Random Fields in Bayesian Inference: Effects of the Random Field Discretization

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Abstract: Probabilistic analysis of engineering systems often requires models that account for the random spatial variability of their parameters. Information about a target set of parameters can be obtained using mathematical models in combination with observational data. Bayesian inference handles this task by computing a posterior probability distribution that quantifies the combined effects of prior knowledge and observations. However, the complexity of the inference process is increased when the spatial variation of the parameters is considered. Spatially variable quantities are usually modeled by random fields that are discretized with a high number of random variables. In this paper, the challenge is addressed by representing the random field with the Karhunen-Loève expansion with the purpose of evaluating its effects on the outcome of the Bayesian inference. To this end, the influence of the number of terms in the expansion and the correlation length of the prior random field are assessed. The analytical study is carried out on a cantilever beam with spatially variable flexibility. We show that it requires more terms in the series expansion to identify the flexibility random field with the same accuracy as the deflection solution. Furthermore, the decay of the variance error when estimating the posterior flexibility is slower than in the posterior deflection.

1 Introduction

In order to realistically model the behavior of engineering systems, uncertainties in material properties and loading conditions must be taken into account. Monitoring and other observational data can be used in combination with mathematical models to obtain information on uncertain parameters. Bayesian inference provides a framework to deal with this task by specifying a prior probability distribution for the uncertain parameters and a likelihood function for the representation of the data. The Bayesian inference process is often carried out numerically since analytical solutions are difficult to compute in most cases. The computational cost involved in the updating process is usually high, primarily because the underlying deterministic model needs to be solved many times.

An additional level of complexity is included if the random spatial variability of the model parameters is considered. The spatial variation is generally modeled using random fields, whose parameter space contains an infinite number of elements describing the uncertain quantity at each point of the continuous domain of the system [10]. A major issue is the proper representation of spatially variable quantities with random fields defined by a finite number of random variables. Therefore, random field discretization methods are applied to reduce the dimensionality of the original random field and to obtain more tractable computations.

A detailed investigation of the effects of the random field discretization on the Bayesian inference has not been conducted so far [9]. Among the several existing methods for random field discretization, the Karhunen-Loève expansion is especially suitable to perform Bayesian inference on random fields, as it often leads to efficient representations with small numbers of random variables. The present contribution uses this method to parametrize the random field so that the inference can be carried out on the random variables arising from the representation. By varying the number of terms in the expansion, a dependence of the dimensionality on the prior and likelihood can be established in order to evaluate its impact on the posterior solution. Furthermore, a study about the influence of the correlation length of the prior random field on the posterior statistics is also provided. In this way, different prior random field assumptions are incorporated into the inference process. These aspects are investigated in the learning of the flexibility of a cantilever beam based on measurements of the displacement along its length. The advantage of this example lies in the possibility to derive closed-form expressions of the posterior random field. This analytical representation allows us to efficiently carry out an error analysis in order to evaluate how the Karhunen-Loève discretization of the prior random field influences the posterior statistics.

2 Modeling and Discretization of Random Fields

2.1 Overview

Random fields are mathematical models used to represent phenomena in which the quantity of interest fluctuates discretely or continuously through space in a non-deterministic pattern. Traditional applications of random fields include materials properties with random spatial variability, geometrical imperfections, among others.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\xi \in \Omega$ a random event. A *random field* [10] can be understood as an indexed collection of random variables $\{H(\mathbf{x}, \xi) : \mathbf{x} \in D\}$ representing the evolution of uncertain values that are spatial coordinates on some bounded domain $D \subset \mathbb{R}^d$.

A random field is said to be Gaussian if all finite-dimensional distributions of the set of random variables $\{H(\mathbf{x}, \xi) : \mathbf{x} \in D\}$ are Gaussian [10]. Gaussian random fields are completely characterized by their first- and second-order moments, i.e., the mean function $\mu_H(\mathbf{x}) : D \to \mathbb{R}$ and the autocovariance function $C_{HH}(\mathbf{x}, \mathbf{x}') : D \times D \to \mathbb{R} = \sigma_H(\mathbf{x})\sigma_H(\mathbf{x}')\rho_{HH}(\mathbf{x}, \mathbf{x}')$, where $\sigma_H(\mathbf{x})$ and $\rho_{HH}(\mathbf{x}, \mathbf{x}')$ are the standard deviation and autocorrelation coefficient functions of the random field, respectively [2].

The parameter space of a random field usually contains an infinite number of elements [10]. For the numerical representation, random fields must be approximated in terms of a finite set of random variables using stochastic discretization methods. According to the underlying probability distribution of the random field, those methods can be grouped in two categories: those that seek the representation of Gaussian random fields and those that deal with non-Gaussian ones. Particularly, several methods have been proposed over the past decades for the representation of Gaussian random fields [2]. Among the different alternatives, the so-called series expansion methods [7] represent the random field as a finite series expansion of random variables and deterministic functions. This contribution focuses on this approach, and more precisely on the Karhunen-Loève expansion, which is introduced next.

2.2 Karhunen-Loève Expansion

Consider a real-valued random field $H(\mathbf{x}, \xi)$ with mean $\mu_H(\mathbf{x})$ and autocovariance function $C_{HH}(\mathbf{x}, \mathbf{x}')$. It can be proved that the autocovariance function is symmetric, bounded and positive semi-definite [1]. Under these conditions, $H(\mathbf{x}, \xi)$ can be approximated by $\hat{H}(\mathbf{x}, \xi)$ using a linear combination of orthogonal functions resulting from the spectral decomposition of a certain

integral operator with the autocovariance function as kernel. This representation is known as the *Karhunen-Loève* (K-L) expansion [4] and is expressed after truncating the series at the M-th term as,

$$H(\mathbf{x},\xi) \approx \hat{H}(\mathbf{x},\xi) = \mu_H(\mathbf{x}) + \sum_{k=1}^{M} \sqrt{\lambda_k}\phi_k(\mathbf{x})\theta_k(\xi), \tag{1}$$

where, $\{\theta_k(\xi)\}$ is a set of mutually uncorrelated random variables with zero mean and unit variance (i.e. $\mathbb{E}[\theta_k(\xi)] = 0$ and $\mathbb{E}[\theta_k(\xi)\theta_l(\xi)] = \delta_{kl}$), and $\lambda_k \in [0,\infty)$ and $\phi_k(\mathbf{x}) : D \to \mathbb{R}$ are the eigenvalues and eigenfunctions of the associated integral operator of the autocovariance function. It is assumed that the eigenvalues are arranged in a non-increasing sequence [1].
The set of eigenpairs $\{\lambda_k, \phi_k\}$ is obtained through the solution of the homogeneous Fredholm integral equation of the second kind [4], $\int_D C_{HH}(\mathbf{x},\mathbf{x}')\phi_k(\mathbf{x}')\mathrm{d}\mathbf{x}' = \lambda_k\phi_k(\mathbf{x})$, for which analytical solutions exist only for specific cases of autocovariance functions [6]. In general, the solution must be estimated numerically, e.g., by approximating the eigenfunctions as a linear combination of complete basis functions [2].
The approximation error arising from the K-L discretization can be quantified by means of different types of measures. This allows one to evaluate the number of terms in the expansion that must be included in the series to achieve certain error margins. For instance, the local approximation is usually quantified in terms of the mean squared error $\mathbb{E}[(H(\mathbf{x},\xi) - \hat{H}(\mathbf{x},\xi))^2]$. Another alternative is to consider global error measures that average local pointwise errors over the domain [2]. These measures are functions of the number of terms in the K-L expansion and are defined by the relative errors of the mean and variance, given respectively by the following weighted integrals (where $|D| = \int_D \mathrm{d}\mathbf{x}$):

$$\bar{\varepsilon}_\mu(M) = \frac{1}{|D|}\int_D \left|\frac{\mathbb{E}[H(\mathbf{x},\xi)] - \mathbb{E}[\hat{H}(\mathbf{x},\xi)]}{\mathbb{E}[H(\mathbf{x},\xi)]}\right| \mathrm{d}\mathbf{x}, \quad \bar{\varepsilon}_\sigma(M) = \frac{1}{|D|}\int_D \left|\frac{\mathrm{Var}[H(\mathbf{x},\xi)] - \mathrm{Var}[\hat{H}(\mathbf{x},\xi)]}{\mathrm{Var}[H(\mathbf{x},\xi)]}\right| \mathrm{d}\mathbf{x}. \tag{2}$$

3 Inverse Problems and Bayesian Inference

In science and engineering, the estimation of uncertain model parameters based on indirect observations is a common task. These type of inferences are known as *inverse problems* [5]. The goal is to find a set of parameters such that, $\tilde{\mathbf{y}} \approx \mathbf{y} = g(\boldsymbol{\theta})$, where $g : \mathbb{R}^M \to \mathbb{R}^m$ is the model or *observation function* which describes the relationship between the data $\tilde{\mathbf{y}} \in \mathbb{R}^m$ and the model parameters $\boldsymbol{\theta} \in \mathbb{R}^M$. When the relationship between $\tilde{\mathbf{y}}$ and $\boldsymbol{\theta}$ is linear and deterministic, the model observation can be simply obtained as the solution of the system of equations $\mathbf{y} = \mathbf{A}\boldsymbol{\theta}$, where $\mathbf{A} \in \mathbb{R}^{m\times M}$ is called the *observation matrix* [5].
In real applications, the measured quantity $\tilde{\mathbf{y}}$ always contains some noise representing the natural discrepancy between the model and the available data. Thus, the classical linear inverse problem is generally written in the form, $\tilde{\mathbf{y}} = \mathbf{A}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$, where the noise $\boldsymbol{\varepsilon} \in \mathbb{R}^m$ is generally assumed to be independent of $\boldsymbol{\theta}$ and has a certain joint probability density function $f_{\boldsymbol{\varepsilon}}$. There exists several approaches for computing the solution of inverse problems. One class of methods determines point estimates of the quantities of interest based on the available data, e.g., regularization techniques [5]. Another common approach is the Bayesian framework [3, 5], which is the focus of this paper and is briefly described in the following.
Let $\boldsymbol{\theta}$ be a set of uncertain variables and suppose that a set of observed or measured data points $\tilde{\mathbf{y}}$ are given. Using Bayes' theorem, the belief about $\boldsymbol{\theta}$ can be updated by,

$$f(\boldsymbol{\theta}|\tilde{\mathbf{y}}) = \frac{f(\tilde{\mathbf{y}}|\boldsymbol{\theta})\, f(\boldsymbol{\theta})}{f(\tilde{\mathbf{y}})},$$

where the *prior distribution* $f(\boldsymbol{\theta})$ represents the initial knowledge about the parameters before performing any measurement; the *likelihood function* $L(\boldsymbol{\theta}) = f(\tilde{\mathbf{y}}|\boldsymbol{\theta})$ is a measure of the plausability of observing data $\tilde{\mathbf{y}}$ conditional on the parameter vector $\boldsymbol{\theta}$; the *model evidence* $c_{\mathrm{E}} = f(\tilde{\mathbf{y}}) = \int f(\tilde{\mathbf{y}}|\boldsymbol{\theta})\, f(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta}$ acts as a normalizing constant; and the *posterior distribution* $f(\boldsymbol{\theta}|\tilde{\mathbf{y}})$ represents the updated belief about $\boldsymbol{\theta}$ after performing the measurement $\tilde{\mathbf{y}}$ [3]. The resulting posterior probability distribution is the solution of the inverse problem.

4 Example Problem

4.1 Exact Prior and Posterior Solution

In this study, the spatially variable flexibility $F(x)$ of a cantilever beam is updated [8]. The beam has length $L = 5$ m and is subjected to a deterministic point load $P = 20$ kN at the free end, as shown in Figure 1. A set of $m = 50$ deflection measurements is taken at equally spaced points $\tilde{x} = \{0.1, 0.2, ..., 5\}$ of the domain.

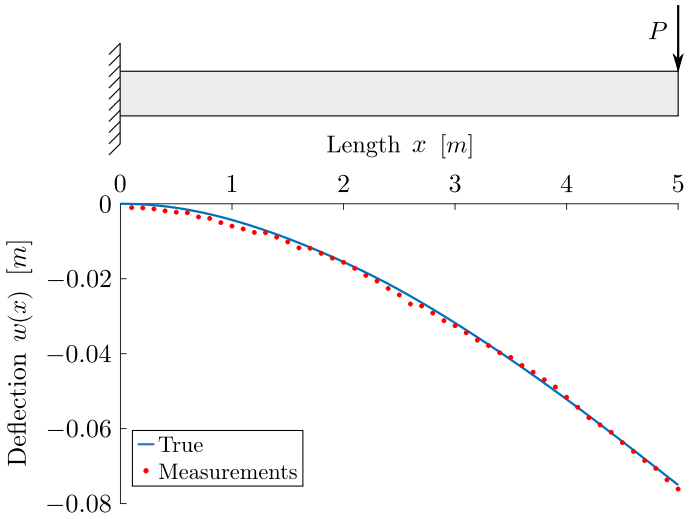

Figure 1: Cantilever beam: True and measured deflection. The measurements are generated by simulation assuming a true *(but in real applications unknown) deflection of the beam.*

The flexibility is defined as the reciprocal of the bending rigidity of the beam, i.e. $F(x) = 1/(E(x)I)$, where $E(x)$ is the Young's modulus and I is the moment of inertia. The prior distribution of $F(x)$ is described by a homogeneous Gaussian random field with mean $\mu_F = 1 \times 10^{-7}\ \mathrm{N^{-1}m^{-2}}$ and exponential autocovariance kernel, $C_{FF}(x, x') = \sigma_F^2 \exp\left(-\frac{|x-x'|}{l_F}\right)$, with standard deviation $\sigma_F = 3.5 \times 10^{-8}\ \mathrm{N^{-1}m^{-2}}$ and correlation length $l_F = 2$ m.
The associated prior mean and autocovariance functions of the deflection $w(x)$ can be evaluated using the prior information about $F(x)$ and assuming an Euler-Bernoulli beam model. As $F(x)$ is Gaussian and $w(x)$ is a linear function of $F(x)$, the prior distribution of $w(x)$ is also Gaussian [5]. The bending moment $M(x)$ is expressed in terms of the deflection $w(x)$ as,

$$M(x) = -E(x)I\,\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \quad \Longrightarrow \quad -M(x)F(x) = \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}. \tag{3}$$

Integrating Eq. 3 twice and noting that the bending moment of the cantilever beam can be simply calculated as $M(x) = P(L - x)$, an analytical expression for the deflection (to be used as the observation function in the Bayesian inverse setting) can be obtained by solving the following equation,

$$w(x, F) = P \int_0^x \int_0^s (L - t)F(t)\, \mathrm{d}t\mathrm{d}s. \tag{4}$$

Consequently, the mean of $w(x)$ can be computed using $\mu_{_F}(x)$ as,

$$\mu_w(x) = P\int_0^x \int_0^s (L-t)\mu_{_F}(t)\,\mathrm{d}t\mathrm{d}s = \frac{P\mu_{_F}}{6}x^2(3L-x)$$

and similarly, the autocovariance function of $w(x)$ can be deduced using $C_{_{FF}}(x,x')$ as the solution of the following quadruple integral (which can be computed analytically):

$$C_{ww}(x,x') = P^2\sigma_{_F}^2 \int_0^{x'} \int_0^x \int_0^{s'} \int_0^s (L-t)(L-t')\exp\left(-\frac{|t-t'|}{l_{_F}}\right)\,\mathrm{d}t\,\mathrm{d}t'\mathrm{d}s\,\mathrm{d}s'.$$

The mean and autocovariance functions for each prior random field are shown in Figure 2.

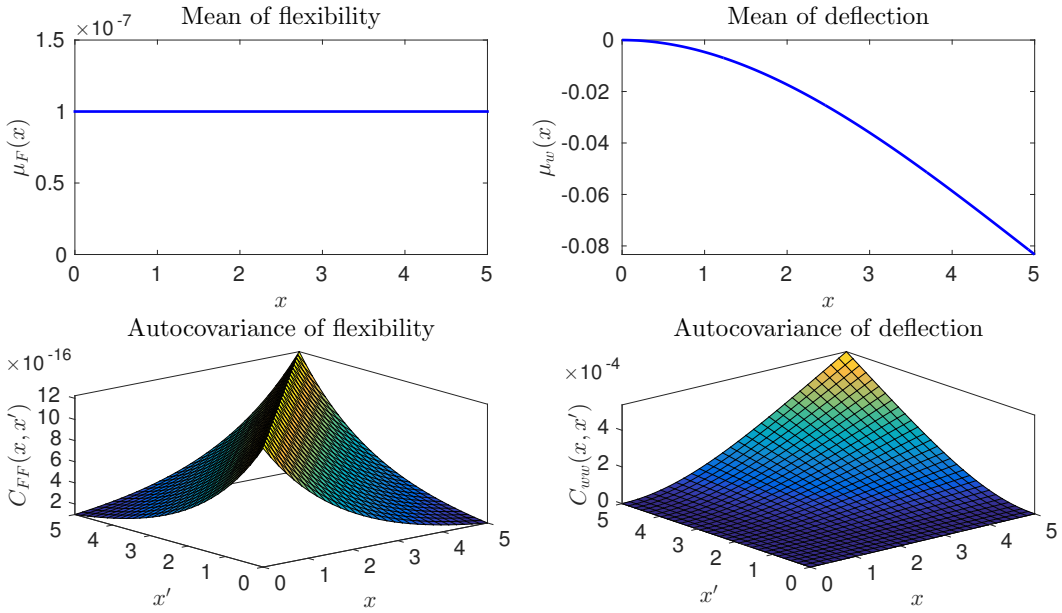

Figure 2: Mean and autocovariance functions of the prior flexibility and prior deflection random fields ($l_{_F} = 2$).

It is assumed that the measurements are subjected to an additive noise which is described by a joint Gaussian PDF $f_{_\varepsilon}$ with zero mean vector $\boldsymbol{\mu}_{_\varepsilon} = [0,...,0]$ and covariance matrix with components $\boldsymbol{\Sigma}_{_{\varepsilon\varepsilon}}(i,j) = \sigma_{_\varepsilon}^2 \exp\left(-\left|\tilde{x}_i - \tilde{x}_j\right|/l_{_\varepsilon}\right)$ for $i,j = 1,...,m$. The standard deviation of the measurement error is $\sigma_{_\varepsilon} = 0.001$ m and its correlation length $l_{_\varepsilon} = 1$ m. These considerations yield a Gaussian likelihood function [8].

Since the prior and likelihood functions are modeled as Gaussian, the posterior distribution will also be Gaussian [3]. Now, assume the vectors of Gaussian random variables $[\mathbf{F}, \tilde{\mathbf{w}}]$ and $[\mathbf{w}, \tilde{\mathbf{w}}]$, where $\mathbf{F}$ and $\mathbf{w}$ correspond to the flexibility and the deflection at a discrete number of spatial points. Note that the mean vectors and covariance matrices of the formed random vectors can be partitioned accordingly in terms of the individual and crossed components [10]. Hence, the exact posterior of the flexibility $f\left(\mathbf{F}|\tilde{\mathbf{w}}\right)$ and the deflection $f\left(\mathbf{w}|\tilde{\mathbf{w}}\right)$ can be calculated analytically as conditional PDFs from their corresponding joint PDFs using Bayes' theorem (see [5] for further details). These posterior random fields given by multivariate Gaussian distributions have the following mean and autocovariance parameters,

$$\boldsymbol{\mu}_{_{F|\tilde{w}}} = \boldsymbol{\mu}_{_F} + \boldsymbol{\Sigma}_{_{F\tilde{w}}}\boldsymbol{\Sigma}_{_{\tilde{w}\tilde{w}}}^{-1}\left(\tilde{\mathbf{w}} - \boldsymbol{\mu}_{_{\tilde{w}}}\right) \qquad \boldsymbol{\Sigma}_{_{FF|\tilde{w}}} = \boldsymbol{\Sigma}_{_{FF}} - \boldsymbol{\Sigma}_{_{F\tilde{w}}}\boldsymbol{\Sigma}_{_{\tilde{w}\tilde{w}}}^{-1}\boldsymbol{\Sigma}_{_{F\tilde{w}}}^{\mathsf{T}} \tag{5}$$

$$\boldsymbol{\mu}_{_{w|\tilde{w}}} = \boldsymbol{\mu}_{_w} + \boldsymbol{\Sigma}_{_{w\tilde{w}}}\boldsymbol{\Sigma}_{_{\tilde{w}\tilde{w}}}^{-1}\left(\tilde{\mathbf{w}} - \boldsymbol{\mu}_{_{\tilde{w}}}\right) \qquad \boldsymbol{\Sigma}_{_{ww|\tilde{w}}} = \boldsymbol{\Sigma}_{_{ww}} - \boldsymbol{\Sigma}_{_{w\tilde{w}}}\boldsymbol{\Sigma}_{_{\tilde{w}\tilde{w}}}^{-1}\boldsymbol{\Sigma}_{_{w\tilde{w}}}^{\mathsf{T}}, \tag{6}$$

where most of the quantities are known from the prior random fields and it can be seen that $\mathbf{\Sigma}_{\tilde{w}\tilde{w}} = \mathbf{\Sigma}_{ww} + \mathbf{\Sigma}_{\varepsilon\varepsilon}$, $\mathbf{\Sigma}_{w\tilde{w}} = \mathbf{\Sigma}_{ww}$ and the covariance matrix $\mathbf{\Sigma}_{F\tilde{w}}$ is obtained by computing the corresponding covariance function $C_{F\tilde{w}}(x,x') = P\sigma_F^2 \int_0^{x'} \int_0^s (L-t) C_{FF}(x,t)\ \mathrm{d}t\mathrm{d}s$ on a given $x, x' \in D$. Figure 3 shows the exact mean and autocovariance functions for each posterior random field.

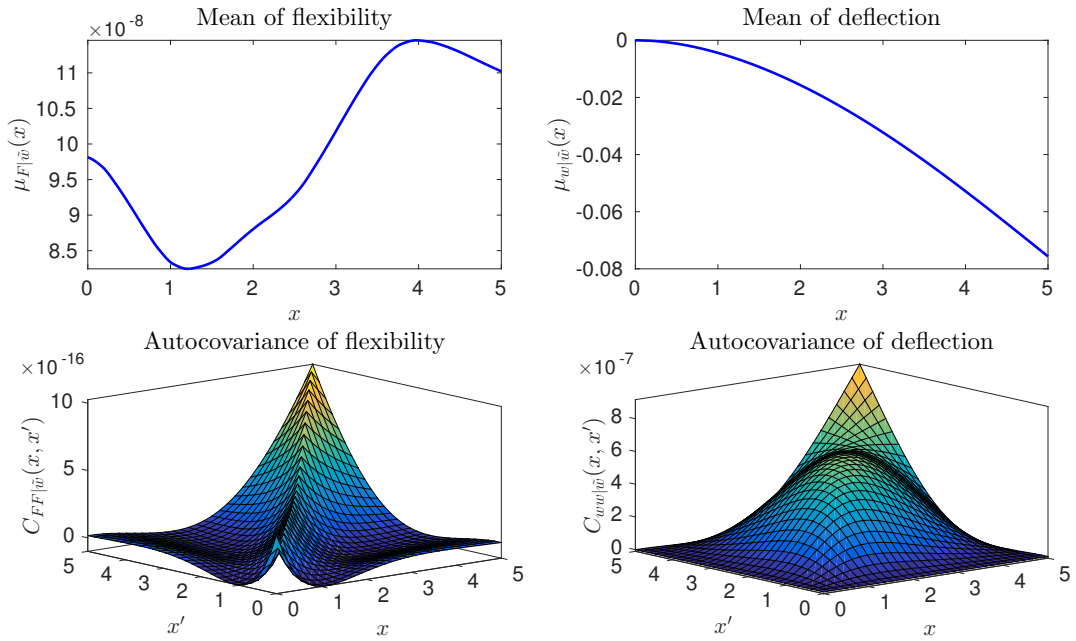

Figure 3: Exact mean and autocovariance functions of the posterior flexibility and deflection random fields ($l_F = 2$).

4.2 Approximated Posterior Solution Using the K-L Expansion

The K-L expansion of the prior flexibility random field (Eq. 1) can be replaced into the forward model (Eq. 4) to obtain,

$$w(x,\boldsymbol{\theta}) = \int_0^x \int_0^s P(L-t)\left[\mu_F(t) + \sigma_F \sum_{k=1}^{M} \sqrt{\lambda_k}\phi_k(t)\theta_k\right] \mathrm{d}t\mathrm{d}s = \mu_w(x) + P\sigma_F \sum_{k=1}^{M} \Phi_k(x)\sqrt{\lambda_k}\theta_k$$

with, $\Phi_k(x) = \int_0^x \int_0^s (L-t)\phi_k(t)\ \mathrm{d}t\mathrm{d}s$. Alternatively, this equation can be expressed in matrix form as,

$$\mathbf{w} = \boldsymbol{\mu}_w + \mathbf{\Phi}\mathbf{\Lambda}\boldsymbol{\theta} = \boldsymbol{\mu}_w + \mathbf{A}\boldsymbol{\theta}$$

where the matrix $\mathbf{\Lambda} = P\sigma_F \mathrm{diag}(\sqrt{\boldsymbol{\lambda}}) \in \mathbb{R}^{M\times M}$ ($\boldsymbol{\lambda}$ is the vector of eigenvalues), the matrix $\mathbf{\Phi} \in \mathbb{R}^{n\times M}$ is computed by the solution of $\Phi_k(x)$ evaluated at a given partition of the domain $x = [x_1, x_2, ..., x_n] \in D = [0, L]$ (for $k = 1, ..., M$), and the mean vector $\boldsymbol{\mu}_w \in \mathbb{R}^{n\times 1}$ is given by the prior mean deflection evaluated in such a partition. Notice that $\Phi_k(x)$ can be solved numerically given that the eigenpairs of the target autocovariance kernel are available. However, since the exponential autocovariance kernel has closed-form solution for its eigenpairs [4], the double integral $\Phi_k(x)$ can be solved analytically.

Now, proceeding in a similar way to the exact posterior case, we assume a Gaussian random vector composed by the two random vectors $\boldsymbol{\theta}$ and $\tilde{\mathbf{w}}$. Again, the posterior distribution can be calculated as the conditional PDF of $\boldsymbol{\theta}$ given $\tilde{\mathbf{w}}$ from the joint PDF $f(\boldsymbol{\theta}, \tilde{\mathbf{w}})$ using the Bayes' theorem [5], then

$$f\left(\boldsymbol{\theta}|\tilde{\mathbf{w}}\right) = \frac{1}{\sqrt{(2\pi)^M \det(\boldsymbol{\Sigma}_{\theta\theta|\tilde{w}})}} \exp\left(-\frac{1}{2}\left[\boldsymbol{\theta} - \boldsymbol{\mu}_{\theta|\tilde{w}}\right]^{\mathsf{T}} \boldsymbol{\Sigma}_{\theta\theta|\tilde{w}}^{-1}\left[\boldsymbol{\theta} - \boldsymbol{\mu}_{\theta|\tilde{w}}\right]\right)$$

where the conditional mean vector $\boldsymbol{\mu}_{\theta|\tilde{w}}$ and the conditional autocovariance matrix $\boldsymbol{\Sigma}_{\theta\theta|\tilde{w}}$ are in analogy to Eqs. 5 and 6, given by

$$\boldsymbol{\mu}_{\theta|\tilde{w}} = \boldsymbol{\mu}_{\theta} + \boldsymbol{\Sigma}_{\theta\tilde{w}}\boldsymbol{\Sigma}_{\tilde{w}\tilde{w}}^{-1}\left(\tilde{\mathbf{w}} - \boldsymbol{\mu}_{\tilde{w}}\right) \qquad \boldsymbol{\Sigma}_{\theta\theta|\tilde{w}} = \boldsymbol{\Sigma}_{\theta\theta} - \boldsymbol{\Sigma}_{\theta\tilde{w}}\boldsymbol{\Sigma}_{\tilde{w}\tilde{w}}^{-1}\boldsymbol{\Sigma}_{\theta\tilde{w}}^{\mathsf{T}}$$

here, it is seen that $\boldsymbol{\mu}_{\tilde{w}} = \boldsymbol{\mu}_{w}$, $\boldsymbol{\mu}_{\theta} = \mathbf{0}$, $\boldsymbol{\Sigma}_{\theta\theta} = \mathbf{I}$ (where $\mathbf{I} \in \mathbb{R}^{M\times M}$ denotes the identity matrix) and the remaining covariance terms are $\boldsymbol{\Sigma}_{\theta\tilde{w}} = \mathbb{E}\left[(\boldsymbol{\theta} - \boldsymbol{\mu}_{\theta})(\tilde{\mathbf{w}} - \boldsymbol{\mu}_{\tilde{w}})^{\mathsf{T}}\right] = \mathbf{A}^{\mathsf{T}}$ and $\boldsymbol{\Sigma}_{\tilde{w}\tilde{w}} = \mathbb{E}\left[(\tilde{\mathbf{w}} - \boldsymbol{\mu}_{\tilde{w}})(\tilde{\mathbf{w}} - \boldsymbol{\mu}_{\tilde{w}})^{\mathsf{T}}\right] = \mathbf{A}\mathbf{A}^{\mathsf{T}} + \boldsymbol{\Sigma}_{\varepsilon\varepsilon}$. Therefore, the mean and covariance matrix of the posterior distribution for the K-L parameters $\boldsymbol{\theta}$ can be computed respectively as,

$$\boldsymbol{\mu}_{\theta|\tilde{w}} = \mathbf{A}^{\mathsf{T}}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}} + \boldsymbol{\Sigma}_{\varepsilon\varepsilon}\right)^{-1}\left(\tilde{\mathbf{w}} - \boldsymbol{\mu}_{w}\right) \quad \text{and} \quad \boldsymbol{\Sigma}_{\theta\theta|\tilde{w}} = \mathbf{I} - \mathbf{A}^{\mathsf{T}}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}} + \boldsymbol{\Sigma}_{\varepsilon\varepsilon}\right)^{-1}\mathbf{A}.$$

The posterior distribution of the flexibility and deflection random fields can be obtained by propagating the posterior of $\boldsymbol{\theta}$ through the K-L expansion and the observation function. In this case, both approximated posteriors random fields are also multivariate Gaussian described by the following approximated mean and autocovariance functions,

$$\hat{\mu}_{F|\tilde{w}}(x) = \mu_F(x) + \sigma_F \sum_{k=1}^{M} \sqrt{\lambda_k}\phi_k(x)\boldsymbol{\mu}_{\theta|\tilde{w}}^{(k)} \qquad \hat{C}_{FF|\tilde{w}}(x,x') = \sigma_F^2 \sum_{k=1}^{M}\sum_{l=1}^{M} \sqrt{\lambda_k\lambda_l}\phi_k(x)\phi_l(x')\boldsymbol{\Sigma}_{\theta\theta|\tilde{w}}^{(k,l)}$$

$$\hat{\mu}_{w|\tilde{w}}(x) = \mu_w(x) + P\sigma_F \sum_{k=1}^{M} \sqrt{\lambda_k}\,\Phi_k(x)\boldsymbol{\mu}_{\theta|\tilde{w}}^{(k)} \quad \hat{C}_{ww|\tilde{w}}(x,x') = (P\sigma_F)^2 \sum_{k=1}^{M}\sum_{l=1}^{M} \sqrt{\lambda_k\lambda_l}\,\Phi_k(x)\Phi_l(x')\boldsymbol{\Sigma}_{\theta\theta|\tilde{w}}^{(k,l)}.$$

Note that the approximated posterior deflection parameters can be written in a compact matrix form as $\hat{\boldsymbol{\mu}}_{w|\tilde{w}} = \boldsymbol{\mu}_{w} + \mathbf{A}\boldsymbol{\mu}_{\theta|\tilde{w}}$ and $\hat{\boldsymbol{\Sigma}}_{ww|\tilde{w}} = \mathbf{A}\boldsymbol{\Sigma}_{\theta\theta|\tilde{w}}\mathbf{A}^{\mathsf{T}}$.

5 Numerical Investigations

Figures 4 and 5 present the 95% confidence intervals (CI) of the posterior flexibility and deflection for different correlation lengths of the prior random field $l_F = [0.5, 2.5, 4.5]$ and number of terms in the K-L expansion $M = [5, 10, 20]$. The exact 95% CI of the posterior solution is highlighted (blue shaded area). Notice that all approximated CIs are within the exact interval. As expected, in both cases the results are more accurate for larger correlation lengths.

It can be seen that the number of terms in the K-L expansion of the posterior flexibility (Figure 4) must be increased for small correlation lengths in order to obtain a better approximation ($M \approx 20$). As the correlation length increases, this number can be reduced since the estimation improves considerably. A similar study was conducted to illustrate the approximation in the posterior deflection (Figure 5). In this case, the K-L representation requires a smaller number of terms ($M \approx 5$), since the approximation is almost in agreement with the exact solution even when the variability of the random field is high. This behavior can be attributed to the fact that the posterior deflection is computed by an averaging of the K-L expansion of the flexibility random field. As a result of this process, the influence of the higher K-L eigenfunctions is almost negligible, and hence, only the first modes have a large contribution to the random field representation.

The error measures given in Eq. 2 are computed for the prior and posterior flexibility and deflection random fields as a function of the number of terms in the expansion ($M = 1, ..., 100$) and for different correlation lengths of the prior random field ($l_F = [0.5, 2.5, 4.5]$); the results are shown in log-log scale plots on Figures 6 and 7.

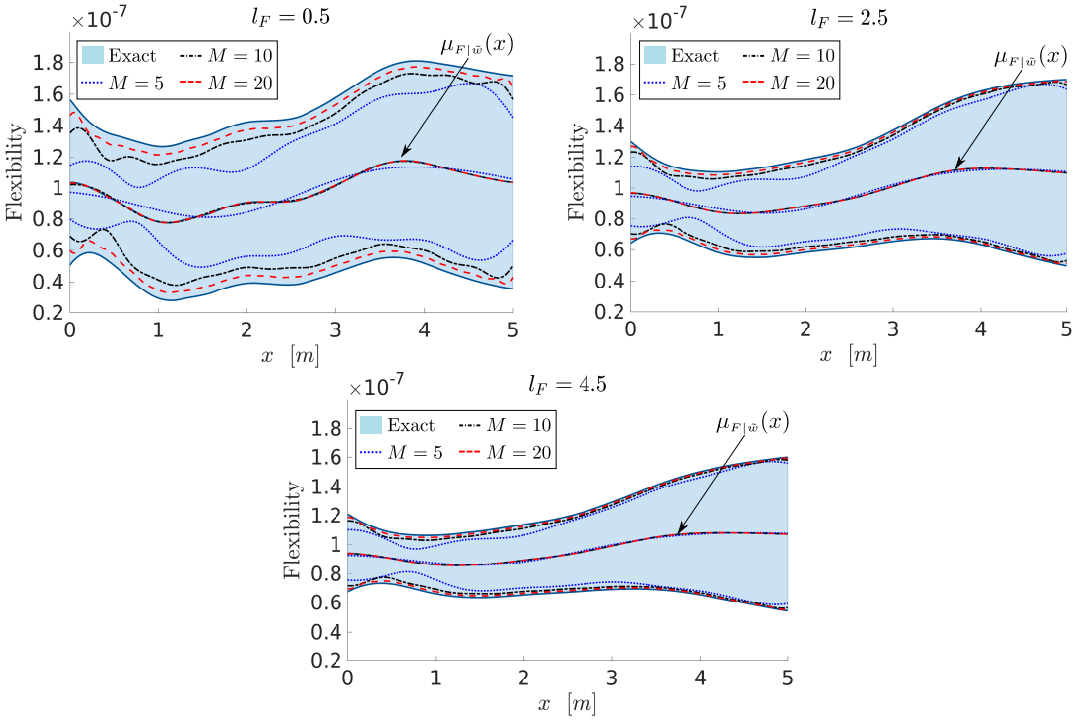

Figure 4: Posterior flexibility: 95% CI for different correlation lengths and terms in the K-L expansion. The shaded area corresponds to the exact CI.

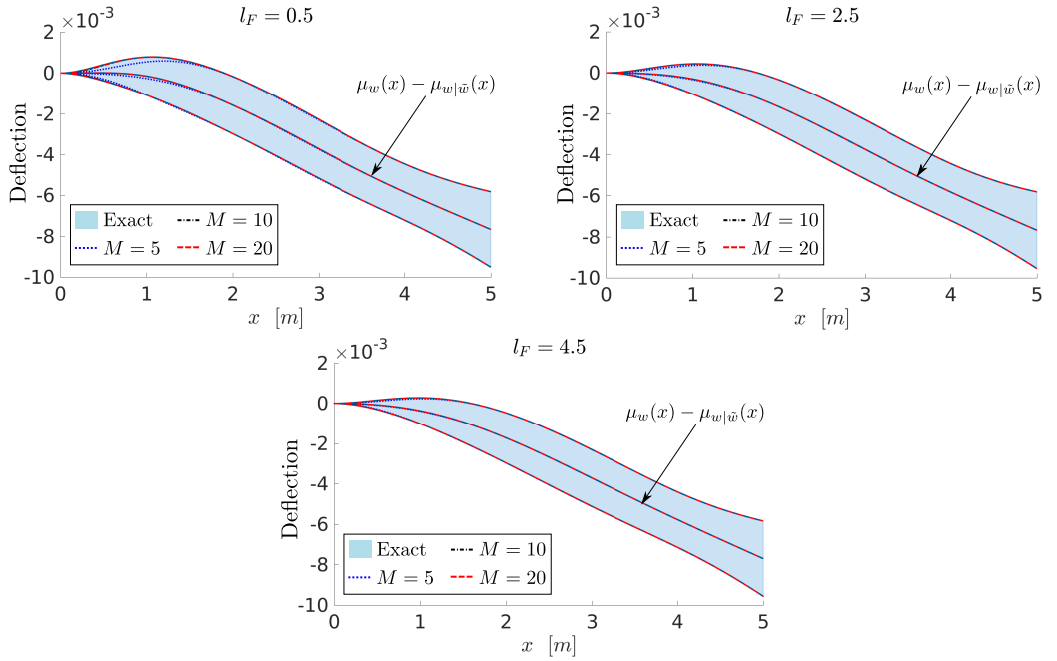

Figure 5: Posterior deflection (relative to the prior mean): 95% CI for different correlation lengths and terms in the K-L expansion. The shaded area corresponds to the exact CI.

The comparison between the averaged relative mean error $\bar{\varepsilon}_\mu(M)$ in the posterior random fields is shown in Figure 6. For the prior random field, this error measure becomes zero since the exact and approximated mean function values coincide. However, for the posterior random field, the truncation in the K-L expansion induces an error in the mean approximation. This error is controlled primarily by the number of terms in the expansion and its magnitude is considerable small as this number increases. Despite the fact that $\bar{\varepsilon}_\mu(M)$ is small in this particular application,

it can be seen that it is at least one order of magnitude higher for the posterior flexibility than for the posterior deflection.

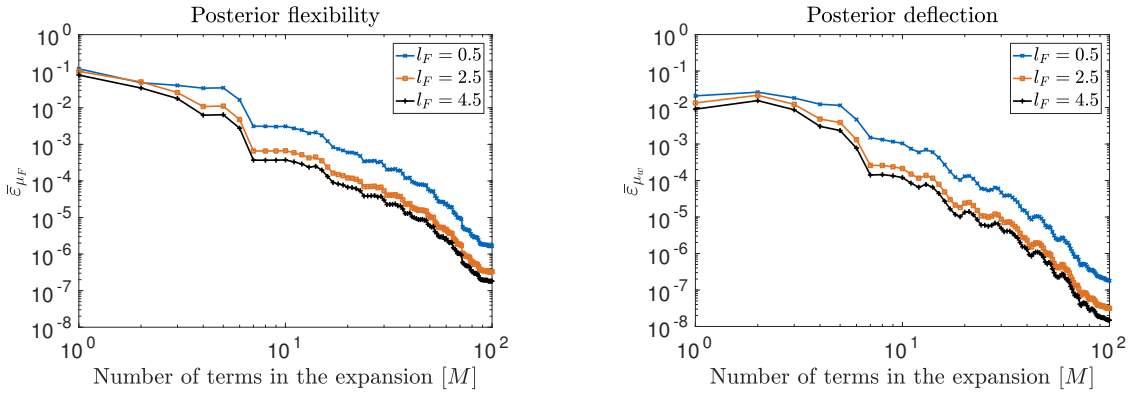

Figure 6: Averaged relative mean error in the posterior flexibility and deflection random fields for different correlation lengths and terms in the K-L expansion.

Next, the comparison between the averaged relative variance error $\bar{\varepsilon}_\sigma(M)$ in the prior and posterior random fields are shown in Figure 7, which illustrates the prior (first column) and the posterior (second column) approximations. It is interesting to see that the influence of the correlation length in the posterior random field is not as large as for the prior random field. The variance error decay of the flexibility (proportional to $1/M$) is slower than in the deflection. Hence, the uncertainty about the posterior flexibility random field is more difficult to reduce as more terms in the K-L expansion are required to obtain the same variance error levels as for the posterior deflection. This characteristic behavior is again a consequence of the averaging, since the decay in the deflection is proportional to some power of $1/M$ (this can be verified analytically and is left as a further study).

6 Conclusions

The present paper investigates the influence of the random field discretization on the outcome of the Bayesian inference based on an example with analytical solution. In particular, the prior random field has been approximated using the truncated K-L expansion in order to reduce the dimensionality and to efficiently parametrize the random field. The advantage of the proposed example lies in the possibility to derive the exact and approximated posterior random fields analytically. This allows us to efficiently carry out an error analysis. We have shown through this study that the K-L discretization has different influence on the prior and posterior random fields of the inverse and forward problems. In the *forward problem* (deflection), the approximation errors of the posterior mean and variance are small. Due to the averaging process that is carried out during the computation of the deflection, the higher K-L modes are essentially canceled out, and therefore, only the first modes of the expansion contribute to the representation. This behavior is different when the quantity of interest is highly dependent on the local fluctuations, e.g. strains and stresses (this is left as a further study). In the *inverse problem* (flexibility), the variance error decrease is slow compared to the deflection, and thus a larger number of terms in the K-L expansion are required for its accurate representation. Furthermore, in spite of the fact that the error levels in both quantities are not considerably drastic in this example, the results indicate that it might be desirable in complex engineering applications the use of a finer K-L discretization when dealing with inverse problems.

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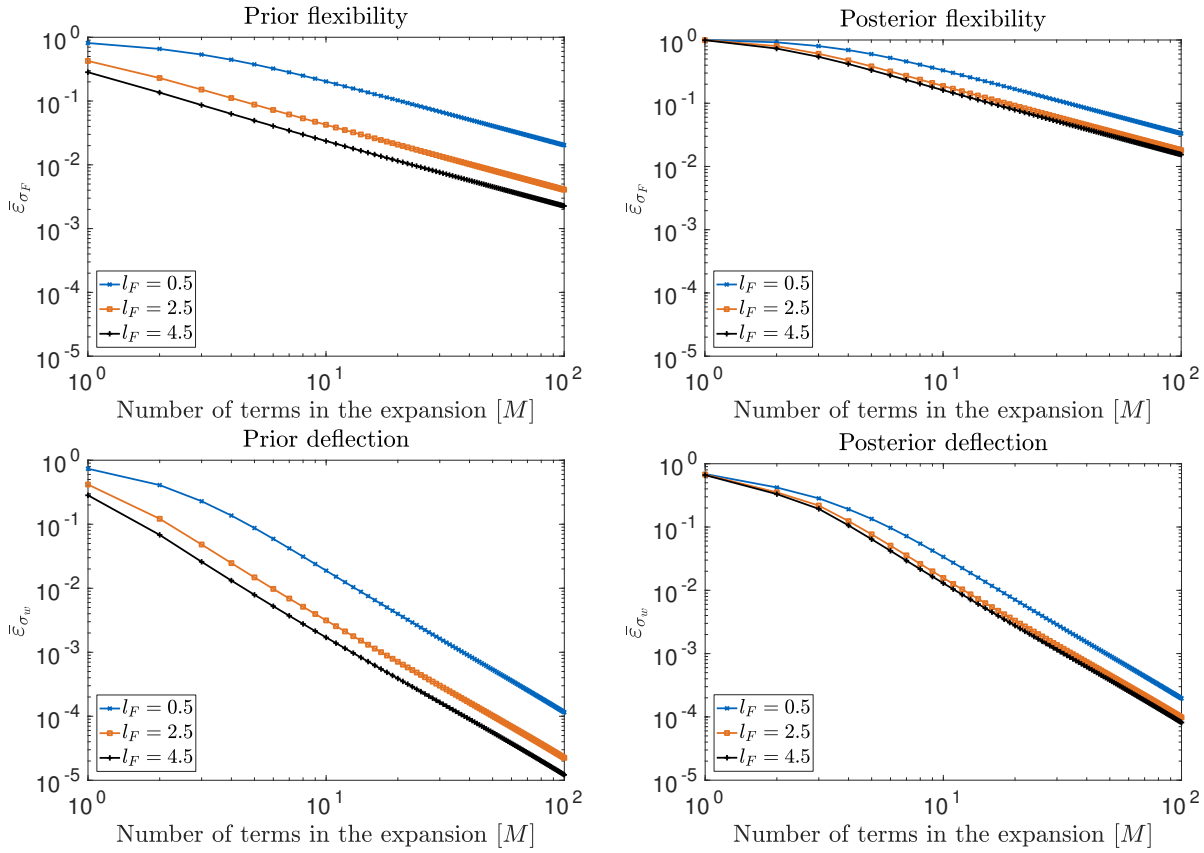

Figure 7: Averaged normalized variance error in the prior and posterior flexibility and deflection random fields for different correlation lengths and terms in the K-L expansion.

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