Efficient estimation of variance-based reliability sensitivities in the presence of multi-uncertainty

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ABSTRACT

In reliability analysis, next to an estimate of the probability of failure, one is often interested in the sensitivity of this estimate to changes in the model input. Moreover, in the presence of multi-uncertainty, e.g. when information can be used to refine the probabilistic model of a subset of the inputs, sensitivity measures with respect to a subset of the input parameters are of particular interest. In this work, we propose a new sensitivity measure for the probability of failure conditional on such a subset of input parameters. Since the failure event of interest is typically a rare event, evaluating reliability sensitivities with sampling-based approaches becomes expensive. This motivates us to develop a method to efficiently compute the new conditional reliability sensitivity measures by means of polynomial basis surrogates. In a two-step procedure, conditional reliability samples are initially obtained from an auxiliary surrogate to circumvent possibly expensive limit-state function evaluations. Based on these samples, a second surrogate is computed which directly yields estimates of the proposed sensitivity indices. The approach performance is tested on a numerical 2-D truss example.

1 Introduction

Reliability analysis is concerned with the evaluation of the probability of failure of an engineering system. The system can be described probabilistically in terms of the input random vector $\boldsymbol{\Theta}$ with joint CDF $F_{\boldsymbol{\Theta}}$ and a deterministic model $\mathcal{Y}$ mapping each $\boldsymbol{\Theta}$ to an output $Y = \mathcal{Y}(\boldsymbol{\Theta})$. The performance of the system can be assessed in terms of its limit-state function g. The limit-state function defines the failure modes of a system and by convention takes values below 0 in the failure domain of the input parameter space $\Omega_{\boldsymbol{\Theta}}$. The system probability of failure is given by (Ditlevsen and Madsen, 1996)

$$\mathbb{P}(F) = \mathbb{E}_{\boldsymbol{\Theta}}\left[\mathrm{I}(g(\boldsymbol{\Theta}) \leq 0)\right] = \int_{\Omega_{\boldsymbol{\Theta}}} \mathrm{I}[g(\boldsymbol{\theta}) \leq 0] f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}, \tag{1}$$

where the indicator function I equals 1 on the failure domain $\{\boldsymbol{\Theta} : g(\boldsymbol{\Theta}) \leq 0\}$ and 0 on its complement and $f_{\boldsymbol{\Theta}}$ is the joint probability density of $\boldsymbol{\Theta}$.
We consider a segmentation of the random input vector in two disjunct subsets $\boldsymbol{\Theta} = [\boldsymbol{\Theta}_A, \boldsymbol{\Theta}_B]^T$. In practice, the type B-variables may represent random inputs on which information can be collected. This information can then be used to refine the probabilistic model of $\boldsymbol{\Theta}_B$ via a Bayesian formulation and, ultimately, improve the estimate of $\mathbb{P}(F)$ (reliability updating). The variables $\boldsymbol{\Theta}_B$ are often termed *reducible* or *epistemic*. In contrast, the uncertainty on variables $\boldsymbol{\Theta}_A$ can not be reduced, which is why $\boldsymbol{\Theta}_A$ are termed *irreducible* or *aleatory*. It is desirable to obtain information on the potential influence of each component of the type B-variables in the reliability updating prior to running the updating e.g. to provide decision support for data acquisition. To this end, we consider the probability of failure conditional on

the prior type B-variable space (that is, before any information on $\boldsymbol{\Theta}_B$ is accounted for to change $f_{\boldsymbol{\Theta}_B}$) (Der Kiureghian and Ditlevsen, 2009):

$$P_F(\boldsymbol{\Theta}_B) = \mathbb{P}(F|\boldsymbol{\Theta}_B) = \mathbb{E}_{\boldsymbol{\Theta}_A}\left[\mathrm{I}(g(\boldsymbol{\Theta}_A, \boldsymbol{\Theta}_B) \leq 0)|\boldsymbol{\theta}_B\right]$$
$$= \int_{\Omega_{\boldsymbol{\Theta}_A}} \mathrm{I}[g(\boldsymbol{\theta}_A, \boldsymbol{\theta}_B) \leq 0] f_{\boldsymbol{\Theta}_A|\boldsymbol{\Theta}_B}(\boldsymbol{\theta}_A|\boldsymbol{\theta}_B)\mathrm{d}\boldsymbol{\theta}_A. \tag{2}$$

Note, that $P_F(\boldsymbol{\Theta}_B)$ is a scalar function of a random vector and hence is itself a random variable. The goal of this work is to quantify the sensitivity of P_F to the elements of $\boldsymbol{\Theta}_B$. To this end we propose a novel reliability sensitivty measure based on a variance decomposition of the conditional probability of failure. Moreover, we discuss an efficient approach for computing the sensitivities by means of surrogate modelling techniques.
The structure of the paper is as follows: In Section 2 we discuss the basics of variance-based sensitivity analysis and introduce the new reliability sensitivty index. Section 3 introduces polynomial basis surrogate modelling (polynomial chaos expansions (PCEs) and low-rank approximations (LRAs)) and their relation to sensitivity measures. In Section 4 we give an outline of the two-level surrogate-based sensitivity estimation procedure which is subsequently tested on a numerical 2-D truss example in Section 5. Section 6 contains a brief discussion on the found results.

2 Sensitivity Analysis

Various sensitivity measures to assert significance across an input-output relation have been proposed. General methodological distinctions can be made with respect to the inputs for which sensitivities are computed (parametric vs. non-parametric), as well as the range over which they are computed (local vs. global). A review of the different methods can be found in (Saltelli et al., 2000) and (Iooss and Lemaître, 2015). The sensitivity measures considered here belong to the realm of variance-based sensitivity (or Analysis Of Variance - ANOVA) methods, which is a class of non-parametric global sensitivity methods that can effectively provide input variable importance rankings.

2.1 Variance-Based Sensitivity Analysis

ANOVA provides measures to estimate the contribution of a certain combination of input variables $\boldsymbol{\Theta}_{\mathcal{A}}, \mathcal{A} \subseteq \mathcal{P}(\{1,2,\ldots,d\})$ to the output variance of a quantity of interest (QOI) $Q = \mathcal{Q}(\boldsymbol{\Theta})$. A first-order measure for this contribution is one that considers the influence of effects for which solely Θ_i is relevant and neglects interaction contributions of Θ_i with other inputs. This is the first-order Sobol' index of Q with respect to the i-th component Θ_i of the model input $\boldsymbol{\Theta}$, which is given by

$$S_{Q,i} = \frac{\mathbb{V}_{\Theta_i}\left[\mathbb{E}_{\boldsymbol{\Theta}_{\sim i}}\left[Q|\Theta_i\right]\right]}{\mathbb{V}\left[Q\right]}, \tag{3}$$

where $\boldsymbol{\Theta}_{\sim i}$ indicates the vector of all components of $\boldsymbol{\Theta}$ but the i-th. $S_{Q,i}$ is the output variance fraction caused by Θ_i only. Analogously, the total sensitivity index (Homma and Saltelli, 1996) measures the combined variance contribution of Θ_i including any interactions with other variables and thus reads

$$S^T_{Q,i} = \frac{\mathbb{E}_{\boldsymbol{\Theta}_{\sim i}}\left[\mathbb{V}_{\Theta_i}\left[Q|\boldsymbol{\Theta}_{\sim i}\right]\right]}{\mathbb{V}\left[Q\right]} = 1 - \frac{\mathbb{V}_{\boldsymbol{\Theta}_{\sim i}}\left[\mathbb{E}_{\Theta_i}\left[Q|\boldsymbol{\Theta}_{\sim i}\right]\right]}{\mathbb{V}\left[Q\right]}. \tag{4}$$

Originally, these expressions follow from a projection of $\mathcal{Q}$ onto a unique orthogonal functional basis with respect to the d-dimensional uniform space $\boldsymbol{U}$ and computing the partial variances related to the summands. This projection is the so-called Sobol' decomposition (Sobol', 1993), which reads:

$$f(\boldsymbol{U}) = f_0 + \sum_{i=1}^{d} f_i(U_i) + \sum_{i=1}^{d}\sum_{j=i+1}^{d} f_{ij}(U_i, U_j) + \cdots + f_{12\ldots d}(\boldsymbol{U}). \tag{5}$$

Given the marginal CDFs of Θ_i, F_i, and assuming pairwise independence amongst all Θ_i, the isoprobabilistic marginal transformation $T : \Theta_i \to F_i(\Theta_i), 1 \leq i \leq d$ can be defined and the decomposition can be generalized as $f(T(\boldsymbol{\Theta}))$.

2.2 The proposed reliability sensitivity index

In reliability analysis, the quantity of interest is the failure event F and the associated probability of failure. Since F is defined via the indicator function of the failure domain, Luyi et al. (2012) propose to compute importance rankings through a variance decomposition of the indicator function $\mathrm{I}(g \leq 0)$. They do so by means of a surrogate modelling technique as the MC-estimators of these rare event-related typically exhibit relatively slow convergence and thus require large amounts of samples and g-evaluations. Here we extend this approach to a multi-uncertainty setting and propose to perform a variance decomposition of the conditional probability of failure defined in Equation (2). More precisely, we will use the log-transformed version, $\ln P(\boldsymbol{\Theta}_B)$. This can be understood as a measure for the magnitude of the conditional probability of failure. In this way, we focus the sensitivity analysis on possibly substantial/magnitude-altering changes in the estimate of $P(\boldsymbol{\Theta}_B)$. The novel sensitivity indices are given by

$$
\begin{aligned}
S_{\ln P,i} &= \frac{\mathbb{V}_{\Theta_{B,i}}\left[\mathbb{E}_{\boldsymbol{\Theta}_{B\sim i}}\left[\ln\left\{\mathbb{E}_{\boldsymbol{\Theta}_A}\left[\mathrm{I}(g \leq 0)|\boldsymbol{\Theta}_B\right]\right\}|\Theta_{B,i}\right]\right]}{\mathbb{V}_{\boldsymbol{\Theta}_B}\left[\ln\left\{\mathbb{E}_{\boldsymbol{\Theta}_A}\left[\mathrm{I}(g \leq 0)|\boldsymbol{\Theta}_B\right]\right\}\right]} \\
S^T_{\ln P,i} &= 1 - \frac{\mathbb{V}_{\boldsymbol{\Theta}_{B\sim i}}\left[\mathbb{E}_{\Theta_{B_i}}\left[\ln\left\{\mathbb{E}_{\boldsymbol{\Theta}_A}\left[\mathrm{I}(g \leq 0)|\boldsymbol{\Theta}_B\right]\right\}|\boldsymbol{\Theta}_{B\sim i}\right]\right]}{\mathbb{V}_{\boldsymbol{\Theta}_B}\left[\ln\left\{\mathbb{E}_{\boldsymbol{\Theta}_A}\left[\mathrm{I}(g \leq 0)|\boldsymbol{\Theta}_B\right]\right\}\right]}.
\end{aligned}
$$

While these expressions appear cumbersome, they exhibit key features of the new indices.

1. The variance decomposition of the total variance contributed by $\boldsymbol{\Theta}_B$ rather than $\boldsymbol{\Theta}$ is performed, which is reflected by the normalizing constants.

2. The variance due to $\boldsymbol{\Theta}_A$ is accounted for as a weight of the contribution of each outcome $\boldsymbol{\theta}_B$.

3. Due to the expectation $\mathbb{E}_{\boldsymbol{\Theta}_A}$, the employed QOI is smooth over $\Omega_{\boldsymbol{\Theta}_B}$. In particular, it is non-binary as opposed to the QOI underlying the indices proposed by Luyi et al. (2012).

In the following section, we discuss common problems when tackling reliability sensitivities with sampling methods. Thereafter, we introduce the means to circumvent sampling almost entirely in the computation of the novel sensitivity indices.

2.3 Monte-Carlo estimators

Saltelli et al. (2000) provide Monte Carlo-estimators for expressions (3) and (4). Based on a set of n_s d-dimensional $\boldsymbol{\Theta}$-samples, $n_s \cdot (d+2)/2$ model evaluations are necessary to compute them. Therefore, these estimators may be intractable if a model evaluation is computationally expensive, d is large or Q is given by a failure event with small associated probability of failure for which $n_s \approx 100/\mathbb{P}(F)$ at an estimator coefficient of variation of 10%. Therefore, since typically $\mathbb{P}(F)$ is very small, n_s becomes prohibitively large. Note, that for the novel sensitivity indices the computational burden would even amount to a multiple of what is needed for the computation of sensitivity indices of the indicator function of g. This is due to the need to solve Equation (2) $n_s \cdot (d+2)/2$ times, which may in turn require many g-evaluations per solution. Conversely, computing the indices associated with $\mathrm{I}(g \leq 0)$ requires a single g-evaluation at each sample to determine whether $g \leq 0$. Therefore, the computational effort approximately scales as the average number of g-calls necessary to solve Equation (2). However, the smoothness in our choice of Q is key to an entirely surrogate-driven sensitivity computation, which facilitates the use of only a small fraction of the samples required in the sampling-based procedure. Two types of surrogate models have been tested and are detailed in the subsequent section.

3 Polynomial Basis Surrogate Modelling

Let $\boldsymbol{\Theta}$ be a random vector on the outcome space $\mathbb{R}^d$ with joint CDF $F_{\boldsymbol{\Theta}}$ whose elements are mutually independent and $\mathcal{Y}(\boldsymbol{\Theta}) = Y \in \mathbb{R}$. If Y has finite mean-square, i.e. $\mathbb{E}_{\boldsymbol{\Theta}}[Y(\boldsymbol{\Theta})^2] < \infty$, it belongs in a Hilbert space $\mathcal{H}$ on which an inner product of any two functions $g,h \in \mathcal{H}$ is defined as

$$\langle g(\boldsymbol{\Theta}), h(\boldsymbol{\Theta})\rangle_{\mathcal{H}} = \mathbb{E}_{\boldsymbol{\Theta}}[g(\boldsymbol{\Theta})h(\boldsymbol{\Theta})] = \int_{\mathbb{R}^d} g(\boldsymbol{\theta})h(\boldsymbol{\theta})f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}, \tag{6}$$

where $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the joint PDF of $\boldsymbol{\Theta}$. g and h are orthogonal if

$$\langle g(\boldsymbol{\theta}), h(\boldsymbol{\theta})\rangle_{\mathcal{H}} = \mathbb{E}_{\boldsymbol{\Theta}}[g(\boldsymbol{\Theta})h(\boldsymbol{\Theta})] = 0. \tag{7}$$

Note, that if g and h can be written as products of univariate functions of the components of $\boldsymbol{\Theta}$, the following holds:

$$\langle g(\boldsymbol{\theta}), h(\boldsymbol{\theta})\rangle_{\mathcal{H}} = \prod_{i=1}^{d} \mathbb{E}_{\Theta_i}[g_i(\Theta_i)h_i(\Theta_i)]. \tag{8}$$

Given a complete and orthonormal basis of $\mathcal{H}$, $\{h_i(\boldsymbol{\Theta}), i \in \mathbb{N}\}$, Y may be expressed as a linear combination of the basis functions:

$$Y = \mathcal{Y}(\boldsymbol{\Theta}) = \sum_{i=0}^{\infty} a_i h_i(\boldsymbol{\Theta}). \tag{9}$$

Then, since $Y \in \mathcal{H}$, the approximation

$$\widehat{Y} = \widehat{\mathcal{Y}}(\boldsymbol{\Theta}) = \sum_{i=0}^{p} a_i h_i(\boldsymbol{\Theta}) \tag{10}$$

asymptotically ($p \to \infty$) converges to Y in the mean-square sense. For $d = 1$, a possible choice of basis functions related to certain standard distribution types of $f_{\boldsymbol{\Theta}}$ are known polynomial families $\{\psi_i(\Theta), i = 0,\dots,p\}$, which can be found systematically by means of the Askey scheme (Xiu and Karniadakis, 2002). This lays the foundation for both PCEs and LRAs. They differ with respect to how the multi-dimensional base polynomials are defined and how the expansion coefficients a_i are determined. For $d > 1$, due to Eq. (8), multi-dimensional basis polynomials Ψ_k can be easily constructed as products of the one-dimensional canonical polynomials $\psi_k^{(i)}$.

3.1 Polynomial Chaos Expansions

Given the polynomial family of the i-th input Θ_i up to p_i-th order $\{\psi_j^{(i)}(\Theta_i), j = 0,\dots,p_i\}$, the j-th multi-dimensional basis function reads

$$\Psi_j = \prod_{i=1}^{d} \psi_{\alpha_{ji}}^{(i)}(\Theta_i), \tag{11}$$

where $\boldsymbol{\alpha}$ contains all combinations of d-dimensional index sets each assigning a polynomial order to each input Θ_i such that the total polynomial order $|\boldsymbol{\alpha}_j| = \sum_{i=1}^{d} \alpha_{ji} \leq p, 0 \leq j \leq P-1$. The number of basis functions P is given by

$$P = \binom{d+p}{p} \tag{12}$$

and the PCE format reads

$$\widehat{\mathcal{Y}}^{PCE}(\boldsymbol{\Theta}) = \sum_{j=0}^{P-1} a_j \prod_{i=1}^{d} \psi_{\alpha_{ji}}^{(i)}(\Theta_i). \tag{13}$$

The coefficients $\boldsymbol{a}$ are found through a projection of $\mathcal{Y}$ onto the space spanned by $\{\Psi_j, j = 0,\dots,P-1\}$. In this work,we evaluate $\boldsymbol{a}$ using an ordinary least-squares (OLS) procedure, which approximates the projection. Equation (12) indicates a fast growth of the OLS problem size with increasing dimension d. This motivates the use of sparse PCE methods which is also the method of choice in this work. Sparse PCE reduces P by penalizing the number of terms in the PCE through solving a modified, $\mathcal{L}_1$-regularized least-squares problem (Blatman and Sudret, 2011). In this way, the method elicits a minimal number of basis functions such as to best explain the output variance.

3.2 Canonical Decomposition

Low-rank approximations have been introduced originally to represent high-dimensional tensors by means of lower-dimensional tensors (Grasedyck et al., 2013). A specific format of such approximations are canonical decompositions, in which tensors are approximated by means of a linear combination of products of one-dimensional tensors (Hitchcock, 1927). The idea extends to continuous spaces where a multivariate function is approximated by a linear combination of products of univariate functions:

$$\widehat{\mathcal{Y}}^{LRA}(\boldsymbol{\Theta}) = \sum_{j=1}^{r} a_j \prod_{i=1}^{d} \sum_{k=1}^{p_i} z_{ijk} \psi_k^{(i)}(\Theta_i). \tag{14}$$

Therein, an additional set of coefficients $\boldsymbol{z}$ appears, which is can be efficiently determined by solving reduced, univariate least squares problems over the directions $i = 1,\ldots,d$ repeatedly (while keeping all remaining directions constant in each step; this is often referred to as alternating least squares). In a second step, the coefficients $\boldsymbol{a}$ are determined via OLS. A detailed description of the procedure may be found in (Chevreuil et al., 2015) and (Konakli and Sudret, 2016b).

3.3 Surrogate-Based sensitivity indices

Both PCEs and LRAs can be used to infer first-order and total sensitivity indices directly from the computed model coefficients. Rather than searching estimates of expressions (3) and (4), the similarity of the underlying orthogonal Sobol' decomposition in Eq. (5) with Eqs. (13) and (14) is exploited. Sudret (2008) showed that the Sobol' decomposition of the PCE can be found by collecting any multi-dimensional Hermite polynomials depending on identical variable subsets $\boldsymbol{\Theta}_{\mathcal{A}}$ into $f_{\mathcal{A}}(\boldsymbol{\Theta}_{\mathcal{A}})$. Therefore, computing the partial variance of the PCE model associated with a subset of variables $\boldsymbol{\Theta}_{\mathcal{A}}$ amounts to summing the squared coefficients of the respective multi-dimensional basis polynomials in which the elements of $\boldsymbol{\Theta}_{\mathcal{A}}$ occur (exclusively for Sobol' indices and greedily for total indices). The same concept can be applied to LRAs even though the compressed format (product) renders the evaluation somewhat more tedious. The expressions for LRA-based first-order and total indices are derived in (Konakli and Sudret, 2016a).

4 Conditional Surrogate-Based Reliability Sensitivities

The computation of sensitivity indices via polynomial surrogates requires the QOI to be sufficiently smooth. In particular, any attempts to obtain surrogate-based indices of the indicator function of the failure domain $\mathrm{I}(g \leq 0)$ directly in such a manner must fail due to the discontinuity in $\mathrm{I}(g \leq 0)$. However, the log-transformed conditional probability of failure is continuous in the space of $\boldsymbol{\Theta}_B$ so that one may compute the proposed sensitivity indices with polynomial surrogates. To this end, we devise a two-level surrogate modelling procedure. Building a surrogate of $\ln P$ (level 2) requires an experimental design that consists of samples of $\boldsymbol{\Theta}_B$ and the associated probabilities of failure given each of these samples. That is, in order to obtain the experimental design one has to solve n_2 reliability problems, where n_2 is the experimental design size for the final surrogate. Therefore, an auxiliary (level 1) surrogate is built for the actual model $\mathcal{Y}$, based on which the reliability computations can be conducted. For the level 1-surrogate, n_1 samples of $\boldsymbol{\Theta}$ as well as the evaluation of the original model at these samples are required. The overall number of original model evaluations is thereby limited to n_1 because any subsequent computations are surrogate-based. The analysis proceeds in the following way:

1. Perform latin hypercube sampling (LHS) to obtain n_1 $\boldsymbol{\Theta}$-samples and evaluate the model $\mathcal{Y}$ at these samples. Build the level 1-surrogate for the model, $\widehat{\mathcal{Y}}$.

2. Elicit a variable subset of interest $\boldsymbol{\Theta}_B$ (possibly repeat the analysis for various subsets). Perform latin hypercube sampling (LHS) to obtain n_2 $\boldsymbol{\Theta}_B$-samples. For each sample, use a structural reliability method and the level 1-surrogate-based limit-state function $\widehat{g}$ to approximate

$$\widehat{P}_F^{(i)} = \mathbb{E}_{\boldsymbol{\Theta}_A}\left[\mathrm{I}(\widehat{g}\left(\boldsymbol{\Theta}_A, \boldsymbol{\theta}_B^{(i)}\right) \leq 0) \middle| \boldsymbol{\theta}_B^{(i)}\right] = \int_{\Omega_A} \mathrm{I}\left[\widehat{g}\left(\boldsymbol{\theta}_A, \boldsymbol{\theta}_B^{(i)}\right) \leq 0\right] f_{\boldsymbol{\Theta}_A|\boldsymbol{\Theta}_B}\left(\boldsymbol{\theta}_A|\boldsymbol{\theta}_B^{(i)}\right) \mathrm{d}\boldsymbol{\theta}_A.$$

3. From the set $\{\boldsymbol{\theta}_B^{(i)}, \ln(\widehat{P}_F^{(i)})\}_{i=1,\cdots,n_2}$, build the level-2 surrogate $\widehat{\ln P_F}(\boldsymbol{\Theta}_B)$.

4. Obtain first-order and total sensitivity indices of $\widehat{\ln P_F}$ by means of the model coefficients.

The procedure outlined above is also sketched in Figure (1).

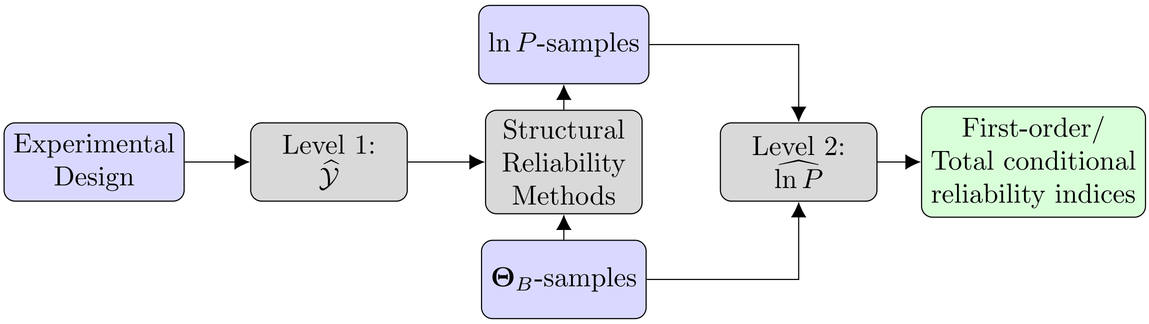

Figure 1: Flow diagram of 2-level surrogate-based conditional reliability sensitivities.

5 Example Application

The considered example is a 2-D truss (Figure (2)) consisting of 13 rods, where horizontal and diagonal rods have log-normally distributed cross-sections A_1, A_2 and Young's moduli E_1, E_2, respectively. The truss sustains 6 vertical point loads P_1 - P_6 which are modelled as Gumbel-distributed (Lee and Kwak, 2006; Konakli and Sudret, 2016a). The parameters of the input variables are given in Table (1). Failure is defined by $g(\boldsymbol{\Theta}) = u_{lim} - u_{max}(\boldsymbol{\Theta})$, where $u_{lim} = \{10cm, 12cm, 14cm\}$ are considered.

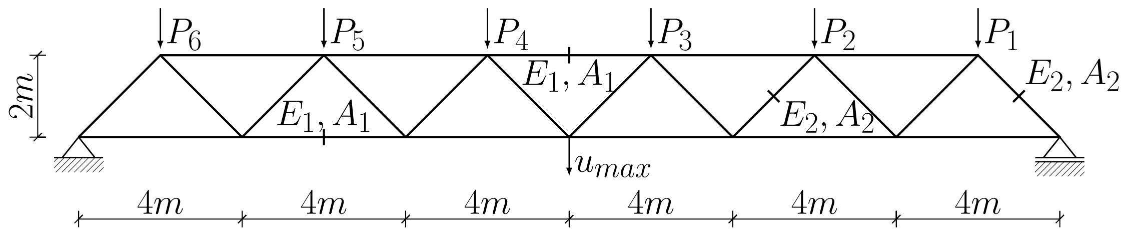

Figure 2: 2-D truss example.

In view of the Bayesian interpretation of the variable sets $\boldsymbol{\Theta}_A$ (irreducible) and $\boldsymbol{\Theta}_B$ (data available, reducible), the latter is chosen to comprise any material properties $\boldsymbol{\Theta}_B = \{E_1, A_1, E_2, A_2\}$ and thus $\boldsymbol{\Theta}_A = \{P_1, \ldots, P_6\}$.

Table 1: Input variable definitions of the truss example.

Random Variable	Distribution	Mean	Standard deviation
A_1 $[m^2]$	Log-Normal	$2 \cdot 10^{-3}$	$2 \cdot 10^{-4}$
A_2 $[m^2]$	Log-Normal	$1 \cdot 10^{-3}$	$1 \cdot 10^{-4}$
E_1, E_2 $[Pa]$	Log-Normal	$2.1 \cdot 10^{11}$	$2.1 \cdot 10^{10}$
P_1 - P_6 $[N]$	Gumbel	$5.0 \cdot 10^4$	$7.5 \cdot 10^4$

Both level-1 and level-2 experimental designs have been obtained via latin hypercube sampling. 100 samples have been used for both PCE and LRA level 1-surrogates. In the first level, the LRA approach yields consistently smaller global and conditional (on samples for which $u \leq u_{lim}$) model errors, which is why all level-2 surrogates are built from a level-1 LRA approach. This is in accordance with the findings of Konakli and Sudret (2016b). The analysis has been repeated for different random level-1 experimental designs 20 times, which yields the estimator statistics given in Figures 3 & 4. They are given for the relative error ε with respect to the direct Monte-Carlo (DMC) reference solution, which is defined as

$$\varepsilon_Q = \frac{Q - Q_{ref}}{Q_{ref}}. \tag{15}$$

All reliability analyses haven been performed with the first-order reliability method (FORM). For the reference solution, $n_{DMC} = 10^5$ samples have been used implying the solution of $3 \cdot 10^5$ reliability problems of dimension 6 (since the reduced B-variable space has dimension 4).
The high sensitivity indices of E_1 and A_1 are computed accurately with 1-2 % relative error with 100 model evaluations. For the variables E_2 and A_2, which have low influence, relative errors can become as large as 40% in single cases. This may partly be due to the small magnitude of the sought numerical values in combination with the slow convergence rate of the DMC-approach for the proposed reliability sensitivities. While the LRAs are superior in predicting the model behaviour (level 1) in the tails of the input joint density, PCE and LRA-based level-2 surrogates perform comparably well. Evidently, the logarithmic transformation yields very similar first- and total-order indices. This is not the case for the reliability indices proposed in (Luyi et al., 2012), in which first-order contributions are often negligibly small and total-order indices have to be considered.

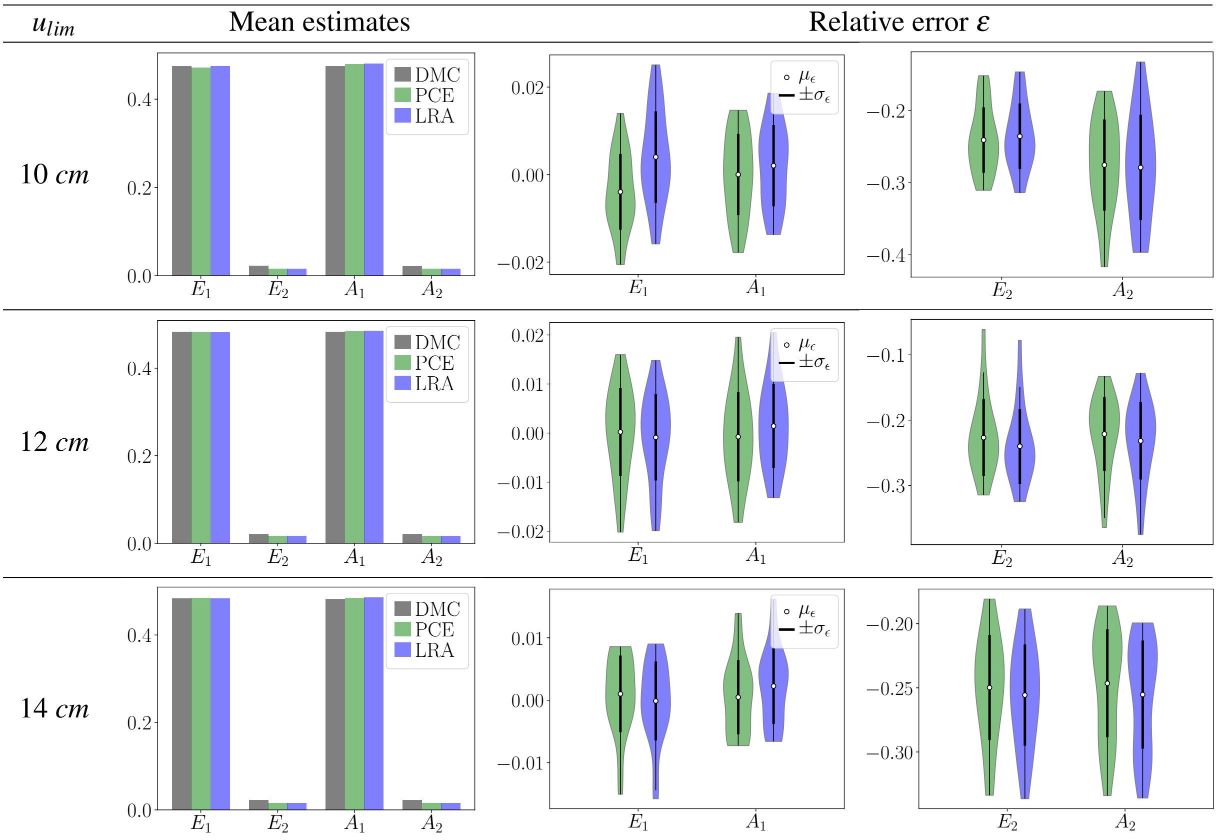

Figure 3: $\ln P_F$ first-order Sobol' indices: mean estimates and relative errors ($n_1 = 100, n_2 = 1000$).

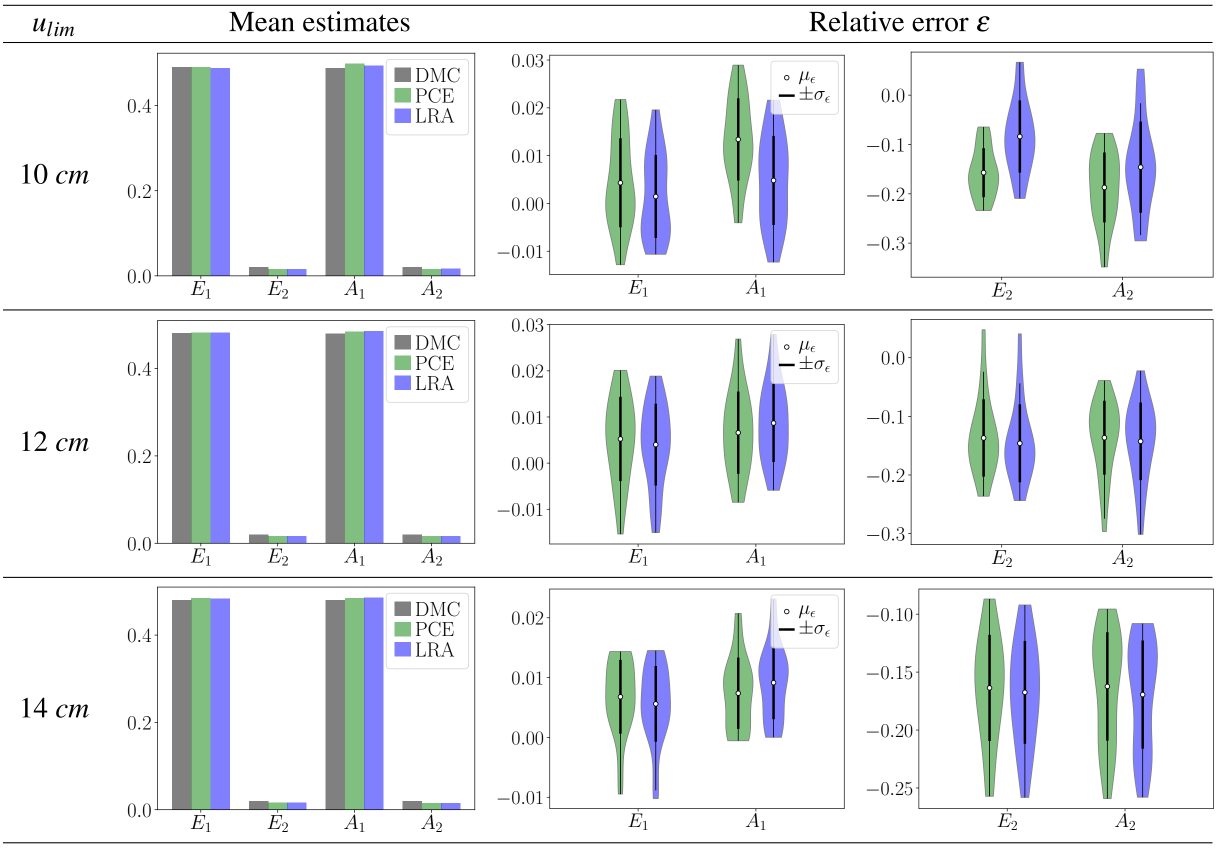

Figure 4: $\ln P_F$ total indices: mean estimates and relative errors ($n_1 = 100, n_2 = 1000$).

6 Concluding Remarks

In this paper we propose a variance-based sensitivity index tailored to reliability analysis in the presence of multi-uncertainty. Moreover, we devise a two-level polynomial surrogate modelling strategy to compute a model input variable importance ordering based on the proposed indices. It was found that for a numerical 2-D truss example as few as 100 original model evaluations suffice to obtain accurate importance rankings. Amongst the two considered surrogates, namely PCEs and LRAs, the LRAs have been found to be more suitable for surrogate-driven reliability computations (level 1). Both PCEs and LRAs perform comparably in level 2.

The contribution of the novel index is threefold. First, it focusses the sensitivity analysis on a variable subset of interest which is specifically interesting in the presence of multi-uncertainty and data assimilation applications. Secondly, it represents a direct sensitivity measure for the probability of failure magnitude as opposed to indices based on the indicator function which rank influence on the failure domain shape. Finally, the new index facilitates the entirely surrogate-driven computation of reliability sensitivities by smoothening the indicator function discontinuity through an integral formulation.

For the investigated numerical example, the estimator variances due to the random selection of the experimental design are relatively small. However, this need not be the case for other models and a possible solution to this issue could be a more guided way to select the level-1 experimental design with respect to the QOI.

The dimensionality of the models, that can be handled by the approach is mainly limited by the surrogate modelling techniques. Using LRAs it is applicable up to several hundred input variables. Additionally, as a byproduct of the analysis within a Bayesian inference problem, the obtained surrogate models may be used to compute approximate reliability updates from posterior samples at negligible additional computational cost.

Acknowledgments

This project was supported by the German Research Foundation (DFG) through Grant STR 1140/6-1 under SPP 1886.

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