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Non-standard Analysis in Projective Geometry

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Contents

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> "Alle guten Dinge haben etwas Lässiges und liegen wie Kühe auf der Wiese."
>
> Friedrich Nietzsche, [100]

First of all I want to thank J. Richter-Gebert for the freedom of choice in the subject and the means of my thesis and for being a lighthouse in the sea of science. And also I want to thank Jacques Fleuriot for examining this thesis and the numerous helpful remarks.

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1. Geometry and Infinity

> "Wahrlich es ist nicht das Wissen, sondern das Lernen, nicht das Besitzen, sondern das Erwerben, nicht das Da-Seyn, sondern das Hinkommen, was den grössten Genuss gewährt. Wenn ich eine Sache ganz ins Klare gebracht und erschöpft habe, so wende ich mich davon weg, um wieder ins Dunkle zu gehen; so sonderbar ist der nimmersatte Mensch, hat er ein Gebäude vollendet so ist es nicht um nun ruhig darin zu wohnen, sondern um ein anderes anzufangen."
>
> Carl Friedrich Gauß, [49] p. 94

In this thesis we will integrate concepts of non-standard analysis into projective geometry. One major application of the developed theory will be the automatic removal of singularities in geometric constructions.

Non-standard analysis allows the precise handling of infinitely small (infinitesimal) and infinitely large (unlimited) numbers which admits statements like: "there is an $\epsilon > 0$ such that $\epsilon < r \ \forall r \in \mathbb{R}^+$" or "there is an H such that $H > r \ \forall r \in \mathbb{R}^+$". This obviously violates the Archimedean axiom. Therefore such fields that contain infinitesimal and unlimited numbers are also called "non-Archimedean". To embed this theory into the "standard" theory, one may use field extensions of $\mathbb{R}$ and $\mathbb{C}$. We will call them $\mathbb{R}^*$ (hyperreal numbers) and $\mathbb{C}^*$ (hypercomplex numbers), and one may seamlessly integrate these numbers into well-established (we will refer to it by the adjective "standard") analysis in the sense of Weierstrass. Although the construction of these fields is highly non-trivial, its mere application is very intuitive.

One strength of projective geometry is the natural and comprehensive integration of infinity. For example: consider the intersection of two (disjoint) lines. Euclidean geometry has to distinguish two cases: the lines intersect or they do not intersect (if they are parallel). In projective geometry there is *always* a point of intersection, it just may be infinitely far way.

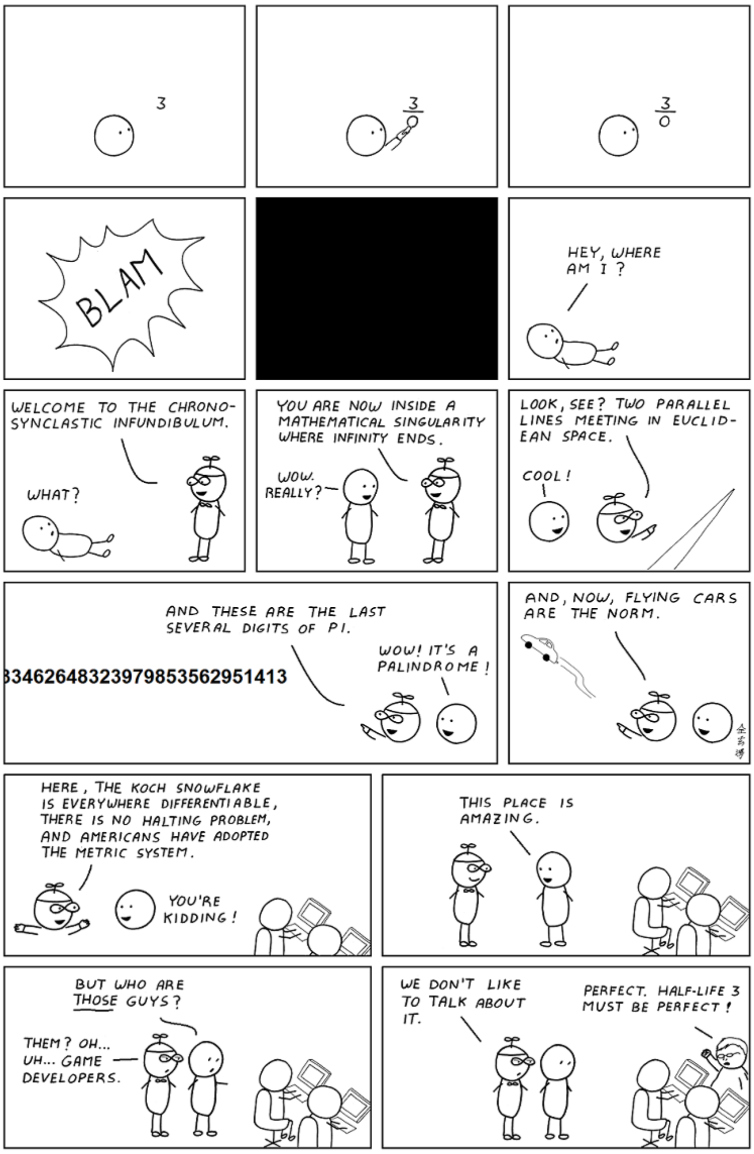
In practice, projective geometry is usually carried out over $\mathbb{R}$ or $\mathbb{C}$, but its not restricted to these canonical fields. One may consider arbitrary fields for geometric operations and this will enable us to combine the mathematical branches of projective geometry and non-standard analysis.

1. Geometry and Infinity

Using infinitesimal elements in geometry is not a new concept, au contraire it is one of the oldest concepts. Already Leibniz and Newton used them for geometric reasoning.

Dynamic (projective) geometry allows us to describe the movement of geometric constructions and implement them on a computer. In dynamic constructions, singularities naturally arise. We will look into methods for resolving singularities and situations in which standard theory does not offer concise and consistent solutions. This will naturally lead to the integration of non-standard analysis into projective geometry. Non-standard analysis in projective geometry is quite an exciting topic on its own and we will explore basic properties that are sometimes surprising. Using this extension of projective geometry, we will be able to develop the theory of proper desingularization. After expanding this theory we will also develop algorithms and a real time capable implementation to integrate the theory into a dynamic geometry system.

2. Site Map


3
3
0
3
0
BLAM
HEY, WHERE AM I ?
WELCOME TO THE CHRONO-SYNCLASTIC INFUNDIBULUM.
WHAT?
YOU ARE NOW INSIDE A MATHEMATICAL SINGULARITY WHERE INFINITY ENDS.
WOW. REALLY?
LOOK, SEE? TWO PARALLEL LINES MEETING IN EUCLID-EAN SPACE.
COOL!
AND THESE ARE THE LAST SEVERAL DIGITS OF PI.
33462648323979853562951413
WOW! IT'S A PALINDROME!
AND, NOW, FLYING CARS ARE THE NORM.
HERE, THE KOCH SNOWFLAKE IS EVERYWHERE DIFFERENTIABLE, THERE IS NO HALTING PROBLEM, AND AMERICANS HAVE ADOPTED THE METRIC SYSTEM.
YOU'RE KIDDING!
THIS PLACE IS AMAZING.
BUT WHO ARE THOSE GUYS?
THEM? OH... UH... GAME DEVELOPERS.
WE DON'T LIKE TO TALK ABOUT IT.
PERFECT. HALF-LIFE 3 MUST BE PERFECT!

If you are looking for a more streamlined version of this thesis, we already published two preprints that cover most of the novelties: [128, 129].

This thesis is essentially divided into five parts: Singularities of Geometric Constructions, an introduction to Non-standard Analysis, Non-standard Projective Geometry, Removal of Singularities & Numerics and Open Problems & Future Work.

We will assume that the reader is familiar with projective geometry, if this is not the case we refer to the books of J. Richter-Gebert: "Geometriekalküle" [114] (in German, with T. Orendt) or "Perspectives on Projective Geometry" [111]. Also, we assume the reader to be familiar with dynamic geometry, the theory of computer aided deformation of geometric constructions, that is formally treated in U. Kortenkamp's dissertation "Foundations of Dynamic Geometry" [77] or the paper "Grundlagen Dynamischer Geometrie" [76] by U. Kortenkamp and J. Richter-Gebert.

In the chapter "Singularities of Geometric Constructions", we will slightly extend the model of [76] to a projective setting and describe the basic problem of singularities in geometric constructions. Then we will discuss classical methods to desingularize such situations in the differentiable (more precisely analytical) case. Unfortunately, we cannot assume that were are always dealing with analytic functions since geometric constructions involving for example circle intersections have to admit radical expressions. The complex n-root cannot be analytically extended at 0 since 0 is a branch point of the function. Finally, we will discuss several classical methods to resolve singularities involving radical expression and their limitations.

In chapter "Non-standard Analysis", we will introduce the formal treatment of a field extension of the real and complex numbers that admits infinitesimal and unlimited numbers, the so called "hyperreal" and "hypercomplex" numbers. We will discuss the construction of these fields and that the "transfer principle", which bridges the real and complex to hyperreal and hypercomplex world, can be applied to prove theorems. After discussing the algebra, we will see how easily differentiation can be handled in these fields and then define some series expansions that will come in handy in later chapters.

Projective geometry is not restricted to the real or complex numbers, it can be defined over every field. Therefore, in the chapter "Non-standard Projective Geometry" we will consider projective geometry over the hyperreal and hypercomplex numbers and introduce adapted versions of basic operations like join and meet to handle the characteristics of this non-standard version of projective geometry. Furthermore, we will discuss basic concepts of projective geometry like incidences, projective transformations, cross-ratios, conics and their non-standard specifics.

Afterwards, we will return to singularities of geometric constructions in the chapter

"Removal of Singularities & Numerics". We will discuss how non-standard analysis might be used to resolve such singularities and the stability of such solutions. We will show how this is related to perturbation theory, which is a well known concept of computational geometry. Since we are interested in dynamic geometry, we also want to implement the theoretical considerations on a computer. It turns out that a full-blown implementation of the hyperreal and hypercomplex numbers is very expensive. But we are lucky that already a much smaller subfield, the so-called "Levi-Civita" field is sufficient for a practical implementation. We will analyze the theory of Levi-Civita numbers and how an implementation can be achieved. Then we will discuss several practical examples like a degenerate von-Staudt construction. Furthermore, we will debate the limitations of the Levi-Civita field and give a construction that reaches these limitations. Finally, we will show how one can avoid singularities in the first place but also examine a construction that shows that such an avoidance is not always possible.

In the end, we will give some hints how the next steps in the field may look like in "Open Problems & Future Work".

3. Singularities of Geometric Constructions

To Infinity and Beyond!

Buzz Lightyear

Building up geometric constructions using primitive operations, like the connecting line of two points or the intersection of two circles, relies on certain non-degeneracy assumptions like the two points or circles being disjoint. Even if primitive operations are implemented in a way that does not introduce artificial degenerate situations, combination of those can produce such situations. Consider a geometric construction that should create a certain element depending on some free elements. As an example consider the calculation of an orthogonal bisector of two points A and B. Very often, a user will implement such a construction by applying a sequence of geometric primitive operations. One possible (perhaps not the most clever) way to do this is by drawing two circles of radius 1 around the points, intersecting them and then joining the two intersections. How should such a construction behave when the user moves the original points? See Figure 3.1 for a visualization of the situation.

It is clear that the two real points of intersection and their connecting line should be shown when A and B are at a distance less then two. At a distance greater than two one could argue (and this is a modeling step) what the desired behavior should be. We propose the following behavior: the intersection points have become complex. Joining them creates a line with complex homogeneous coordinates. However a common factor can be extracted from this complex coordinate vector and the line can be interpreted as a real line: The orthogonal bisector of A and B. But what should happen if A and B are exactly in a situation where their distance equals 2? In this case the two circle intersections coincide and we do not have a well defined connecting line. However, in a sense the situation behaves like a removable singularity. We can consider the situation as a limit process: in an epsilon neighborhood of the singularity the position of the line stably converges to the same situation. The question is how to automatically detect such cases and automatically *desingularize* them.

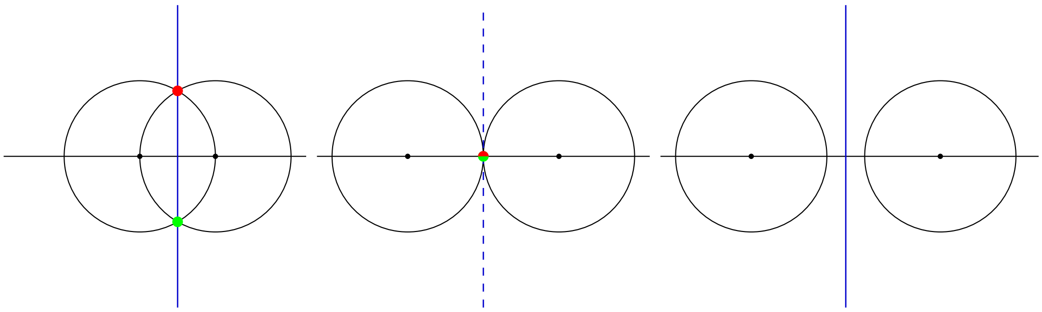


Figure 3.1.: Construction of an orthogonal bisector with removable singularity (middle).

3.1. Modelling Dynamic Geometry

> Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit.
>
> Albert Einstein, Geometrie und Erfahrung [33]

First, we have to define our model for the removal of singularities. U. Kortenkamp and J. Richter-Gebert have already laid the foundations in their article "Grundlagen Dynamischer Geometrie" [76].

In this paper, U. Kortenkamp and J. Richter-Gebert use Euclidean representations for geometric objects to obtain a classical notion of continuity. Here, we will define continuity in a topological sense and preserve the benefits of a projective space at the expense of the introduction of topological machinery.

Definition 3.1.1 (Topology Notation, [69] p. 39)
Let X be a topological space and $\sim$ an equivalence relation on X. We denote by $X_{/\sim}$ the set of all equivalence classes, by $[x] \in X_{/\sim}$ the equivalence class of $x \in X$ and by $\pi : X \to X_{/\sim}$ the canonical projection, *i.e.* $\pi(x) = [x]$. $\diamond$

Definition 3.1.2 (Quotient Topology and Quotient Space, [69] p. 39)
Let X be a topological space and $\sim$ an equivalence relation on X. A subset $U \subset X_{/\sim}$ is called *open in the quotient topology*, if $\pi^{-1}(U)$ is open in X. The space $X_{/\sim}$ equipped with the hereby declared topology is called the *quotient space* of X by $\sim$. $\diamond$

Remark. For example the projective space $\mathbb{RP}^2$ is defined as the quotient space $(\mathbb{R}^3 \setminus \{0\})_{/\sim}$ with the equivalence relation $v \sim w \Leftrightarrow \exists \lambda \in \mathbb{R} \setminus \{0\} : v = \lambda \cdot w$.

Definition 3.1.3 (Continuity, [69])
Let X and Y be topological spaces. A function $f : X \to Y$ is called *continuous*, if the

preimages of open sets are open sets. $\diamond$

Lemma 3.1.4 (Topological Continuity, [69] p. 41)
Let X, Y be two topological spaces and $X_{/\sim}$ a quotient space of X by an equivalence relation $\sim$ on X. We give two criteria for continuity:

1. A function $f : X_{/\sim} \to Y$ is continuous, if and only if $f \circ \pi : X \to Y$ is continuous.
2. Let $\phi : Y \to X_{/\sim}$ be a function. If there is a continuous function $\Phi : Y \to X$ with $\phi = \pi \circ \Phi$, then ϕ is continuous.

Remark. Especially the second point is very interesting for us: if we can find a continuous function $f : \mathbb{R} \to \mathbb{C}^3 \setminus \{0\}$, then its counterpart $[f]$ in $\mathbb{CP}^2$ is automatically continuous. We will exploit this fact very often, since this gives us a simple recipe to find continuous extension, we will call them C^0–continuations, of a function.

Remark. Let X be a topological space and $X_{/\sim}$ be the quotient space defined by an equivalence relation $\sim$ on X. By abuse of notation, we will write $x = y$ for all $[x], [y] \in X_{/\sim}$ if we mean $[x] = [y]$.

The term “singularity” is used to describe a point of a function where the function is undefined or not well-behaved. We will encounter singularities in at least two variations: for functions $f : [0,1] \to \mathbb{C}^{d+1}$ for which the preimage of 0 is not the empty set (we will motivate this after the definition) and the more classical variant from complex analysis: if a fraction of two functions attains 0 for nominator and the denominator at the same time. We will see that these notions are closely related for our purposes.

Definition 3.1.5 (Singularity)
Let $f : [0,1] \to \mathbb{C}^{d+1}$ be a continuous function and let $t_0 \in [0,1]$. We call f *singular at t_0* if $f(t_0) = 0$. And we call t_0 a *singularity.* $\diamond$

Remark. This definition is motivated by the fact that the projection of the zero vector is not an element of the complex projective space $\mathbb{CP}^d$. While a function $f : [0,1] \to \mathbb{C}^{d+1}$ is well defined if $f(t_0) = 0$ the same function seen as a function in the projective space $[f] : [0,1] \to \mathbb{CP}^d, t \mapsto [f(t)]$ is not a well-defined function for t_0.

As mentioned before we want do develop techniques to remove such singularities if possible. Therefore we will now define a C^0-continuation which is a notion of a continuous extension of a function.

Definition 3.1.6 ($\boldsymbol{C^0}$-continutation)
Let $I \subset [0,1]$ be a subset of the unit interval with no isolated points (in standard topology

of $\mathbb{R}$), $s_1, \ldots, s_n \in I$ and let $I' := I \setminus \{s_1, \ldots, s_n\}$. Let $\phi : I' \to \mathbb{CP}^d$ be continuous. If there is a continuous function $\hat{\phi} : \hat{I} \to \mathbb{CP}^d$ with $I' \subsetneq \hat{I} \subset I$ and $\phi(t) = \hat{\phi}(t)\, \forall t \in I'$ (note that "=" here means the same equivalence class), then we call ϕ *C^0–continuable on $\hat{I}$* and $\hat{\phi}$ the *C^0–continuation (or desingularization) of ϕ on $\hat{I}$*. And all $s \in \hat{I} \setminus I'$ are called *removable singularities.* ◇

Remark. In other words: in the previous definition we consider a function ϕ which is singular at t_0. If we can find another function $\hat{\phi}$ which coincides with ϕ on the domain of ϕ but is also continuous at t_0, then we can continuously extend the function ϕ at t_0, with the value of $\hat{\phi}$ at t_0, and remove the singularity.

Remark. If not mentioned otherwise we will denote by $\langle \cdot, \cdot \rangle$ the complex standard scalar product of $\mathbb{C}^n$: for $x, y \in \mathbb{C}^n$

$$\langle x, y \rangle := \sum_{i=1}^{n} \overline{x_i} y_i.$$

Definition 3.1.7 (Quasi Continuous)
Let $I \subset [0, 1]$ be a subset of the unit interval $[0, 1]$ with no isolated points and $I' := I \setminus \{s_1, \ldots, s_n\}$ for $s_1, \ldots, s_n \in [0, 1]$ $(n \in \mathbb{N})$. We will define quasi continuity for functions which map from an interval to the set of points $\mathcal{P}$, lines $\mathcal{L}$ and conics $\mathcal{C}$ in $\mathbb{CP}^2$ respectively:

- $\phi : I' \to \mathcal{P}$ is called *quasi continuous*, if there exists a continuous function $\psi : I \to \mathcal{P}$ such that $\phi(t) = \psi(t)$ for all $t \in I'$.
- $\phi : I' \to \mathcal{L}$ is called *quasi continuous*, if there exists a continuous function $\psi : I \to \mathcal{L}$ such that $\{p \in \mathcal{P} \mid \langle p, \phi(t) \rangle = 0\} = \{p \in \mathcal{P} \mid \langle p, \psi(t) \rangle = 0\}$ for all $t \in I'$.
- $\phi : I' \to \mathcal{C}$ is called *quasi continuous*, if there exists a continuous function $\psi : I \to \mathcal{C}$ such that $\{p \in \mathcal{P} \mid p^T A_{\phi(t)} p = 0\} = \{p \in \mathcal{P} \mid p^T A_{\psi(t)} p = 0\}$ where $A_{\phi(t)}$ and $A_{\psi(t)}$ denote the associated matrices of $\phi(t)$ and $\psi(t)$ for all $t \in I'$.

In each case we will call the function ψ the *quasi continuation.* ◇

Remark. In essence, a quasi continuous function can be extended continuously on a larger domain. Furthermore quasi continuity here is a generalized version of the quasi continuity in "Grundlagen Dynamischer Geometrie" [76].

Lemma 3.1.8 (Conics in $\mathbb{CP}^{\mathbf{2}}$ interpreted as Points in $\mathbb{CP}^{\mathbf{5}}$)
Every conic section $\mathcal{C}_A$ with associated non–zero matrix $A \in \mathbb{C}^{3\times3}$ can be interpreted as a point in $\mathbb{CP}^5$ and the other way round.

Proof. As for example J. Richter-Gebert proved in "Perspectives on Projective Geometry" ([111], p. 146 ff.), we can assume symmetry (*i.e.* $\mathcal{C}_A = \mathcal{C}_A^T$) and homogeneity for the associated conic matrix $\mathcal{C}_A$ for all entries: $\lambda \cdot \mathcal{C}_A$ describes the same conic section for all $\lambda \in \mathbb{C} \setminus \{0\}$. By the symmetry of the matrix, we have six distinct entries and write these in a vector $x = (a, b, c, d, e, f)^T$. By homogeneity, x is an object of $\frac{\mathbb{C}^6 \setminus \{0\}}{\mathbb{C} \setminus \{0\}}$ which is the definition of $\mathbb{CP}^5$.

And for a given vector $x = (a, b, c, d, e, f)^T \in \mathbb{CP}^5$ the associated matrix $\mathcal{C}_A$ is given by

$$\mathcal{C}_A = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

□

Remark. From now on, we will refer to conic sections as objects in $\mathbb{C}^6$ or $\mathbb{CP}^5$, respectively.

Lemma 3.1.9 (Quasi Continuity and $\boldsymbol{C^0}$-Continuations)
Let $I \subset [0, 1]$ be a subset of the unit interval $[0, 1]$ with no isolated points and $I' := I \setminus \{s_1, \dots, s_n\}$ for $s_1, \dots, s_n \in [0, 1]$ $(n \in \mathbb{N})$. Let $\phi : I' \to \mathbb{CP}^d$ be a continuous function. $\phi(t)$ is quasi continuous if and only if for all $s \in I \setminus I'$ there is a C^0–continuation.

Proof. If ϕ is quasi continuous then the claim is clear just by definition.

Now let there be a C^0-continuation for all $s_i \in I \setminus I'$. By the prerequisites, we know that for every s_i there is a C^0–continuation $\psi_i : I' \cup \{s_i\} \to \mathbb{CP}^d$ for $i \in \{1, \dots, n\}$.

We can now "glue" the C^0–continuations ψ_i to one continuous function on I: define $\psi(t) : I \to \mathbb{CP}^d$ by

$$\psi(t) := \begin{cases} \phi(t), & \text{if } t \in I' \\ \psi_i(t), & \text{if } t = s_i \text{ for } i \in \{1, \dots, n\} \end{cases}$$

By construction $\psi(t)$ coincides with $\phi(t)$ for all $t \in I'$. Furthermore by assumption $\psi(t)$ is continuous for all $t \in I$ since a C^0–continuation exists for all $t \in I \setminus I'$ and is given by ψ_i. But this means that ϕ is quasi continuous. □

Example 3.1.10 ($\boldsymbol{C^0}$-Continutation and removable Singularities)

Let $I = [0,1]$ and $I' := I \setminus \{\frac{1}{2}\}$. Define $\phi(t) : I' \to \mathbb{CP}^2$ by

$$\phi(t) := \left[\begin{pmatrix} \sin\left(t - \frac{1}{2}\right) \\ t - \frac{1}{2} \\ t - \frac{1}{2} \end{pmatrix}\right]$$

First note that $\phi(t)$ is continuous on I' by Lemma 3.1.4 (Topological Continuity).

Now define $\psi : I \to \mathbb{C}^3 \setminus \{0\}$ with

$$\psi(t) := \begin{pmatrix} \frac{\sin\left(t-\frac{1}{2}\right)}{t-\frac{1}{2}} \\ 1 \\ 1 \end{pmatrix} \text{ if } t \neq \frac{1}{2} \text{ and } \psi\left(\frac{1}{2}\right) := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Obviously, ψ is continuous on I', but is it also continuous on I by de L'Hospital's rule. Now define $\hat{\phi} := \pi \circ \psi = [\psi]$, then $\phi(t) = \hat{\phi}(t) \, \forall t \in I'$, which can easily be seen if one divides ϕ by its z–coordinate. Hence $\hat{\phi}$ is a topological C^0–continuation and $t = \frac{1}{2}$ is a removable singularity. Furthermore, this also shows that ϕ is quasi continuous: the C^0-continuation here coincides with the quasi continuation since there is only one singularity to resolve. $\diamond$

The next theorem will give us a way to resolve singularities: given a function $f : [0,1] \to \mathbb{C}^d$ which attains 0, and therefore has a singularity, we might find a C^0-continuation of $[f]$ by dividing the whole image by one of the components of the function f.

Theorem 3.1.11 (Existence of a Continuous Path, [76])
Let $\Psi : [0,1] \to \mathbb{C}^d$ be a continuous path and $t_0 \in [0,1]$. Let $A \subset \{1,\ldots,d\}$ be the set of indices of Ψ–components which are not constantly 0 on a neighborhood of t_0. If

1. $A \neq \emptyset$
2. for all $i,j \in A$: $\Psi_i(t)/\Psi_j(t)$ or $\Psi_j(t)/\Psi_i(t)$ has a removable singularity or is continuous at t_0,

then there are an $\epsilon > 0$, a continuous path $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ and a function $\lambda : B_\epsilon(t_0) \to \mathbb{C}$ such that $\lambda(t) \cdot \Theta(t) = \Psi(t)$ for all $t \in B_\epsilon(t_0) \setminus \{t_0\}$.

The described functions Θ, λ have the form

$$\Theta(t) = \frac{\Psi(t)}{\Psi_i(t)}, \quad \lambda(t) = \Psi_i(t)$$

where $i \in A$ is appropriately chosen such that all coordinate entries of Θ have a removable singularity or are continuous.

Remark. The last theorem might seem a bit unimposing at first look, but au contraire, it is actually a criterion for the situation when a singularity is removable! If the prerequisites are fulfilled, we can split Ψ into two parts: a vanishing (λ) and a non-vanishing (Θ) part at t_0. Since $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ is non–vanishing at t_0 we can use $[\Theta]$ as a C^0-continuation of $[\Psi]$ at t_0. We will make note of this fact in Lemma 3.1.13 (Fractionally Continuable $\Rightarrow$ C^0-continuable).

First we will establish more notation which is implied by the previous theorem.

Definition 3.1.12 (Fractionally Continuable)
If a continuous function $\Psi : [0,1] \to \mathbb{C}^d$ fulfills the requirements of the Theorem 3.1.11 (Existence of a Continuous Path) at $t_0 \in [0,1]$, we call Ψ *fractionally continuable at t_0*. And we call the fractional function $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ as defined in Theorem 3.1.11 the *fractional continuation of Ψ around t_0*. We call the function $\lambda : B_\epsilon(t_0) \to \mathbb{C}$ as defined in Theorem 3.1.11 *removable prefactor* or, more shortly, *prefactor*. We call the component $i \in A$, which defines λ in Theorem 3.1.11, the *selected component at t_0*.

If additionally λ is analytic, we call λ an *analytic removable prefactor* or shortly *analytic prefactor*.

If the function Θ is analytic on $B_\epsilon(t_0)$, then we call the function Ψ *analytical fractionally continuable at t_0*. Furthermore we call Θ the *analytic fractional continuation of Ψ around t_0*. ◇

The next lemma shows that if a function is fractionally continuable then it is also C^0-continuable.

Lemma 3.1.13 (Fractionally Continuable $\Rightarrow$ $\boldsymbol{C^0}$-continuable)
Let $I \subset [0,1]$ be a subset of the unit interval with no isolated points (in standard topology of $\mathbb{R}$), $\Psi : I \to \mathbb{C}^{d+1}$ be continuous and the preimage of zero, denoted by $S := \Psi^{-1}(0)$, be discrete (*i.e.* $S = \{s_1, \ldots, s_n\}$ for an $n \in \mathbb{N}$) and define $I' := I \setminus S$. Let Ψ be fractionally continuable for all $s_i \in \hat{I} \cap S$ where $I' \subsetneq \hat{I} \subset I$. Then $[\Psi] : I' \to \mathbb{CP}^d$ with $t \mapsto [\Psi(t)]$ is C^0–continuable on $\hat{I}$. And if it furthermore holds true that $\hat{I} = I$, then $[\Psi]$ is quasi continuous.

Proof. Let $i \in \{1, \ldots, n\}$. Since Ψ is fractionally continuable for all $s_i \in \hat{I} \cap S$, by Theorem 3.1.11 (Existence of a Continuous Path) there exist $\epsilon_i > 0$ and a continuous functions $\Theta_i : B_{\epsilon_i}(s_i) \to \mathbb{C}^{d+1} \setminus \{0\}$ and $\lambda_i : B_{\epsilon_i}(s_i) \to \mathbb{C}$ such that

$$[\lambda_i(t) \cdot \Theta_i(t)] = [\Psi(t)] \text{ if and only if } [\Theta_i(t)] = [\Psi(t)] \text{ for all } t \in B_{\epsilon_i}(s_i) \setminus \{s_i\}.$$

As Θ_i is continuous and non-vanishing on whole $B_{\epsilon_i}(s_i)$, this means that Ψ has a C^0-continuation for all $s_i \in \hat{I} \cap S$. Finally, by definition the property that $\hat{I} = I$ is equivalent for $[\Psi]$ being quasi continuous. □

In their proof of Theorem 3.1.11 (Existence of a Continuous Path) U. Kortenkamp and J. Richter-Gebert assume that possibly occurring singularities may be removed, although they do not state how this desingularization should be achieved. From a mathematical point of view the problem is quite easy: either one checks for the classical epsilon-delta criterion or equivalently uses a limit process.

From a computational point of view, the problem is completely different. D. Richardson proved in [110] that zero testing for certain classes of functions is undecidable. As D. Gruntz points out in his Ph.D. Thesis [57], this problem can be reformulated in order to check for the continuity of a function and therefore the problem is also undecidable (for certain functions). This result is very devastating at first sight, but the situation is not that bad if one assumes certain regularity of functions. We are in the lucky position that the functions we are considering are even analytic, which is a very strong property. Nevertheless, we will face problems at essential singularities, as we will see later on.

3.2. The Analytic Case

> How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?
>
> Sherlock Holmes, The Sign of the Four [28]

The problem of handling removable singularities in geometric constructions is closely related to the problem of removable singularities in complex analysis since one can analyze quotients of functions, as we saw in the theorem by U. Kortenkamp and J. Richter-Gebert (Theorem 3.1.11 (Existence of a Continuous Path)). For quotients of holomorphic functions, this is classic theory and can be found in every textbook on complex variables (see *e.g.* [12, 68, 80]). The standard procedure is to extend the function in a continuous way and this is equivalent to analytic continuation by Riemann's theorem ([80] p. 42-43). More concretely, one usually expands the functions to its unique power series and cancels common zeros at the singularity.

Firstly, we will analyze the connection of roots and singularities. If we speak of roots we usually mean a point where a function attains 0. We will refer to functions like $\sqrt[n]{z}$ as "radical expressions" or more shortly "radicals" to avoid confusion.

Lemma 3.2.1 (Roots and Singularities, adapted version of [68] p. 37)
Let $D \subset \mathbb{C}$ be a domain and $z_0 \in D$. Let $f, g : D \to \mathbb{C}$ be analytic. If both f possess a root of order $k \in \mathbb{N}$ at z_0 and g a root of order $l \in \mathbb{N}$ at z_0, $k \geq l$ then $h(z) := f(z)/g(z)$ has a removable singularity at z_0. Furthermore h can be written by

$$h(z) = \frac{(z - z_0)^k \cdot \hat{f}(z)}{(z - z_0)^l \cdot \hat{g}(z)}$$

where both $\hat{f}, \hat{g}$ are analytic and $\hat{g}$ does not vanish at z_0.

We will now apply the previous insights to our setting.

Lemma 3.2.2 (Component Decomposition)
Let $\Psi : I \subset (0,1) \to \mathbb{C}^d$ be analytic in every component and analytically fractionally continuable at $t_0 \in I$. Then we can decompose the components of $\Psi(t)$ around t_0 into non-vanishing and (possibly) vanishing parts: there exist $\epsilon > 0$ and $k \in \mathbb{N}$ such that:

$$\Psi(t) = (t - t_0)^k \cdot \tilde{\Psi}(t) \text{ for all } t \in B_\epsilon(t_0)$$

where $\tilde{\Psi}(t) : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ is an analytic function. The function $[\Psi]\big|_{B_\epsilon(t_0)} : B_\epsilon(t_0) \to \mathbb{CP}^{d-1}, t \mapsto [\Psi(t)]$ is C^0–continuable on at t_0 with C^0-continuation $[\tilde{\Psi}]$.

Remark. Note that in the previous lemma k is finite but might also be zero. This describes the case when the function Ψ has at least one component which does not vanish on $B_\epsilon(t_0)$.

Proof. The first part of the claim is obvious by Lemma 3.2.1 (Roots and Singularities). We will only prove the C^0-continuation statement: since it holds true that

$$\Psi(t) = (t - t_0)^k \cdot \tilde{\Psi}(t) \text{ for all } t \in B_\epsilon(t_0)$$

and we know that $\tilde{\Psi}$ and $(t - t_0)^k$ are non-vanishing on $B_\epsilon(t_0) \setminus \{t_0\}$, we find:

$$[\Psi(t)] = [(t - t_0)^k \cdot \tilde{\Psi}(t)] = [\tilde{\Psi}(t)] \text{ for all } t \in B_\epsilon(t_0) \setminus \{t_0\}$$

So the equivalence classes of $[\Psi(t)]$ and $[\tilde{\Psi}(t)]$ coincide everywhere on $B_\epsilon(t_0) \setminus \{t_0\}$. Since $[\tilde{\Psi}(t)]$ is continuous (even analytic) and non-vanishing on whole $B_\epsilon(t_0)$ we conclude that

$[\tilde{\Psi}(t)]$ is a C^0-continuation of $[\Psi]$ at t_0. $\square$

Remark. More figuratively speaking, we factor out the minimum amount of roots at the singularity t_0 such that not all components vanish.

Definition 3.2.3 (Assotication Polynomial and Order of Removable Singularity)
The function $(t - t_0)^k$ ($k \in \mathbb{N}$) in Lemma 3.2.2 (Component Decomposition) is called *t_0–associated polynomial of* Ψ or shortly *associated polynomial*, if it is obvious at which point we consider the series expansion. Usually we will denote the associated polynomial by $p(t)$. We call k the *order of the removable singularity.* $\diamond$

Remark. As we saw in the two preceding statements the associated polynomial a removable prefactor which "pushes" functions unnecessarily to zero. So removing the associated polynomial removes also the singularity.

The assumption that the functions have to be analytic is not far fetched. Although we want to model continuous movement, a much weaker premise, U. Kortenkamp and J. Richter-Gebert show that problems in dynamic geometry can be formulated as algebraic systems [76]. Employing classic theory of algebraic curves as described in the book by Brieskorn and Knörrer [13, 14] (or in a more condensed form in the book by G. Fischer [37]), one can describe these algebraic curves locally as generalized power series, so called Puiseux series which are complex analytic functions.

3.3. Singularities and Derivatives

> At ubi materia, ibi Geometria. – Wo Materie ist, dort ist auch Geometrie.
>
> Johannes Kepler, De fundamentis astrologiae certioribus [73]

Now we want to investigate the connection between analytic prefactors and derivatives. We will find that out removable singularities that do not involve radical expression can be easily resolved using derivatives. From the viewpoint of an implementation one particular concept can be useful: automatic differentiation (AD). Automatic differentiation combines the best of symbolic and numerical wolds: stability and speed.

Theorem 3.3.1 (Complex de L'Hospital)
Let $D \subset \mathbb{C}$ be a domain, $f, g : D \to \mathbb{C}$ be analytic functions with roots of order k for f at $z_0 \in D$ and l ($k \geq l$) for g at z_0. Then $f(z)/g(z)$ possess a removable singularity at z_0 and it holds:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}$$

Remark. The real version of the theorem is well known and usually proven by Rolle's Theorem (see [43], p. 186 ff). Unfortunately, Rolle's Theorem does not hold for complex functions.

Proof. A slightly less general version of the theorem is stated as an exercise in [44], p. 138. The proof possess some insight that will be useful later on so we will carry it out here.

Since f and g are analytic functions and both functions have a root of (at least) order k we can expand f and g to their power series and can factor out the roots, so there exist other analytic functions $\hat{f}$ and $\hat{g}$ such that:

$$\frac{f(z)}{g(z)} = \frac{(z-z_0)^k \hat{f}(z)}{(z-z_0)^k \hat{g}(z)} = \frac{\hat{f}(z)}{\hat{g}(z)}$$

By assumption $\hat{g}$ does not possess a root at z_0. Deriving nominator and denominator of the middle term separately and using the product rule the claim follows immediately. If $k > l$ then the f/g will vanish at z_0. □

Remark. Although the result was published by de L'Hospital in 1696 ([58]), the result should actually be called Johann Bernoulli's rule. By a contract Bernoulli was obliged to sell certain results to de L'Hospital which can be read up in detail in [126] p. 442–443.

Lemma 3.3.2 (Removal of Analytic Prefactors)
Let $\Psi : I \subset (0,1) \to \mathbb{C}^d$ be analytic in every component. Furthermore, let Ψ be analytic fractionally continuable at $t_0 \in I$, with a root of order $k \in \mathbb{N}$ at t_0 in component i, where i is the selected component at t_0. Then an analytic fractional continuation $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d$ on an open subset around t_0 Θ is given by:

$$\Theta_j(t) := \begin{cases} \Psi_j(t)/\Psi_i(t), & \text{if } t \neq t_0 \\ \Psi_j^{(k)}(t)/\Psi_i^{(k)}(t), & \text{if } t = t_0 \end{cases}$$

for all $j \in \{1, \ldots, d\}$.

Proof. Direct application of the complex version of de L'Hospital's rule (Theorem 3.3.1 (Complex de L'Hospital)). □

The last Lemma motivates a more direct way to the remove singularities. Instead of evaluating the derivatives of quotients of components one can also apply the derivatives directly to the components, which will be our next statement.

Theorem 3.3.3 (Direct Derivation)
Let $\Psi : I \subset (0,1) \to \mathbb{C}^d$ be analytic in every component and analytical fractionally continuable at $t_0 \in I$, with $p(t) = (t - t_0)^k$ an analytic prefactor of order $k \in \mathbb{N}$. Then the k'th derivative of Ψ at t_0 is a non–zero scalar multiple of the analytic fractional continuation Θ at t_0: there is $\tau \in \mathbb{C} \setminus \{0\}$:

$$\Theta(t_0) = \tau \cdot \Psi^{(k)}(t_0)$$

Furthermore, the continuous function $[\Psi] : I \setminus \{\Psi^{-1}(0)\} \to \mathbb{CP}^{d-1}, t \mapsto [\Psi(t)]$ has a C^0–continuation at t_0.

Proof. The proof is essentially based on the product rule and rules of derivation for polynomials. By Lemma 3.2.2 (Component Decomposition) we know that we can find a decomposition of Ψ such that $\Psi(t) = p(t) \cdot \tilde{\Psi}(t) = (t - t_0)^k \cdot \tilde{\Psi}(t)$ with $\tilde{\Psi}(t)$ analytic and non-vanishing at t_0, then

$$\begin{aligned}
&\Psi(t_0) = p(t_0) \cdot \tilde{\Psi}(t_0) \\
\Rightarrow\ &\Psi^{(1)}(t_0) = p^{(1)}(t_0) \cdot \tilde{\Psi}(t_0) + \underbrace{p(t_0)}_{=0 \text{ by definition}} \cdot \tilde{\Psi}^{(1)}(t_0) \\
\Rightarrow\ &\Psi^{(1)}(t_0) = p^{(1)}(t_0) \cdot \tilde{\Psi}(t_0) \\
&\vdots \\
\Rightarrow\ &\Psi^{(k)}(t_0) = p^{(k)}(t_0) \cdot \tilde{\Psi}(t_0) = k! \cdot \tilde{\Psi}(t_0)
\end{aligned}$$

On the other, hand we know that the j'th component of the analytic fractional continuation Θ is given by

$$\Theta_j(t_0) = \Psi_j^{(k)}(t_0)/\Psi_i^{(k)}(t_0),$$

where $i \in \{1, \ldots, d\}$ is the selected component at t_0, which we proved in Lemma 3.3.2 (Removal of Analytic Prefactors). Using Lemma 3.2.2 (Component Decomposition) we can write this as:

$$\begin{aligned}
\Theta_j(t_0) &= \Psi_j^{(k)}(t_0)/\Psi_i^{(k)}(t_0) \\
&= \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(p(t) \cdot \tilde{\Psi}_j(t)\right) \right)\Bigg|_{t=t_0} \Bigg/ \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(p(t) \cdot \tilde{\Psi}_i(t)\right) \right)\Bigg|_{t=t_0} \\
&= \left(k! \cdot \tilde{\Psi}_j(t_0)\right) / \left(k! \cdot \tilde{\Psi}_i(t_0)\right) \\
&= \tilde{\Psi}_j(t_0))/\tilde{\Psi}_i(t_0)
\end{aligned}$$

So we can set, $\tau = \frac{k!}{\tilde{\Psi}_i(t_0)}$ where $\tilde{\Psi}_i(t_0) \neq 0$ by construction, and have:

$$\Theta(t_0) = \tau \cdot \Psi^{(k)}(t_0).$$

Finally, this also means that

$$[\Theta(t_0)] = [\tau \cdot \Psi^{(k)}(t_0)] = [\Psi^{(k)}(t_0)]$$

which shows the C^0-continuation property. $\square$

Now we will give a series of examples which illustrate the concept discussed.

Remark. We defined quasi continuity only for functions whose pre-image is a subset of the unit interval. We will deviate from this assumption from time to time if examples can be easier written down using different (bounded) intervals. Of course this is without loss of generality, one can simple re-parameterize the functions at any time to fit into a subset of the unit interval.

Example 3.3.4 (Farpoint of Parallel Lines)
Let $a = (0,1,0)^T$ and $b(t) := (0,1,(t-\frac{1}{2})^3),\ t \in [0,1]$ be families of two lines in $\mathbb{RP}^2$. Denote their point of intersection as $P := a \wedge b$ with

$$P(t) = \begin{pmatrix} (t-\frac{1}{2})^3 \\ 0 \\ 0 \end{pmatrix} = (t-\frac{1}{2})^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

So it is easy to see that their point of intersection is either a far point or not defined (for $t = \frac{1}{2}$). The singularity at $t = \frac{1}{2}$ has order 3 and we can apply the developed theory. By Theorem 3.3.3 (Direct Derivation) we have to evaluate three derivatives of P:

$$P'(t) = 3 \cdot (t-\frac{1}{2})^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ P''(t) = 6 \cdot (t-\frac{1}{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ P'''(t) = 6 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Then the function $[P(t)] : [0,1] \setminus \{\frac{1}{2}\} \to \mathbb{RP}^2$ is quasi continuous with quasi continuation

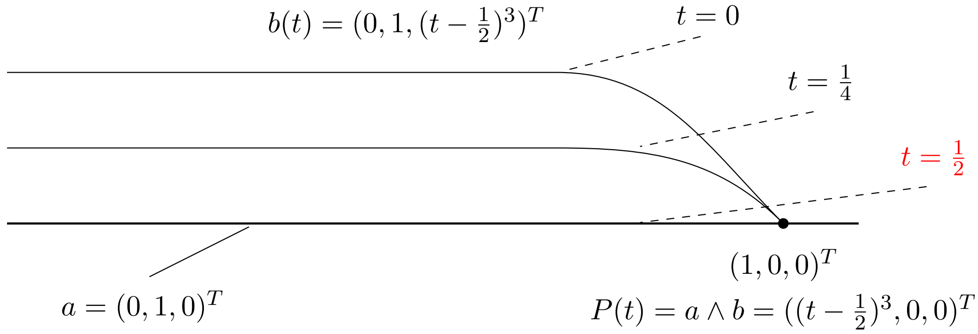


Figure 3.2.: Intersection of two lines with a removable singularity for $t = \frac{1}{2}$.

continuation $[\hat{P}(t)] : [0, 1] \to \mathbb{RP}^2$

$$[\hat{P}(t)] := \left\{ \begin{array}{ll} [P(t)], & \text{if } t \neq \dfrac{1}{2}, \\ [(1, 0, 0)^T], & \text{if } t = \dfrac{1}{2} \end{array} \right\} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \forall t \in [0, 1].$$

One can find an illustration in Figure 3.2. ◇

Example 3.3.5 (Rotating Line)
Consider the lines $a := (1, 1, 0)^T$ and $b(t) := (1, -\frac{2}{\pi^2} \cdot (\sin(\pi \cdot t) - 1), 0)^T$ for $t \in [0, 1]$. Their point of intersection is $P(t) := a \wedge b = (0, 0, -\frac{2}{\pi^2} \cdot (\sin(\pi \cdot t) - 1))^T$ with a removable singularity at $t = 1/2$.

We learned multiple options to remove the singularity: series expansion, de L'Hospital or direct application of the derivative. The first two options are closely related as we saw in the proof of the complex de L'Hospital: we used the series expansion and the product rule to prove L'Hostpital's rule.

Define $f(t) := -\frac{2}{\pi^2}(\sin(\pi \cdot t) - 1)$ then we can expand the series around $t = 1/2$:

$$f(t) = (t - \frac{1}{2})^2 + O((t - \frac{1}{2})^4)$$

When we apply Theorem 3.1.11 (Existence of a Continuous Path) and normalize the term $P(t)$ by its z–component, we find a removable singularity in the classical complex analysis setting: we divide $f(t)$ by itself. The term f/f has a singularity at $t = \frac{1}{2}$. We can apply complex de L'Hospital twice, either to the series expansion or the terms itself, which both removes the singularity: $\frac{f''(\frac{1}{2})}{f''(\frac{1}{2})} = \frac{2}{2} = 1$. Assembling the vector back together we can continuously extend P at $t = \frac{1}{2}$ with $(0, 0, 1)^T$.

More directly we can apply Theorem 3.3.3 (Direct Derivation) to the components and derive the function:

$$P'(\frac{1}{2}) = (0,0,-\frac{2}{\pi}\cdot\cos(\frac{\pi}{2}))^T = (0,0,0)^T, \quad P''(\frac{1}{2}) = (0,0,2\cdot\sin(\frac{\pi}{2})) \sim (0,0,1)^T.$$

Both ways allow it to to define a C^0–continuation $[\hat{P}(t)]$ for $[P(t)]$ at $t = 1/2$:

$$[\hat{P}(\frac{1}{2})] = (0,0,1)^T.$$

$\diamond$

Example 3.3.6 (Midpoint of a Circle)
Let $A = (-1,0,1)^T, B = (1,0,1)^T$ and $C = (0,1,1)^T$ be points in $\mathbb{RP}^2$. Define another point $C' = (0,0,1)^T$. Then we can define two circles (interpreted as conic section) being incident to A, B, C or A, B, C' respectively, and call their associated conic matrices X and Y. Spelling that out yields:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since A, B and C' are collinear the circle $\mathcal{C}_Y$ will degenerate to a line.

Consider now the linear combination, $f(t) := (1-t)\cdot X + t\cdot Y$ for $t \in [0,2]$: this describes the unit circle for $t = 0$, a line for $t = 1$ and a linear interpolation of the circle and the line for $t \in (0,1)$. If we calculate the center $M(t)$ of the conic given by $\mathcal{C}_{f(t)}$ one finds the following:

$$M(t) = \begin{pmatrix} 0 \\ (t-1)t \\ (t-1)^2 \end{pmatrix} = (t-1)\begin{pmatrix} 0 \\ t \\ (t-1) \end{pmatrix}$$

which yields $(0,0,0)^T$ for $t = 1$, which means that there is a (removable) singularity. We can again apply Theorem 3.3.3 (Direct Derivation) and take the derivatives of $M(t)$: if one takes the first derivative, one recovers the correct solution even for $t = 1$ which is $(0,1,0)^T$ the far point of the y axis. This can of course be again interpreted as a continuous movement with a C^0–continuation at $t = 1$. See Figure 3.3 for a picture. $\diamond$

Remark. For a practical implementation of automatic differentiation we refer the reader to [55] or Section 6.4 on page 103.

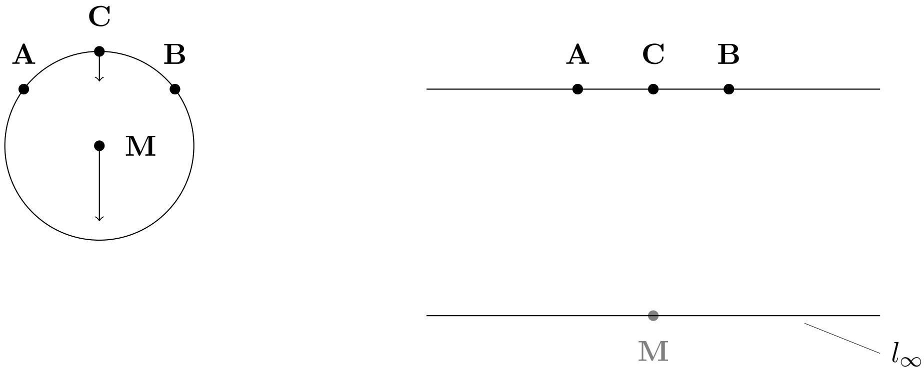


Figure 3.3.: **Degenerate Midpoint:** moving C to the connecting line of A and B yields a removable singularity along a fixed path.

3.4. The Radical of all Evil

> Auch eine Enttäuschung, wenn sie nur gründlich und endgültig ist, bedeutet einen Schritt vorwärts, und die mit der Resignation verbundenen Opfer würden reichlich aufgewogen werden durch den Gewinn an Schätzen neuer Erkenntnis.
>
> Max Planck, Physikalische Randblicke [108]

In the last section we investigated the role of derivatives in the process of removing singularities. The basic idea was an unpretentious one: singularities of the form $0/0$ can be dissipated using de L'Hospital's rule. All the theorems had one prerequisite: the analyticity of the functions around t_0. In geometry, a lot of constructions are accessible using only ruler constructions. These can be carried out analytical everywhere, but not for nothing the classic constructions are often called "ruler and compass constructions". And the utilization of a compass means that algebraically one has to admit radical expressions ([91], p. 35). The complex radical functions is intrinsically monodromic and has a branch point at 0 (and one at ∞). But more down to earth: complex radical functions cannot be defined analytical at branch points, which is the reason why Lemma 3.3.2 (Removal of Analytic Prefactors) or Theorem 3.3.3 (Direct Derivation) cannot be applied, since analyticity was a crucial premiss for these. Eventually this leads to the beautiful theory of Riemann surfaces.

We will illustrate this with an example which U. Kortenkamp already has mentioned in his thesis [77].

Example 3.4.1 (Disjoint Circle Intersection)
As circles are conic sections, in general two of them have four points of intersection.

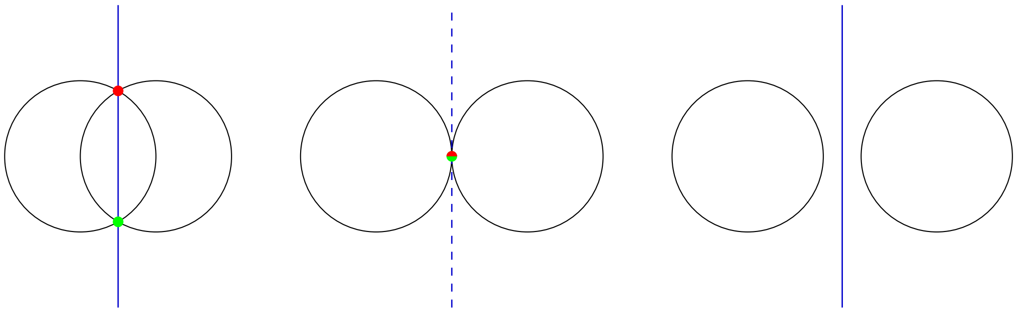


Figure 3.4.: **Disjoint Circle Intersection:** The connecting line of the intersection of two circles. The line is undefined for the tangential situation in the middle.

It is well known that two points are always incident to all circles in $\mathbb{CP}^2$: namely the points $\mathrm{I} = (-\mathrm{i}, 1, 0)^T$ and $\mathrm{J} = (\mathrm{i}, 1, 0)^T$. We will neglect these two and concentrate on the remaining two points, which are usually considered in Euclidean geometry. It might also occur that the points of intersection become complex. The details and an algorithm how to implement the intersection can be found in J. Richter-Gebert's "Perspectives on Projective Geometry" ([111] p. 196 ff.).

Usually these two "other points" (*i.e.* not being I or J) of intersection are disjoint, but there are two exceptions: if the two circles are the same and if the two circles are tangential.

We will now analyze the latter: the basic idea is to reduce the problem of intersecting two circles to the task of intersection a line and a conic. Intersecting a conic and a line involves using a square root operation, since essentially we are solving a quadratic equation ([111] p. 194 ff.). In the tangential situation the two points of intersection will merge to one. Seen from the point of view of quadratic equations: the discriminant in the solution of the quadratic expression vanishes, leaving only one solution.

Now if one examines the connecting line of those two merging points the operation will be inherently undefined. But actually the singularity is removable: if one considers a limit process one can assign a unique continuous extension of the function. We will now examine the situation along a fixed path.

Given two circles C, D with radius 1 and centers $M_C = (0, 0, 1)^T$ and $M_D(t) := (t, 0, 1)^T$ for $t \in (0, 2]$. D is moving along the x-axis and one obtains two points of intersection of

C and D, call them A and B:

$$A = \begin{pmatrix} t \\ \sqrt{1-t^2} \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} t \\ -\sqrt{1-t^2} \\ 1 \end{pmatrix},$$

We will consider the connecting line of A and B and call this line l:

$$l(t) = \begin{pmatrix} 2\sqrt{1-t^2} \\ 0 \\ 2t\sqrt{1-t^2} \end{pmatrix} = (2\sqrt{1-t^2}) \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix} \sim \sqrt{1-t^2} \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}$$

A drawing of the situation can be found in Figure 3.4. The connecting line l is well defined for all t expect for $t = 1$: there $l(t)$ has a singularity (and also at $t = -1$, which is not part of the domain of the function). The theory we developed so far would suggest to apply de L'Hospital's rule or similar techniques based on derivatives. So if we apply Theorem 3.3.3 (Direct Derivation) we find the following:

$$\begin{aligned} l'(t) &= \begin{pmatrix} \frac{-t}{\sqrt{1-t^2}} \\ 0 \\ \frac{1-2t^2}{\sqrt{1-t^2}} \end{pmatrix} = \frac{1}{\sqrt{1-t^2}} \begin{pmatrix} -t \\ 0 \\ 1-2t^2 \end{pmatrix} \\ l''(t) &= \begin{pmatrix} \frac{-1}{(1-t^2)^{\frac{3}{2}}} \\ 0 \\ \frac{t(2t^2-3)}{(1-t^2)^{\frac{3}{2}}} \end{pmatrix} = \frac{1}{(1-t^2)^{\frac{3}{2}}} \begin{pmatrix} -1 \\ 0 \\ t(2t^2-3) \end{pmatrix} \\ l'''(t) &= \dots \end{aligned}$$

We run into the same singularity over and over again. This is of course not remarkable at all since the derivatives of the complex square radical have an essential singularity at 0. A picture of the Riemann surface of the first derivative of $\sqrt{z}$ can be found in Figure 3.5. ◇

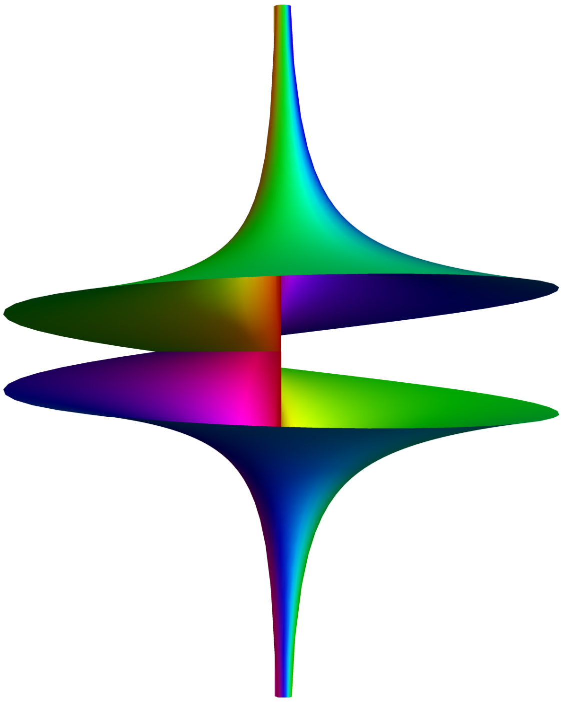


Figure 3.5.: Riemann surface of the first derivative of $\sqrt{z}$ with an essential singularity at $z = 0$.

Remark. Note that we excluded $t = 0$ in at which the two circles would merge. We will analyze the difference later on and argue that there are situations where even the merging of two circles can yield a meaningful connecting line but only for a given path. But the solution is **only** meaningful along this path and behaves discontinuous over spacial perturbations. We will investigate this later on in Section 6.2 on page 97.

So we can see that derivatives cannot resolve the singularity. We have to come up with different techniques to handle the situation. We will briefly discuss the possible candidates now and then propose our solution.

3.5. Methods for Non-differentiable Functions

> Wer etwas Großes will, der muß sich, wie Goethe sagt, zu beschränken wissen. Wer dagegen alles will, der will in der Tat nichts und bringt es zu nichts. Es gibt eine Menge interessante Dinge in der Welt; spanische Poesie, Chemie, Politik, Musik, ist alles sehr interessant, und man kann es keinem übel nehmen, der sich dafür interessiert; um aber als ein Individuum in einer bestimmten Lage etwas zustande zu bringen, muß man sich an etwas Bestimmtes halten und seine Kraft nicht nach vielen Seiten hin zersplittern.
>
> Georg Wilhelm Friedrich Hegel, Enzyklopädie der philosophischen Wissenschaften im Grundrisse [61]

We will now discuss methods we considered for the non-differentiable case of removable singularities.

3.5.1. Classical Power Series

Firstly, we tried to expand our problem into **Taylor series**, which worked quite well for the differentiable case. This not very surprising in the light of Theorem 3.3.3 (Direct Derivation), where we explored the relation of singularities and derivatives. There is also a fast implementation available using automatic differentiation. We ran into serious problems when we tried to apply the series argument to radical expressions. Although we can define holomorphic radical expressions we always have to exclude the connecting segment between the two essential singularities: 0 and ∞ (as proven in [1]). It is easy to see that there is no hope to expand these techniques to our problems as the Example 3.4.1 (Disjoint Circle Intersection) shows: there cannot be an ordinary Taylor expansion in a singularity.

Our next candidate in line was the formal **Laurent series**, which naturally arise in the study of singularities in complex analysis. We dismissed the idea quite quickly due to the fact that this ring is not algebraically closed. Radical expressions arise in geometric constructions, as soon as conic sections are involved, and Laurent series cannot approximate these around the origin.

Based on that we looked into the algebraic closure of Laurent series, the so called **Puiseux series**. They already arise in the study of geometric constructions with their relation to algebraic curves (see [76]). Using the so called Newton polygons ([37, 13, 14]) one can transform multi valued algebraic curves locally to Puiseux series in one variable. But still the resolution of numerical quotients of vanishing radical expressions is not easily addressed. Nevertheless, the idea of series with non-integer powers will be one solution to the problem of removable singularities.

We will go into greater details on the topic of generalized power-series when discussing implementation details on page 105.

3.5.2. Law of Large Numbers

The weak law of large numbers (see for example: Hans-Otto Georgii - Stochastik [50], p. 120 ff) dictates that for a sequence of independent, identically distributed, Lebesgue integrable random variables $(X_n)_{n\in\mathbb{N}}$ with expectation $\mathrm{E}(|X_n|) < \infty \ \forall\, n \in \mathbb{N}$ and their mean value $\overline{X}_n$:

$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathrm{E}(X_i))$$

it holds true that

$$\lim_{n\to\infty} P\left(\left|\overline{X}_n\right| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0.$$

Hence for a continuous function $f : [-1,1] \setminus \{0\} \to \mathbb{C}^n$ with a removable singularity at $t = 0$ one could approximate the function value $f(0)$ by a mean value of function evaluations on a δ-ball ($\delta > 0$) around zero. By setting

$$f(0) := \frac{1}{n}\left(\sum_{i=1}^{n} f(X_i)\right)$$

with $X_1, \ldots, X_n$ independent and identically distributed random variables on $[-\delta, \delta]$, $\mathrm{E}(X_i) = 0, X_i \neq 0$ and variance $0 < \mathrm{Var}(X_i) < \infty$ for all $i \in \{1, \ldots, n\}$.

If one defines $\nu := \sup_{\{i\leq 1\}} \mathrm{Var}(X_i)$ one can use Chebyshev's inequality to derive a decay rate:

$$P\left(\left|\overline{X}_n\right| \geq \epsilon\right) \leq \frac{\nu}{n\epsilon^2}$$

Since the singularities are isolated in our problem, one can choose $\delta > 0$ arbitrarily small and accelerate the convergence significantly.

Our experimental results on the disjoint circle intersection (Example 3.4.1 (Disjoint Circle Intersection)) showed for a uniformly distributed random variable on $[-10^{-3}, 10^{-3}]$ and 10 sample points a precision of 3 significant digits. For a uniformly distributed random variable on $[-10^{-6}, 10^{-6}]$ we obtained 6 significant digits. An implementation in CindyScript can be found in Appendix C (Law of Large Numbers).

Nevertheless one has to evaluate a non-negligible amount of function values and is still prone to numerical and statistical error. Furthermore, we will not get a guarantee that the value we estimate is the function value we are actually looking for.

3.5.3. Cauchy's Integral Formula

A very well known theorem from complex analysis is Cauchy's integral formula. It can be used to obtain function values using a contour integral.

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Let B be an open disk with $\bar{B} \subset U$ then it holds true (see [12] p. 29 f.):

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi, \quad z \in B.$$

Since we are considering functions with removable singularities consider a holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$ with removable singularity at z_0. If we can continuously extend the function at z_0. This will yield a holomorphic function on whole U (including z_0) by Riemann's theorem on removable singularities (see for example the book by Krantz: "Handbook of complex variables" [80] p. 42 f).

Hence, one can use a contour integral to define f at z_0:

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(\xi)}{\xi - z_0} \, \mathrm{d}\xi$$

This is a very appealing approach since the theory is very concise. The drawback is that we would have to calculate the contour integral. Either symbolic, which would be quite slow, or numerically, which is error prone. Numerically one usually considers quadrature formulas like Newton–Cotes or Gaussian quadrature (for details see Stoer and Bulirsch: "Introduction to Numerical Analysis" [127], p. 145 ff) or Monte Carlo integration (see [115], p. 79 ff).

We did not pursue this approach since the computational expense is generally high to obtain fairly exact results.

3.5.4. Algebraic Curves: Blow Up

A classical method from algebraic curve theory are blow ups. The basic idea is to take a two dimensional curve with a singularity, which here means a double point, and "blow it up" into dimension three in order to resolve the singularity. Then the projection back to the plane is isomorphic to the original curve. Unfortunately, this will not help for our problem.

If we take the disjoint circle intersection example Example 3.4.1 (Disjoint Circle Intersection): for $t = 1$ the points of intersection merge. Essentially, the movement of the two points is defined by the function $\pm\sqrt{1 - t^2}$ for $t \in [0, 2]$. Blowing up the functions separately may result in two non–intersecting blown up functions.

Nevertheless, the connecting line of the points, even in dimension three, will be projected to a single point in the plane. By construction of the blow up, the projection of the two points have to coincide which means that all points of the connecting line also will be projected to this particular point for all linear projections. Nonlinear projection might even not generate a line at all.

A visualization can be found in Figure 3.6. For further details we refer the reader to the book on algebraic curves by Brieskorn and Knörrer: [14] p. 455 ff.

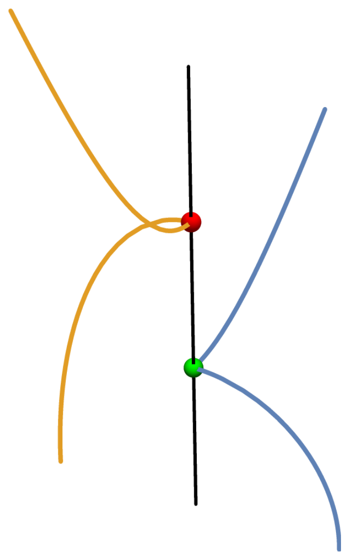


Figure 3.6.: Visualization of the connecting line of the disjoint circle intersection of Example 3.4.1 (Disjoint Circle Intersection). The plot shows $(\mathrm{Re}(f(t)), \mathrm{Im}(f(t)), t)$ (blue) and $(-\mathrm{Re}(f(t)), -\mathrm{Im}(f(t)), 2t)$ (orange) for $f(t) := \sqrt{1-t^2}$ and $t \in [0, 2]$. The connecting line of the intersections, in the tangential position $t = 1$, would always be projected to a single point and not to a (plane) line. Note that this is not an actual blow up, it is not differentiable. Differentiability would not change the situation since the projection of the red and green point always do coincide.

Computer Algebra Systems

A viable way to go would have been the usage of a computer algebra system.

The sketch of an algorithm to resolve singularities would look essentially like this: define your free and dependent objects using algebraic techniques as described in “Perspectives on Projective Geometry” [111] and the previous sections here. Then check whether the resulting objects have common factors attaining zero for certain input values. Factoring out and canceling these common factors can resolve a singularity.

Or, more directly, one could also utilize the build-in limit functions and compute the limit of a fraction of components like we did in the Theorem 3.1.11 (Existence of a Continuous Path). We looked into several implementations: for example Sage [118] uses de L'Hospital's rule for a fixed number of times and if this fails, it switches to Taylor series expansion. This has some drawbacks for certain highly varying function which can be read up in [57]. SymPy [92] uses the Gruntz's algorithm as proposed in [57, 56] which overcomes most of the obstacles and seems to be still the most widespread algorithm for evaluating limits. Unfortunately, Wolfram's Mathematica [134] does not explain which algorithm they actually implemented, just stating "Limits are found from series and using other methods" [133].

Generally computer algebra systems are quite slow: for example evaluating the limit

$$\lim_{t \to -1} \frac{t \cdot \sqrt{1-t^2}^{100}}{|\sqrt{1-t^2}|}$$

takes around 4 seconds in SymPy and 2 seconds in Mathematica.

We do not pursue that path due to the vast amount of time such operations take and a implementation of a fully featured CAS, like Mathematica or Sage, is not planned for Cinderella and CindyJS in the near future.

4. Non-standard Analysis

> Arithmetic starts with the integers and proceeds by successively enlarging the number system by negative and rational numbers, irrational numbers, etc. But the next quite natural step after the reals, namely the introduction of infinitesimals, has simply been omitted. I think, in incoming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus.
>
> Kurt Gödel, [117]

Infinitesimal numbers have been widely used by mathematicians like Leibniz, Descartes or Newton in a very successful way to exercise calculus. Due to the lack of proper formalization the theory contained contradictions and became discredited. Later on, infinitesimal calculus was superseded by the "ϵ-δ" formalism of Weierstrass and then regarded as imprecise and error prone.

An important step towards rigorous treatment of infinitesimal quantities was the paper by A. Schmieden and D. Laugwitz: "Eine Erweiterung der Infinitesimalrechnung" [119] in 1958. Their construction had the drawback that it wasn't actually a field and therefore elements with no multiplicative inverse. Nevertheless, this paper was a huge step towards the formal treatment of the infinitesimals and therefore Schmieden and Laugwitz can be seen as the founders of modern non-standard analysis, which also Werner Heisenberg noted in a letter to Wolfgang Pauli ([106], p. 1300-1301):

> Und man kann doch sicher nicht behaupten, daß ein Abelscher Grenzwert "schlechter" definiert sei als eine "normale" Distribution. Es kommt ja nur darauf an, daß man die Konvergenzfragen sauber behandelt, aber nicht darauf, was die Mathematiker zufällig bisher sorgfältig behandelt haben. Ich möchte Dich bei dieser Gelegenheit auf eine Arbeit von Schmieden und Laugwitz (Mathematische Zeitschrift 69, S. 1, 1958) hinweisen, in der die δ- Funktionen völlig anders als durch Distributionen legitimiert werden. Wie weit man das für die Physik brauchen kann, ist eine andere Frage.
>
> Viele Grüße! Dein W. Heisenberg

Eventually a thorough treatment of non-standard analysis was given by A. Robinson in 1961 [116, 117] which overcame the drawbacks of the ansatz by Laugwitz and Schmieden and constructed a proper field with infinitesimal and unlimited members.

4.1. Constructing the Hyperreal and Hypercomplex Numbers

> To those who ask what the infinitely small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be.
>
> Leonhard Euler

We will give a very brief construction of the hyperreal and hypercomplex numbers. They were introduced by A. Robinson in the 1960s using a model theoretic approach [116, 117]. Since model theory is not usual part of the curriculum (at least not in German universities) we will here follow a different approach using filters, which is more accessible for mathematicians.

For further reading, I can recommend the books by Goldblatt [52], Arkeryd *et al.* [2] or Keisler [72].

Geometric reasoning, especially with application in physics, using infinitesimal numbers has already been investigated by J. Fleuriot [39, 41, 42]. His implementation of the hyperreals was done in Isabelle [101] which can implement higher order logic, another approach to non-standard analysis.

Remark. While literature on the Hyperreals has been developed quite broadly there is surprisingly few work done on the Hypercomplex numbers. The original work of Robinson [116] has a chapter on some important theorems of complex analysis and the book by Diener *et al.* [27] is another example. Robinson concisely introduces the hypercomplex space using the isomorphism of $\mathbb{C} \cong \mathbb{R}^2$ (see [116] p. 147).

We will mostly follow the book of Goldblatt [52] for the real case and extend the theory for the complex case when necessary. We will generally refer to the current field as $\mathbb{K}$ and put $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 4.1.1 (Filters, [52] p. 18)
Let I be a set, $I \neq \emptyset$ and $\mathcal{P}(I)$ be the power set of I. A subset $\mathcal{F}$ of $\mathcal{P}(I)$ is called *filter*,

if it satisfies the following conditions:

$$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$$
$$A \in \mathcal{F} \text{ and } A \subset B \subset I \Rightarrow B \in \mathcal{F}.$$

We call $\mathcal{F}$ *proper*, if $\emptyset \notin \mathcal{F}$. And $\mathcal{F}$ is called an *ultrafilter* if it is proper and

$$\forall A \subset I : A \in \mathcal{F} \text{ or } A^c \in \mathcal{F}.$$

A ultrafilter $\mathcal{F}$ is called *principal*, if $\mathcal{F}$ can be written as

$$\mathcal{F} = \mathcal{F}^i := \{A \subset I \mid i \in A\}$$

otherwise it is called *nonprincipal*. ⋄

Lemma 4.1.2 (Nonprincipal Ultrafilter, [52] p. 21)
Any infinite set has a nonprincipal ultrafilter on it.

Definition 4.1.3 (Ring of sequences, partially [52] p. 23)
Let $\mathbb{K}^{\mathbb{N}}$ be the set of all sequences on $\mathbb{K}$. One can write $r \in \mathbb{K}^{\mathbb{N}}$ as $r = \langle r_1, r_2, r_3, \ldots \rangle := \langle r_n \rangle$. For $r, s \in \mathbb{K}^{\mathbb{N}}$ define

$$r \oplus s := \langle r_n + s_n \mid n \in \mathbb{N} \rangle$$
$$r \odot s := \langle r_n \cdot s_n \mid n \in \mathbb{N} \rangle$$

where the summation and multiplication are defined in the in the corresponding field. It is well known that $(\mathbb{K}^{\mathbb{N}}, \oplus, \odot)$ is a commutative ring with $\mathbf{0} = \langle 0, 0, 0, \ldots \rangle$ and $\mathbf{1} = \langle 1, 1, 1, \ldots \rangle$. The additive inverse of $r \in \mathbb{K}^{\mathbb{N}}$ is given by

$$-r := \langle -r_n \mid n \in \mathbb{N} \rangle.$$

This is of course not a field since it includes zero divisors:

$$\langle 1, 0, 1, 0, \ldots \rangle \odot \langle 0, 1, 0, 1, \ldots \rangle = \mathbf{0}.$$

⋄

Definition 4.1.4 (Equivalence Relation, partially [52] p. 24)

Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Define a relation $\equiv$ on $\mathbb{K}^{\mathbb{N}}$ by

$$\langle r_n \rangle \equiv \langle s_n \rangle \quad \Leftrightarrow \quad \{n \in \mathbb{N} \mid r_n = s_n\} \in \mathcal{F}$$

This defines a equivalence relation on $\mathbb{K}^{\mathbb{N}}$ (for the real case: [52] p. 24, complex analogously). $\diamond$

Definition 4.1.5 (Hyperreal and Hypercomplex numbers, partially [52] p. 25)
For $r \in \mathbb{K}^{\mathbb{N}}$ define an equivalence class

$$[r] = \{s \in \mathbb{K}^{\mathbb{N}} \mid r \equiv s\}.$$

Then we define the *hyperreal numbers* $\mathbb{R}^*$ and *hypercomplex numbers* $\mathbb{C}^*$ by

$$\mathbb{R}^* = \{[r] \mid r \in \mathbb{R}^{\mathbb{N}}\}, \quad \mathbb{C}^* = \{[r] \mid r \in \mathbb{C}^{\mathbb{N}}\}.$$

Define summation and multiplication by:

$$[r] + [s] = [r \oplus s], \quad [r] \cdot [s] = [r \odot s].$$

An order on $\mathbb{K} = \mathbb{R}$ is defined by:

$$[r] < [s] \quad \Leftrightarrow \quad \{n \in \mathbb{N} \mid r_n < s_n\} \in \mathcal{F}.$$

$\diamond$

Remark. If we don't want to specify the field we refer to, either the hyperreal $\mathbb{R}^*$ or hypercomplex $\mathbb{C}^*$ numbers, we will simply write $\mathbb{K}^*$.

Theorem 4.1.6 (Hyperreals are a Field, [52] p. 25)
The structure $(\mathbb{R}^*, +, -, \cdot, <)$ is an ordered field with zero $\mathbf{0}$ and unity $\mathbf{1}$.

Theorem 4.1.7 (Hypercomplex numbers are a Field)
The structure $(\mathbb{C}^*, +, -, \cdot)$ is an field with zero $\mathbf{0}$ and unity $\mathbf{1}$.

Proof. Analogously to [52] p. 25. $\square$

Remark. Note that the proof uses Zorn's lemma. This is of course highly non-constructive and we will later see that this will cause severe problem when implementing hyperreal numbers on a computer. For a practical implementation of a subfield of the hyperreal- and hypercomplex numbers see Section 6.4 on page 103.

Definition 4.1.8 (Enlarging Sets, partially [52] p. 28)
A subset A of $\mathbb{K}$ can be enlarged to the subset A^* of $\mathbb{K}^*$ by putting

$$[r] \in A^* \quad \Leftrightarrow \quad \{n \in \mathbb{N} \mid r_n \in A\} \in \mathcal{F}.$$

And define

$$[\![r \in A]\!] := \{n \in \mathbb{N} \mid r_n \in A\}.$$

◇

Theorem 4.1.9 (Nonstandard Members, partially [52] p. 29)
Any enlargement A^* infinite subset of $A \subset \mathbb{K}$ has nonstandard members.

Proof. Again analogously to [52] p. 29. □

Definition 4.1.10 (Extended Functions, partially [52] p. 30-31)
Let $f : \mathbb{K} \to \mathbb{K}$ be a function. Then we define $f^* : \mathbb{K}^* \to \mathbb{K}^*$ by

$$f^*([\langle r_1, r_2, \ldots]\rangle) := [\langle f(r_1), f(r_2), \ldots\rangle].$$

For partial functions, defined on a subset of $\mathbb{K}$, we have to put a little more effort: Let $A \subset \mathbb{K}$ with its enlargement A^*. For $[r] \in A^*$ and $[\![r \in A]\!]$ define

$$s_n := \begin{cases} f(r_n) & \text{if } n \in [\![r \in A]\!], \\ 0 & \text{if } n \notin [\![r \in A]\!]. \end{cases}$$

Then we define with $s = [s_1, s_2, \ldots]$

$$f^*([r]) = [s].$$

This handles the case that $f(r_n)$ might be not defined for some n. Of course 0 here is arbitrary, but the filter property holds almost everywhere so it does not matter which particular number one chooses. ◇

Definition 4.1.11 (Enlarging relations, partially [52] p. 31-32)
Let P be a k-ary relation on $\mathbb{K}$ (thus a subset of $\mathbb{K}^k$). For a sequence $r^1, \ldots, r^k \in \mathbb{K}^n$ define

$$[\![P(r^1, \ldots, r^k)]\!] = \{n \in \mathbb{N} \mid P(r^1_n, \ldots, r^k_n)\}.$$

Then we can extend this to a k-ary relation P^* on $\mathbb{K}^*$ (*i.e.* to a subset of $(\mathbb{K}^*)^k$) by

$$P^*([r^1], \dots, [r^k]) \Leftrightarrow [\![P(r^1, \dots, r^k)]\!] \in \mathcal{F}.$$

One can show that if $A_1, \dots, A_k$ are subsets of $\mathbb{K}$ and we put $A = A_1 \times \cdots \times A_k$ that

$$(A_1 \times \cdots \times A_k)^* = A_1^* \times \cdots \times A_k^*.$$

$\diamond$

Lemma 4.1.12 (Enlarging Rules, partially [52] p. 29)
Let $A, B \subset \mathbb{K}$, then

- if A is a finite set, then $A^* = A$, *i.e.* A^* has no non–standard members
- $A \subset B$, if and only if $A^* \subset B^*$,
- $A = B$, if and only if $A^* = B^*$,
- $(A \cup B)^* = A^* \cup B^*$,
- $(A \cap B)^* = A^* \cap B^*$,
- $(A \setminus B)^* = A^* \setminus B^*$,
- $\emptyset^* = \emptyset$

and for $a, b \in \mathbb{R}$ let $[a, b]$ be the closed interval from a to b, then it holds true that

$$[a, b]^* = \{x \in \mathbb{R}^* \mid a \leq x \leq b\}.$$

Proof. The real case is shown in [52] p. 29. The complex variant follows from the isomorphism $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$. In Definition 4.1.11 (Enlarged relations) it is shown that $(A \times B)^* = A^* \times B^*$ which yields the claim. $\square$

4.2. Mathematical Logic and the Transfer Principle

Je pense, donc je suis.

René Descartes, Discours de la méthode [25]

In this section, we will make a small detour to mathematical logic. This will culminate in the so called "Transfer principle" which makes the hyper (real or complex) numbers extremely powerful. We will state the theorems in an intuitive way first and then work our way around the cliffs of logic introduce the theory thoroughly. We will mostly follow Goldblatt's book ([52], chapter 4). For further reading, I can recommend for example the book "Einführung in die mathematische Logik" by Ebbinghaus *et al.* [30].

Intuition 4.2.1 (Universal Transfer, partially [52] p. 45)
If a property holds for all real or complex numbers, then it holds for all hyperreal or hypercomplex numbers.

Intuition 4.2.2 (Existential Transfer, partially [52] p. 45)
If there is a hyperreal or hypercomplex number satisfying a certain property, then there exists a real or complex number with this property.

To make these intuitions rigorous we have to introduce some logical machinery.

Definition 4.2.3 (Relational structure of $\mathbb{K}$, partially [52] p. 38)
Let S an arbitrary nonempty set and define

$$\mathcal{S} := \langle S, \{P^* \mid P \in Rel_{\mathcal{S}}\}, \{f^* \mid f \in Fun_{\mathcal{S}}\}\rangle.$$

Where $Rel_{\mathcal{S}}$ are all relations on S and $Fun_{\mathcal{S}}$ are all (possibly partial) functions on S. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we name

$$\mathfrak{K}^* := \langle \mathbb{K}^*, \{P^* \mid P \in Rel_{\mathbb{K}}\}, \{f^* \mid f \in Fun_{\mathbb{K}}\}\rangle.$$

$\diamond$

Definition 4.2.4 (Formal language, [52] p. 38-39)
Associated with $\mathcal{S}$ is a language $\mathcal{L}_{\mathcal{S}}$ with the alphabet

Logical Connectives:		
	$\wedge$	and
	$\vee$	or
	$\neg$	not
	$\Rightarrow$	implies
	$\Leftrightarrow$	if and only if
Quantifier Symbols:		
	$\forall$	forall
	$\exists$	there exists
Parentheses:	$(\cdot), [\cdot]$	
Variables:	x, y, z	countable collection of letters

$\diamond$

Definition 4.2.5 (Terms, [52] p. 39)
We define a *term of* $\mathcal{L}_{\mathcal{S}}$ as strings of the form:

- each variable is an $\mathcal{L}_{\mathcal{S}}$-term,
- each element $s \in S$ is an $\mathcal{L}_{\mathcal{S}}$-term, called a *constant*,
- if $f \in Fun_{\mathcal{S}}$ is an m-ary function and $\tau_1, \ldots, \tau_m$ are $\mathcal{L}_{\mathcal{S}}$-terms, then $f(\tau_1, \ldots, \tau_m)$ is an $\mathcal{L}_{\mathcal{S}}$-term.

And we call a term *closed* if does not contain a variable. $\diamond$

Definition 4.2.6 (Undefined Terms, [52] p. 39)
A closed term *names* something by the following rules:

- The constant s names itself.
- If $\tau_1, \ldots, \tau_m$ name elements $s_1, \ldots, s_m$ respectively and the m-tuple $(s_1, \ldots, s_m)$ is in the domain of $f \in Fun_{\mathcal{S}}$ then $f(\tau_1, \ldots, \tau_m)$ names $f(s_1, \ldots, s_m)$.
- $f(s_1, \ldots, s_m)$ is undefined, if one of τ_i is undefined, or if the m-tuple is not in the domain of f.

A closed term is *undefined* if it does not name anything. $\diamond$

Definition 4.2.7 (Formulae and Sentences, [52] p. 40-41)
For $\mathcal{L}_\mathcal{S}$ terms τ_i we call strings of the form

$$P(\tau_1, \ldots, \tau_k) \in Rel_\mathcal{S}$$

atomic formulae of $\mathcal{L}_\mathcal{S}$.

Inductively, we define *formulae*:

- Every atomic $\mathcal{L}_\mathcal{S}$-formula is an $\mathcal{L}_\mathcal{S}$-formula.
- If φ and ψ are $\mathcal{L}_\mathcal{S}$-formulae, then so are $\varphi \wedge .\psi, \varphi \vee \psi, \neg\varphi, \varphi \Rightarrow \psi, \varphi \Leftrightarrow \psi$
- If φ is an $\mathcal{L}_\mathcal{S}$-formula, x a variable, $P \in Rel_\mathcal{S}$ an unary relation, then

  $$(\forall x \in P)\varphi, \quad (\exists x \in P)\varphi$$

  are $\mathcal{L}_\mathcal{S}$-formulae. We call P the *bound* of the quantifier.

$\diamond$

Definition 4.2.8 (Bound Variable, [52] p. 41)
We call a variable x *bound* within a formula ψ if it is located within a formula of the form $(\forall x \in P)\varphi$ or $(\exists x \in P)\varphi$ that is part of ψ. If x is not bound, we call it *free.* $\diamond$

Definition 4.2.9 (Sentence, [52] p. 41)
We call a formula *sentence*, if all variables are bound. A defined sentence can be **true** or **false**. We call an atomic formula $P(\tau_1, \ldots, \tau_k)$ an *atomic sentence*, if it is a sentence. $\diamond$

Definition 4.2.10 (Truth, [52] p. 41–42)
Let x be a free variable of a formula φ, then we define

- $(\forall x \in P)\varphi$ is true, if and only if for all $s \in P$ the sentence $\varphi(s)$ is defined and true.
- $(\exists x \in P)\varphi$ is true,if and only if there exists an $s \in P$ such that the sentence $\varphi(s)$ is defined and true.

We define the symbolic connections in the usual way. For formulae φ, ψ

- $\varphi \wedge \psi$ is true, if and only if φ is true and ψ is true
- $\varphi \vee \psi$ is true, if and only if φ is true or ψ is true
- $\neg\varphi$ is true if and only if φ is is not true (*i.e.* false)

- $\varphi \Rightarrow \psi$ is true, if and only if φ implies ψ (*i.e.* $(\neg\varphi) \vee \psi$ is true)
- $\varphi \Leftrightarrow \psi$ is true, if and only if φ implies ψ and ψ implies φ

and finally

- $P(\tau_1, \ldots, \tau_k)$ is true, if and only if the closed terms $\tau_1, \ldots, \tau_k$ are all defined and the k-tuple they name is element of P.

◇

Example 4.2.11
The following sentence is true:

$$(\forall x \in \mathbb{R})\ [\exp(x + x) = \exp(x) \cdot \exp(x)]$$

◇

Definition 4.2.12 (Star Transform, partially [52] p. 42-43)
We define the *star transform* $*$ inductively:

- If τ is a variable on $\mathcal{L}_{\mathfrak{K}}$, then τ^* is just τ
- If τ is a function $f(\tau_1, \ldots, \tau_m)$ then τ^* is $f^*(\tau_1^*, \ldots, \tau_m^*)$

The $*$-transform φ^* of an $\mathcal{L}_{\mathfrak{K}}$-formula φ is defined by

- replace each term τ in φ by τ^*,
- replace the relation P of an atomic formula in φ by P^*,
- replace the bound P of any quantifier in φ by P^*.

Spelling this out more explicitly:

$$\begin{aligned}
(P(\tau_1,\dots,\tau_k))^* &:= P(\tau_1^*,\dots,\tau_k^*)\\
(\varphi \wedge \psi)^* &:= \varphi^* \wedge \psi^*\\
(\varphi \vee \psi)^* &:= \varphi^* \vee \psi^*\\
(\neg\varphi)^* &:= \neg(\varphi^*)\\
(\varphi \Rightarrow \psi)^* &:= \varphi^* \Rightarrow \psi^*\\
(\varphi \Leftrightarrow \psi)^* &:= \varphi^* \Leftrightarrow \psi^*\\
((\forall x \in P)\varphi)^* &:= (\forall x \in P^*)\varphi^*\\
((\exists x \in P)\varphi)^* &:= (\exists x \in P^*)\varphi^*
\end{aligned}$$

$\diamond$

Remark. We will drop the $*$ whenever the context admits to do so. For example for the transform f^* of a function f:

$$(\forall x \in \mathbb{R} : \exp(x + x) = \exp(x) \cdot \exp(x))^*$$

we will simply write

$$\forall x \in \mathbb{R}^* : \exp(x + x) = \exp(x) \cdot \exp(x).$$

Theorem 4.2.13 (Transfer Principle, [52] p. 47)
For any $\mathcal{L}_{\mathfrak{K}}$-formula $\varphi(x_1,\dots,x_p)$ and any $z^1,\dots,z^p \in \mathbb{K}^{\mathbb{N}}$ the sentence $\varphi([z^1],\dots,[z^p])^*$ is true, if and only if $\varphi(z_n^1,\dots,z_n^p)$ is true for almost all $n \in \mathbb{N}$.

Proof. The proof uses Łoś theorem (pronounced "wash") which can be found in its general form in the book of Burris and Sankappanavar [15] p. 210. $\square$

Remark. This theorem might look inconspicuous in the first place but this is very deep and powerful result. It is the justification for the universal (Intuition 4.2.1 (Universal Transfer)) and existential transfer (Intuition 4.2.2 (Existential Transfer)).

Definition 4.2.14 (Universal and existential transfer)
In Theorem 4.2.13 (Transfer Principle) we call the implication "$\Leftarrow$" *universal transfer* and the implication "$\Rightarrow$" *existential transfer*. This is just the more formal version of Intuition 4.2.1 (Universal Transfer) and Intuition 4.2.2 (Existential Transfer). $\diamond$

We will show the power of the transfer principle in an easy example: the fundamental theorem of algebra.

Theorem 4.2.15 (Fundamental Theorem of Algebra)
Every non–constant polynomial over the hypercomplex numbers has at least one root.

Proof. The following sentence is true:

$$\forall f \in \mathbb{C}[z] \text{ with } \deg(f) > 0 : \exists z_0 \in \mathbb{C} \text{ with } f(z_0) = 0.$$

This is the fundamental theorem of algebra (see for example [38] p. 66). Then by transfer the following sentence is true:

$$\forall f \in \mathbb{C}^*[z] \text{ with } \deg(f) > 0 : \exists z_0 \in \mathbb{C}^* \text{ with } f(z_0) = 0.$$

$\square$

4.3. Basic Operations on $\mathbb{K}^*$

> Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können.
>
> David Hilbert, Über das Unendliche [63]

In the following we will introduce basic concepts of the algebra of non-standard analysis.

Definition 4.3.1 ($\mathbb{K}^*$ sets, partially [52] p. 50)

$$
\begin{aligned}
\mathbb{I}_{\mathbb{R}} &:= \{\epsilon \in \mathbb{R}^* : \; |\epsilon| < |r| \quad \forall r \in \mathbb{R}\}, \quad \textit{real infinitesimal numbers} \\
\mathbb{I}_{\mathbb{C}} &:= \{\epsilon \in \mathbb{C}^* : \; |\epsilon| < |r| \quad \forall r \in \mathbb{R}\}, \quad \textit{complex infinitesimal numbers} \\
\mathbb{A}_{\mathbb{R}} &:= \{r^* \in \mathbb{R}^* : \exists n \in \mathbb{N} : \frac{1}{n} < |r^*| < n\}, \quad \textit{real appreciable numbers} \\
\mathbb{A}_{\mathbb{C}} &:= \{r^* \in \mathbb{C}^* : \exists n \in \mathbb{N} : \frac{1}{n} < |r^*| < n\}, \quad \textit{complex appreciable numbers} \\
\mathbb{R}_{\infty}^{+} &:= \{H \in \mathbb{R}^* : \; H > r \quad \forall r \in \mathbb{R}\}, \quad \textit{positive unlimited numbers} \\
\mathbb{R}_{\infty}^{-} &:= \{H \in \mathbb{R}^* : \; H < r \quad \forall r \in \mathbb{R}\}, \quad \textit{negative unlimited numbers} \\
\mathbb{R}_{\infty} &:= \mathbb{R}_{\infty}^{+} \cup \mathbb{R}_{\infty}^{-}, \quad \textit{real unlimited numbers} \\
\mathbb{C}_{\infty} &:= \mathbb{C}^* \setminus \{\mathbb{I}_{\mathbb{C}} \cup \mathbb{A}_{\mathbb{C}}\}, \quad \textit{complex unlimited numbers} \\
\mathbb{L}_{\mathbb{R}} &:= \mathbb{R}^* \setminus \mathbb{R}_{\infty}, \quad \textit{real limited numbers} \\
\mathbb{L}_{\mathbb{C}} &:= \mathbb{C}^* \setminus \mathbb{C}_{\infty}, \quad \textit{complex limited numbers}
\end{aligned}
$$

$\diamond$

Lemma 4.3.2 (Subsets)

$$\mathbb{I}_{\mathbb{R}} \subset \mathbb{I}_{\mathbb{C}}, \quad \mathbb{A}_{\mathbb{R}} \subset \mathbb{A}_{\mathbb{C}}, \quad \mathbb{L}_{\mathbb{R}} \subset \mathbb{L}_{\mathbb{C}}$$

Proof. Remember the construction of $\mathbb{R}^*$ and $\mathbb{C}^*$ in Definition 4.1.5 (Hyperreal and Hypercomplex Numbers):

$$\mathbb{R}^* = \{[r] \mid r \in \mathbb{R}^{\mathbb{N}}\}, \quad \mathbb{C}^* = \{[r] \mid r \in \mathbb{C}^{\mathbb{N}}\}.$$

since $\mathbb{R}^{\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ it is obvious that $\mathbb{R}^* \subset \mathbb{C}^*$. The other properties follow immediately by definition. $\square$

Remark. We will refer to $\mathbb{I}, \mathbb{A}$ or $\mathbb{L}$ as the corresponding complex or real sets when the context is obvious.

Theorem 4.3.3 (Arithmetics, partially [52] p. 50-51)
Let $n \in \mathbb{N}$, ϵ, δ be infinitesimal, b, c appreciable and H, K unlimited hypercomplex numbers. Then it holds true:

- Sums

  $\epsilon + \delta$ is infinitesimal

$b + \epsilon$ is appreciable

$b + c$ is limited

$H + \epsilon$ and $H + b$ are unlimited

- Additive inverse

  $-\epsilon$ is infinitesimal

  $-b$ is appreciable

  $-H$ is unlimited

- Products

  $\epsilon \cdot \delta$ and $\epsilon \cdot b$ are infinitesimal

  $b \cdot c$ is appreciable

  $b \cdot H$ and $H \cdot K$ are unlimited

- Reciprocals

  $\frac{1}{\epsilon}$ is unlimited if $\epsilon \neq 0$

  $\frac{1}{b}$ is appreciable

  $\frac{1}{H}$ is infinitesimal

- Quotiens

  $\frac{\epsilon}{b}$, $\frac{\epsilon}{H}$ and $\frac{b}{H}$ are infinitesimal

  $\frac{b}{c}$ is appreciable

  $\frac{b}{\epsilon}$, $\frac{H}{\epsilon}$ and $\frac{H}{b}$ are unlimited

- Real roots, let $\epsilon \in \mathbb{I}_{\mathbb{R}}, b \in \mathbb{A}_{\mathbb{R}}, H \in \mathbb{R}_{\infty}^{+}$

  If $\epsilon > 0$, $\sqrt[n]{\epsilon}$ is infinitesimal

  If $b > 0$, $\sqrt[n]{b}$ is appreciable

  If $H > 0$, $\sqrt[n]{H}$ is unlimited

- Undetermined forms

  $\frac{\epsilon}{\delta}, \frac{H}{K}, \epsilon \cdot H, H + K$ are undetermined

Remark. Undetermined does not mean undefined in this context. Take for example the hyperreals $\epsilon = \langle \frac{1}{n} \mid n \in \mathbb{N} \rangle$ and $\delta = \langle \frac{1}{n^2} \mid n \in \mathbb{N} \rangle$: both are obviously infinitesimal. On the one hand we have $\frac{\epsilon}{\delta} = \frac{1}{\epsilon}$, and on the other we find $\frac{\epsilon}{\epsilon} = \mathbf{1}$. So for the undetermined forms it depends not only on the class of the numbers (infinitesimal or unlimited) but also on the concrete number itself.

Definition 4.3.4 (Infinitely Close and Limited Distance, partially [52] p. 52)
Define for $b, c \in \mathbb{K}^*$ the equivalence relation

$$b \simeq c \text{ if and only if },b - c \in \mathbb{I}$$

and we call b is *infinitely close* to c. Furthermore, we write

$$b \sim c \text{ if and only if },b - c \in \mathbb{L}$$

and we say b has *limited distance* to c. ◇

Definition 4.3.5 (Halo and Galaxy, partially [52] p. 52)
Define the *halo* of an arbitrary $b \in \mathbb{K}^*$ as

$$\mathbf{hal}(b) := \{c \in \mathbb{K}^* : b \simeq c\}$$

and the *galaxy* of $b \in \mathbb{K}^*$ as

$$\mathbf{gal}(b) := \{c \in \mathbb{K}^* : b \sim c\}.$$

◇

Theorem 4.3.6 (Shadow)
Every limited hyperreal (hypercomplex) z^* is infinitely close to exactly one real (complex) number. We call this the *shadow* of z^* denoted by $\mathbf{sh}(z^*)$.

Proof. The real case is proven in [52] p. 53. We will prove the complex case using the constructions of the real case for both real and imaginary part. Then we use an estimate to conclude.

Write $z^* = a + \mathrm{i} \cdot b$, then we have $a, b \in \mathbb{L}_{\mathbb{R}}$ (otherwise z^* would not be limited). Define the following sets:

$$A := \{r \in \mathbb{R} \mid r < a\}, \quad B := \{r \in \mathbb{R} \mid r < b\}.$$

And set $\alpha := \sup A$ and $\beta := \sup B$. By the completeness of $\mathbb{R}$ we know that $\alpha, \beta \in \mathbb{R}$. Let $z := \alpha + \mathrm{i} \cdot \beta$. We first show that $z^* - z \in \mathbb{L}_{\mathbb{C}}$: take any $\epsilon > 0$. Since α is an upper bound of A we know $\alpha + \epsilon \notin A \Rightarrow a \leq \alpha + \epsilon$. Furthermore, $\alpha - \epsilon < a$, hence otherwise we have $a \leq \alpha - \epsilon$ which would be a lower upper bound of A which is a contradiction to the construction of α. So we have

$$\alpha - \epsilon < a \leq \alpha + \epsilon \Leftrightarrow |a - \alpha| \leq \epsilon.$$

We can argue the same way for b and have $|b - \beta| \leq \epsilon'$ for some $\epsilon' > 0$. Finally, we can conclude

$$|z^* - z| = \sqrt{(a-\alpha)^2 + (b-\beta)^2} \leq |a - \alpha| + |b - \beta| \leq \epsilon + \epsilon' =: \hat{\epsilon},$$

where we used that $\|x\|_1 \geq \|x\|_2 \, \forall x \in \mathbb{R}$ and by universal transfer also for all $x \in \mathbb{R}^*$. Since this holds for all $\hat{\epsilon} > 0$ we know that z^* and z are infinitesimal close.

We still have to show the uniqueness of z. Assume there is another $z' \in \mathbb{C}$ with the same property. Then $z^* \simeq z'$ and therefore $z \simeq z'$. Since both z and z' are complex numbers this means $z = z'$. □

Lemma 4.3.7 (Complex Shadow)
For $z \in \mathbb{L}_{\mathbb{C}}$ and $z = a + \mathrm{i} \cdot b$ we have

$$\mathbf{sh}(z) = \mathbf{sh}(a) + \mathrm{i} \cdot \mathbf{sh}(b)$$

Proof. Direct consequence of the construction in the proof of Theorem 4.3.6 (Shadow). □

Lemma 4.3.8 (Complex Shadow and Conjugation)
Let $z \in \mathbb{L}$, then it holds true that $\mathbf{sh}(\overline{z}) = \overline{\mathbf{sh}(z)}$.

Proof. Apply Lemma 4.3.7 (Complex Shadow). □

Definition 4.3.9 (Almost Real)
We call a number $z \in \mathbb{C}^*$ *almost real*, if z is infinitely close to a real number. That means there exists a $r \in \mathbb{R} : z \simeq r$. ◇

Lemma 4.3.10 (Shadow Properties)
Let $a, b \in \mathbb{L}$ and $n \in \mathbb{N}$ then

1. $\mathbf{sh}(a \pm b) = \mathbf{sh}(a) \pm \mathbf{sh}(b)$
2. $\mathbf{sh}(a \cdot b) = \mathbf{sh}(a) \cdot \mathbf{sh}(b)$

3. $\mathbf{sh}(\frac{a}{b}) = \frac{\mathbf{sh}(a)}{\mathbf{sh}(b)}$, if $\mathbf{sh}(b) \neq 0$
4. $\mathbf{sh}(b^n) = \mathbf{sh}(b)^n$
5. $\mathbf{sh}(|b|) = |\,\mathbf{sh}(b)|$
6. for $a, b \in \mathbb{L}_{\mathbb{R}}$: if $a \leq b$ then $\mathbf{sh}(a) \leq \mathbf{sh}(b)$

Proof. For the real case see [52] p. 53-54. For the complex case we can reduce this to the real case by Lemma 4.3.7 (Complex Shadow). □

Theorem 4.3.11 (Isomorphism, partially [52] p. 54)
The quotient ring $\mathbb{L}_{\mathbb{R}}/\mathbb{I}_{\mathbb{R}}$ ($\mathbb{L}_{\mathbb{C}}/\mathbb{I}_{\mathbb{C}}$) is isomorphic to the field of the real (complex) numbers by $\mathbf{hal}(b) \mapsto \mathbf{sh}(b)$. Therefore $\mathbb{I}$ is a maximal ideal of the ring $\mathbb{L}$.

Proof. For the real case [52] p. 54. For the complex case, note that $\mathbb{C}^* \simeq \mathbb{R}^* \times \mathbb{R}^*$ and apply the same arguments as for the real case. □

Theorem 4.3.12 (Continuity, partially [52] p. 75)
The function $f : \mathbb{C} \to \mathbb{C}$ is continuous at $c \in \mathbb{C}$, if and only if $f(c) \simeq f(x)$ for all $x \in \mathbb{C}^*$ such that $x \simeq c$. In other words if and only if

$$f(\mathbf{hal}(c)) \subset \mathbf{hal}(f(c)).$$

Proof. The real case is again in [52] p. 75. The complex case reads analogously since the definition of continuity is essentially the same only with a different definition of the absolute value. □

Remark. This property will be crucial later on to resolve singularities in geometric constructions.

Lemma 4.3.13 (Real Limits, [52] p. 78)
For $c, L \in \mathbb{R}$ and f be defined on $A \subset \mathbb{R}$ then it holds true:

$$\begin{aligned}
\lim_{x \to c} f(x) = L &\Leftrightarrow f(x) \simeq L \quad \forall x \in A^* : x \simeq c,\ x \neq c \\
\lim_{x \to c^+} f(x) = L &\Leftrightarrow f(x) \simeq L \quad \forall x \in A^* : x \simeq c,\ x > c \\
\lim_{x \to c^-} f(x) = L &\Leftrightarrow f(x) \simeq L \quad \forall x \in A^* : x \simeq c,\ x < c
\end{aligned}$$

Remark. For a subset A of a topological space X we denote by $\mathbf{Int}(A)$ the *interior of* A, *i.e.* the union of all open set contained in A. For details see [97] p. 95ff.

Lemma 4.3.14 (Squeezing Limits)
Let $A \subset \mathbb{R}, c \in \mathbf{Int}(A)$, $L \in \mathbb{C}$ and let $f : A \setminus \{c\} \to \mathbb{C}$ be continuous. If there exists a $\Delta x \in \mathbb{I}_{\mathbb{R}} \setminus \{0\}, \Delta x > 0$ such that $f(c + \Delta x) \simeq f(c - \Delta x) \simeq L$, then the function f can be continuously extended on A with $f(c) = L$.

Proof. Define the sets $A^+ = \{a \in A \mid a > c\}$, $A^- = \{a \in A \mid a < c\}$ and $H^\circ := \mathbf{hal}(c) \setminus \{c\}$. Since f is continuous it holds true by Lemma 4.3.13 (Real Limits) for all $h^+ \in f(H^\circ \cap (A^+)^*)$ that $h^+ \simeq f(c + \Delta x)$ and furthermore for $h^- \in f(H^\circ \cap (A^-)^*)$ that $h^- \simeq f(c - \Delta x)$. As by assumption $f(c + \Delta x) \simeq f(c - \Delta x)$ it holds true that $h \simeq f(c \pm \Delta x)$ for all $h \in f(H^\circ)$.

Now we are almost done: by Lemma 4.1.12 (Enlarging Rules) it holds true that $(A \setminus \{c\})^* = A^* \setminus \{c\}$. So if we want to extend f to whole A^* we only have to define a value for $f(c)$: we define $f(c) := L$. Since $f(c \pm \Delta x) \simeq L = f(c)$ it holds true that $f(x) \simeq L$ for all $x \in \mathbf{hal}(c)$. That is equivalent with the continuity of f at c by Theorem 4.3.12 (Continuity). □

Remark. The last lemma is very useful since it gives us a recipe to continuously extend a function! If we have a discontinuity of f at some point c, it is removable if and only if the shadows of $f(c + \Delta x)$ and $f(c - \Delta x)$ coincide and we can continue the function with the calculated shadow.

Lemma 4.3.15 (Complex Limits, partially [52] p. 78)
Let $c, L \in \mathbb{C}$ and f be defined on $A \subset \mathbb{C}$. Then

$$\lim_{x \to c} f(x) = L \Leftrightarrow f(x) \simeq L \quad \forall x \in A^* : x \simeq c,\ x \neq c.$$

Proof. Essentially, the proof is based on the equivalence of the Weierstrass $\epsilon - \delta$ continuity and the limit definition of continuity. The theorem is stated for the real case in [52] p. 78. The proof by transfer can be found on p. 75–76. The complex version reads completely analogously. □

The next theorem will give important facts about the topology. It turns out that a set is open if and only if it includes the halo of all its points.

Remember that a set A is open if and only if $\mathbf{Int}\, A = A$ (see [97] p. 95).

Theorem 4.3.16 (Topology, partially [52] p. 114-117)
If $A \subset \mathbb{C}$ and $z \in A$, then

$$
\begin{aligned}
z \in \mathbf{Int}\, A \;\Leftrightarrow\; \forall x : z \simeq x \Rightarrow x \in A^* \;\Leftrightarrow\; \mathbf{hal}(z) \subset A^* \\
\mathbf{hal}(z) = \bigcap \{A^* \mid z \in A \text{ and } A \text{ is open}\} \\
A \text{ is compact } \Leftrightarrow A^* \subset \bigcup_{a \in A} \mathbf{hal}(a)
\end{aligned}
$$

Definition 4.3.17 (Topological Halo, [21] chapter 3)
Let $(X, \mathcal{O})$ be a standard topological space. For $p \in X$ denote by $\mathcal{O}_p := \{O \in \mathcal{O} \mid p \in O\}$ the open neighborhood of p. We define the *topological halo of* p by

$$\mathbf{hal}(p) := \bigcap \{O^* \mid O \in \mathcal{O}_p\}$$

and we write $q \simeq p$ for $q \in \mathbf{hal}(p)$. ◇

Lemma 4.3.18 (Halo and a Metric, [21] p. 89)
If d is a standard metric on X then it holds true

$$\mathbf{hal}(x) := \{y \in X \mid d(x, y) \simeq 0\}.$$

And so we can write $q \simeq p$ for $p, q \in X^*$ if and only if $d(p, q) \simeq 0$.

Theorem 4.3.19 (Topological Non-standard Continuity, [21] p. 79)
Let f map X to Y where X, Y are both topological spaces. Let $p \in X$. Then f is continuous at p if and only if

$$q \simeq p \Rightarrow f(q) \simeq f(p).$$

Remark. The characterization of continuity in the previous theorem reads exactly analogously to the definition of continuity in Theorem 4.3.12 (Continuity) and this is completely canonical since the previous theorem generalizes infinitesimal proximity to arbitrary topological spaces. This can be used later on to define continuity directly in the projective space.

Lemma 4.3.20 (Constant on Open Sets)
Let $A \subset \mathbb{C}$ open, $c \in A$ and $f : A \to \mathbb{C}$ be a function on A.

1. If f is constant on $\mathbf{hal}(c)$ then there is an open set $A' \subset A$ with $c \in A'$ such that f is constant on A'.

2. If f is constant on on A then f is also constant on A^*.

Proof. We prove 1: write the statement using the formal language $\mathcal{L}_{\mathfrak{C}}$. If f is constant on $\mathbf{hal}(c)$ the following statement is true:

$$\exists\delta \in (\mathbb{R}^+)^* : \forall x \in \mathbb{C}^* : (|x-c| < \delta \Rightarrow f(x) = f(c)) .$$

Now we apply existential transfer and find the next statement also true:

$$\exists\delta \in \mathbb{R}^+ : \forall x \in \mathbb{C} : (|x-c| < \delta \Rightarrow f(x) = f(c))$$

Which says that there is an open set $A' := \{x \in A \mid |x-c| < \delta\}$ such that f is constant on A'.

Now we prove 2: if f is constant on A, then the following sentence is true: there is $d \in \mathbb{C}$ such that

$$\forall x \in A : f(x) = d$$

Then apply universal transfer:

$$\forall x \in A^* : f(x) = d$$

which is the claim. □

Remark. As the converse argument this also implies that if a function f does not vanish on the $\mathbf{hal}(c)$ then there is an $\epsilon > 0, \epsilon \in \mathbb{R}$ such that the function f also does not vanish on $B_\epsilon(c)$.

Theorem 4.3.21 (Identity Theorem, [1] p. 122)
Let each of two functions $f(z)$ and $g(z)$ be analytic on a common domain D. If $f(z)$ and $g(z)$ coincide on a open subset $D' \subset D$ or on a curve Γ interior to D, then $f(z) = g(z)$ everywhere in D.

Remark. A similar statement holds for real analytic function, see the book by Krantz and Parks: "A Primer of Real Analytic Functions" ([81] p. 14).

Lemma 4.3.22 (Constant)
Let D be a domain and $z_0 \in D$. If an analytic function $f : D \to \mathbb{C}$ is constant on $\mathbf{hal}(z_0)$ then it constant on D.

Proof. By Lemma 4.3.20 (Constant on Open Sets) we know that if f is constant on a $\mathbf{hal}(z_0)$ then there is an open set $D' \subset D$ where f is constant. By Theorem 4.3.21

(Identity Theorem) the claim follows immediately. □

4.4. Derivatives and Newton Quotients

> You take a function of x and you call it y,
> Take any x-nought that you care to try,
> Make a little change and call it delta-x,
> The corresponding change in y is what you find nex',
> And then you take the quotient, and now carefully
> Send delta-x to zero and I think you'll see,
> That what the limit gives us, if our work all checks,
> Is what we call dy/dx,
> it's just dy/dx.
>
> Tom Lehrer, The Derivative Song [83]

One particular interesting property of non-standard analysis is that the differential quotient of a differentiable function f with an infinitesimal ϵ

$$\frac{f(z+\epsilon)-f(z)}{\epsilon}$$

is infinitely close to the derivative of a function. In the following we will illustrate how differentiation, as practiced by Newton and Leibniz, can be put on a solid mathematical (or better say logical) basis.

Theorem 4.4.1 (Derivative, partially [52] p. 91)
If f is defined at $z \in \mathbb{C}$ then $L \in \mathbb{C}$ is the *derivative* at of f at z if and only if for every nonzero infinitesimal $\epsilon \in \mathbb{I}_{\mathbb{C}}$ the term $f(x+\epsilon)$ is defined and

$$\frac{f(z+\epsilon)-f(z)}{\epsilon} \simeq L \tag{4.1}$$

When f is complex differentiable at z we have

$$f'(z) = \mathbf{sh}\left(\frac{f(z+\epsilon)-f(z)}{\epsilon}\right)$$

Proof. [52] p. 91/92. □

Definition 4.4.2 (Newton Quotient and Increment, [52] p. 92)
We define the *increment* of a function f for an arbitrary infinitesimal Δx induced by

the variable x as

$$\Delta f := f(x + \Delta x) - f(x)$$

Then the *Newton quotient (NQ)* is defined by $\frac{\Delta f}{\Delta x}$ and furthermore

$$\frac{\Delta f}{\Delta x} \simeq f'(x)$$

if f is differentiable at x for arbitrary Δx. $\diamond$

Theorem 4.4.3 (Taylor Series, [52] p. 102)
If the n'th derivative $f^{(n)}$ exists on an open set containing the complex number z and $f^{(n)}$ is continuous at z, then for any complex infinitesimal Δz it holds true

$$f(x + \Delta z) = f(z) + f'(z)\Delta z + \frac{f''(z)}{2!} + \cdots + \frac{f^{(n)}(z)}{n!} + \epsilon \Delta x^n$$

for some infinitesimal ϵ.

Proof. The real version is shown in [52] p. 100 ff. The complex proof is completely analogously. $\square$

4.5. Some Useful Series Expansions

> Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.
>
> Benoît Mandelbrot, The Fractal Geometry of Nature [90]

As we will see later, it can become useful to consider a certain class of hypercomplex numbers. Let us assume we are given an ascending ordered set of rational numbers $M := \{q_i \mid q_i \in \mathbb{Q}, i \in \mathbb{N}, q_i < q_j \forall i < j\}$, where we assume that M is left–finite, *i.e.* it has a least element q_0. For given complex coefficients $\{a_{q_i} \mid a_{q_i} \in \mathbb{C}, i \in \mathbb{N}\}$ and for an arbitrary positive infinitesimal hyperreal Δx we write $z = \sum_{k=0}^{\infty} a_{q_k} \cdot \Delta x^{q_k}$. If we factor out the (in absolute value) largest nonzero element of the sum (*w.l.o.g.* let $a_{q_0} \neq 0$) we see that

$$z = a_{q_o} \cdot \Delta x^{q_0}(1 + \sum_{k=1}^{\infty} \frac{a_{q_k}}{a_{q_0}} \cdot \Delta x^{q_k - q_0}) =: a_{q_o} \cdot \Delta x^{q_0}(1 + x).$$

So we can write $z = y \cdot (1+x)$ where $y \in \mathbb{C}^*$ and $x \in \mathbb{I}_{\mathbb{C}}$ is a sum of decreasing infinitesimal hypercomplex numbers. We will exploit that x is infinitesimal several times.

Theorem 4.5.1 (Homogeneity for Certain Radical Expressions)
Let $n \in \mathbb{N}$, $x, y \in \mathbb{C}^*$ with the properties that x is infinitely close to 0 and y is not in the halo of a negative real number, *i.e.* $x \in \mathbf{hal}(0)$ and $y \in \mathbb{C}^* \setminus \left(\bigcup_{r \in (-\infty,0)} \mathbf{hal}(r)\right)$. Then it holds true:

$$\sqrt[n]{y \cdot (1+x)} = \sqrt[n]{y} \cdot \sqrt[n]{1+x}$$

Proof. Since $x \in \mathbf{hal}(0)$ we can write $1+x$ using polar coordinates by $1+x = r_x \cdot \mathrm{e}^{\theta_x}$ with $r_x \simeq 1$ and $\theta_x \simeq 0$. Analogously we can write $y = r_y \cdot \mathrm{e}^{\theta_y}$ for $r_y \in \mathbb{R}^*, r_y \geq 0$ and $\theta_y \in (-\pi, \pi)^*$. Then the product yields:

$$y \cdot (1+x) = r_y \cdot r_x \cdot \mathrm{e}^{\mathrm{i} \cdot \theta_y} \cdot \mathrm{e}^{\mathrm{i} \cdot \theta_x} = r_y \cdot r_x \cdot \mathrm{e}^{\mathrm{i}(\theta_y + \theta_x)}.$$

What does the polar angle of this look like? By assumption, we know that $|\theta_y| < \pi$ (since we excluded the negative real numbers and their halos) and $\epsilon := |\theta_x| \in \mathbb{I}$ (since $1+x$ with $x \in \mathbf{hal}(0)$ has an infinitesimal polar angle).

So it holds true that: $|\theta_y + \theta_x| \leq |\theta_y| + |\theta_x| = |\theta_y| + \epsilon < \pi$ since by construction θ_y can't be infinitely close to $\pm\pi$. This means that we can expand the exponents of the exponential function to two separate exponential functions and do not exceed $|\pi|$.

Taking the n'th root:

$$\begin{aligned} \sqrt[n]{y \cdot (1+x)} &= \sqrt[n]{r_y \cdot r_x \cdot \mathrm{e}^{\mathrm{i}(\theta_y + \theta_x)}} = \sqrt[n]{r_y \cdot r_x} \cdot \mathrm{e}^{\mathrm{i}\frac{(\theta_y + \theta_x)}{n}} = \sqrt[n]{r_y \cdot r_x} \cdot \mathrm{e}^{\mathrm{i}\frac{\theta_y}{n}} \cdot \mathrm{e}^{\mathrm{i}\frac{\theta_x}{n}} \\ &= \left(\sqrt[n]{r_y} \cdot \mathrm{e}^{\mathrm{i}\frac{\theta_y}{n}}\right)\left(\sqrt[n]{r_x} \cdot \mathrm{e}^{\mathrm{i}\frac{\theta_x}{n}}\right) = \sqrt[n]{y} \cdot \sqrt[n]{1+x} \end{aligned}$$

where we used the homogeneity of the hyperreal root and the argument that we cannot exceed $|\pi|$ and cross a branch cut. $\square$

To proceed it is useful to have a closer look on the series expansion of $(1+x)^s, s \in \mathbb{C}$.

Lemma 4.5.2 (Binomial Expressions, [75] p. 115)
For all $s \in \mathbb{C}$ and $x \in (-1, 1)$ holds true:

$$(1+x)^s = \sum_{n=0}^{\infty} \binom{s}{n} x^n$$

where the generalized binomial coefficient is defined, as in [75] p. 34, for all $z \in \mathbb{C}$ and

$k \in \mathbb{Z}$ by:

$$\binom{z}{k} := \begin{cases} \dfrac{z(z-1)\dots(z-k+1)}{k!}, & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ 0, & \text{if } k < 0, \end{cases}$$

The original idea of exploiting the homogeneity, which we will use in the following two lemmas, was already given without proof by B. Crowell and M. Khafateh in their implementation of the Levi-Civita field [19] over the real numbers.

Corollary 4.5.3 (Expansion of Certain Radical Expressions)
Let $x, y \in \mathbb{C}^*$ with $x \in \mathbf{hal}(0)$ and $y \in (\mathbb{C} \setminus [0, \infty))^*$. Then we can write

$$\sqrt[n]{y \cdot (1+x)} = \sqrt[n]{y} \left(\sum_{k=0}^{\infty} \binom{1/n}{k} x^k \right)$$

Proof. Choose x, y according to the prerequisites. Now use Theorem 4.5.1 (Homogeneity for Certain Radical Expressions) to write $\sqrt[n]{y \cdot (1+x)} = \sqrt[n]{y} \cdot \sqrt[n]{1+x}$. Then apply Lemma 4.5.2 (Binomial Expressions) to $(1+x)^{\frac{1}{n}}$ which then is the claim. □

Lemma 4.5.4 (Inversion)
Let $y \in \mathbb{C}^* \setminus \{0\}$ and $x \in \mathbb{I}_{\mathbb{C}}$. Then the inverse of $z := y \cdot (1+x)$ denoted by z^{-1} is given by:

$$z^{-1} = y^{-1} \left(\sum_{k=0}^{\infty} x^k \right)$$

Proof. Let z be defined as in the assumptions. Then $z^{-1} = y^{-1} \cdot (1+x)^{-1}$. Since x is infinitesimal we know that $|x| < 1$. So we can apply Lemma 4.5.2 (Binomial Expressions), for $s = -1$ and find

$$(1+x)^{-1} = \frac{1}{1+x} = \sum_{k=0}^{\infty} x^k$$

multiplying this with the inverse of y gives us the claim. □

Remark. Of course we would not have to employ Lemma 4.5.2 (Binomial Expressions) in the last proof. For $s = -1$ this describes the well known geometric series. Interestingly the partial sum formula of the geometric series was already known to Euclid ([60], proposition 35, p. 420).

One particular interesting and rigorous proof of the theorem on geometric series was given by Grégoire de Saint–Vincent (1584—1667). He showed that the value of the geometric series can be obtained using properties of similar triangles. Details can be read up in [126] p. 244 ff.

5. Non-standard Projective Geometry

Doch manches Rätsel knüpft sich auch.
Laß du die große Welt nur sausen,
Wir wollen hier im stillen hausen.
Es ist doch lange hergebracht,
Daß in der großen Welt man kleine Welten macht.
...
Komm mit! Komm mit! Es kann nicht anders sein,
Ich tret heran und führe dich herein,
Und ich verbinde dich aufs neue.
Was sagst du, Freund? das ist kein kleiner Raum.
Da sieh nur hin! du siehst das Ende kaum.

Johann Wolfgang von Goethe: "Faust. Der Tragödie erster Teil" [51]

We will now introduce the usage of hyperreal and hypercomplex numbers in projective geometry. Surprisingly there is some preliminary work by A. Leitner [84]. Leitner introduces projective space over the hyperreal numbers (not considering complex space) and uses this to prove properties of the diagonal Cartan subgroup of $\mathrm{SL}_n(\mathbb{R})$.

Definition 5.0.1 ($\mathbb{K}^*\mathbb{P}^d$)
Let $d \in \mathbb{N}$. We define $\mathbb{K}^*\mathbb{P}^d$ analogously to $\mathbb{KP}^d$:

$$\mathbb{K}^*\mathbb{P}^d := \frac{\mathbb{K}^{*d} \setminus \{0\}}{\mathbb{K}^* \setminus \{0\}}$$

where $\mathbb{K}^*$ denote the hyper (real or complex) numbers. $\diamond$

Definition 5.0.2 (Standard and Non-standard Projective Geometry)
If we talk about projective geometry in $\mathbb{KP}^d$, we will refer to it as *standard projective geometry* and for results in the extended field $\mathbb{K}^*\mathbb{P}^d$ we will refer to as *non-standard projective geometry.* $\diamond$

Definition 5.0.3 (Similary)
If two vectors $x, y \in \mathbb{KP}^2$ represent the same equivalence class, we will write $x \sim_{\mathbb{KP}^2} y$

and analogously for the enlarged space $\mathbb{K}^*\mathbb{P}^2$ we will write $x \sim_{\mathbb{K}^*\mathbb{P}^2} y$. ◇

Remark. If the situation allows, we will sometimes also drop the "$\sim$" relation symbol and just write "$=$".

Definition 5.0.4 (Representatives)
We call a representative x of $[x] \in \mathbb{K}^*\mathbb{P}^d$

- *limited*, if and only if for all components x_i of x it holds true that $x_i \in \mathbb{L}$.
- *infinitesimal*, if and only if for all components x_i of x it holds true that $x_i \in \mathbb{I}$.
- *appreciable*, if and only if for all components x_i of x it holds true that $x_i \in \mathbb{L}$ and there is at least one component $x_{i'}$ which is appreciable.
- *unlimited*, if and only if for at least one x_i of x it holds true that $x_i \in \mathbb{K}_\infty$.

◇

Remark. We will denote by $\|x\|$ the Euclidean norm $\|x\|_2$ of a vector in $x \in \mathbb{K}^d$ or $x \in \mathbb{K}^{*d}$, if not stated otherwise.

Lemma 5.0.5 (Representatives and Norms)
Let $x \in \mathbb{C}^*\mathbb{P}^d$, then it holds true:

- x is limited, if and only if $\|x\| \in \mathbb{L}$.
- x is infinitesimal, if and only if $\|x\| \in \mathbb{I}$.
- x is appreciable, if and only if $\|x\| \in \mathbb{A}$.
- x is unlimited, if and only if $\|x\| \in \mathbb{K}_\infty$.

Proof. Obvious. □

Definition 5.0.6 (Points and Lines in $\mathbb{K}^*\mathbb{P}^2$)
We define the following sets: the set of points $\mathcal{P}_{\mathbb{K}^*}$ and the set of lines $\mathcal{L}_{\mathbb{K}^*}$ in $\mathbb{K}^*\mathbb{P}^2$ by

$$\mathcal{P}_{\mathbb{K}^*} := \frac{\mathbb{K}^{*3} \setminus \{0\}}{\mathbb{K}^* \setminus \{0\}}, \quad \mathcal{L}_{\mathbb{K}^*} := \frac{\mathbb{K}^{*3} \setminus \{0\}}{\mathbb{K}^* \setminus \{0\}}.$$

◇

This definition is is the logical consequence of the introduction of hyperreal and hypercomplex numbers. But this also entails modifications to the standard operations in projective geometry. For example the cross product of two points defines the connecting line, but the definition will not be independent of the representative any more (as we will see later in Example 5.1.4 (Geometriekalküle)). This is not as bad as it seems on the first sight: it turns out that if representatives are of reasonable length (meaning appreciable), a lot of properties transfer from standard geometry to the non-standard version. Therefore, we will introduce adapted definitions for the basic operations in projective geometry.

Firstly, we want to measure whether two numbers are "approximately" of the same size. We already know the concept of galaxies but this is not that useful in our case. We rather would like to have something which measures if the quotient of two numbers yields an appreciable number, *i.e.* whether the shadow of the quotient is in $\mathbb{C}$ and not 0. Hence, we define the magnitude of complex numbers:

Definition 5.0.7 (Magnitude)
We define the *magnitude* of a number $r \in \mathbb{C}^* \setminus \{0\}$ by

$$\mathbf{mag}(r) := \{s \in \mathbb{C}^* \setminus \{0\} \mid \exists A \in \mathbb{A} : r = A \cdot s\}.$$

$\diamond$

Remark. A. Leitner has a similar concept and calls two numbers of same order if their quotient is appreciable [84].

Example 5.0.8 (Magnitude Examples)
Let $x \in \mathbb{C} \setminus \{0\}, H \in \mathbb{C}_\infty, \epsilon \in \mathbb{I} \setminus \{0\}$

$$\begin{aligned}
&5 \cdot H \notin \mathbf{gal}(H) \quad (5 \cdot H - H = 4 \cdot H \notin \mathbb{L})\\
&5 \cdot H \in \mathbf{mag}(H) \quad (\frac{5 \cdot H}{H} = 5 \in \mathbb{A})\\
&\epsilon^2 \notin \mathbf{mag}(\epsilon) \quad (\frac{\epsilon}{\epsilon^2} = \epsilon \notin \mathbb{A})\\
&\mathbb{L} \setminus \{0\} = \mathbf{mag}(x)
\end{aligned}$$

$\diamond$

Lemma 5.0.9 (Appreciable and Magnitude)
Let $r \in \mathbb{C}^* \setminus \{0\}$ then

$$s \in \mathbf{mag}(r) \Leftrightarrow \mathbf{sh}(\frac{r}{s}) \in \mathbb{C} \setminus \{0\} \Leftrightarrow \mathbf{sh}(\frac{s}{r}) \in \mathbb{C} \setminus \{0\}.$$

Proof. Let $s \in \mathbf{mag}(r)$ then there is an $A \in \mathbb{A}$ such that $r = A \cdot s$, so $A = \frac{r}{s}$. By definition there is a $n \in \mathbb{N}$ such that $\frac{1}{n} < |A| < n$ *i.e.* there is a $x \in \mathbb{C} \setminus \{0\}$ such that $\mathbf{sh}(A) = x$. Furthermore the inverse $A^{-1} \in \mathbb{A}$ since $\frac{1}{n} < |\frac{1}{A}| < n$ just by definition of the reciprocal. □

Lemma 5.0.10 (Magnitude Lemmas)
Let $r, \lambda \in \mathbb{C}^* \setminus \{0\}$ and $\sigma \in \mathbb{A}$, then

$$\mathbf{mag}(r) = \mathbf{mag}(\sigma \cdot r), \tag{5.1}$$

$$\mathbf{mag}(r)^{-1} := \{\frac{1}{s} \mid s \in \mathbf{mag}(r)\} = \mathbf{mag}(r^{-1}), \tag{5.2}$$

$$\mathbf{mag}(\lambda \cdot r) = \lambda \cdot \mathbf{mag}(r). \tag{5.3}$$

Proof. (5.1): $s \in \mathbf{mag}(r) \Leftrightarrow \exists A \in \mathbb{A} : r = A \cdot s \Leftrightarrow \sigma \cdot r = \sigma \cdot A \cdot s$. Since $A, \sigma \in \mathbb{A}$ their product $A' := \sigma \cdot A \in \mathbb{A}$ too, which means $\sigma \cdot r = A' \cdot s$. Therefore $s \in \mathbf{mag}(\sigma \cdot r) \Leftrightarrow s \in \mathbf{mag}(r)$.
(5.2): Let $s \in \mathbf{mag}(r^{-1}) \Leftrightarrow \exists A \in \mathbb{A} : r^{-1} = A \cdot s \Leftrightarrow r = A^{-1}s^{-1} \Leftrightarrow s^{-1} \in \mathbf{mag}(r)$.
(5.3):

$$\begin{aligned}
\lambda \cdot \mathbf{mag}(r) &= \lambda \cdot \{s \in \mathbb{C}^* \setminus \{0\} \mid \exists A \in \mathbb{A} : r = s \cdot A\} \\
&= \{\underbrace{\lambda \cdot s}_{=:s'} \in \mathbb{C}^* \setminus \{0\} \mid \exists A \in \mathbb{A} : r = s \cdot A\} \\
&= \{s' \in \mathbb{C}^* \setminus \{0\} \mid \exists A \in \mathbb{A} : r = s' \cdot \lambda^{-1} \cdot A\} \\
&= \{s' \in \mathbb{C}^* \setminus \{0\} \mid \exists A \in \mathbb{A} : \lambda \cdot r = s' \cdot A\} \\
&= \mathbf{mag}(\lambda \cdot r)
\end{aligned}$$

□

Lemma 5.0.11 (Magnitude and Limited Numbers)
Let $r, \lambda \in \mathbb{C}^* \setminus \{0\}$ and $s \in \mathbf{mag}(r)$ then

$$s \in \mathbf{mag}(\lambda \cdot r) \Leftrightarrow \lambda \in \mathbb{A}$$

Proof. "$\Leftarrow$" we already showed in Lemma 5.0.10 (Magnitude Lemmas).
"$\Rightarrow$" Let $s \in \mathbf{mag}(\lambda \cdot r) \Leftrightarrow \exists A \in \mathbb{A} : \lambda \cdot r = A \cdot s \Leftrightarrow \frac{r}{s} = \frac{A}{\lambda}$. If now $\lambda \in \mathbb{I}$ the quotient $\frac{A}{\lambda} \in \mathbb{C}_\infty$ and if $\lambda \in \mathbb{C}_\infty$ we have $\frac{A}{\lambda} \in \mathbb{I}$ (both by Theorem 4.3.3 (Arithmetics)). This is a contradiction to the assumption.

The only case left is the appreciable one: since the reciprocal of a appreciable number is appreciable, also Theorem 4.3.3 (Arithmetics), the claim follows immediately. $\square$

Theorem 5.0.12 (Limited and Appreciable Entries)
Let $x = (x_1, \dots, x_d)^T \in \mathbb{C}^*\mathbb{P}^d$ and let $\lambda \in \mathbf{mag}(\|x\|)$ then all entries of $\lambda^{-1}x$ are limited and at least one is appreciable.

Proof. Let $\hat{x}$ be the (entry wise) absolute maximum of x. Then we know that by Theorem 4.2.13 (Transfer Principle) that the Cauchy-Schwarz inequality holds true:

$$|\hat{x}| = \sqrt{\hat{x} \cdot \overline{\hat{x}}} \leq \sqrt{x_1 \cdot \overline{x_1} + \ldots + x_d \cdot \overline{x_d}} = \|x\| \,.$$

Since $\lambda \in \mathbf{mag}(\|x\|)$ there is an $A \in \mathbb{A}$ such that $\|x\| = |A| \cdot |\lambda|$. So $|\hat{x}\lambda^{-1}| \leq \|x\| \, |\lambda|^{-1} = |A|$ which means the $\hat{x}\lambda^{-1}$ has to be limited. Since $\hat{x}$ was the absolute maximum all other entries have to be limited, too.

Now we prove that at least one entry has to be appreciable and at least $\hat{x}\lambda^{-1}$ has to have this property (otherwise all entries are infinitesimal). By the arguments above we know that $\hat{x}\lambda^{-1}$ is limited, therefore we have to show that $\hat{x}\lambda^{-1}$ can't be infinitesimal. Assume the converse. Then all entries of the vector $x\lambda^{-1}$ have to be infinitesimal by Lemma 5.0.5 (Representatives and Norms). And so the norm $\|x\lambda^{-1}\|$ is also infinitesimal equal to an $\epsilon \in \mathbb{I}$. Then consider

$$1 = \left\| \frac{x\lambda^{-1}}{\|x\lambda^{-1}\|} \right\| = \left\| \frac{x\lambda^{-1}}{\|x\| \, |\lambda^{-1}|} \right\| = \left\| \frac{x\lambda^{-1}}{|A||\lambda||\lambda^{-1}|} \right\| = \left\| \frac{x\lambda^{-1}}{|A|} \right\|$$

Therefore it holds true $|A| = \|x\lambda^{-1}\| = |\epsilon|$ which is a contradiction to $A \in \mathbb{A}$. $\square$

The last theorem motivates the definition of an appreciable representative:

Definition 5.0.13 (Appreciable Representative)
For an element $x = (x_1, \dots, x_d)^T \in \mathbb{C}^*\mathbb{P}^d$ a *appreciable representative* is defined as

$$x_{\mathbb{A}} := \frac{1}{\lambda} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

with $\lambda \in \mathbf{mag}(\|x\|)$. Then all entries of $x_{\mathbb{A}}$ are limited and at least one entry is appreciable. ◇

Remark. While the notion of the magnitude might look a bit cumbersome, at first it is actually a fairly easy concept. The magnitude of the norm of a vector is just a way to ensure that we rescale the vector in a way that it is appreciable, meaning that there is a shadow in $\mathbb{C}^{d+1} \setminus \{0\}$ which yield a viable representative of a equivalence class in $\mathbb{CP}^d$.

It would be possible to divide by the Euclidean norm of the vector and have a vector of length one, but the magnitude gives you more flexibility. Still it is sometimes preferable to divide, for example, by the absolute maximum of a vector, which is in the magnitude of the Euclidean norm.

Lemma 5.0.14 (Equivalence of Euclidean and Maximum Norm)
For $z \in \mathbb{C}^{*d}$ the Euclidean and the maximum norm are standard equivalent. This means that there are $a, A, b, B \in \mathbb{R}^+ \setminus \{0\}$ such that

$$a \left\| z \right\|_\infty \leq \left\| z \right\|_2 \leq A \left\| z \right\|_\infty$$

and

$$b \left\| z \right\|_2 \leq \left\| z \right\|_\infty \leq B \left\| z \right\|_2$$

Proof. Analogously to the direct proofs in $\mathbb{C}^d$, see for example [65]. □

Remark. The important thing here is that $a, A, b, B \in \mathbb{R}^+ \setminus \{0\}$ are real and not hyperreal.

Lemma 5.0.15 (Maximum Norm and Magnitude)
For a vector $z \in \mathbb{C}^{*d} \setminus \{0\}$ it holds true that $\left\| z \right\|_\infty \in \mathbf{mag}(\left\| z \right\|)$.

Proof. By Lemma 5.0.14 (Equivalence of Euclidean and Maximum Norm) there are $a, A \in \mathbb{R}^+ \setminus \{0\}$ such that

$$\begin{aligned} a \left\| z \right\|_\infty \leq \left\| z \right\|_2 &\leq A \left\| z \right\|_\infty \\ \Leftrightarrow a \leq \frac{\left\| z \right\|_2}{\left\| z \right\|_\infty} &\leq A \end{aligned}$$

which means that with $c := \frac{\left\| z \right\|_2}{\left\| z \right\|_\infty}$ it holds true

$$c = \frac{\left\| z \right\|_2}{\left\| z \right\|_\infty} \Leftrightarrow \ c \cdot \left\| z \right\|_\infty = \left\| z \right\|_2 \Rightarrow \left\| z \right\|_\infty \in \mathbf{mag}(\left\| z \right\|_2)$$

□

Remark. So instead of normalizing with the Euclidean norm we can also use the absolute maximum value for the projective shadow. Even more conveniently, we can use the $\arg\max z_i$ of a hypercomplex vector $z \in {\mathbb{C}^*}^d \setminus \{0\}$ since the absolute value function and the $\arg\max$ differ only by an appreciable phase value, *i.e.* $\arg\max z_i = \mathrm{e}^{\phi} \cdot |z_i|$ for an $\phi \in [0, 2\pi)$.

Lemma 5.0.16 (Different Appreciable Representatives)
Let $x \in \mathbb{C}^*\mathbb{P}^d$, $x_{\mathbb{A}}$ and $\hat{x}_{\mathbb{A}}$ be appreciable representatives. Then there is a unique $c \in \mathbb{A}$ such that $\hat{x}_{\mathbb{A}} = c \cdot x_{\mathbb{A}}$.

Proof. By the definition of equivalence classes in $\mathbb{C}^*\mathbb{P}^d$, we know that there is a unique $c \in \mathbb{C}^* \setminus \{0\}$ such that $\hat{x}_{\mathbb{A}} = c \cdot x_{\mathbb{A}}$. We have to prove that $c \in \mathbb{A}$: assume the converse. Then $c \in \mathbb{I}$ or $c \in \mathbb{C}_\infty$. We know by Theorem 5.0.12 (Limited and Appreciable Entries) that all entries are limited and at least one is appreciable. Since all entries are limited we know that c can't be unlimited, otherwise the product of a limited entry with c would be unlimited. But c cannot be infinitesimal either, since the product of limited entries with an infinitesimal is infinitesimal by Theorem 4.3.3 (Arithmetics) which is a contradiction to appreciability of at least one component. □

Lemma 5.0.17 (Magnitude and Appreciability)
Let $x \in \mathbb{C}^*\mathbb{P}^d$ and let $\lambda \in \mathbb{C}^* \setminus \{0\}$. If all components of $\lambda^{-1}x$ are limited and at least one component is appreciable, then $\lambda \in \mathbf{mag}\,(\|x\|)$.

Proof. We know that all entries of $x_{\mathbb{A}} := \frac{x}{\|x\|}$ are limited and at least one component is appreciable. Furthermore we know there is a $c \in \mathbb{C}^* \setminus \{0\}$ such that:

$$\lambda^{-1}x = c \cdot \frac{x}{\|x\|}$$

and additionally $c \in \mathbb{A}$, since otherwise $\lambda^{-1}x$ would not have an appreciable component. It follows that

$$\begin{aligned} \lambda &= \frac{1}{c}\,\|x\| \\ \Leftrightarrow \frac{\lambda}{\|x\|} &= \frac{1}{c}. \end{aligned}$$

Since $\frac{1}{c} \in \mathbb{A}$ this means that $\lambda \in \mathbf{mag}(\|x\|)$, which is the claim. □

Remark. This argument can be easily generalized to matrix norms which will be useful later on when we discuss conic sections and projective transformations (see Definition 5.2.1 (Appreciable Matrix)).

Definition 5.0.18 (Projective Shadow)
The *projective shadow* of $[x] = [(x_1, \dots, x_d)^T] \in \mathbb{C}^*\mathbb{P}^d$ is defined by

$$\mathbf{psh}([x]) := [\mathbf{sh}(x_{\mathbb{A}})]$$

where $x_{\mathbb{A}}$ is an appreciable representative of x. ◇

Remark. Writing out the definition above:

$$\mathbf{psh}([x]) = \left[\mathbf{sh}\left(\frac{1}{\lambda}\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right)\right] = \left[\begin{pmatrix} \mathbf{sh}\,\frac{x_1}{\lambda} \\ \vdots \\ \mathbf{sh}\,\frac{x_d}{\lambda} \end{pmatrix}\right]$$

with $\lambda \in \mathbf{mag}(\|x\|)$ and $\mathbf{sh}$ the standard shadow of $\mathbb{C}^*$.

Remark. A slightly less general definition was already given by A. Leitner for the real non-standard projective space [84].

Remark. One might be tempted to use Lemma 4.3.10 (Shadow Properties), especially something like: $\mathbf{sh}\left(\frac{x_i}{\lambda}\right) = \frac{\mathbf{sh}(x_i)}{\mathbf{sh}(\lambda)}$ for the projective shadow. This it is wrong in a general setting! First of all, Lemma 4.3.10 (Shadow Properties) is only valid for limited numbers and especially the statement of fractions only holds true, if the shadow of λ is not zero, *i.e.* λ is not infinitesimal.

Theorem 5.0.19 ($\mathbb{C}\mathbb{P}^d$ Isomorphism)
The space $\mathbb{C}^*\mathbb{P}^d/\mathbb{I}$ is isomorphic to $\mathbb{C}\mathbb{P}^d$ with the correspondence map **psh**.

Proof. Let $x \in \mathbb{C}^*\mathbb{P}^d$. We know by Theorem 5.0.12 (Limited and Appreciable Entries) that all entries of $x_{\mathbb{A}}$ are limited. Therefore we already know that for every component we can apply Theorem 4.3.11 (Isomorphism) and use that $\mathbb{L}/\mathbb{I}$ is isomorphic to $\mathbb{C}$. Furthermore we know that at least one component is appreciable and not infinitesimal again by Theorem 5.0.12 (Limited and Appreciable Entries). This ensures that at least one component is not in the halo of 0. Therefore we know the shadow of an appreciable representative $x_{\mathbb{A}}$ is element of $\mathbb{C}\mathbb{P}^d$. □

Incidences play a very important role in projective geometry. For points and lines these can conveniently be determined by the standard scalar product in $\mathbb{C}\mathbb{P}^2$. We will define a generalization of the scalar product here in order to embrace infinitesimal and unlimited representatives.

Definition 5.0.20 ($\mathbb{C}^*\mathbb{P}^\mathbf{2}$ Appreciable Scalar Product)
We define the *appreciable scalar product* $\langle\cdot,\cdot\rangle_*$ for objects $x, y \in \mathbb{C}^*\mathbb{P}^2$ as

$$\langle x, y\rangle_* := \langle x_\mathbb{A}, y_\mathbb{A}\rangle,$$

with arbitrary appreciable representatives $x_\mathbb{A}, y_\mathbb{A}$, where $\langle\cdot,\cdot\rangle$ denotes the standard complex scalar product. $\diamond$

Remark. For the case of $\mathbb{R}^*\mathbb{P}^2$ one would of course use the real scalar product.

The value of the scalar product is not independent of the representatives, for example the scalar product of infinitesimal vectors is always infinitesimal. But one can show that an incidence relation for appreciable representatives is well-defined. That means $\langle x, y\rangle_* \in \mathbb{I}$ is independent of the appreciable representative.

Theorem 5.0.21 (Well-defined $\mathbb{I}$ Relation)
For $x, y \in \mathbb{C}^*\mathbb{P}^2$ the relation

$$\langle x, y\rangle_* \in \mathbb{I}$$

is well defined.

Proof. Let $x_\mathbb{A}, \hat{x}_\mathbb{A}$ be limited representatives of x and $y_\mathbb{A}, \hat{y}_\mathbb{A}$ be limited representatives of y. By Lemma 5.0.16 (Different Appreciable Representatives) we can choose $\iota, \kappa \in \mathbb{A}$ such that $x = \iota \cdot \hat{x}$, $y = \kappa \cdot \hat{y}$. Then it holds true:

$$\langle x, y\rangle_* = \langle x_\mathbb{A}, y_\mathbb{A}\rangle = \langle \iota \cdot \hat{x}_\mathbb{A}, \kappa \cdot \hat{y}_\mathbb{A}\rangle = \underbrace{\bar{\iota} \cdot \kappa}_{=:c} \langle \hat{x}_\mathbb{A}, \hat{y}_\mathbb{A}\rangle$$

Since $\kappa, \iota \in \mathbb{A}$ it holds true that also $c \in \mathbb{A}$. And so

$$\langle x_\mathbb{A}, y_\mathbb{A}\rangle \in \mathbb{I} \Leftrightarrow \langle \hat{x}_\mathbb{A}, \hat{y}_\mathbb{A}\rangle \in \mathbb{I},$$

which is the claim. $\square$

Theorem 5.0.22 (Appreciable Scalar Product and Shadows)
For $x, y \in \mathbb{C}^*\mathbb{P}^2$ it holds true

$$\mathbf{sh}\left(\langle x, y\rangle_*\right) = 0 \Leftrightarrow \langle \mathbf{psh}(x), \mathbf{psh}(y)\rangle = 0$$

Proof. Let $x_{\mathbb{A}}, \hat{x}_{\mathbb{A}}$ be appreciable representatives of x and $y_{\mathbb{A}}, \hat{y}_{\mathbb{A}}$ be appreciable representatives of y. Then it holds true:

$$\begin{aligned}
\langle \mathbf{psh}(x), \mathbf{psh}(y) \rangle &= \langle \mathbf{sh}(\hat{x}_{\mathbb{A}}), \mathbf{sh}(\hat{y}_{\mathbb{A}}) \rangle = \overline{\mathbf{sh}(x_{\mathbb{A}})}^T \cdot \mathbf{sh}(y_{\mathbb{A}}) \\
&= \sum_{i=1}^{3} \mathbf{sh}(\overline{x_{\mathbb{A}_i}}) \cdot \mathbf{sh}(y_{\mathbb{A}_i}) = \sum_{i=1}^{3} \mathbf{sh}\left(\overline{x_{\mathbb{A}_i}} \cdot y_{\mathbb{A}_i}\right) \\
&= \mathbf{sh}\left(\sum_{i=1}^{3} \overline{x_{\mathbb{A}_i}} \cdot y_{\mathbb{A}_i} \right) = \mathbf{sh}(\langle x_{\mathbb{A}}, y_{\mathbb{A}} \rangle).
\end{aligned}$$

□

Lemma 5.0.23 (psh = sh($\langle \cdot, \cdot \rangle_*$))
Let $x, y \in \mathbb{C}^*\mathbb{P}^2$, if we choose the same appreciable representative in $\mathbf{sh}\left(\langle x, y \rangle_*\right)$ and $\langle \mathbf{psh}(x), \mathbf{psh}(y) \rangle$ respectively it even holds true:

$$\mathbf{sh}\left(\langle x, y \rangle_*\right) = \langle \mathbf{psh}(x), \mathbf{psh}(y) \rangle$$

Proof. We only have to realize that in the proof of Theorem 5.0.22 (Appreciable Scalar Product and Shadows) $c = 1$ if we choose the same representatives. Then the claim immediately follows. □

Remark. Some of the following "almost relations" in the following sections were already investigated by J. Fleuriot in an Euclidean setting (see for example [39, 41]). We have to do a little more work in projective space than in the Euclidean setting since the representation of a vector plays an important role.

Definition 5.0.24 (Almost Orthogonal)
We two vectors $l, p \in \mathbb{C}^{*3}$ *almost orthogonal* and write $[p] \perp_{\mathbb{I}} [l]$ if and only if

$$\langle p, l \rangle_* = \epsilon \in \mathbb{I}.$$

◇

Remark. Note that if two vectors in $\mathbb{C}^3$ are orthogonal, they are also almost orthogonal since 0 is the only infinitesimal in $\mathbb{C}$.

5.1. Incidence of Points and Lines

> Der Unterschied zwischen Kant und Einstein besteht nicht darin, daß der eine einen euklidischen und der andere einen nicht-euklidischen Raum annahm, sondern vor allem in der Beziehung, die sie zwischen der Mathematik und der Wirklichkeit herstellten.
>
> Friedrich Dürrenmatt, Albert Einstein [29]

Now we will start out with basic projective geometry: we define incidence relations of points and lines and analyze well known properties from the standard world.

Definition 5.1.1 (Almost Incident)
Define relation $\mathcal{I}_{\mathbb{C}^*} \subset \mathcal{P}_{\mathbb{C}^*} \times \mathcal{L}_{\mathbb{C}^*}$ for a point $p \in \mathcal{P}_{\mathbb{C}^*}$ and a line $l \in \mathcal{L}_{\mathbb{C}^*}$ by

$$[p]\mathcal{I}_{\mathbb{C}^*}[l] \Leftrightarrow [p] \perp_{\mathbb{I}} [l].$$

Then we call p *almost incident* to l. $\diamond$

Remark. The incidence relation of $\mathbb{CP}^2$ is a special case of the almost incidence relation above *i.e.* if p and l are incident in $\mathbb{CP}^2$ then they are almost incident in $\mathbb{C}^*\mathbb{P}^2$.

Lemma 5.1.2 (Well-defined Incidence)
The incidence relation is well defined.

Proof. Special case of Theorem 5.0.21 (Well-defined $\mathbb{I}$ Relation). $\square$

Lemma 5.1.3 (Incident and Almost Incident)
A point p and a line l in $\mathbb{C}^*\mathbb{P}^2$ are almost incident if and only of their projective shadow $\mathbf{psh}(p)$ and $\mathbf{psh}(l)$ are incident in $\mathbb{CP}^2$.

Proof. Note that the shadow of an infinitesimal number is 0. Then the claim it a direct application of Theorem 5.0.22 (Appreciable Scalar Product and Shadows). $\square$

Example 5.1.4 (Geometriekalküle)
In "Geometriekalküle" by J. Richter-Gebert and T. Orendt [114] there is a nice example on the introduction of farpoints (see p. 4). Let $x, y \in \mathbb{C}$, and define a point $P(t)$ by:

$$[P(t)] = \left[\begin{pmatrix} x \cdot t \\ y \cdot t \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} x \\ y \\ 1/t \end{pmatrix}\right].$$

If one takes the limit of t to infinity, one finds:

$$\lim_{t\to\infty}[P(t)] = \lim_{t\to\infty}\left[\begin{pmatrix} x \\ y \\ 1/t \end{pmatrix}\right] = \left[\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\right].$$

One can easily check that this point is incident with the line at infinity l_∞ of the standard embedding.

We can give an analogous example using hyperreal numbers. Let H be an arbitrary real unlimited number and define P':

$$[P'] := \left[\begin{pmatrix} x \cdot H \\ y \cdot H \\ 1 \end{pmatrix}\right].$$

P' is a point which should be incident to l_∞. However, then the usual check of incidence, the standard scalar product, fails:

$$\langle \begin{pmatrix} x \cdot H \\ y \cdot H \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle = 0 \cdot x \cdot H + 0 \cdot y \cdot H + 1 = 1 \notin \mathbb{I}.$$

Now if we note that H is in $\mathbf{mag}(\|P\|)$, then we find:

$$\langle \begin{pmatrix} x \cdot H \\ y \cdot H \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle_* = \langle \begin{pmatrix} x \\ y \\ \frac{1}{H} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle = \frac{1}{H}$$

Since H is unlimited, its inverse is infinitesimal and therefore the point is almost incident to the line at infinity. This means that the projection of P' to $\mathbb{CP}^2$ is incident to l_∞ and hence a farpoint as expected. ◇

Remark. As for the appreciable scalar product in Definition 5.0.20 ($\mathbb{C}^*\mathbb{P}^2$ Appreciable Scalar Product), we also have to say how we find the connecting line of two points or the intersection of two lines, respectively. The standard cross-product is fine, if we want to work in $\mathbb{C}^*\mathbb{P}^2$, but if we want to draw conclusions about $\mathbb{CP}^2$ (*i.e.* taking the show) we have to redefine the operator and normalize beforehand.

Definition 5.1.5 (Appreciable Cross-Product)
Let x, y be in two vectors in $\mathbb{C}^*\mathbb{P}^2$. We define the *appreciable cross-product* $\times_*$: by:

$$x \times_* y := x_{\mathbb{A}} \times y_{\mathbb{A}}$$

For appreciable representatives $x_{\mathbb{A}}, y_{\mathbb{A}}$ of x and y. $\diamond$

Lemma 5.1.6 (Well-defined Cross-Product)
The appreciable cross-product defined in Definition 5.1.5 (Appreciable Cross-Product) is well defined.

Proof. By the bilinearity of the cross-product (see for example [38] p. 282) the claim is obvious. $\square$

Remark. We will see in Example 5.1.21 (Appreciable Cross-Product) that we have to be careful with the application of shadow function if it is applied to the cross-product.

Definition 5.1.7 (Almost Far Point)
We call $p \in \mathcal{P}_{\mathbb{C}^*}$ an *almost far point*, if it holds true for an appreciable representative $p_{\mathbb{A}}$ of p with

$$p_{\mathbb{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and } x, y, z \in \mathbb{L}$$

that $z \in \mathbb{I}$, *i.e.* that the z-component of the appreciable representative is infinitesimal. $\diamond$

Definition 5.1.8 (Appreciable Join and Meet)
Analogously to the standard definition of the join in $\mathbb{CP}^2$ we define the *appreciable join* for two points $p, q \in \mathcal{P}_{\mathbb{C}^*}$ by

$$\mathbf{join}_*(p, q) := p \times_* q = p_{\mathbb{A}} \times q_{\mathbb{A}}.$$

And for two lines $l, m \in \mathcal{L}_{\mathbb{C}^*}$ we define the *appreciable meet* by

$$\mathbf{meet}_*(l, m) := l \times_* m = l_{\mathbb{A}} \times m_{\mathbb{A}}.$$

$\diamond$

Remark. The result of an appreciable **join** or **meet** is not necessarily appreciable! Take for example the two points $p = (0, 0, 1)$ and $q = (\epsilon, 0, 1)$, with $\epsilon \in \mathbb{I}$, *i.e.* ϵ is infinitesimal. Then their appreciable **join**, which is equivalent to the standard join since p and q are appreciable, is

$$\mathbf{join}_*(p, q) = \mathbf{join}(p, q) = \begin{pmatrix} 0 \\ \epsilon \\ 0 \end{pmatrix}$$

which is not an appreciable vector!

Definition 5.1.9 (Almost Parallel)
Let $l, m \in \mathcal{L}_{\mathbb{C}^*}$ be two lines. We call l and m *almost parallel* if an appreciable representative of $\mathbf{meet}_*(l, m)$ is an almost far point. $\diamond$

Lemma 5.1.10 (Almost Parallel is Well-defined)
The definition of almost parallel in Definition 5.1.9 (Almost Parallel) is well-defined.

Proof. Special case of Theorem 5.0.21 (Well-defined $\mathbb{I}$ Relation). $\square$

Lemma 5.1.11 (Almost Far Points on $\boldsymbol{l_\infty}$)
A point p is an almost far point if and only if p is almost incident to $l_\infty = (0, 0, 1)^T$.

Proof. If p is an almost far point then we can write an appreciable representative $p_{\mathbb{A}}$ with $p_{\mathbb{A}} = (x, y, \epsilon)$ where $x, y \in \mathbb{L}$ and $\epsilon \in \mathbb{I}$. Then the appreciable scalar product $\langle p_{\mathbb{A}}, l_\infty \rangle = 0 + 0 + \epsilon = \epsilon \in \mathbb{I}$. This means that p is almost incident to l_∞.

If p is almost incident to l_∞ we know that for an appreciable representative $p_{\mathbb{A}} = (x, y, z)^T$ it holds true $\langle p_{\mathbb{A}}, l_\infty \rangle = z \in \mathbb{I}$. Therefore p is an almost far point. $\square$

Lemma 5.1.12 (Almost Parallel Lemma)
Two lines $l, m \in \mathcal{L}_{\mathbb{C}^*}$ are almost parallel if and only if it holds true:

$$\mathbf{meet}_*(l, m) \perp_{\mathbb{I}} l_\infty$$

where $l_\infty = (0, 0, 1)^T$ denotes the line at infinity. In other words the intersection of both lines is almost incident to the line at infinity.

Proof. Special case of Lemma 5.1.11 (Almost Far Points on l_∞). $\square$

Definition 5.1.13 (Shadow Cross-Product)
Let x, y be two objects in $\mathbb{K}^*\mathbb{P}^2$. We define the *shadow cross-product* $\times_{\mathbf{sh}}$ by:

$$x \times_{\mathbf{sh}} y := \mathbf{sh}(x \times_* y) = \mathbf{sh}(x_{\mathbb{A}} \times y_{\mathbb{A}}).$$

◇

Lemma 5.1.14 (Well-defined Shadow Cross-Product)
The shadow cross-product defined in Definition 5.1.13 (Shadow Cross-Product) is well-defined.

Proof. Let $x, y \in \mathbb{K}^*\mathbb{P}^2$, pick now two appreciable representatives $x_{\mathbb{A}}, \hat{x}_{\mathbb{A}}$ and $y_{\mathbb{A}}, \hat{y}_{\mathbb{A}}$. Then we know by Lemma 5.0.16 (Different Appreciable Representatives) that there are appreciable numbers c, d such that

$$x_{\mathbb{A}} = c \cdot \hat{x}_{\mathbb{A}}, \quad y_{\mathbb{A}} = d \cdot \hat{y}_{\mathbb{A}}.$$

$$x \times_* y = x_{\mathbb{A}} \times y_{\mathbb{A}} = c \cdot \hat{x}_{\mathbb{A}} \times d \cdot \hat{y}_{\mathbb{A}} = (c \cdot d) \cdot (\hat{x}_{\mathbb{A}} \times \cdot \hat{y}_{\mathbb{A}})$$

Where we used the bilinearity of the cross-product. Then, if we apply the shadow function and use the linearity of the shadow for limited numbers we know:

$$\mathbf{sh}(x \times_* y) = \underbrace{\mathbf{sh}(c \cdot d)}_{\in \mathbb{K} \setminus \{0\}} \cdot \mathbf{sh}(\hat{x}_{\mathbb{A}} \times \hat{y}_{\mathbb{A}}) \sim_{\mathbb{KP}^2} \mathbf{sh}(\hat{x}_{\mathbb{A}} \times \hat{y}_{\mathbb{A}}) = \hat{x} \times_{\mathbf{sh}} \hat{y}$$

where we used that $c \cdot d$ is appreciable. □

Definition 5.1.15 (Almost Linearly Dependent)
A set of objects in $\{v_1, \ldots, v_m\} \subset \mathbb{K}^{*n}$ is called *almost linearly dependent*, if there are $\lambda_1, \ldots, \lambda_m \in \mathbb{K}^* \setminus \mathbb{I}$ such that

$$\sum_{i=1}^{m} \lambda_i \cdot v_i \simeq 0.$$

and we call them *linearly dependent* if it even holds true that $\sum_{i=1}^{m} \lambda_i \cdot v_i = 0$. Otherwise, we call them *linearly independent*. ◇

Definition 5.1.16 (Almost Equivalent)
We call two objects $x, y \in \mathbb{C}^*\mathbb{P}^n$ *almost equivalent*, if $\exists \lambda \in \mathbb{A}_{\mathbb{C}}$ such that

$$x_{\mathbb{A}} \simeq \lambda \cdot y_{\mathbb{A}}$$

which means that the appreciable representatives of x and y are almost linearly dependent. We write again $x \simeq y$. ◇

Definition 5.1.17 (Projective Halo)

For a point $x \in \mathbb{C}^*\mathbb{P}^n$ we define the *projective halo* $\mathbb{P}$-**hal** *of* x by

$$\mathbb{P}\text{-}\mathbf{hal}(x) := \{y \in \mathbb{C}^*\mathbb{P}^n \mid x \simeq y\}.$$

◇

Lemma 5.1.18 (Appreciable Scaling Factor)
Let $x, y \in \mathbb{C}^*\mathbb{P}^n$ be almost equivalent, *i.e.* $\exists \lambda \in \mathbb{C}^* \setminus \mathbb{I}$ such that $x_{\mathbb{A}} \simeq \lambda \cdot y_{\mathbb{A}}$. Then it holds true that λ is appreciable, *i.e.* $\lambda \in \mathbb{A}$.

Proof.

$$x_{\mathbb{A}} \simeq \lambda \cdot y_{\mathbb{A}} \Rightarrow \|x_{\mathbb{A}}\| \simeq |\lambda| \, \|y_{\mathbb{A}}\| \Rightarrow |\lambda| \simeq \frac{\|x_{\mathbb{A}}\|}{\|y_{\mathbb{A}}\|} \in \mathbb{A}.$$

□

Lemma 5.1.19 (Almost Equivalent and Angles)
Two elements $p, p' \in \mathbb{C}^*\mathbb{P}^2$ are almost equivalent, if and only if the angle α between p and p', interpreted as vectors in $\mathbb{C}^{*3}$, is infinitesimal or infinitely close to π.

Proof. Pick appreciable representatives $p_{\mathbb{A}}$ and $p'_{\mathbb{A}}$ of p and p'. Then by definition there are $c, c' \in \mathbb{A}$ such that

$$p_{\mathbb{A}} = \frac{p}{c\,\|p\|} \quad \text{and} \quad p'_{\mathbb{A}} = \frac{p}{c'\,\|p'\|}.$$

Since p, p' are almost equivalent there is a $\lambda \in \mathbb{A}$ such that $p'_{\mathbb{A}} \simeq \lambda p_{\mathbb{A}}$. Then it holds true

$$p_{\mathbb{A}} \times p'_{\mathbb{A}} = \frac{p}{c\,\|p\|} \times \frac{p'}{c'\,\|p'\|} = \frac{1}{\underbrace{c \cdot c'}_{=:d}} \left(\frac{p}{\|p\|} \times \frac{p'}{\|p'\|} \right).$$

Note that $d \in \mathbb{A}$ since the product of appreciable numbers is appreciable and also its inverse.

By the properties of the scalar product and transfer we know:

$$\|p_{\mathbb{A}} \times p'_{\mathbb{A}}\| = |d| \left\| \frac{p}{\|p\|} \right\| \left\| \frac{p'}{\|p'\|} \right\| \sin(\alpha) = |d| \sin(\alpha).$$

The sine function is infinitesimal if and only of $\alpha \simeq 0$ or $\alpha \simeq \pi$, which can be easily seen by the series expansion. This is the claim. □

Lemma 5.1.20 (Cross-Product and Almost Equivalent)
For $x, y \in \mathbb{C}^*\mathbb{P}^2$ the appreciable cross-product $x \times_* y$ is infinitesimal if and only if $x \simeq y$.

Proof. Let $x \simeq y$ then there exists $\lambda \in \mathbb{C}^* \setminus \mathbb{I}$ such that $x_{\mathbb{A}} \simeq \lambda \cdot y_{\mathbb{A}}$. Then it follows:

$$\begin{aligned} x \times_* y = x_{\mathbb{A}} \times y_{\mathbb{A}} \simeq \lambda \cdot y_{\mathbb{A}} \times y_{\mathbb{A}} = 0 \\ \Rightarrow x \times_* y \in \mathbb{I} \end{aligned}$$

The other direction of the proof follows from Lemma 5.1.19 (Almost Equivalent and Angles): two vectors are almost linearly dependent, if the angle between the vectors is either infinitesimal or infinitely close to π, thus if their appreciable cross-product is an infinitesimal vector. □

The next example shows that we cannot neglect the representative of an equivalence class for the cross-product.

Example 5.1.21 (Appreciable Cross-Product)
Let $H \in \mathbb{R}^*{}_\infty$ and define

$$x = \begin{pmatrix} H \\ 0 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Both represent the same equivalence class in $\mathbb{RP}^2$ which we see by their projective shadows

$$\mathbf{psh}(x) = \mathbf{sh} \begin{pmatrix} 1 \\ 0 \\ \frac{1}{H} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{psh}(y) = y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So the unnormalized cross-product should resemble the zero vector, but as we can see

$$x \times y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x \times_{\mathbf{sh}} y = \mathbf{sh} \begin{pmatrix} 0 \\ \frac{1}{H} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

the cross-product yields a wrong result, while the shadow cross-product is correct. ◇

Now we will come back to relations of points and lines. One important concept is collinearity of three points. We will generalize this to a further "almost" relation.

Definition 5.1.22 (Almost Collinear)
We call three points $x, y, z \in \mathbb{C}^*\mathbb{P}^2$ *almost collinear*, if the point x is almost incident to

the line $l := \mathbf{join}_*(y,z)$, *i.e.*

$$\langle x, l\rangle_* = \langle x_{\mathbb{A}}, l_{\mathbb{A}}\rangle \in \mathbb{I}.$$

◇

Lemma 5.1.23 (Almost Collinear is Well-defined)
The definition of being almost collinear in Definition 5.1.22 (Almost Collinear) is well defined.

Proof. Special case of Theorem 5.0.21 (Well-defined $\mathbb{I}$ Relation). □

The canonical way to check for collinearity is usually the determinant, which we will now define in an normalized version.

Definition 5.1.24 (Normalized Determinant)
Let x, y, z be in three distinct objects in $\mathbb{C}^*\mathbb{P}^2$. We define the *normalized determinant of* x, y, z by

$$\det_{\|\cdot\|}[x,y,z] = \frac{\det[x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}]}{\lambda}, \quad \lambda \in \mathbf{mag}\left(\|y_{\mathbb{A}} \times z_{\mathbb{A}}\|\right)$$

where $x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}$ are appreciable representatives of x, y, z and det the standard definition of the determinant. ◇

Remark. The formula might look a bit asymmetric but the term we are dividing by assures that the cross-product of $y_{\mathbb{A}}$ and $z_{\mathbb{A}}$ is again appreciable.

Lemma 5.1.25 (Determinants, Scalar-Product and Cross-Product)
For $x, y, z \in \mathbb{C}^*\mathbb{P}^2$ and fixed appreciable representatives of $x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}$ of x, y, z. Then it holds true: $\exists c \in \mathbb{A}$ such that

$$\det_{\|\cdot\|}[x,y,z] = c \cdot \langle x, y \times_* z\rangle_*$$

where $\langle\cdot,\cdot\rangle_*$ denotes the appreciable scalar product and $(\cdot \times_* \cdot)$ the appreciable cross product.

Proof. Since

$$\det_{\|\cdot\|}[x,y,z] = \frac{\det[x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}]}{\lambda}, \quad \lambda \in \mathbf{mag}\left(\|y_{\mathbb{A}} \times z_{\mathbb{A}}\|\right)$$

we can rewrite the latter, using the rewrite rule for the “normal” determinant, to

$$\det[x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}] = \langle x_{\mathbb{A}}, y_{\mathbb{A}} \times z_{\mathbb{A}}\rangle$$

Then it holds true for $\lambda \in \mathbf{mag}\left(\|y_{\mathbb{A}} \times z_{\mathbb{A}}\|\right)$:

$$\begin{aligned}\det_{\|\cdot\|}[x,y,z] &= \frac{\det[x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}]}{\lambda} = \frac{\langle x_{\mathbb{A}}, y_{\mathbb{A}} \times z_{\mathbb{A}}\rangle}{\lambda} \\ &= \langle x_{\mathbb{A}}, \frac{y_{\mathbb{A}} \times z_{\mathbb{A}}}{\lambda}\rangle = c \cdot \langle x, y \times_* z\rangle_*.\end{aligned}$$

The equality $\frac{y_{\mathbb{A}} \times z_{\mathbb{A}}}{\lambda} = c(y \times_* z)$ can be explained as follows: if y and z are almost equivalent then their cross-product is infinitesimal (see Lemma 5.1.20 (Cross-Product and Almost Equivalent)) and is rescaled by λ to appreciable length. Otherwise the cross-product has appreciable length and therefore the $\lambda \in \mathbb{A}$, then the claim follows from Lemma 5.0.16 (Different Appreciable Representatives). ☐

Corollary 5.1.26 (Collinearity and the Normalized Determinat)
The distinct points x, y, z in $\mathbb{C}^*\mathbb{P}^2$ are almost are almost collinear if and only if

$$\det_{\|\cdot\|}[x,y,z] \in \mathbb{I}.$$

Proof. By Lemma 5.1.25 (Determinants, Scalar-Product and Cross-Product) and the definition of being almost collinear. ☐

Definition 5.1.27 (Appreciable Determinant)
Let x, y, z be in three distinct vectors in $\mathbb{C}^*\mathbb{P}^2$. We define the *appreciable determinant of* x, y, z by

$$\det_*[x,y,z] := \det[x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}]$$

where $x_{\mathbb{A}}, y_{\mathbb{A}}, z_{\mathbb{A}}$ are appreciable representatives of x, y, z and det the standard definition of the determinant. ◇

Lemma 5.1.28 (Appreciable Determinant and Almost Collinear)
Let x, y, z be three points in $\mathbb{C}^*\mathbb{P}^2$ which are pairwise not almost equivalent. Then x, y, z are almost collinear if and only if $\det_*[x,y,z] \in \mathbb{I}$.

Proof. Special case of the proof of Lemma 5.1.25 (Determinants, Scalar-Product and Cross-Product). The lack of the normalization using $\lambda \in \mathbf{mag}\left(\|y_{\mathbb{A}} \times z_{\mathbb{A}}\|\right)$ is negligible since if y and z are not almost equivalent then their cross product yields an appreciable vector. ☐

One might wonder why we defined two different types of determinants: the appreciable and the normalized one. Both have their purpose and are can be used to determine if points are almost collinear, but the appreciable determinant might give false positives if two (or more) points are almost equivalent. We will illustrate this with an example:

Example 5.1.29 (Almost Collinear and Normalized Determinants)
Pick an arbitrary $\epsilon \in \mathbb{I}$ and define the points

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad y = \begin{pmatrix} \epsilon \\ 0 \\ 1 \end{pmatrix} \quad z = \begin{pmatrix} 0 \\ \epsilon \\ 1 \end{pmatrix}$$

These points are *not* almost collinear, the join of y and z is the line $(\epsilon, \epsilon, -\epsilon^2)^T \sim (1, 1, -\epsilon)^T$, which is almost equivalent to the angle bisector of x- and y-axis. In particular, observe that the appreciable join of y and z is infinitesimal and so y and z are almost equivalent (see Lemma 5.1.19 (Almost Equivalent and Angles)).

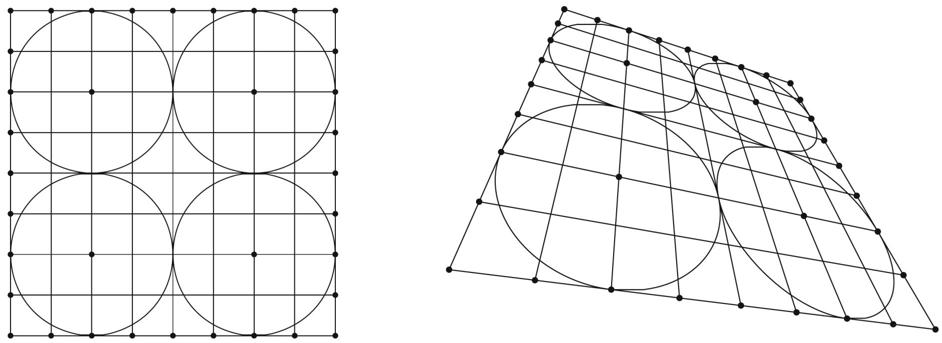
The appreciable determinant gives a false positive incidence relation here: first note that x, y, z are already appreciable and the appreciable determinant yields $\det_*[x, y, z] = -\epsilon(1-\epsilon)$, which is infinitesimal. This is not a contradiction to Lemma 5.1.28 (Appreciable Determinant and Almost Collinear) since y and z are almost equivalent. This resembles the fact that y and z have the same shadow and therefore the join of their shadows is not an element of the (standard) projective space.

The normalized determinant yields the correct result: $\det_{\|\cdot\|}[x, y, z] = \frac{\det_*[x,y,z]}{\epsilon} = 1 - \epsilon \notin \mathbb{I}$ where we used that $\epsilon \in \mathbf{mag}(\|y \times z\|)$. This also corresponds to the definition of being almost incident:

$$\langle x, l \rangle_* = \langle x_{\mathbb{A}}, l_{\mathbb{A}} \rangle = 1 - \epsilon \notin \mathbb{I}.$$

◇

5.2. Non-standard Projective Transformations



The image of a grid under a projective transformation, [111] p. 63.

This section will start out with basic properties of linear algebra over a hyper field. Since we are interested in a projective setting, we will directly identify a matrix M with its appreciable representative, *i.e.* its normalized version $\frac{M}{\|M\|}$.

Definition 5.2.1 (Appreciable Matrix)
Let $M \in \mathbb{C}^{*m\times n}$ and not all entries of M equal zero. Then we define an *appreciable matrix representation $M_{\mathbb{A}}$ of M* by

$$M_{\mathbb{A}} := \frac{1}{\lambda}M, \quad \lambda \in \mathbf{mag}\left(\|A\|\right)$$

where $\|\cdot\|$ is defined as an arbitrary submultiplicative and self-adjoint matrix norm. If not stated otherwise we will use, for convenience, the Frobenius norm:

$$\|M\|_F := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|M_{i,j}|^2}.$$

◇

Remark. It is important to have a submultiplicative matrix norm in order to use estimations like $\|Ax\| \leq \|A\|\,\|x\|$. All important matrix norms are submultiplicative, *e.g.* the spectral norm, natural norm, Frobenius norm or the (scaled) maximum norm. Note that only taking the absolute value of the absolute maximum of a matrix it *not* submultiplicative! The self-adjoint property is also important: then transposition (and complex conjugation) do not change the matrix norm.

Lemma 5.2.2 (Appreciable Matrix Properties)
Let $M_{\mathbb{A}}$ be an appreciable matrix. Then all entries of $M_{\mathbb{A}}$ are limited and at least one is appreciable.

Proof. Analogously to the proof of Theorem 5.0.12 (Limited and Appreciable Entries). $\square$

Next we will define a conic section, or shortly conic, in the same way as J. Richter-Gebert's perpectives on projective geometry [111] p. 145 ff. Our conics are enlarged versions of the original definitions.

Lemma 5.2.3 (Appreciable Matrix-Vector Product)
Let $M_{\mathbb{A}} \in \mathbb{C}^{*m \times n}$ be an appreciable matrix and $x_{\mathbb{A}} \in \mathbb{CP}^{n-1}$ an appreciable representative of $x \in \mathbb{CP}^{n-1}$. Then their the matrix–vector product

$$M_{\mathbb{A}} \cdot x_{\mathbb{A}}$$

is limited.

Proof. All entries of $M_{\mathbb{A}}$ and $x_{\mathbb{A}}$ are appreciable, products and sums of appreciable numbers are limited according to Theorem 4.3.3 (Arithmetics). $\square$

Definition 5.2.4 (Almost Singular)
We call a matrix $M \in \mathbb{K}^{*3\times 3} \setminus \{0\}$ *almost singular* it the determinant of its appreciable representative $M_{\mathbb{A}} = \frac{1}{\lambda} M$, $\lambda \in \mathbf{mag}\left(\|M\|\right)$ is infinitesimal but not zero, *i.e.*

$$\det(M_{\mathbb{A}}) \in \mathbb{I} \setminus \{0\}.$$

If the $\det(M_{\mathbb{A}}) = 0$, we call M *singular*, and *non-singular* if $\det(M_{\mathbb{A}})$ is appreciable. $\diamond$

Remark. If $M \in \mathrm{GL}_3(\mathbb{K}^*)$ then M is non-singular or almost singular. One can easily see that if one applies the Gaussian algorithm to decompose M into an upper and lower triangular matrix and uses the equivalence of full rank and $\det(\cdot) \neq 0$ (see [38]).

Remark. A. Leitner chose a different approach for (in our terms) non-singular projective transformations: she called a hyperreal projective transformations $A \in \mathrm{PGL}_n(\mathbb{R}^*)$ (where PGL denotes the projective linear group) *finite* if there is a standard real projective transformation $B \in \mathrm{PGL}_n(\mathbb{R})$ and a $\lambda \in \mathbb{R}^*$ such that $B - \lambda A$ is infinitesimal [84]. This is equivalent to our definition of non-singular matrices: if a projective transformation is finite, it holds true that:

$$B \simeq \lambda A \Rightarrow \det(B) \simeq \det(\lambda A)$$

by the continuity of the determinant. We know that $\det(B) \in \mathbb{R} \setminus \{0\}$, so it holds true that $\det(\lambda A) \in \mathbb{A}$. And by the same arguments as in Lemma 5.0.17 (Magnitude and Appreciability) we know that $\lambda^{-1} \in \mathbf{mag}(\|A\|)$ and therefore λA is non-singular

On the other hand, for a non-singular projective transformation M we know that the determinant of an appreciable representative $M_{\mathbb{A}}$ is appreciable. Then it holds true that the shadow of $M_{\mathbb{A}}$ is a standard projective transformation. So the objects $B := \mathbf{sh}(M_{\mathbb{A}}), \lambda = \frac{1}{\|M\|}$, the matrix $B - \lambda A$ is infinitesimal, which is Leitners definition.

Theorem 5.2.5 (Appreciable Image)
Let $M \in \mathrm{GL}_3(\mathbb{K}^*)$, let M be regular and $p \in \mathbb{K}^{*3} \setminus \{0\}$. Then the matrix vector multiplication of $M_{\mathbb{A}} = \frac{1}{\|M\|}M$ and $p_{\mathbb{A}} = \frac{1}{\|p\|}p$ is an appreciable vector, *i.e.* $\|M_{\mathbb{A}} p_{\mathbb{A}}\| \in \mathbb{A}$.

Proof. We know that all entries of both $M_{\mathbb{A}}$ and $p_{\mathbb{A}}$ are limited. The matrix-vector multiplication consists of multiplication and summation, which preserve the property of being limited (see Theorem 4.3.3 (Arithmetics)). So the norm of $M_{\mathbb{A}} p_{\mathbb{A}}$ cannot be unlimited.

$M_{\mathbb{A}} p_{\mathbb{A}}$ cannot be an infinitesimal vector either. Assume the contrary, then we can use Lemma 4.3.10 (Shadow Properties) and find

$$0 = \mathbf{sh}(M_{\mathbb{A}} p_{\mathbb{A}}) = \mathbf{sh}(M_{\mathbb{A}})\,\mathbf{sh}(p_{\mathbb{A}}).$$

Since $p_{\mathbb{A}}$ has appreciable length, it holds true that $\mathbf{sh}(p_{\mathbb{A}}) \in \mathbb{K}^3 \setminus \{0\}$. But this means that $\mathbf{sh}(p_{\mathbb{A}})$ is an element of the kernel of $\mathbf{sh}(M_{\mathbb{A}})$ and so $\mathbf{sh}(M_{\mathbb{A}})$ is singular. But this implies that either M is singular or almost singular, which is a contradiction. □

Definition 5.2.6 (**ϵ**-kernel)
Let $M \in \mathbb{K}^{*3\times 3}$. We define the ϵ-*kernel* by

$$\epsilon(M) := \{v \in \mathbb{C}^*\mathbb{P}^2 \mid M_{\mathbb{A}} v_{\mathbb{A}} \simeq 0\}$$

where $M_{\mathbb{A}}$ and $v_{\mathbb{A}}$ are appreciable representatives of M and v. ◇

Remark. By Theorem 5.2.5 (Appreciable Image) we know that for regular matrices the ϵ-kernel consists only of vectors which are almost equivalent to the zero vector, which are not part of a projective space.

We define the adjugate of a matrix in the usual way:

Definition 5.2.7 (Adjugate non–standard matrix)

For an appreciable matrix $M_{\mathbb{A}} \in \mathbb{C}^{*3\times 3}$ with

$$M_{\mathbb{A}} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

we define the *(non-standard) adjugate matrix* $M_{\mathbb{A}}^{\triangle}$ by

$$M_{\mathbb{A}}^{\triangle} := \begin{pmatrix} \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} & -\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} & \det\begin{pmatrix} d & e \\ g & h \end{pmatrix} \\ -\det\begin{pmatrix} b & c \\ h & i \end{pmatrix} & \det\begin{pmatrix} a & c \\ g & i \end{pmatrix} & -\det\begin{pmatrix} a & b \\ g & h \end{pmatrix} \\ \det\begin{pmatrix} b & c \\ e & f \end{pmatrix} & -\det\begin{pmatrix} a & c \\ d & f \end{pmatrix} & \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} \end{pmatrix}^T$$

where the determinant det is defined in the usual way, just with non–standard arithmetic.

◇

Lemma 5.2.8 (Adjugate non–standard Matrix is limited)
Let $M_{\mathbb{A}}$ be an appreciable matrix. The entries of the adjugate matrix $M_{\mathbb{A}}^{\triangle}$ are limited.

Proof. A. Leitner gave a similar proof in [84] for the real case. The entries of $M_{\mathbb{A}}^{\triangle}$ are generated by multiplication and substraction of appreciable numbers. This yields limited (if the sum is infinitesimal) or appreciable results by Theorem 4.3.3 (Arithmetics). □

Lemma 5.2.9 (Norm of the Inverse)
Let $M \in \mathrm{GL}_3(\mathbb{K}^*)$ and let M be regular. Then it holds true that $\left\| M_{\mathbb{A}}^{-1} \right\|$ is appreciable.

Proof. It is obvious that $\|M_{\mathbb{A}}\|$ has appreciable norm just by definition. Since $\|\cdot\|$ is submultiplicative we know that there is a $c \in \mathbb{R} \setminus \{0\}$ such that

$$1 = \|\mathrm{Id}\| = \left\| M_{\mathbb{A}}(M_{\mathbb{A}})^{-1} \right\| \leq \|M_{\mathbb{A}}\| \left\| (M_{\mathbb{A}})^{-1} \right\|.$$

Then we know that $\frac{1}{\|M_{\mathbb{A}}\|} \leq \|(M_{\mathbb{A}})^{-1}\|$ and since $\|M_{\mathbb{A}}\|$ is appreciable its inverse is also appreciable and so $\|(M_{\mathbb{A}})^{-1}\|$ cannot be infinitesimal.

We still have to show that the norm of $(M_{\mathbb{A}})^{-1}$ is not unlimited:

$$\begin{aligned}(M_{\mathbb{A}})^{-1} &= \frac{1}{\det(M_{\mathbb{A}})}(M_{\mathbb{A}})^{\triangle}\\ \Rightarrow \left\|(M_{\mathbb{A}})^{-1}\right\| &= \frac{1}{|\det(M_{\mathbb{A}})|}\left\|(M_{\mathbb{A}})^{\triangle}\right\|\end{aligned}$$

By Lemma 5.2.8 (Adjugate non–standard Matrix is limited) we know that all entries of the adjugate matrix $M_{\mathbb{A}}^{\triangle}$ are appreciable and furthermore $|\det(M_{\mathbb{A}})|^{-1}$ is appreciable since M is non-singular. The product of an appreciable and a limited number is limited.

Combining both arguments the norm of the inverse has to be appreciable. □

Completely analogously to projective geometry over $\mathbb{R}$ or $\mathbb{C}$ we define a projective transformation.

Definition 5.2.10 (Appreciable Projective Transformation)
Let $M \in \mathrm{GL}_3(\mathbb{K}^*)$. For $p \in \mathcal{P}^*$ and $l \in \mathcal{L}^*$ we define the *appreciable projective transformation by M* as follows:

$$p \mapsto M_{\mathbb{A}} p_{\mathbb{A}} \quad \text{and} \quad l \mapsto (M_{\mathbb{A}})^{-H} l_{\mathbb{A}}$$

with $M^{-H} := (\overline{M^{-1}})^T$. We call $M_{\mathbb{A}}$ an *appreciable transformation matrix*. If M is non-singular, we call the projective transformation *non-singular* as well. ◇

Theorem 5.2.11 (Non-singular Projective Transformations and Almost Equivalent)
Let M be a non-singular projective transformation and let $p, p' \in \mathcal{P}^*$ be almost equivalent. Then the images of p and p' under the regular projective transformation M are also almost equivalent.

Proof. We have to show that

$$\frac{1}{\|M_{\mathbb{A}} p_{\mathbb{A}}\|} M_{\mathbb{A}} p_{\mathbb{A}} \times \frac{1}{\|M_{\mathbb{A}} p'_{\mathbb{A}}\|} M_{\mathbb{A}} p'_{\mathbb{A}} \in \mathbb{I}^3.$$

Equivalently we show that the norm of the cross-product above is infinitesimal:

$$\begin{aligned}\frac{1}{\|M_{\mathbb{A}}p_{\mathbb{A}}\|}M_{\mathbb{A}}p_{\mathbb{A}} \times \frac{1}{\|M_{\mathbb{A}}p'_{\mathbb{A}}\|}M_{\mathbb{A}}p'_{\mathbb{A}} &= \left\| \frac{\det(M_{\mathbb{A}})}{\|M_{\mathbb{A}}p_{\mathbb{A}}\| \|M_{\mathbb{A}}p'_{\mathbb{A}}\|} M_{\mathbb{A}}^{-T}\left(p_{\mathbb{A}} \times p'_{\mathbb{A}}\right) \right\| \\ &\leq \frac{|\det(M_{\mathbb{A}})|}{\|M_{\mathbb{A}}p_{\mathbb{A}}\| \|M_{\mathbb{A}}p'_{\mathbb{A}}\|} \left\|M_{\mathbb{A}}^{-T}\right\| \|p_{\mathbb{A}} \times p'_{\mathbb{A}}\| \\ &= \frac{|\det(M_{\mathbb{A}})|}{\|M_{\mathbb{A}}p_{\mathbb{A}}\| \|M_{\mathbb{A}}p'_{\mathbb{A}}\|} \left\|M_{\mathbb{A}}^{-1}\right\| \|p_{\mathbb{A}} \times p'_{\mathbb{A}}\|\end{aligned}$$

Where we used that $\|\cdot\|$ is submultiplicative and that the transposition does not change the norm of a matrix for self-adjoint matrix norms. By assumption M is regular and therefore the determinant of $M_{\mathbb{A}}$ is appreciable (and so is its absolute value). By Theorem 5.2.5 (Appreciable Image), we know that the terms $\|M_{\mathbb{A}}p_{\mathbb{A}}\|, \|M_{\mathbb{A}}p'_{\mathbb{A}}\|$ are appreciable and so are their inverse fractions. Then by Lemma 5.2.9 (Norm of the Inverse), we know that the norm of the inverse of $M_{\mathbb{A}}$ is also appreciable. So the first two terms yield an appreciable result. Finally, by assumption we know that $\|p_{\mathbb{A}} \times p'_{\mathbb{A}}\|$ is infinitesimal, which multiplied with an appreciable number is infinitesimal, which proves the claim. □

Theorem 5.2.12 (Regular Projective Transformations and Almost Incident)
Let $l \in \mathcal{P}^*$ and $p \in \mathcal{L}^*$ be almost incident and let M be an non-singular projective transformation. Then the images of l and p under M are also almost incident.

Proof. Since l and p are almost incident there is an $\epsilon \in \mathbb{I}$ such that

$$\begin{aligned}&\epsilon = \langle l, p\rangle_* = \langle l_{\mathbb{A}}, p_{\mathbb{A}}\rangle = \langle (M_{\mathbb{A}})^{-H} l_{\mathbb{A}}, M_{\mathbb{A}}p_{\mathbb{A}}\rangle \qquad \Big| \cdot \|M_{\mathbb{A}}p_{\mathbb{A}}\| \left\|(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}\right\| \\ \Leftrightarrow &\epsilon \cdot \|M_{\mathbb{A}}p_{\mathbb{A}}\| \left\|(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}\right\| = \langle \frac{(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}}{\|(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}\|}, \frac{M_{\mathbb{A}}p_{\mathbb{A}}}{\|M_{\mathbb{A}}p_{\mathbb{A}}\|}\rangle\end{aligned}$$

By Theorem 5.2.5 (Appreciable Image), we know that $M_{\mathbb{A}}p_{\mathbb{A}}$ is appreciable and so is its norm.

We chose M to be a regular matrix and so the inverse of $M_{\mathbb{A}}$ exists and its matrix norm is appreciable by Lemma 5.2.9 (Norm of the Inverse). Analogously to the proof of Theorem 5.2.5 (Appreciable Image) we can bound the term $\left\|(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}\right\| \leq \left\|(M_{\mathbb{A}})^{-H}\right\| \|l_{\mathbb{A}}\| = \|(M_{\mathbb{A}})^{-1}\| \|l_{\mathbb{A}}\|$ which is a product of appreciable numbers and therefore appreciable. Then $\epsilon' := \epsilon \cdot \left\|(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}\right\| \|M_{\mathbb{A}}p_{\mathbb{A}}\|$ is infinitesimal as product of an infinitesimal and two appreciable numbers. Then, we can conclude for arbitrary $c, c' \in \mathbb{A}$

$$\Leftrightarrow \epsilon'' := \epsilon \cdot c \cdot c' = \langle \frac{(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}}{c' \|(M_{\mathbb{A}})^{-H}l_{\mathbb{A}}\|}, \frac{M_{\mathbb{A}}p_{\mathbb{A}}}{c \|M_{\mathbb{A}}p_{\mathbb{A}}\|}\rangle = \langle M^{-H}l, Mp\rangle_*$$

which shows that the appreciable projective transformation of l and p are almost incident, that was the claim. $\square$

Definition 5.2.13 (Almost Affine Projective Transformation)
Let $M \in \mathbb{GL}_3(\mathbb{R}^*)$ be a projective transformation. We call M *almost affine*, if there is an appreciable representative $M_\mathbb{A}$ of M which can be written as

$$M_\mathbb{A} = \begin{pmatrix} c & s & a \\ -s & c & b \\ \epsilon & \delta & 1 \end{pmatrix}$$

with $\epsilon, \delta \in \mathbb{I}$ and $c^2 + s^2 \notin \mathbb{I}$. $\diamond$

Lemma 5.2.14 (Almost Collinear and Regular Projective Transformations)
Let $p, q, r \in \mathcal{P}^*$ be three distinct points almost conlinear points in $\mathbb{R}^*\mathbb{P}^2$ and let M be a regular non-standard projective transformation. Then it holds true that the transformed points Mp, Mq, Mr are also almost collinear.

Proof. Consider the line $l := \mathbf{join}(p, q)$ and apply Theorem 5.2.12 (Regular Projective Transformations and Almost Incident) to l and r. $\square$

Lemma 5.2.15 (Almost Singular Transformations and Almost Relations)
There are almost singular projective transformations which do *not* preserve the property of being almost equivalent and almost collinear.

Proof. By example: let ϵ be a positive hyperreal and define the appreciable projective transformation (which is already an appreciable representative)

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}.$$

Then $\det(M) = \epsilon \in \mathbb{I}$, so M is almost singular. Pick the following points $a, b, c \in \mathcal{P}^*$:

$$a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad b = \begin{pmatrix} \epsilon \\ 0 \\ 1 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ \epsilon \\ 1 \end{pmatrix}$$

It's easy to see that the points are almost equivalent. Additionally the connecting line l of a and b is almost incident to c:

$$\mathbf{join}_*(a,b) = \frac{a \times b}{\|a \times b\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \langle c, l \rangle_* = 0 + \epsilon + 0 = \epsilon \in \mathbb{I}$$

But the appreciable projective transformations of a, b and c are

$$Ma = \begin{pmatrix} 0 \\ 0 \\ \epsilon \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad Mb = \begin{pmatrix} \epsilon \\ 0 \\ \epsilon \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad Mc = \begin{pmatrix} 0 \\ \epsilon \\ \epsilon \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

which are obviously neither almost collinear nor almost equivalent! □

Remark. This is quite remarkable since linear functions as they are described by matrices are continuous everywhere and so one would expect the transformations to preserve "almost"-relations. The crucial point here is not the transformation itself but the normalization afterwards: the function $\frac{x}{\|x\|}$ is undefined for zero vector and is discontinuous for infinitesimal vectors and this is where the points are "pushed apart".

The points a, b and c in the previous statement were chosen such that they are in the ϵ-kernel of M. Hence, Ma, Mb and Mc yield vectors with infinitesimal length that are after normalization not almost equivalent anymore.

5.3. Non-standard Cross-Ratios

All geometry is projective geometry.

Arthur Cayley

We will start with the basic definition of a cross-ratio in $\mathbb{K}^{*2}$.

Definition 5.3.1 (Cross-Ratio in $\mathbb{K}^{*2}$)
For $A, B, C, D \in \mathbb{K}^{*2}$ we define the cross-ratio in the standard way (see *e.g.* "Geometriekalküle" [114] p. 42).

$$(A, B; C, D) := \frac{[A, C][B, D]}{[A, D][B, C]}$$

◇

Remark. It is a bit surprising that the cross-ratio is analogously defined as in the standard setting, one would assume that due the appearance of infinitesimal and unlimited values a normalization would be necessary (for example the employment of the appreciable or the normalized determinant). But the cross-ratio is completely independent of the representative of a vector as we will see in the next statement.

Lemma 5.3.2 (Cross-ratios and Representatives)
Let $\lambda_A, \lambda_B, \lambda_C, \lambda_D \in \mathbb{K}^* \setminus \{0\}$. Then it holds true that

$$(A, B; C, D) = (\lambda_A \cdot A, \lambda_B \cdot B; \lambda_C \cdot C, \lambda_D \cdot D).$$

Proof. Completely analogously to "Geometriekalküle" [114] p. 42. □

Remark. Particularly interesting is the fact that we did only exclude 0 from the possible scaling factors of A, B, C, D and not all infinitesimal numbers. The reason for this is that the cross-ratio is completely independent of the representative even for non-Archimedean fields.

Lemma 5.3.3 (Projective Transformations and Cross-ratios)
Let $M \in \mathbb{K}^{*3\times 3}$ be regular or almost singular then

$$(A, B; C, D) = (M \cdot A, M \cdot B; M \cdot C; M \cdot D).$$

Proof. Similar to "Geometriekalküle" [114] p. 42. □

Remark. Again it is at least a bit astonishing that even if a projective transformation is almost singular it still preserves cross-ratios.

Theorem 5.3.4 (Almost Equivalent Cross-Ratios)
Let A, B, C, D and A', B', C', D' be vectors with $A \simeq A', B \simeq B', C \simeq C', D \simeq D'$ then it holds true:

$$\mathbf{sh}((A, B; C, D)) = \mathbf{sh}((A', B', C', D'))$$

Proof. By the continuity of the determinant. □

Remark. One has to be careful with the relation above: it does *not* hold true that

$$(A, B; C, D)) \simeq ((A', B'; C', D').$$

Take for example three disjoint points A, B, C and consider the cross-ratio $R := (A, B; C, D)$ for $D = A$. The value R of this cross-ratio is ∞ since we divide by zero. If one now takes $D \simeq A, D \neq A$ then the cross-ratio r will be unlimited, but $R \not\simeq \infty$.

5.4. Non-standard Conics

> The Circle is a Geometrical Line, not because it may be express'd by an Æquation, but because its Description is a Postulate. It is not the Simplicity of the Æquation, but the Easiness of the Description, which is to determine the Choice of our Lines for the Construction of Problems.
>
> Sir Isaac Newton in "Arithmetica Universalis", [99] p. 228

Definition 5.4.1 (Non-standard Conic)
For a given matrix $M \in \mathbb{C}^{*3\times 3}$ whose entries are not all equal to zero. We define a *non-standard conic with associated matrix M* by

$$\mathcal{C}_M := \{\, [p] \in \mathcal{P}_{\mathbb{C}^*} \mid p_{\mathbb{A}}^T M_{\mathbb{A}} p_{\mathbb{A}} \simeq 0\}$$

where $p_{\mathbb{A}}$ denotes an appreciable representative of p and $M_{\mathbb{A}}$ is an appreciable matrix representation of M. ◇

Definition 5.4.2 (Point and Conic Almost Incidence Relation)
Let a point $[p] \in \mathcal{P}_{\mathbb{C}^*}$ and a non–standard conic $\mathcal{C}_M$ with associated matrix M fulfill the relation of Definition 5.4.1 (Non-standard Conic): $p_{\mathbb{A}}^T M_{\mathbb{A}} p_{\mathbb{A}} \simeq 0$. Then we call p and $\mathcal{C}_M$ *almost incident* and write

$$[p]\, \mathcal{I}_{\mathbb{C}^*}\, [\mathcal{C}_M]$$

◇

Remark. Of course the standard incidence relation of a point and a conic is a special case of the almost incidence relation due to the fact that 0 is infinitesimal (and also the only infinitesimal value in $\mathbb{C}$).

Lemma 5.4.3 (Well-defined Conic)
The almost incidence relation of points and conics in Definition 5.4.2 (Point and Conic Almost Incidence Relation) is well defined.

Proof. We have to show that the relation $p_{\mathbb{A}}^T M_{\mathbb{A}} p_{\mathbb{A}} \simeq 0$ is well defined. Define $l_{\mathbb{A}} := M_{\mathbb{A}} p_{\mathbb{A}}$, then

$$p_{\mathbb{A}}^T M_{\mathbb{A}} p_{\mathbb{A}} = \langle p_{\mathbb{A}}, M_{\mathbb{A}} p_{\mathbb{A}}\rangle = \langle p_{\mathbb{A}}, l_{\mathbb{A}}\rangle$$

We can interpret $l_{\mathbb{A}}$ as element of $\mathcal{L}_{\mathbb{C}^*}$ (the polar) which is appreciable by Lemma 5.2.3 (Appreciable Matrix-Vector Product). Use Theorem 5.0.21 (Well-defined $\mathbb{I}$ Relation) where we showed that the property of being infinitesimal in the appreciable scalar product is well defined. This is the claim. □

Lemma 5.4.4 (Conics and Projective Halo)
Let $\mathcal{C}$ be a non-standard conic and $p \in \mathbb{CP}^2$ be incident to C. Then a point p' which almost equivalent $p' \simeq p$ is almost incident to $\mathcal{C}$.

In other words: the projective halo $\mathbb{P}$-**hal**(p) is almost incident to C.

Proof. Let $M \in \mathbb{C}^{*3\times 3}$ be the associated matrix to C. Since p' is almost equivalent to p there is $\lambda \in \mathbb{A}$ such that $p_{\mathbb{A}} \simeq \lambda \cdot p'_{\mathbb{A}}$ (since p is standard it's appreciable) and due to the incidence of p and $\mathcal{C}$ it holds true:

$$0 = p_{\mathbb{A}}^T M_{\mathbb{A}} p_{\mathbb{A}} \simeq \lambda^2 p_{\mathbb{A}}'^T M_{\mathbb{A}} p'_{\mathbb{A}}.$$

Since $\lambda^2 \in \mathbb{A}$ it holds true that $p_{\mathbb{A}}'^T M_{\mathbb{A}} p'_{\mathbb{A}} \in \mathbb{I}$ and thus $p' \, \mathcal{I}_{\mathbb{C}^*} \, \mathcal{C}$. □

We will now analyze cocircularity and therefore we need to define two special points of $\mathbb{CP}^2$.

Definition 5.4.5 (The points **I** and **J**, [111] p. 330)
We define the points I and J in $\mathbb{CP}^2$ by

$$\mathrm{I} := \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{J} := \begin{pmatrix} \mathrm{i} \\ 1 \\ 0 \end{pmatrix}.$$

◇

Remark. As proven in "Perspectives on Projective Geometry" ([111] p. 330 ff.) the points I and J are incident to all circles and a conic section is a circle if it passes through I and J.

Definition 5.4.6 (Almost Cocircular)
We call four points $A, B, C, D \in \mathbb{R}^*\mathbb{P}^2$ *almost cocircular*, if J is almost incident to the conic section defined by A, B, C, D, I. ◇

Theorem 5.4.7 (Cocircluarity)
Four points $A, B, C, D \in \mathbb{R}^*\mathbb{P}^2$ are almost cocircular, if and only if the following relations

holds true:

$$[CA\mathrm{I}][DB\mathrm{I}][DA\mathrm{J}][CB\mathrm{J}] - [CA\mathrm{J}][DB\mathrm{J}][DA\mathrm{I}][CB\mathrm{I}] \simeq 0.$$

Proof. Essentially the same proof as in [111] page 331 ff. if one replaces $=$ with $\simeq$. $\square$

Theorem 5.4.8 (Almost Cocircular and Almost Affine Transformations)
An almost affine projective transformation preserves the almost cocircular property and all non-singular projective transformations that leave I' and J', with $\mathrm{I} \simeq \mathrm{I}'$ and $\mathrm{J}' \simeq \mathrm{J}$, almost invariant (*i.e.* $I' \simeq M\mathrm{I}', \mathrm{J}' \simeq M\mathrm{J}'$) are almost affine.

Proof. By Definition 5.2.13 (Almost Affine Projective Transformation) an almost affine projective transformation M has an appreciable representative of the form

$$M = \begin{pmatrix} c & s & a \\ -s & c & b \\ \epsilon & \delta & 1 \end{pmatrix}$$

with $\epsilon, \delta \in \mathbb{I}$ and $c^2 + s^2 \notin \mathbb{I}$. Let I' be infinitely close to I which means there is $\lambda \in \mathbb{A}$ and $\mathrm{T} := (\tau_1, \tau_2, \tau_3)^T \in \mathbb{I}^3$ such that

$$\mathrm{I} \simeq \mathrm{I}' \Leftrightarrow \mathrm{I}' = \lambda \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$$

By Theorem 5.2.5 (Appreciable Image) we know that the product of $\lambda M \cdot I$ is appreciable and the product of $M \cdot \mathrm{T}$ is infinitesimal, so $\lambda M \cdot \mathrm{I} + M \cdot \mathrm{T}$ almost equivalent to $\lambda M \cdot \mathrm{I}$, then we find

$$\begin{aligned} M \cdot \mathrm{I}' &= \begin{pmatrix} c & s & a \\ -s & c & b \\ \epsilon & \delta & 1 \end{pmatrix} \left(\lambda \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \right) \\ &\simeq \lambda \begin{pmatrix} c & s & a \\ -s & c & b \\ \epsilon & \delta & 1 \end{pmatrix} \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\mathrm{i} \cdot c + s \\ \mathrm{i} \cdot s + c \\ -\epsilon \cdot \mathrm{i} + \delta \end{pmatrix} \simeq \begin{pmatrix} -\mathrm{i} \cdot c + s \\ \mathrm{i} \cdot s + c \\ 0 \end{pmatrix} = (c + \mathrm{i} \cdot s)\mathrm{I} \sim \mathrm{I} \end{aligned}$$

Analogously we can show the same property for J.

Conversely let M be a matrix with the property $\mathrm{I} \simeq M \cdot \mathrm{I}'$. We show the claim analogously to “Perspectives on Projective Geometry” [111] p. 336 ff. We start with the first two entries of the last row of $M_{\mathbb{A}}$: we claim that they are infinitesimal. Remember that all entries of an appreciable matrix are limited and at least one is appreciable.

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ x & y & \bullet \end{pmatrix} \left(\begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{pmatrix} \right) \simeq \lambda \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix}$$

This means that $-\mathrm{i}x + y \simeq 0$ but since x, y are real and limited this means that both entries have to be infinitesimal. The rest of the argumentation is analogous to the one in the mentioned source. $\square$

6. Removal of Singularities & Numerics

> Only geometry can hand us the thread [which will lead us through] the labyrinth of the continuum's composition, the maximum and the minimum, the infinitesimal and the infinite; and no one will arrive at a truly solid metaphysic except he who has passed through this.
>
> Gottfried Wilhelm von Leibniz

6.1. Non-standard Analysis and Removal of Singularities

Our original motivation to study non-standard analysis was the promise to resolve singularities in geometric constructions. So we follow up to that promise and will provide the reader with methods to do so. We will now combine non-standard analysis and non-standard projective geometry to achieve the goal.

Imagine the situation of an isolated removable singularity: on every epsilon ball around the singularity the function is well behaved and even analytic. By the notion of non-standard continuity every perturbation by in infinitesimal displacement will be infinitely close the analytic continuation of the function. We will now analyze Theorem 3.1.11 (Existence of a Continuous Path) by U. Kortenkamp and J. Richter-Gebert and its prerequisites from a non-standard viewpoint. The basic idea of the Theorem was that if a function Ψ does not vanish on an epsilon ball around a singularity t_0 and furthermore if there is a component Ψ_i of Ψ such that the fraction $\frac{\Psi_j(t_0)}{\Psi_i(t_0)}$ has a removable singularity then we can continuously extend the function at t_0. Recall the Theorem:

Theorem 3.1.11 (Existence of a Continuous Path, [76])
Let $\Psi : [0,1] \to \mathbb{C}^d$ be a continuous path and $t_0 \in [0,1]$. Let $A \subset \{1, \ldots, d\}$ be the set of indices of Ψ–components which are not constantly 0 on a neighborhood of t_0. If

1. $A \neq \emptyset$
2. for all $i, j \in A$: $\Psi_i(t)/\Psi_j(t)$ or $\Psi_j(t)/\Psi_i(t)$ has a removable singularity or is continuous at t_0,

then there are an $\epsilon > 0$, a continuous path $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ and a function $\lambda : B_\epsilon(t_0) \to \mathbb{C}$ such that $\lambda(t) \cdot \Theta(t) = \Psi(t)$ for all $t \in B_\epsilon(t_0) \setminus \{t_0\}$.

The described functions Θ, λ have the form

$$\Theta(t) = \frac{\Psi(t)}{\Psi_i(t)}, \quad \lambda(t) = \Psi_i(t)$$

where $i \in A$ is appropriately chosen such that all coordinate entries of Θ have a removable singularity or are continuous.

Firstly, we will generalize the non-vanishing property from an epsilon ball to the halo around t_0.

Lemma 6.1.1 (Non-vanishing on the Halo)
Assume the setting of Theorem 3.1.11 (Existence of a Continuous Path): let $\Psi : [0,1] \to \mathbb{C}^d$ be a continuous path and fractionally continuable at $t_0 \in (0,1)$. Denote by $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ the fractional continuation of Ψ at t_0, $\lambda : B_\epsilon(t_0) \to \mathbb{C}$ the removable prefactor, $A \subset \{1, \dots, d\}$ the set of all indices such that Ψ is not constantly zero on a neighborhood of t_0.

Define A' as the set of indices of Ψ–components which are not constantly 0 on $\mathbf{hal}(t_0) \setminus \{t_0\}$. Then the sets coincide: $A' = A$.

Proof. First we show that $j \in A \Rightarrow j \in A'$: if $j \in A$ there is a fixed $\epsilon \in \mathbb{R}, \epsilon > 0$ such that the following sentence is true:

$$\forall z \in \mathbb{C} : (|z - t_0| < \epsilon \wedge z \neq t_0 \Rightarrow \Psi_j(z) \neq 0).$$

By universal transfer of Theorem 4.2.13 we know that

$$\forall z \in \mathbb{C}^* : (|z - t_0| < \epsilon \wedge z \neq t_0 \Rightarrow \Psi_j^*(z) \neq 0).$$

By definition the distance of every member z' of $\mathbf{hal}(t_0)$ to t_0 is infinitesimal. Hence, every z' trivially fulfills the bound $|z' - t_0| < \epsilon$. Therefore $\Psi_j^*(z') \neq 0$, which is just differently phrased that $j \in A'$.

Now we show $j \in A' \Rightarrow j \in A$: we apply existential transfer: if $j \in A'$ the following sentence is true:

$$\exists \epsilon \in (\mathbb{R}^*)^+ \, \forall z \in \mathbb{C}^* : |t_0 - z| < \epsilon \Rightarrow |\Psi_j^*(z)| > 0.$$

Then by existential transfer of Theorem 4.2.13 also the following sentence is true

$$\exists \epsilon \in \mathbb{R}\, \forall z \in \mathbb{C} : |t_0 - z| < \epsilon \Rightarrow |\Psi_j(z)| > 0.$$

which means nothing else then $j \in A$. □

Lemma 6.1.2 (Limited Fraction)
Assume the setting of Theorem 3.1.11 (Existence of a Continuous Path): let $\Psi : [0,1] \to \mathbb{C}^d$ be a continuous path and fractionally continuable at $t_0 \in (0,1)$. Denote by $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ the fractional continuation of Ψ at t_0, $\lambda : B_\epsilon(t_0) \to \mathbb{C}$ the removable prefactor, $A \subset \{1,\ldots,d\}$ the set of all indices such that Ψ is not constantly zero on a neighborhood of t_0 and $i \in A$ the selected component as defined in Definition 3.1.12 (Fractionally Continuable).

Then it holds true for all $\Delta t \in \mathbb{I}, \Delta t \neq 0$:

$$\frac{\Psi_j^*(t_0 + \Delta t)}{\Psi_i^*(t_0 + \Delta t)} \in \mathbb{L}$$

$\forall j \in \{1,\ldots,d\}$. In other words the fraction is limited and has a complex number as shadow.

Proof. Firstly, we note that we do not divide by 0 due to Lemma 6.1.1 (Non-vanishing on the Halo). U. Kortenkamp and J. Richter-Gebert argue in [76] that there is a $g \in \mathbb{C}$ such that

$$g = \lim_{t \to t_0} \frac{\Psi_j(t)}{\Psi_i(t)} \text{ for all } t \in B_\epsilon(t_0)$$

Now we apply the rules on limits in Lemma 4.3.15 (Complex Limits):

$$\begin{aligned}
g &= \lim_{t \to t_0} \frac{\Psi_j(t)}{\Psi_i(j)} \\
\Leftrightarrow g &\simeq \frac{\Psi_j^*(t)}{\Psi_i^*(t)}, \quad t \simeq t_0, t \neq t_0 \\
\Leftrightarrow g &\simeq \frac{\Psi_j^*(t_0 + \Delta t)}{\Psi_i^*(t_0 + \Delta t)}, \quad \Delta t \in \mathbb{I}, \Delta t \neq 0 \\
\Rightarrow g &= \mathbf{sh}\left(\frac{\Psi_j^*(t_0 + \Delta t)}{\Psi_i^*(t_0 + \Delta t)}\right), \quad \Delta t \in \mathbb{I}, \Delta t \neq 0
\end{aligned}$$

Which is equivalent to the claim. □

Now we are able to formulate a first main result on the removal of singularities. The next theorem will give us a proper way to desingularize functions using methods of non–standard analysis.

Theorem 6.1.3 (Shadow of a Singularity)
Assume the setting of Theorem 3.1.11 (Existence of a Continuous Path): let $\Psi : [0,1] \to \mathbb{C}^d$ be a continuous path and fractionally continuable at $t_0 \in (0,1)$. Denote by $\Theta : B_\epsilon(t_0) \to \mathbb{C}^d \setminus \{0\}$ the fractional continuation of Ψ at t_0, $A \subset \{1,\dots,d\}$ the set of all indices such that Ψ is not constantly zero on a neighborhood of t_0 and $i \in A$ the selected component as defined in Definition 3.1.12 (Fractionally Continuable).

Then for an arbitrary $\Delta t \in \mathbb{I} \setminus \{0\}$ it holds true, for $j = 1,\dots,d$:

$$\Theta_j(t_0) = \mathbf{sh}\left(\frac{\Psi_j^*(t_0+\Delta t)}{\Psi_i^*(t_0+\Delta t)}\right).$$

Remark. In other words we can continuously extend the function Ψ at t_0 with

$$\Theta_j(t_0) = \mathbf{sh}\left(\frac{\Psi_j^*(t_0+\Delta t)}{\Psi_i^*(t_0+\Delta t)}\right).$$

Proof. By assumption $\Theta(t)$ is continuous, so we remember Theorem 4.3.12 (Continuity) which says that $\Theta(t_0) \simeq \Theta(t_0+\Delta t)$ for all $\Delta t \in \mathbb{I}$. With this argument, choose $\Delta t \neq 0$. For non-zero values of Δt Lemma 6.1.1 (Non-vanishing on the Halo) guarantees proper non-vanishing values.

$$\Theta_j(t_0) \simeq \Theta_j(t_0+\Delta t) \overset{\text{Def.}}{=} \frac{\Psi_j^*(t_0+\Delta t)}{\Psi_i^*(t_0+\Delta t)}$$

Now we apply Lemma 6.1.2 (Limited Fraction):

$$\frac{\Psi_j^*(t_0+\Delta t)}{\Psi_i^*(t_0+\Delta t)} \in \mathbb{L} \Rightarrow \mathbf{sh}\left(\frac{\Psi_j^*(t_0+\Delta t)}{\Psi_i^*(t_0+\Delta t)}\right) \in \mathbb{C}.$$

With the uniqueness theorem of shadows, Theorem 4.3.6 (Shadow), and the properties on limits, Lemma 4.3.15 (Complex Limits), we can conclude that:

$$\Theta_j(t_0) = \lim_{t\to t_0} \Theta_j(t) = \lim_{t\to t_0} \frac{\Psi_j(t)}{\Psi_j(t)} = \mathbf{sh}\left(\frac{\Psi_j^*(t_0+\Delta t)}{\Psi_i^*(t_0+\Delta t)}\right).$$

□

Example 6.1.4 (Circle Intersection Revisited)

We will now analyze the circle intersection of Example 3.4.1 (Disjoint Circle Intersection) with our new method. Essentially, the example reads like this: let $\Psi : [-2,2] \to \mathbb{C}$ be defined by

$$\Psi(t) := \begin{pmatrix} \sqrt{1-t^2} \\ 0 \\ \sqrt{1-t^2} \cdot t \end{pmatrix}$$

The function Ψ is singular for $t_0 = 1$ and $t_1 = -1$ since $\Psi(t_0) = \Psi(t_1) = 0$. Consider the situation for $t_0 = 1$:

Firstly, we observe that the premisses of Theorem 3.1.11 (Existence of a Continuous Path) hold true. In the notion of the theorem: $A = \{1,3\}$ and both $\Psi_i(t_0)/\Psi_j(t_0)$ and $\Psi_j(t_0)/\Psi_i(t_0)$ have a removable singularity, so the choice for the selected component i is arbitrary. Without loss of generality choose $i = 3$. Then we have for $\Delta t \in \mathbb{I} \setminus \{0\}, t_0 = 1$:

$$\begin{aligned} \frac{\Psi_1^*(t_0+\Delta t)}{\Psi_3^*(t_0+\Delta t)} &= \frac{\Psi_1^*(1+\Delta t)}{\Psi_3^*(1+\Delta t)} = \frac{\sqrt{1-(1+\Delta t)^2}}{\sqrt{1-(1+\Delta t)^2} \cdot (1+\Delta t)} \\ &= \frac{\sqrt{-\Delta t^2 - 2\Delta t}}{\sqrt{-\Delta t^2 - 2\Delta t} \cdot (1+\Delta t)} = \frac{1}{1+\Delta t} \end{aligned}$$

Where we used that $\sqrt{-\Delta t^2 - 2\Delta t}$ is a proper (infinitesimal) non-zero hypercomplex number, and we can simply cancel the fraction.

Then we apply the shadow:

$$\Theta_1(t_0) = \mathbf{sh}\left(\frac{\Psi_1^*(t_0+\Delta t)}{\Psi_3^*(t_0+\Delta t)}\right) = \mathbf{sh}\left(\frac{1}{1+\Delta t}\right) = \frac{1}{1} = 1.$$

Analogously we can calculate the other components:

$$\begin{aligned} \Theta_2(t_0) &= \mathbf{sh}\left(\frac{\Psi_2^*(t_0+\Delta t)}{\Psi_3^*(t_0+\Delta t)}\right) = \mathbf{sh}\left(\frac{0}{1+\Delta t}\right) = 0 \\ \Theta_3(t_0) &= \mathbf{sh}\left(\frac{\Psi_3^*(t_0+\Delta t)}{\Psi_3^*(t_0+\Delta t)}\right) = \mathbf{sh}\left(\frac{1+\Delta t}{1+\Delta t}\right) = 1. \end{aligned}$$

So the C^0–continuation of our function Ψ, which we denoted by $[\Theta]$ is given in the singularity $t_0 = 1$ by:

$$[\Theta(t_0)] = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

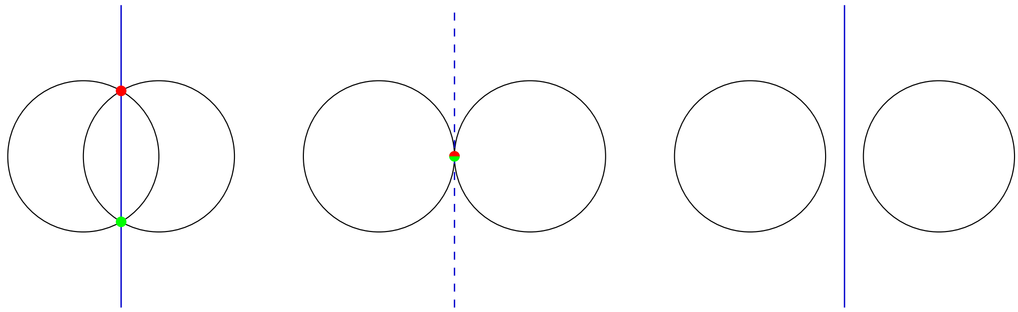


Figure 6.1.: C^0-continuation of the disjoint circle intersection.

and $[\Psi(t)]$ for all $t \neq 1$. A visualization of the example can be found in Figure 6.1. $\diamond$

Next stop will be a more direct approach to removing singularities. While Theorem 3.1.11 (Existence of a Continuous Path) goes the way of dehomogenization there is a more direct way to obtain the desired desingularized function using hypercomplex numbers.

Theorem 6.1.5 (Normalization and Continous Paths)
Let $\Psi : (0,1) \to \mathbb{C}^{d+1}$ be a continuous function and $t_0 \in (0,1)$. Let Ψ fulfill the following premisses

1. $\Psi(t)$ is not vanishing on $\mathbf{hal}(t_0) \setminus \{t_0\}$
2. $\exists L \in \mathbb{C}^{d+1} \setminus \{0\} : L \simeq \frac{\Psi(t_0+\Delta t)}{\|\Psi(t_0+\Delta t)\|} \simeq \frac{\Psi(t_0+\Delta t')}{\left\|\Psi(t_0+\Delta t')\right\|} \quad \forall \Delta t, \Delta t' \in \mathbb{I} \setminus \{0\}$.

Then there exists $\epsilon \in \mathbb{R}, \epsilon > 0$ and a continuous function $\Theta(t) : B_\epsilon(t_0) \to \mathbb{C}^{d+1} \setminus \{0\}$ such that

$$[\Psi(t)] = [\Theta(t)]$$

for all $t \in B_\epsilon \setminus \{t_0\}$.

This means that $[\Theta]$ is a C^0–continuation of $[\Psi]$ at t_0.

Proof. We will only consider the case where $\Psi(t)$ vanishes at t_0. Otherwise the function Θ is continuous as quotient of continuous non–vanishing functions. The functions $[\Psi]$ and $[\Theta]$ coincide in projective space since $\|\Psi(t)\|$ is a non-zero scalar factor for all $t \neq t_0$ and so the equivalence classes coincide in a projective setting.

So let Ψ vanish at t_0. Since the function does not vanish on $\mathbf{hal}(t_0) \setminus \{t_0\}$, by assumption 1), we can apply Lemma 4.3.20 (Constant on Open Sets) which says that there is a $\mathbb{R} \ni \epsilon > 0$ such that Ψ does not vanish on $B_\epsilon \setminus \{t_0\}$. So the function

$\Theta(t) : B_\epsilon \setminus \{t_0\} \to \mathbb{C}, t \mapsto \frac{\Psi(t)}{\|\Psi(t)\|}$ is defined and as fraction of continuous functions again continuous.

The continuous function $\Theta(t) : B_\epsilon(t_0) \to \mathbb{C}^{d+1} \setminus \{0\}$ is defined, for an arbitrary $\Delta t \in \mathbb{I} \setminus \{0\}$, as follows:

$$\Theta(t) := \begin{cases} \frac{\Psi(t)}{\|\Psi(t)\|}, & \text{if } t \neq t_0 \\ \mathbf{sh}\left(\frac{\Psi(t+\Delta t)}{\|\Psi(t+\Delta t)\|}\right), & \text{if } t = t_0 \end{cases}$$

Now we can use Lemma 4.3.15 (Complex Limits), the hypercomplex notion of limits, that says

$$\begin{aligned} &\lim_{t \to t_0} \Theta(t) = L \in \mathbb{C},\ t \in B_\epsilon(t_0) \setminus \{t_0\} \\ &\Leftrightarrow \Theta(t) \simeq L\ \forall t \in (B_\epsilon)^* : t \simeq t_0,\ t \neq t_0 \end{aligned}$$

which is equivalent to prerequisite 2) since:

$$\begin{aligned} &\Theta(t) \simeq L\ \forall t \simeq t_0, t \neq t_0 \\ \Leftrightarrow\ &\Theta(t) \simeq L\ \forall t \in \mathbf{hal}(t_0) \setminus \{t_0\} \\ \Leftrightarrow\ &\Theta(t_0 + \Delta t) = \Theta(t_0 + \Delta t')\ \forall \Delta t, \Delta t' \in \mathbb{I} \setminus \{0\} \end{aligned}$$

this means we can continuously extend the function Θ to t_0 with $\Theta(t_0) = L$.

Putting everything together: by construction Θ is continuous, with continuous extension $\Theta(t_0) = L$. Since the projection $[\cdot] : \mathbb{C}^{d+1} \to \mathbb{CP}^d$ is continuous by definition, the function $[\Theta] : B_\epsilon(t_0) \to \mathbb{CP}^d$ is continuous as composition of continuous functions. Since rescaling Ψ by its (non-zero) norm does not change the equivalence class in a projective setting it holds true:

$$[\Psi(t)] = [\Theta(t)] \quad \forall t \in B_\epsilon \setminus \{t_0\}$$

So $[\Theta]$ is a C^0–continuation of $[\Psi]$ at t_0. □

Remark. In the previous theorem, the domain for t_0 can easily be extended from $(0, 1)$ to its closure $[0, 1]$ with the same arguments. For the sake of readability we omitted the generalization.

Definition 6.1.6 (Normalized Function)

Let $t_0 \in \mathbb{R}$ and $f : B_\epsilon(t_0) \to \mathbb{C}^n \setminus \{0\}$ be continuous. We call the function

$$f_{\|\cdot\|}(t) := \frac{f(t)}{\|f(t)\|}$$

the *normalization of f*. $\diamond$

Remark. We usually define normalization via the Euclidean norm $\|\cdot\|_2$ but actually it is not important which norm we use. By the equivalence of the norms in $\mathbb{C}^d$ we always will obtain appreciable representatives with unit length one (according to the chosen norm). For illustrating examples it is very practical to use the maximum norm and normalize the vector by the absolute value of the maximal element. Even more conveniently we can use the magnitude machinery we introduced before and actually do not have to divide by the absolute value of the maximum but by the value itself! The norm of the maximal value and the maximal value itself only differ by the complex phase, which is an appreciable number. Therefore, we simply can pick the arg max of the maximum norm and divide by the maximal component.

Corollary 6.1.7 (Projective Shadow and Normalized Functions)
Let $t_0 \in \mathbb{R}$ and $f : B_\epsilon(t_0) \to \mathbb{C}^{d+1} \setminus \{0\}$ be continuous. Then it holds true:

$$[f_{\|\cdot\|}(t)] = \mathbf{psh}([f(t + \Delta t)]) \; \forall \Delta t \in \mathbb{I}$$

Proof. Obvious by Definition 5.0.18 (Projective Shadow). $\square$

Remark. This means that condition 2. of Theorem 6.1.5 (Normalization and Continous Paths)

$$\exists L \in \mathbb{C}^{d+1} : L \simeq \frac{\Psi(t_0 + \Delta t)}{\|\Psi(t_0 + \Delta t)\|} \simeq \frac{\Psi(t_0 + \Delta t')}{\|\Psi(t_0 + \Delta t')\|} \quad \forall \Delta t, \Delta t' \in \mathbb{I} \setminus \{0\}$$

can be written as

$$\exists L \in \mathbb{CP}^d : [L] = \mathbf{psh}(\Psi(t_0 + \Delta t)) = \mathbf{psh}(\Psi(t_0 + \Delta t')) \quad \forall \Delta t, \Delta t' \in \mathbb{I} \setminus \{0\}.$$

Instead taking limits in the space $\mathbb{C}^{d+1}$ and then use a projection to $\mathbb{CP}^d$ one can also directly use the quotient space continuity.

Remark. This is an important step towards a more algorithmic approach to resolve singularities. The theorem allows us to directly compute the singularity free representation of the continuous path. If we know that a function is continuous up to a singular point we can use Lemma 4.3.14 (Squeezing Limits) to check whether the right hand side and left hand side limits coincide with an arbitrary Δt. This gives a practical algorithm to test whether a singularity can be resolved.

Remark. Just to make the important point very clear: since our perturbations are infinitesimal, the results we obtain from the proposed method in Theorem 6.1.5 (Normalization and Continous Paths) are **exact**. Our perturbation does **not** entail (additional) numeric error.

Example 6.1.8 (Yet Another Circle Intersection Example)
Remember our circle intersection example which essentially is encapsulated in

$$\Psi(t) := \begin{pmatrix} \sqrt{1-t^2} \\ 0 \\ \sqrt{1-t^2}\cdot t \end{pmatrix}$$

for $t \in [-2,2]$ with singularities at $t_0 = 1$ and $t_1 = -1$. Now we want to utilize Theorem 6.1.5 (Normalization and Continous Paths) and Corollary 6.1.7 (Projective Shadow and Normalized Functions).

Pick $\Delta t \in \mathbb{I} \setminus \{0\}$ arbitrarily and consider the normalization of $\Psi(t_0 + \Delta t)$ for which we use the arg max:

$$\begin{aligned} \Psi_{\|\cdot\|}(1+\Delta t) &= \frac{1}{\sqrt{1-(1+\Delta t)^2}\cdot(1+\Delta t)} \begin{pmatrix} \sqrt{1-(1+\Delta t)^2} \\ 0 \\ \sqrt{1-(1+\Delta t)^2}\cdot(1+\Delta t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{1+\Delta t} \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

The shadow of the term above is obviously independent of Δt and equals $(1,0,1)^T$ which then can be used to define a C^0-continuation at t_0. $\diamond$

Example 6.1.9 (Premisses cannot be neglected)
Here is an example that should explain why we had to stipulate the second premise of Theorem 6.1.5 (Normalization and Continous Paths), the congruency of the projective

shadows. Define $\Psi : [-1, 1] \to \mathbb{R}^2$ as

$$\Psi(t) := \begin{pmatrix} t \\ |t| \end{pmatrix}$$

Although both components are continuous we cannot apply Theorem 6.1.5 (Normalization and Continous Paths) and Corollary 6.1.7 (Projective Shadow and Normalized Functions) because the projective shadows do note coincide. Pick an arbitrary positive infinitesimal hyperreal number Δt. Then for $t_0 = 0$ it holds true:

$$\mathbf{psh}(\Psi(t_0 + \Delta t)) = \mathbf{sh}\left(\frac{1}{|\Delta t|}\begin{pmatrix} \Delta t \\ |\Delta t| \end{pmatrix}\right) = \mathbf{sh}\left(\frac{1}{\Delta t}\begin{pmatrix} \Delta t \\ \Delta t \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

But for the negation of Δt:

$$\mathbf{psh}(\Psi(t_0 - \Delta t)) = \mathbf{sh}\left(\frac{1}{|-\Delta t|}\begin{pmatrix} -\Delta t \\ |-\Delta t| \end{pmatrix}\right) = \mathbf{sh}\left(\frac{1}{\Delta t}\begin{pmatrix} -\Delta t \\ \Delta t \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This is not really astonishing since $t/|t|$ has a non-removable singularity at $t_0 = 0$. $\diamond$

6.2. Stability of a Solution

> Wie im Leben der Völker das einzelne Volk nur dann gedeihen kann, wenn es auch allen Nachbarvölkern gut geht, und wie das Interesse der Staaten es erheischt, dass nicht nur innerhalb jedes einzelnen Staates Ordnung herrsche, sondern auch die Beziehungen der Staaten unter sich gut geordnet werden müssen, so ist es auch im Leben der Wissenschaften.
>
> David Hilbert, Axiomatisches Denken [62]

The concept of quasi continuity was so far only developed along a predefined and only time dependent path. Unfortunately, this is not the whole picture. There are situations where one can resolve a singularity of a construction along a fixed path but as soon as the defining free or semi–free elements on which an object may depend are perturbed slightly the construction may become unstable or even discontinuous. One particular situation is again the intersection of two circles: this time we do not investigate the tangential situations as in Example 3.4.1 (Disjoint Circle Intersection) but rather an even more degenerate one: the unification of the two circles.

Example 6.2.1 (Unified Circles Intersection)
Again, we will intersect two circles of the same radius and draw the connecting line of the two intersections. Consider two (non-degenerate) circles C, D with radius $r > 0$ and disjoint midpoints M_C and M_D. If one now moves M_D along a straight segment (parameterized using the unit inverval) passing though M_C at some time t_0, the two circles unify and the intersections of C and D expand from 4 points (counting I and J) to an infinite set. Along a linear path it is reasonable to resolve the singularity occurring at t_0. But if one chooses to turn at t_0 and move M_D continuously to a different direction the intersection points will behave discontinuous. See the pictures in Figure 6.3 and Figure 6.4 on page 101 for an illustration of the situation. $\diamond$

As U. Kortenkamp and J. Richter-Gebert show in [76], geometric constructions can be partitioned to (semi–)free and dependent elements. Then (semi–)free elements can be used to alter the configuration. As argued in [76] we will encapsulate the movement of (semi–)free elements into a movement alone a straight line in a (possibly high–dimensional) space. We will denote the start point of the movement by A and the end point by B. Diverging from [76], we will view A and B not as points in $\mathbb{C}^{k+1}$ but rather as projective objects in $\mathbb{CP}^k$. Dynamic movements can then be modeled using the function $P : [0, 1] \to \mathbb{CP}^k$ with $v(t) = t \cdot A + (t - 1) \cdot B$. For further details, we refer the reader to the paper [76]. We will now discuss the continuity of a construction on the neighborhood around the elements of the image of P.

Definition 6.2.2 (Extended $\boldsymbol{C^0}$-continuation)
Let $V \subset \mathbb{CP}^k$ be open and let $\Psi : W \subset V \to \mathbb{CP}^d$ continuous. If there is an open set Y with $W \subsetneq Y \subset V$ and a continuous function $\hat{\Psi} : Y \to \mathbb{CP}^d$ with $\hat{\Psi}(w) = \Psi(w) \forall w \in W$, then we call Ψ *extended C^0–continuable on Y* and $\hat{\Phi}$ the *extended C^0–continuation of Ψ on Y*.

If even holds true that $Y = V$ then we call Ψ *extended quasi continuous on V*. $\diamond$

An illustration of the situation can be found in Figure 6.2.

Remark. It it obvious that if a function is extended C^0-continuable on a certain domain, then it is also C^0-continuable on a path inside the domain.

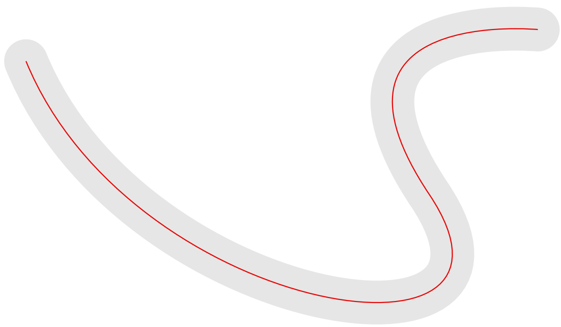


Figure 6.2.: **The difference between a C^0–continuation and an extended C^0–continuation quasi continuity:** while the first defines continuity only along a curve (red in the picture), the extended C^0–continuation requires continuity in an open neighborhood (grey) around the curve. The neighborhood might be restricted by additional constraints for semi–free elements.

Remark. In their paper "Grundlagen dynamischer Geometrie" [76] U. Kortenkamp and J. Richter-Gebert show that the movement of dependent elements induced by free elements along a fixed path in $\mathbb{V}$ can be modeled using only the time dimension.

Additionally Definition 6.2.2 (Extended C^0-continuation) also ensures continuity of the movement of a dependent element if the (semi–)free elements move continuously not only depending on a time parameter, along a fixed path, but even more generally in in space. This covers also cases like in Figure 6.3 and Figure 6.4 on page 101.

Essentially v models the movement of the (semi–)free elements and Ψ the movement of a dependent element implied by the variation of the (semi–)free elements. The function composition $\phi = \Psi \circ v$ then models the complete movement only dependent on a time parameter.

Theorem 6.2.3 (Normalization and Extended Continous Paths)
Let $v : [0,1] \to \mathbb{C}^{k+1}$ be continuous, Y be open with $v([0,1]) \subset Y \subset \mathbb{C}^{k+1}$, $\Psi : Y \to \mathbb{C}^{d+1}$ a function that is continuous on $v([0,1])$, *i.e.* $\Psi|_{v([0,1])} : v([0,1]) \to \mathbb{C}^{d+1}$ is a continuous function, and let $t_0 \in [0,1]$. Let Ψ fulfill the following premisses:

1. $\exists \epsilon \in \mathbb{R}, \epsilon > 0$: $\Psi(v(t))$ does not vanish on a dense subset X of $B_\epsilon(v(t_0)) \cap Y$

2. Ψ can be extended on $(\overline{X})^*$ such that $\exists L \in \mathbb{C}^{d+1} \setminus \{0\}$:

$$L \simeq \frac{\Psi(v(t_0) + \Delta v)}{\|\Psi(v(t_0) + \Delta v)\|} \simeq \frac{\Psi(v(t_0) + \Delta v')}{\|\Psi(v(t_0) + \Delta v')\|}$$
$$\forall \Delta v, \Delta v' \in \mathbb{I}^{k+1} : v(t_0) + \Delta v, v(t_0) + \Delta v' \in (\overline{X})^*.$$

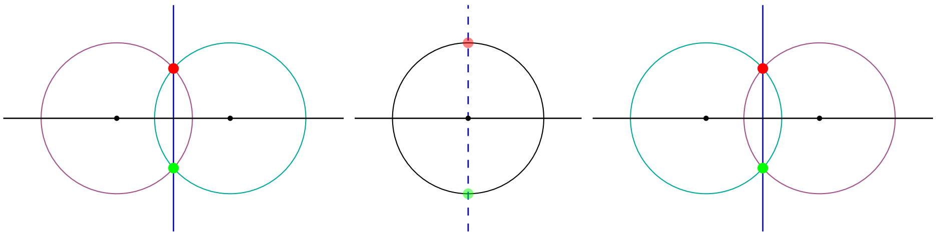


Figure 6.3.: **The Unification of Circles (semi-free)**: the connecting line of the two circle intersections is extended C^0-continuable if one assumes the circle centers bound to a common line.

Then there exists a continuous function $\Theta(t) : B_\epsilon(v(t_0)) \cap Y \to \mathbb{C}^{d+1} \setminus \{0\}$ such that

$$[\Psi(v(t))] = [\Theta(v(t))]$$

for all $t \in X$.

This means that $[\Theta]$ is an extended C^0–continuation of $[\Psi]$ on the open set $\mathbf{Int}(\overline{B_\epsilon(v(t_0)) \cap Y})$.

Proof. By assumption 1. the function

$$\Theta(v) := \begin{cases} \dfrac{\Psi(v)}{\|\Psi(v)\|}, & \text{if } v \in X \\ \lim_{n\to\infty} \dfrac{\Psi(v_n)}{\|\Psi(v_n)\|}, & \text{if } v \in \overline{X}, \text{ with } v_n \to v \ (n \to \infty) \end{cases}$$

is well defined. By transfer we know that Ψ does not vanish on $\mathbf{hal}(v(t_0)) \cap Y^*$ and so Θ does not vanish either. By Theorem 4.3.12 (Continuity), assumption 2. assures the continuity of Θ at $v(t_0)$ and we can continuously extend Θ at $v(t_0)$ with $\mathbf{sh}(\Psi(v(t_0)+\Delta v))$ with an arbitrary Δv that fulfills the requirements of assumption 2. Back in projective space, it holds true

$$[\Psi(v)] = [\Theta(v)] \quad \forall v \in X.$$

and therefore $[\Theta]$ is a C^0-continuation of $[\Psi]$ on the open set $\mathbf{Int}(\overline{B_\epsilon(v(t_0)) \cap Y})$. □

Remark. Again we can rewrite condition 2. of Theorem 6.2.3 (Normalization and Extended Continous Paths) as follows:

$$\exists L \in \mathbb{CP}^d : [L] = \mathbf{psh}(\Psi(v(t_0) + \Delta v)) = \mathbf{psh}(\Psi(v(t_0) + \Delta v')).$$

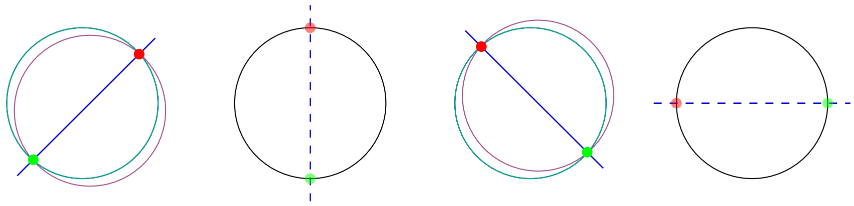


Figure 6.4.: **The Unification of Circles (free)**: if the centers of the circles are not bound to to a one dimensional subspace the connecting line of the intersection behaves discontinuously, thus the problem is not extended continuable.

6.3. The Relation to Perturbation Theory

> So that in the nature of man, we find three principal causes of quarrel. First, competition; secondly, diffidence; thirdly, glory.
>
> Thomas Hobbes, Leviathan [64]

While the algorithmic removal of singularities in dynamic geometry is relatively unexplored, the methods presented are related to the so called "perturbation theory" in the field of computational geometry. We will briefly discuss the concepts and integrate our theory of non-standard analysis in projective geometry.

One and probably the most important difference is that the field of computational geometry usually does not employ projective geometry. Devadoss *et al.* [26] do not mention the concepts of projective geometry at all and De Berg *et al.* [22] just in an exercise on page 333. Only the CGAL project [130] employs affine homogeneous coordinates to avoid divisions but prohibits the usage of far points. Furthermore, many problems in computational geometry arise from degenerate input and numerical instability, although these are often removable singularities which are usually computationally stable and "the true value" is only shadowed (pun certainly intended). But there are also symbolical approaches, like by Yap *et al.* [135], which are for sure related to our method.

We will follow the nice survey article "The Nature and Meaning of Perturbations in Geometric Computing" by R. Seidel [120]. Most problems of computational geometry can be described using *extended algebraic decision trees*. These ternary trees are structured as follows: each interior node v is labeled by a test function $f_v : \mathbb{R}^n \to \mathbb{R}$ and its branches labeled $-1, 0$ and $+1$ respectively. Using these test functions the label of the root node is computed and then recursively propagated through the sub trees. The task of perturbation here is to slightly deflect the input parameters such that certain assumption that the algorithm makes are fulfilled, *e.g.* general position of the input points. Several methods have been developed over the years: by Edelsbrunner *et al.* [31], Emiris *et al.* [35, 34] and most general by Yap [135]. The use the derivatives of functions, analogously to the methods presented in Section 3.3 (Singularities and Derivatives). Interestingly, Yap already was aware that non–standard analysis would be the proper model for treating perturbation. He states:

> Given a point $a \in \mathbb{R}^n$ and a set $U \subset \mathbb{R}[x]$ of polynomials, the idea of "U–perturbation at a" is to choose a sufficiently small $d \in \mathbb{R}^n$ such that
>
> $$p(a) \text{ and } p(a+d) \text{ have the same sign whenever } p(a) \neq 0 \text{ and}$$
> $$p(a+d) \neq 0 \quad \forall p \in U$$
>
> If U is the entire ring $\mathbb{R}[x]$ then it not hard to see that there cannot exist a choice of d to achieve this perturbation. Yet our evaluation scheme appears to be a perturbation of the entire set $\mathbb{R}[x]$. The resolution of this paradox must therefore lie in choosing a non–standard (infinitesimal) d. This is precisely our solution although we prefer a more direct and informative approach instead of invoking non–standard analysis.

According to Seidel [120], Canny already proposed the usage of automatic differentiation in Yap's approach, which also would applicable to the differentiable case as analyzed in Section 3.2 ff.

D. Michelucci proposed an ϵ arithmetic [93] to implement Yap's approach. Introducing a polynomial ring over one infinitesimal ϵ and using logarithmic lexicographical ordering. Operating over the rational numbers he does not treat radical expressions, though. He also proposes the usage of functional programming and especially "lazy evaluation" (see for example [66] for an overview of functional concepts) which can operate even on unbounded expressions.

One more recent method is presented by G. Irving and F. Green [67]. They introduce several infinitesimal numbers with relative ordering of the monomials of the infinitesimal numbers which form a ring but no field as the hyperreals (or the Levi-Civita field) do. They mainly treat signs of polynomial predicates and just mention that they also could handle rational functions of polynomials. Unlimited or infinitesimal results are used as indicators for faulty results, while we treat them as first class citizen (as seen in Example 5.1.4 (Geometriekalküle) and Example 6.1.4 (Circle Intersection Revisited)). Furthermore they state that they implemented a square root function which could handle rational expressions, but again not for the unlimited case like $\sqrt{\frac{1}{\Delta t}}$. This is of course fine for their use case but unsuitable for enlarged projective spaces like $\mathbb{C}^*\mathbb{P}^2$.

Often it is argued that perturbation in computational geometry entails a post processing step (mentioned in [120, 67, 53]). This is not true for our use case, the only thing we apply is the shadow function to have a projection back to the projective space.

The topic is quite extensive and for further reading we can recommend the "Handbook of Discrete and Computational Geometry" by Goodman *et al.* [53] which devotes a complete chapter to the robustness of algorithms (chapter 45: "Robust Geometric Computation").

6.4. Fast Numerical Methods

> Ich glaube nichts ohne eine Implementierung!
>
> J. Richter-Gebert, 2017

6.4.1. Derivatives Revisited

As we have seen before in the circle intersection example (Example 3.4.1 (Disjoint Circle Intersection)) there is no way standard derivatives could resolve singularities involving radical expressions. But there also an even more severe practical obstacle: computing time.

Consider the following functions, which we borrowed from U. Kortenkamp [76]:

Example 6.4.1 (Non–standard Arithmetics)
Given two function $f, g : \mathbb{R} \to \mathbb{R}$ with $f : x \mapsto x^{1001}$ and $g : x \mapsto x^{1000}$. For $x_0 = 0$ both functions coincide for 1000 derivatives and only the 1001 derivatives makes them distinguishable from a numerical perspective. So every algorithm which would use the classical de L'Hospital's rule to evaluate $f(x_0)/g(x_0)$ would fail with a reasonable depth of derivation. The example is not far fetched, it would be easily constituted with von-Staudt constructions.

In the hyperreal setting one could perturb the function evaluating $x_0' = 0 + \Delta x$ for $\Delta x \in \mathbb{I} \setminus \{0\}$, then by using continuity (see Theorem 4.3.12 (Continuity)) it holds true that $f(x_0')/g(x_0') \simeq f(0)/g(0)$ after resolving the singularity. Furthermore for an appropriate implementation one could simply evaluate the expression $f(x_0')/g(x_0') = \Delta x^{1001}/\Delta x^{1000}$. Which would yield Δx and therefore the continuation would be $\mathbf{sh}(\Delta x) = 0$. $\diamond$

One could probably come up with a solution for the problem of the example above without evoking non-standard analysis. Still this would not be a viable solution for removable essential singularities for expressions like $\lim_{x\to 0}\frac{\sqrt{x}}{\sqrt{x}}$. The automatic differentiation community realized that this problem should be addressed and proposes the ansatz of a "Laurent Model with variable Exponent Bounds" and stating in the book "Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation" by A. Griewank and A. Walther, [55] p. 256:

> "For simplicity we will shy away from such fully blown Levi-Civita numbers."

Which is quite remarkable since this is the second time that people are not willing to introduce nonstandard analysis, as stated in the chapter about the relation to perturbation theory (see page 102). Essentially in the book a simpler model for the Levi-Civita numbers is developed, which we will introduce later on. For details see [55] p. 269ff.

We already mentioned the Algorithm by D. Gruntz [57, 56] in Section 3.5.4 (Computer Algebra Systems). Still the performance of the algorithm is suboptimal since it is tuned for a larger set of problems at the expense of slower execution speed. For example the evaluation of

$$\lim_{t\to -1}\frac{t\cdot\sqrt{1-t^2}^{100}}{|\sqrt{1-t^2}|}$$

takes around 4 seconds in the implementation of SymPy [92], as already mentioned on page 29.

6.4.2. There is no such Thing as a Free Non-standard Lunch

When we looked into the implementation of non–standard analysis, we ran into quite some difficulties. Due to the non–constructive ansatz using ultrafilters and the axiom of choice a numerical implementation is generally very hard to achieve. One particular curious thing is the following: even though the existence is guaranteed, it is undecidable for certain expressions which sign they attain [6].

Furthermore the choice of the ultrafilter is not unique: consider for example the sequences $\langle 1, 0, 1, 0, \ldots\rangle$ and $\langle 0, 1, 0, 1, \ldots\rangle$: an ultrafilter could identify both sequences with either $\mathbf{1}$ or $\mathbf{0}$. Nevertheless, by the continuum hypotheses the choice of an particular ultrafilter is irrelevant: the fields determined by the filter are isomorphic ([52] p. 33).

J. Fleuriot [39] uses higher–order logic in Isabelle [101] to implement non–standard analysis. Since we are interested in a practical implementation for Cinderella [112] and CindyJS [47, 48, 95] this was not an option.

There are also constructive approaches to the hyperreals, for example by E. Palmgren [103, 104, 105]. Furthermore, there are the so called "Harthong-Reeb" numbers that rely on the existence of an infinitely large integer ω. These were used to explore possible non-standard discretizations of geometric objects, also in a software implementation using Coq [3]. For details see [88] and for a similar (but non-constructive) approach by J. Fleuriot see [40].

We decided to go a different way for a good reason: we wanted to implement a non-Archimedean field which allows us to perform the arithmetical operations we introduced, while achieving maximal performance for a real time implementation. As it turns out: most of real-world applications can be implemented on a much smaller field, which is implementable with good performance.

6.4.3. Implementation of a Non–Archimedean Field

> But if the question, "What are the infinitesimals?" arises, we do not have a really good answer, except for something like, "Mathematical logic takes care of that."
>
> Roman Kossak, [78]

On the quest for the proper field to implement we considered several candidates.

First we looked into **Dual Numbers**. They can be used to define automatic differentiation (see for example [36]) which would be useful for differentiable removable singularities. We will follow the book by Benz: "Vorlesungen über Geometrie der Algebren" [5]. Dual numbers are tuples of the form (a, b) for $a, b \in \mathbb{R}$. One can define summation and multiplication by

$$\begin{aligned} (a,b) + (a',b') &:= (a + a', b + b') \\ (a,b) \cdot (a',b') &:= (a \cdot a', a \cdot a' + b \cdot a') \end{aligned}$$

Define $\epsilon := (0, 1)$ then one can write $(a, b) = (a, 0) + (b, 0) \cdot (0, 1) = a + b \cdot \epsilon$. And for ϵ it holds true that $\epsilon^2 = 0$. Unfortunately dual numbers do not form a field, ϵ is a zero divisor (assume there is an inverse element ϵ^{-1} of ϵ, then $\epsilon \cdot \epsilon^{-1} = 1 \Rightarrow \epsilon^2 \cdot \epsilon^{-1} = \epsilon \Rightarrow 0 = \epsilon$, a contradiction). This is not a huge surprise, every other result would contradict Frobenius theorem on real division algebras [46] which states that finite-dimensional associative division algebras have to be isomorphic to either: the real numbers, the complex numbers or the quaternions), so we can't invert certain expressions. This is very unhandy, if we want to consider projective shadows, which involves a normalization by division.

So dual numbers could not be the answer. There are a lot of other non–archimedean field out there: Surreal Numbers [17] or the Dehn plane [23] name just a few. A good overview can be found in [107]. We formulated requirements which a potential algebraic structure should fulfill:

1. it should be non–archimedean, *i.e.* contain unlimited numbers
2. it has to be a field, so we would have infinitesimal members (with 1.)
3. it ought to be easy implementable on a computer and have efficient operations in order to achieve real time computations
4. it should be an algebraically closed
5. $\mathbb{C}$ should be a subset of the field, so we can express complex projective geometry
6. the field of Puiseux series be should a subset, so that dynamic geometry can be expressed in the way U. Kortenkamp and J. Richter-Gebert proposed in [76].

Power Series - an Overview

In more details the last point of our demands suggests that the field that come into consideration should be a generalized power series.

We will now give an overview of (generalized) power series and their properties. We will follow the nice overview article by M. Berz and K. Shamseddine [121]:

The ring of **formal power series** $[[\mathbb{R}]]$ is not a field, *e.g.* for $f(T) := T$ the inverse $f'(T) := T^{-1}$ is not in $[[\mathbb{R}]]$. Also it does not contain infinitesimal elements, this violates our requests 1 and 2.

The canonical next candidate would be the **formal Laurent series**, which are isomorphic to $[[\mathbb{R}^{\mathbb{Z}}]]$. They are not closed, *e.g.* for $f(T) := T^2 - 1$ the equation $f(T) = 0$ does not have a solution in $[[\mathbb{R}^{\mathbb{Z}}]]$ or $[[\mathbb{C}^{\mathbb{Z}}]]$.

The natural generalization of Laurent series are the **Puiseux series**. They are the algebraic closure of the Laurent series ([16] p. 27). They are of special importance to us, due to their relation to algebraic curves (see [76, 13, 14, 37]).

One drawback of the Puiseux series is that it is not Cauchy complete, see *e.g.* [71] and with real or complex coefficients they do not contain infinitesimal numbers a priori.

The completion of the Puiseux series are the **Hahn series**. Hans Hahn showed in 1907 [59] that if $\mathbb{K}$ is an ordered field and G is an Abelian group, then the generalized power series in the variable T with coefficients $a_g \in \mathbb{K}$ given by

$$[[\mathbb{K}^G]] := \left\{ \sum_{g \in G} a_g T^g \mid \{g \mid a_g \neq 0\} \text{ is a well–ordered group subset of } G \right\}$$

with the natural summation and multiplication, can be given a lexicographical total order. These structures are called **Hahn series**. Furthermore, any ordered non–Archimedean field extension of $\mathbb{R}$ is isomorphic to a subfield of a generalized power series field $[[\mathbb{R}^G]]$ for a certain abelian group G.

We will consider two special cases of the Hahn series: it can be shown that $[[\mathbb{R}^{\mathbb{Q}}]]$ is real closed, see *e.g.* [109] and is isomorphic to $\cup_{n\in\mathbb{N}}[[\mathbb{R}^{\frac{1}{n}\mathbb{Z}}]]$ [71]. In the book of Basu *et al.*, [4] p. 34, it is shown that $[[\mathbb{C}^{\mathbb{Q}}]]$ algebraically complete.

Our interest lies not so much in completeness but rather in an efficient implementation. Infinite series, which cannot be properly truncated (we will see later that that means), are not particularly useful for this purpose. This applies to the fields $[[\mathbb{R}^{\mathbb{Q}}]]$ and $[[\mathbb{C}^{\mathbb{Q}}]]$.

The field which fulfills these requirements the best is in our opinion the **Levi-Civita field**. It is the smallest non–archimedian ordered field extension of $\mathbb{R}$ that is both real closed and complete (see [121] for details).

M. Berz and K. Shamseddine showed in [121] that the real and complex Levi-Civita fields, $\mathfrak{R}$ and $\mathfrak{C}$ are subfields of $[[\mathbb{C}^{\mathbb{Q}}]]$. They have left–finite support, which allows us to implement them, using a proper truncation, in a computer system (details in the following sections and especially in Subsection 6.4.5 (Implementation of the Levi-Civita Field)).
Finally we get the inclusion ([121]):

$$\mathbb{R} \subset [[\mathbb{R}]] \subset [[\mathbb{R}^{\mathbb{Z}}]] \subset \bigcup_{n\in\mathbb{N}} \left[\left[\mathbb{R}^{\frac{1}{n}\mathbb{Z}}\right]\right] \subset \mathfrak{R}$$

Furthermore, the ordered non–archimedian field extensions of $\mathbb{R}$ have to be infinite dimensional, if viewed as vector space over $\mathbb{R}$, in order to not contradict Frobenius theorem, as mentioned before. It is also easy to see that for a positive infinitesimal hyperreal Δx the set $\{\Delta x^i \mid i \in \mathbb{Q}\}$ is linearly independent over $\mathbb{R}$ and $\mathbb{C}$.

6.4.4. Levi-Civita Field

Tullio Levi-Civita laid foundation to the field later named after him in two papers: [85, 86]. He showed that the field is ordered and Cauchy complete. His work was picked up and refined by A. Ostrowski [102], L. Neder [98] and D. Laugwitz [82]. A. H. Lightstone and A. Robinson, the father of modern non–standard analysis, also recapitulated his work in [87]. Starting in the 1990s the Levi-Civita field was extensively studied by M. Berz and K. Shamseddine [11, 125, 123, 121, 122, 7, 9, 124, 8]. They could not simply use the results of the field of the hyperreals due to the lack of the powerful transfer principle of Theorem 4.2.13 (Transfer Principle). As we will see in a minute the (real or complex) Levi-Civita field is a subset of the hyperreal or hypercomplex number. So every statement that holds true for hyper numbers holds also true for the Levi-Civita numbers. Although we cannot employ the transfer principle we are lucky since M. Berz and K. Shamseddine worked out all the nifty details for us.

The advantage of the Levi-Civita field is that it implementable on computer with real time performance.

We will follow the lecture notes by M. Berz published in the proceedings of a summer school ([6]) and refer to the given references for the proofs.

Definition 6.4.2 (Left-Finite Sets, [6])
A subset $M \subset \mathbb{Q}$ is called *left-finite*, if and only if for every number $r \in \mathbb{Q}$ there are only finitely many elements of M that are smaller than r. The set of all left-finite sets of $\mathbb{Q}$ will be denoted by $\mathcal{F}$. $\diamond$

Remark. This $\mathcal{F}$ must not be confounded with a filter we defined in Definition 4.1.1 (Filters).

Lemma 6.4.3 (Arrangement of Left-Finite Sets, [6])
Let $M \in \mathcal{F}$. If $M \neq \emptyset$, the elements of M can be arranged in ascending order and there exists a minimum of M.

Remark. The preceding lemma will later assure that we have a maximal component in a series expansion and give an easy example that the Levi-Civita field is a true subset of the hyperreal and hypercomplex numbers.

Lemma 6.4.4 (Left-Finite Properties, [6])
Let $M, N \subset \mathcal{F}$ then we have:

1. $X \subset M, \Rightarrow X \in \mathcal{F}$,
2. $M \cup N \in \mathcal{F},\ M \cap N \in \mathcal{F}$,
3. $M + N = \{x + y \mid x \in M, y \in N\} \in \mathcal{F}$,
4. for every $x \in M + N$, there are only finitely many pairs $(a, b) \in M \times N$ such that $x = a + b$.

Definition 6.4.5 (The Levi-Civita Fields: $\mathfrak{R}$ and $\mathfrak{C}$, [6])
We define the *real Levi-Civita field* $\mathfrak{R}$ and the *complex Levi-Civita field* $\mathfrak{C}$ by

$$\mathfrak{R} := \{f : \mathbb{Q} \to \mathbb{R} \mid \{x \mid f(x) \neq 0\} \in \mathcal{F}\},$$
$$\mathfrak{C} := \{f : \mathbb{Q} \to \mathbb{C} \mid \{x \mid f(x) \neq 0\} \in \mathcal{F}\}.$$

So the elements of $\mathfrak{R}$ and $\mathfrak{C}$ are those real or complex valued functions on $\mathbb{Q}$ that are non-zero only on a left-finite set, *i.e.* have a left finite support. $\diamond$

Definition 6.4.6 (Notation for Elements in $\boldsymbol{\mathfrak{R}}$ and $\boldsymbol{\mathfrak{C}}$, [6])
An element x of $\mathfrak{R}$ or $\mathfrak{C}$ is uniquely characterized by an ascending (finite or infinite) sequence (q_n) of support points and a corresponding sequence $(x[q_n])$ of function values. We will refer to the pair of sequences $((q_n), (x[q_n]))$ as the *table* of x. $\diamond$

Remark. Later on we will implement the Levi-Civita field saving essentially two items: a leading coefficient and the table.

Definition 6.4.7 ($\mathbf{supp}, \boldsymbol{\lambda}, \boldsymbol{\sim}_{\mathrm{LC}}, \boldsymbol{\approx}_{\mathrm{LC}}, \boldsymbol{=}_{\boldsymbol{r}}$, [6])
For $x, y \in \mathfrak{R}$ we define

1. $\operatorname{supp}(x) := \{q \in \mathbb{Q} \mid x[q] \neq 0\}$ and call it the *support* of x,
2. $\lambda(x) := \min \operatorname{supp}(x)$ for $x \neq 0$ and $\lambda(0) := \infty$.

Comparing two elements we say:

1. $x \sim_{\mathrm{LC}} y$ if and only if $\lambda(x) = \lambda(y)$,
2. $x \approx_{\mathrm{LC}} y$ if and only if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$,
3. $x =_r y$ if and only if $x[q] = y[q]$ for all $q \leq r$.

$\diamond$

Remark. The notion of $\sim_{\mathrm{LC}}$ is the Levi-Civita counterpart of the concept of a magnitude for hyperreal and hypercomplex numbers in Definition 5.0.7 (Magnitude).

Definition 6.4.8 (Summation and Multiplication on $\boldsymbol{\mathfrak{R}}$ and $\boldsymbol{\mathfrak{C}}$, [6])
We define the summation on $\mathfrak{R}$ and $\mathfrak{C}$ componentwise:

$$(x + y)[q] := x[q] + y[q]$$

And the multiplication is defined for $q \in \mathbb{Q}$ by:

$$(x \cdot y)[q] := \sum_{q_x, q_y \in \mathbb{Q},\ q_x + q_y = q} x[q_x] \cdot y[q_y]$$

$\diamond$

Theorem 6.4.9 (Embedding, [6])
$\mathbb{R}$ and $\mathbb{C}$ can be embedded in $\mathfrak{R}$ and $\mathfrak{C}$ under their preservation of their arithmetic

structures with the embedding function

$$\Pi(x)[q] = \begin{cases} x, & \text{if } q = 0, \\ 0, & \text{else.} \end{cases}$$

Theorem 6.4.10 (Fields, [6])
$(\mathfrak{R}, +, \cdot)$ and $(\mathfrak{C}, +, \cdot)$ are fields.

Theorem 6.4.11 (Fundamental Theorem of Algebra for $\mathfrak{C}$, [6])
Every polynomial of positive degree with coefficients in $\mathfrak{C}$ has a root in $\mathfrak{C}$.

Definition 6.4.12 (The set $\mathfrak{R}^+$, [6])
Let $\mathfrak{R}^+$ be the set of all non–vanishing elements x of $\mathfrak{R}$ which satisfy $x[\lambda(x)] > 0$. ◇

Definition 6.4.13 (Ordering in $\mathfrak{R}$, [6])
Let x, y be elements of $\mathfrak{R}$. We say $x > y$ if and only if $x - y \in \mathfrak{R}^+$. ◇

Theorem 6.4.14 (Property of Ordering, [6])
With the ordering of Definition 6.4.13 (Ordering in $\mathfrak{R}$) $\mathfrak{R}$ is a totally ordered field.

Definition 6.4.15 ($\gg, \ll$, partially [6])
Lat a, b be positive. We say *a is infinitely smaller than b* if and only if $n \cdot a < b$ for all $n \in \mathbb{N}$. We say *a is infinitely smaller than b* and write $a \gg b$ if and only if $b \ll a$. If $|a| \ll 1$ we say that *a is infinitesimal* and if $|a| \gg 1$ we say that *a is unlimited.* ◇

Remark. Berz uses other terms for limited, unlimited and appreciable numbers. We rather stick to the terms which resemble properties if hyperreal numbers.

Definition 6.4.16 (The number d, [6])
Define the element d as

$$d[q] := \begin{cases} 1, & \text{if } q = 1, \\ 0, & \text{else.} \end{cases}$$

◇

Remark. In the terms of hyperreals d is an arbitrary positive infinitesimal hyperreal number.

Lemma 6.4.17 (Comparisons, [6])
For all $a, b, c \in \mathfrak{R}$ we have

1. $a \ll b \Rightarrow a < b$
2. $a \ll b, b \ll c \Rightarrow a \ll c$

3. $d^q \ll 1$ if and only if $q > 0$, $d^q \gg 1$ if and only if $q < 0$

Lemma 6.4.18 ($\mathfrak{R}$ and $\mathfrak{C}$ are non-Archimedean, [6])
The fields $\mathfrak{R}$ and $\mathfrak{C}$ are non-Archimedean, *i.e.* there are elements which are not exceeded by any natural number.

Definition 6.4.19 (Absolute Value on $\mathfrak{R}$, [6])
Let $x \in \mathfrak{R}$. We define the *absolute value* $|x|$ *of* x by: x if $x \geq 0$ and $-x$ if $x < 0$. $\diamond$

Definition 6.4.20 (Absolute Value on $\mathfrak{C}$ and $\mathfrak{R}^n$, [6])
For any element $z \in \mathfrak{C}$ we can write $z = a + \mathrm{i} \cdot b$ with $a, b \in \mathfrak{R}$. Then we define $|z| := \sqrt{a^2 + b^2}$ and for any $(a_1, \ldots, a_n) \in \mathfrak{R}^n$ define $|(x_1, \ldots, x_n)| := \sqrt{x_1^2 + \ldots + x_n^2}$. $\diamond$

Theorem 6.4.21 (Expansion in Powers, [6])
One can write every $x \in \mathfrak{R}$ or $\mathfrak{C}$ as

$$x = \sum_{i=1}^{\infty} x[q_i] \cdot d^{q_i}.$$

Remark. This property comes in very handy: we can write every Levi-Civita number as a sum of posynomial over the variable d.

Theorem 6.4.22 (Point Formula a la Cauchy, [6])
Let $f(z) = \sum_{a=0}^{\infty} a_i (z - z_0)^i$ be a complex power series on $\mathfrak{C}$. Then the function is completely determined by its value at $z_0 + h$, where h is an arbitrary nonzero infinitesimal. Furthermore

$$f(z_0 + h) = \sum_{i=0}^{\infty} a_i h^i.$$

For $h = d$ it holds true: $a_i = (f(z_0 + d))[i]$.

Formally the hyperreal and hypercomplex numbers do not have any relation to the Levi-Civita fields. We will now show that the latter is isomorphic to a sub field of the hyperreal or hypercomplex numbers.

Definition 6.4.23 (Levi-Civita Fields as Subfield)
Pick an arbitrary positive real infinitesimal Δx. We define the following subsets of $\mathbb{R}^*$ and $\mathbb{C}^*$:

$$\mathfrak{R}_{\mathbb{R}^*} := \{x \in \mathbb{R}^* \mid \exists M \subset \mathbb{Q}, M \text{ is left finite } : x = \sum_{q \in M} a_q \cdot \Delta x^q \text{ where } a_q \in \mathbb{R}\}$$
$$\mathfrak{C}_{\mathbb{C}^*} := \{x \in \mathbb{C}^* \mid \exists M \subset \mathbb{Q}, M \text{ is left finite } : x = \sum_{q \in M} a_q \cdot \Delta x^q \text{ where } a_q \in \mathbb{C}\}$$

◇

Remark. Obviously it does not matter which Δx we pick in the previous definition, the subsets are all isomorphic. We will show that the previously defined subsets of $\mathbb{R}^*$ and $\mathbb{C}^*$ are isomorphic to the Levi-Civita fields $\mathfrak{R}$ and $\mathfrak{C}$ and therefore are fields as well.

Theorem 6.4.24 (Subfields)
$\mathfrak{R}$ and $\mathfrak{C}$ are isomorphic to the sets $\mathfrak{R}_{\mathbb{R}^*} \subset \mathbb{R}^*$ and $\mathfrak{C}_{\mathbb{C}^*} \subset \mathbb{C}^*$. Furthermore $\mathfrak{R}_{\mathbb{R}^*}$ and $\mathfrak{C}_{\mathbb{C}^*}$ are fields.

Proof. By Theorem 6.4.21 (Expansion in Powers) we can write every x in $\mathfrak{R}$ or $\mathfrak{C}$ as

$$x = \sum_{i=1}^{\infty} x[q_i] \cdot d^{q_i}$$

with coefficients $x[q_i]$ in $\mathbb{R}$ or $\mathbb{C}$. The number $d = 1 \cdot d^1$ is infinitesimal by Lemma 6.4.17 (Comparisons). So pick an arbitrary positive real infinitesimal $\Delta x \in \mathbb{I}_{\mathbb{R}}^+$ and write

$$x' := \sum_{i=1}^{\infty} x[q_i] \cdot \Delta x^{q_i}$$

We will only prove the claim for $\mathfrak{R}$, the case of $\mathfrak{C}$ reads completely analogously.

Define $f : \mathfrak{R} \to \mathfrak{R}_{\mathbb{R}^*}$ to be the function which maps x to x'. More formally: Let $M \subset \mathbb{Q}$ be left finite and appropriately chosen according to an element $x \in \mathfrak{R}$ with table $(q_i, x[q_i]), i \in \mathbb{N}$, $f : M \times \mathbb{R} \to \mathfrak{R}_{\mathbb{R}^*}, x \mapsto \sum_{i \in M} x[q_i] \cdot \Delta x^{q_i}$. We will define the ring generated by f: $(f(\mathfrak{R}), +_{\mathbb{R}^*}, \cdot_{\mathbb{R}^*})$ and show that it is isomorphic to the Levi-Civita field. Note that we can also write an elements x of $\mathfrak{R}_{\mathbb{R}^*}$ by $x = \sum_{i=1}^{\infty} a_{q_i} \cdot \Delta x^{q_i}$ by the left finiteness of the support (see the proof of Lemma 6.4.3 (Arrangement of Left-Finite Sets)). f is bijective: surjective is clear by construction, injective by the linear independence of $\{\Delta x^m \mid m \in M\}$. Choose now $a, b \in \mathfrak{R}$. Now we have to show the following field isomorphism axioms:

$$f(0_{\mathfrak{R}}) = 0_{\mathbb{R}^*} \text{ and } f(1_{\mathfrak{R}}) = 1_{\mathbb{R}^*}$$
$$\forall a; b \in \mathfrak{R}\colon f(a +_{LC} b) = f(a) +_{\mathbb{R}^*} f(b)$$
$$\forall a; b \in \mathfrak{R}\colon f(a \cdot_{LC} b) =_{LC} f(a) \cdot_{\mathbb{R}^*} f(b)$$

The first property is trivial due to the fact that both use an isomorphism of the real zero and one as neutral elements of summation and multiplication. We can evaluate f

componentwise, since M is a set and therefore $q_i \neq q_j$ for $i \neq j$ and again the linear independence of $\{\Delta x^m \mid m \in M\}$.

$$f((a +_{LC} b)[q]) \overset{\text{Def.}}{=} f(a[q] +_{\mathbb{R}} b[q]) = a[q] \cdot \Delta x^q +_{\mathbb{R}^*} b[q] \cdot \Delta x^q = f(a[q]) +_{\mathbb{R}^*} f(b[q])$$

And for the multiplication: first of all note that the sums in the multiplication of Levi-Civita numbers are finite due to the left–finiteness by Lemma 6.4.4 (Left-Finite Properties). Then we find:

$$\begin{aligned}
f((a \cdot_{LC} b)[q]) &\overset{\text{Def.}}{=} f\left(\sum_{q_a, q_b \in \mathbb{Q}, q_a + q_b = q} a[q_a] \cdot b[q_b]\right) \\
&= \left(\sum_{q_a, q_b \in \mathbb{Q}, q_a + q_b = q} (a[q_a] \cdot b[q_b])\right) \Delta x^q \\
&= \sum_{q_a, q_b \in \mathbb{Q}, q_a + q_b = q} (a[q_a] \cdot b[q_b]) \Delta x^{q_a + q_b} \\
&= \sum_{q_a, q_b \in \mathbb{Q}, q_a + q_b = q} (a[q_a] \cdot \Delta x^{q_a}) \cdot (b[q_b] \cdot \Delta x^{q_b}) \\
&= f(a[q]) \cdot_{\mathbb{R}^*} f(b[q])
\end{aligned}$$

Formally we still would have to show that the constructed subfield of the hyperreals is actually a field. Using the constructed isomorphisms we could just mimic the proofs of Berz and use the field properties of $\mathbb{R}^*$. $\square$

Remark. Since $\mathfrak{R}_{\mathbb{R}^*}$ and $\mathfrak{C}_{\mathbb{C}^*}$ are isomorphic to the corresponding Levi-Civita fields $\mathfrak{R}$ and $\mathfrak{C}$ we will now drop the subscripts and simply write $\mathfrak{R}$ and $\mathfrak{C}$ when we mean the hyperreal and hypercomplex counterparts.

Theorem 6.4.25 (Proper Subsets)
$\mathfrak{R}$ and $\mathfrak{C}$ are proper subset of $\mathbb{R}^*$ and $\mathbb{C}^*$.

Proof. There are elements in $\mathbb{R}^* \setminus \mathfrak{R}$, for example: pick an arbitrary positive infinitesimal $\Delta x \in \mathbb{R}^*$. Then $\Delta x^{\sqrt{2}} \in \mathbb{R}^*$ while $d^{\sqrt{2}} \notin \mathfrak{R}$ due to the fact that the exponents have to be rational.

Further examples would be sums which have no left finite support: for $H := \Delta x^{-1}$, which is clearly element of the Levi-Civita field, the exponential function would yield a hyperreal member which cannot be element of the Levi-Civita field:

$$\exp(H) = \sum_{k=0}^{\infty} \frac{(\Delta x^{-1})^k}{k!} = 1 + \Delta x^{-1} + \frac{\Delta x^{-2}}{2} + \frac{\Delta x^{-3}}{6} + \dots$$

simply $\exp(H)$ does not have left–finite support. □

6.4.5. Implementation of the Levi-Civita Field

When we looked for implementations of the Levi-Civita field the first practical implementation was COSY infinity by Berz *et al.* [10, 89]. Surprisingly there is also a online version of the (real) Levi-Civita numbers: `http://www.lightandmatter.com/calc/inf/` [18]. It accompanies the book "Brief Calculus" by B. Crowell [20] which uses non-standard analysis to teach first semester calculus. Although the implementation of Crowell is open source it comes with small drawbacks: it is only implemented for real numbers and it is licensed under GPL v2 [GPL]. Since we are using the MIT license [MIT] in the CindyJS project we cannot adopt the code and thus wrote our own version using complex numbers. We will give an overview of the algorithms we use in the following, a full implementation in CindyScript [113] can be found in Appendix A (CindyScript Implementation of the Levi-Civita Field).

Remark. We will only analyze the field $\mathfrak{C}$ and neglect $\mathfrak{R}$. Of course all propositions hold analogously for the real field.

Lemma 6.4.26 (Normalization of Levi-Civita Numbers)
Let z be in $\mathfrak{C}$ with support $M \subset \mathbb{Q}$, M left–finite. Then we can write z in the following way:

$$z = a_0 \cdot d^{q_0}(1 + \sum_{i=1}^{\infty} \frac{a_i}{a_0} d^{q_i - q_0}) =: a_0 \cdot d^{q_0}(1 + \epsilon_z)$$

where $q_i \in M$, $a_i \in \mathbb{C}$ for all $i \in \mathbb{N}$, $q_0 = \min M$ and, furthermore ϵ_z is infinitesimal.

Proof. First we analyze the case of $M = \emptyset$, this means that z is the zero element of $\mathfrak{C}$. Then we can write $z = 0 = 0 \cdot d^0 = 0 \cdot d^0(1 + 0)$, which fulfills the requirements.

Now assume that $M \neq \emptyset$: by Theorem 6.4.21 (Expansion in Powers) we know that we can write every Levi-Civita number by $z = \sum_{i=0}^{\infty} a_i \cdot d^{q_i}$. Without loss of generality we can assume the q_i to be ordered, *i.e.* $q_i < q_{i+1}$ for all $i \in \mathbb{N}$. We are not affected by summation order, since the d^{q_i} are pairwise linearly independent over $\mathbb{R}$ and $\mathbb{C}$. Since M is left-finite it's minimum is q_0. The real or complex number a_0 does not vanish, since q_0

is part of the support of z. So we can write

$$z = a_0 \cdot d^{q_0}(1 + \sum_{i=1}^{\infty} \frac{a_i}{a_0} d^{q_i - q_0})$$

which is the claimed series representation. We still have to show that ϵ_z is infinitesimal: which by definition means $|\epsilon_z| \ll 1$. First of all note that $q_i - q_0 > 0$ for all $i \in \mathbb{N}$, since q_0 is the minimum of M. We will argue with induction over i. For $i = 1$: since a_1 is a real or complex number there is an $n \in \mathbb{N}$ such that $|a_1| < n$ by the Archimedean axiom. Then $|a_1 \cdot d^{q_1 - q_0}| \leq n \cdot d^{q_1 - q_0}$. By Lemma 6.4.17 (Comparisons) we know $d^q \ll 1$ if and only if $q > 0$ which is the claim for $i = 1$. By induction step we know that

$$\epsilon_{k-1} := |\sum_{i=1}^{k-1} \frac{a_i}{a_0} d^{q_i - q_0}| \ll 1$$

then the sum

$$|\epsilon_{k-1} + a_k \cdot d^{q_k - q_0}| \leq |\epsilon_{k-1}| + |a_k \cdot d^{q_k - q_0}| \leq 2 \cdot \max\{|\epsilon_{k-1}|, |a_k \cdot d^{q_k - q_0}|\} = 2 \cdot |\epsilon_{k-1}| \ll 1$$

where we used that the sequence q_i is ascending and so $d^{q_k} > d^{q_{k+1}}$ for all $k \in \mathbb{N}$. □

Remark. We did the proof in detail because it is important to note that infinite summation can lead to overflow to a different magnitude: Let $N \in \mathbb{N}^* \backslash \mathbb{N}$. Then $\epsilon := \frac{1}{N}$ is infinitesimal. The hyperfinite sum (which are essentially sums over elements in $\mathbb{N}^*$, see [52] p. 71) $\sum_{i=0}^{N} \epsilon$ is not infinitesimal, since $\sum_{i=0}^{N} \epsilon = N \cdot \epsilon = N \frac{1}{N} = 1$.

Definition 6.4.27 (Normalized Form)
Lemma 6.4.26 (Normalization of Levi-Civita Numbers) showed that we can write $z \in \mathfrak{C}$ in a special form:

$$z = a_0 \cdot d^{q_0}(1 + \epsilon_z)$$

where $a_0 \in \mathbb{C}$, $q_0 \in \mathbb{Q}$ and ϵ_z infinitesimal. When we write z in the form above we call z *normalized.* Furthermore we call $a_0 \cdot d^{q_0}$ *the leading coefficient of* z and ϵ_z the *ϵ–tail of* z. ◇

Definition 6.4.28 (Shadow of Levi-Civita numbers)
Let z be in $\mathfrak{C}$ and normalized: $z = a_0 \cdot d^{q_0}(1 + \epsilon_z)$. Then we define $\mathbf{sh}(z) : \mathfrak{C} \to \mathbb{C} \cup \{\infty\}$,

the *shadow* of z, by

$$\mathbf{sh}(z) = \begin{cases} a_0, & \text{if } q_0 = 0 \\ \infty, & \text{if } q_0 < 0 \\ 0, & \text{if } q_0 > 0 \end{cases}$$

◇

Lemma 6.4.29 (Shadows Coincide)
The definition of the shadow of Levi-Civita of Definition 6.4.28 (Shadow of Levi-Civita numbers) and the hypercomplex numbers of Theorem 4.3.6 (Shadow) coincide.

Proof. By Theorem 6.4.24 (Subfields) we know that the complex Levi-Civita is isomorphic to a subfield of the hypercomplex numbers. Therefore we can apply the shadow to the isomorphic elements.

Let z in $\mathfrak{C}$ in normalized form: $z = a_0 \cdot d^{q_0}(1+\epsilon_z)$. Then we can interpret d as an arbitrary strictly positive hyperreal number. Then we can apply the hypercomplex shadow to z:

$$\mathbf{sh}(z) = \mathbf{sh}(a_0 \cdot d^{q_0}(1+\epsilon_z)) = \mathbf{sh}(a_0 \cdot d^{q_0}) \cdot \mathbf{sh}(1+\epsilon_z) = \mathbf{sh}(a_0 \cdot d^{q_0}) \cdot 1$$

where we used Lemma 4.3.10 (Shadow Properties) and the fact that ϵ_z is infinitesimal. Then we can distinguish three cases: if $q_0 > 0$ then d^{q_0} is infinitesimal and its shadow is zero. If $q_0 < 0$ then d^{q_0} is unlimited and its shadow is ∞. The only case left is $q_0 = 0$, then $a_0 \cdot d^0 = a_0$ and $\mathbf{sh}(a_0) = a_0$ since a_0 is a complex number. □

Lemma 6.4.30 (Summation)
For normalized x, y in $\mathfrak{C} \setminus \{0\}$ with $x = a_0 \cdot d^{q_0}(1+\epsilon_x)$ and $y = b_0 \cdot d^{p_0}(1+\epsilon_y)$ we can write

$$x + y = a_0 \cdot d^{q_0}\left(1 + \epsilon_x + \frac{b_0}{a_0}d^{p_0-q_0} + \epsilon_y \cdot \frac{b_0}{a_0}d^{p_0-q_0}\right)$$

Proof. Simple expansion. □

Remark. The result of the summation is normalized if $p_0 - q_0 > 0$. For an implementation it is beneficial to either calculate $x + y$ or $y + x$ accordingly such that the result stays normalized.

Lemma 6.4.31 (Negation)
For a normalized x in $\mathfrak{C}$ with $x = a_0 \cdot d^{q_0}(1+\epsilon_x)$ the inverse element of summations is $-x = -a_0 \cdot d^{q_0}(1+\epsilon_x)$.

Proof. Obvious. $\square$

Lemma 6.4.32 (Subtraction)
For normalized x, y in $\mathfrak{C} \setminus \{0\}$ with $x = a_0 \cdot d^{q_0}(1 + \epsilon_x)$ and $y = b_0 \cdot d^{p_0}(1 + \epsilon_y)$ we can write

$$x - y = a_0 \cdot d^{q_0} \left(1 + \epsilon_x - \frac{b_0}{a_0} d^{p_0 - q_0} - \epsilon_y \cdot \frac{b_0}{a_0} d^{p_0 - q_0}\right)$$

Proof. Write $x - y$ in the form $x + (-y)$ and apply the preceding lemmas. $\square$

Lemma 6.4.33 (Multiplication)
For normalized x, y $\mathfrak{C}$ with $x = a_0 \cdot d^{q_0}(1 + \epsilon_x)$ and $y = b_0 \cdot d^{p_0}(1 + \epsilon_y)$ we can write the multiplication

$$x \cdot y = a_0 \cdot b_0 \cdot d^{q_0 + p_0}(1 + \epsilon_x + \epsilon_y + \epsilon_x \cdot \epsilon_y).$$

Proof. Simple expansion of the terms. $\square$

Lemma 6.4.34 (Inversion)
For a normalized x in $\mathfrak{C} \setminus \{0\}$ with $x = a_0 \cdot d^{q_0}(1 + \epsilon_x)$ the inverse element of multiplication x^{-1} can be written

$$x^{-1} = \frac{1}{a_0} d^{-q_0} \sum_{k=0}^{\infty} (-\epsilon_x)^k$$

Proof. We interpret the Levi-Civita fields as subfields of the hyperreal or hypercomplex numbers as in Theorem 6.4.24 (Subfields): the claim was shown in Lemma 4.5.4 (Inversion). $\square$

Lemma 6.4.35 (Radical Expressions)
Let x in $\mathfrak{C}$ be a normalized and not infinitely close to a real negative number, *i.e.* $x \not\approx_{\mathrm{LC}} y \; \forall y \in (-\infty, 0]$, with $x = a_0 \cdot d^{q_0}(1 + \epsilon_x)$. Then one solution of $\sqrt[n]{x}$ is:

$$\sqrt[n]{a_0 \cdot d^{q_0} \cdot (1 + \epsilon_x)} = \sqrt[n]{a_0} \cdot d^{\frac{q_0}{n}} \left(\sum_{k=0}^{\infty} \binom{1/n}{k} \epsilon_x^k \right)$$

Proof. Again we interpret the Levi-Civita numbers as a subfield of the hypercomplex numbers: the claim was proven in Theorem 4.5.1 (Homogeneity for Certain Radical Expressions). $\square$

Remark. Of course one could pick a different branch cut in Theorem 4.5.1 (Homogeneity for Certain Radical Expressions) and further solutions can be obtained by multiplication with (standard) roots of unity.

Theorem 6.4.36 (Truncation of Series)
Let $f(z) = \sum_{a=0}^{\infty} a_i(z - z_0)^i$ be the continuation of a complex power series on $\mathfrak{C}$. Let $\nu \in \mathbb{N} \setminus \{0\}$ and $f_\nu(z) = \sum_{a=0}^{\nu} a_i(z - z_0)^i$. Then for an infinitesimal h it holds true:

$$f(z_0 + h) =_\nu f_\nu(z_0 + h)$$

i.e. the coefficients of the truncated series f_ν coincides with f up to the order ν.

Proof. We use Theorem 6.4.22 (Point Formula a la Cauchy): let h be infinitesimal. Then according to the Theorem we can write $f(z_0 + h)$ as

$$\begin{aligned}
f(z_0 + h) &= \sum_{i=0}^{\infty} a_i h^i \\
\Rightarrow f(z_0 + h) - f_\nu(z_0 + h) &= \sum_{i=\nu+1}^{\infty} a_i h^i \\
\Rightarrow f(z_0 + h) &=_\nu f_\nu(z_0 + h).
\end{aligned}$$

$\square$

Remark (Trunction of arithmetic Operations). The functions $\{+, -, \cdot, \div, \sqrt[n]{\cdot}\}$ utilized in dynamic geometry are analytic around 1 in every variable with radius of convergence > 0. By the previous theorem we can expand the power series for arguments of the form $1 + \epsilon$ with $|\epsilon| \ll 1$. The power series then coincide with their truncated series up to the order ν of the truncation.

We will now have a practical example of the truncated series.

Example 6.4.37 (Numerical Examples of Levi-Civita Arithmetic)
Let $z = 1 + d + d^2 + d^3 = 1 \cdot d^0(1 + d + d^2 + d^3)$. We will give some truncated numerical results for truncation order $\nu = 4$:

$$\begin{aligned}
z &= 1 \cdot d^0 \cdot (1 \cdot d^0 + 1 \cdot d^1 + 1 \cdot d^2 + 1 \cdot d^3) \\
\sqrt{z} &= 1 \cdot d^0 \cdot (1 \cdot d^0 + 0.5 \cdot d^1 + 0.375 \cdot d^2 + 0.3125 \cdot d^3 - 0.2266 \cdot d^4) \\
\sqrt{z} \cdot \sqrt{z} &= 1 \cdot d^0 \cdot (1 \cdot d^0 + 1 \cdot d^1 + 1 \cdot d^2 + 1 \cdot d^3 + 0.0078 \cdot d^5) \\
\frac{1}{\sqrt{z}} &= 1 \cdot d^0 \cdot (1 \cdot d^0 - 0.5 \cdot d^1 - 0.125 \cdot d^2 - 0.0625 \cdot d^3 + 0.4609 \cdot d^4) \\
\frac{\sqrt{z}}{\sqrt{z}} &= 1 \cdot d^0 \cdot (1 \cdot d^0 + 0.2813 \cdot d^5)
\end{aligned}$$

So the results for $\sqrt{z} \cdot \sqrt{z}$ and $\frac{\sqrt{z}}{\sqrt{z}}$ are exact up to the order of $\nu = 4$ since the first incorrect coefficient is $c \cdot d^5$ for $c \neq 0$. ◇

Remark. We want to underline again that although we truncate the series this does **not** introduce numerical error in the shadow! If we take z as above

$$\begin{aligned}\mathbf{sh}\left(\frac{\sqrt{z}}{\sqrt{z}}\right) &= \mathbf{sh}(1 \cdot d^0 \cdot (1 \cdot d^0 + 0.2813 \cdot d^5)) \\ &= \mathbf{sh}(1 \cdot d^0) + \mathbf{sh}(0.2813 \cdot d^5) = 1 + 0 = 1.\end{aligned}$$

Example 6.4.38 (Removable singularities)
How can we now remove singularities? The function $f(x) = \frac{x^{1000}}{x^{1000}}$ has a removable singularity at $x = 0$. Since $d \simeq 0$ we can remove the singularity using the shadow projection

$$f(d) = \frac{d^{1000}}{d^{1000}} = d^{1000} \cdot d^{-1000} = 1 \Rightarrow \lim_{x \to 0} f(x) = \mathbf{sh}(f(d)) = 1$$

Another classic example is the function $g(x) = \frac{\sin(x)}{x}$, which has a removable singularity at $x = 0$. Utilization of the series definition of $\sin(x)$ at $x = d$ yields

$$\sin(d) = d - 0.1667 \cdot d^3 + 0.0083 \cdot d^5 - 0.0002 \cdot d^7 + 2.7557 \cdot 10^{-6} \cdot d^9$$

then for the fraction $\frac{\sin(d)}{d}$ we have

$$\frac{\sin(d)}{d} = 1 - 0.1667 \cdot d^2 + 0.0083 \cdot d^4 - 0.0002 \cdot d^6 + 2.7557 \cdot 10^{-6} \cdot d^8$$

Then the shadow yields

$$\mathbf{sh}\left(\frac{\sin(d)}{d}\right) = 1$$

which is the correct value (as one can see using de L'Hospital's rule). ◇

6.4.6. Application to Singularities in Geometric Constructions

We will now describe two algorithms which will use Theorem 6.1.5 (Normalization and Continous Paths) and Theorem 6.2.3 (Normalization and Extended Continous Paths) to automatically remove singularities in geometric constructions.

First we will analyze the less general C^0-continuation: Algorithm 1 (C^0-Continuation Algorithm) takes as input a function Ψ with $\Psi(t_0) = 0$ and returns a C^0-continuation at the given point t_0 if possible. The correctness of the algorithm is obvious by Theorem 6.1.5 (Normalization and Continous Paths), if the non-vanishing property of the theorem is fulfilled and as U. Kortenkamp and J. Richter-Gebert show in "Grundlagen Dynamischer Geometrie" [76], singularities of geometric constructions are isolated and therefore we only have to check whether the left and right side limit coincide (see Lemma 4.3.14 (Squeezing Limits)).

Algorithm 1: C^0-Continuation Algorithm

Input: A continuous function $\Psi : (0,1) \to \mathbb{C}^{d+1}$ and $t_0 \in (0,1)$
Output: C^0-Continuation of $[\Psi]$ at t_0

begin
    Pick arbitrary $\Delta t \in \mathbb{I} \setminus \{0\}$
    $L \leftarrow \mathbf{sh}\left(\frac{\Psi(t_0+\Delta t)}{\|\Psi(t_0+\Delta t)\|}\right)$
    $L^+ \leftarrow \frac{\Psi(t_0+\Delta t)}{\|\Psi(t_0+\Delta t)\|}$
    $L^- \leftarrow \frac{\Psi(t_0-\Delta t)}{\|\Psi(t_0-\Delta t)\|}$
    if $L = NaN$ **then**
        return *ERROR: vanishing* Ψ
    end
    else if $L \not\approx L^+$ *or* $L \not\approx L^-$ **then**
        return *ERROR: discontinuous*
    end
    else
        return $[\Psi(t_0)] = L$
    end
end

Algorithm 2 (Extended C^0-Continuation Algorithm) checks additionally whether the solution is stable under spatial perturbations and therefore a continuation is extended continuous. Since this is a randomized algorithm one might develop a probability estimation on how certain one can be that the solutions is actually continuous. Usually this is done using the Schwartz–Zippel lemma (see for example "Probability and Computing: Randomized Algorithms and Probabilistic Analysis" by M. Mitzenmacher and E. Upfal [94], or including radical expressions, "Randomized Zero Testing of Radical Expressions and Elementary Geometry Theorem Proving" by Tulone *et al.* [131]). We will defer the development of randomized continuity testing using infinitesimal deflection to future work and use Algorithm 2 as heuristic algorithm. In practice it turns out that the algorithm quickly can decide if a given singularity t_0 is removable or not by only checking a few randomized inputs.

Algorithm 2: Extended C^0-Continuation Algorithm

Input: Continuous functions $v : [0,1] \to \mathbb{C}^{k+1}$, $\Psi : Y \to \mathbb{C}^{d+1}$ with open set Y, $v([0,1]) \subset Y \subset \mathbb{C}^{k+1}$ and $t_0 \in [0,1]$.

Output: Extended C^0-Continuation of $[\Psi]$ at t_0

begin
- Pick $n \in \mathbb{N}$ arbitrarily $\Delta v_i \in \mathbb{I}^{k+1} \setminus \{0\}$ $(i \in 1 \ldots n)$, $\Delta v_i \neq \Delta v_j$ $(i \neq j)$ such that $v(t_0) + \Delta v_i$ is in the preimage of Ψ
- Evaluate $\Psi_i := \frac{\Psi(v(t_0)+\Delta v_i)}{\|\Psi(v(t_0)+\Delta v_i)\|}$ for all $i \in \{1, \ldots, n\}$
- $L \leftarrow \mathbf{sh}(\Psi_i)$ for arbitrary $i \in \{1, \ldots, n\}$
- **if** *any* $\Psi_i = 0$ **then**
  - **return** *ERROR: vanishing* Ψ
- **end**
- **else if** *any* $\Psi_i \not\simeq \Psi_j$ **then**
  - **return** *ERROR: discontinuous*
- **end**
- **else**
  - **return** $[\Psi(v(t_0))] = L$
- **end**

end

6.5. Experimental Results & Examples

> Alle Pädagogen sind sich darin einig: man muß vor allem tüchtig Mathematik treiben, weil ihre Kenntnis fürs praktische Leben den größten direkten Nutzen gewährt.
>
> Felix Klein, [74]

In this section we will discuss several examples like von-Staudt constructions and the disjoint circle intersection. We will give concrete numerical values to illustrate our method.

6.5.1. Von-Staudt Construction

Our first example will be a von-Staudt construction. These constructions can be used to encode summation and multiplication employing geometry, and use auxiliary points on which restrictions are imposed (which are essentially removable singularities, as we will find out later). We will use the summation example to show how our developed methods can be used to resolve the singularities such that these restrictions can be omitted.

Example 6.5.1 (Classical von-Staudt Contruction)

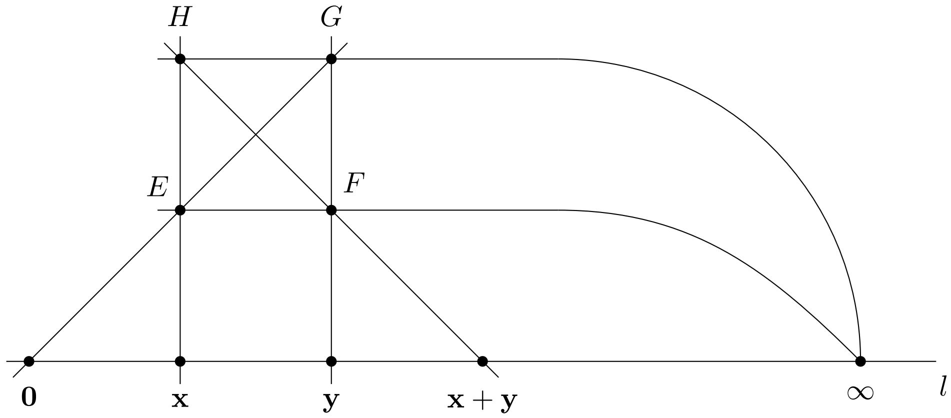


Figure 6.5.: non–degenerate von-Staudt

We will follow J. Richter-Gebert's "Perspectives on Projective Geometry" [111] p. 89 ff.: for given points $\mathbf{x} = (x, 0, 1)^T$ and $\mathbf{y} = (y, 0, 1)^T$ we will construct the sum $\mathbf{x} + \mathbf{y}$ according to the projective scale $\mathbf{0} = (0, 0, 1)^T$ and $\infty = (1, 0, 0)^T$. The choice of the projective scale is arbitrary, but for the sake for simplicity, we use a classical setting in which ∞ is a far point. Hence, the homogeneous coordinates of $x + y$ are equal to $\mathbf{x} + \mathbf{y} = (x + y, 0, 1)^T$, up to scalar multiplication.

The constructions reads a follows: denote the connecting line of $\mathbf{0}$ and ∞ by l. Then, choose an auxiliary point E, which is not incident to l. Choose a further auxiliary point F on $\mathbf{join}(\infty, E)$, which is unequal to E. Then we further proceed with the following operations:

$$\begin{aligned} G &= \mathbf{meet}(\mathbf{join}(\mathbf{0}, E), \mathbf{join}(\mathbf{y}, F)) \\ H &= \mathbf{meet}(\mathbf{join}(\infty, G), \mathbf{join}(\mathbf{x}, E)) \\ m &= \mathbf{join}(F, H) \\ \mathbf{x} + \mathbf{y} &= \mathbf{meet}(l, m) \end{aligned}$$

See Figure 6.5 (non–degenerate von-Staudt) for a picture of the construction. $\diamond$

As we can see the von-Staudt construction has two requirements:

1. E and F have to be disjoint,
2. the point E must not be incident to l (hence F is also not incident to l)

If one violates these constraints, the construction breaks down in standard geometry, because essential construction steps are undefined.

We will now violate both prerequisites of the construction in standard geometry. Then, we will use our method for resolving singularities by applying non-standard projective geometry and show how our method behaves numerically. We will always use the following coordinates for our construction: $\mathbf{0} = (0,0,1)^T, \mathbf{x} = (2,0,1)^T, \mathbf{y} = (4,0,1)^T, \infty = (1,0,0)^T$. This should result in $\mathbf{x}+\mathbf{y} = (6,0,1)^T \sim (1,0,1/6)^T$.

Example 6.5.2 (Von-Staudt Construction - non-disjoint auxiliary Points)
Let break the first requirement: the auxiliary points E and F must not coincide. It is easy to see that if $E = F$, then also $H = F$ and $G = F$, just by definition. In particular, the line $m = \mathbf{join}(H, F)$ which defines $x + y = \mathbf{meet}(l, m)$ is not defined (and thus equal to $(0,0,0)^T \notin \mathbb{RP}^2$). This results in $x + y$ not being defined (*i.e.* being equal to $(0,0,0)^T \notin \mathbb{RP}^2$).

The operators **join** and **meet** are defined using the cross-product in $\mathbb{RP}^2$, which is a continuous function $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. The whole construction is build up using cross product, so the construction steps are compositions of continuous functions and therefore continuous. So we can apply Theorem 6.2.3 (Normalization and Extended Continous Paths): we infinitesimally perturb the free point E and replace the point by $E' := E + (\epsilon_x, \epsilon_y, 0)^T$ and infinitesimally perturb F to F' with $F' := F + (\delta_x, \delta_y, 0)$ such that $F' \neq E'$, but $F' \simeq E'$.

The actual perturbation is arbitrary as long as it fulfills the requirements of the theorem. According to Algorithm 2 (Extended C^0-Continuation Algorithm) we should actually test several perturbations to get an indicator if the function is actually continuous. Furthermore, note that we will use the appreciable cross-product of Definition 5.1.5 (Appreciable Cross-Product) for all non–standard projective joins and meets. A picture of the construction can be found in Figure 6.6 (Degenerate von-Staudt: Merging Auxiliary Points) and the computational values in Table 6.1. Using the perturbation one finds

$$m' = d^1 \begin{pmatrix} -0.125 \\ -0.125 \\ 0.75 \end{pmatrix}$$

especially m' is infinitesimal (since d^1 is infinitesimal). If one would apply the shadow function it would yield $\mathbf{sh}(m') = (0,0,0)^T$, which resembles the degeneracy of the

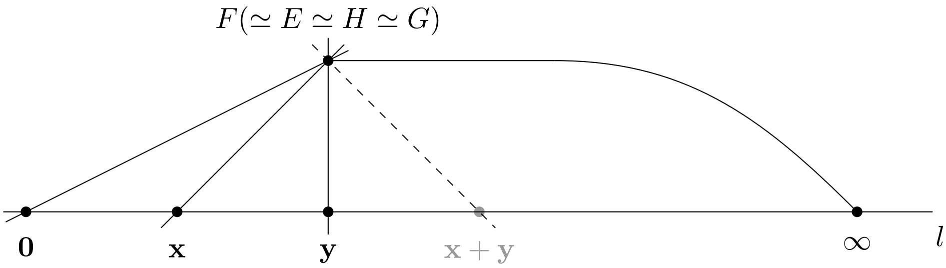


Figure 6.6.: Degenerate von-Staudt: Merging Auxiliary Points

	E/E'	F/F'	H/H'	m/m'	$x+y,(x+y)'$
CindyJS	$\begin{pmatrix}1\\0.5\\0.25\end{pmatrix}$	$\begin{pmatrix}1\\0.5\\0.25\end{pmatrix}$	$\begin{pmatrix}1\\0.5\\0.25\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$
NSA	$\begin{pmatrix}1\cdot d^0+\epsilon_x\\0.5\cdot d^0+\epsilon_y\\0.25\cdot d^0\end{pmatrix}$	$\begin{pmatrix}1\cdot d^0+\delta_x\\0.5\cdot d^0+\delta_y\\0.25\cdot d^0\end{pmatrix}$	$\begin{pmatrix}1\cdot d^0+\gamma_x\\0.5\cdot d^0+\gamma_y\\0.25\cdot d^0\end{pmatrix}$	$\begin{pmatrix}-0.125\cdot d^1\\-0.125\cdot d^1\\0.75\cdot d^1\end{pmatrix}$	$\begin{pmatrix}-1\cdot d^0\\0\cdot d^0\\-0.1667\cdot d^0\end{pmatrix}$
psh	$\begin{pmatrix}1\\0.5\\0.25\end{pmatrix}$	$\begin{pmatrix}1\\0.5\\0.25\end{pmatrix}$	$\begin{pmatrix}1\\0.5\\0.25\end{pmatrix}$	$\begin{pmatrix}-0.1667\\-0.1667\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0.1667\end{pmatrix}$

Table 6.1.: **Merging auxiliary points**. Perturbation: $\epsilon_x = \epsilon_y = (1+d)$, $\delta_{x,y}$ and $\gamma_{x,y}$ calculated accordingly. Therefore $F' \neq H'$ but $F' \simeq H'$ (tails omitted for readability), $m' = \mathbf{join}(F', H')$ is properly defined and infinitesimal.

construction in standard geometry, but applying the projective shadow the result is

$$\mathbf{psh}(m') = \begin{pmatrix}-0.1667\\-0.1667\\1\end{pmatrix}$$

which is the correct line, hence results in the desired point:

$$\mathbf{meet}(l, \mathbf{psh}\, m') = \begin{pmatrix}1\\0\\\frac{1}{6}\end{pmatrix} \sim \begin{pmatrix}6\\0\\1\end{pmatrix} = \mathbf{x}+\mathbf{y}$$

◇

Remark. In the non-degenerate non-standard construction it was actually not important for F to be incident (and not only almost incident) to $\mathbf{join}(\infty, E)$. By continuity of all operations and the non-degeneracy, we could actually perturb all points and generate the same result via projective shadows.

In the degenerate case the situation is different: if F' and H' are only chosen to be in the projective halo of E', we can generate arbitrary lines m (which are almost incident to E') but with arbitrarily chosen direction!

We will shed some light on this using Theorem 6.2.3 (Normalization and Extended Continous Paths). The perturbations are chosen to be element of $(\mathbb{I}^{k+1} \setminus \{0\}) \cap V^*$, where V denotes the subspace of a (semi-)free element, which models the restrictions a semi-free element is subjected. This means the perturbations must be chosen such that these restrictions of the dependent elements are not violated.

Now we will degenerate the construction even more: we move the point E onto the line $l = \mathbf{join}(\mathbf{0}, \infty)$.

Example 6.5.3 (Von-Staudt construction – $\boldsymbol{E}$ incident to $\boldsymbol{l}$)
We break the second requirement: E must not be incident to l. If we move E to $\mathbf{x}$ then practically the whole construction breaks down: G, since $x = E$ and their join is undefined, H because G is undefined, m since H is undefined and of course so is $\mathbf{x} + \mathbf{y}$. A picture of the situation can be found in Figure 6.7.

Again we infinitesimally perturb the free point E replacing the point by $E' := E + (\epsilon_x, \epsilon_y, \epsilon_z)^T$ and the dependent elements F', G', H', m' and $(x + y)'$ accordingly. The values of the resulting vectors can be read up in Table 6.2.

	E/E'	F/F'	G/G'	H/H'	$x+y, (x+y)'$
CindyJS	$\begin{pmatrix}1\\0\\0.5\end{pmatrix}$	$\begin{pmatrix}1\\0\\0.25\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$
NSA	$\begin{pmatrix}1\cdot d^0\\1\cdot d^1\\0.5\cdot d^0\end{pmatrix}$	$\begin{pmatrix}1\cdot d^0\\0.5\cdot d^1\\0.25\cdot d^0\end{pmatrix}$	$\begin{pmatrix}1\cdot d^0\\1\cdot d^1\\0.25\cdot d^0\end{pmatrix}$	$\begin{pmatrix}-1\cdot d^0\\-2\cdot d^1\\-0.5\cdot d^0\end{pmatrix}$	$\begin{pmatrix}6\cdot d^1\\0\cdot d^0\\1\cdot d^1\end{pmatrix}$
psh	$\begin{pmatrix}1\\0\\0.5\end{pmatrix}$	$\begin{pmatrix}1\\0\\0.25\end{pmatrix}$	$\begin{pmatrix}1\\0\\0.25\end{pmatrix}$	$\begin{pmatrix}1\\0\\0.5\end{pmatrix}$	$\begin{pmatrix}1\\0\\0.1667\end{pmatrix}$

	$\mathbf{join}(\infty, E/E')$	$\mathbf{join}(\mathbf{0}, E/E')$	$\mathbf{join}(y, F/F')$	$\mathbf{join}(\infty, G/G')$	$\mathbf{join}(\mathbf{x}, E/E')$	$m/m' = \mathbf{join}(H/H', F/F')$
CindyJS	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$
NSA	$\begin{pmatrix}0\cdot d^0\\-0.5\cdot d^0\\1\cdot d^1\end{pmatrix}$	$\begin{pmatrix}-1\cdot d^1\\1\cdot d^0\\0\cdot d^0\end{pmatrix}$	$\begin{pmatrix}-0.125\cdot d^1\\0\cdot d^0\\0.5\cdot d^1\end{pmatrix}$	$\begin{pmatrix}0\cdot d^0\\-0.25\cdot d^0\\1\cdot d^1\end{pmatrix}$	$\begin{pmatrix}-0.5\cdot d^1\\0.5\cdot d^1\\1\cdot d^1\end{pmatrix}$	$\begin{pmatrix}0.25\cdot d^1\\0.25\cdot d^0\\-1.5\cdot d^1\end{pmatrix}$
psh	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}-0.25\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}-0.5\\0.5\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$

Table 6.2.: **Auxiliary points on $\mathbf{join}(\mathbf{0},\boldsymbol{\infty})$**. Infinitesimal entries marked in red. Remember the construction: $G = \mathbf{meet}(\mathbf{join}(\mathbf{0}, E), \mathbf{join}(\mathbf{y}, F))$, $H = \mathbf{meet}(\mathbf{join}(\infty, G), \mathbf{join}(\mathbf{x}, E))$, $m = \mathbf{join}(F, H)$, $\mathbf{x}+\mathbf{y} = \mathbf{meet}(l, m)$ with $l = (0,1,0)^T$. The ϵ-tails are omitted for reasons of readability.

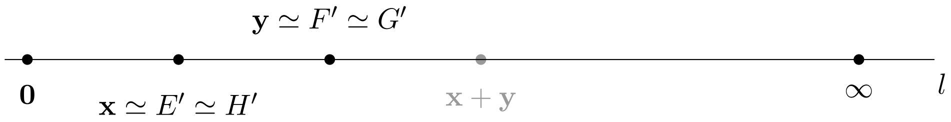


Figure 6.7.: Degenerate von-Staudt: auxiliary points on $l = \mathbf{join}(\mathbf{0}, \infty)$. Here $x = E = H$ and $y = F = G$.

Again, we can see that classical projective geometry over the real numbers yields a lot of undefined objects (as expected). Using non–standard analysis in projective geometry in the form of a Levi-Civita field can still provide us with the correct results.

Also, Table 6.2 shows some interesting effects: firstly, the application of the projective shadow when a vector is appreciable. For example the point $E' = (1 \cdot d^0, 1 \cdot d^1, 0.5 \cdot d^0)^T$ (where we neglected the ϵ–tail). Then the y–component of $(E')_y = d^1$ is discarded by the **psh** function, since it is one magnitude smaller than the x and z component. Therefore, the projective shadow and the unperturbed point E coincide as desired.

But even more interesting is the line $\mathbf{join}(\mathbf{x}, E') = (-0.5 \cdot d^1, 0.5 \cdot d^1, 1 \cdot d^1)^T$. The vector is infinitesimal, since all entries are infinitesimal. As expected, standard projective geometry, as used in CindyJS, will yield the zero vector. If we divide the line representative by d^1, which is in the magnitude of the whole vector, one can find the correct result with removed singularity: $\mathbf{psh}(\mathbf{join}(\mathbf{x}, E')) = (-0.5, 0.5, 1)^T$.

The bottom line of the example is that we can resolve singularities in geometric constructions, even if they are highly degenerate, using methods of non–standard analysis. Again, we emphasize that our method does **not** introduce numerical error due to its symbolic character, while retaining a real time execution speed. $\diamond$

6.5.2. Disjoint Circle Intersection

Example 6.5.4 (Disjoint Circle Intersection)
The "disjoint circle intersection" problem was introduced in Example 3.4.1 (Disjoint Circle Intersection). Essentially we are considering the **join** of two points of intersection of two disjoint circles in a tangential situation, which turns out to be removable singularity. We already analyzed the problem using non–standard arithmetic in Example 6.1.4 (Circle Intersection Revisited) from a theoretical point of view. Now we will also give computational values as well. As a quick reminder consider Figure 6.8 (middle).

If one examines the connecting line of those two merging points the operation will be inherently undefined. But if one perturbs the construction infinitesimally the operation is well defined again. A closer look at the operation reveals that we have a removable singularity: in Appendix B (Computer Algebra System Code Tangential Circle Intersection) we analyze the situation with a computer algebra system and show the claim symbolically, taking the limit for an arbitrary perturbation.

Using non–standard analysis we can perturb the construction at any reasonable point. We chose to do so in the algorithm which intersects a line and a conic. The line $l = (\lambda, \tau, \mu)^T$ is defined as described in [111] p. 195 ff.. Algorithm 2 (Extended C^0-Continuation Algorithm) actually tells us to perturb the (semi-) free elements of a construction but for the sake of simplicity we assume l to be free although it is dependent. This is without loss of generality since we could always choose the circle centers to generate a line with arbitrary coordinates. We perturb l to l' with $l' = (\lambda - d, \tau - d, \mu - d)^T$, where d is the infinitesimal number of the Levi-Civita field as defined in Definition 6.4.16 (The number d).

The leading coefficients of p_1 and p_2 read as follows:

$$p_1 = \begin{pmatrix} 1 \cdot d^0 \\ 2.4495 \cdot d^{\frac{1}{2}} \\ 1 \cdot d^0 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 1 \cdot d^0 \\ -2.4495 \cdot d^{\frac{1}{2}} \\ 1 \cdot d^0 \end{pmatrix}$$

As one can see the y–components of p_1 and p_2 are infinitesimal: they are the result of the square root operation in the Levi-Civita field and so they are in the magnitude of $\sqrt{d}$. If one takes the shadow of every component (or here equivalently the projective shadow since the vector is appreciable) one finds the double point of intersection $\mathbf{psh}(p_1) = \mathbf{psh}(p_2) = (1, 0, 1)^T$. Then of course the connecting line $\mathbf{join}(\mathbf{psh}(p_1), \mathbf{psh}(p_2))$ is the zero vector. This is essentially what standard projective geometry software would do.

Now consider the non-standard connecting line of p_1 and p_2:

$$\mathbf{join}(p_1, p_2) = \begin{pmatrix} 4.899 \cdot d^{\frac{1}{2}} \\ 0 \cdot d^0 \\ -4.899 \cdot d^{\frac{1}{2}} \end{pmatrix}$$

As one can see the vector is infinitesimal, since all entries are infinitesimal. Taking the shadow of all components would therefore lead to the same result as before: $\mathbf{sh}(\mathbf{join}(p_1, p_2)) = (0, 0, 0)^T$. Taking the projective shadow, and normalize the vector to an appreciable length in advance one finds the desired result:

$$\mathbf{psh}(\mathbf{join}(p_1, p_2)) = \mathbf{sh}\left(\frac{1}{4.899 \cdot d^{\frac{1}{2}}}\begin{pmatrix} 4.899 \cdot d^{\frac{1}{2}} \\ 0 \cdot d^{0} \\ -4.899 \cdot d^{\frac{1}{2}} \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The untruncated values can be found in Table 6.3.

Just another word on performance: the operations over the Levi-Civita field can be performed in real time, which is essential for a dynamic geometry system. For comparison: it took the computer algebra system seconds to find the same result (see Appendix B (Computer Algebra System Code Tangential Circle Intersection)).

	p_1	p_2	$\mathbf{join}(p_1, p_2)$
CindyJS	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$
NSA	$\begin{pmatrix}1 \cdot d^0 \cdot (1 \cdot d^0 - 0.1875 \cdot d^1 + 4.7076 \cdot d^{1.5} + 2.8125 \cdot d^2)\\ 2.4495 \cdot d^{0.5} \cdot (1 \cdot d^0 + 0.6124 \cdot d^{0.5} - 3 \cdot d^1 - 1.8371 \cdot d^{1.5})\\ 1 \cdot d^0 \cdot (1 \cdot d^0 - 3.1875 \cdot d^1 + 1.0334 \cdot d^{1.5} + 1.6875 \cdot d^2)\end{pmatrix}$	$\begin{pmatrix}1 \cdot d^0 \cdot (1 \cdot d^0 - 0.1875 \cdot d^1 - 4.7076 \cdot d^{1.5} + 2.8125 \cdot d^2)\\ -2.4495 \cdot d^{0.5} \cdot (1 \cdot d^0 - 0.6124 \cdot d^{0.5} - 3 \cdot d^1 + 1.8371 \cdot d^{1.5})\\ 1 \cdot d^0 \cdot (1 \cdot d^0 - 3.1875 \cdot d^1 - 1.0334 \cdot d^{1.5} + 1.6875 \cdot d^2)\end{pmatrix}$	$\begin{pmatrix}4.899 \cdot d^{0.5} \cdot (1 \cdot d^0)\\ 0 \cdot d^0 \cdot (1 \cdot d^0)\\ -4.899 \cdot d^{0.5} \cdot (1 \cdot d^0)\end{pmatrix}$
psh	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\end{pmatrix}$

Table 6.3.: Computational values for tangential circles: points of intersection and their connecting line. For two circles C and D, both with radius 1, centered at the origin and at $(2, 0, 1)^T$ respectively. A picture of the situation can be found in Figure 6.8 (middle). Their points of intersection p_1 and p_2 coincide. Without methods of non–standard analysis the connecting line is undefined. The perturbed situation on the other hand yields an infinitesimal vector whose projective shadow **psh** is a well defined line. We marked infinitesimal entries in the leading coefficients in red.

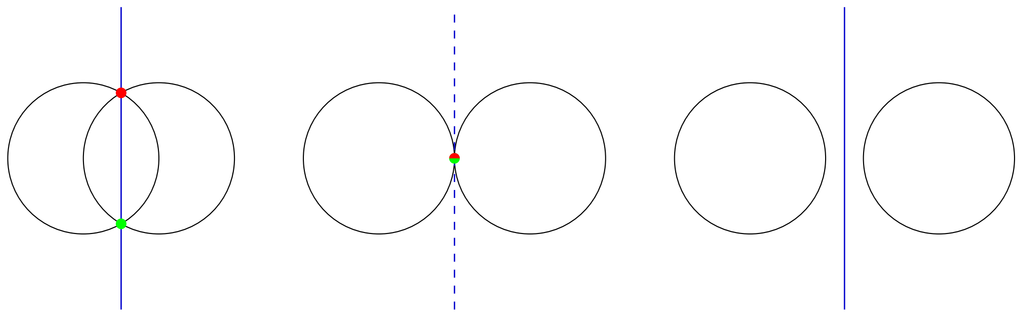


Figure 6.8.: **join** of two intersection points of circles. Left: the two real intersection points of two circles in red and green and their connecting line. Right: the so called “powerline”: a real line connecting the (now complex) intersection points red and green. Middle: the connecting line of red and green is undefined (since they are equal) and an example of a removable singularity. Numerical values can be found in Table 6.3.

◇

Example 6.5.5 (The Unification of Circles)
We already encountered the intersection of identical circles in Example 6.2.1 (Unified Circles Intersection) on p. 98. We will again consider the (Euclidean) points of intersection of two circles that have the same radius and (almost) the same center and consider their connecting line. When the centers coincide then the situation is singular.

We employ Algorithm 2 (Extended C^0-Continuation Algorithm) to check if a C^0-continuation is extended continuous or not. If for example the centers of the circles are bound to a common line, then the solution will be stable under infinitesimal perturbation. The unique C^0-continuation will yield a line that is orthogonal to the line that binds the centers and passes through the center.

On the other hand, if the centers of the circles are free, then random infinitesimal perturbations will lead to random intersections of the circles. The connecting lines of these points are thus random lines which even could not intersect the circle at all (if we consider complex solutions). Algorithm 2 verifies whether the results of the perturbations are infinitely close, for example by comparing the angles between the resulting vectors. The probability of a false positive result is negligible.

We will not plot any numerical values since they are essentially random noise, nevertheless see Figure 6.9 for an illustration. ◇

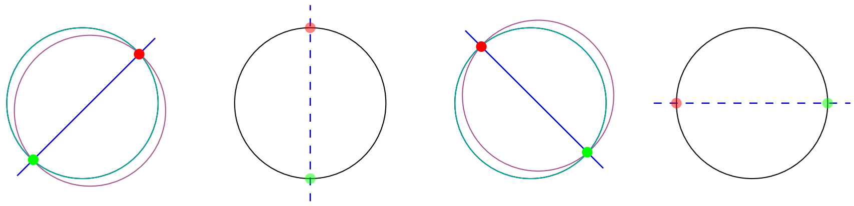


Figure 6.9.: **The Unification of Circles (free)**: the intersection of two identical circles yields unstable results under infinitesimal perturbation. Algorithm 2 (Extended C^0-Continuation Algorithm) quickly detects that the singularity is not removable.

6.6. Limitation of the Levi-Civita Field

> Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist.
>
> Kenneth Boulding (attributed, [54] p. 248)

The next section will show the limitations of the implementation presented in the previous section. It turns out that the proposed methods work well in most practical cases but there are also constructions which will result in exponential growth of support points of the Levi-Civita numbers.

One fundamental building block of the Levi-Civita field was the left-finiteness of support points in the construction of a Levi-Civita number (see Definition 6.4.2 (Left-Finite Sets) and Definition 6.4.5 (The Levi-Civita Fields: $\mathfrak{R}$ and $\mathfrak{C}$)). With the following arithmetical example and a corresponding construction we will push the support exponentially fast towards minus infinity.

We will give a recursive definition of a sequence which will force the support of the Levi-Civita numbers in exponential manner. More precisely: to calculate the n'th sequence member exactly, up to infinitesimal error, one has to preserve at least 2^n support points.

Definition 6.6.1 (Square unlimited Sequence)
Define the *square unlimited sequence* by:

$$\begin{aligned} a_0 &:= 1 + \frac{1}{d} = d^{-1} + d^0 \\ a_{i+1} &:= (a_i)^2 \end{aligned}$$

Writing out the first few members:

$$
\begin{aligned}
a_0 &= d^{-1} + d^0 \\
a_1 &= d^{-2} + 2 \cdot d^{-1} + d^0 \\
a_2 &= d^{-4} + 4 \cdot d^{-3} + 6 \cdot d^{-2} + 4 \cdot d^{-1} + d^0 \\
a_3 &= d^{-8} + 8 \cdot d^{-7} + 28 \cdot d^{-6} + 56 \cdot d^{-5} + 70 \cdot d^{-4} + 56 \cdot d^{-3} + 28 \cdot d^{-2} + 8 \cdot d^{-1} + d^0 \\
a_4 &= d^{-16} + 16 \cdot d^{-15} + 120 \cdot d^{-14} + 560 \cdot d^{-13} + \dots
\end{aligned}
$$

◇

We will give an explicit formula for the recursive definition before.

Lemma 6.6.2 (Explicit Formula)
The explicit formulation of the square unlimited sequence

$$
\begin{aligned}
a_0 &:= 1 + \frac{1}{d} = d^{-1} + d^0 \\
a_{i+1} &:= (a_i)^2
\end{aligned}
$$

is given by

$$a_n = (d^{-1} + 1)^{2^n} = \sum_{k=0}^{2^n} \binom{2^n}{k} d^{-(2^n - k)}$$

Proof. Easy proof by induction and application of the binomial formula. □

Theorem 6.6.3 (Exponential Growth)
The number of non–infinitesimal support points of the square unlimited sequence a_n grows with 2^n for all $n \in \mathbb{N}$.

Proof. By Lemma 6.6.2 (Explicit Formula) we know that $a_n = \sum_{k=0}^{2^n} \binom{2^n}{k} d^{k-2^n}$. In particular, all exponents d^{k-2^n} are not infinitesimal (for $k = 2^n$ the term d^{k-2^n} is appreciable, for all other k unlimited). The binomial factor $\binom{2^n}{k}$ is non–vanishing. This means a_n has 2^n non–infinitesimal support points. □

Remark. One might wonder why this should bother us at all. One advantage of non–standard arithmetic was that we can simply neglect some factors by using the shadow function and exploit this for the previously developed perturbation theory of geometric constructions. There we also just cut off infinite sequences and used the shadow function to recover the correct result.

The difference is that we only neglected infinitesimal components which have no influence on the shadow. Just remember that for an arbitrary number $z \in \mathbb{A}$ it holds true that $\mathbf{sh}(z+\epsilon) = \mathbf{sh}(z)$ for all $\epsilon \in \mathbb{I}$. If we now neglect non-infinitesimal this is not valid anymore. We will illustrate this with a numerical and a geometric example.

Example 6.6.4 (Arithmetical Bound)
An implementation which only saves a fixed amount of support points will fail calculations for the square unlimited sequence a_n at a $n \in \mathbb{N}$. We will show an example for $n = 3$.

$$a_3 = \frac{1}{d^8} + \frac{8}{d^7} + \frac{28}{d^6} + \frac{56}{d^5} + \frac{70}{d^4} + \frac{56}{d^3} + \frac{28}{d^2} + \frac{8}{d} + 1$$

We now define a truncation operation $(\hat{z})_k : \mathfrak{C} \to \mathfrak{C}$ which returns only the first k leading, meaning the smallest exponents of d, support points.

If we truncate a_3 we find

$$(\hat{a_3})_5 = \frac{1}{d^8} + \frac{8}{d^7} + \frac{28}{d^6} + \frac{56}{d^5} + \frac{70}{d^4}$$

Now it is quite obvious that certain relations do not hold anymore:

$$\Delta x := a_3 - (\hat{a_3})_5 = \frac{56}{d^3} + \frac{28}{d^2} + \frac{8}{d} + 1 \neq 0$$

We already encountered that truncations lead to results which are not exactly the same, but this was not really an obstacle since they all were in the same halo. But Δx is **not** infinitesimal! Δx is an unlimited number, all operations which rely on a_3 can not be replaced using $(\hat{a_3})_5$, they are not in the same halo (not even in the same galaxy!). Then taking simply a further multiplication and shadowing will result in completely wrong results:

$$\mathbf{sh}(x \cdot d^3) = \mathbf{sh}\left(56 \cdot d^0 + 28 \cdot d^1 + 8 \cdot d^2 + 1 \cdot d^3\right) = 56$$

while

$$\mathbf{sh}\left((\hat{a_3})_5 \cdot d^3\right) = \mathbf{sh}\left(0 \cdot d^3\right) = \mathbf{sh}(0) = 0.$$

This means that at some point we truncated relevant coefficients. ◇

After having an arithmetical example we will also consider a geometric construction which will illustrate the same effect.

Example 6.6.5 (Unlimited von-Staudt Construction)
Arithmetics can be encoded in geometry using von-Staudt constructions (see [111] p. 89 ff.). We already saw the summation in Example 6.5.1 (Classical von-Staudt Contruction),

now we will use the multiplication.

For the almost far point $\mathbf{x} = (d^{-1}+1, 0, 1)^T$ we will construct its square $\mathbf{x}^2$ according to the projective scale $\mathbf{0} = (0, 0, 1)^T$ and $\infty = (1, 0, 0)^T$. As neutral element of multiplication we use $\mathbf{1} = (1, 0, 1)^T$.

Denote by l the connecting line of $\mathbf{0}$ and ∞. Choose another line $m \neq l$ parallel to l and an arbitrary point Q not incident to l. Proceed with the following steps:

$$\begin{aligned}
P &:= \mathbf{meet}_*(m, \mathbf{join}_*(\mathbf{0}, Q)) \\
R &:= \mathbf{meet}_*(\mathbf{join}_*(\mathbf{x}, Q), m) \\
S &:= \mathbf{meet}_*(\mathbf{join}_*(\mathbf{1}, Q), m) \\
Q_1 &:= \mathbf{meet}_*(\mathbf{join}_*(\mathbf{x}, S), \mathbf{join}_*(P, Q)) \\
\mathbf{x}^2 &:= \mathbf{meet}_*(m, \mathbf{join}_*(Q_1, R))
\end{aligned}$$

A drawing can be found in Figure 6.10. The remarkable fact about this construction

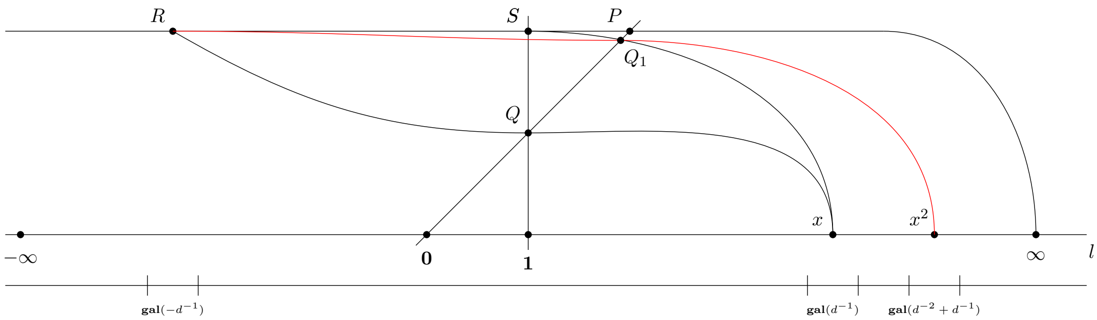


Figure 6.10.: **Unlimited von-Staudt construction** the construction generates x^2 of an unlimited value x.

is that all points incident to l, beside $\mathbf{1}$ and $\mathbf{0}$, are almost far points. Their projective shadow maps them all to the far point of the x-axis: $(1, 0, 0)^T$. The lines $\mathbf{join}_*(r, Q_1)$, $\mathbf{join}_*(Q, \mathbf{x})$ and $\mathbf{join}_*(\mathbf{x}, Q_1)$ are almost parallel to l and m and their projective shadow would be mapped to l or m. This means that in standard projective geometry the construction is highly degenerate while it is well defined in a non-standard setting.

Iterating the construction will successively yield the square unlimited sequence and therefore is a construction whose support points grow exponentially.

We give a table of values generated using a Levi-Civita field with the following fixed points: $Q = (1, 1, 1)^T$ and $m = (0, -1, 2)^T$. The numerical values in Table 6.4 indicate that one needs to preserve the first $3, 7$ and 13 leading support points to obtain results infinitely close to the correct results of x^2, x^4 and x^8.

The growth of factors is even faster than predicted in Theorem 6.6.3 (Exponential Growth). This is caused by the von-Staudt construction which heavily relies on the cross product. Every cross product itself consists of 6 multiplications and 3 subtractions which are also prone to truncation errors.

	$\mathbf{x^2}$	$\mathbf{x^4}$	$\mathbf{x^8}$
$\nu<1$	$1\cdot d^{-2}(1\cdot d^0)$	NaN	NaN
$\nu<2$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1)$	NaN	NaN
$\nu<3$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1+1\cdot d^2)$	$1\cdot d^{-4}(1\cdot d^0+2\cdot d^2)$	NaN
$\nu<5$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1+1\cdot d^2)$	$1\cdot d^{-4}(1\cdot d^0+4\cdot d^1+6\cdot d^2-28\cdot d^3-174\cdot d^4)$	$1\cdot d^{-8}(1\cdot d^0+2\cdot d^4)$
$\nu<6$	$1\cdot d^{-2}(1\cdot d^0)$	$1\cdot d^{-4}*(1\cdot d^0+4\cdot d^1+6\cdot d^2+4\cdot d^3+82\cdot d^4)$	$1\cdot d^{-8}*(1\cdot d^0+8\cdot d^1+64\cdot d^2+512\cdot d^3+4098\cdot d^4+32776\cdot d^5)$
$\nu<7$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1+1\cdot d^2)$	$1\cdot d^{-4}(1\cdot d^0+4\cdot d^1+6\cdot d^2+4\cdot d^3+1\cdot d^4)$	$1\cdot d^{-8}(1\cdot d^0+8\cdot d^1+28\cdot d^2-64\cdot d^3-1518\cdot d^4-9848\cdot d^5-24116\cdot d^6)$
$\nu<8$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1+1\cdot d^2)$	$1\cdot d^{-4}(1\cdot d^0+4\cdot d^1+6\cdot d^2+4\cdot d^3+1\cdot d^4)$	$1\cdot d^{-8}(1\cdot d^0+8\cdot d^1+28\cdot d^2+56\cdot d^3+402\cdot d^4+4552\cdot d^5+28684\cdot d^6+113800\cdot d^7)$
$\nu<10$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1+1\cdot d^2)$	$1\cdot d^{-4}(1\cdot d^0+4\cdot d^1+6\cdot d^2+4\cdot d^3+1\cdot d^4)$	$1\cdot d^{-8}(1\cdot d^0+8\cdot d^1+28\cdot d^2+56\cdot d^3+70\cdot d^4+56\cdot d^5+1900\cdot d^6+25800\cdot d^7+167250\cdot d^8)$
$\nu<12$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1+1\cdot d^2)$	$1\cdot d^{-4}(1\cdot d^0+4\cdot d^1+6\cdot d^2+4\cdot d^3+1\cdot d^4)$	$1\cdot d^{-8}(1\cdot d^0+8\cdot d^1+28\cdot d^2+56\cdot d^3+70\cdot d^4+56\cdot d^5+28\cdot d^6+8\cdot d^7+9170\cdot d^8)$
$\nu<13$	$1\cdot d^{-2}(1\cdot d^0+2\cdot d^1+1\cdot d^2)$	$1\cdot d^{-4}(1\cdot d^0+4\cdot d^1+6\cdot d^2+4\cdot d^3+1\cdot d^4)$	$1\cdot d^{-8}(1\cdot d^0+8\cdot d^1+28\cdot d^2+56\cdot d^3+70\cdot d^4+56\cdot d^5+28\cdot d^6+8\cdot d^7+1\cdot d^8)$

Table 6.4.: **Von-Staudt Multiplication: Construction of the square unlimited Sequence**. The table plots the dehomogenized x-component with wrong results, due to truncations, marked in red.

◇

6.7. A priori Avoidance of Singularities

Die Grenzen meiner Sprache bedeuten die Grenzen meiner Welt.

Ludwig Wittgenstein, [132]

In the previous sections we discussed how to algorithmically remove singularities in geometric constructions. Although the presented methods are real time capable they still involved quite some machinery and advanced techniques and so it would be beneficial, if we did not have to employ the proposed resolutions in the first place. In the following chapter we will give a blueprint to analyze geometric algorithms and optimize their algebraic properties to remove avoidable prefactors. These prefactors are superfluous in a projective setting since they generate just another representative in the same equivalence class of an projective object.

6.7.1. Modeling the Avoidance of Singularities

In the spirit of Theorem 3.1.11 (Existence of a Continuous Path), where we split a function $\Psi : [0,1] \to C^d$ into a vanishing and a non-vanishing part in order to resolve a singularity, we will generalize this ansatz to a more abstract level.

Definition 6.7.1 (Removable Prefactor)
Let $\mathbb{V} \subset \mathbb{CP}^{d_1} \times \ldots \times \mathbb{CP}^{d_n}$ model n homogenious input elements. And let $f : \mathbb{V} \to \mathbb{C}^{d+1}, f \not\equiv 0$ be a continuous function, which models an output element. If we can write f such that there is a non-vanishing continuous function $\tilde{f} : \mathbb{V} \to \mathbb{C}^{d+1} \setminus \{0\}$ and a function $C : \mathbb{V} \to \mathbb{C}$ with

$$f(I) = C(I) \cdot \tilde{f}(I) \quad \forall I \in \mathbb{V}$$

then we call C a *removable prefactor*. ◇

Analogously to the property of being extended C^0-continuous of Definition 6.2.2 (Extended C^0-continuation), we define the property of being universal C^0-continuous. The major difference between these concepts is that the first is only defined for one input element while the second is defined for multiple input elements.

Definition 6.7.2 (Universal $\boldsymbol{C^0}$-continous)
Let $\mathbb{V} \subset \mathbb{CP}^{d_1} \times \ldots \times \mathbb{CP}^{d_n}$ and let $f : \mathbb{V} \to \mathbb{CP}^d$ be a continuous function. If there

is $\hat{\mathbb{V}} \supsetneq \mathbb{V}$ and a continuous $\hat{f} : \hat{\mathbb{V}} \to \mathbb{CP}^d$ such that $f(I) = \hat{f}(I) \forall I \in \mathbb{V}$ then we call f *universal C^0-continuable* and $\hat{f}$ the *universal C^0-continuation* on $\hat{\mathbb{V}}$. $\diamond$

The next theorem will provide a way to find universal C^0-continuations if we can split a function into a non-vanishing part and a prefactor.

Theorem 6.7.3 (Avoiding Singularities)
Let $\mathbb{V} \subset \mathbb{CP}^{d_1} \times \ldots \times \mathbb{CP}^{d_n}$ and let $f : \mathbb{V} \to \mathbb{C}^{d+1}$ have a removable prefactor $C : \mathbb{V} \to \mathbb{C}$. Call $\mathbb{V}' := \{v \in \mathbb{V} \mid f(v) \neq 0\}$. Then $[f] : \mathbb{V}' \to \mathbb{CP}^d$ has an universal C^0-continuation $[\hat{f}]$ on $\mathbb{V}' \cup C^{-1}(0)$.

Proof. Since f has a removable prefactor we can write

$$f(I) = C(I) \cdot \tilde{f}(I) \quad \forall I \in \mathbb{V}$$

Now define: $\hat{f} : \mathbb{V}' \cup C^{-1}(0) \to \mathbb{C}^{d+1}, I \mapsto \tilde{f}(I)$ then it holds true that

$$[f(I)] = [\hat{f}(I)] \forall I \in \mathbb{V}'.$$

Furthermore, since $\tilde{f}$ is continuous and non-vanishing we have a universal continuation of $[f]$ on $\mathbb{V}' \cup C^{-1}(0)$ given by $[\tilde{f}]$. $\square$

6.7.2. From Plücker's μ to Efficient Implementation

Plücker's μ is very nice trick to construct geometric objects that fulfill multiple constraints. An example would be a line l that passes through the intersection of two other lines l_1 and l_2 and additionally through a third point q [111]. For further reading we refer the reader to [111] p. 96ff.

Theorem 6.7.4 (Midpoint via Plücker's $\boldsymbol{\mu}$)
The Euclidean midpoint of two point X and $Y \in \mathbb{RP}^2$ is given by:

$$M = [Y, P_\infty]_L X + [X, P_\infty]_L Y$$

where $P_\infty = \mathbf{meet}(\mathbf{join}(X, Y), l_\infty)$ and L is chosen such that it is linearly independent of X, P_∞ and Y, P_∞.

Remark. For objects X, Y, Z in $\mathbb{CP}^2$ we write $[X, Y]_Z = [X, Y, Z] = \det(X, Y, Z)$.

Proof. According to “Perspectives on Projective Geometry” [111] p. 81 X, Y, the midpoint

M and P_∞ are in harmonic position (seen from an appropriate point L):

$$(X, Y; M, P_\infty)_L = -1$$

Furthermore we know that we can construct the midpoint via Plücker's μ as a linear combination of X and Y. This means there are $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$ such that $M = \lambda X + \mu Y$. In the following, we will derive those parameters using the cross-ratio property:

$$\begin{aligned}
&(X, Y; M, P_\infty)_L = -1 \\
\Leftrightarrow\ &\frac{[X, M]_L [Y, P_\infty]_L}{[X, P_\infty]_L [Y, M]_L} = -1 \\
\Leftrightarrow\ &[X, M]_L [Y, P_\infty]_L = -[X, P_\infty]_L [Y, M]_L \\
\Leftrightarrow\ &\mu [X, Y]_L [Y, P_\infty]_L = -\lambda [X, P_\infty]_L [Y, X]_L \\
\Leftrightarrow\ &[X, Y]_L (\mu [Y, P_\infty]_L - \lambda [X, P_\infty]_L) = 0
\end{aligned}$$

An appropriate choice for $\mu[Y, P_\infty]_L - \lambda[X, P_\infty]_L$ to be zero is $\lambda = [Y, P_\infty]_L$ and $\mu = [X, P_\infty]_L$. And therefore we can write M as

$$M = [Y, P_\infty]_L X + [X, P_\infty]_L Y$$

which is the claim. □

Remark. An appropriate choice for L in the previous theorem is $L = X \times Y$. This is fairly easy to see as we are looking for a point L which is not incident to the line $l = \mathbf{join}(X, Y)$ since X, Y and P_∞ are all incident to l. Hence L, which is the line l interpreted as point, is an appropriate choice since $\langle L, l \rangle = \langle L, L \rangle = \|L\|^2 \neq 0$: therefore L (interpreted as point) is not incident to l.

Lemma 6.7.5 (Algebra of Midpoint via Plücker's $\boldsymbol{\mu}$)
The algebraic representation of the midpoint $M = [Y, P_\infty, L]X + [X, P_\infty, L]Y$ of Theorem 6.7.4 (Midpoint via Plücker's μ) of $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ with $L = X \times Y$ has the greatest common divisor

$$C := x_2^2 y_1^2 + x_3^2 y_1^2 - 2x_1 x_2 y_1 y_2 + x_1^2 y_2^2 + x_3^2 y_2^2 - 2x_1 x_3 y_1 y_3 - 2x_2 x_3 y_2 y_3 + x_1^2 y_3^2 + x_2^2 y_3^2$$

Proof. Simple calculation using a computer algebra system: see Appendix D. □

Theorem 6.7.6 (Efficient Midpoint, [111] p. 433)

For two points $X = (x_1, x_2, x_3)^T, Y = (y_1, y_2, y_3)^T \in \mathbb{RP}^2$ their Euclidean midpoint M is given by:

$$M = y_3 X + x_3 Y.$$

Proof. Using a computer algebra system we can show (see Appendix D) that the midpoint M_μ of Theorem 6.7.4 (Midpoint via Plücker's μ) can be written with C of Lemma 6.7.5 (Algebra of Midpoint via Plücker's μ)

$$M_\mu = C \cdot (y_3 X + x_3 Y).$$

Therefore the solutions are equivalent and the efficient formula has less singularities. □

Lemma 6.7.7 (Removable Singulariy of Theorem 6.7.4 (Midpoint via Plücker's $\boldsymbol{\mu}$))
Theorem 6.7.4 (Midpoint via Plücker's μ) has a removable singularity with continuation $M = X = Y$ if $X = Y$, while Theorem 6.7.6 (Efficient Midpoint, [111] p. 433) is well defined.

Proof. Since $X = Y$ the operation $\mathbf{join}(X, Y)$ will yield the zero vector. Because the definition of M contains $P_\infty = \mathbf{meet}(\mathbf{join}(X, Y), l_\infty)$ it is also the zero vector. Thus, the C of Lemma 6.7.5 (Algebra of Midpoint via Plücker's μ) is attaining 0 for $X = Y$.

It is easy to see that Theorem 6.7.6 (Efficient Midpoint, [111] p. 433) is well defined for $X = Y$. □

6.7.3. Limits of a Priori Resolution

While it is wise to scrutinize the formulas of geometric objects for superfluous pre factors this is unfortunately not always possible. We will give a construction that has a singularity which cannot be resolved a priori.

Theorem 6.7.8 (No a Priori Resolution)
There is a geometric construction whose singularities cannot be resolved a priori.

Proof. By example: take two finite lines $l = (l_1, l_2, l_3)^T$ and $g = (g_1, g_2, g_3)^T$. Denote their point of intersection by $P = \mathbf{meet}(l, g)$, this operation is always well defined if $l \neq g$. A simple calculation shows that the far point F of all lines perpendicular to l is given by $F = (l_1, l_2, 0)^T$, which is always well defined since we assumed $l \neq l_\infty$. Then the connecting line l' of F and P is algebraically given by

$$l' = \left(-l_2^2 g_1 + l_1 l_2 g_2,\ l_1 l_2 g_1 - l_1^2 g_2,\ l_1 l_3 g_1 + l_2 l_3 g_2 - l_1^2 g_3 - l_2^2 g_3\right).$$

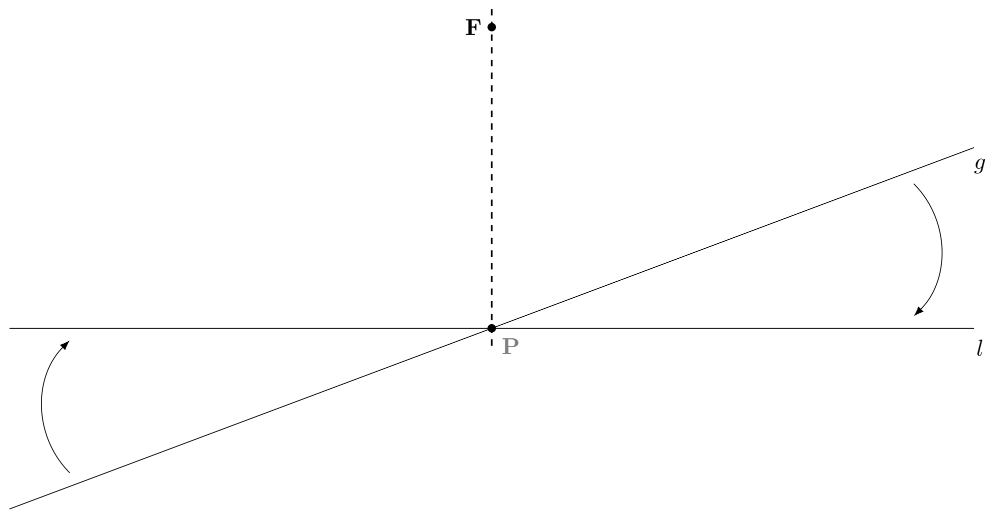


Figure 6.11.: **No a priori resolution:** the singularity in the construction if $l = g$ is not a priori removable. It depends on the position of P which is determined by the position of l and g. P is not fixed, it may vary heavily on the movement of l and g, if P would be fixed, then an a priori resolution would be possible.

Using a computer algebra system one can verify that the greatest common divisor of the components of l' is 1 and therefore there is no removable prefactor which could be canceled out in a priori.

Nevertheless the operation $\mathbf{meet}(l, g)$ is undefined if $l = g$, then $l' = \mathbf{join}(F, P)$ is also undefined and so the operation has a singularity for $l = g$ that cannot be removed a priori.

□

7. Open Problems & Future Work

> So fängt denn alle menschliche Erkenntnis mit Anschauungen an, geht von da zu Begriffen, und endigt mit Ideen.
>
> Immanuel Kant: “Kritik der reinen Vernuft” [70]

We will conclude this thesis with some remaining problems and future work.

7.1. Randomized Proving using Infinitesimal Deflection

> Der Gedanke, daß ein einem Strahl ausgesetztes Elektron aus freiem Entschluß den Augenblick und die Richtung wählt, in der es fortspringen will, ist mir unerträglich. Wenn schon, dann möchte ich lieber Schuster oder gar Angestellter einer Spielbank sein als Physiker.
>
> Albert Einstein, Brief an Max Born [32]

One particularly interesting problem for further investigation would be randomized continuity proving. While in Algorithm 1 (C^0-Continuation Algorithm) (see page 120) we exploited that singularities along a given path are isolated and we only had to check if the right and left hand side limit coincide, the situation is different for unrestricted situations. Algorithm 2 (Extended C^0-Continuation Algorithm) (see page 121) heuristically checks if a function is stable under (random) perturbations and therefore fulfills the non-standard notion of continuity. To be sure that a multivariate complex function (even if the directional derivatives exists) is continuous we would have to check infinitely many coefficients.

In practice being 100% sure that the function is actually continuous is usually unnecessary and we can content ourself with a high probability of continuity. One classic result is the Schwartz-Zippel theorem for multivariate polynomials.

Theorem 7.1.1 (Schwartz-Zippel, [96] p. 165 ff.)
Let $p(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$ be a multivariate polynomial of degree $d \in \mathbb{N}$ over the

field F. Fix any finite set $S \subset F$ and let $r_1, \ldots, r_n$ be chosen independently and uniformly at random from S. Then

$$\Pr[p(r_1, \ldots, r_n) = 0 \mid p(x_1, \ldots, x_n) \not\equiv 0] \leq \frac{d}{|S|}.$$

Since the hyperreal- and hypercomplex numbers form a field, the Schwartz-Zippel theorem also holds true there. Although the objects in dynamic geometry can be modeled by multivariate polynomials (for details see [77]) we cannot apply Schwartz-Zippel here. From Theorem 6.2.3 (Normalization and Extended Continous Paths) (see p. 99) we know that we can find a extended C^0-continuation, if it fulfills the following criterion:

$$\exists L \in \mathbb{C}^{d+1}: \; L \simeq \frac{\Psi(v(t_0) + \Delta v)}{\|\Psi(v(t_0) + \Delta v)\|} \simeq \frac{\Psi(v(t_0) + \Delta v')}{\|\Psi(v(t_0) + \Delta v')\|}$$
$$\forall \Delta v, \Delta v' \in \mathbb{I}^{k+1} : v(t_0) + \Delta v, v(t_0) + \Delta v' \in (\overline{X})^*.$$

We actually have no guarantee that we are testing polynomials here but rather analytic functions in the case of a C^0-continuation. There is some generalization of the Schwartz-Zippel theorem by Tulone *et al.* [131] that can treat radical expressions and division but the paper only examines the real case and was not designed to test for continuity in the non-standard sense. Further research here would be quite interesting to deduce probability bounds for continuity in Algorithm 2 (Extended C^0-Continuation Algorithm).

Open problem 7.1.2 (Randomized Continuity Testing)
Given an analytic black box function $f : \mathbb{C}^m \to \mathbb{C}^n$. Can one deduce a probability bound for the continuity of the function $\frac{f(z)}{\|f(z)\|}$ using the non-standard notion of continuity?

7.2. Möbius Transformations

Möbi! Möbi! Möbi!

Münchner Kammerspiele, Die Physiker (Friedrich Dürrenmatt)

Möbius transformations are a basic building block of complex analysis. They are conformal automorphisms of the Riemann sphere and have many applications. We refer to “Perspectives on Projective Geometry” [111] for further details.

Definition 7.2.1 (Interpreting Points in $\mathbb{RP}^2$ in $\mathbb{CP}^1$ and vice versa, [111] p. 330 f.)
A given finite point $P \in \mathbb{RP}^2$ with $p = [p_1, p_2, 1]$ can be interpreted as a finite point

$P_C \in \mathbb{CP}^1$ with $P_C = [p_1 + ip_2, 1]$ and the other way round. $\diamond$

Remark. So from a topological point of view there is a bijection of finite points in $\mathbb{RP}^2$ and $\mathbb{CP}^1$. This does not hold true for far points since $\mathbb{CP}^1$ possesses only one far point while $\mathbb{RP}^2$ contains a whole line of far points.

Theorem 7.2.2 (Möbius and Projective Transformations, [111] p. 315 f.)
A projective transformation $M : \mathbb{CP}^1 \to \mathbb{CP}^1$ can be uniquely identified with a Möbius transformation $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ and vice versa.

Proof. We can write every projective transformation in $\mathbb{CP}^1$ by a matrix $M \in \mathrm{GL}_2(\mathbb{C})$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consider the following sequence of homogenization, transformation and dehomogenization

$$z \quad \mapsto \quad \begin{pmatrix} z \\ 1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} a \cdot z + b \\ c \cdot z + d \end{pmatrix} \quad \mapsto \quad \frac{a \cdot z + b}{c \cdot z + d}$$

where the last term is the formula for a Möbius transformation of z. $\square$

It is a bit unhandy to perform the chain of homogenization, transformation and dehomogenization and the other way round. As it turns out one can also apply a Möbius transformation to point in $\mathbb{RP}^2$ directly.

Theorem 7.2.3 (Applying Möbius Transformations to Points in $\mathbb{RP}^2$)
Given a Möbius transformation with coefficients $a, b, c, d \in \mathbb{C}$ with $a = a_1 + ia_2, b = b_2 + ib_2 \ldots$ one can apply the transformation to a finite point $P \in \mathbb{RP}^2$ with two matrix-vector multiplications and a meet:

$$\hat{P} := \mathbf{meet}(M_1 \cdot P, M_2 \cdot P)$$

with

$$M_1 := \begin{bmatrix} -c_1 & c_2 & -d_1 \\ c_2 & c_1 & d_2 \\ a_1 & -a_2 & b_1 \end{bmatrix}, \qquad M_2 := \begin{bmatrix} -c_2 & -c_1 & -d_2 \\ -c_1 & c_2 & -d_1 \\ a_2 & a_1 & b_2 \end{bmatrix}.$$

Proof. For a finite point $P \in \mathbb{RP}^2$ with $P = [p_1, p_2, 1]$ we write out the formula from above:

$$P_1 := M_1 \cdot P = \begin{bmatrix} -c_1p_1 + c_2p_2 - d_1 \\ c_1p_2 + c_2p_1 + d_2 \\ a_1p_1 - a_2p_2 + b_1 \end{bmatrix} \quad P_2 := M_2 \cdot P \begin{bmatrix} -c_1p_2 - c_2p_1 - d_2 \\ -c_1p_1 + c_2p_2 - d_1 \\ a_1p_2 + a_2p_1 + b_2 \end{bmatrix}$$

applying the meet yields the final point $\hat{P}$:

$$\hat{P} = \begin{bmatrix} (a_1p_1 - a_2p_2 + b_1)(c_1p_1 - c_2p_2 + d_1) + (a_1p_2 + a_2p_1 + b_2)(c_1p_2 + c_2p_1 + d_2) \\ -(a_1p_1 - a_2p_2 + b_1)(c_1p_2 + c_2p_1 + d_2) + (a_1p_2 + a_2p_1 + b_2)(c_1p_1 - c_2p_2 + d_1) \\ (c_1p_1 - c_2p_2 + d_1)^2 + (c_1p_2 + c_2p_1 + d_2)^2 \end{bmatrix}$$

Now on the complex number level: $P_C = p_1 + ip_2$:

$$M(P_C) := \frac{P_C \cdot a + b}{P_C \cdot c + d} = \frac{(a_1 + ia_2)(p_1 + ip_2) + b_1 + ib_2}{(c_1 + ic_2)(p_1 + ip_2) + d_1 + id_2}$$

Evaluating $\frac{1}{2}(M(P_C) + \overline{M(P_C)})$ yields the real part of $M(P_C)$:

$$\mathrm{Re}(M(P_C)) = \frac{(a_1p_1 - a_2p_2 + b_1)(c_1p_1 - c_2p_2 + d_1) + (a_1p_2 + a_2p_1 + b_2)(c_1p_2 + c_2p_1 + d_2)}{(c_1p_1 - c_2p_2 + d_1)^2 + (c_1p_2 + c_2p_1 + d_2)^2}$$

and for the imaginary part $\frac{1}{2i}(M(P_C) + \overline{M(P_C)})$:

$$\mathrm{Im}(M(P_C)) = \frac{-(a_1p_1 - a_2p_2 + b_1)(c_1p_2 + c_2p_1 + d_2) + (a_1p_2 + a_2p_1 + b_2)(c_1p_1 - c_2p_2 + d_1)}{(c_1p_1 - c_2p_2 + d_1)^2 + (c_1p_2 + c_2p_1 + d_2)^2}$$

If one has a close look one finds that the real and imaginary part are the z-normalized version of the x and y component of $\hat{P}$, which is the claim. □

Open problem 7.2.4 (Möbius Transformations as Conic Sections)
Consider the formulas of Theorem 7.2.3 (Applying Möbius Transformations to Points in $\mathbb{RP}^2$):

$$\hat{P} := \mathbf{meet}(M_1 \cdot P, M_2 \cdot P)$$

with two matrices $M_1, M_2 \in \mathbb{R}^{3\times 3}$. From a projective point of view one can interpret M_1 and M_2 as conic sections.

The matrix-vector multiplication of a point P and a conic section matrix M can be seen as polar line of the conic section at a point P and if P is incident to the conic

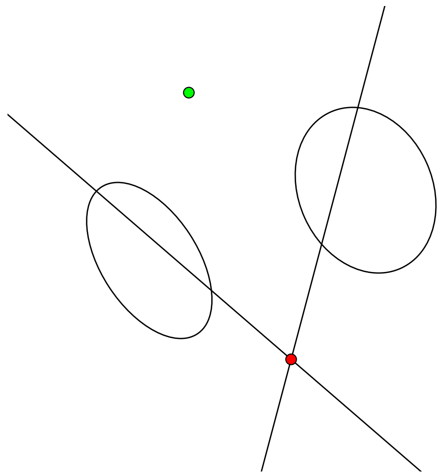


Figure 7.1.: **Möbius transformations in $\mathbb{RP}^2$:** Two conic sections with polar lines to the green point and their intersection (red point).

section, this yields the tangent of the conic at the point P.

With the given formula above one can interpret the Möbius transformation of P as the intersection of two polar lines $l_1 := M_1 \cdot P$ and $l_2 := M_2 \cdot P$. It would be highly interesting to see if there is a geometric interpretation of these two conic sections. See Figure 7.1 for an illustration.

7.3. New Approaches to Tracing

> Let us consider an arbitrary figure, in a general position which in a certain sense is indeterminate among all positions it can assume without violating the laws, the conditions, the bonds that exist between the different parts of the system; let us suppose according to these data one has found one or more relations or properties, which may be metric or descriptive, belonging to the figure, by way of ordinary explicit reasoning, that is, the procedure which is in certain cases considered as the only rigorous one. Is it not obvious that if while preserving those data one undertakes to vary the original figure ever so slightly and subjects parts of it to an arbitrary but continuous motion – is it not obvious that the properties and relations, found in the first system, remain valid in its successive stages, provided that due account is attributed to the particular modifications that may arise, for example if certain magnitudes vanish or change their direction or sign, and so on, modifications that can easily be recognized *a priori* and by sure rules?
>
> Jean-Victor Poncelet, [45]

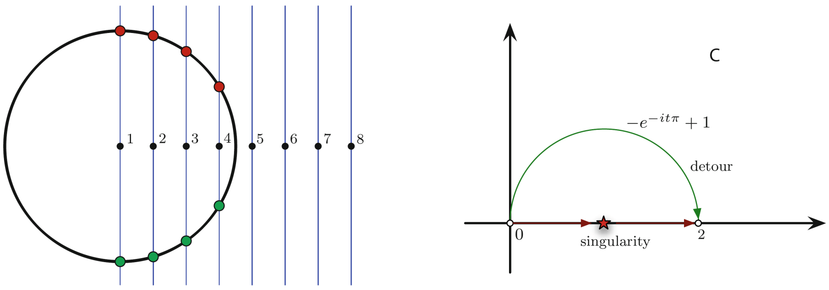


Figure 7.2.: Intersections of a vertical line and a circle under a movement (left). The space of the controlling parameter of the line (right) [111] p. 553.

One major topic of dynamic geometry is the so called "tracing" problem. Many constructions in geometry involve situations where singularities arise. In the previous chapters, we explored how to assign meaningful resolutions to these singularities. Another questions would be how a construction should behave when we move through such a singularity. We will only give a brief introduction (following [111] p. 550 ff.) and refer to [24, 79] for further details.

Consider the following example (borrowed from [111] p. 553). Take a unit circle C and a vertical line l controlled by a free point P (with initial position inside C) and assign colors to the points of intersection of l and C. Then by moving p horizontally to the right for example, the two points of intersection eventually get closer and closer. When l is tangential to C the two points of intersection coincide. If we move P horizontally further away from from C then the two points are distinguishable again (although they are complex) but no meaningful assignment of colors is possible anymore. The main idea of tracing is to avoid the singularity and to perform a complex detour around the singularity such that the two points never coincide. See Figure 7.2 for an illustration.

7.3.1. Infinitesimal Tracing

In the previous section we described how tracing works in principle. One major problem of the ansatz is that there is no guarantee for the complex detour to be free of singularities. One possible way to avoid the problem is to exploit a nice fact of geometric constructions: under reasonable assumptions geometric constructions have only isolated singularities (see [76]). Instead of detouring on a standard semi circle like

$$c(t) := -\mathrm{e}^{-\mathrm{i}t\pi} + 1 \quad (t \in [0,1]),$$

one could simple use a semi circle with an infinitesimal radius like

$$c'(t) := \Delta x \cdot c(t) = \Delta x(-\mathrm{e}^{-\mathrm{i}t\pi} + 1) \text{ for } \Delta x \in \mathbb{I}_{\mathbb{R}} \setminus \{0\}.$$

A problem of this ansatz is how we pass the barrier from the standard world to the non-standard infinitesimal world, analogously to the transition from classical- to quantum physics.

Open problem 7.3.1 (Infinitesimal Tracing)
Develop a tracing strategy that uses infinitesimal radii for complex detours.

7.3.2. Tracing and Automatic Differentiation

In the current tracing process the assignment of objects (think of color mapping of points) is decided by proximity. This is very reasonable: by continuity objects should not move far if we alter the input elements only slightly. The semi circle is traversed non-linearly: if the objects are far away from each other the traversing speed is increased because it is unlikely that a singularity is close. On the other hand if objects are close to each other the speed is decreased to make proper decisions and intermediate evaluation steps are required. To avoid such steps one could use Taylor's formula to have a better estimate the position of the objects. This requires that one can determine the Taylor coefficients fast and stable. The de facto standard for numerical differentiation is nowadays automatic differentiation (see [55]). So one way to obtain the Taylor coefficients would be to use an automatic differentiation library within a dynamic geometry software.

But there is a further use case for non-standard analysis: from Theorem 4.4.3 (Taylor Series) we know that we can expand an analytic function f using $\Delta z \in \mathbb{I}$ as follows: there exists $\epsilon \in \mathbb{I}$ such that

$$f(x + \Delta z) = f(z) + f'(z)\Delta z + \frac{f''(z)}{2!}\Delta z^2 + \cdots + \frac{f^{(n)}(z)}{n!}\Delta z^n + \epsilon\Delta x^n$$

This is one powerful usage of non-standard analysis: we can simply read up the derivatives as prefactors of infinitesimal numbers! In the Levi-Civita world M. Berz already explored the connection of non-standard analysis and automatic differentiation. We refer the reader to [8] for details.

Open problem 7.3.2
Improve tracing with automatic differentiation using a non-standard implementation.

A. CindyScript Implementation of the Levi-Civita Field

```
degree = 5; // degree of LCNum
debug = false;

// generates a levi civita number of the form aq0*d^q0(1+eps)
genLC(aq0, q0, arr):=(
    // everything is dict
    obj = dict();
    obj = put(obj, "aq0", aq0);
    obj = put(obj, "q0", q0);

    // sort by exponent
    sortarr = sort(arr, #_2);
    obj = put(obj, "arr", sortarr);
    // adjust exponents - fix that later
    //if(sortarr_1_2 != 0, obj = colLC(obj));
    obj;
);

// clone a LC number
cloneLC(x):=(
    aq0 = get(x, "aq0");
    q0 = get(x, "q0");
    arr = get(x, "arr");

    genLC(aq0, q0, arr);
);
```

```
// 1-element - 1*d^0
genOne():=(
    genLC(1,0,[[1,0]]);
);
// 0-element - 0*d^0
genZero():=(
    genLC(0,0,[[1,0]]);
);

isZero(x):=(
   xx = colLC(x);
   aq0= get(x, "aq0");
   aq0 == 0;
);

// collect components
colLC(x):=(
    arr = get(x, "arr");
    arr = sort(arr, #_2);

    minq = arr_1_2; // minimal exponent
    n = 0; // adjust exponent such that the array has only infinitesimal members
    if(minq != 0,
        tmp = [];
        n = -minq; // how much do we have to add?
        forall(arr, tmp = tmp ++ [[#_1, #_2 + n]]);
        arr = tmp;
    );

    narr = [];

    sumUp(a):=(
            sum = 0;
            forall(a, sum = sum + #_1);
            sum;
    );
```

```
while(length(arr) > 0,
        first = arr_1;
    els = select(arr, #_2 == first_2);
    arr = arr -- els;
    narr = narr ++ [[sumUp(els), first_2]];
);

vanished = select(narr, (#_1) ~= 0);
narr = narr -- vanished;

// normalize first entry to 1
c = 1;
if(length(narr) > 1, // no empty arrays
    narr = sort(narr, #_2);
    c = narr_1_1;
);
if((c != 1) & (c ~!= 0),
tmp = [];
forall(narr, tmp = tmp ++ [[(#_1)/c, #_2]]);
narr = tmp;
);
if(c ~= 0, c = 1); // dont divide by 0


// truncate series to degree
if(narr_(length(narr))_2 > degree,
        tmp = [];
        forall(narr, if(#_2 < degree, tmp = tmp ++ [#]));
        narr = tmp;
);

// add n to adjust exponent, and divide by c to normalize to 1 for aq1

if(length(narr) > 0, // if all terms vanished return 0
    genLC(get(x, "aq0")*c, get(x, "q0")-n, narr);,
```

```
        genZero();
    );
);

// convert to TeX Strings
stringLC(dic):=(
    str = get(dic, "aq0") + "*" + "d^{" + get(dic, "q0") + "}";
    arr = get(dic, "arr");

    str = str + "*(";
    forall(arr, str = str + #_1 + "*d^{" + #_2 + "}+");
    // remove last (+
    str_(length(str)) = "";
    str = str + ")";
    str;
);

// Output function
printLC(dic):=(
    les = stringLC(dic);
    println(les);
);

texLC(str):=(
    "$"+str+"$";
);

// negate LC number
negLC(x):=(
    old = get(x, "arr");
    new = [];
    forall(old, new = new ++ [[-(#_1), #_2]]);
    erg = genLC(get(x, "aq0"), get(x, "q0"), new);
    colLC(erg);
);
```

```
// x + y
addLC(x,y):=(
    xq0 = get(x, "q0");
    q0diff = xq0 - get(y, "q0");

    xaq0 = get(x, "aq0");
    yaq0 = get(y, "aq0");

    // adjust exponents
    yarr = get(y, "arr");
    zarr = [];
    if(xaq0 == 0, zarr = yarr,
    forall(yarr, zarr = zarr ++ [[yaq0*(#_1)/xaq0, #_2 - q0diff]]);
    );

    // add x arr
    erg = y;
    if(!isZero(x),
    erg = genLC(xaq0, xq0, zarr ++ get(x, "arr"));
    );
    colLC(erg);
);

// x - y
subLC(x,y):=(
    if(isZero(y), x,
        addLC(x, negLC(y)
    ));
);

// x * y
multLC(xx,yy):=(
    x = colLC(xx);
    y = colLC(yy);
    if(debug, println(["x", x, "y", y]));
    back = [];
```

```
    xaq0 = get(x, "aq0");
    yaq0 = get(y, "aq0");
    zaq0 = xaq0*yaq0;
    back = back ++ [zaq0];

    xq0 = get(x, "q0");
    yq0 = get(y, "q0");
    zq0 = xq0+yq0;
    back = back ++ [zq0];
    if(debug, println(["back", back]));

    xarr = get(x, "arr");
    yarr = get(y, "arr");
    tmp = directproduct(xarr, yarr);

    zarr = [];
    forall(tmp,
            val = [[(#_1_1) * (#_2_1),(#_1_2) + (#_2_2)]];
            if(val_1_2 <= degree, zarr = zarr ++  val);
    );

    if(debug, println(["zarr", zarr]));
    z = genLC(back_1, back_2, zarr);
    z = colLC(z);
);

// integer pow
intPow(x,r):=(
    erg = x;
    repeat(r-1, erg = multLC(erg, x));
    erg;
);

// x/y
divLC(x,y):=(
    colLC(multLC(x,invLC(y)));
```

```
);

// mult with reel/complex fact
multLCFac(x, r):=(
    xaq0 = get(x, "aq0");
    xq0 = get(x, "q0");
    xarr = get(x, "arr");
    genLC(xaq0*r, xq0, xarr);
);

// x + c*d^0
addLCFac(x, r):=(
    y = genLC(r, 0, [[1,0]]);
    colLC(addLC(x,y));
);

// generate Taylor series
genTaylor(Num, koeff):=(
    tayOrd = 8;
    if(isZero(Num), tayOrd = 0); // for zero we know the result
    erg = genOne();
    if(koeff == "log", erg = genZero()); // log does not have 1

    tmpx = Num;
    repeat(tayOrd, i,
      if(koeff == "sqrt", // p is degree of root
                  p = 2;
                  tmpx = multLCFac(tmpx,(1/p-i+1)/i);
                      );
        erg = colLC(erg);
        erg = addLC(erg, tmpx);
        tmpx = multLC(tmpx, Num);

        if(i < tayOrd & koeff == "exp", tmpx = multLCFac(tmpx, 1/(i+1)));
        if(i < tayOrd & koeff == "log", tmpx = multLCFac(tmpx, ((-1)^i)*i/(i+1)));
     ); // end repeat
```

```
    erg;
);


// get infinitesimal part compared to leading term
getInf(x):=(
    xx = colLC(x);
    txarr = get(xx, "arr");
    taq0 = get(xx, "aq0");

    erg = genZero(); // default form for eps part = 1
    if(txarr != [[1,0]],   // we have really something to find
            fOne = genLC(1,0,txarr);
            erg = subLC(fOne,genOne());
    );

    colLC(erg);
);

// 1/x
invLC(xx):=(
    if(isZero(xx), println("LC Division by 0!"));
    x = colLC(xx);
    xaq0 = get(xx, "aq0");
    yaq0 = 1/(xaq0);
    xq0 = get(x, "q0");
    yq0 = -xq0;

    invbackup = [yaq0, yq0];

    eps = getInf(x);
    eps = negLC(eps);
    eps = colLC(eps);
    ser = genTaylor(eps,"1");
    ser = colLC(ser);
    yarr = get(ser, "arr");
```

```
    erg = genLC( invbackup_1, invbackup_2, yarr);

    colLC(erg);
);

expLC(xx):=(
   x = colLC(xx);
   genTaylor(x,"exp");
);

// log(xx)
// assume that LC Num is of form a + b*eps + c*eps^2
logLC(xx):=(
    x = colLC(xx);
    xaq0 = get(xx, "aq0");
    xq0 = get(x, "q0");
    if(xq0 != 0, println("log is undefined!"));
    logAq0 = log(xaq0);

    eps = getInf(x);
    tay = genTaylor(eps, "log");

    erg = addLCFac(tay,logAq0);
    colLC(erg);
);

// sqrt(x)
sqrtLC(xx):=(
    sx = colLC(xx);
    //yaq0 = (get(sx, "aq0"))^(1/2);
    yaq0 = sqrt((get(sx, "aq0")));
    yq0 = get(sx, "q0")/2;
    sqback = [yaq0, yq0];

    seps = getInf(sx);
```

```
    tay = genTaylor(seps, "sqrt");
    yyy = genLC(sqback_1,sqback_2, [[1,0]]);
    erg = multLC(yyy, tay);
    colLC(erg);
);
```

B. Computer Algebra System Code Tangential Circle Intersection

The following code uses the SAGE [118] computer algebra system and shows that one special tangential situation of the circle intersection is extended continuous. By translation it also holds for all other tangential situations. The computation took about 40 seconds Intel Core i7-4712HQ CPU.

```
from time import time
start = time();

var("x_m, y_m, x, y", domain=CC);

# circle with radius 1 centered at the origin
A = Matrix([[1,0,0],[0,1,0], [0,0,-1]]);
# Circle tangential to A
B = Matrix([[1,0,-x_m -2],[0,1,-y_m],
    [-x_m-2,-y_m,(x_m-2)*(x_m-2)+y_m*y_m-1]]);

p = vector([x,y,1]);
eqA = p*A*p == 0;
eqB = p*B*p == 0;

sol = solve([eqA,eqB], x, y); # intersection of the two Conics

p1 = vector([sol[0][0].rhs(), sol[0][1].rhs(), 1]); #p1
p2 = vector([sol[1][0].rhs(), sol[1][1].rhs(), 1]); #p2

l = p1.cross_product(p2); # join of the intersection

l2 = l/l[2]; # normalize by third component
```

```
l3 = vector([l2[0].full_simplify(),
    l2[1].full_simplify(), l2[2].full_simplify()]);

res = l3.subs(x_m=0,y_m=0); res # apply "shadow"

(-1, 0, 1) # correct solution!

# measured time in seconds
elapsed_time = time() - start; elapsed_time

38.82769203186035
# about 40 seconds for one solution on a Intel Core i7-4712HQ CPU
```

C. Law of Large Numbers

This is a CindyScript implementation of an approximation using the weak law of large numbers. For a removable singularity we approximate the function value by a mean value around the singularity. We used the disjoint circle intersection (Example 3.4.1 (Disjoint Circle Intersection)) as an exemplar problem. More details can be found in Subsection 3.5.2 (Law of Large Numbers).

```
delta = 10^(-3);
n = 10;  // number of samples

larr = [];

repeat(n,
t = delta*2*(random()-0.5) + 1; // rescale uniform distribution to [1-delta, 1+delta]
// removable singularity at t = 1
p1 = [t, sqrt(1-t^2), 1];
p2 = [t, -sqrt(1-t^2), 1];
l = cross(p1,p2);
larr = larr ++ [l/l_3]; // normalize to z-coordinate
);

meanLine = sum(larr)/n; // mean value

trueLine = [-1,0,1];
println(format(|meanLine-trueLine|,16)); // output: 0.000148323842304
```

D. Projective Midpoint via Plücker's μ

The calculation of the Euclidean midpoint of $X = (x1, x2, x3)$ and $Y = (y1, y2, y3)$ and the it's greatest common divisor, which is an avoidable prefactor. See Section 6.7 for further details.

```
sage: PR.<x1,x2,x3,y1,y2,y3> = QQ[]
sage: x = vector([x1,x2,x3]); y = vector([y1,y2,y3]); linf = vector([0,0,1])
sage: line = x.cross_product(y);
sage: infp = line.cross_product(linf);
sage: ix = det(Matrix([x, infp, line]));
sage: iy = det(Matrix([y, infp, line]));
sage: z1 = x*iy;
sage: z2 = y*ix;
sage: M = z1 + z2;
sage: C = gcd(M.list())                         # C = gratest common divisor
sage: C
x2^2*y1^2 + x3^2*y1^2 - 2*x1*x2*y1*y2 + x1^2*y2^2 + x3^2*y2^2
- 2*x1*x3*y1*y3 - 2*x2*x3*y2*y3 + x1^2*y3^2 + x2^2*y3^2

sage: M1 = simplify(M/C); M1                    #remove prefactor
(x3*y1 + x1*y3, x3*y2 + x2*y3, 2*x3*y3)

sage: M2 = x*y[2] + y*x[2]; M2                  # efficient solution
(x3*y1 + x1*y3, x3*y2 + x2*y3, 2*x3*y3)
sage: (M == C * M2)                             # equivalence
True
```

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