An Optimal LQG Controller for Stochastic Event-triggered Scheduling over a Lossy Communication Network ⋆

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Abstract: We consider a networked control loop in which the sensors acquire partial state information and communicate to a remote controller through a lossy communication network. A scheduler, collocated with the sensors, decides to transmit a locally estimated state to the controller based on an event-triggered transmission policy with stochastic thresholds. Assuming that the local estimator either senses the communication channel or receives an ideal acknowledgment from the remote estimator, then the optimal control law can be shown to be a linear function of the conditional expectation of the state. However, the probability distribution of the state conditioned on the information available to the controller based on the mentioned transmission policy and network is not Gaussian, but rather described by a sum of Gaussians with an increasing number of terms at every time-step. We show that the optimal LQG control law can be determined without tracking this probability distribution for finding its expected value. Moreover, we establish that the stochastic event-triggered scheduler can be appropriately regulated in order to achieve a desired triggering probability at every time-step.

Keywords: Optimal LQG controller, Stochastic event-triggered scheduler, Lossy communication network, Triggering rate

1. INTRODUCTION

Event triggered control (ETC) was introduced to decrease the communication burden of feedback control loops and the energy consumption of remote wireless sensors in networked control systems (NCS) (Tabuada (2007); Molin (2014); Han et al. (2013); Dolk et al. (2014); Mamduhi et al. (2017); Khashooei et al. (2017); Balaghi and Antunes (2017); Demirel et al. (2017); Mamduhi and Hirche (2018)). In recent years, much attention has been devoted to the design of ETC policies not only guaranteing stability, but also preserving a desired control performance for the system (Antunes and Khashooei (2016)).

The ETC design problem can be approached from an optimal control perspective. A common optimal control formulation of ETC is to jointly design a scheduler, collocated with the sensors, and a controller, collocated with the actuators, in order to minimize a cost function, typically an LQG-type cost. One of the challenges arising is whether an optimal design of the controller and the event-based scheduler is tractable. Several works in the literature considered this problem from different approaches (e.g. Molin (2014); Goldenshluger and Mirkin (2017)). In these works, it is proved that under some assumptions the optimal controller and event-based scheduler are separable where the optimal controller for linear systems is the classical LQG and the optimal event generator is the solution of an optimal stopping time problem.

However, this statement is valid only when there is an ideal communication network between the event-based scheduler and the controller. Within a realistic scenario, however, when several control loops share a common communication network, then other phenomena such as data collision and data loss can affect the optimality of the designed scheduler and controller. For example, when the elements of the control loop communicate through a shared slotted-ALOHA communication network, then there is always a probability of data collision which results in data loss. Data loss may also occur due to bit detection errors in the physical layer, or buffer overflow in the network layer. Another example is in the context of cyber-security control in which a jamming signal may try to interrupt the communication randomly. Therefore, it is important to investigate whether an optimal design is

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possible when data transmission is carried out through a lossy communication network.

In this work, we approach this problem by *fixing a stochastic policy* for the scheduler and searching for an optimal controller. This stochastic policy is similar to the one proposed in Brunner et al. (2018) for the case of ideal networks and specifies that transmissions occur when the error between state measurements and the estimated state exceeds a stochastic threshold. One of the advantages of this approach in the context of ideal networks is that the probability distribution of the state conditioned on the information available at the controller remains Gaussian. Note that although the optimal scheduling policy for linear systems should be deterministic just like the *Lebesgue sampling* proposed for the first-order stochastic systems (Bernhardsson and Åström (1999)), in some of the situations, it makes more sense to consider stochastic policies. In fact, for example in the slotted-ALOHA communication protocol data transmission is often governed by a stochastic policy. Moreover, in the context of cyber-security the optimal transmission policy can be designed by using game theory which typically results in stochastic (or mixed) policies.

We consider a linear time-invariant system with partial state information (output feedback) and the mentioned stochastic threshold policy for the scheduler transmitting data over a lossy network to a remote controller, as depicted in Figure 1. We show that, in the absence of an ideal communication, the probability distribution of the state conditioned on the information available to the controller can be described by a sum of Gaussians, rather than a single Gaussian, that is observed in the case when the network is ideal. The number of Gaussians needed to represent this probability distribution in the controller doubles every time-step after the last successful transmission. Still, we establish that the optimal policy for the controller does not need to keep track of these Gaussians and is actually a linear function of the state estimate running at the controller side. In other words, one can compute the state estimate given the information available at the controller and the control input without keeping track of the Gaussians. Moreover, we show that it is possible to regulate the triggering rate of the proposed event-based scheduler at every time-step when the previous transmissions have not been received by the controller.

The remainder of the paper is organized as follows. In Section 2 we provide the problem setting. In Section 3 the optimal linear controller is proposed. We establish how to regulate the triggering rate of the scheduler in Section 4. In Section 5 we validate our results by providing numerical simulation, and we provide some concluding remarks in Section 6.

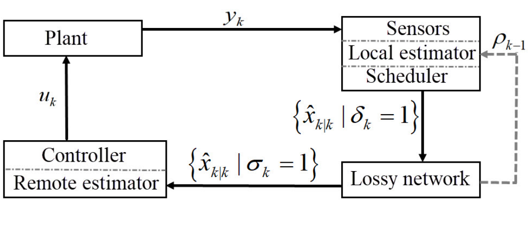

Fig. 1. A schematic view of the ETC loop with lossy communication network

Notation: $f(y|\mathcal{I}) = \mathcal{N}(y;\bar{y},Y) = \det(2\pi Y)^{-\frac{1}{2}} \exp\big((y-\bar{y})^\intercal Y^{-1}(y-\bar{y})\big)$ indicates that conditioned on the information set $\mathcal{I}$, y is a Gaussian random variable with mean $\bar{y}$ and covariance Y. $\Pr(.)$ denotes the probability of an event. Let $\varrho(A)$ denote the spectral radius of the square matrix A and we set $T_k = \{0, 1, \dots, k\}$.

2. PROBLEM SETTING

Consider an LTI system with output feedback modeled by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_k &= Cx_k + v_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^m$ are, respectively, the state, the control input and the output vectors at time-step $k \in \mathbb{N} \cup \{0\}$. Moreover, let $(w_0, w_1, \dots)$ and $(v_0, v_1, \dots)$ be sequences of i.i.d. Gaussian random variables with zero means and covariances $W = \mathbb{E}[w_k w_k^\intercal]$ and $V = \mathbb{E}[v_k v_k^\intercal]$, $\forall k \in \mathbb{N} \cup \{0\}$. A scheduler, collocated with the sensors, decides when to transmit the estimated state to a remote controller, collocated with the actuators, over a lossy communication network. The pairs (A, B) and (A, C) are assumed to be controllable and observable, respectively.

The communication network is assumed to be lossy due to collisions, drop-outs or jammed signals. For each time-step k we define a random variable $\rho_k \in \{0, 1\}$ where $\rho_k = 1$ indicates successful communication between the scheduler and the controller of the corresponding control loop, and $\rho_k = 0$ otherwise. Moreover, we assume that ρ_k for $k \in \mathbb{N} \cup \{0\}$ are i.i.d. random variables.

We define a random variable $\delta_k \in \{0, 1\}$ for the scheduler which has access to the output y_k at every time-step k. It decides whether to transmit the estimated state to the controller, in which case $\delta_k = 1$, or $\delta_k = 0$ otherwise. The binary random variable $\sigma_k = \rho_k \delta_k$ denotes whether a transmission has been successful at every time-step k. Let

$$\mathcal{I}_k = \{\delta_\ell, \rho_\ell, y_\ell | \ell \in T_{k-1}\} \cup \{y_k\} \tag{2}$$

and

$$\mathcal{J}_k = \{\sigma_\ell | \ell \in T_k\} \cup \{\hat{x}_{\ell|\ell} | \sigma_\ell = 1, \ell \in T_k\} \tag{3}$$

denote the available information sets in the local and remote estimators, respectively, and $\hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{I}_k]$, $\bar{x}_{k|k} = \mathbb{E}[x_k|\mathcal{J}_k]$ are the estimated states in these units. We assume that the local estimator either senses the communication channel or receives an acknowledgment from the remote estimator via an ideal dedicated channel, therefore, it has access to ρ_{k-1} at every time-step k. The control loop performance is evaluated by the following average LQG cost

$$J = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}[\sum_{k=0}^{T-1} x_k^\intercal Q x_k + u_k^\intercal R u_k] \tag{4}$$

in which Q is a positive semi-definite matrix and R is a positive definite matrix with appropriate dimensions.

The scheduler of the control loop is fixed to operate based on the stochastic threshold policy as

$$\delta_k = \begin{cases} 1, & \text{if } \frac{1}{2} e_{k|k-1}^\intercal \Psi_{k|k-1}^{-1} e_{k|k-1} \geq r_k \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

in which $r_k \sim \exp(\lambda_k)$ is an exponentially distributed random threshold, $e_{k|k-1} = \hat{x}_{k|k} - \bar{x}_{k|k-1}$ is the state estimation error between the local and remote estimator where $\bar{x}_{k|k-1} = \mathbb{E}[x_k|\mathcal{J}_{k-1}]$ and $\Psi_{k|k-1} = \mathbb{E}[e_{k|k-1}e_{k|k-1}^\intercal|\mathcal{I}_k]$. We will show in Section 4 how to compute $\Psi_{k|k-1}$ for $k \in \mathbb{N}$ in real time. Moreover, the optimal local state estimator at the scheduler side is the well-known Kalman filter as

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k, \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L(y_k - C\hat{x}_{k|k-1}) \end{aligned} \tag{6}$$

where

$$\begin{aligned} L &= \Theta C^\intercal (C\Theta C^\intercal + V)^{-1}, \\ \Theta &= A\Theta A^\intercal + W - A\Theta C^\intercal (C\Theta C^\intercal + V)^{-1} C\Theta A^\intercal. \end{aligned} \tag{7}$$

For simplicity we assume $\mathbb{E}[(x_0 - \hat{x}_{0|0})(x_0 - \hat{x}_{0|0})^\intercal] = \Theta$ which implies that $\mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^\intercal] = \Theta$, $\forall k \in \mathbb{N}$.

2.1 Problem statement

Consider a control loop using a lossy communication network for the data transmission between the scheduler and the controller with the fixed scheduling policy (5). Firstly, the goal is to find the optimal controller for this transmission configuration for a given sequence of λ_k, $k \in \mathbb{N} \cup \{0\}$. Secondly, it is desirable to determine the λ_k of the scheduling policy (5) that leads to a desired data triggering probability ($\Pr[\delta_k = 1]$) at every time-step k.

3. OPTIMAL CONTROLLER

In this section, we propose the optimal controller for the fixed scheduling policy (5) when the communication between the scheduler and the controller is carried out through a lossy communication network.

Theorem 1. Consider the LTI system (1) and an ETC setting in which the communication between the sensor and the controller is governed based on the stochastic threshold policy (5) through a lossy communication network. Assume that the sequence $\{\Psi_{k|k-1}|k \in \mathbb{N}\}$ is uniformly bounded (U.B.). The optimal control input is then

$$u_k = K\bar{x}_{k|k} \tag{8}$$

where

$$\begin{aligned} K &= -(B^\intercal PB + R)^{-1}B^\intercal PA, \\ P &= A^\intercal PA + Q - K^\intercal (B^\intercal PB + R)K, \end{aligned}$$

in which Q and R are given in (4) and

$$\begin{aligned} \bar{x}_{k+1|k} &= A\bar{x}_{k|k} + Bu_k, \\ \bar{x}_{k|k} &= \begin{cases} \hat{x}_{k|k}, & \text{if } \sigma_k = 1 \\ \bar{x}_{k|k-1}, & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

where $\hat{x}_{k|k}$ is given in (6). □

The proof is given in the Appendix. We will later give a condition under which $\{\Psi_{k|k-1}|k \in \mathbb{N}\}$ is guaranteed to be U.B. for constant triggering rate at all time-steps (see Assumption 1) when the communication channel is ideal.

As discussed in the proof, the pdf of the state conditioned on $\mathcal{J}_k$ can be described by a sum of Gaussians, with a growing number of terms at every time-step. On the other hand, due to the structure of the scheduler which prevents the generation of the control input's dual effect in the control loop (Molin (2014)) it is possible to show that the optimal control policy is $K\mathbb{E}[x_k|\mathcal{J}_k]$. Having this said, Theorem 1 facilitates finding the optimal controller u_k without tracking the conditional pdf in order to find $\bar{x}_{k|k} = \mathbb{E}[x_k|\mathcal{J}_k]$.

4. REGULATION OF THE TRIGGERING RATE

In this section, we show that contrary to the conditioned state pdf in the remote controller, the number of Gaussian terms of the state estimation error pdf in the scheduler $f(e_{k|k-1}|\mathcal{I}_k)$ doubles only at time instances that a data loss occurs. Therefore, one can keep track of this pdf for determining the triggering condition (5) at every time-step. We first consider a simpler setting in which the communication network is ideal and there is no data loss, i.e. $\rho_k = 1$ for all time-steps k. Then, we extend the result to the case when the communication network is lossy.

4.1 Ideal communication network

When the communication network is ideal and the triggered data is not prone to loss, it can be successfully delivered to the controller at every triggering time-step. In the following lemma, a recursive Lyapunov equation is given for determining the covariance of the state estimation error between the local and remote estimators when the communication network is ideal.

Lemma 1. Consider the aforementioned control loop using an ideal communication network for data transmission between the scheduler and the controller. The distribution of $e_{k|k-1}$ remains Gaussian at every time-step k, i.e. $f(e_{k|k-1}|\mathcal{I}_k) = \mathcal{N}(e_{k|k-1}; 0, \Psi_{k|k-1})$ with the covariance matrix evolving as follows:

$$\begin{aligned} \Psi_{k+1|k} &= A\Psi_{k|k}A^\intercal + \Phi, \\ \Psi_{k|k} &= \begin{cases} 0, & \text{if } \sigma_k = 1 \\ \dfrac{1}{1+\lambda_k}\Psi_{k|k-1}, & \text{otherwise.} \end{cases} \end{aligned} \tag{10}$$

where $\Phi = A\Theta A^\intercal - \Theta + W$. Moreover, the triggering probability of the scheduler at every time-step k is

$$p_k = \Pr(\delta_k = 1|\mathcal{I}_k) = 1 - (1+\lambda_k)^{-\frac{n}{2}}. \tag{11}$$

□

According to (11), if the parameter λ_k of the random threshold is set to a constant value λ at all time-steps, then the probability of transmission (11) is always constant. The following theorem provides the performance of the control loop operating based on the given event triggering policy (5) with a constant transmission rate p at all time-steps. To state the next theorem, we need the following assumption which holds when A is stable or p is sufficiently large (Sinopoli et al. (2004)).

Assumption 1. $(1-p)^{1+\frac{2}{n}}\varrho(A)^2 < 1$.

Theorem 2. Consider the aforementioned control loop using an ideal communication network for the data transmission between the scheduler and the controller. Moreover, assume that $\lambda_k = \lambda$ for all time-steps k. Suppose that Assumption 1 holds. Then the average covariance of the state estimation error between the scheduler and the controller ($\bar{\Psi} = \limsup_{t\to\infty} \mathbb{E}[\Psi_{t|t}|\mathcal{I}_t]$) satisfies

$$\bar{\Psi} = gA\bar{\Psi}A^\intercal + g\Phi \tag{12}$$

where $g := (1-p)^{1+\frac{2}{n}}$ for $p = 1-(1+\lambda)^{-\frac{n}{2}}$ is the constant data triggering rate at every time-step and $\Phi = A\Theta A^\intercal - \Theta + W$. Moreover, the control loop performance is

$$J = \mathrm{tr}(PW + (1-g)\Theta Y) + \sum_{i=0}^{\infty} g^{i+1}\mathrm{tr}\Big(A^i\big((1-g)A\Theta A^\intercal + W\big)A^{\intercal i}Y\Big) \quad (13)$$

where $Y = K^\intercal(B^\intercal PB + R)K$. □

4.2 Lossy communication network

Now we consider the condition in which the communication network is lossy. In Balaghi et al. (2018), it is proved that the stochastic threshold policy does not preserve the Gaussianity of the propagated state estimation error between the scheduler and the controller if there is a probability of data loss in the communication network.

More specifically, in between every two successive successful transmissions, the distribution of the state estimation error in the local estimator remains Gaussian until the first data loss ($\delta_k = 1 \wedge \rho_k = 0$). However, a data loss changes the state estimation error pdf to the sum of Gaussians. In the following Lemma, we determine the pdf of the updated state estimation error assuming the pdf of the predicted state estimation error follows a single Gaussian at time-step k as

$$f(e_{k|k-1}|\mathcal{I}_k) = \mathcal{N}(e_{k|k-1}; 0, \Psi_{k|k-1}). \quad (14)$$

Lemma 2. Assume the pdf of the predicted state estimation error follows (14). Then

$$p_k = 1 - (1+\lambda_k)^{-\frac{n}{2}} \quad (15)$$

is the triggering probability of the scheduler at time-step k and $q_k = 1 - p_k$. Moreover,

$$f(e_{k|k}|\delta_k = 0, \mathcal{I}_k) = \mathcal{N}(e_{k|k}; 0, \Psi_{k|k})$$

and

$$f(e_{k|k}|\delta_k = 1, \rho_k = 0, \mathcal{I}_k) = \frac{\mathcal{N}(e_{k|k}; 0, \Psi_{k|k-1}) - q_k\mathcal{N}(e_{k|k}; 0, \Psi_{k|k})}{p_k}$$

where $\Psi_{k|k} = \Psi_{k|k-1}/(1+\lambda_k)$, are the pdfs of the updated state estimation errors in case of no data triggering ($\delta_k = 0$) and data loss ($\delta_k = 1 \wedge \rho_k = 0$), respectively. □

Based on Lemma 2, the pdf of the updated state estimation error remains Gaussian up to the time-step before the first data loss in between every two successive successful transmissions. However, the first data loss changes the pdf of the updated state estimation error to the sum of two Gaussians. Moreover, the triggering probability of the scheduler is independent of the state estimation error covariance up to the first data loss time-step. In the following Theorem, we derive the propagation pattern for the pdf of the updated state estimation error after it becomes a sum of Gaussian terms (after the first data loss time-step). Assume the pdf of the updated state estimation error at time-step $k-1$ is the sum of Gaussians as follows

$$f(e_{k-1|k-1}|\mathcal{I}_k) = \sum_{l=1}^{h_k} \alpha_k^l \mathcal{N}(e_{k-1|k-1}; 0, \Psi_{k-1|k-1}^l) \quad (16)$$

in which h_k is the number of Gaussian terms at time-step k. Then the pdf of the predicted state estimation error at time-step k will be

$$f(e_{k|k-1}|\mathcal{I}_k) = \sum_{l=1}^{h_k} \alpha_k^l \mathcal{N}(e_{k|k-1}; 0, \Psi_{k|k-1}^l) \quad (17)$$

where the covariance of the Gaussian terms of the predicted state estimation error is determined as

$$\Psi_{k|k-1}^l = A^\intercal \Psi_{k-1|k-1}^l A + W, \ \forall l \in \{1, \dots, h_k\}.$$

Since all the Gaussian terms have zero means, the total covariance used in the scheduling law (5) is

$$\Psi_{k|k-1} = \sum_{l=1}^{h_k} \alpha_k^l \Psi_{k|k-1}^l. \quad (18)$$

Theorem 3. Assume that the distribution of the predicted state estimation error follows (17) and the event-based scheduling law (5) is determined based on (18), then

$$p_k = 1 - \sum_{l=1}^{h_k} q_k^l \quad (19)$$

is the triggering probability of the scheduler in which

$$q_k^l = \alpha_k^l \det\big(I + \lambda_k \Psi_{k|k-1}^{-1} \Psi_{k|k-1}^l\big)^{-\frac{1}{2}}$$

for $l \in \{1, \dots, h_k\}$. Moreover,

$$f(e_{k|k}|\delta_k = 0, \mathcal{I}_k) = \sum_{l=1}^{h_k} \frac{q_k^l}{1-p_k} \mathcal{N}(e_{k|k}; 0, \Psi_{k|k}^l) \quad (20)$$

and

$$f(e_{k|k}|\delta_k = 1, \rho_k = 0, \mathcal{I}_k) = \frac{1}{p_k}\Big(\sum_{l=1}^{h_k} \alpha_k^l \mathcal{N}(e_{k|k}; 0, \Psi_{k|k-1}^l) - q_k^l \mathcal{N}(e_{k|k}; 0, \Psi_{k|k}^l)\Big) \quad (21)$$

are the pdfs of the updated state estimation errors in case of no data triggering ($\delta_k = 0$) and data loss ($\delta_k = 1, \rho_k = 0$), respectively, where

$$\Psi_{k|k}^l = \big(I + \lambda_k (\Psi_{k|k-1})^{-1} \Psi_{k|k-1}^l\big)^{-1} \Psi_{k|k-1}^l. \quad (22)$$

□

Based on Theorem 1, in case of no attempt of data transmission ($\delta_k = 0$), the number of Gaussian terms of the updated state estimation error pdf remains the same as that for the predicted state estimation error pdf. However, every data loss in between every two successive successful transmissions doubles the number of Gaussian terms of the updated state estimation error pdf. Moreover, the probability of the data triggering becomes dependent on the Gaussian terms' covariance of the predicted state estimation error pdf. Therefore, the triggering probability cannot be set to a constant value by using a constant threshold parameter λ after the first collision time instance. It is needed to solve the nonlinear equation (19) at every time-step after the first collision instance based on the desired triggering probability p_k^d to determine an appropriate threshold parameter λ_k^d. If one needs to keep the triggering rate constant at all time-steps, the value of λ_k should become smaller after every data loss time instance. This is due to an increase in the covariance of the state estimation error and as a consequence, a decrease in the left-hand side of the triggering condition in (5). Therefore, the probability of the right-hand side of the triggering condition (or equivalently λ_k) should also be decreased in order to keep the triggering rate constant as the previous time-step. The following Lemma, illustrates the existence of the solution for this equation.

Lemma 3. For any desired triggering probability p_k^d, there exists a real positive threshold parameter ($\lambda_k^d \in \mathbb{R}_{\geq 0}$) for the scheduling policy (5) obtained by solving (19). □

5. NUMERICAL SIMULATIONS

In this section, we consider a scalar system in which $A = 0.9$, $B = 1$, $C = 1.5$, $W = 1$ and $V = 0.5$ and take $Q = 1$ and $R = 0.1$ as the parameters of the controller. We assume the network is lossy based on a Bernoulli distribution where $\mathbb{E}[\rho_k] = 0.7$. The LQG control performance is determined using Monte-Carlo simulations when the scheduler is operating based on a desired constant probability at all time-steps. We use Matlab's numerical methods to regulate the threshold parameter of the event-based scheduler by solving (19) at every time-step. In Figure 2, we show the control performance of the stochastic event-based scheduler is improved in comparison with that of its non-event-based counterpart in which the scheduler transmits randomly with the same constant probability at all time-steps. Moreover, Figure 3 shows the value of λ_k and the number of Gaussian terms of the state estimation error pdf in the local estimator (h_k) for the mentioned ETC policy with $p_k = 0.75$ for all time-steps over the lossy network. As it can be seen, at every data loss time instance in which the number of Gaussian terms of the updated state estimation error pdf in the scheduler doubles, the value of λ_k is decreased in order to keep the triggering rate constant.

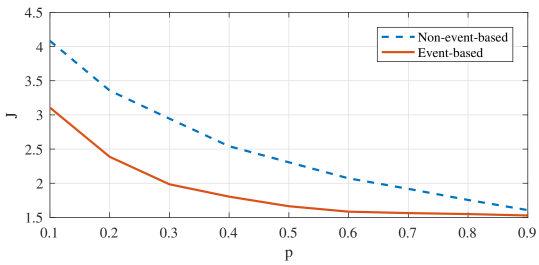

Fig. 2. Comparison of the average control performance of the stochastic threshold event-triggered scheduling and its non-event-based counterpart scheduling policies in which transmissions occur randomly with the same constant probability at all time-steps.

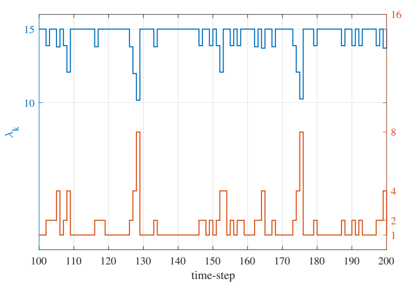

Fig. 3. The value of λ_k and the number of Gaussian terms of the state estimation error pdf in the local estimator (h_k) with $p_k = 0.75$ for all time-steps.

6. CONCLUSIONS

This work considers a networked control loop in which an event-based scheduler decides when to transmit the estimated state to the controller via a lossy communication network. The scheduler is fixed to operate based on the stochastic threshold event-triggered policy. The optimal controller for this transmission configuration is a linear function of the conditional expectation of the state which as established here, can be determined without tracking its conditional pdf in the controller. Moreover, the scheduler is adapted to trigger data with a desired probability by tuning its threshold parameter.

7. APPENDIX

7.1 Proof of Theorem 1

The finite horizon optimal LQG event-based output-feedback controller for the linear systems has the certainty equivalent property when the scheduler prevents the generation of the control input's dual effect as the one proposed in (5) (Molin (2014)). The result can be extended to the average cost problem (4) provided that the cost is bounded (Molin, 2014, Ch. 6, Sec. 2). This can be shown under the assumption that $\{\Psi_{k|k-1} | k \in \mathbb{N}\}$ is uniformly bounded and (A, B), (A, C) are controllable and observable, respectively, following the same steps as in Brunner et al. (2018). The details are omitted due to space constraints. It shall be noted that the acknowledgment of ρ_{k-1} to the local estimator at every time-step k provides a nested information structure for the controller, therefore, $u_k = K\mathbb{E}[x_k|\mathcal{J}_k]$ where K is determined by (7). In order to determine $\bar{x}_{k|k} = \mathbb{E}[x_k|\mathcal{J}_k]$, we follow an induction proof. Without loss of generality assume that at $t = 0$, $\sigma_0 = 1$, then the predicted pdf of the state in the remote estimator at the next time-step is Gaussian, i.e.

$$f(x_1|\mathcal{J}_0, \bar{x}_{1|0}) = \mathcal{N}(x_1; \bar{x}_{1|0}, \Gamma_{1|0}) \tag{23}$$

where $\mathcal{J}_0 = \{\sigma_0 = 1, \hat{x}_{0|0}\}$, $\bar{x}_{1|0} = A\hat{x}_{0|0} + Bu_0$ and $\Gamma_{1|0} = A\Theta A^\intercal + W$. Moreover, the updated pdf is

$$f(x_1|\mathcal{J}_1) = f(x_1|\mathcal{J}_0, \bar{x}_{1|0}, \sigma_1, \{\hat{x}_{1|1}|\sigma_1 = 1\}).$$

If $\sigma_1 = 1$, then $f(x_1|\mathcal{J}_0, \sigma_1 = 1, \hat{x}_{1|1}, \bar{x}_{1|0}) = \mathcal{N}(x_1; \hat{x}_{1|1}, \Theta)$ in which $\hat{x}_{1|1}$ is the updated state estimation determined by the Kalman-filter running in the local estimator at $t = 1$. However, if $\sigma_1 = 0$, then by using the Bayes law of conditional probability

$$\begin{aligned} &f(x_1|\mathcal{J}_0, \bar{x}_{1|0}, \sigma_1 = 0) \\ &\quad = \frac{\Pr(\sigma_1 = 0|\mathcal{J}_0, \bar{x}_{1|0}, x_1)}{\Pr(\sigma_1 = 0|\mathcal{J}_0, \bar{x}_{1|0})} f(x_1|\mathcal{J}_0, \bar{x}_{1|0}). \end{aligned} \tag{24}$$

Moreover,

$$\begin{aligned} \Pr(\sigma_1 = 0|\mathcal{J}_0, \bar{x}_{1|0}, x_1) &= \Pr(\delta_1 = 0|\mathcal{J}_0, \bar{x}_{1|0}, x_1) \\ &\quad + \Pr(\rho_1 = 0)\Pr(\delta_1 = 1|\mathcal{J}_0, \bar{x}_{1|0}, x_1). \end{aligned} \tag{25}$$

Now consider $\hat{z}_1 = x_1 - \hat{x}_{1|1}$, $\bar{z}_1 = x_1 - \bar{x}_{1|0}$, $z_1 = \hat{x}_{1|1} - \bar{x}_{1|0}$ where $f(\hat{z}_1) = \mathcal{N}(\hat{z}_1; 0, \Theta)$ and $\mathbb{E}[z_1 z_1^\intercal|\sigma_0 = 1] = \Phi_{1|0}$ in which $\Phi_{1|0} = A^\intercal \Theta A - \Theta + W$. Then

$$\begin{aligned} \Pr(\delta_1 = 0|\mathcal{J}_0, x_1, \bar{x}_{1|0}) &= \Pr(\delta_1 = 0|\mathcal{J}_0, \bar{z}_1) \\ &= \int\limits_{\hat{z}_1 \in \mathbb{R}^n} \int\limits_{r_0,}^{\infty} \frac{(\lambda_1 e^{-\lambda_1 r}) e^{-\frac{1}{2}\hat{z}_1^\intercal \Theta^{-1} \hat{z}_1}}{\det(2\pi\Theta)^{\frac{1}{2}}} dr d\hat{z}_1 \end{aligned}$$

where $r_0 = \frac{1}{2} z_1^\intercal \Phi_{1|0}^{-1} z_1 = \frac{1}{2}(\bar{z}_1 - \hat{z}_1)^\intercal \Phi_{1|0}^{-1}(\bar{z}_1 - \hat{z}_1)$, then

$$\Pr(\delta_1 = 0 | \mathcal{J}_0, x_1, \bar{x}_{1|0}) = \xi_1 e^{-\frac{1}{2}\bar{z}_1^\intercal \Pi_{1|0}^{-1} \bar{z}_1} \tag{26}$$

where

$$\Pi_{1|0} = \left(\lambda_1 \Phi_{1|0}^{-1} - \lambda_1 \Phi_{1|0}^{-1} (\lambda_1 \Phi_{1|0}^{-1} + \Theta^{-1})^{-1} \lambda_1 \Phi_{1|0}^{-1}\right)^{-1}$$

and $\xi_1 = 1/\big(\det(\lambda_1 \Phi_{1|0}^{-1} + \Theta^{-1}) \det(\Theta)\big)$. Moreover,

$$\Pr(\delta_1 = 1 | \mathcal{J}_0, x_1, \bar{x}_{1|0}) = 1 - \xi_1 e^{-\frac{1}{2}\bar{z}_1^\intercal \Pi_{1|0}^{-1} \bar{z}_1}. \tag{27}$$

First substitute (26) and (27) into (25) which results in

$$\Pr(\sigma_1 = 0 | \mathcal{J}_0, \bar{x}_{1|0}, x_1) = 1 - q + q\xi_1 e^{-\frac{1}{2}\bar{y}_1^\intercal \Pi_{1|0}^{-1} \bar{y}_1}$$

where $q = \Pr(\rho_1 = 1)$, then substitute the result into (24)

$$\begin{aligned} f(x_1 | \mathcal{J}_0, \bar{x}_{1|0}, \sigma_1 = 0) &= \frac{1 - q + q\xi_1 e^{-\frac{1}{2}\bar{z}_1^\intercal \Pi_{1|0}^{-1} \bar{z}_1}}{1-p} \frac{e^{-\frac{1}{2}\bar{z}_1^\intercal \Gamma_{1|0}^{-1} \bar{z}_1}}{\det(2\pi \Gamma_{1|0}^{-1})} \\ &= \sum_{i=1}^{2} \beta_i' \mathcal{N}(x_k; \bar{x}_{1|0}, \Gamma_{1|1}^i) \end{aligned} \tag{28}$$

where $\sum_{i=1}^{2} \beta_i' = 1$. As it is seen, at the first time-step after the successful transmission the updated state estimation pdf is the sum of two Gaussian terms with different covariances. However, their means are equal to the one obtained at the prediction stage. Therefore,

$$\bar{x}_{1|1} = \mathbb{E}[x_1 | \hat{x}_{0|0}, \sigma_0 = 1, \sigma_1 = 0] = \bar{x}_{1|0}.$$

Now let us assume that at $t = k$ time-step after the last successful transmission the pdf of the updated state estimation is the sum of h_k' Gaussian terms as follows

$$f(x_k | \mathcal{J}_{k-1}, \bar{x}_{k|k-1}, \sigma_{k-1} = 0) = \sum_{i=1}^{h_k'} \beta_i \mathcal{N}(x_k; \bar{x}_{k|k-1}, \Gamma_{k|k}^i) \tag{29}$$

where their means are equal and their covariances are different. Then the predicted state estimation pdf at $t = k+1$ will be

$$f(x_{k+1} | \mathcal{J}_k, \bar{x}_{k+1|k}) = \sum_{i=1}^{h_k'} \beta_i \mathcal{N}(x_{k+1}; \bar{x}_{k+1|k}, \Gamma_{k+1|k}^i) \tag{30}$$

where

$$\begin{aligned} &\bar{x}_{k+1|k} = A\bar{x}_{k|k-1} + Bu_k, \\ &\Gamma_{k+1|k}^i = A\Gamma_{k|k}^i A^\intercal + W, \quad \forall i \in \{1, \dots, h_k'\}. \end{aligned} \tag{31}$$

It is clear if $\sigma_{k+1} = 1$, then the updated state estimate at $t = k+1$ is equal to $\hat{x}_{k+1|k+1}$ whose distribution is $\mathcal{N}(x_{k+1}; \hat{x}_{k+1|k+1}, \Theta)$. However, if $\sigma_{k+1} = 0$, then by using the same method as the one used for $t = 1$ in (24)-(28) we can prove that

$$\begin{aligned} &f(x_{k+1} | \mathcal{J}_k, \bar{x}_{k+1|k}, \sigma_{k+1} = 0) = \\ &\qquad \sum_{i=1}^{2h_k'} \beta_i' \mathcal{N}(x_{k+1}; \bar{x}_{k+1|k}, \Gamma_{k+1|k+1}^i) \end{aligned} \tag{32}$$

As it is seen, the number of Gaussian terms of the updated state estimation pdf is twice that of the updated state estimation at the previous time-step. However, the mean of all Gaussian terms is equal to the mean of the Gaussian terms of the predicted state estimation i.e. $\bar{x}_{k+1|k+1} = \bar{x}_{k+1|k}$ when $\sigma_{k+1} = 0$. Therefore, the assumption of induction is correct and we can conclude that $\bar{x}_{k|k}$ in (9) is the state conditional expectation given the information set available for the controller.

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