1}\bar{e}_{t+1} + \mathrm{tr}(\Gamma_{t+1}\bar{P}_{t+1}), \end{aligned}$$

where in the first equality we used the definition of $V_{k+1}^{\bar{\pi}}$ and the fact that $p_t = \bar{p}_t$ for all $t \geq k+1$ and in the second equality the definitions $\bar{e}_t = \mathsf{E}[e_t|\mathcal{I}_k^s, \gamma_k]$ and $\bar{P}_t = \mathsf{cov}[e_t | \mathcal{I}_k^s, \gamma_k]$ for all $t \geq k+1$.

From the dynamics of the estimation error in (3.37), given the fact that $\gamma_t = 1$ with $\mathsf{PSR}_t \simeq 1$ for all $t \geq k+1$, we obtain

$$e_{t+1} = (A_t - K_t C_t)e_t + w_t - K_t v_t.$$

Accordingly, when $\gamma_k = 0$, we have

$$\bar{e}^0_{t+1} = (A_t - K^0_t C_t)\bar{e}^0_t,$$

$$\bar{P}^0_{t+1} = (A_t - K^0_t C_t)\bar{P}^0_t (A_t - K^0_t C_t)^T + W_t + K^0_t V_t K^{0T}_k,$$

and

$$P^0_{t+1} = A_t P^0_t A^T_t + W_t - K^0_t C_t P^0_t A^T_t,$$

$$K^0_t = A_t P^0_t C^T_t (C_t P^0_t C^T_t + V_t)^{-1},$$

for $t \geq k+1$ with initial conditions $\bar{e}^0_{k+1} = A_k \varepsilon_k$, $\bar{P}^0_{k+1} = A_k \Sigma_k A^T_k + W_k$, and $P^0_{k+1} = A_k P_k A^T_k + W_k$. The initial conditions $\bar{e}^1_{k+1}$ and $\bar{P}^1_{k+1}$ were obtained by using (3.37). Hence, we find

$$\mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 0] = \textstyle\sum_{t=k+1}^N \theta_t \bar{p}_t + \bar{e}^{0T}_{t+1} \Gamma_{t+1} \bar{e}^0_{t+1} + \operatorname{tr}(\Gamma_{t+1} \bar{P}^0_{t+1}).$$

Moreover, when $\gamma_k = 1$, we have

$$\bar{e}^1_{t+1} = (A_t - K^1_t C_t)\bar{e}^1_t,$$

$$\bar{P}^1_{t+1} = (A_t - K^1_t C_t)\bar{P}^1_t (A_t - K^1_t C_t)^T + W_t + K^1_t V_t K^{1T}_k,$$

and

$$P^1_{t+1} = A_t P^1_t A^T_t + W_t - K^1_t C_t P^1_t A^T_t,$$

$$K^1_t = A_t P^1_t C^T_t (C_t P^1_t C^T_t + V_t)^{-1},$$

for $t \geq k+1$ with initial conditions $\bar{e}^1_{k+1} = A_k \varepsilon_k - K_k \nu_k$, $\bar{P}^1_{k+1} = A_k \Sigma_k A^T_k + W_k$, and $P^1_{k+1} = A_k P_k A^T_k + W_k - K_k C_k P_k A^T_k$. The initial conditions $\bar{e}^1_{k+1}$ and $\bar{P}^1_{k+1}$ were obtained by using (3.37). Hence, we find

$$\mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 1] = \textstyle\sum_{t=k+1}^N \theta_t \bar{p}_t + \bar{e}^{1T}_{t+1} \Gamma_{t+1} \bar{e}^1_{t+1} + \operatorname{tr}(\Gamma_{t+1} \bar{P}^1_{t+1}).$$

Finally, following the definition of $\hat{\varrho}^{\bar{\pi}}_k$, we have

$$\begin{aligned}
\hat{\varrho}^{\bar{\pi}}_k &= (1 - \mathsf{PSR}_k)\, \mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 0] + \mathsf{PSR}_k\, \mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 1] \\
&= \mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 0] - \mathsf{PSR}_k \left(\mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 0] - \mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 1] \right) \\
&= \hat{\varrho}_0 - \mathsf{PSR}_k \Big(\textstyle\sum_{t=k+1}^N \Big\{ \bar{e}^{0T}_{t+1} \Gamma_{t+1} \bar{e}^0_{t+1} + \operatorname{tr}(\Gamma_{t+1} \bar{P}^0_{t+1}) \\
&\qquad - \bar{e}^{1T}_{t+1} \Gamma_{t+1} \bar{e}^1_{t+1} - \operatorname{tr}(\Gamma_{t+1} \bar{P}^1_{t+1} \Big\} \Big) \\
&= \hat{\varrho}_0 - \mathsf{PSR}_k \Big(\textstyle\sum_{t=k+2}^N \bar{e}^{0T}_t \Gamma_t \bar{e}^0_t + \operatorname{tr}(\Gamma_t \bar{P}^0_t) - \bar{e}^{1T}_t \Gamma_t \bar{e}^1_t - \operatorname{tr}(\Gamma_t \bar{P}^1_t) \Big),
\end{aligned}$$

where in the third equality $\hat{\varrho}_0 = \mathsf{E}[V^{\bar{\pi}}_{k+1} | \mathcal{I}^s_k, \gamma_k = 0]$ is a constant and in the last equality we used the fact that $\Gamma_{N+1} = 0$. Incorporating this into (3.34), we obtain the result. $\square$

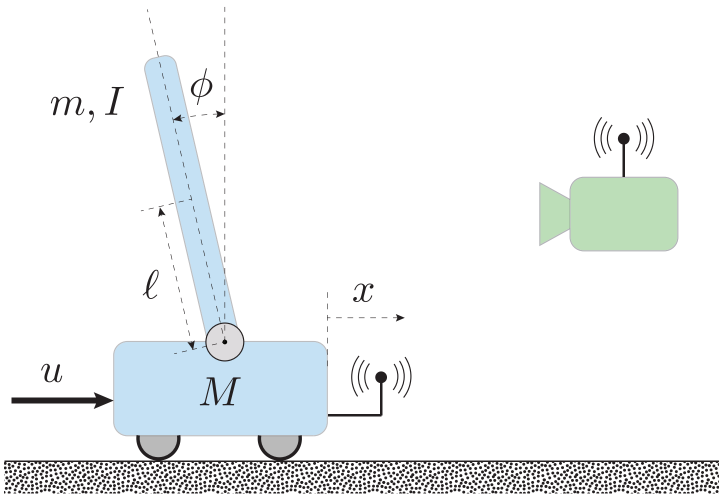

FIGURE 3.4. Model of an inverted pendulum on a cart. The controller is insourced, but the sensor is outsourced to a wireless node.

3.3. Numerical Example

In this section, we show an application of the theoretical framework we developed in this chapter. Consider an inverted pendulum on a cart observed by a remote sensor (see Fig. 3.4). The parameters A_k, B_k, C_k, W_k, V_k, m_0, and M_0 are similar to those used in Chapter 2.

The sensor is connected to an internal controller through a wireless fading channel that is assumed to be a discrete-time AWGN channel. The fading distribution is chosen according to a combined path loss and shadowing model [**25**], which is given by

$$g_k = \left(\frac{4\pi f d_0}{c}\right)^{-2} \left(\frac{d}{d_0}\right)^{-\eta} 10^{\psi_k/10},$$

where f is the carrier frequency, d_0 is the reference distance, c is the speed of light, d is the distance between the receiver and transmitter, η is the path loss exponent, and ψ_k is a Gaussian variable with zero mean and variance Σ_ψ. The channel is characterized by the following parameters: the noise bandwidth $W = 2$ MHz, data rate $R = 250$ Kbps, carrier frequency $f = 2.45$ GHz, receiver distance $d = 10.00$ m, reference distance $d_0 = 1$ m, path loss exponent $\eta = 3.6$, noise power spectral density $N_0 = 10^{-18}$ watt, and shadowing variance $\Sigma_\psi = 8$. We suppose that the channel performance is specified by the functions in (3.6) and (3.8) with $\rho_0 = 1$, $\rho_1 = 2$, and $\ell = 128$ bits.

For this system, we are interested in designing a power scheduler that is employed at the sensor and a controller that is collocated with the actuator. The cost function of form (3.16) is specified by the weights $Q_{N+1} = \operatorname{diag}\{1, 1, 1000, 1\}$, $Q_k = \operatorname{diag}\{1, 1, 1000, 1\}$, $R_k = 1$, and $\theta_k = 9 \times 10^6$ for all $0 \leq k \leq N$ where $N = 500$. The state estimates $\check{x}_k$ and $\hat{x}_k$ are provided by Proposition 3.1 and Proposition 3.2 respectively. From Theorem 3.1, it follows that the optimal power policy is $p_k^* = \frac{N_0 R}{g_k}\phi^{-1}(\mathsf{PSR}_k^*)$ and the optimal control policy is $u_k^* = -L_k \hat{x}_k$. We approximated PSR_k^* according to Proposition 3.3 and by limiting PSR_k in the interval $[0.00001, 0.99999]$, and obtained the control gain L_k by solving the Riccati equation in Lemma A.1. We carried out a simulation experiment. For a realization of the system, Fig. 3.5 and Fig. 3.6 illustrate the trajectories of the main variables of the system. In particular, the trajectories of the position, velocity, pitch angle, and pitch rate are shown in Fig. 3.5. Moreover, the trajectories of the channel gain, packet success rate, arrival process, power, and control are shown in Fig. 3.6. As it is observed, the packet success rate is at its lowest value at many times, and the wireless sensor avoids a transmission at those times to preserve energy further. Moreover, it is seen that a high transmission power is used when the wireless sensor decides to transmit an observation.

3.4. Summary

In this chapter, we provided a theoretical framework for the analysis and design of wireless control systems with power schedulers. We formulated the problem as a dynamic game, and characterized the optimal power policy and optimal control policy such that the corresponding policy profile represents a Nash equilibrium. We proved, under certain assumptions, that the optimal closed-loop power policy is a nonlinear policy with packet success rate that particularly depends on the estimation innovation at the controller, mismatch estimation error, and channel gain; and that the optimal closed-loop control policy is a certainty-equivalence policy. Moreover, we synthesized a closed-form suboptimal power policy with a performance guarantee. We propose that as future research one should extend the present framework to networks of interacting systems in which the effect of the channel interference is taken into account. In such a case, the interference imposes a constraint that must be incorporated in the underlying optimization problem.

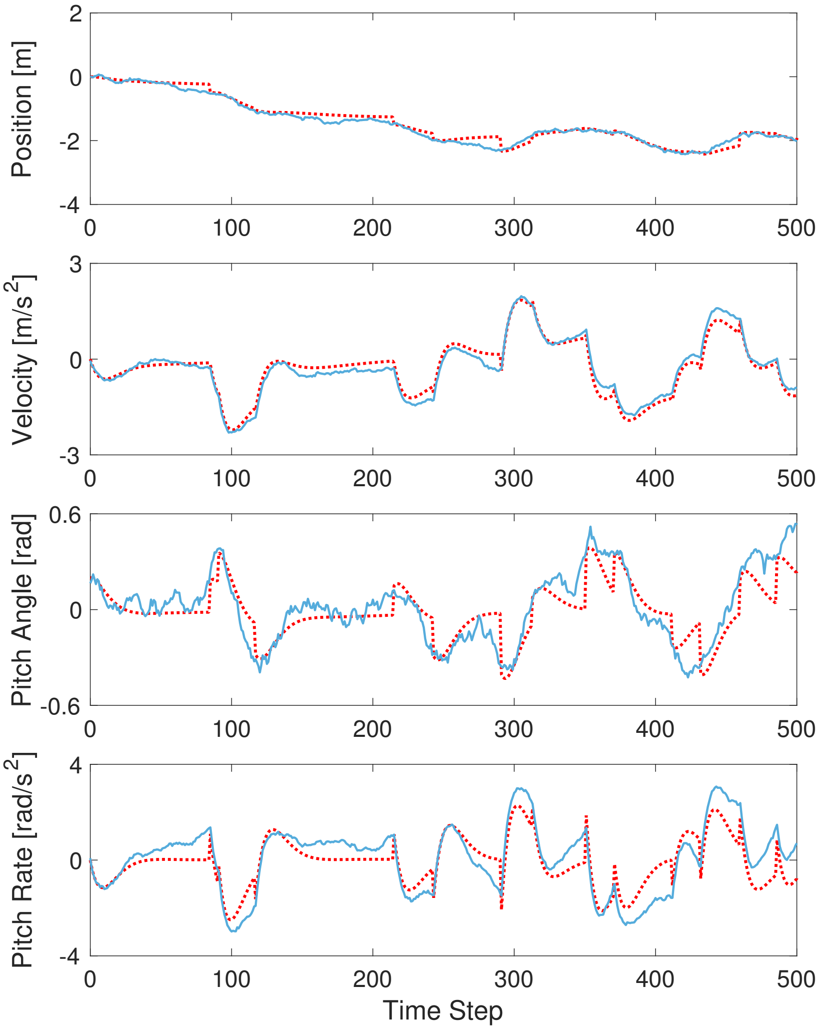

FIGURE 3.5. Trajectories of the position, velocity, pitch angle, and pitch rate. The solid lines represent the state components and the dotted lines represent the state estimate components at the controller.

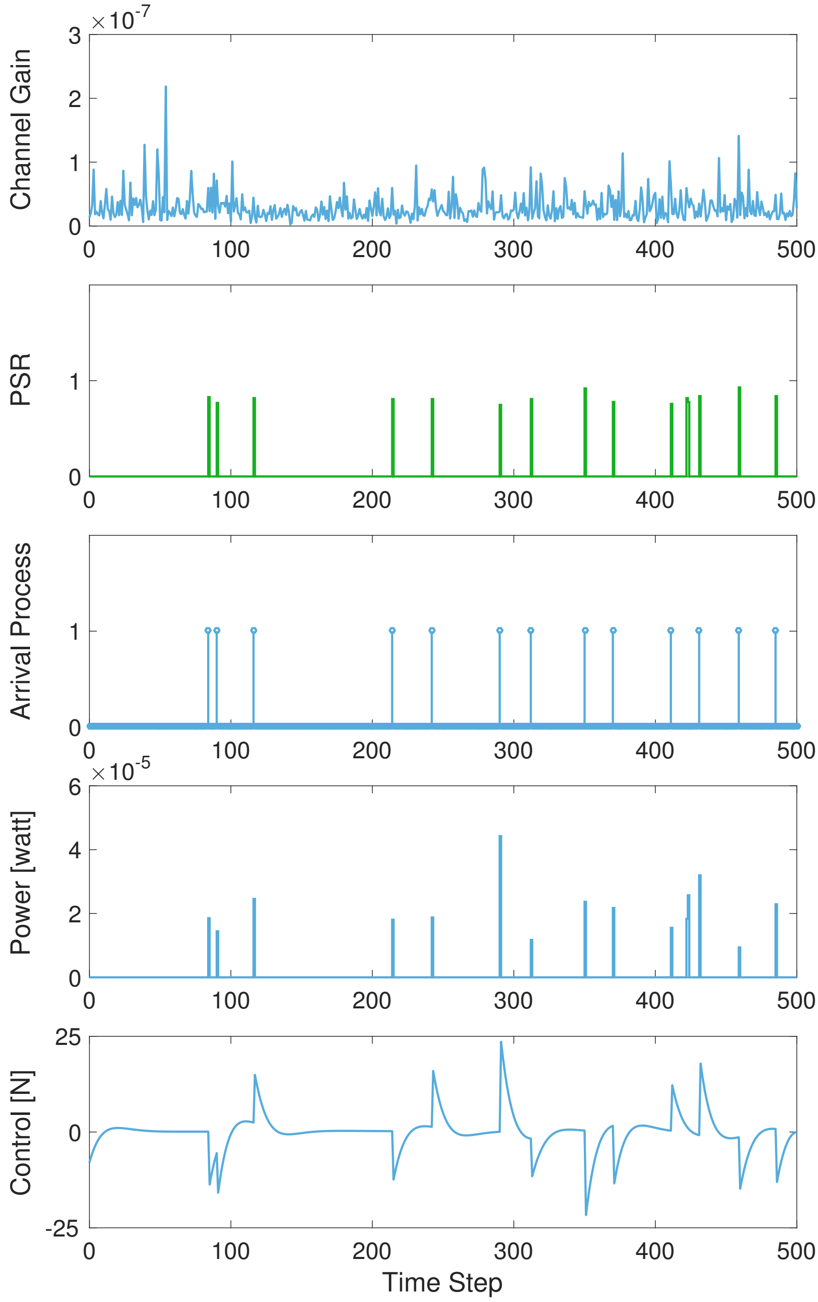

FIGURE 3.6. Trajectories of the channel gain, packet success rate, arrival process, transmit power, and control.

CHAPTER 4

Optimal Privacy Filter in Cloud Control

In this chapter, we consider a cloud control system, and seek to jointly design a privacy filter and a controller by constructing a trade-off between the privacy level and control performance. Based on our unified framework, we develop the optimal estimators and characterize the optimal policies such that the corresponding policy profile represents a Nash equilibrium. Moreover, we define the value of information in the trade-off between the privacy level and control performance. This chapter is organized in the following way. We formulate the problem in Section 4.1. We provide the main results of the chapter in Section 4.2. We present a numerical example in Section 4.3. Finally, we give a summary in Section 4.4.

4.1. Problem Formulation

In this section, we describe the system model. We specify the privacy filter, and define the indices for the privacy level and control performance. To measure the amount of information leakage, we adopt the causally-conditioned directed information. Then, we formulate the main problem of the chapter. The material here requires basic knowledge of information theory, stochastic control theory, and game theory.

4.1.1. Process Model. Consider a stochastic process with dynamics generated by the following linear discrete-time time-varying state system:

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{4.1}$$

for $0 \leq k \leq N$ and with initial condition x_0 where $x_k \in \mathbb{R}^n$ is the state of the process, $A_k \in \mathbb{R}^{n \times n}$ is the state matrix, $B_k \in \mathbb{R}^{n \times m}$ is the input matrix, $u_k \in \mathbb{R}^m$ is the control input to be decided by a *cloud provider*, $w_k \in \mathbb{R}^n$ is a white Gaussian noise with zero mean and covariance $W_k \succ 0$, and N is a finite terminal time. It is assumed that x_0 is a Gaussian vector with mean m_0 and covariance M_0. At each time, a noisy output of the process is measured by a sensor, which given by

$$y_k = C_k x_k + v_k, \tag{4.2}$$

for $0 \leq k \leq N$ where $y_k \in \mathbb{R}^p$ is the output of the process, $C_k \in \mathbb{R}^{p \times n}$ is the output matrix, and $v_k \in \mathbb{R}^p$ is a white Gaussian noise with zero mean and covariance $V_k \succ 0$. It is assumed that x_0, w_k, and v_k are mutually independent for all $0 \leq k \leq N$. In addition, it is assumed that (A_k, B_k) is controllable and (A_k, C_k) is observable.

A privacy filter is employed at the sensor, which obfuscates the private information (i.e., the sate of the process) and transmits public information to the controller (see Fig. 4.1). At time k, the output of the process y_k is available at the privacy filter instantly, and a perturbed output is transmitted to the controller with one-step delay. Next, we describe the structure of the privacy filter.

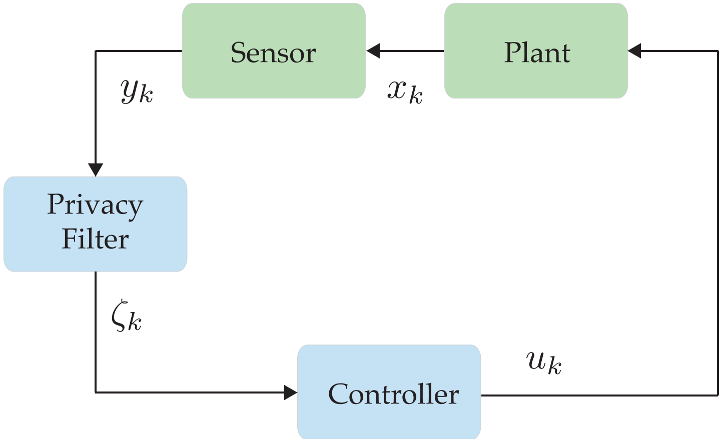

FIGURE 4.1. Schematic view of a cloud control system with a privacy filter. At each time, the output of the process is available at the privacy filter instantly, and a perturbed output is transmitted to the controller with one-step delay.

4.1.2. Privacy Protection and Information Leakage. Consider a privacy filter with an additive Gaussian noise structure in which the perturbed output is given by

$$z_k = E_k y_k + n_k, \tag{4.3}$$

where $z_k \in \mathbb{R}^q$ is the perturbed output, $E_k \in \mathbb{R}^{q \times p}$ is the privacy filter's output matrix, and $n_k \in \mathbb{R}^q$ is a Gaussian noise generated with zero mean and covariance N_k. We assume that n_k is independent of the state given N_k for all $0 \leq k \leq N$. In the sequel, we assume that $E_k = I$ and $q = p$. In this case, the perturbation covariance N_k is the only decision variable of the privacy filter. Moreover, we define $\zeta_{k+1} = z_k$ as the perturbed output subject to one-step delay.

Remark 4.1. *Similar to the analysis in [74], one can show that for the problem considered here the optimal test channel is in fact an additive Gaussian noise channel. This justifies the chosen structure of the privacy filter.*

Now, consider the sequences $\mathbf{x}_N$, $\mathbf{z}_N$, and $\mathbf{u}_N$ over the horizon N. The casually-conditioned entropy of $\mathbf{z}_N$ given $\mathbf{u}_N$ is defined by

$$h(\mathbf{z}_N || \mathbf{u}_N) = \textstyle\sum_{k=0}^{N} h(z_k | \mathbf{z}_{k-1}, \mathbf{u}_k), \tag{4.4}$$

and the casually-conditioned entropy of $\mathbf{z}_N$ given $\mathbf{x}_N$ and $\mathbf{u}_N$ is defined by

$$h(\mathbf{z}_N || \mathbf{x}_N, \mathbf{u}_N) = \textstyle\sum_{k=0}^{N} h(z_k | \mathbf{z}_{k-1}, \mathbf{x}_k, \mathbf{u}_k). \tag{4.5}$$

Then, the information leakage by disclosure of the public information to the controller can be measured by the causally-conditioned directed information [**62**] from $\mathbf{x}_N$ to $\mathbf{z}_N$ given $\mathbf{u}_N$, which is obtained as

$$I(\mathbf{x}_N \to \mathbf{z}_N || \mathbf{u}_N) = h(\mathbf{z}_N || \mathbf{u}_N) - h(\mathbf{z}_N || \mathbf{x}_N, \mathbf{u}_N). \tag{4.6}$$

Furthermore, given the structure of the privacy filter in (4.3) and from the properties of the mutual information, we can write:

$$
\begin{aligned}
I(\mathbf{x}_N \to \mathbf{z}_N || \mathbf{u}_N) &= \textstyle\sum_{k=0}^N I(\mathbf{x}_k; z_k | \mathbf{z}_{k-1}, \mathbf{u}_k) \\
&= \textstyle\sum_{k=0}^N I(x_k; z_k | \mathbf{z}_{k-1}, \mathbf{u}_k) \\
&\quad + \textstyle\sum_{k=0}^N I(\mathbf{x}_{k-1}; z_k | x_k, \mathbf{z}_{k-1}, \mathbf{u}_k) \\
&= \textstyle\sum_{k=0}^N I(x_k; z_k | \mathbf{z}_{k-1}, \mathbf{u}_k) \\
&= \textstyle\sum_{k=0}^N h(x_k | \mathbf{z}_{k-1}, \mathbf{u}_k) - h(x_k | \mathbf{z}_k, \mathbf{u}_k),
\end{aligned} \tag{4.7}
$$

where the first equality comes from the properties of the mutual information and $I(\mathbf{x}_k; z_k|\mathbf{z}_{k-1}, \mathbf{u}_k)$ is the conditional mutual information between $\mathbf{x}_k$ and z_k given $\mathbf{z}_{k-1}$ and $\mathbf{u}_k$, the second equality comes from the chain rule, the third equality from the fact that z_k is independent of $\mathbf{x}_{k-1}$ given x_k, and the last equality from the definition of the mutual information.

In the sequel, we assume that the following assumption holds.

Assumption 4.1. *The information that the perturbation covariance N_k carries about the process is discarded at the controller.*

Remark 4.2. *Assumption 4.1 leads to a Gaussian conditional distribution at the controller, and allows us to obtain the optimal estimator at the controller in a tractable way. Instead of the above assumption, one can assume that state estimates under perturbed observations, which will be derived later, are transmitted to the controller.*

4.1.3. Trade-off Problem. Consider a perturbation policy $\pi = \{z_0, \ldots, z_N\}$ and a control policy $\mu = \{u_0, \ldots, u_N\}$. Let $\mathcal{I}_k^f$ and $\mathcal{I}_k^c$ denote the admissible information set of the privacy filter and of the controller at time k respectively, and $\mathcal{P}$ and $\mathcal{M}$ denote the admissible set of perturbation policies and of control policies respectively. Then, $\pi \in \mathcal{P}$ if z_k is a measurable function of $\mathcal{I}_k^f$ for all $0 \le k \le N$, and $\mu \in \mathcal{M}$ if u_k is a measurable function of $\mathcal{I}_k^c$ for all $0 \le k \le N$.

We would like to make a trade-off between the privacy level and control performance. We measure the privacy level by a weighted causally-conditioned directed information from the private information to the public information, i.e.,

$$
P(\mu, \pi) = \tfrac{1}{N+1} \mathsf{E}\left[\textstyle\sum_{k=0}^N \alpha_k I(\mathbf{x}_k; z_k | \mathbf{z}_{k-1}, \mathbf{u}_k)\right], \tag{4.8}
$$

where α_k is a weighting coefficient. Moreover, we measure the control performance by an average cost function penalizing the state deviation and control effort, i.e.,

$$
J(\pi, \mu) = \tfrac{1}{N+1} \mathsf{E}\left[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^N x_k^T Q_k x_k + u_k^T R_k u_k\right], \tag{4.9}
$$

where $Q_k \succeq 0$ and $R_k \succ 0$ are weighting matrices. The desired trade-off is formulated by the following optimization problem:

$$
\text{minimize} \quad \lambda J(\pi, \mu) + (1-\lambda) P(\pi, \mu), \tag{4.10}
$$

for $\lambda \in (0, 1)$ over $\pi \in \mathcal{P}$ and $\mu \in \mathcal{M}$. Equivalently, we can solve the following optimization problem:

$$
\text{minimize} \quad \Psi(\pi, \mu), \tag{4.11}
$$

where

$$\Psi(\pi,\mu) = \mathsf{E}\Big[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u_k^T R_k u_k \\ + \textstyle\sum_{k=0}^{N} \theta_k I(\mathbf{x}_k; z_k | \mathbf{z}_{k-1}, \mathbf{u}_k)\Big], \tag{4.12}$$

where $\theta_k = \alpha_k(1-\lambda)/\lambda$.

Now, we can formally define the value of information.

Definition 4.1. *The value of information* VoI_k *in the trade-off between the privacy level and control performance is the variation in the value function associated with the optimization problem in (4.11) with respect to an unperturbed observation at the controller.*

In the sequel, we shall obtain π^* and μ^* such that (π^*, μ^*) represents a Nash equilibrium, and characterize the value of information VoI_k.

4.2. Main Results

In this section, our goal is to characterize the optimal policies. Let us define the admissible information set of the privacy filter at time k as the set of the current and prior outputs and prior perturbations, i.e.,

$$\mathcal{I}_k^f = \Big\{ y_t, n_{t'} \,\Big|\, t \le k, t' < k \Big\}, \tag{4.13}$$

and the admissible information set of the controller at time k as the set of the prior perturbed outputs and prior perturbation covariances, i.e.,

$$\mathcal{I}_k^c = \Big\{ z_t, N_t \,\Big|\, t < k \Big\}. \tag{4.14}$$

We see that the latter is a subset of the former, i.e., $\mathcal{I}_k^c \subseteq \mathcal{I}_k^f$. Note that the information set $\mathcal{I}_k^c$ satisfies Assumption 4.1.

4.2.1. Optimal Estimators.

Given that the process is partially observable, both privacy filter and controller have to estimate the state of the process. We shall derive the optimal estimators based on the Bayesian analysis. The next two propositions give the optimal estimators with respect to the information sets $\mathcal{I}_k^f$ and $\mathcal{I}_k^c$ respectively, and show that such estimators are linear.

Proposition 4.1. *The conditional expectation* $\mathsf{E}[x_k|\mathcal{I}_k^f]$ *with the following dynamics minimizes the mean-square error at the privacy filter:*

$$\check{x}_k = A_{k-1}\check{x}_{k-1} + B_{k-1}u_{k-1} + H_k\big(y_k - C_k(A_{k-1}\check{x}_{k-1} + B_{k-1}u_{k-1})\big), \tag{4.15}$$

$$\Sigma_k = \big((A_{k-1}\Sigma_{k-1}A_{k-1}^T + W_{k-1})^{-1} + C_k^T V_k^{-1} C_k\big)^{-1}, \tag{4.16}$$

where

$$H_k = \Sigma_k C_k^T V_k^{-1}, \tag{4.17}$$

for $1 \le k \le N$ *with initial conditions* $\check{x}_0 = m_0 + \Sigma_0 C_0^T V_0^{-1}(y_0 - C_0 m_0)$ *and* $\Sigma_0 = (M_0^{-1} + C_0^T V_0^{-1} C_0)^{-1}$ *where* $\check{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^f]$ *and* $\Sigma_k = \mathsf{cov}[x_k|\mathcal{I}_k^f]$.

PROOF. The output y_k is available at the privacy filter at each time. Hence, it is clear that given the information set $\mathcal{I}_k^f$ at the privacy filter, the state estimate minimizing the mean-square error is the conditional expectation $\mathsf{E}[x_k|\mathcal{I}_k^f]$, and the optimal estimator is the Kalman filter (see e.g., [**71**]). □

Proposition 4.2. *The conditional expectation $\mathsf{E}[x_k|\mathcal{I}_k^c]$ with the following dynamics minimizes the mean-square error at the controller:*

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + K_{k-1}(z_{k-1} - C_{k-1}\hat{x}_{k-1}), \tag{4.18}$$

$$P_k = A_{k-1}P_{k-1}A_{k-1}^T + W_{k-1} - K_{k-1}C_{k-1}P_{k-1}A_{k-1}^T, \tag{4.19}$$

where

$$K_{k-1} = A_{k-1}P_{k-1}C_{k-1}^T(C_{k-1}P_{k-1}C_{k-1}^T + V_{k-1} + N_{k-1})^{-1}, \tag{4.20}$$

for $1 \leq k \leq N$ with initial conditions $\hat{x}_0 = m_0$ and $P_0 = M_0$ where $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^c]$ and $P_k = \mathsf{cov}[x_k|\mathcal{I}_k^c]$.

PROOF. Given the information set $\mathcal{I}_k^c$ at the controller, the state estimate minimizing the mean-square error is clearly the conditional expectation $\mathsf{E}[x_k|\mathcal{I}_k^c]$. From the definition, $\hat{x}_{k+1} = \mathsf{E}[x_{k+1}|\mathcal{I}_{k+1}^c]$ and $P_{k+1} = \mathsf{cov}[x_{k+1}|\mathcal{I}_{k+1}^c]$. Taking the conditional expectation of (4.1), we get

$$\hat{x}_{k+1} = A_k\,\mathsf{E}[x_k|\mathcal{I}_{k+1}^c] + B_k u_k, \tag{4.21}$$

$$P_{k+1} = A_k\,\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c]A_k^T + W_k. \tag{4.22}$$

The controller receives $\zeta_{k+1} = z_k$ at time $k+1$. Let us define $\xi_k = [x_k^T \;\; z_k^T]^T$. We can easily show that

$$\mathsf{E}[\xi_k|\mathcal{I}_k^c] = \begin{bmatrix} \hat{x}_k \\ C_k\hat{x}_k \end{bmatrix}, \tag{4.23}$$

$$\mathsf{cov}[\xi_k|\mathcal{I}_k^c] = \begin{bmatrix} P_k & P_kC_k^T \\ C_kP_k & C_kP_kC_k^T + V_k + N_k \end{bmatrix}, \tag{4.24}$$

where we used the fact that n_k is independent of x_k given N_k. Now, we can use Lemma A.2 together with the conditional distribution specified by the mean and covariance in (4.23), (4.24), and get

$$\mathsf{E}[x_k|\mathcal{I}_{k+1}^c] = \hat{x}_k + K_k'(z_k - C_k\hat{x}_k), \tag{4.25}$$

$$\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c] = P_k - K_k'C_kP_k, \tag{4.26}$$

where $K_k' = P_kC_k^T(C_kP_kC_k^T + V_k + N_k)^{-1}$ and we used the definition of $\mathcal{I}_{k+1}^c$ given N_k at the controller at time $k+1$. Moreover, from Assumption 4.1, apart from (4.25), (4.26), no inference is possible about x_k given z_k and N_k. We obtain the results by substituting (4.25), (4.26) in (4.21), (4.22) respectively. □

As it becomes evident later, in addition to the Kalman filter given in Proposition 4.1, the privacy filter needs to construct a copy of the estimator given in Proposition 4.2. This is possible because $\mathcal{I}_k^c \subseteq \mathcal{I}_k^f$.

4.2.2. Optimal Policies. We shall design the optimal policies using backward induction. Let $e_k = x_k - \hat{x}_k$ be the estimation error and $\nu_k = y_k - C_k\hat{x}_k$ be the innovation both associated with the estimator at the controller. Moreover, let $\varepsilon_k = \check{x}_k - \hat{x}_k$ be the mismatch estimation error associated with the estimators at the privacy filter and controller. We can obtain

$$\mathsf{E}[e_k|\mathcal{I}_k^f] = \mathsf{E}[x_k - \hat{x}_k|\mathcal{I}_k^f] = \check{x}_k - \hat{x}_k = \varepsilon_k, \tag{4.27}$$

$$\mathsf{cov}[e_k|\mathcal{I}_k^f] = \mathsf{cov}[x_k|\mathcal{I}_k^f] = \Sigma_k. \tag{4.28}$$

Given the optimal estimator in Proposition 4.2, the following proposition can be used to calculate the weighted causally conditioned directed information.

Proposition 4.3. *The causally conditioned directed information is equal to*

$$I(\mathbf{x}_N \to \mathbf{z}_N||\mathbf{u}_N) = -\textstyle\sum_{k=0}^N \frac{1}{2}\ln\det D_k, \tag{4.29}$$

where $D_k = I - P_kC_k^T(C_kP_kC_k^T + V_k + N_k)^{-1}C_k$.

Proof. Since the conditional distributions $\mathsf{P}(x_k|\mathcal{I}_k^c)$ and $\mathsf{P}(x_k|\mathcal{I}_{k+1}^c)$ are Gaussian, the entropies associated with the optimal estimator in Proposition 4.2 are obtained as

$$h(x_k|\mathcal{I}_k^c) = \frac{1}{2}\ln\big((2\pi e)^n \det\mathsf{cov}[x_k|\mathcal{I}_k^c]\big),$$
$$h(x_k|\mathcal{I}_{k+1}^c) = \frac{1}{2}\ln\big((2\pi e)^n \det\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c]\big),$$

where $\mathsf{cov}[x_k|\mathcal{I}_k^c] = P_k$ and $\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c] = P_k - K_k'C_kP_k$ where $K_k' = P_kC_k^T(C_kP_kC_k^T + V_k + N_k)^{-1}$. Therefore, given $\mathbf{u}_k$, we have

$$\begin{aligned} I(\mathbf{x}_N \to \mathbf{z}_N||\mathbf{u}_N) &= \textstyle\sum_{k=0}^N h(x_k|\mathbf{z}_{k-1},\mathbf{u}_k) - h(x_k|\mathbf{z}_k,\mathbf{u}_k) \\ &= \textstyle\sum_{k=0}^N h(x_k|\mathcal{I}_k^c) - h(x_k|\mathcal{I}_{k+1}^c) \\ &= \textstyle\sum_{k=0}^N \frac{1}{2}\ln\det P_k - \frac{1}{2}\ln\det\big(P_k - K_k'C_kP_k\big) \\ &= -\textstyle\sum_{k=0}^N \frac{1}{2}\ln\det(I - K_k'C_k), \end{aligned}$$

where the first equality comes from (4.7) and the second equality from the definition of $\mathcal{I}_k^c$ given N_{k-1} at the controller at time k. This completes the proof. □

Remark 4.3. *Following Proposition 4.3, the privacy filter can control the amount of information leakage by means of the perturbation covariance* N_k. *The two extreme cases of the perturbation covariances are* $N_k \to 0$ *and* $N_k \to \infty$, *which correspond to the maximum and minimum information leakage.*

The next theorem characterizes the structures of the optimal perturbation policy and optimal control policy, and shows that there exists a separation between the optimal designs of the privacy filter and controller.

Theorem 4.1. *Let* $S_k \succeq 0$ *be a matrix that satisfies the condition in Lemma A.1. The optimal closed-loop perturbation policy is an affine policy given by*

$$z_k^* = y_k + n_k, \tag{4.31}$$

where n_k is a Gaussian noise generated with zero mean and covariance N_k^* that is given by

$$\begin{aligned} N_k^* = \underset{N_k}{\operatorname{argmin}} \Big\{ -\tfrac{\theta_k}{2} \ln\det D_k + \nu_k^T K_k^T \Gamma_{k+1}(K_k \nu_k - 2A_k \varepsilon_k) \\ + \operatorname{tr}(K_k^T \Gamma_{k+1} K_k N_k) + \hat{\varrho}_k \Big\}, \end{aligned} \tag{4.32}$$

where $\hat{\varrho}_k$ is a variable that depends on ε_k and ν_k, and the optimal closed-loop control policy is a certainty-equivalence policy given by

$$u_k^* = -L_k \hat{x}_k, \tag{4.33}$$

where L_k is the control gain defined as

$$L_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k. \tag{4.34}$$

Proof. We need to show that (π^*, μ^*) represents a Nash equilibrium. Using the optimal control policy μ^* in the cost function $\Psi(\pi, \mu)$ given by Lemma A.1, we obtain

$$\begin{aligned} \Psi(\pi, \mu^*) = \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{ -\tfrac{\theta_k}{2} \ln\det D_k + w_k^T S_{k+1} w_k \\ + e_k^T L_k^T (B_k^T S_{k+1} B_k + R_k) L_k e_k \Big\}\Big], \end{aligned}$$

where we used the definition of the estimation error e_k. Following the fact that x_0 and w_k are independent of the perturbation policy, associated with $\Psi(\pi, \mu^*)$, we define the value function V_k^f as

$$V_k^f = \min_{\mathbf{N}^k} \mathsf{E}\Big[\textstyle\sum_{t=k}^N -\tfrac{\theta_k}{2} \ln\det D_k + e_{t+1}^T \Gamma_{t+1} e_{t+1} \Big| \mathcal{I}_k^f\Big],$$

where $\Gamma_k = L_k^T(B_k^T S_{k+1} B_k + R_k) L_k$ with the exception of $\Gamma_{N+1} = 0$. From the additivity of the value function V_k^f, we have

$$\begin{aligned} V_k^f &= \min_{N_k} \mathsf{E}\Big[-\tfrac{\theta_k}{2} \ln\det D_k + e_{k+1}^T \Gamma_{k+1} e_{k+1} \\ &\quad + \min_{N_{k+1}} \mathsf{E}\Big[-\tfrac{\theta_{k+1}}{2} \ln\det D_{k+1} + e_{k+2}^T \Gamma_{k+2} e_{k+2} + \dots \Big| \mathcal{I}_{k+1}^f\Big] \Big| \mathcal{I}_k^f\Big] \\ &= \min_{N_k} \mathsf{E}\Big[-\tfrac{\theta_k}{2} \ln\det D_k + e_{k+1}^T \Gamma_{k+1} e_{k+1} + V_{k+1}^f \Big| \mathcal{I}_k^f\Big], \end{aligned}$$

with initial condition $V_{N+1}^f = 0$. We prove by induction that the value function V_k^f is independent of the control policy. Clearly, the claim is satisfied for time $N+1$. We assume that the claim holds at time $k+1$, and we shall prove that it also holds at time k. We can write the dynamics of the estimation error at the controller as

$$e_{k+1} = A_k e_k + w_k - K_k \nu_k - K_k n_k. \tag{4.35}$$

Thus, we find

$$
\begin{aligned}
&\mathsf{E}[e_{k+1}^T\Gamma_{k+1}e_{k+1}|\mathcal{I}_k^f] \\
&\quad= \mathsf{E}\Big[e_k^TA_k^T\Gamma_{k+1}A_ke_k + w_k^T\Gamma_{k+1}w_k + \nu_k^TK_k^T\Gamma_{k+1}K_k\nu_k + n_k^TK_k^T\Gamma_{k+1}K_kn_k \\
&\qquad + 2e_k^TA_k^T\Gamma_{k+1}w_k - 2\nu_k^TK_k^T\Gamma_{k+1}w_k - 2n_k^TK_k^T\Gamma_{k+1}w_k \\
&\qquad - 2\nu_k^TK_k^T\Gamma_{k+1}A_ke_k - 2n_kK_k^T\Gamma_{k+1}A_ke_k + 2\nu_k^TK_k^T\Gamma_{k+1}K_kn_k\Big|\mathcal{I}_k^f\Big] \\
&\qquad = \varepsilon_k^TA_k^T\Gamma_{k+1}A_k\varepsilon_k + \operatorname{tr}(A_k^T\Gamma_{k+1}A_k\Sigma_k) + \operatorname{tr}(\Gamma_{k+1}W_k) + \nu_k^TK_k^T\Gamma_{k+1}K_k\nu_k \\
&\qquad + \operatorname{tr}(K_k^T\Gamma_{k+1}K_kN_k) - 2\nu_k^TK_k^T\Gamma_{k+1}A_k\varepsilon_k,
\end{aligned}
$$

where in the second equality we used the definitions of ε_k and Σ_k and the facts that ν_k is $\mathcal{I}_k^f$-measurable, that w_k is independent of e_k, and that n_k is independent of e_k and w_k given N_k. Hence, we have

$$
\begin{aligned}
V_k^f = \min_{N_k}\Big\{ &-\tfrac{\theta_k}{2}\ln\det D_k + \varepsilon_k^TA_k^T\Gamma_{k+1}A_k\varepsilon_k + \operatorname{tr}(A_k^T\Gamma_{k+1}A_k\Sigma_k) \\
&+ \operatorname{tr}(\Gamma_{k+1}W_k) + \nu_k^TK_k^T\Gamma_{k+1}K_k\nu_k + \operatorname{tr}(K_k^T\Gamma_{k+1}K_kN_k) \\
&- 2\nu_k^TK_k^T\Gamma_{k+1}A_k\varepsilon_k + \mathsf{E}[V_{k+1}^f|\mathcal{I}_k^f]\Big\}.
\end{aligned} \tag{4.36}
$$

The minimizer in (4.36) is obtained as

$$
\begin{aligned}
N_k^* = \operatorname*{argmin}_{N_k}\Big\{ &-\tfrac{\theta_k}{2}\ln\det D_k + \nu_k^TK_k^T\Gamma_{k+1}(K_k\nu_k - 2A_k\varepsilon_k) \\
&+ \operatorname{tr}(K_k^T\Gamma_{k+1}K_kN_k) + \hat{\varrho}_k\Big\},
\end{aligned}
$$

where $\hat{\varrho}_k = \mathsf{E}[V_{k+1}^f|\mathcal{I}_k^f]$. From the hypothesis assumption $\hat{\varrho}_k$ is independent of the control policy. Hence, we conclude that V_k^f is independent of the control policy. This complete the induction.

Now, using the the perturbation policy π^* in the cost function $\Psi(\pi,\mu)$ given by Lemma A.1, we obtain

$$
\begin{aligned}
\Psi(\pi^*,\mu) = \mathsf{E}\Big[x_0^TS_0x_0 + \textstyle\sum_{k=0}^N\Big\{ &-\tfrac{\theta_k}{2}\ln\det D_k + w_k^TS_{k+1}w_k \\
&+ (u_k + L_kx_k)^T\Lambda_k(u_k + L_kx_k)\Big\}\Big],
\end{aligned}
$$

where $\Lambda_k = B_k^TS_{k+1}B_k + R_k$. Following the fact that x_0, D_k, and w_k are independent of the control policy, associated with $\Psi(\pi^*,\mu)$, we define the auxiliary value function V_k^c as

$$
V_k^c = \min_{\mathbf{u}^k}\mathsf{E}\Big[\textstyle\sum_{t=k}^N(u_t + L_tx_t)^T\Lambda_t(u_t + L_tx_t)\Big|\mathcal{I}_k^c\Big].
$$

From the additivity of the auxiliary value function V_k^c, we obtain

$$\begin{aligned} V_k^c &= \min_{u_k} \mathsf{E}\Big[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \\ &\quad + \min_{u_{k+1}} \mathsf{E}\Big[(u_{k+1} + L_{k+1} x_{k+1})^T \Lambda_{k+1} \\ &\quad \times (u_{k+1} + L_{k+1} x_{k+1}) + \dots \Big| \mathcal{I}_{k+1}^c \Big] \Big| \mathcal{I}_k^c \Big] \\ &= \min_{u_k} \mathsf{E}\Big[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) + V_{k+1}^c \Big| \mathcal{I}_k^c \Big], \end{aligned}$$

with initial condition $V_{N+1}^c = 0$. We prove by induction that the auxiliary value function V_k^c is a function of P_k. Clearly, the claim is satisfied for time $N+1$. We assume that the claim holds at time $k+1$, and we shall prove that it also holds at time k. Using the identity $x_k = \hat{x}_k + e_k$, we find

$$\begin{aligned} &\mathsf{E}\left[(u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) \middle| \mathcal{I}_k^c\right] \\ &\quad = \mathsf{E}\Big[(u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) \\ &\qquad + e_k^T L_k^T \Lambda_k L_k e_k + 2(u_k + L_k \hat{x}_k)^T \Lambda_k L_k e_k \Big| \mathcal{I}_k^c \Big] \\ &\quad = (u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) + \operatorname{tr}(L_k^T \Lambda_k L_k P_k), \end{aligned}$$

where in the second equality we used the fact that $\hat{x}_k$ is $\mathcal{I}_k^c$-measurable and $\mathsf{E}[e_k | \mathcal{I}_k^c] = 0$. Hence, we have

$$\begin{aligned} V_k^c = \min_{u_k} \Big\{ &(u_k + L_k \hat{x}_k)^T \Lambda_k (u_k + L_k \hat{x}_k) \\ &+ \operatorname{tr}(L_k^T \Lambda_k L_k P_k) + \mathsf{E}[V_{k+1}^c | \mathcal{I}_k^c] \Big\}. \end{aligned} \tag{4.37}$$

The minimizer in (4.37) is obtained as $u_k^* = -L_k \hat{x}_k$. Moreover, we conclude that V_k^c is a function of P_k. This completes the induction and also the proof. □

According to Theorem 4.1, the optimal perturbation policy depends on ε_k and ν_k, and is independent of the control policy. Besides, the error covariance P_k in (4.19) does not depend on $\mathbf{u}_{k-1}$. Hence, the control has no dual effect. In Theorem 4.1, we expressed the value of information based on the value function V_k^f and corresponding to the transmission of the private information. This is emphasized in the following definition.

Definition 4.2. *The value of information* VoI_k *in the trade-off between the privacy level and control performance is given by*

$$\mathrm{VoI}_k = \nu_k^T K_k^{nT} \Gamma_k (2 A_k \varepsilon_k - K_k^n \nu_k) + \tfrac{\theta_k}{2} \ln\det D_k^n + \varrho_k, \tag{4.38}$$

where $\varrho_k = \mathsf{E}[V_{k+1}^f | \mathcal{I}_k^f, N_k \to \infty] - \mathsf{E}[V_{k+1}^f | \mathcal{I}_k^f, N_k \to 0]$, $D_k^n = I - P_k C_k^T (C_k P_k C_k^T + V_k)^{-1} C_k$, $K_k^n = A_k P_k C_k^T (C_k P_k C_k^T + V_k)^{-1}$.

Remark 4.4. *The optimal perturbation policy provided above depends on the variable* $\hat{\varrho}_k$. *Although* $\hat{\varrho}_k$ *can be computed with an arbitrary accuracy by solving recursively the optimality equation in (4.36), its computation is expensive. Next, we shall introduce a procedure for approximation of this variable.*

4.2.3. Approximation Algorithm. We here provide a rollout algorithm [**72**] for approximation of the variable $\hat{\varrho}_k$ and the value of information VoI_k, and accordingly synthesize a closed-form suboptimal perturbation policy with a performance guarantee that can readily be implemented. Let $\bar{\pi} = \{\bar{z}_0, \ldots, \bar{z}_N\}$ be a perturbation policy with perturbation covariances $N_k \simeq 0$ for all $0 \leq k \leq N$. The following algorithm gives an approximation of the variable $\hat{\varrho}_k$.

Algorithm 4.1. *An approximation of the variable $\hat{\varrho}_k$ associated with the policy $\bar{\pi}$ is given by*

$$\hat{\varrho}_k^{\bar{\pi}} = \mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^f],$$

where

$$\mathsf{E}[V_{k+1}^{\bar{\pi}} | \mathcal{I}_k^f] = \mathsf{E}\left[\textstyle\sum_{t=k}^{N} -\frac{\theta_k}{2} \ln\det D_k + e_{t+1}^T \Gamma_{t+1} e_{t+1} \middle| \mathcal{I}_k^f \right],$$

with $\mathbf{z}^{k+1} = \bar{\mathbf{z}}^{k+1}$.

Similar to Theorem 2.2, we can prove that it is possible to synthesize a suboptimal perturbation policy that outperforms the policy $\bar{\pi}$. In the next proposition, we synthesize a closed-form suboptimal perturbation policy with a performance guarantee.

Proposition 4.4. *A suboptimal perturbation policy that outperforms the policy $\bar{\pi}$ is given by*

$$z_k^+ = y_k + n_k, \tag{4.39}$$

where n_k is a Gaussian noise generated with zero mean and covariance N_k^+ that is given by

$$\begin{aligned} N_k^+ = \operatorname*{argmin}_{N_k} \Big\{ &-\tfrac{\theta_k}{2} \ln\det D_k + \nu_k^T K_k^T \Gamma_{k+1} (K_k \nu_k - 2 A_k \varepsilon_k) + \mathrm{tr}(K_k^T \Gamma_{k+1} K_k N_k) \\ &+ \textstyle\sum_{t=k+1}^{N} -\frac{\theta_t}{2} \ln\det D_t^n + \bar{e}_{t+1}^T \Gamma_{t+1} \bar{e}_{t+1} + \mathrm{tr}(\Gamma_{t+1} \bar{P}_{t+1}) \Big\}, \end{aligned} \tag{4.40}$$

and

$$\bar{e}_{t+1} = (A_t - K_t^n C_t) \bar{e}_t,$$

$$\bar{P}_{t+1} = (A_t - K_t^n C_t) \bar{P}_t (A_t - K_t^n C_t)^T + W_t + K_t^n V_t K_t^{nT},$$

$$P_{t+1}^n = A_t P_t^n A_t^T + W_t - K_t^n C_t P_t^n A_t^T,$$

where

$$K_t^n = A_t P_t^n C_t^T (C_t P_t^n C_t^T + V_t)^{-1},$$

$$D_t^n = I - P_t^n C_t^T (C_t P_t^n C_t^T + V_t)^{-1} C_t,$$

for $t \geq k+1$ with initial conditions $\bar{e}_{k+1} = A_k \varepsilon_k - K_k \nu_k$, $\bar{P}_{k+1} = A_k \Sigma_k A_k^T + W_k + K_k N_k K_k^T$, and $P_{k+1}^n = A_k P_k A_k^T + W_k - K_k C_k P_k A_k^T$.

Proof. For the proof, it is enough to derive $\hat{\varrho}_k^{\bar{\pi}}$ based on the policy $\bar{\pi}$. First, note that

$$\begin{aligned}\mathsf{E}[V_{k+1}^{\bar{\pi}}|\mathcal{I}_k^f, N_k] &= \mathsf{E}\left[\textstyle\sum_{t=k+1}^{N} -\frac{\theta_t}{2}\ln\det D_t + e_{t+1}^T\Gamma_{t+1}e_{t+1}\Big|\mathcal{I}_k^f, N_k\right]\\ &= \textstyle\sum_{t=k+1}^{N} -\frac{\theta_t}{2}\ln\det D_t^n + \bar{e}_{t+1}^T\Gamma_{t+1}\bar{e}_{t+1} + \mathrm{tr}(\Gamma_{t+1}\bar{P}_{t+1}),\end{aligned}$$

where in the first equality we used the definition of $V_{k+1}^{\bar{\pi}}$ and in the second equality the fact that $z_t = \bar{z}_t$ for all $t \geq k+1$ and the definitions $\bar{e}_t = \mathsf{E}[e_t|\mathcal{I}_k^f, N_k]$, $\bar{P}_t = \mathsf{cov}[e_t|\mathcal{I}_k^f, N_k]$, and $D_t^n = I - P_t^n C_t^T(C_t P_t^n C_t^T + V_t)^{-1}C_t$ for all $t \geq k+1$.

From the dynamics of the estimation error in (4.35), given the fact that $N_t \simeq 0$ for all $t \geq k+1$, we obtain

$$e_{t+1} = (A_t - K_t^n C_t)e_t + w_t - K_t^n v_t,$$

where

$$\begin{aligned}P_{t+1}^n &= A_t P_t^n A_t^T + W_t - K_t^n C_t P_t^n A_t^T,\\ K_t^n &= A_t P_t^n C_t^T(C_t P_t^n C_t^T + V_t)^{-1},\end{aligned}$$

for $t \geq k+1$ with initial condition $P_{k+1}^n = A_k P_k A_k^T + W_k - K_k C_k P_k A_k^T$. Accordingly, we have

$$\begin{aligned}\bar{e}_{t+1} &= (A_t - K_t^n C_t)\bar{e}_t,\\ \bar{P}_{t+1} &= (A_t - K_t^n C_t)\bar{P}_t(A_t - K_t^n C_t)^T + W_t + K_t^n V_t K_t^{nT},\end{aligned}$$

for $t \geq k+1$ with initial conditions $\bar{e}_{k+1} = A_k\varepsilon_k - K_k\nu_k$ and $\bar{P}_{k+1} = A_k\Sigma_k A_k^T + W_k + K_k N_k K_k^T$. The initial conditions $\bar{e}_{k+1}^1$ and $\bar{P}_{k+1}^1$ were obtained by using (4.35).

Finally, following the definition of $\hat{\varrho}_k^{\bar{\pi}}$, we have

$$\begin{aligned}\hat{\varrho}_k^{\bar{\pi}} &= \mathsf{E}[V_{k+1}^{\bar{\pi}}|\mathcal{I}_k^f, N_k]\\ &= \textstyle\sum_{t=k+1}^{N} -\frac{\theta_t}{2}\ln\det D_t^n + \bar{e}_{t+1}^T\Gamma_{t+1}\bar{e}_{t+1} + \mathrm{tr}(\Gamma_{t+1}\bar{P}_{t+1}).\end{aligned}$$

Incorporating this into (4.32), we obtain the result. □

4.3. Illustrative Example

In this section, we show an application of the theoretical framework we developed in this chapter. Consider an inverted pendulum on a cart observed by an internal sensor (see Fig. 4.2). The parameters A_k, B_k, C_k, W_k, V_k, m_0, and M_0 are similar to those used in Chapter 2.

For this system, we are interested in designing a privacy filter that is employed at the cart and a controller that is placed at a cloud. The cost function of form (4.12) is specified by the weights $Q_{N+1} = \mathrm{diag}\{1, 1, 1000, 1\}$, $Q_k = \mathrm{diag}\{1, 1, 1000, 1\}$, $R_k = 1$, and $\theta_k = 100$ for all $0 \leq k \leq N$ where $N = 500$. The state estimates $\check{x}_k$ and $\hat{x}_k$ are provided by Proposition 4.1 and Proposition 4.2 respectively. From Theorem 4.1, it follows that the optimal perturbation policy is specified by $z_k^* = y_k + n_k$, and that the optimal control policy is $u_k^* = -L_k\hat{x}_k$. We approximated the variable N_k^* according to Proposition 4.4 assuming that N_k^* is a diagonal matrix with similar elements whose values are limited in the interval $[0, 0.2]$, and obtained the control gain L_k by solving the Riccati equation in Lemma A.1. We carried out a simulation experiment. For a realization of the system, Fig. 4.3 and Fig. 4.4 illustrate the

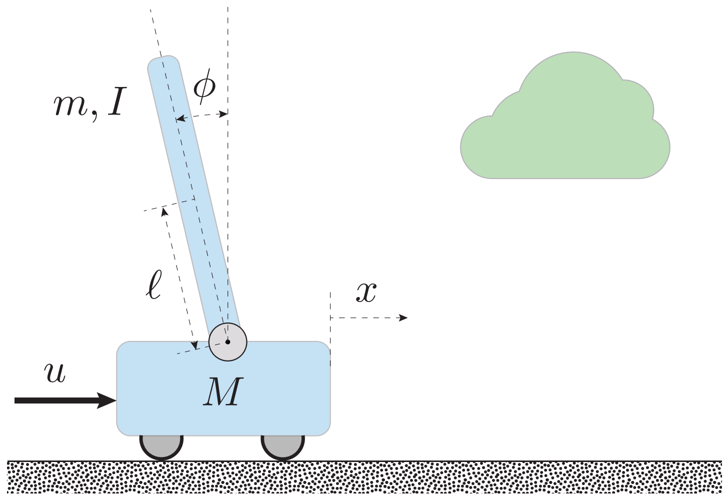

FIGURE 4.2. Model of an inverted pendulum on a cart. The sensor is insourced, but the controller is outsourced to a cloud.

trajectories of the main variables of the system. In particular, the trajectories of the position, velocity, pitch angle, and pitch rate are shown in Fig. 4.3. Moreover, the trajectories of the perturbation covariance's diagonal element, position perturbation, pitch angle perturbation, and control are shown in Fig. 4.4. Although the public information is noisy due to privacy preservation, the controller could obtain a satisfactory performance.

4.4. Summary

In this chapter, we provided a theoretical framework for the analysis and design of cloud control systems with privacy filters. We formulated the problem as a dynamic game, and characterized the optimal perturbation policy and optimal control policy such that the corresponding policy profile represents a Nash equilibrium. We proved, under certain assumptions, that the optimal closed-loop perturbation policy is an affine policy with additive Gaussian noise whose covariance particularly depends on the estimation innovation at the controller and mismatch estimation error; and that the optimal closed-loop control policy is a certainty-equivalence policy. Moreover, we synthesized a closed-form suboptimal perturbation policy with a performance guarantee. The results here can be useful for the rate-distortion problem in causal systems. In such a problem, the directed information measures the data rate in the communication channel with feedback, and the privacy filter developed here should be seen as a test channel.

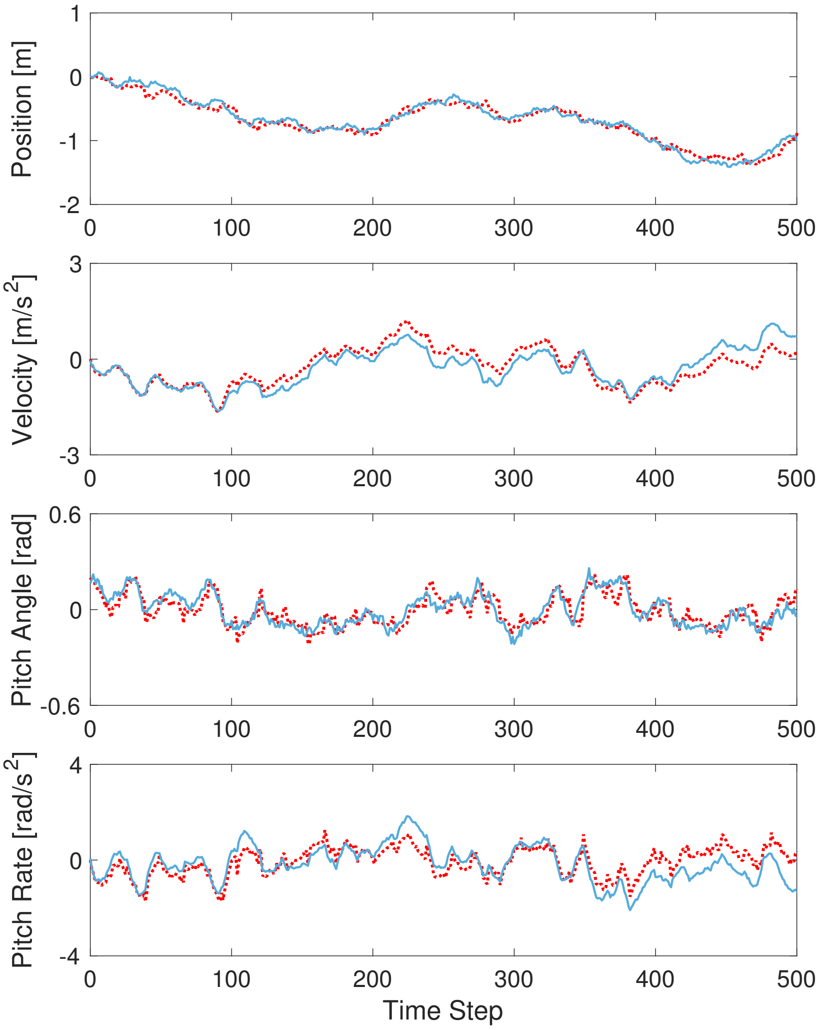

FIGURE 4.3. Trajectories of the position, velocity, pitch angle, and pitch rate. The solid lines represent the state components and the dotted lines represent the state estimate components at the controller.

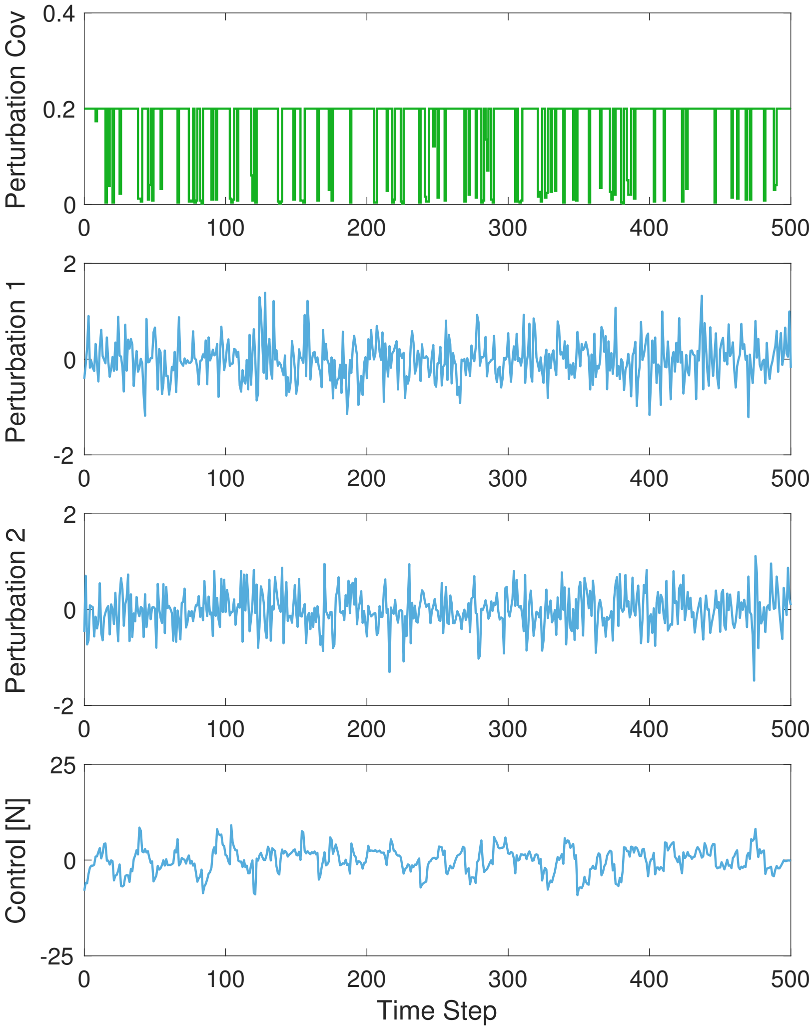

FIGURE 4.4. Trajectories of the perturbation covariance's diagonal element, position perturbation, pitch angle perturbation, and control.

CHAPTER 5

Conclusions and Future Work

Ubiquitous communication, sensing, and computing have led to the emergence of a new kind of complex systems. Our main goal in this thesis was to answer two questions pertaining to control of such systems: 1) how does the control performance change with respect to the quality of information? 2) what is the optimal information collection policy? In this regard, we developed a unified theoretical framework in the context of linear quadratic Gaussian control for the maximization of the control performance subject to different constraints that restrict the information flow in the system. The results have implications for decision making in causal systems. We have already reviewed the results in the previous chapters. Here, we discuss the common points, and propose general steps that are required for the extension of our framework.

5.1. Conclusions

In this thesis, we introduced our framework and tackled three different problems. We formulated each problem as a dynamic game with a controller and a mechanism that use closed-loop policies to influence a partially observable stochastic process. We demonstrated that the controller and mechanism have to construct dedicated estimators, and that the mechanism requires to construct the estimator of the controller as well. We characterized the structures of the optimal closed-loop control policy and optimal closed-loop information collection policy. We proved that a separation between the optimal policies is achievable. In particular, we showed that the former is a certainty-equivalence policy and the latter is a feedback policy that depends on the sample path of the system, i.e., on an estimation innovation and an estimation mismatch error.

5.2. Future Work

There are several directions for the extension of the framework presented in this thesis. First, we developed our framework under some assumptions that reduce the information set of the controller. One should study the optimal policies when these assumptions do not hold. Furthermore, our framework was presented in the context of linear quadratic Gaussian control. This can be extended to partially observable Markov decision processes for instance. Moreover, in our framework we used rollout algorithms for approximation of the optimal information collection polices. Other approximation algorithms should be developed, and a comparison between the results should be made. Finally, we considered different constraints on the observation channel, and carried out a value of information analysis. We suggest that future research should consider different constraints on the control channel, and accordingly a value of control analysis should be carried out.

Appendix

Here, we provide some definitions and lemmas that are used throughout this thesis.

Definition A.1. *Consider a team game with two decision makers. Let* $\gamma^1 \in \mathcal{G}^1$ *and* $\gamma^2 \in \mathcal{G}^2$ *be the policies of the first and the second decision makers respectively, and* $J(\gamma^1, \gamma^2)$ *be the cost function. A policy profile* $(\gamma^{1*}, \gamma^{2*})$ *represents a* Nash equilibrium *[**75**] if and only if*

$$J(\gamma^{1*}, \gamma^{2*}) \leq J(\gamma^1, \gamma^{2*}), \text{ for all } \gamma^1 \in \mathcal{G}^1,$$

$$J(\gamma^{1*}, \gamma^{2^*}) \leq J(\gamma^{1*}, \gamma^2), \text{ for all } \gamma^2 \in \mathcal{G}^2.$$

The optimality considered in this study is in the above sense.

Definition A.2. *Let* $\mathcal{I}_k^c$ *be the information set of the controller and* $\tilde{\mathcal{I}}_k^c$ *be the information set of the controller when controls are equal to zero. The control has* no dual effect *[**76**] of order* r *(*$r \geq 2$*) if*

$$\mathsf{E}[M_{k,i}^r | \mathcal{I}_k^c] = \mathsf{E}[M_{k,i}^r | \tilde{\mathcal{I}}_k^c],$$

where $M_{k,i}^r = (x_{k,i} - \mathsf{E}[x_{k,i} | \mathcal{I}_k^c])^r$ *is the* r*th central moment of the* i*th component of the state conditioned on* $\mathcal{I}_k^c$*. In other words, the control has no dual effect if the expected future uncertainty is not affected by the prior controls.*

Definition A.3. *Let a bandpass signal* $s(t)$ *at carrier frequency* f_c *be represented by:*

$$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t),$$

where $x(t)$ *and* $y(t)$ *are real baseband signals of bandwidth* $B \ll f_c$*. The energy of the signal* $s(t)$ *is defined as*

$$E = \textstyle\int_{-\infty}^{\infty} s^2(t) \text{ d}t.$$

Definition A.4. *The Q-function is defined as the tail function of the standard Gaussian distribution:*

$$Q(z) = \tfrac{1}{\sqrt{2\pi}} \textstyle\int_x^{\infty} e^{-x^2/2} \text{ d}x.$$

Notice that the Q-function cannot be solved for in closed form.

Definition A.5. *Let* $\mathsf{P}(x)$ *and* $\mathsf{Q}(x)$ *be two probability measures defined on a measurable space. The* Kullback-Leibler divergence *[**77**] is defined as*

$$D(\mathsf{P}||\mathsf{Q}) = \begin{cases} \int \ln \frac{\mathrm{d}\mathsf{P}(x)}{\mathrm{d}\mathsf{Q}(x)} \mathrm{d}\mathsf{P}(x), & \textit{if } \frac{\mathrm{d}\mathsf{P}(x)}{\mathrm{d}\mathsf{Q}(x)} \textit{ exists}, \\ \infty, & \textit{otherwise}, \end{cases}$$

where $\frac{\mathsf{dP}(x)}{\mathsf{dQ}(x)}$ *is the Radon-Nikodym derivative. In practical terms the Kullback-Leibler divergence is the amount of information lost when* $\mathsf{Q}(x)$ *is used to approximate* $\mathsf{P}(x)$*. The Kullback-Leibler divergence is always non-negative.*

Definition A.6. *Let* $\mathsf{P}(x)$ *be a probability measure defined on a measurable space. The* differential entropy *[**77**] of the random variable* x *with distribution* $\mathsf{P}(x)$ *is defined as*

$$h(x) = -D(\mathsf{P}||\mathsf{R}),$$

where $\mathsf{R}(x)$ *is the Lebesgue measure.*

Definition A.7. *Let* x *and* y *be jointly distributed random variables according to the joint measure* $\mathsf{P}(x,y)$ *with marginal distributions* $\mathsf{P}(x)$ *and* $\mathsf{P}(y)$ *respectively. The* mutual information *[**77**] of* x *and* y *is defined as*

$$I(x;y) = D\Big(\mathsf{P}(x,y)||\mathsf{P}(x)\mathsf{P}(y)\Big).$$

We can equivalently show that $I(x;y) = h(x) - h(x|y)$*.*

Lemma A.1. *Consider a stochastic process with the following dynamics:*

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{A.1}$$

for $0 \leq k \leq N$ *and with initial condition* x_0 *where* $x_k \in \mathbb{R}^n$ *is the state of the process,* $A_k \in \mathbb{R}^{n\times n}$ *is the state matrix,* $B_k \in \mathbb{R}^{n\times m}$ *is the input matrix,* $u_k \in \mathbb{R}^m$ *is the control input, and* $w_k \in \mathbb{R}^n$ *is a white Gaussian noise with zero mean and covariance* $W_k \succ 0$*. We assume that* w_k *is independent of* x_0*. Moreover, consider the following cost function:*

$$\Psi = \mathsf{E}\Big[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u_k^T R_k u_k + \theta_k \sigma_k\Big], \tag{A.2}$$

where $Q_k \succeq 0$*,* $R_k \succ 0$*, and* $\theta_k \geq 0$ *are weighting parameters and* $\sigma_k \in \mathbb{R}$ *is a variable that can depend on* $\mathbf{x}_k$ *and* $\mathbf{u}_k$*. Let* $S_k \succeq 0$ *be a matrix that satisfies the following algebraic Riccati equation:*

$$S_k = Q_k + A_k^T S_{k+1} A_k - L_k^T (B_k^T S_{k+1} B_k + R_k) L_k, \tag{A.3}$$

$$L_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k, \tag{A.4}$$

for $0 \leq k \leq N$ *with initial condition* $S_{N+1} = Q_{N+1}$*. Then, the cost function is equal to*

$$\begin{aligned}\Psi = \mathsf{E}\Big[&x_0^T S_0 x_0 + \textstyle\sum_{k=0}^{N} \theta_k \sigma_k + w_k^T S_{k+1} w_k \\ &+ (u_k + L_k x_k)^T (B_k^T S_{k+1} B_k + R_k)(u_k + L_k x_k)\Big].\end{aligned} \tag{A.5}$$

Proof. Using the process dynamics (A.1) and the Riccati equation (A.3), we can write

$$\begin{aligned}x_{k+1}^T S_{k+1} x_{k+1} = {}&(A_k x_k + B_k u_k + w_k)^T \\ &\times S_{k+1}(A_k x_k + B_k u_k + w_k),\end{aligned} \tag{A.6}$$

$$\begin{aligned}x_k^T S_k x_k = {}&x_k^T \big(Q_k + A_k^T S_{k+1} A_k \\ &- L_k^T (B_k^T S_{k+1} B_k + R_k) L_k\big) x_k.\end{aligned} \tag{A.7}$$

Consequently, we find

$$\begin{aligned}
&x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0 \\
&\quad = \textstyle\sum_{k=0}^N x_{k+1}^T S_{k+1} x_{k+1} - x_k^T S_k x_k \\
&\quad = \textstyle\sum_{k=0}^N \Big\{ w_k^T S_{k+1} w_k + 2(A_k x_k + B_k u_k)^T S_{k+1} w_k \\
&\qquad + x_k^T L_k^T (B_k^T S_{k+1} B_k + R_k) L_k x_k \\
&\qquad - x_k^T Q_k x_k - u_k^T R_k u_k + 2 x_k^T A_k^T S_{k+1} B_k u_k \\
&\qquad + u_k^T (B_k^T S_{k+1} B_k + R_k) u_k \Big\},
\end{aligned}$$

where the first equality is an identity, and in the second equality we used (A.6) and (A.7) and also added and subtracted the term $\sum_{k=0}^N u_k^T R_k u_k$ to and from the right-hand side. Rearranging the terms in the above relation, we find

$$\begin{aligned}
&x_{N+1}^T S_{N+1} x_{N+1} + \textstyle\sum_{k=0}^N x_k^T Q_k x_k + u_k^T R_k u_k \\
&\quad = x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{ w_k^T S_{k+1} w_k \\
&\qquad + 2(A_k x_k + B_k u_k)^T S_{k+1} w_k \\
&\qquad + (u_k + L_k x_k)^T (B_k^T S_{k+1} B_k + R_k)(u_k + L_k x_k) \Big\}.
\end{aligned}$$

Adding the term $\sum_{k=0}^N \theta_k \sigma_k$ to both sides of the above relation and taking expectation, we obtain the result:

$$\begin{aligned}
\Psi &= \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{ \theta_k \sigma_k + w_k^T S_{k+1} w_k \\
&\quad + 2(A_k x_k + B_k u_k)^T S_{k+1} w_k \\
&\quad + (u_k + L_k x_k)^T (B_k^T S_{k+1} B_k + R_k)(u_k + L_k x_k) \Big\} \Big] \\
&= \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^N \Big\{ \theta_k \sigma_k + w_k^T S_{k+1} w_k \\
&\quad + (u_k + L_k x_k)^T (B_k^T S_{k+1} B_k + R_k)(u_k + L_k x_k) \Big\} \Big],
\end{aligned}$$

where in the second equality we used the fact that w_k is independent of x_k. □

Lemma A.2. *Let x and y be two random vectors that are jointly Gaussian with the following mean and covariance:*

$$\mathsf{E}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}, \quad \mathsf{cov}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{bmatrix}.$$

Then, the conditional distribution of x given y is also Gaussian with the following mean and covariance:

$$\mathsf{E}[x|y] = m_x + R_{xy} R_y^{-1} (y - m_y), \tag{A.8}$$

$$\mathsf{cov}[x|y] = R_x - R_{xy} R_y^{-1} R_{yx}. \tag{A.9}$$

PROOF. Let us define the variables $\xi = x - m_x - R_{xy}R_y^{-1}(y - m_y)$ and $R_\xi = R_x - R_{xy}R_y^{-1}R_{yx}$. We have

$$\begin{bmatrix} \xi \\ y - m_y \end{bmatrix} = \begin{bmatrix} I & -R_{xy}R_y^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} x - m_x \\ y - m_y \end{bmatrix},$$

and

$$\begin{bmatrix} x - m_x \\ y - m_y \end{bmatrix} = \begin{bmatrix} I & R_{xy}R_y^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ y - m_y \end{bmatrix}.$$

As the Jacobian of the transformation is one, we find that the joint distribution of x and y can be written as

$$\begin{aligned} \mathsf{P}(x, y) = {} & (2\pi)^{-(n+p)/2}(\det R)^{-1/2} \\ & \times \exp\Big(-\tfrac{1}{2}\xi^T R_\xi^{-1}\xi - \tfrac{1}{2}(y - m_y)^T R_y^{-1}(y - m_y)\Big), \end{aligned}$$

where

$$R = \begin{bmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{bmatrix}.$$

Moreover, the distribution of y is

$$\begin{aligned} \mathsf{P}(y) = {} & (2\pi)^{-p/2}(\det R_y)^{-1/2} \\ & \times \exp\Big(-\tfrac{1}{2}(y - m_y)^T R_y^{-1}(y - m_y)\Big). \end{aligned}$$

From determinant properties, we can write

$$\det R = \det(R_x - R_{xy}R_y^{-1}R_{yx}) \det R_y = \det R_\xi \det R_y.$$

Now, we can compute the conditional distribution of x given y as

$$\mathsf{P}(x|y) = \frac{\mathsf{P}(x, y)}{\mathsf{P}(y)} = (2\pi)^{-n/2}(\det R_\xi)^{-1/2} \exp\Big(-\tfrac{1}{2}\xi^T R_\xi^{-1}\xi\Big).$$

This completes the proof. □

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