Technische Universität München

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Master’s Thesis

Asymptotic properties of the empirical spatial extremogram observed on an irregular grid

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I hereby declare that this thesis is my own work and that no other sources have been used except those clearly indicated and referenced.

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Abstract

The extremogram is a widely common measure to assess extremal dependence for random processes and is applicable throughout different fields in extreme value theory. For example, the tail dependence coefficient, which is often considered in the finance industry, is one of its special cases. We investigate the asymptotic properties of the empirical version of the extremogram, which is based on a kernel estimator, and assume that the observations of the random process come from a multidimensional Poisson process inducing a so-called irregular grid. We present the proof of a central limit theorem for the empirical spatial extremogram in full detail. In particular, we show that this central limit theorem holds when the underlying distribution of the random process is that of a Brown-Resnick process with an isotropic dependence function. Moreover, in this case, we give a new bias corrected version of the empirical extremogram to obtain better convergence rates for the asymptotic normality. The results are then applied to real precipitation data. We compare the empirical extremogram with its bias corrected version under different kernel functions and apply a transformation of the observation space to justify the assumption of isotropic data.

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1 Introduction

Extremal events affect our daily lives in various ways. Extraordinary losses or gains in the financial markets such as during the financial crises 2007 brought the financial system to the edge all over the globe, provoked a worldwide recession and made governmental action necessary for years to cope with the consequences. Moreover, extreme rainfalls and floodings cause hundreds of fatalities and billions in material damage as seen during the hurricane season 2017 when hurricanes Harvey, Irma and Jose devastated large areas around the Gulf of Mexico. In 2015, an earthquake with a magnitude of 7.8 to 8.1 killed 9,000 people and left about 22,000 injured in Nepal. Even 30 months after the catastrophe, only 12% of the destroyed buildings were reconstructed, affecting the Nepalis life for years and leaving many homeless. So, it is of major interest, not only for underwriting in the insurance business and pricing of catastrophe bonds but as well for natural disaster prevention, to model such events as accurately as possible. In almost every field dealing with extremes, history shows that such events tend to occur simultaneously which consequently results in a growing interest in modelling extremal dependence.

An essential tool to investigate extremal dependence for a stationary regularly varying random process X in $\mathbb{R}^d$ is the so-called extremogram, introduced by Davis and Mikosch [9] for time series. Assuming stationarity means that the probabilistic structure of the underlying random process is invariant under spatial translations. So stationary random processes give no space for modelling directional dependence. Even though this assumption may exclude some physical applications it is not as restrictive as imposing isotropy, i.e. invariance under rotation implying stationarity. In this setting the extremogram can be seen as the covariance function of indicator functions of exceedance events in an asymptotic sense. It allows to measure extremal dependence on various combinations of extremal sets that are bounded away from 0. This means that, by choosing the extremal sets accordingly, a wide range of measures of extremal dependence like dependence among large absolute values or the (upper and lower) tail dependence coefficient can be seen as special cases of the extremogram. We will make use of this fact when showing a central limit theorem (CLT) for the tail dependence coefficient under an isotropic Brown-Resnick process. The tail dependence coefficient can be interpreted as the conditional probability of given an extreme event at one location s_1 to simultaneously observe an extreme event at a different location s_2 and is therefore of the utmost importance to assess extremal dependence in a spatial setting.

Since in real life the extremogram needs to be estimated on the basis of observed data, we consider the empirical extremogram. We formulate the estimator in line with Cho et al. [7] who did this for a spatial d-dimensional random process. Buhl et al. [4] as well as Steinkohl [21] investigate the empirical spatial extremogram, when having observations on a regular grid. Cho et al. [7] prove a CLT for the empirical extremogram centered by its pre-asymptotic version for a finite set of spatials lags. Bolthausen [2] set the stage for their work and the CLT for the space-time covariance estimators, proved by Li et al. [16].

Cho et al. [7] showed a CLT when observing the random process on locations that are assumed to follow a homogeneous d-dimensional Poisson process. For that purpose, they employ a kernel estimator for the extremogram, transferring the idea of the estimate of autocorrelation from Li et al. [16] to their setting. However, parts of their proof lack details and contain several obscurities which is why we developed these arguments in a more explicit scope.

We note that the assumption of observation locations following a homogeneous Poisson process, that is independent of the random process X, implies that not only the locations are randomly distributed on the observation space but also that the number of such locations is random. Furthermore, Karr [13] emphasized that this hypothesis comes with an important statistical argument as there is only one parameter, namely the intensity of the Poisson process, to estimate in the sampling part of the model. Additionally, it is a common assertion within the fields of geography and geology that locations of cities fit Poisson models. The Poisson assumption does not only allow to model measuring stations of rain- or snowfall, but also enables us to consider various other applications like test borings or strokes of lightning. On the other hand, Karr [13] stressed that Poisson processes inherit a major disadvantage when it comes to studying extremal dependence in rainfall since measuring stations do not vary over time and thus temporal dependence is enforced in the model. We avoid this issue in our rainfall study by taking only isolated and independent single periods into account.

The main goal of this work is to present a complete proof of a CLT for the empirical spatial extremogram on an irregular grid and a new bias correction when considering a Brown-Resnick process. The results are then applied to estimate the tail dependence coefficient for precipitation data.

This thesis is organized as follows.

In Chapter 2, we formulate the general setting, present the concept of regular variation and α-mixing coefficients and give the formal definition of the extremogram. Furthermore, in line with Cho et al. [7], we introduce a kernel estimator for the extremogram consisting of two separate estimators, one estimating the numerator, $\hat{\tau}$ and another one being the empirical version of the denominator, $\hat{\rho}$. The resulting estimator can be seen as an estimator of a conditional probability.

Chapter 3 contains the proof of the CLT for the empirical extremogram. We first show some technical details, beginning with the continuity of the limit measure τ in section 3.1. Then we proceed with proving asymptotic unbiasedness and consistency for the estimator of the denominator $\hat{\rho}$, chapter 3.2, and the estimator of the numerator $\hat{\tau}$, chapter 3.3, including the derivation of its asymptotic covariance matrix. In chapter 3.4, we infer a CLT for $\hat{\tau}$ which is employed along with previous results to give a CLT for the estimator of the extremogram centered by the pre-asymptotic version, chapter 3.5.

Chapter 4 considers the empirical extremogram when the underlying random process X follows the law of a Brown-Resnick process, see Brown and Resnick [3]. We introduce the Brown-Resnick process in chapter 4.1 and show that the CLT is applicable for an isotropic Brown-Resnick process in particular, section 4.2. Furthermore, we establish conditions under which the empirical extremogram can be centered by its theoretical version in chapter 4.3 and introduce a new bias correction to give a better range of rates so that asymptotic normality holds.

In chapter 5, we apply our results to spatial rainfall data in Germany from 1971 to 2010 and employ the empirical extremogram to investigate the effect of extremal dependence in precipitation. We apply different kernel densities and transform the observation space to obtain an isotropic random field. Moreover, we compare the empirical extremogram with the new bias corrected version presented chapter 4.

2 General setting- Regular variation and the extremogram

Before investigating asymptotic properties of the empirical extremogram, we need to establish some background on regular variation for stochastic processes and vectors, for this purpose also see Hult and Lindskog [11]. Resnick [19], [20] gives a more detailed insight into this field. Let $\{X_s : s \in \mathbb{R}^d\}$ be a strictly stationary random process, $d \in \mathbb{N}$. For every finite set $I \subset \mathbb{R}^d$, we define the vector

$$X_I := (X_s : s \in I)^\top .$$

In the following, $|I|$ denotes the cardinality of I and $f(n) \sim g(n)$ means that the positive functions $f(\cdot)$ and $g(\cdot)$ are asymptotically equivalent as $n \to \infty$, i.e. $f(n)/g(n) \to 1$ as $n \to \infty$.

Definition 2.1 (Regularly varying process)**.** *A strictly stationary random process $\{X_s : s \in \mathbb{R}^d\}$ is called regularly varying, if there exists some normalizing sequence $0 < a_n \to \infty$ such that $\mathbb{P}(|X(0)| > a_n) \sim n^{-1}$ as $n \to \infty$ and for every finite set $I \subset \mathbb{R}^d$*

$$n\mathbb{P}\left(\frac{X_I}{a_n} \in \cdot\right) \xrightarrow{v} \mu_I(\cdot), \qquad n \to \infty,$$

for some non-null Radon measure μ_I on the Borel sets in $\bar{\mathbb{R}}^{|I|}\backslash\{0\}$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $\xrightarrow{v}$ denotes vague convergence. In that case, there exists the so-called index of regular variation $\beta > 0$ such that

$$\mu_I(xC) = x^{-\beta}\mu_I(C), \qquad x > 0,$$

for every Borel set $C \subset \bar{\mathbb{R}}^{|I|}\backslash\{0\}$. We call such a limit measure homogeneous of order $-\beta$.

By vague convergence $\mu_{I,n}(\cdot) \xrightarrow{v} \mu_I(\cdot)$ as $n \to \infty$, we mean that

$$\int_{\bar{\mathbb{R}}^{|I|}\backslash\{0\}} f(x)\mu_{I,n}(dx) \xrightarrow{n\to\infty} \int_{\bar{\mathbb{R}}^{|I|}\backslash\{0\}} f(x)\mu_I(dx)$$

for all continuous, non-negative functions $f : \mathbb{R}^{|I|}\backslash\{0\} \to (0, \infty)$ with compact support. Note that by stationarity for every $s \in \mathbb{R}^d$, we have $\mu_{\{s\}}(\cdot) = \mu_{\{0\}}(\cdot) =: \mu(\cdot)$. We now introduce the extremogram for random processes, on which we focus in this thesis, see also Davis and Mikosch [9] for time series.

Definition 2.2 (Extremogram)**.** *Let $\{X_s : s \in \mathbb{R}^d\}$ be a strictly stationary regularly varying random process in $\mathbb{R}^d$. We define the extremogram by*

$$\rho_{AB}(h) := \lim_{n\to\infty} \mathbb{P}\left(\frac{X_h}{a_n} \in B \,\middle|\, \frac{X_0}{a_n} \in A\right) = \lim_{n\to\infty} \frac{\mathbb{P}\left(\frac{X_0}{a_n} \in A, \frac{X_h}{a_n} \in B\right)}{\mathbb{P}\left(\frac{X_0}{a_n} \in A\right)}, \qquad h \in \mathbb{R}^d,$$

where A and B are two μ-continuous Borel sets in $\bar{\mathbb{R}}\backslash\{0\}$, i.e. $\mu(\partial A) = \mu(\partial B) = 0$, where ∂A denotes the boundary of the set A, and $\mu(A) > 0$. In the special case of $A = B = (1, \infty)$, we define the tail dependence coefficient

$$\chi(h) := \rho_{(1,\infty)(1,\infty)}(h), \qquad h \in \mathbb{R}^d.$$

Remark 2.3. *In the upcoming proofs we make frequent use of the pre-asymptotic version of the extremogram $\rho_{AB,m}(\cdot)$ given by*

$$\rho_{AB}(h) = \lim_{n\to\infty} \rho_{AB,m}(h) = \lim_{n\to\infty} \frac{\tau_{AB,m}(h)}{p_m(A)}, \qquad h \in \mathbb{R}^d,$$

*where the sequence m_n is later specified in condition **(M1)** and*

$$\begin{aligned} \rho_{AB,m}(h) &:= \mathbb{P}\left(\frac{X_h}{a_m} \in B \middle| \frac{X_0}{a_m} \in A \right) \\ \tau_{AB,m}(h) &:= m_n \mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_h}{a_m} \in B \right) \\ p_m(A) &:= m_n \mathbb{P}\left(\frac{X_0}{a_m} \in A \right). \end{aligned}$$

The goal of this work is to establish asymptotic properties of the empirical version of the extremogram. This will include consistency and a multivariate central limit theorem (CLT). In contrast to Buhl and Klüppelberg [6], we assume that the random process is not observed on a regular grid, but the observations are given by a $d-$dimensional Poisson process. Cho et al. [7] already provided essential results. However, we present the proofs in more detail and prove a new CLT for a bias corrected version of the estimator. In order to do this, we need to control dependence within the random process. For this purpose, we introduce the concept of α-mixing processes, also see Bolthausen [2].

Definition 2.4 (α-mixing coefficients)**.** *Let $\{X_s : s \in \mathbb{R}^d\}$ be a strictly stationary regularly varying random process in $\mathbb{R}^d$. Let $d(\cdot,\cdot)$ be some metric introduced by a norm $|\cdot|$ on $\mathbb{R}^d$. For $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ set*

$$d(\Lambda_1, \Lambda_2) := \inf\{|s_1 - s_2| : s_1 \in \Lambda_1, s_2 \in \Lambda_2\}$$

and denote for $i = 1, 2$ the σ-algebra generated by $\{X_s : s \in \Lambda_i\}$ by $F_{\Lambda_i} = \sigma\{X_s : s \in \Lambda_i\}$. Then, we define the α-mixing coefficients for $k, l \in \mathbb{N} \cup \{\infty\}$ and $r > 0$ by

$$\alpha_{k,l}(r) = \sup\{|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| : A_1 \in F_{\Lambda_1}, A_2 \in F_{\Lambda_2}, |\Lambda_1| \le k, |\Lambda_2| \le l, d(\Lambda_1, \Lambda_2) \ge r\}.$$

Here, $|\Lambda_i|$ denotes the number of elements contained in the set Λ_i.

In this work, the considered norm $|\cdot|$ is the L^2-norm, if not stated differently.

For ease of notation, we consider a 2-dimensional strictly stationary regularly varying random process $\{X_s : s \in \mathbb{R}^2\}$ throughout this thesis. We emphasize that all results can be generalized to higher dimensions by analogous arguments. We assume that the sampling locations are generated by a 2-dimensional Poisson process N with intensity parameter $\nu > 0$ that is independent of X and define the product measure $N^{(2)}(ds_1, ds_2) := N(ds_1)N(ds_2)\mathbb{1}\{s_1 \neq s_2\}$. Furthermore, we consider a sequence of convex sets $S_n \subset \mathbb{R}^2$ with Lebesgue measure $|S_n| = O(n^2)$ and boundary ∂S_n such that $|\partial S_n| = O(n)$. These sets S_n model the spaces where the Poisson observations occur.
In line with Karr [14], chapter 10.3, we introduce a kernel-based estimator for the spatial extremogram $\hat{\rho}_{AB,m}(h) = \hat{\tau}_{AB,m}(h)/\hat{p}_m(A)$, where

$$\hat{p}_m(A) := \frac{m_n}{\nu |S_n|} \int_{S_n} \mathbb{1}\left\{ \frac{X_{s_1}}{a_m} \in A \right\} N(ds_1)$$

$$\hat{\tau}_{AB,m}(h) := \frac{m_n}{\nu^2|S_n|}\int_{S_n}\int_{S_n} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}N^{(2)}(ds_1,ds_2).$$

The sequence of weight functions $w_n(\cdot) = \frac{1}{\lambda_n^2}w\left(\frac{\cdot}{\lambda_n}\right)$ is chosen in such way that $w(\cdot)$ is a positive, bounded, isotropic probability density function on $\mathbb{R}^2$, the bandwidth λ_n satisfies $\lambda_n \to 0$, and $\lambda_n^2|S_n| \to \infty$ as $n \to \infty$. Our goal is to derive a CLT for the empirical extremogram $\hat{\rho}_{AB,m}(h)$. In order to prove consistency of the estimator $\hat{p}_m(A)$, we need the subsequent conditions to hold:

(M1) There exist increasing sequences $m_n \to \infty$ and $r_n \to \infty$, as $n \to \infty$, with $m_n = o(n)$ and $r_n^2 = o(m_n)$ such that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_{B[k,r_n]} m_n\mathbb{P}\left(|X_y| > \epsilon a_m, |X_0| > \epsilon a_m\right)dy = 0, \quad \forall\epsilon > 0 \tag{1}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^2\setminus B[0,r_n)} m_n\alpha_{1,1}(|y|)dy = 0 \tag{2}$$

$$\int_{\mathbb{R}^2}\tau_{AA}(y)dy < \infty \tag{3}$$

where a_m satisfies $\mathbb{P}\left(|X_0| > a_m\right) \sim \frac{1}{m}$, $B[a,b) := \{s \in \mathbb{R}^2 : a \le |s| < b\}$ and $\tau_{AA}(y) := \lim_{n\to\infty}\tau_{AA,m}(y)$.

A CLT for $\hat{\tau}_{AB,m}(\cdot)$ requires to assume the following conditions.

(M2) Let $B_n \subset S_n$ be a cube with $|B_n| = O(n^{2\gamma})$ and $|\partial B_n| = O(n^\gamma)$ for some $\gamma \in (0,1)$. Furthermore, assume there exists an increasing sequence $m_n \to \infty$, as $n \to \infty$, with $m_n = o(n^\gamma)$ and $\lambda_n^2 m_n \xrightarrow{n\to\infty} 0$ such that for every $h \in \mathbb{R}^2$

$$\sup_n \mathbb{E}\left[\left(\sqrt{\frac{|B_n|\lambda_n^2}{m_n}}\left|\hat{\tau}_{AB,m}(h:B_n) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h:B_n)\right]\right|\right)^{2+\delta}\right] \le C_\delta, \quad \text{for some } \delta > 0,\ C_\delta < \infty \tag{4}$$

$$\int_{\mathbb{R}^2}\tau_{AB}(y)dy < \infty \tag{5}$$

$$\int_{\mathbb{R}^2}\alpha_{2,2}(|y|)dy < \infty \tag{6}$$

$$\sup_{l\in\mathbb{N}}\frac{\alpha_{l,l}(|h|)}{l^2} = O(|h|^{-\epsilon}), \quad \text{for some } \epsilon > 0. \tag{7}$$

Finally, we need a smoothness condition for the random process.

Definition 2.5 (local uniform negligibility condition (LUNC))**.** *A strictly stationary regularly varying random process $\{X_s : s \in \mathbb{R}^d\}$ satisfies the local uniform negligibility condition (LUNC), if for an increasing sequence a_n such that $\mathbb{P}(|X| > a_n) \sim \frac{1}{n}$ and for all $\epsilon, \delta > 0$, there exists $\delta' > 0$ such that*

$$\limsup_{n\to\infty} n\mathbb{P}\left(\sup_{|s|<\delta'}\frac{|X_s - X_0|}{a_n} > \delta\right) < \epsilon.$$

3 A central limit theorem for the empirical extremogram on an irregular grid

In this chapter, we prove a central limit theorem (CLT) for the empiricial extremogram observed on a grid, which is generated by a 2-dimensional Poisson process N on a space $S_n \subseteq \mathbb{R}^2$. We assume $|S_n| = O(n^2)$ with $|\partial S_n| = O(n)$, and denote the intensity parameter of N by ν. Before presenting the CLT in Theorem 3.6, we show continuity of the measure of the vector of finite length $(X_0, \dots, X_{s_k}) \in \mathbb{R}^{k+1}$, Proposition 3.1. Furthermore, we need some auxiliary results on the asymptotic expectation and covariance of the empirical denominator $\hat{p}_m(\cdot)$ and numerator $\hat{\tau}_m(\cdot)$, Proposition 3.2 and Proposition 3.3. This enables us to prove a multivariate CLT for the numerator, Proposition 3.5, which we will use to show the final result of Theorem 3.6. These results were already shown in Cho et al. [7]. However, we will give a more detailed insight into the proofs, especially those of Proposition 3.5 and Theorem 3.6.

3.1 Continuity of the measure

We show a continuity property of the limit measure τ that we will constantly apply when proving asymptotic properties of the estimators.

Proposition 3.1. *Consider a strictly stationary, regularly varying random process $\{X_s : s \in \mathbb{R}^d\}$ with index $\beta > 0$ satisfying the LUNC. Let k be a positive integer. For a continuity set $A_0 \times A_1 \times \cdots \times A_k$ of the limit measure*

$$\tau_{A_0 A_1 \dots A_k}(s_1, \dots, s_k) := \lim_{n \to \infty} n\mathbb{P}\left(\frac{X_0}{a_n} \in A_0, \frac{X_s}{a_n} \in A_1, \dots, \frac{X_{s_k}}{a_n} \in A_k\right), \tag{8}$$

where $s_1, \dots, s_k$ are arbitrary elements in $\mathbb{R}^d$, we define the pre-asymptotic limit measure by

$$\tau^{(n)}_{A_0 A_1 \dots A_k}(s_1, \dots, s_k) := n\mathbb{P}\left(\frac{X_0}{a_n} \in A_0, \frac{X_{s_1+\lambda_n}}{a_n} \in A_1, \dots, \frac{X_{s_k+\lambda_n}}{a_n} \in A_k\right). \tag{9}$$

Then if $\lambda_n \to 0$ as $n \to \infty$, it holds that

$$\lim_{n \to \infty} \tau^{(n)}_{A_0 A_1 \dots A_k}(s_1, \dots, s_k) = \tau_{A_0 A_1 \dots A_k}(s_1, \dots, s_k).$$

Proof. We will apply a Portmanteau theorem, see [15]. For this purpose, let f be a continuous function with compact support on $\mathbb{R}^{k+1} \backslash \{0\}$.
Then f is bounded and uniformly continuous, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x_1, \dots, x_{k+1}) - f(y_1, \dots, y_{k+1})|_1 < \epsilon$ whenever $|(x_1, \dots, x_{k+1}) - (y_1, \dots, y_{k+1})|_1 < \delta$, where $|\cdot|_1$ denotes the L^1-norm.
We define $\overline{X}_n := (X_0, X_{s_1+\lambda_n}, \dots, X_{s_k+\lambda_n})$ and $\overline{X} := (X_0, X_{s_1}, \dots, X_{s_k})$ and consider

$$n\mathbb{E}\left[\left|f\left(\frac{\overline{X}_n}{a_n}\right) - f\left(\frac{\overline{X}}{a_n}\right)\right|_1\right] = n\mathbb{E}\left[|\cdot|_1 \mathbb{1}\left\{\frac{|\overline{X}_n - \overline{X}|_1}{a_n} > \delta\right\}\right] + n\mathbb{E}\left[|\cdot|_1 \mathbb{1}\left\{\frac{|\overline{X}_n - \overline{X}|_1}{a_n} \leq \delta\right\}\right] =: C_n + D_n.$$

By boundedness of f we see $M := \sup_{\overline{x} \in \mathbb{R}^{k+1}} \left|f\left(\frac{\overline{x}}{a_n}\right)\right|_1 < \infty$. This gives $\left|f\left(\frac{\overline{X}_n}{a_n}\right) - f\left(\frac{\overline{X}}{a_n}\right)\right|_1 \leq 2M$.
Without loss of generality let

$$\max_{i \in \{1, \dots, k\}} \mathbb{P}\left(|X_{s_i+\lambda_n} - X_{s_i}|_1 > \frac{\delta a_n}{k}\right) = \mathbb{P}\left(|X_{s_1+\lambda_n} - X_{s_1}|_1 > \frac{\delta a_n}{k}\right) = \mathbb{P}\left(|X_{\lambda_n} - X_0|_1 > \frac{\delta a_n}{k}\right),$$

where the second equality follows from stationarity of the random process. Then we can find an upper bound for C_n by

$$\begin{aligned}
\limsup_{n\to\infty} C_n &= \limsup_{n\to\infty} n\mathbb{E}\left[\left|f\left(\frac{\overline{X}_n}{a_n}\right) - f\left(\frac{\overline{X}}{a_n}\right)\right|_1 \mathbb{1}\left\{\frac{\left|\overline{X}_n - \overline{X}\right|_1}{a_n} > \delta\right\}\right] \\
&\leq \limsup_{n\to\infty} 2Mn\mathbb{P}\left(\frac{\left|\overline{X}_n - \overline{X}\right|_1}{a_n} > \delta\right) \\
&\leq \limsup_{n\to\infty} 2Mn\mathbb{P}\left(|X_{s_i+\lambda_n} - X_{s_i}|_1 > \frac{\delta a_n}{k} \text{ for at least one } i \in \{1,\dots,k\}\right) \\
&= \limsup_{n\to\infty} 2Mn\mathbb{P}\left(\bigcup_{i=1}^{k}\left\{|X_{s_i+\lambda_n} - X_{s_i}|_1 > \frac{\delta a_n}{k}\right\}\right) \\
&\leq \limsup_{n\to\infty} 2Mn\sum_{i=1}^{k}\mathbb{P}\left(|X_{s_i+\lambda_n} - X_{s_i}|_1 > \frac{\delta a_n}{k}\right) \\
&\leq 2Mk\limsup_{n\to\infty} n\mathbb{P}\left(|X_{\lambda_n} - X_0|_1 > \frac{\delta a_n}{k}\right) \qquad \text{by stationarity} \\
&\leq 2Mk\limsup_{n\to\infty} n\mathbb{P}\left(\sup_{|s|\leq\delta'}\frac{|X_s - X_0|_1}{a_n} > \frac{\delta}{k}\right) \\
&< 2M\epsilon
\end{aligned}$$

where the second last inequality follows from the fact that $\lambda_n \to 0$ and thus for n large enough there is $\delta' > \lambda_n > 0$ such that $\mathbb{P}\left(\frac{|X_{\lambda_n}-X_0|_1}{a_n} > \frac{\delta}{k}\right) \leq \mathbb{P}\left(\sup_{|s|\leq\delta'}\frac{|X_s-X_0|_1}{a_n} > \frac{\delta}{k}\right)$. Applying the LUNC with ϵ and δ from Definition 2.5 corresponding to ϵ/k and δ/k, respectively, gives the last inequality.
Now, we turn to D_n and find by continuity of f

$$\begin{aligned}
\limsup_{n\to\infty} D_n &= \limsup_{n\to\infty} n\mathbb{E}\left[\left|f\left(\frac{\overline{X}_n}{a_n}\right) - f\left(\frac{\overline{X}}{a_n}\right)\right|_1 \mathbb{1}\left\{\frac{\left|\overline{X}_n - \overline{X}\right|_1}{a_n} \leq \delta\right\}\right] \\
&\leq \epsilon\limsup_{n\to\infty} n\mathbb{P}\left(\frac{\left|\overline{X}_n - \overline{X}\right|_1}{a_n} \leq \delta\right).
\end{aligned}$$

As we only consider events that are bounded away from $0 \in \mathbb{R}^{k+1}$ we may assume the existence of some $K > 0$ such that the support of f is a subset of $\{\overline{x} \in \mathbb{R}^{k+1} : |\overline{x}|_1 > K\}$. In particular, we only have to consider $|\overline{X}_n|_1/a_n > K$ and $|\overline{X}|_1/a_n > K$, giving

$$\begin{aligned}
\epsilon\limsup_{n\to\infty} n\mathbb{P}\left(\frac{\left|\overline{X}_n - \overline{X}\right|_1}{a_n} \leq \delta\right) &\leq \epsilon\limsup_{n\to\infty} n\mathbb{P}\left(\left\{\frac{\left|\overline{X}_n\right|_1}{a_n} > K\right\} \cup \left\{\frac{\left|\overline{X}\right|_1}{a_n} > K\right\}\right) \\
&\leq \epsilon\limsup_{n\to\infty} n\mathbb{P}\left(\frac{\left|\overline{X}_n\right|_1}{a_n} > K\right) + \mathbb{P}\left(\frac{\left|\overline{X}\right|_1}{a_n} > K\right) \\
&\leq \epsilon\limsup_{n\to\infty} n\left[2\mathbb{P}\left(\frac{|X_0|_1}{a_n} > \frac{K}{k+1}\right) + \sum_{i=1}^{k}\left(\mathbb{P}\left(\frac{|X_{s_i+\lambda_n}|_1}{a_n} > \frac{K}{k+1}\right) + \mathbb{P}\left(\frac{|X_{s_i}|_1}{a_n} > \frac{K}{k+1}\right)\right)\right]
\end{aligned}$$

$$= 2\epsilon(k+1) \limsup_{n\to\infty} n\mathbb{P}\left(\frac{|X_0|_1}{a_n} > \frac{K}{k+1}\right), \qquad \text{by stationarity}$$
$$= 2\epsilon(k+1)\tau_{BB}(0) < \infty$$

where $B := \left\{x : |x|_1 > \frac{K}{k+1}\right\}$. The last inequality is due to the fact that

$$\{|\,\overline{x}\,|_1 > K\} \subseteq \left\{|x_0|_1 > \frac{K}{k+1}\right\} \cup \cdots \cup \left\{|x_k|_1 > \frac{K}{k+1}\right\}, \quad \text{for every } \overline{x} = (x_0, x_1, \ldots, x_k)^\top \in \mathbb{R}^{k+1}.$$

Combining these results gives

$$n\mathbb{E}\left[\left|f\left(\frac{X}{a_n}\right) - f\left(\frac{\overline{X}}{a_n}\right)\right|_1\right] \leq 2\epsilon(M + (k+1)\tau_{BB}(0)).$$

Since $\epsilon > 0$ may be chosen arbitrarily, letting $\epsilon \to 0$ and recalling the definitions of the limit measure τ from (8) and the pre-asymptotic measure $\tau^{(n)}$ from (9), we find

$$\int f(y_0, y_1, \ldots, y_k)\, \tau^{(n)}_{dy_0, dy_1, \ldots, dy_k}(s_1, \ldots, s_k) = n\mathbb{E}\left[f\left(\frac{\overline{X}_n}{a_n}\right)\right] = n\mathbb{E}\left[f\left(\frac{\overline{X}}{a_n}\right)\right]$$
$$\xrightarrow{n\to\infty} \int f(y_0, y_1, \ldots, y_k)\, \tau_{dy_0, dy_1, \ldots, dy_k}(s_1, \ldots, s_k)$$

for any arbitrary continuous and bounded function f with compact support. Then Portemanteau's theorem implies

$$\tau^{(n)}_{A_0 A_1 \ldots A_k}(s_1, \ldots, s_k) \xrightarrow{n\to\infty} \tau_{A_0 A_1 \ldots A_k}(s_1, \ldots, s_k)$$

which concludes the proof. □

3.2 Asymptotic results for the estimator of the denominator

In this section, we prove the asymptotic unbiasedness of $\hat{p}_m(\cdot)$ and infer an asymptotic result for its variance.

Proposition 3.2. *Under condition **(M1)** and the LUNC it holds that*

$$\mathbb{E}\left[\hat{p}_m(A)\right] = p_m(A) \xrightarrow{n\to\infty} \mu(A)$$
$$\text{and } \frac{|S_n|}{m_n} \mathbb{V}\text{ar}\left(\hat{p}_m(A)\right) \xrightarrow{n\to\infty} \frac{\mu(A)}{\nu} + \int_{\mathbb{R}^2} \tau_{AA}(y)dy.$$

In particular, it holds that $\hat{p}_m(A) \xrightarrow{P} \mu(A)$.

Proof. We begin with the proof of asymptotic unbiasedness of $\hat{p}_m(A)$. Using the definition of $\hat{p}_m(A)$, we obtain by the law of total expectation and independence of X and N

$$
\begin{aligned}
\mathbb{E}\left[\hat{p}_m(A)\right] &= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}N(ds_1)\right] && (10)\\
&= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\mathbb{E}\left[\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}N(ds_1)\middle| X\right]\right]\\
&= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{E}\left[N(ds_1)|\, X\right]\right], && \text{by measurability}\\
&= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{E}\left[N(ds_1)\right]\right], && \text{by independence}\\
&= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\nu ds_1\right], && \text{since } \mathbb{E}\left[N(ds_1)\right]=\nu ds_1\\
&= \frac{m_n}{|S_n|}\int_{S_n}\mathbb{E}\left[\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\right]ds_1, && \text{by Fubini}\\
&= \frac{m_n}{|S_n|}\int_{S_n}\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A\right)ds_1\\
&= \frac{m_n}{|S_n|}\int_{S_n}\mathbb{P}\left(\frac{X_0}{a_m}\in A\right)ds_1, && \text{by stationarity}\\
&= m_n\mathbb{P}\left(\frac{X_0}{a_m}\in A\right) = p_m(A) \xrightarrow{n\to\infty} \mu(A), && \text{by regular variation.}
\end{aligned}
$$

This concludes the asymptotic unbiasedness of the estimator $\hat{p}_m(A)$ and we turn to its variance. Recall from Karr [14], equation (1.6), that $N^{(2)}(ds_1, ds_2) = N(ds_1)N(ds_2)\mathbb{1}\{s_1 \neq s_2\}$. We compute the second moment

$$
\begin{aligned}
&\mathbb{E}\left[\hat{p}_m(A)^2\right] = \frac{m_n^2}{|S_n|^2}\frac{1}{\nu^2}\mathbb{E}\left[\int_{S_n}\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in A\right\}N(ds_1)N(ds_2)\right]\\
&= \frac{m_n^2}{|S_n|^2}\frac{1}{\nu^2}\left[\mathbb{E}\left[\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}N(ds_1)\right] + \mathbb{E}\left[\int_{S_n}\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in A\right\}N^{(2)}(ds_1, ds_2)\right]\right]\\
&= \frac{m_n}{|S_n|}\frac{1}{\nu}\,\mathbb{E}\left[\hat{p}_m(A)\right] + \frac{m_n^2}{|S_n|^2}\frac{1}{\nu^2}\mathbb{E}\left[\int_{S_n}\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in A\right\}\mathbb{E}\left[N^{(2)}(ds_1, ds_2)\middle| X\right]\right]
\end{aligned}
$$

using (10) for the first and the law of total expectation for the second summand. By independence of X and N the last equation is equivalent to

$$
\begin{aligned}
&= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \frac{m_n^2}{|S_n|^2}\frac{1}{\nu^2}\mathbb{E}\left[\int_{S_n}\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in A\right\}\underbrace{\mathbb{E}\left[N^{(2)}(ds_1, ds_2)\right]}_{=\nu^2 ds_1 ds_2,\text{ by Proposition A.1}}\right]\\
&= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \frac{m_n^2}{|S_n|^2}\mathbb{E}\left[\int_{S_n}\int_{S_n}\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in A\right\}ds_1 ds_2\right]
\end{aligned}
$$

$$= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \frac{m_n^2}{|S_n|^2}\int_{S_n}\int_{S_n}\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A, \frac{X_{s_2}}{a_m}\in A\right)ds_1ds_2 - p_m(A)^2 + \underbrace{p_m(A)^2}_{=\mathbb{E}[\hat{p}_m(A)]^2}$$

$$= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \frac{m_n^2}{|S_n|^2}\int_{S_n}\int_{S_n}\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A, \frac{X_{s_2}}{a_m}\in A\right) - \frac{1}{m_n^2}p_m(A)^2ds_1ds_2 + \mathbb{E}[\hat{p}_m(A)]^2$$

$$= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \frac{m_n^2}{|S_n|^2}\int_{S_n-S_n}\int_{S_n\cap(S_n-y)}\left(\mathbb{P}\left(\frac{X_0}{a_m}\in A, \frac{X_y}{a_m}\in A\right) - \frac{1}{m_n^2}p_m(A)^2\right)dsdy + \mathbb{E}[\hat{p}_m(A)]^2,$$

where in the last step we substituted $y = s_2 - s_1$ and $s = s_1$. Note that we intepret $S_n - y$ as the set $\{x - y : x \in S_n\}$. Recalling that $\mathbb{P}\left(\frac{X_0}{a_m}\in A, \frac{X_y}{a_m}\in A\right) = \frac{1}{m_n}\tau_{AA,m}(y)$, we can proceed with the calculations and obtain since the integrand does not depend on s

$$= \frac{m_n}{|S_n|}\frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \frac{m_n^2}{|S_n|}\int_{S_n-S_n}\frac{|S_n\cap(S_n-y)|}{|S_n|}\left(\frac{1}{m_n}\tau_{AA,m}(y) - \frac{1}{m_n^2}p_m(A)^2\right)dy + \mathbb{E}[\hat{p}_m(A)]^2$$

$$= \frac{m_n}{|S_n|}\left[\frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \int_{S_n-S_n}\frac{|S_n\cap(S_n-y)|}{|S_n|}\left(\tau_{AA,m}(y) - \frac{1}{m_n}p_m(A)^2\right)dy\right] + \mathbb{E}[\hat{p}_m(A)]^2.$$

Let $k > 0$ and r_n sucht that $r_n^2 = o(m_n)$ (by (**M1**)). Then

$$\frac{|S_n|}{m_n}\mathbb{V}\text{ar}\left(\hat{p}_m(A)\right) = \frac{|S_n|}{m_n}\left(\mathbb{E}[\hat{p}_m(A)^2] - \mathbb{E}[\hat{p}_m(A)]^2\right)$$

$$= \frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \int_{S_n-S_n}\frac{|S_n\cap(S_n-y)|}{|S_n|}\left(\tau_{AA,m}(y) - \frac{1}{m_n}p_m(A)^2\right)dy$$

$$= \frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + \int_{B[0,k)\cap(S_n-S_n)}\frac{|S_n\cap(S_n-y)|}{|S_n|}\left(\tau_{AA,m}(y) - \frac{1}{m_n}p_m(A)^2\right)dy$$

$$+ \int_{B[k,r_n)\cap(S_n-S_n)}\frac{|S_n\cap(S_n-y)|}{|S_n|}\left(\tau_{AA,m}(y) - \frac{1}{m_n}p_m(A)^2\right)dy$$

$$+ \int_{(S_n-S_n)\setminus B[0,r_n)}\frac{|S_n\cap(S_n-y)|}{|S_n|}\left(\tau_{AA,m}(y) - \frac{1}{m_n}p_m(A)^2\right)dy$$

$$=: \frac{1}{\nu}\mathbb{E}\left[\hat{p}_m(A)\right] + A_{1n} + A_{2n} + A_{3n}.$$

Note that $S_n - S_n \xrightarrow{n\to\infty} \mathbb{R}^2$. We consider the limit behaviour of $A_{1n},\ A_{2n}$ and A_{3n} and find by dominated convergence

$$\lim_{n\to\infty} A_{1n} = \lim_{n\to\infty}\int_{B[0,k)\cap(S_n-S_n)}\underbrace{\frac{|S_n\cap(S_n-y)|}{|S_n|}}_{\to 1 \text{ by Lemma A.2}}\left(\underbrace{\tau_{AA,m}(y)}_{\to\tau_{AA}(y)} - \underbrace{\frac{1}{m_n}}_{\to 0}\underbrace{p_m(A)^2}_{\to\mu(A)^2}\right)dy$$

$$= \int_{B[0,k)}\tau_{AA}(y)dy \xrightarrow{k\to\infty} \int_{\mathbb{R}^2}\tau_{AA}(y)dy.$$

Concerning A_{2n}, recall that the set A is bounded away from 0 i.e. there is $\epsilon > 0$ such that $A \subseteq \mathbb{R}^2 \backslash B[0, \epsilon)$ and

$$\lim_{k\to\infty} \limsup_{n\to\infty} |A_{2n}|$$

$$\leq \lim_{k\to\infty} \limsup_{n\to\infty} \left(\int_{B[k,r_n)} \underbrace{\frac{|S_n \cap (S_n - y)|}{|S_n|}}_{\leq 1} m_n \mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_y}{a_m} \in A\right) dy + \int_{B[k,r_n)} \frac{1}{m_n} p_m(A)^2 dy \right)$$

$$\leq \lim_{k\to\infty} \limsup_{n\to\infty} \left(\underbrace{\int_{B[k,r_n)} m_n \mathbb{P}\left(|X_0| > \epsilon a_m, |X_y| > \epsilon a_m\right) dy}_{\to 0 \text{ by } (\mathbf{M1})} + \pi \underbrace{\frac{r_n^2}{m_n}}_{\to 0} \underbrace{p_m(A)^2}_{\to \mu(A)^2} \right) = 0,$$

And finally for A_{3n} by the triangular inequality and taking the definition of the mixing rate $\alpha_{11}(\cdot)$ into account

$$\begin{aligned}
\limsup_{n\to\infty} |A_{3n}| &\leq \limsup_{n\to\infty} \int_{(S_n - S_n)\backslash B[0,r_n)} \frac{|S_n \cap (S_n - y)|}{|S_n|} \left| \tau_{AA,m}(y) - \frac{1}{m_n} p_m(A)^2 \right| ds \\
&\leq \limsup_{n\to\infty} \int_{\mathbb{R}^2 \backslash B[0,r_n)} \underbrace{\frac{|S_n \cap (S_n - y)|}{|S_n|}}_{\leq 1} \left| \tau_{AA,m}(y) - \frac{1}{m_n} p_m(A)^2 \right| dy \\
&\leq \limsup_{n\to\infty} \int_{\mathbb{R}^2 \backslash B[0,r_n)} \left| \tau_{AA,m}(y) - \frac{1}{m_n} p_m(A)^2 \right| dy \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^2 \backslash B[0,r_n)} m_n \underbrace{\left| \mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_y}{a_m} \in A\right) - \mathbb{P}\left(\frac{X_0}{a_m} \in A\right) \underbrace{\mathbb{P}\left(\frac{X_0}{a_m} \in A\right)}_{= \mathbb{P}\left(\frac{X_y}{a_m} \in A\right)} \right|}_{\leq \alpha_{11}(y), \text{ where } \Lambda_1 = \{0\},\ \Lambda_2 = \{y\} \text{ such that } |\Lambda_1| = |\Lambda_2| = 1} dy \\
&\leq \limsup_{n\to\infty} \int_{\mathbb{R}^2 \backslash B[0,r_n)} m_n \alpha_{11}(|y|) dy = 0, \quad \text{by condition } (\mathbf{M1}).
\end{aligned}$$

Combining the results for A_{2n} and A_{3n} gives that

$$\lim_{k\to\infty} \lim_{n\to\infty} A_{2n} = 0 = \lim_{k\to\infty} \lim_{n\to\infty} A_{3n}.$$

Putting all results together, we can derive the statement of the proposition

$$\begin{aligned}
\lim_{n\to\infty} \frac{|S_n|}{m_n} \mathbb{V}\mathrm{ar}\left(\hat{p}_m(A)\right) &= \lim_{n\to\infty} \left(\frac{1}{\nu} \mathbb{E}\left[\hat{p}_m(A)\right] + A_{1n} + A_{2n} + A_{3n} \right) \\
&= \frac{\mu(A)}{\nu} + \lim_{k\to\infty} \lim_{n\to\infty} \left(A_{1n} + A_{2n} + A_{3n}\right) = \frac{\mu(A)}{\nu} + \int_{\mathbb{R}^2} \tau_{AA}(y) dy.
\end{aligned}$$

□

3.3 Asymptotic results for the estimator of the numerator

Similarly to the previous section, we show asymptotic unbiasedness of $\hat{\tau}_{AB,m}(\cdot)$ and derive asymptotics for its covariance.

Proposition 3.3. *Assume the strictly stationary random process $\{X_s : s \in \mathbb{R}^2\}$ satisfies the LUNC. Further let (**M2**) hold. Then*

(i) $\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right] \xrightarrow{n\to\infty} \tau_{AB}(h)$, *for every arbitrary spatial lag* $h \in \mathbb{R}^2$,

(ii) $\frac{|S_n|\lambda_n^2}{m_n}\mathbb{C}\mathrm{ov}\left(\hat{\tau}_{AB,m}(h_1), \hat{\tau}_{AB,m}(h_2)\right)$

$$\xrightarrow{n\to\infty} \frac{1}{\nu^2}\left[\tau_{AB}(h_1)\mathbb{1}\{h_1 = h_2\} + \tau_{A\cap B,A\cap B}(h_1)\mathbb{1}\{h_1 = -h_2\}\right]\int_{\mathbb{R}^2} w(y)^2 dy,$$

(iii) $\frac{|S_n|\lambda_n^2}{m_n}\mathbb{V}\mathrm{ar}\left(\hat{\tau}_{AB,m}(h)\right) \xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2} w(y)^2 dy.$

Proof. (i) From the stationarity of $\{X_s : s \in \mathbb{R}^2\}$ and by independence of X and N, we obtain for a fixed spatial lag $h \in \mathbb{R}^2$

$$\begin{aligned}
&\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right]\\
&\quad= \mathbb{E}\left[\frac{m_n}{|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}N^{(2)}(ds_1, ds_2)\right]\\
&\quad= \frac{m_n}{|S_n|}\frac{1}{\nu^2}\mathbb{E}\left[\int_{S_n}\int_{S_n} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}\underbrace{\mathbb{E}\left[N^{(2)}(ds_1, ds_2)\middle| X\right]}_{=\mathbb{E}\left[N^{(2)}(ds_1,ds_2)\right]=\nu^2 ds_1 ds_2,\text{ by Lemma A.1}}\right]\\
&\quad= \frac{m_n}{|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w_n(h+s_1-s_2)\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A, \frac{X_{s_2}}{a_m}\in B\right)\nu^2 ds_1 ds_2\\
&\quad= \frac{m_n}{|S_n|}\int_{S_n}\int_{S_n}\frac{1}{\lambda_n^2}w\left(\frac{h+s_1-s_2}{\lambda_n}\right)\mathbb{P}\left(\frac{X_0}{a_m}\in A, \frac{X_{s_2-s_1}}{a_m}\in B\right)ds_1 ds_2. \qquad (11)
\end{aligned}$$

Next, we set $y = \frac{h+s_1-s_2}{\lambda_n}$, $u = s_2$ and define the integral transformation $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $s_1 = \phi_1(y,u) = \lambda_n y + u - h$ and $s_2 = \phi_2(y,u) = u$. Then the determinant of the Jacobian is given by

$$|D\phi(y,u)| = \left|\begin{pmatrix}\lambda_n\mathbb{1}_2 & \mathbb{1}_2\\ 0 & \mathbb{1}_2\end{pmatrix}\right| = \lambda_n^2,$$

where $\mathbb{1}_2$ denotes the 2-dimensional identity matrix. For the integration we need that $s_2 = s_1 + h - \lambda_n y$ and $s_2 - s_1 = h - \lambda_n y$. Applying this transformation to (11) gives

$$\frac{m_n}{|S_n|}\int_{\frac{h+(S_n-S_n)}{\lambda_n}}\int_{S_n\cap(h+S_n-\lambda_n y)}\frac{1}{\lambda_n^2}w\left(y\right)\underbrace{\mathbb{P}\left(\frac{X_0}{a_m}\in A, \frac{X_{h-\lambda_n y}}{a_m}\in B\right)}_{=\frac{1}{m_n}\tau_{AB,m}(h-\lambda_n y)}\lambda_n^2 du dy$$

$$= \int_{\frac{h+(S_n-S_n)}{\lambda_n}} \frac{|S_n \cap (h+S_n-\lambda_n y)|}{|S_n|} w\,(y)\, \tau_{AB,m}(h-\lambda_n y)dy, \tag{12}$$

since the inner integral does not depend on u. Note that

$$\underbrace{\frac{|S_n \cap (h+S_n-\lambda_n y)|}{|S_n|}}_{\leq 1} \tau_{AB,m}(h-\lambda_n y) \leq m_n \mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_{h-\lambda_n y}}{a_m} \in B\right)$$

$$\leq m_n \mathbb{P}\left(\frac{X_0}{a_m} \in A\right) = p_m(A)$$

and, since $w(\cdot)$ is a probability density on $\mathbb{R}^2$,

$$\int_{\mathbb{R}^2} w(y) p_m(A) dy = p_m(A) \int_{\mathbb{R}^2} w(y) dy = p_m(A) \xrightarrow{n\to\infty} \mu(A) < \infty.$$

Hence, monotone convergence is applicable and by the virtue of Proposition 3.1,

$$\int_{\frac{h+(S_n-S_n)}{\lambda_n}} \underbrace{\frac{|S_n \cap (h+S_n-\lambda_n y)|}{|S_n|}}_{\to 1} w\,(y) \underbrace{\tau_{AB,m}(h-\lambda_n y)}_{\to \tau_{AB}(h),\text{ by Proposition 3.1}} dy \xrightarrow{n\to\infty} \int_{\mathbb{R}^2} w(y)\tau_{AB}(h)dy = \tau_{AB}(h).$$

For the convergence of the integrating area, we employed that, for n large enough, $h+(S_n-S_n)$ contains an open ball around $0 \in \mathbb{R}^2$. Then together with $\lambda_n \to 0$ as $n \to \infty$, we obtain that $\frac{h+(S_n-S_n)}{\lambda_n} \xrightarrow{n\to\infty} \mathbb{R}^2$. This concludes the proof of part (i).

(ii) For sets A and B we define

$$\tau_m^*(s_1,s_2,s_3,s_4) := m_n \mathbb{P}\left(\frac{X_{s_1}}{a_m} \in A, \frac{X_{s_2}}{a_m} \in B, \frac{X_{s_3}}{a_m} \in A, \frac{X_{s_4}}{a_m} \in B\right)$$

and compute by conditioning and independence of N and X

$$\begin{aligned}
&\frac{|S_n|\lambda_n^2}{m_n} \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\hat{\tau}_{AB,m}(h_2)\right] \\
&= \frac{|S_n|\lambda_n^2}{m_n} \mathbb{E}\Bigg[\frac{m_n^2}{|S_n|^2}\frac{1}{\nu^4} \int_{S_n}\int_{S_n}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2) w_n(h_2+s_3-s_4) \\
&\mathbb{1}\left\{\frac{X_{s_1}}{a_m} \in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m} \in B\right\}\mathbb{1}\left\{\frac{X_{s_3}}{a_m} \in A\right\}\mathbb{1}\left\{\frac{X_{s_4}}{a_m} \in B\right\} N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\Bigg] \\
&= \frac{m_n\lambda_n^2}{|S_n|}\frac{1}{\nu^4} \int_{S_n}\int_{S_n}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2) w_n(h_2+s_3-s_4) \\
&\underbrace{\mathbb{P}\left(\frac{X_{s_1}}{a_m} \in A, \frac{X_{s_2}}{a_m} \in B, \frac{X_{s_3}}{a_m} \in A, \frac{X_{s_4}}{a_m} \in B\right)}_{=\frac{1}{m_n}\tau_m^*(s_1,s_2,s_3,s_4)} \mathbb{E}\left[N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\right].
\end{aligned} \tag{13}$$

According to equation (4.7) in Karr [13] the expectation is given by

$$\mathbb{E}\left[N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\right] = \nu^4 ds_1ds_2ds_3ds_4 + \nu^3 ds_1ds_2\delta_{s_1}(ds_3)ds_4 + \nu^3 ds_1ds_2\delta_{s_2}(ds_3)ds_4$$
$$+\nu^3 ds_1ds_2ds_3\delta_{s_1}(ds_4) + \nu^3 ds_1ds_2ds_3\delta_{s_2}(ds_4) + +\nu^2 ds_1ds_2\delta_{s_1}(ds_3)\delta_{s_2}(ds_4) + \nu^2 ds_1ds_2\delta_{s_2}(ds_3)\delta_{s_1}(ds_4) \tag{14}$$

where $\delta_x(A) = \mathbb{1}\{x \in A\}$ denotes the Dirac-measure. We denote the integrals in (13) corresponding to (14) by $I_j, j \in \{1,\dots,7\}$ and we compute the different terms, beginning with I_2

$$\begin{aligned}
I_2 &= \frac{\lambda_n^2}{|S_n|}\frac{1}{\nu^4}\int_{S_n}\int_{S_n}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2)w_n(h_2+s_3-s_4)\tau_m^*(s_1,s_2,s_3,s_4)\nu^3 ds_1ds_2\delta_{s_1}(ds_3)ds_4\\
&= \frac{\lambda_n^2}{|S_n|}\frac{1}{\nu}\int_{S_n}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2)w_n(h_2+s_1-s_4)\tau_m^*(s_1,s_2,s_1,s_4)ds_1ds_2ds_4\\
&= \frac{1}{\lambda_n^2|S_n|}\frac{1}{\nu}\int_{S_n}\int_{S_n}\int_{S_n} w\left(\frac{h_1+s_1-s_2}{\lambda_n}\right)w\left(\frac{h_2+s_1-s_4}{\lambda_n}\right)\tau_m^*(0,s_2-s_1,0,s_4-s_1)ds_1ds_2ds_4.
\end{aligned}$$

Note that the last equality results from stationarity and definition of $w_n(\cdot)$. In the next step, we substitute $x = \frac{h_1+s_1-s_2}{\lambda_n}$, $y = \frac{h_2+s_1-s_4}{\lambda_n}$ and $u = s_4$. The corresponding integral transform $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ and the determinant of its Jacobian reads

$$s_2 = \phi_1(x,y,u) = u + \lambda_n(x+y) - h_1 - h_2, \quad s_1 = \phi_2(x,y,u) = u + \lambda_n y - h_2, \quad s_4 = \phi_3(x,y,u) = u$$
$$|D\phi(x,y,u)| = \left|\begin{pmatrix}\lambda_n\mathbb{1}_2 & \lambda_n\mathbb{1}_2 & \mathbb{1}_2\\ 0 & \lambda_n\mathbb{1}_2 & \mathbb{1}_2\\ 0 & 0 & \mathbb{1}_2\end{pmatrix}\right| = \lambda_n^4$$

such that I_2 becomes

$$\begin{aligned}
&\frac{\lambda_n^2}{|S_n|}\frac{1}{\nu}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\int_{\frac{(S_n-S_n)+h_2}{\lambda_n}}\int_{S_n\cap(S_n-\lambda_n(x+y)+h_1+h_2)\cap(S_n-\lambda_n y+h_2)} w(x)\,w(y)\\
&\qquad\qquad\times\tau_m^*(0,h_1-\lambda_n x,0,h_2-\lambda_n y)dudydx\\
&= \frac{\lambda_n^2}{\nu}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\int_{\frac{(S_n-S_n)+h_2}{\lambda_n}} \frac{|S_n\cap(S_n-\lambda_n(x+y)+h_1+h_2)\cap(S_n-\lambda_n y+h_2)|}{|S_n|}w(x)\,w(y)\\
&\qquad\qquad\times\underbrace{\tau_m^*(0,h_1-\lambda_n x,0,h_2-\lambda_n y)}_{=m_n\mathbb{P}\left(\frac{X_0}{a_m}\in A,\frac{X_{h_1-\lambda_n x}}{a_m}\in B,\frac{X_{h_2-\lambda_n x}}{a_m}\in B\right)\le m_n\mathbb{P}\left(\frac{X_0}{a_m}\in A\right)=p_m(A)}dydx\\
&\le \frac{\lambda_n^2}{\nu}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\int_{\frac{(S_n-S_n)+h_2}{\lambda_n}} w(x)w(y)p_m(A)dydx\\
&\le p_m(A)\frac{\lambda_n^2}{\nu}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(x)w(y)dydx = \underbrace{p_m(A)}_{\to\mu(A)}\underbrace{\frac{\lambda_n^2}{\nu}}_{\to 0}\xrightarrow{n\to\infty} 0.
\end{aligned}$$

Analogously, we can establish $I_3 \to 0,\ I_4 \to 0$ and $I_5 \to 0$ as $n \to \infty$.
So, we turn to I_6 and first elucidate the case when $h_1 = h_2$. We find by substituting $y = \frac{h_1+s_1-s_2}{\lambda_n}$ and $u = s_2$ as in (i)

$$\begin{aligned}
I_6 &= \frac{\lambda_n^2}{|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2)^2\tau_m^*(s_1,s_2,s_1,s_2)ds_1ds_2 \\
&= \frac{1}{\lambda_n^2|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w\left(\frac{h_1+s_1-s_2}{\lambda_n}\right)^2\tau_m^*(s_1,s_2,s_1,s_2)ds_1ds_2 \\
&= \frac{1}{\lambda_n^2|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w\left(\frac{h_1+s_1-s_2}{\lambda_n}\right)^2\tau_{AB,m}(s_2-s_1)ds_1ds_2 \\
&= \frac{1}{|S_n|}\frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\int_{S_n\cap(S_n+h_1-\lambda_n y)} w\left(y\right)^2\tau_{AB,m}(h_1-\lambda_n y)dudy \\
&= \frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\frac{|S_n\cap(S_n+h_1-\lambda_n y)|}{|S_n|}w\left(y\right)^2\tau_{AB,m}(h_1-\lambda_n y)dy.
\end{aligned}\tag{15}$$

At this point, recall that $w(\cdot)$ is a bounded probability density function on $\mathbb{R}^2$ and let $0 < M < \infty$ be its upper bound. Then it follows that

$$\begin{aligned}
\int_{\mathbb{R}^2} w(y)^2dy &= \int_{\{y\in\mathbb{R}^2:w(y)<1\}} w(y)^2dy + \int_{\{y\in\mathbb{R}^2:w(y)\geq 1\}} w(y)^2dy \\
&\leq \int_{\{y\in\mathbb{R}^2:w(y)<1\}} w(y)^2dy + M^2\underbrace{\left|\{y\in\mathbb{R}^2: w(y)\geq 1\}\right|}_{\leq C<\infty} \\
&\leq \int_{\mathbb{R}^2} w(y)dy + M^2C = 1 + M^2C < \infty.
\end{aligned}$$

Hence, together with Proposition 3.1, we may apply dominated convergence to (15) such that for $h_1 = h_2$

$$I_6 = \frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\frac{|S_n\cap(S_n+h_1-\lambda_n y)|}{|S_n|}w\left(y\right)^2\tau_{AB,m}(h_1-\lambda_n y)dy \xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h_1)\int_{\mathbb{R}^2} w(y)^2dy.$$

In the consecutive step, let $h_1 \neq h_2$. Thus, I_6 becomes, using the substitution $y = \frac{h_1+s_1-s_2}{\lambda_n}$ and $u = s_2$

$$\begin{aligned}
I_6 &= \frac{\lambda_n^2}{|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2)w_n(h_2+s_1-s_2)\tau_m^*(s_1,s_2,s_1,s_2)ds_1ds_2 \\
&= \frac{1}{\lambda_n^2|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w\left(\frac{h_1+s_1-s_2}{\lambda_n}\right)w\left(\frac{h_2+s_1-s_2}{\lambda_n}\right)\tau_{AB,m}(s_2-s_1)ds_1ds_2 \\
&= \frac{1}{|S_n|}\frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\int_{S_n\cap(S_n+h_1-\lambda_n y)} w\left(y\right)w\left(y+\frac{h_2-h_1}{\lambda_n}\right)\tau_{AB,m}(h_1-\lambda_n y)dudy \\
&= \frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\frac{|S_n\cap(S_n+h_1-\lambda_n y)|}{|S_n|}w\left(y\right)w\left(y+\frac{h_2-h_1}{\lambda_n}\right)\tau_{AB,m}(h_1-\lambda_n y)dy.
\end{aligned}$$

We want to apply dominated convergence again. For this purpose, we estimate

$$\begin{aligned}
&\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}} w\left(y+\frac{h_2-h_1}{\lambda_n}\right) w(y)dy \leq \int_{\mathbb{R}^2} \max\{w(y), w\left(y+\frac{h_2-h_1}{\lambda_n}\right)\}dy \\
&= \int_{\left\{y\in\mathbb{R}^2: w(y)>w\left(y+\frac{h_2-h_1}{\lambda_n}\right)\right\}} w(y)^2 dy + \int_{\left\{y\in\mathbb{R}^2: w(y)\leq w\left(y+\frac{h_2-h_1}{\lambda_n}\right)\right\}} w\left(y+\frac{h_2-h_1}{\lambda_n}\right)^2 dy \\
&\leq 2\int_{\mathbb{R}^2} w(y)^2 dy < \infty
\end{aligned}$$

and as in part (i)

$$\frac{|S_n\cap(S_n+h_1-\lambda_n y)|}{|S_n|}\tau_{AB,m}(h_1-\lambda_n y) \leq p_m(A) \xrightarrow{n\to\infty} \mu(A) < \infty.$$

Note that $w(y)\to 0$ as $y\to\infty$ and $\lambda_n\to 0$ as $n\to\infty$. Using these arguments and Proposition 3.1, I_6 reads for $h_1\neq h_2$

$$I_6 = \frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}} \underbrace{\frac{|S_n\cap(S_n+h_1-\lambda_n y)|}{|S_n|}}_{\to 1} w(y) \underbrace{w\left(y+\frac{h_2-h_1}{\lambda_n}\right)}_{\to 0} \underbrace{\tau_{AB,m}(h_1-\lambda_n y)}_{\to\tau_{AB,m}(h_1)} dy \xrightarrow{n\to\infty} 0.$$

In the following, the asymptotics for I_7 are considered. We distinguish the cases $h_1\neq -h_2$ and $h_1=-h_2$ and begin with the latter such that I_7 can be written as

$$\begin{aligned}
I_7 &= \frac{\lambda_n^2}{|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2)w_n(-h_1+s_2-s_1) \underbrace{\tau_m^*(s_1,s_2,s_2,s_1)}_{=m_n\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A,\frac{X_{s_2}}{a_m}\in B,\frac{X_{s_2}}{a_m}\in A,\frac{X_{s_1}}{a_m}\in B\right)} ds_1 ds_2 \\
&= \frac{\lambda_n^2}{|S_n|}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w_n(h_1+s_1-s_2)w_n(-h_1+s_2-s_1)\tau_{A\cap BA\cap B,m}(s_2-s_1)ds_1 ds_2 \\
&= \frac{1}{|S_n|}\frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}}\int_{S_n\cap(S_n+h_1-\lambda_n y)} w(y)w(-y)\tau_{A\cap BA\cap B,m}(h_1-\lambda_n y)du dy \\
&= \frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}} \frac{|S_n\cap(S_n+h_1-\lambda_n y)|}{|S_n|} w(y)^2\tau_{A\cap BA\cap B,m}(h_1-\lambda_n y)dy,
\end{aligned}$$

where the last equality follows from isotropy of $w(\cdot)$ and the independence of the inner integral of u. Then by dominated convergence and Proposition 3.1

$$\begin{aligned}
I_7 &= \frac{1}{\nu^2}\int_{\frac{(S_n-S_n)+h_1}{\lambda_n}} \underbrace{\frac{|S_n\cap(S_n+h_1-\lambda_n y)|}{|S_n|}}_{\to 1} w(y)^2 \underbrace{\tau_{A\cap BA\cap B,m}(h_1-\lambda_n y)}_{\to\tau_{A\cap BA\cap B}(h_1)} dy \\
&\xrightarrow{n\to\infty} \tau_{A\cap BA\cap B}(h_1)\frac{1}{\nu^2}\int_{\mathbb{R}^2} w(y)^2 dy.
\end{aligned}$$

If $h_1 \neq -h_2$ we will obtain, by the same arguments as for I_6 in the $h_1 \neq h_2$ case,

$$I_7 \xrightarrow{n\to\infty} 0.$$

Lastly, we turn to I_1. We will prove

$$\left| I_1 - \frac{|S_n|\lambda_n^2}{m_n} \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \right| \xrightarrow{n\to\infty} 0$$

and find using the same reasoning as in (11), stationarity and the triangle inequality

$$\begin{aligned}
&\left| I_1 - \frac{|S_n|\lambda_n^2}{m_n} \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \right| \\
&= \frac{m_n\lambda_n^2}{|S_n|\nu^4} \left| \int_{S_n^4} w_n(h_1 + s_1 - s_2) w_n(h_2 + s_3 - s_4) \frac{\tau_m^*(s_1, s_2, s_3, s_4)}{m_n} \nu^4 ds_1 ds_2 ds_3 ds_4 \right. \\
&\left. - \int_{S_n^4} w_n(h_1 + s_1 - s_2) w_n(h_2 + s_3 - s_4) \mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_{s_2 - s_1}}{a_m} \in B \right) \mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_{s_4 - s_3}}{a_m} \in B \right) \nu^4 ds_1 ds_2 ds_3 ds_4 \right| \\
&\leq \frac{m_n\lambda_n^2}{|S_n|} \int_{S_n^4} w_n(h_1 + s_1 - s_2) w_n(h_2 + s_3 - s_4) \\
&\qquad \left| \frac{1}{m_n} \tau_m^*(0, s_2 - s_1, s_3 - s_1, s_4 - s_1) - \frac{1}{m_n^2} \tau_{AB,m}(s_2 - s_1) \tau_{AB,m}(s_4 - s_3) \right| ds_1 ds_2 ds_3 ds_4 \\
&= \frac{m_n\lambda_n^2}{|S_n|} \int_{(S_n - S_n)^3} \int_{S_n \cap (S_n - v_1) \cap (S_n - v_2) \cap (S_n - v_3)} w_n(h_1 - v_1) w_n(h_2 - (v_3 - v_2)) \\
&\qquad \left| \frac{1}{m_n} \tau_m^*(0, v_1, v_2, v_3) - \frac{1}{m_n^2} \tau_{AB,m}(v_1) \tau_{AB,m}(v_3 - v_2) \right| dv_4 dv_1 dv_2 dv_3. \qquad (16)
\end{aligned}$$

In the last step we substituted $v_1 = s_2 - s_1,\ v_2 = s_3 - s_1,\ v_3 = s_4 - s_1$ and $v_4 = s_1$. Again, the integral does not depend on v_4 and $\frac{|S_n \cap (S_n - v_1) \cap (S_n - v_2) \cap (S_n - v_3)|}{|S_n|} \leq 1$ such that (16) is bounded by

$$\begin{aligned}
&m_n\lambda_n^2 \int_{(S_n - S_n)^3} w_n(h_1 - v_1) w_n(h_2 - (v_3 - v_2)) \left| \frac{1}{m_n} \tau_m^*(0, v_1, v_2, v_3) - \frac{1}{m_n^2} \tau_{AB,m}(v_1) \tau_{AB,m}(v_3 - v_2) \right| dv_1 dv_2 dv_3 \\
&= m_n\lambda_n^2 \int_{\frac{h_2 - (S_n - S_n) - (S_n - S_n)}{\lambda_n}} \int_{\frac{h_1 - (S_n - S_n)}{\lambda_n}} \int_{S_n - S_n} \frac{1}{\lambda_n^4} w(y_1) w(y_2) \\
&\qquad \left| \frac{1}{m_n} \tau_m^*(0, h_1 - \lambda_n y_1, u, u + h_2 - \lambda_n y_2) - \frac{1}{m_n^2} \tau_{AB,m}(h_1 - \lambda_n y_1) \tau_{AB,m}(h_2 - \lambda_n y_2) \right| \lambda_n^4 du dy_1 dy_2, \qquad (17)
\end{aligned}$$

where we used $y_1 = \frac{h_1 - v_1}{\lambda_n},\ y_2 = \frac{h_2 - (v_3 - v_2)}{\lambda_n}$ and $u = v_2$ and the corresponding integral transformation $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ given by $v_1 = \phi_1(y_1, y_2, u) = h_1 - \lambda_n y_1,\ v_3 = \phi_2(y_1, y_2, u) = h_2 + u - \lambda_n y_2$ and $v_2 = \phi_3(y_1, y_2, u) = u$ with the Jacobian

$$|D\phi(y_1, y_2, u)| = \left| \begin{pmatrix} -\lambda_n \mathbb{1}_2 & 0 & 0 \\ 0 & -\lambda_n \mathbb{1}_2 & \mathbb{1}_2 \\ 0 & 0 & \mathbb{1}_2 \end{pmatrix} \right| = \lambda_n^4.$$

Note that by stationarity and definitions of $\tau_{AB,m}$ and τ_m^*

$$
\begin{aligned}
&\left|\frac{1}{m_n}\tau_m^*(0, h_1 - \lambda_n y_1, u, u + h_2 - \lambda_n y_2) - \frac{1}{m_n^2}\tau_{AB,m}(h_1 - \lambda_n y_1)\tau_{AB,m}(h_2 - \lambda_n y_2)\right| \\
&= \Bigg|\mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_{h_1-\lambda_n y_1}}{a_m} \in B, \frac{X_u}{a_m} \in A, \frac{X_{u+h_2-\lambda_n y_2}}{a_m} \in B\right) \\
&\qquad -\mathbb{P}\left(\frac{X_0}{a_m} \in A, \frac{X_{h_1-\lambda_n y_1}}{a_m} \in B\right)\mathbb{P}\left(\frac{X_u}{a_m} \in A, \frac{X_{u+h_2-\lambda_n y_2}}{a_m} \in B\right)\Bigg| \\
&\le \sup\{|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| : A_1 \in F_{\Lambda_1}, A_2 \in F_{\Lambda_2}, \Lambda_1 = \{0, h_1 - \lambda_n y_1\}, \Lambda_2 = \{u, u + h_2 - \lambda_n y_2\}\} \\
&\le \sup\{|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| : A_1 \in F_{\Lambda_1}, A_2 \in F_{\Lambda_2}, |\Lambda_1| \le 2, |\Lambda_2| \le 2, d(\Lambda_1, \Lambda_2) \le k\} \\
&= \alpha_{22}(k),
\end{aligned}
$$

where $d(A, B) \coloneqq \inf\{|a - b| : a \in A, b \in B\}$ denotes the minimal distance between two sets $A, B \subset \mathbb{R}^2$, $F_A \coloneqq \sigma\{(X_s : s \in A\}$ the sigma-algebra generated by the random process $\{X_s\}_{s\in A}$. Furthermore, we define $k = \min\{|u|, |u + h_2 - \lambda_n y_2|, |u - h_1 + \lambda_n y_1|, |u + h_2 - \lambda_n y_2 - h_1 + \lambda_n y_1|\}$ and observe

$$
\alpha_{22}(k) \le \alpha_{22}(|u|) + \alpha_{22}(|u + h_2 - \lambda_n y_2|) + \alpha_{22}(|u - h_1 + \lambda_n y_1|) + \alpha_{22}(|u + h_2 - \lambda_n y_2 - h_1 + \lambda_n y_1|)
$$

Thus, (17) is bounded by

$$
\begin{aligned}
&m_n\lambda_n^2\Bigg[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(y_1)w(y_2)\alpha_{22}(|u|)dy_1dy_2du \\
&\qquad + \int_{\frac{h_2-(S_n-S_n)-(S_n-S_n)}{\lambda_n}}\int_{\mathbb{R}^2}\int_{S_n-S_n} w(y_1)w(y_2)\alpha_{22}(|u + h_2 - \lambda_n y_2|)dudy_1dy_2 \\
&\qquad + \int_{\mathbb{R}^2}\int_{\frac{h_1-(S_n-S_n)}{\lambda_n}}\int_{S_n-S_n} w(y_1)w(y_2)\alpha_{22}(|u - h_1 + \lambda_n y_1|)dudy_1dy_2 \\
&\qquad + \int_{\frac{h_2-(S_n-S_n)-(S_n-S_n)}{\lambda_n}}\int_{\frac{h_1-(S_n-S_n)}{\lambda_n}}\int_{S_n-S_n} w(y_1)w(y_2)\alpha_{22}(|u + h_2 - \lambda_n y_2 - h_1 + \lambda_n y_1|)dudy_1dy_2\Bigg] \\
&=: m_n\lambda_n^2\left[A_{1n} + A_{2n} + A_{3n} + A_{4n}\right].
\end{aligned}
$$

Since $w(\cdot)$ is a probability density on $\mathbb{R}^2$, we find by condition (**M2**)

$$
A_{1n} = \int_{\mathbb{R}^2} \alpha_{22}(|u|)du < \infty.
$$

Proceeding with A_{2n} and substituting $x = u + h_2 - \lambda_n y_2$ and $y = y_2$, gives

$$
\begin{aligned}
A_{2n} &= \int_{\frac{h_2-(S_n-S_n)-(S_n-S_n)}{\lambda_n}}\int_{S_n-S_n} w(y_2)\alpha_{22}(|u + h_2 - \lambda_n y_2|)\underbrace{\int_{\mathbb{R}^2} w(y_1)dy_1}_{=1} dudy_2 \\
&\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(y)\alpha_{22}(|x|)dxdy = \int_{\mathbb{R}^2}\alpha_{22}(|x|)dx < \infty, \qquad \text{by } \textbf{(M2)},
\end{aligned}
$$

where the integral transformation function $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ is given by $u = \phi_1(x,y) = x + \lambda_n y - h_2$ and $y_2 = \phi_2(x,y) = y$. The determinant of the Jacobian is

$$|D\phi(x,y)| = \left| \begin{pmatrix} \mathbb{1}_2 & \lambda_n \mathbb{1}_2 \\ 0 & \mathbb{1}_2 \end{pmatrix} \right| = 1.$$

Analogous arguments allow us to infer

$$A_{3n} < \infty.$$

For A_{4n} we find by substituting $x = u + h_2 - \lambda_n z - h_1 + \lambda_n y,\ y = y_1$ and $z = y_2$

$$\begin{aligned} A_{4n} &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{(S_n - S_n) + h_2 - h_1 + \lambda_n y - \lambda_n z} w(y) w(z) \alpha_{22}(|x|) dx dy dz \\ &= \int_{\mathbb{R}^2} \alpha_{22}(|x|) dx \underbrace{\left(\int_{\mathbb{R}^2} w(y) dy \right)^2}_{=1} = \int_{\mathbb{R}^2} \alpha_{22}(|x|) dx < \infty, \qquad \text{by } \textbf{(M2)}, \end{aligned}$$

where the integral transform function is given by $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ such that $u = \phi_(x,y,z) = x - h_2 + \lambda_n z + h_1 - \lambda_n y$, $y_1 = \phi_2(x,y,z) = y$ and $y_2 = \phi_2(x,y,z) = z$. Therefore, the determinant of its Jacobian reads

$$|D\phi(x,y,z)| = \left| \begin{pmatrix} \mathbb{1}_2 & \lambda_n \mathbb{1}_2 & -\lambda_n \mathbb{1}_2 \\ 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & \mathbb{1}_2 \end{pmatrix} \right| = 1.$$

Finally, we combine the results for A_{1n}, A_{2n}, A_{3n} and A_{4n} to obtain

$$\begin{aligned} \left| I_1 - \frac{|S_n| \lambda_n^2}{m_n} \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \right| &\leq m_n \lambda_n^2 \left[A_{1n} + A_{2n} + A_{3n} + A_{4n} \right] \\ &= 4 \underbrace{m_n \lambda_n^2}_{\to 0 \text{ by } \textbf{(M2)}} \int_{\mathbb{R}^2} \alpha_{22}(|x|) dx \xrightarrow{n \to \infty} 0. \end{aligned}$$

This means we may conclude the proof of (ii), since for the covariance we can infer

$$\begin{aligned} \frac{|S_n| \lambda_n^2}{m_n} \mathbb{C}\text{ov}\left(\hat{\tau}_{AB,m}(h_1), \hat{\tau}_{AB,m}(h_2) \right) &= I_6 + I_7 + I_1 - \frac{|S_n| \lambda_n^2}{m_n} \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \\ &\xrightarrow{n \to \infty} \frac{1}{\nu^2} \left[\tau_{AB}(h_1) \mathbb{1}\{h_1 = h_2\} + \tau_{A\cap B, A\cap B}(h_1) \mathbb{1}\{h_1 = -h_2\} \right] \int_{\mathbb{R}^2} w(y)^2 dy. \end{aligned}$$

(iii) This is a direct result of (ii), setting $h_1 = h_2$. □

Remark 3.4. *Consider the estimator $\hat{\tau}_{AB,m}(h : B_n^i)$ of $\tau_{AB}(h)$, $i \in \{1, \dots, k_n\}$, that is confined to observations on the cube $B_n^i \subset D_n^i$ and $|B_n^i| = |D_n^i| = O(n^{2\gamma})$, where D_n^i is a bigger cube, containing B_n^i. Then we have*

$$\begin{aligned} &\frac{|B_n^1|^2 \lambda_n^2}{|D_n^1| m_n} \mathbb{C}\text{ov}(\hat{\tau}_{AB,m}(h_1 : B_n^i), \hat{\tau}_{AB,m}(h_2 : B_n^i)) \\ &\qquad \xrightarrow{n \to \infty} \frac{1}{\nu^2} \left[\tau_{AB}(h_1) \mathbb{1}\{h_1 = h_2\} + \tau_{A\cap B, A\cap B}(h_1) \mathbb{1}\{h_1 = -h_2\} \right] \int_{\mathbb{R}^2} w(y)^2 dy. \end{aligned}$$

Proof. The proof is completely analogous to the proof of Proposition 3.3. We only give a brief outline of the steps. First note that

$$\frac{|B_n^1|^2\lambda_n^2}{|D_n^1|m_n}\mathbb{E}\left[\hat{\tau}_{AB,m}(h_1 : B_n^i)\hat{\tau}_{AB,m}(h_2 : B_n^i)\right]$$
$$= \frac{m_n\lambda_n^2}{|D_n^1|}\frac{1}{\nu^4}\int_{(B_n^i)^4} w_n(h_1+s_1-s_2)w_n(h_2+s_3-s_4)\frac{1}{m_n}\tau_m^*(s_1,s_2,s_3,s_4)\mathbb{E}\left[N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\right].$$

Then again we obtain seven integrals $\tilde{I}_j,\ j\in\{1,\dots,7\}$ according to (14). We observe that since $|B_n^1|$ and $|D_n^1|$ both are in $O(n^{2\gamma})$ and $B_n^i\subset D_n^i$. We obtain for every $y\in\mathbb{R}^2$

$$\frac{|B_n^i\cap(B_n^i-y)|}{|D_n^1|}\xrightarrow{n\to\infty}1$$
$$\frac{|B_n^i\cap(B_n^i-y)|}{|D_n^1|}\leq 1.$$

This allows us to conduct the same steps as in the proof above such that we obtain

$$\left|\tilde{I}_1-\frac{|S_n|\lambda_n^2}{m_n}\mathbb{E}\left[\hat{\tau}_{AB,m}(h_1 : B_n^i)\right]\mathbb{E}\left[\hat{\tau}_{AB,m}(h_2 : B_n^i)\right]\right|\xrightarrow{n\to\infty}0,$$
$$\tilde{I}_j\xrightarrow{n\to\infty}0,\qquad j=2,3,4,5$$
$$\tilde{I}_6\xrightarrow{n\to\infty}\mathbb{1}\{h_1=h_2\}\frac{1}{\nu^2}\tau_{AB}(h_1)\int_{\mathbb{R}^2}w(y)^2dy,$$
$$\tilde{I}_7\xrightarrow{n\to\infty}\mathbb{1}\{h_1=-h_2\}\frac{1}{\nu^2}\tau_{AB}(h_1)\int_{\mathbb{R}^2}w(y)^2dy,$$

and thus the remark follows. □

3.4 A CLT for the estimator of the numerator

Applying the previous results, we are now able to prove a CLT for the estimator of the numerator $\hat{\tau}_{AB,m}(\cdot)$ which then enables us to prove the CLT for the empirical extremogram, Theorem 3.6.

Proposition 3.5. *Assume that the strictly stationary random process $\{X_s : s \in \mathbb{R}^2\}$ satisfies the LUNC. Further let (**M1**), (**M2**) hold. Then for an arbitrary, but fixed, lag $h \in \mathbb{R}^2$*

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\tau}_{AB,m}(h) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right]\right) \xrightarrow{d} N(0,\sigma^2(h)), \qquad n \to \infty,$$

where $\sigma^2(h) = \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2} w(y)^2 dy$. Furthemore, if $\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right] - \tau_{AB}(h) = o\left(\sqrt{\frac{m_n}{|S_n|\lambda_n^2}}\right)$, then

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\tau}_{AB,m}(h) - \tau_{AB}(h)\right) \xrightarrow{d} N(0,\sigma^2(h)), \qquad n \to \infty.$$

Proof. We follow Li et al. [16] and use a blocking technique. For this purpose let D_n^i be non-overlapping equal-sized cubes in S_n for $i = 1,\dots,k_n$, where $k_n = \frac{|S_n|}{|D_n^1|}$ is the number of such cubes. Within each D_n^i there is a smaller inner cube, denoted by B_n^i with $|B_n^i| = |B_n^1|$ for all $i = 1,\dots,k_n$, that shares the same center as D_n^i and $d(\partial D_n^i, B_n^i) \geq n^\eta$. Furthermore, let $|D_n^1| = n^{2\gamma}$ and $|B_n^1| = (n^\gamma - 2n^\eta)^2 = O(n^{2\gamma})$ with $\eta > 0$ such that $\frac{6}{2+\epsilon} < \eta < \gamma < 1$ for some $\epsilon > \frac{2+4\gamma}{\eta}$ that satisfies the last condition of (**M2**). Then we have that $k_n = O\left(n^{2(1-\gamma)}\right)$. To ease notation in the upcoming proof, we introduce

$$\begin{aligned}
A_n &= \sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\int_{S_n}\int_{S_n} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}N^{(2)}(ds_1,ds_2)\\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\hat{\tau}_{AB,m}(h)\\
a_{ni} &= \sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}N^{(2)}(ds_1,ds_2)\\
&= \sqrt{\frac{m_n\lambda_n^2}{|D_n^i|k_n}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}N^{(2)}(ds_1,ds_2)\\
&= \sqrt{\frac{|B_n^i|^2\lambda_n^2}{|S_n|m_n}}\,\hat{\tau}_{AB,m}(h:B_n^i)\\
\tilde{A}_n &= A_n - \mathbb{E}\left[A_n\right]\\
\tilde{a}_{ni} &= a_{ni} - \mathbb{E}\left[a_{ni}\right]\\
a_n &= \sum_{i=1}^{k_n} a_{ni}
\end{aligned}$$

$$\tilde{a}_n = \sum_{i=1}^{k_n} \tilde{a}_{ni}$$
$$\tilde{a}'_n = \sum_{i=1}^{k_n} \tilde{a}'_{ni},$$

where $\tilde{a}'_{ni}$ are independent copies of $\tilde{a}_{ni}$, $i = 1, \dots, k_n$. We divide the proof in three steps.

Step 1

We will show $\mathbb{V}\text{ar}\left(\tilde{A}_n - \tilde{a}_n\right) \xrightarrow{n\to\infty} 0$ which then implies by an application of Chebyshev's inequality that $\tilde{A}_n - \tilde{a}_n \xrightarrow{P} 0$: Note that by definitions of $\tilde{A}_n$ and $\tilde{a}_n$, we obtain that $\mathbb{E}\left[\tilde{A}_n - \tilde{a}_n\right] = 0$. Then Chebyshev's inequality gives for every $\epsilon > 0$

$$\mathbb{P}\left(\tilde{A}_n - \tilde{a}_n > \epsilon\right) \le \frac{1}{\epsilon^2}\mathbb{V}\text{ar}\left(\tilde{A}_n - \tilde{a}_n\right) \xrightarrow{n\to\infty} 0.$$

The result $\mathbb{V}\text{ar}\left(\tilde{A}_n - \tilde{a}_n\right) \xrightarrow{n\to\infty} 0$ is proven by showing:

$$\begin{aligned}
&(i) && \mathbb{V}\text{ar}(\tilde{A}_n) && \xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2} w(y)^2 dy \\
&(ii) && \mathbb{C}\text{ov}(\tilde{A}_n, \tilde{a}_n) && \xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2} w(y)^2 dy \\
&(iii) && \mathbb{V}\text{ar}(\tilde{a}_n) && \xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2} w(y)^2 dy
\end{aligned}$$

(i) This is an immediate result of Proposition 3.3 (ii) since

$$\mathbb{V}\text{ar}\left(\tilde{A}_n\right) = \mathbb{V}\text{ar}\left(A_n\right) = \mathbb{V}\text{ar}\left(\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\hat{\tau}_{AB,m}(h)\right) = \frac{|S_n|\lambda_n^2}{m_n}\mathbb{V}\text{ar}\left(\hat{\tau}_{AB,m}(h)\right) \xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2} w(y)^2 dy.$$

(ii) For part (ii) we consider

$$\begin{aligned}
\mathbb{E}\left[A_n a_n\right] = &\frac{m_n\lambda_n^2}{|S_n|\nu^4}\mathbb{E}\left[\sum_{i=1}^{k_n}\int_{S_n}\int_{S_n}\int_{B_n^i}\int_{B_n^i} w_n\left(h+s_1-s_2\right)w_n\left(h+s_3-s_4\right)\right.\\
&\left.\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}\mathbb{1}\left\{\frac{X_{s_3}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_4}}{a_m}\in B\right\}N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\right]\\
= &\frac{\lambda_n^2}{|S_n|\nu^4}\sum_{i=1}^{k_n}\int_{S_n}\int_{S_n}\int_{B_n^i}\int_{B_n^i} w_n\left(h+s_1-s_2\right)w_n\left(h+s_3-s_4\right)\\
&\underbrace{m_n\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A, \frac{X_{s_2}}{a_m}\in B, \frac{X_{s_3}}{a_m}\in A, \frac{X_{s_4}}{a_m}\in B\right)}_{=\tau_m^*(s_1,s_2,s_3,s_4)}\mathbb{E}\left[N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda_n^2}{|S_n|\nu^4}\sum_{i=1}^{k_n}\Bigg[\int_{S_n\backslash B_n^i}\int_{S_n\backslash B_n^i}\int_{B_n^i}\int_{B_n^i}\big[\,\cdot\,\big]+\int_{S_n\backslash B_n^i}\int_{B_n^i}\int_{B_n^i}\int_{B_n^i}\big[\,\cdot\,\big]+\int_{B_n^i}\int_{S_n\backslash B_n^i}\int_{B_n^i}\int_{B_n^i}\big[\,\cdot\,\big] \\
&\qquad +\int_{B_n^i}\int_{B_n^i}\int_{B_n^i}\int_{B_n^i}\big[\,\cdot\,\big]\Bigg] \qquad (18)\\
&=: \bar{A}_1+\bar{A}_2+\bar{A}_3+\bar{A}_4 =: \sum_{i=1}^{4}\sum_{j=1}^{7}\bar{A}_i^j,
\end{aligned}
$$

where $\bar{A}_i^j$ corresponds to the i-th term of the four integrals in (18) with the j-th term of the seven terms mentioned in the proof of Proposition 3.1 equation (14).
Since $\cup_{i=1}^{k_n}B_n^i \subset S_n$ and $\cup_{i=1}^{k_n}S_n\backslash B_n^i \subset S_n$, we infer that $\sum_{i=1}^4 \bar{A}_j^i \leq I_j \xrightarrow{n\to\infty} 0$ such that $\sum_{i=1}^4 \bar{A}_j^i \xrightarrow{n\to\infty} 0$ for $j\in\{2,3,4,5\}$ and I_j as in the proof of Proposition 3.3.
Also with $\cup_{i=1}^{k_n}B_n^i \subset S_n$ and by the same arguments as for the I_1-term in part (ii) in the proof of Proposition 3.3, we obtain

$$
\left|\sum_{i=1}^{4}\bar{A}_i^1 - \mathbb{E}[A_n]\mathbb{E}[a_n]\right| \xrightarrow{n\to\infty} 0.
$$

Note that if $j=6$ and $j=7$, we have $s_1=s_3$ and $s_2=s_4$, respectively, $s_1=s_4$ and $s_2=s_3$, for the integrating variables in (18). Thus, since $S_n\backslash B_n^i \cap B_n^i=\emptyset$, the terms $\bar{A}_1^j, \bar{A}_2^j$ and $\bar{A}_3^j$ vanish, $j\in\{6,7\}$. Therefore, we only consider $\bar{A}_4^6$ and $\bar{A}_4^7$ to see by stationarity and the known substitutions from the proof of Proposition 3.3

$$
\begin{aligned}
\bar{A}_4^6 &= \frac{\lambda_n^2}{\nu^4|S_n|}\sum_{i=1}^{k_n}\int_{B_n^i}\int_{B_n^i} w_n\left(h+s_1-s_2\right)^2 \quad \underbrace{\tau_m^*(s_1,s_2,s_1,s_2)}_{=m_n\mathbb{P}\left(\frac{X_0}{a_m}\in A,\frac{X_{s_2-s_1}}{a_m}\in B\right)} \quad \nu^2 ds_1 ds_2 \\
&= \frac{\lambda_n^2}{\nu^2|S_n|}\sum_{i=1}^{k_n}\int_{B_n^1}\int_{B_n^1} w_n\left(h+s_1-s_2\right)^2 \tau_{AB,m}(s_2-s_1)ds_1ds_2 \\
&= \frac{k_n}{\nu^2|S_n|}\int_{\frac{(B_n^1-B_n^1)+h}{\lambda_n}}\int_{B_n^1\cap(B_n^1+h-\lambda_n y)} w(y)^2\tau_{AB,m}(h-\lambda_n y)dudy \\
&= \frac{1}{\nu^2}\int_{\frac{(B_n^1-B_n^1)+h}{\lambda_n}}\frac{|B_n^1\cap(B_n^1+h-\lambda_n y)|}{|D_n^1|}w(y)^2\tau_{AB,m}(h-\lambda_n y)dy, \quad \text{using } k_n=\frac{|S_n|}{|D_n^1|} \\
&\xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2} w(y)^2dy,
\end{aligned}
$$

where we used the facts that $\frac{|B_n^1\cap(B_n^1+h-\lambda_n y)|}{|D_n^1|}\leq 1$, $\frac{|B_n^1\cap(B_n^1+h-\lambda_n y)|}{|D_n^1|}\xrightarrow{n\to\infty} 1$, which can be directly inferred from $B_n^1\subset D_n^1$ and $|B_n^1|$ and $|D_n^1|$, both being in $O(n^{2\gamma})$, as well as $\frac{(B_n^1-B_n^1)+h}{\lambda_n}\xrightarrow{n\to\infty}\mathbb{R}^2$. Combining this together with Proposition 3.1, the fact that $\tau_{AB,m}(h-\lambda_n y)\leq p_m(A)\xrightarrow{n\to\infty}\mu(A)$ and applying dominated convergence, the upper result follows. For $\bar{A}_4^7$ we conduct the same steps as for I_7 in the proof of Proposition 3.3 (ii) and find by the known substitutions and isotropy of $w(\cdot)$

$$\begin{aligned}
\bar{A}_4^7 &= \frac{\lambda_n^2}{\nu^4|S_n|}\sum_{i=1}^{k_n}\int_{B_n^i}\int_{B_n^i} w_n\left(h+s_1-s_2\right) w_n\left(h+s_2-s_1\right) \underbrace{\tau_m^*(s_1,s_2,s_2,s_1)}_{=m_n\mathbb{P}\left(\frac{X_0}{a_m}\in A,\frac{X_{s_2-s_1}}{a_m}\in B,\frac{X_{s_2-s_1}}{a_m}\in A,\frac{X_0}{a_m}\in B\right)}\nu^2 ds_1 ds_2 \\
&= \frac{\lambda_n^2}{\nu^4|S_n|}k_n\int_{B_n^1}\int_{B_n^1} w_n\left(h+s_1-s_2\right) w_n\left(h+s_2-s_1\right)\tau_{A\cap BA\cap B}(s_2-s_1)ds_1ds_2 \\
&= \frac{1}{|S_n|}\frac{1}{\nu^2}\int_{\frac{(B_n^1-B_n^1)+h_1}{\lambda_n}}\int_{B_n^1\cap(B_n^1+h_1-\lambda_n y)} w(y)w(-y)\tau_{A\cap BA\cap B,m}(h_1-\lambda_n y)dudy \\
&= \frac{1}{\nu^2}\int_{\frac{(B_n^1-B_n^1)+h_1}{\lambda_n}}\frac{|B_n^1\cap(B_n^1+h_1-\lambda_n y)|}{|S_n|}w(y)^2\tau_{A\cap BA\cap B,m}(h_1-\lambda_n y)dy.
\end{aligned}$$

Now, we note that $\frac{|B_n^1\cap(B_n^1+h_1-\lambda_n y)|}{|S_n|}\xrightarrow{n\to\infty}0$, since $|B_n^1|=O(n^{2\gamma})$ and $|S_n|=O(n^2)$, where $\gamma<1$. Then applying dominated convergence gives

$$\bar{A}_4^7 = \frac{1}{\nu^2}\int_{\frac{(B_n^1-B_n^1)+h_1}{\lambda_n}}\underbrace{\frac{|B_n^1\cap(B_n^1+h_1-\lambda_n y)|}{|S_n|}}_{\to 0}w(y)^2\underbrace{\tau_{A\cap BA\cap B,m}(h_1-\lambda_n y)}_{\to\tau_{AB}(h_1)<\infty}dy \xrightarrow{n\to\infty} 0.$$

Combining all of the results above, we are able to show part (ii) by

$$\begin{aligned}
\mathbb{C}\mathrm{ov}(A_n,a_n) &= \mathbb{E}\left[A_n a_n\right]-\mathbb{E}\left[A_n\right]\mathbb{E}\left[a_n\right]=\sum_{i=1}^{4}\sum_{j=1}^{7}\bar{A}_i^j-\mathbb{E}[A_n]\mathbb{E}[a_n] \\
&= \underbrace{\sum_{i=1}^{4}\sum_{j=2}^{5}\bar{A}_i^j}_{\to 0}+\underbrace{\sum_{i=1}^{3}\bar{A}_i^6}_{=0}+\underbrace{\bar{A}_4^6}_{\to\frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2}w(y)^2dy}+\underbrace{\sum_{i=1}^{3}\bar{A}_i^7}_{=0}+\underbrace{\bar{A}_4^7}_{\to 0}+\underbrace{\sum_{i=1}^{4}\bar{A}_i^1-\mathbb{E}[A_n]\mathbb{E}[a_n]}_{\to 0} \\
&\xrightarrow{n\to\infty}\frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2}w(y)^2dy.
\end{aligned}$$

(iii) First note that $\mathbb{V}\mathrm{ar}(\tilde{a}_n)=\sum_{i=1}^{k_n}\mathbb{V}\mathrm{ar}(\tilde{a}_{ni})+\sum_{i=1}^{k_n}\sum_{\substack{j=1\\j\neq i}}^{k_n}\mathbb{C}\mathrm{ov}(\tilde{a}_{ni},\tilde{a}_{nj})$. By stationarity and Remark 3.4, we derive

$$\begin{aligned}
\sum_{i=1}^{k_n}\mathbb{V}\mathrm{ar}(\tilde{a}_{ni}) &= k_n\mathbb{V}\mathrm{ar}(\tilde{a}_{n1})=k_n\mathbb{V}\mathrm{ar}(a_{n1})=k_n\frac{|B_n^1|^2\lambda_n^2}{|S_n|m_n}\mathbb{V}\mathrm{ar}(\hat{\tau}_{AB,m}(h:B_n^1)) \\
&= \frac{|B_n^1|^2\lambda_n^2}{|D_n^1|m_n}\mathbb{V}\mathrm{ar}(\hat{\tau}_{AB,m}(h:B_n^1))\xrightarrow{n\to\infty}\frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}^2}w(y)^2dy.
\end{aligned}$$

Furthermore, $\tilde{a}_{ni}$ and $\tilde{a}_{nj}$ are integrals over disjoint sets for $i \neq j$, also $\{X_s : s \in \mathbb{R}^2\}$ and N are independent implying that $\mathbb{E}[\tilde{a}_{ni}|N]$ and $\mathbb{E}[\tilde{a}_{nj}|N]$ are independent. This is since a_{ni} only depends on Poisson points in B_n^i. In the following, the argument is explained in more detail:

$$\begin{aligned}
\mathbb{E}[a_{ni}|N] &= \mathbb{E}\left[\sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}N^{(2)}(ds_1,ds_2)\middle|N\right] \\
&= \sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{E}\left[\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}N^{(2)}(ds_1,ds_2)\middle|N\right] \\
&= \sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{E}\left[\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}\middle|N\right]N^{(2)}(ds_1,ds_2) \\
&= \sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{E}\left[\mathbb{1}\left\{\frac{X_{s_1}}{a_m}\in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m}\in B\right\}\right]N^{(2)}(ds_1,ds_2) \\
&= \sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A, \frac{X_{s_2}}{a_m}\in B\right)N^{(2)}(ds_1,ds_2) \\
&= \sqrt{\frac{\lambda_n^2}{|S_n|m_n}}\frac{1}{\nu^2}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\tau_{AB,m}(s_2-s_1)N^{(2)}(ds_1,ds_2).
\end{aligned}$$

Now since B_n^i and B_n^j are disjoint cubes, the conditional expectations only depend on a Poisson process on disjoint sets. Thus, they are independent. Then by the law of total covariance and the independence argument from above

$$\begin{aligned}
\sum_{i=1}^{k_n}\sum_{\substack{j=1\\j\neq i}}^{k_n}|\mathbb{C}\mathrm{ov}(\tilde{a}_{ni},\tilde{a}_{nj})| &= \sum_{i=1}^{k_n}\sum_{\substack{j=1\\j\neq i}}^{k_n}\left|\mathbb{E}\left[\mathbb{C}\mathrm{ov}(\tilde{a}_{ni},\tilde{a}_{nj}|N)\right] + \mathbb{C}\mathrm{ov}\left(\mathbb{E}[\tilde{a}_{ni}|N],\mathbb{E}[\tilde{a}_{nj}|N]\right)\right| \\
&= \sum_{i=1}^{k_n}\sum_{\substack{j=1\\j\neq i}}^{k_n}\left|\mathbb{E}\left[\mathbb{C}\mathrm{ov}(\tilde{a}_{ni},\tilde{a}_{nj}|N)\right]\right|.
\end{aligned}$$

We proceed by finding an almost sure upper bound for a_{ni} given the process N

$$\begin{aligned}
|a_{ni}| &\leq \frac{1}{\nu^2}\sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)N^{(2)}(ds_1,ds_2) \\
&= \frac{1}{\nu^2}\sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\int_{B_n^i}\int_{B_n^i}\frac{1}{\lambda_n^2}w\left(\frac{h+s_1-s_2}{\lambda_n}\right)N^{(2)}(ds_1,ds_2)
\end{aligned}$$

This allows us to apply Lemma A.3 with $U = B_n^i$ and $V = B_n^j$, $i \neq j$ such that $d(U,V) \geq n^\eta > 0$ and $\max\left\{|B_n^i|,|B_n^j|\right\} = |B_n^1| =: M$ resulting in

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{C}\mathrm{ov}(\tilde{a}_{ni},\tilde{a}_{nj}|N)\right]| \\
&\leq \frac{4}{\nu^4}\frac{\lambda_n^2 m_n}{|S_n|}\mathbb{E}\left[\int_{B_n^i}\int_{B_n^i}\int_{B_n^j}\int_{B_n^j}\frac{1}{\lambda_n^4}w\left(\frac{h+s_1-s_2}{\lambda_n}\right)w\left(\frac{h+s_3-s_4}{\lambda_n}\right)N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\right]\alpha_{MM}(n^\eta) \\
&\leq \frac{C}{\nu^4}\frac{n^{-\eta\epsilon}m_n}{\lambda_n^2|S_n|}\int_{B_n^i}\int_{B_n^i}\int_{B_n^j}\int_{B_n^j}w\left(\frac{h+s_1-s_2}{\lambda_n}\right)w\left(\frac{h+s_3-s_4}{\lambda_n}\right)\mathbb{E}\left[N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)\right]M^2 \\
&= \frac{C}{\nu^4}\frac{n^{-\eta\epsilon}m_n}{\lambda_n^2|S_n|}\int_{B_n^i}\int_{B_n^i}\int_{B_n^j}\int_{B_n^j}w\left(\frac{h+s_1-s_2}{\lambda_n}\right)w\left(\frac{h+s_3-s_4}{\lambda_n}\right)\mathbb{E}\left[N^{(2)}(ds_1,ds_2)\right]\mathbb{E}\left[N^{(2)}(ds_3,ds_4)\right]|B_n^1|^2 \\
&= C\frac{n^{-\eta\epsilon}m_n|B_n^1|^2}{\lambda_n^2|S_n|}\int_{B_n^i}\int_{B_n^i}\int_{B_n^j}\int_{B_n^j}w\left(\frac{h+s_1-s_2}{\lambda_n}\right)w\left(\frac{h+s_3-s_4}{\lambda_n}\right)ds_1ds_2ds_3ds_4 \\
&= C\frac{n^{-\eta\epsilon}m_n|B_n^1|^4\lambda_n^2}{|S_n|}\int_{\frac{B_n^i-B_n^i+h}{\lambda_n}}\frac{\left|B_n^i\cap\left(B_n^i-\lambda_n y_1+h\right)\right|}{|B_n^1|}w(y_1)\,dy_1\int_{\frac{B_n^j-B_n^j+h}{\lambda_n}}\underbrace{\frac{\left|B_n^j\cap\left(B_n^j-\lambda_n y_2+h\right)\right|}{|B_n^1|}}_{\leq 1}w(y_2)\,dy_2 \\
&\leq C\frac{n^{-\eta\epsilon}m_n|B_n^1|^4\lambda_n^2}{|S_n|},
\end{aligned}
$$

where the second inequality holds due to condition **(M2)** and $C>0$ is some appropriate constant. For the last equality, we substituted $y_1=\frac{h+s_1-s_2}{\lambda_n},\ u_1=s_2$ and $y_2=\frac{h+s_3-s_4}{\lambda_n},\ u_2=s_4$ such that the integrand becomes independent of u_1 and u_2. Recalling that $k_n=|S_n|/|D_n^1|$, the upper bound derived above allows us to estimate

$$
\sum_{i=1}^{k_n}\sum_{\substack{j=1\\ j\neq i}}^{k_n}|\mathbb{E}\left[\mathbb{C}\mathrm{ov}(\tilde{a}_{ni},\tilde{a}_{nj}|N)\right]|\leq Ck_n^2\frac{n^{-\eta\epsilon}m_n|B_n^1|^4\lambda_n^2}{|S_n|}=C\frac{|S_n|^2}{|D_n^1|^2}\frac{n^{-\eta\epsilon}m_n|B_n^1|^4\lambda_n^2}{|S_n|}=O\left(n^{2+4\gamma-\epsilon\eta}m_n\lambda_n^2\right)
$$

which converges to 0 since $m_n\lambda_n^2\to 0$ and $\epsilon>\frac{2+4\gamma}{\eta}$ by assumption.

Step 2

Let $\phi_n(\cdot),\phi_n'(\cdot)$ be the characteristic functions of $\tilde{a}_n$ and $\tilde{a}_n'$. We follow the idea of Davis and Mikosch [9]

$$
\begin{aligned}
\left|\phi_n(x)-\phi_n'(x)\right| &= \mathbb{E}\left[e^{ix\tilde{a}_n}-e^{ix\tilde{a}_n'}\right] \\
&= \left|\mathbb{E}\left[e^{ix\sum_{l=1}^{k_n}\tilde{a}_{nl}}-e^{ix\sum_{l=1}^{k_n}\tilde{a}_{nl}'}\right]\right| \\
&= \left|\mathbb{E}\left[\sum_{j=1}^{k_n}\left(e^{ix\left(\sum_{l=1}^{j}\tilde{a}_{nl}+\sum_{l=j+1}^{k_n}\tilde{a}_{nl}'\right)}-e^{ix\left(\sum_{l=1}^{j}\tilde{a}_{nl}+\sum_{l=j+1}^{k_n}\tilde{a}_{nl}'\right)}\right)+e^{ix\sum_{l=1}^{k_n}\tilde{a}_{nl}}-e^{ix\sum_{l=1}^{k_n}\tilde{a}_{nl}'}\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \mathbb{E}\left[\sum_{j=1}^{k_n} e^{ix\left(\sum_{l=1}^{j} \tilde{a}_{nl} + \sum_{l=j+1}^{k_n} \tilde{a}'_{nl}\right)} - \sum_{j=0}^{k_n-1} e^{ix\left(\sum_{l=1}^{j} \tilde{a}_{nl} + \sum_{l=j+1}^{k_n} \tilde{a}'_{nl}\right)} \right] \right| \\
&= \left| \mathbb{E}\left[\sum_{j=1}^{k_n} \left(e^{ix\left(\sum_{l=1}^{j} \tilde{a}_{nl} + \sum_{l=j+1}^{k_n} \tilde{a}'_{nl}\right)} - e^{ix\left(\sum_{l=1}^{j-1} \tilde{a}_{nl} + \sum_{l=j}^{k_n} \tilde{a}'_{nl}\right)} \right) \right] \right|, \qquad \text{change of index} \\
&= \left| \mathbb{E}\left[\sum_{j=1}^{k_n} \prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}} \left(e^{ix\tilde{a}_{nj}} - e^{ix\tilde{a}'_{nj}} \right) \prod_{l=j+1}^{k_n} e^{ix\tilde{a}'_{nl}} \right] \right| \\
&\le \sum_{j=1}^{k_n} \left| \mathbb{E}\left[\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}} \underbrace{\left(e^{ix\tilde{a}_{nj}} - e^{ix\tilde{a}'_{nj}} \right)}_{\text{mean zero}} \right] \right| \underbrace{\left| \mathbb{E}\left[\left| \prod_{l=j+1}^{k_n} e^{ix\tilde{a}'_{nl}} \right| \right] \right|}_{=1}, \qquad \text{triangle inequality and independence of } \tilde{a}'_{nl} \\
&= \sum_{j=1}^{k_n} \left| \mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}}, e^{ix\tilde{a}_{nj}} - e^{ix\tilde{a}'_{nj}} \right) \right| \\
&= \sum_{j=1}^{k_n} \left| \mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}}, e^{ix\tilde{a}_{nj}} \right) - \underbrace{\mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}}, e^{ix\tilde{a}'_{nj}} \right)}_{=0,\text{by independence}} \right| \\
&= \sum_{j=1}^{k_n} \left| \mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}}, e^{ix\tilde{a}_{nj}} \right) \right| \\
&= \sum_{j=1}^{k_n} \left| \mathbb{E}\left[\mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}}, e^{ix\tilde{a}_{nj}} \,\middle|\, N \right) \right] + \underbrace{\mathbb{C}\mathrm{ov}\left(\mathbb{E}\left[\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}} \,\middle|\, N \right], \mathbb{E}\left[e^{ix\tilde{a}_{nj}} \,\middle|\, N \right] \right)}_{=0,\text{ by the same reasoning as in step 1 part (iii)}} \right|
\end{aligned}
$$

Now again, we apply the Lemma A.3. This time let $U = \bigcup_{l=1}^{j-1} B_n^l$ and $V = B_n^j$ such that $d(U,V) \ge n^\eta$ and $\max\{|U|,|V|\} = \max\left\{ \left| \left(\bigcup_{l=1}^{j-1} B_n^l \right) \right|, \left| B_n^j \right| \right\} = (j-1)|B_n^1| =: M$. Together with $\left| e^{ix} \right| = 1$ for all $x \in \mathbb{R}$ and condition **(M2)**, we deduce for some constant $C > 0$

$$
\begin{aligned}
\left| \mathbb{E}\left[\mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}}, e^{ix\tilde{a}_{nj}} \,\middle|\, N \right) \right] \right| &\le \mathbb{E}\left[4\alpha_{MM}(n^\eta) \right] = 4\alpha_{MM}(n^\eta) \\
&\le C n^{-\epsilon\eta} M^2 = C n^{-\epsilon\eta} (j-1)^2 |B_n^1|^2.
\end{aligned}
$$

Hence, by using $\sum_{j=1}^{k_n}(j-1)^2 \le k_n^3 = O(n^{6-6\gamma})$ and $|B_n^1| = O(n^{2\gamma})$

$$\sum_{j=1}^{k_n}\left|\mathbb{E}\left[\mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ix\tilde{a}_{nl}}, e^{ix\tilde{a}_{nj}}\middle| N\right)\right]\right| \le Cn^{-\epsilon\eta}k_n^3|B_n^1|^2 = O\left(n^{6-2\gamma-\epsilon\eta}\right),$$

which converges to 0 since $\frac{6}{2+\epsilon} < \eta < \gamma < 1$.
Note that from step 1, we already have $\epsilon > \frac{2+4\gamma}{\eta}$. Choosing $\gamma = \frac{3}{4}$, $\eta = \frac{1}{2}$ and $\epsilon = 11$ shows that the set of possible rates for γ, η and ϵ is non-empty.

Step 3

We will show the central limit theorem for $\tilde{a}_n'$ by applying the theorem of Lindeberg-Feller. For notational ease, we consider

$$I_{ni} := \int_{B_n^i}\int_{B_n^i} w_n(h+s_1-s_2)\mathbb{1}\left\{\frac{X_{s_1}}{a_m} \in A\right\}\mathbb{1}\left\{\frac{X_{s_2}}{a_m} \in B\right\} N^{(2)}(ds_1, ds_2).$$

Then we may infer for $\delta > 0$, according to condition **(M2)**, and some $C_\delta > 0$ chosen accordingly

$$\begin{aligned}
\mathbb{E}\left[\left|\sqrt{k_n}\tilde{a}_{n1}'\right|^{2+\delta}\right] &= \mathbb{E}\left[\left|\sqrt{k_n}\sqrt{\frac{m_n\lambda_n^2}{|S_n|}}\frac{1}{\nu^2}\left(I_{n1} - \mathbb{E}\left[I_{n1}\right]\right)\right|^{2+\delta}\right] \\
&= \mathbb{E}\left[\left|\sqrt{\frac{m_n\lambda_n^2}{|D_n^1|}}\frac{|B_n^1|}{m_n}\left(\hat{\tau}_{AB,m}(h:B_n^1) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h:B_n^1)\right]\right)\right|^{2+\delta}\right] \\
&= \mathbb{E}\left[\left|\sqrt{\frac{|B_n^1|^2\lambda_n^2}{m_n|D_n^1|}}\left(\hat{\tau}_{AB,m}(h:B_n^1) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h:B_n^1)\right]\right)\right|^{2+\delta}\right] \\
&< \mathbb{E}\left[\left|\sqrt{\frac{|B_n^1|\lambda_n^2}{m_n}}\left(\hat{\tau}_{AB,m}(h:B_n^1) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h:B_n^1)\right]\right)\right|^{2+\delta}\right], \qquad \text{since } \frac{|B_n^1|}{|D_n^1|} < 1 \\
&\le C_\delta, \qquad \text{by condition } (\mathbf{M2}).
\end{aligned}$$

This gives by independence and the same argument as in the beginning of Step 1 part (iii)

$$\sigma_n^2 := \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{k_n}\tilde{a}_{ni}'\right) = \sum_{i=1}^{k_n}\mathbb{V}\mathrm{ar}\left(\tilde{a}_{ni}'\right) = k_n\mathbb{V}\mathrm{ar}\left(\tilde{a}_{n1}'\right) = k_n\mathbb{V}\mathrm{ar}\left(\tilde{a}_{n1}\right) \xrightarrow{n\to\infty} \frac{1}{\nu^2}\tau_{AB}(h)\int_{\mathbb{R}} w(y)^2dy =: \sigma^2.$$

Furthermore, we observe for $\delta > 0$

$$\frac{\sum_{i=1}^{k_n}\mathbb{E}\left[|\tilde{a}_{ni}'|^{2+\delta}\right]}{\sigma_n^{2+\delta}} = \frac{k_nk_n^{-1-\frac{\delta}{2}}\mathbb{E}\left[\left|\sqrt{k_n}\tilde{a}_{ni}'\right|^{2+\delta}\right]}{\sigma_n^{2+\delta}} \le C_\delta\frac{k_n^{-\delta/2}}{\sigma_n^{2+\delta}} \xrightarrow{n\to\infty} 0$$

and for the sake of completeness, we know that $\mathbb{E}\left[\tilde{a}_{ni}\right]=0$ and $\mathbb{V}\mathrm{ar}\left(\tilde{a}_{ni}\right)$ are finite. This means Lyapunov's condition holds and we may infer the central limit theorem

$$\frac{\tilde{a}'_n}{\sigma_n} \xrightarrow{d} N(0,1).$$

Then Step 2 and Slutsky's theorem imply that $\frac{\tilde{a}_n}{\sigma_n} \xrightarrow{d} N(0,1)$ or, equivalently, $\tilde{a}_n \xrightarrow{d} N(0,\sigma^2)$. Since convergence in probability implies weak convergence, Step 1 yields $\tilde{A}_n \xrightarrow{d} N(0,\sigma^2)$ which is, by inserting the definitions of $\tilde{A}_n$ and $\hat{\tau}_{AB,m}(h)$, equivalent to

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\tau}_{AB,m}(h)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right]\right) \xrightarrow{n\to\infty} N(0,\sigma^2)$$

and obviously, if $\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right]-\tau_{AB}(h)=o\left(\sqrt{\frac{m_n}{|S_n|\lambda_n^2}}\right)$ we have

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\tau}_{AB,m}(h)-\tau_{AB}(h)\right)-\underbrace{\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right]-\tau_{AB}(h)\right)}_{\to 0} \xrightarrow{d} N(0,\sigma^2), \quad n\to\infty$$

$$\Leftrightarrow \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\tau}_{AB,m}(h)-\tau_{AB}(h)\right) \xrightarrow{d} N(0,\sigma^2), \quad n\to\infty,$$

which proves Proposition 3.5. □

3.5 A CLT for the empirical extremogram

The following theorem is one of the key results of this work. The result is given in Cho et al. [7] but lacks some essential parts of the proof which we provide in full detail.

Theorem 3.6. *Let $\{X_s : s\in\mathbb{R}^2\}$ be a strictly stationary regularly varying random process with index $\beta>0$ satisfying the LUNC. Assume N is a homogeneous 2-dimensional Poisson process with parameter ν and independent of X. Consider a sequence of compact and convex sets $S_n\subset\mathbb{R}^2$ satisfying $|S_n|\to\infty$ as $n\to\infty$. Assume conditions **(M1)** and **(M2)** hold. Then we obtain for every finite set of non-zero lags $H=\{h_1,\dots,h_p\}$ in $\mathbb{R}^2$*

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h)-\rho_{AB,m}(h)\right)_{h\in H} \xrightarrow{d} N(0,\Sigma), \qquad n\to\infty,$$

where Σ is specified in the proof of this theorem. If $\rho_{AB,m}(h)-\rho_{AB}(h)=o\left(\sqrt{\frac{m_n}{|S_n|\lambda_n^2}}\right)$ for every $h\in H$, we have

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h)-\rho_{AB}(h)\right)_{h\in H} \xrightarrow{d} N(0,\Sigma), \qquad n\to\infty.$$

Proof. We have to make the following assumption which is not mentioned in Cho et al. [7] but is still necessary in the idea of the proof they present

$$\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right] - \tau_{AB,m}(h) = o\left(\sqrt{\frac{m_n}{|S_n|\lambda_n^2}}\right). \tag{19}$$

Due to Proposition 3.2, that yields $\hat{p}_m(A) \xrightarrow{P} \mu(A)$ as $n \to \infty$, Proposition 3.5, the continuous mapping theorem and Slutsky's theorem, we may infer for every fixed lag $h \in \mathbb{R}^2$ by employing assumption (19)

$$\begin{aligned}
&\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\frac{\hat{\tau}_{AB,m}(h)}{\hat{p}_m(A)} - \frac{\tau_{AB,m}(h)}{\hat{p}_m(A)}\right) \\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \frac{\tau_{AB,m}(h)}{\hat{p}_m(A)}\right) \\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \frac{\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right]}{\hat{p}_m(A)}\right) + \frac{1}{\hat{p}_m(A)}\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right] - \tau_{AB,m}(h)\right) \\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \frac{\mathbb{E}\left[\hat{\tau}_{AB,m}(h)\right]}{\hat{p}_m(A)}\right) + o_P(1) \xrightarrow{d} N\left(0, \frac{\sigma^2}{\mu(A)^2}\right), \qquad n \to \infty.
\end{aligned} \tag{20}$$

We can compute

$$\begin{aligned}
\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \frac{\tau_{AB,m}(h)}{\hat{p}_m(A)}\right) &= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \rho_{AB,m}(h) + \rho_{AB,m}(h) - \frac{\tau_{AB,m}(h)}{\hat{p}_m(A)}\right) \\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \rho_{AB,m}(h)\right) + \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\frac{\tau_{AB,m}(h)}{p_m(A)} - \frac{\tau_{AB,m}(h)}{\hat{p}_m(A)}\right) \\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \rho_{AB,m}(h)\right) + \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{\tau_{AB,m}(h)\left(\hat{p}_m(A) - p_m(A)\right)}{p_m(A)\hat{p}_m(A)} \\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h) - \rho_{AB,m}(h)\right) + \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{\tau_{AB,m}(h)\left(\hat{p}_m(A) - p_m(A)\right)}{\left(\hat{p}_m(A) - \mu(A)\right)p_m(A) + \mu(A)p_m(A)}.
\end{aligned} \tag{21}$$

We consider the denominator for the second summand first and find by Slutsky's theorem

$$\underbrace{\left(\hat{p}_m(A) - \mu(A)\right)}_{=o_P(1)}\underbrace{p_m(A)}_{\to\mu(A)} + \underbrace{\mu(A)p_m(A)}_{\to\mu(A)^2} \xrightarrow{P} \mu(A)^2, \qquad n \to \infty.$$

Turning to the numerator, including the rate, we obtain for every $\epsilon > 0$ by an application of Chebyshev's inequality (note that by Proposition 3.2 $\mathbb{E}\left[\hat{p}_m(A)\right] = p_m(A)$, which gives the numerator a mean of zero) and the result from Proposition 3.3 (ii)

$$\mathbb{P}\left(\left|\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\tau_{AB,m}(h)\left(\hat{p}_m(A)-p_m(A)\right)\right|>\epsilon\right)\leq\underbrace{\frac{\lambda_n^2}{\epsilon^2}}_{\to 0}\underbrace{\tau_{AB,m}(h)^2}_{\to\tau_{AB}(h)^2}\underbrace{\frac{|S_n|}{m_n}\mathbb{V}\mathrm{ar}\left(\hat{p}_m(A)\right)}_{\to\frac{\mu(A)}{\nu}+\int_{\mathbb{R}^2}\tau_{AA}(y)dy<\infty}\xrightarrow{n\to\infty}0. \quad (22)$$

We conclude that the numerator converges to 0 in probability while the denominator has a limit different from 0 in probability. An application of Slutsky's theorem then gives that (21) equals

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h)-\rho_{AB,m}(h)\right)+o_P(1)$$

proving, in combination with (20), the central limit theorem for $\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h)-\rho_{AB,m}(h)\right)$.

Next, we prove the multivariate normality of all lags in H by application of the Cramér-Wold device. For an arbitrary $z=(z_1,\ldots,z_p)^\top\in\mathbb{R}^p$ we consider k_n i.i.d. copies of

$$T_{nj}(z):=\sum_{i=1}^p z_i\sqrt{\frac{|B_n^1|^2\lambda_n^2}{|S_n|m_n}}\left(\hat{\tau}_{AB,m}(h_i:B_n^j)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i:B_n^j)\right]\right),\quad j=1,\ldots,k_n$$

and denote the independent copies by $\tilde{T}_{nj}$. Then for the finite set of lags $H=\{h_1,\ldots,h_p\}$ we compute

$$\begin{aligned}
&\mathbb{E}\left[\left|\sqrt{k_n}\tilde{T}_{n1}(z)\right|^{2+\delta}\right]\\
&=\mathbb{E}\left[\left|\sqrt{k_n}\sqrt{\frac{|B_n^1|^2\lambda_n^2}{|S_n|m_n}}\sum_{i=1}^p z_i\left(\hat{\tau}_{AB,m}(h_i:B_n^1)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i:B_n^1)\right]\right)\right|^{2+\delta}\right]\\
&\leq\max_{i\in\{1,\ldots,p\}}\{z_i\}\mathbb{E}\left[\left|\sqrt{\frac{|B_n^1|^2\lambda_n^2}{|D_n^1|m_n}}\sum_{i=1}^p\left(\hat{\tau}_{AB,m}(h_i:B_n^1)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i:B_n^1)\right]\right)\right|^{2+\delta}\right]\\
&\leq p^{2+\delta}\max_{i\in\{1,\ldots,p\}}\{z_i\}\mathbb{E}\left[\left|\sqrt{\frac{|B_n^1|^2\lambda_n^2}{|D_n^1|m_n}}\max_{i\in\{1,\ldots,p\}}\left|\hat{\tau}_{AB,m}(h_i:B_n^1)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i:B_n^1)\right]\right|\right|^{2+\delta}\right]\\
&<p^{2+\delta}\max_{i\in\{1,\ldots,p\}}\{z_i\}\mathbb{E}\left[\left|\sqrt{\frac{|B_n^1|\lambda_n^2}{m_n}}\max_{i\in\{1,\ldots,p\}}\left|\hat{\tau}_{AB,m}(h_i:B_n^1)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i:B_n^1)\right]\right|\right|^{2+\delta}\right],\quad\text{since }\frac{|B_n^1|}{|D_n^1|}<1.
\end{aligned}\quad (23)$$

Since condition (**M2**) holds and the set H is finite, (23) is bounded by $C_\delta^\star := p^{2+\delta} \cdot \max_{i \in \{1,\dots,p\}} \{z_i\} \cdot C_\delta$, for some appropriate and finite positive C_δ. Not that $\mathbb{E}\left[\tilde{T}_{nj}(z)\right] = \mathbb{E}\left[T_{nj}(z)\right] = 0$ for all $j \in \{1,\dots,k_n\}$. Then we observe by and independence of the $\tilde{T}_{nj}$ and Remark 3.4,

$$\sum_{j=1}^{k_n} \mathbb{V}\mathrm{ar}\left(\tilde{T}_{n1}(z)\right) = k_n \mathbb{E}\left[\tilde{T}_{n1}(z)^2\right] = k_n \mathbb{V}\mathrm{ar}\left(\tilde{T}_{n1}(z)\right)$$

$$= k_n \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{p} z_i \sqrt{\frac{|B_n^1|^2 \lambda_n^2}{|S_n| m_n}} \left(\hat{\tau}_{AB,m}(h_i : B_n^1) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h_i : B_n^1)\right]\right)\right)$$

$$= \sum_{i=1}^{p} z_i^2 k_n \frac{|B_n^1|^2 \lambda_n^2}{|S_n| m_n} \mathbb{V}\mathrm{ar}\left(\hat{\tau}_{AB,m}(h_i : B_n^1)\right) + \sum_{i=1}^{p} \sum_{\substack{j=1 \\ j \neq i}}^{p} z_i z_j k_n \frac{|B_n^1|^2 \lambda_n^2}{|S_n| m_n} \mathbb{C}\mathrm{ov}\left(\hat{\tau}_{AB,m}(h_i : B_n^1), \hat{\tau}_{AB,m}(h_j : B_n^1)\right)$$

$$= \sum_{i=1}^{p} z_i^2 \frac{|B_n^1|^2 \lambda_n^2}{|D_n^1| m_n} \mathbb{V}\mathrm{ar}\left(\hat{\tau}_{AB,m}\left(h_i : B_n^1\right)\right) + \sum_{i=1}^{p} \sum_{\substack{j=1 \\ j \neq i}}^{p} z_i z_j \frac{|B_n^1|^2 \lambda_n^2}{|D_n^1| m_n} \mathbb{C}\mathrm{ov}\left(\hat{\tau}_{AB,m}\left(h_i : B_n^1\right), \hat{\tau}_{AB,m}\left(h_j : B_n^1\right)\right)$$

$$= z' V_n z \xrightarrow{n \to \infty} z' V z < \infty,$$

where V_n and V are $p \times p$-matrices with the following entries

$$V_n := \left(\frac{|B_n^1|^2 \lambda_n^2}{|D_n^1| m_n} \mathbb{C}\mathrm{ov}\left(\hat{\tau}_{AB,m}\left(h_i : B_n^1\right), \hat{\tau}_{AB,m}\left(h_j : B_n^1\right)\right)\right)_{\substack{i=1,\dots,p \\ j=1,\dots,p}}$$

$$V := \left(\frac{1}{\nu^2}\left[\tau_{AB}(h_i) \mathbb{1}\{h_i = h_j\} + \tau_{A\cap B, A\cap B}(h_i) \mathbb{1}\{h_i = -h_j\}\right] \int_{\mathbb{R}^2} w(y)^2 dy\right)_{\substack{i=1,\dots,p \\ j=1,\dots,p}}.$$

Thus we obtain asymptotic normality of $\sum_{j=1}^{k_n} \tilde{T}_{nj}(z)$ by showing Lyapunov's condition, namely,

$$\frac{\sum_{i=j}^{k_n} \mathbb{E}\left[\left|\tilde{T}_{nj}(z)\right|^{2+\delta}\right]}{(z' V_n z)^{2+\delta}} = \frac{k_n k_n^{-1-\frac{\delta}{2}} \mathbb{E}\left[\left|\sqrt{k_n} \tilde{T}_{n1}(z)\right|^{2+\delta}\right]}{(z' V_n z)^{2+\delta}} < \frac{C_\delta^\star k_n^{-\frac{\delta}{2}}}{(z' V_n z)^{2+\delta}} \xrightarrow{n \to \infty} 0.$$

Hence, we obtain $\sum_{j=1}^{k_n} \tilde{T}_{nj}(z) \xrightarrow{d} N(0, z' V z)$, as $n \to \infty$.
Denote the characteristic functions of $\sum_{j=1}^{k_n} T_{nj}(z)$ and $\sum_{j=1}^{k_n} \tilde{T}_{nj}(z)$ by $\phi(\cdot)$ and $\tilde{\phi}(\cdot)$, respectively. We use the same technique as in the proof of Proposition 3.5 Step 2, and find

$$\left|\phi(x) - \tilde{\phi}(x)\right| \leq \sum_{j=1}^{k_n} \left|\mathbb{E}\left[\mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ixT_{nl}(z)}, e^{ixT_{nj}(z)} \middle| N\right)\right] + \mathbb{C}\mathrm{ov}\left(\mathbb{E}\left[\prod_{l=1}^{j-1} e^{ixT_{nl}(z)} \middle| N\right], \mathbb{E}\left[e^{ixT_{nj}(z)} \middle| N\right]\right)\right|. \tag{24}$$

Since $\left(\bigcup_{l=1}^{j-1} B_n^l\right) \cap B_n^j = \emptyset$, we obtain that $\{T_{nl} : l = 1, \ldots, j-1\}$ and T_{nj} are independent, given N. Hence, the second term vanishes. The first term can be bounded by Lemma A.3. For this virtue, recall that, given N, $T_{nj}(z)$ and $\sum_{l=1}^{j-1} T_{nl}(z)$ are $U := B_n^j$ and $V := \bigcup_{l=1}^{j-1} B_n^l$ measurable, respectively. This gives $d(U, V) \geq n^{\eta}$. Let $M := (j-1)|B_n^1| = \max\left\{|B_n^j|, \left|\bigcup_{l=1}^{j-1} B_n^l\right|\right\}$. Together with $\left|\prod_{l=1}^{j-1} e^{ixT_{nl}(z)}\right| = \left|e^{ixT_{nj}(z)}\right| = 1$ and condition (**M2**), we get for an appropriate $C > 0$

$$\begin{aligned}\mathbb{E}\left[\mathbb{C}\mathrm{ov}\left(\prod_{l=1}^{j-1} e^{ixT_{nl}(z)}, e^{ixT_{nj}(z)}\middle| N\right)\right] &\leq 4\mathbb{E}\left[\alpha_{MM}(n^{\eta})\right] = 4\alpha_{MM}(n^{\eta}) \leq Cn^{-\epsilon\eta}M^2 \\ &= Cn^{-\epsilon\eta}(j-1)^2|B_n^1|^2\end{aligned}$$

Then by $k_n = \frac{|S_n|}{|D_n^1|} = O\left(n^{2(1-\gamma)}\right)$ and $B_n^1 = O(n^{2\gamma})$, we obtain the following uppper bound for (24)

$$\left|\phi(x) - \tilde{\phi}(x)\right| \leq C\sum_{j=1}^{k_n} n^{-\epsilon\eta}(j-1)^2|B_n^1|^2 \leq C{k_n}^3 n^{-\epsilon\eta}|B_n^1|^2 = O(n^{6-2\gamma-\epsilon\eta}),$$

which converges to 0 as already seen in Step 1 of the proof of Proposition 3.5. Hence, we conclude

$$\sum_{j=1}^{k_n} T_{nj}(z) \xrightarrow{d} N(0, z'Vz)$$

for any arbitrary $z \in \mathbb{R}^p$. Applying the Cramér-Wold-device gives

$$\sqrt{\frac{|B_n^1|^2\lambda_n^2}{|S_n|m_n}} \sum_{j=1}^{k_n} \begin{pmatrix} \hat{\tau}_{AB,m}(h_1 : B_n^j) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1 : B_n^j)\right] \\ \vdots \\ \hat{\tau}_{AB,m}(h_p : B_n^j) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h_p : B_n^j)\right] \end{pmatrix} \xrightarrow{d} N(0, V), \quad n \to \infty$$

$$\Leftrightarrow \sum_{j=1}^{k_n} \begin{pmatrix} a_{nj}(h_1) - \mathbb{E}\left[a_{nj}(h_1)\right] \\ \vdots \\ a_{nj}(h_p) - \mathbb{E}\left[a_{nj}(h_p)\right] \end{pmatrix} \xrightarrow{d} N(0, V), \quad n \to \infty$$

$$\Leftrightarrow \begin{pmatrix} \tilde{a}_n(h_1) \\ \vdots \\ \tilde{a}_n(h_p) \end{pmatrix} \xrightarrow{d} N(0, V), \quad n \to \infty$$

$$\Rightarrow \begin{pmatrix} \tilde{A}_n(h_1) \\ \vdots \\ \tilde{A}_n(h_p) \end{pmatrix} = \underbrace{\begin{pmatrix} \tilde{A}_n(h_1) \\ \vdots \\ \tilde{A}_n(h_p) \end{pmatrix} - \begin{pmatrix} \tilde{a}_n(h_1) \\ \vdots \\ \tilde{a}_n(h_p) \end{pmatrix}}_{\xrightarrow{d} 0, \text{ since } \tilde{a}_n(h_i) - \tilde{A}_n(h_i) = o_P(1),\ \forall i \in \{1, \ldots, p\}} + \begin{pmatrix} \tilde{a}_n(h_1) \\ \vdots \\ \tilde{a}_n(h_p) \end{pmatrix} \xrightarrow{d} N(0, V), \quad n \to \infty$$

$$\Leftrightarrow \sqrt{\frac{|S_n|\lambda_n^2}{m_n}} \begin{pmatrix} \hat{\tau}_{AB,m}(h_1) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \\ \vdots \\ \hat{\tau}_{AB,m}(h_p) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h_p)\right] \end{pmatrix} \xrightarrow{d} N(0, V), \quad n \to \infty. \tag{25}$$

Then, we can observe, using similar steps as in the sketch of the proof of Theorem 4.2 in Buhl and Klüppelberg [6],

$$
\begin{aligned}
&\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{\rho}_{AB,m}(h_i)-\rho_{AB,m}(h_i)\right)=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\frac{\hat{\tau}_{AB,m}(h_i)}{\hat{p}_m(A)}-\frac{\tau_{AB,m}(h_i)}{p_m(A)}\right)\\
&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{1}{\hat{p}_m(A)p_m(A)}\left(\hat{\tau}_{AB,m}(h_i)p_m(A)-\tau_{AB,m}(h_i)\hat{p}_m(A)\right)\\
&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{p_m(A)/\hat{p}_m(A)}{p_m(A)^2}\Big[\big(\hat{\tau}_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\big)p_m(A)\\
&\qquad\qquad+\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]p_m(A)-\tau_{AB,m}(h_i)\hat{p}_m(A)\Big]\\
&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{p_m(A)/\hat{p}_m(A)}{p_m(A)^2}\left[\left(\hat{\tau}_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\right)\mu(A)\frac{p_m(A)}{\mu(A)}\right.\\
&\qquad\left.-\left(\hat{p}_m(A)-p_m(A)\right)\tau_{AB}(h_i)\frac{\tau_{AB,m}(h_i)}{\tau_{AB}(h_i)}-\left(\tau_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\right)p_m(A)\right]\\
&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{p_m(A)/\hat{p}_m(A)}{p_m(A)\mu(A)}\left[\left(\hat{\tau}_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\right)\mu(A)\right]\\
&\qquad-\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{p_m(A)/\hat{p}_m(A)}{p_m(A)^2}\left[\left(\hat{p}_m(A)-p_m(A)\right)\tau_{AB}(h_i)\underbrace{\frac{\tau_{AB,m}(h_i)}{\tau_{AB}(h_i)}}_{=1+o(1)}\right]\\
&\qquad-\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{p_m(A)/\hat{p}_m(A)}{p_m(A)^2}\left[\left(\tau_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\right)p_m(A)\right]\\
&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{1+o_P(1)}{\mu(A)^2}\left[\left(\hat{\tau}_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\right)\mu(A)\right]\\
&\qquad-\frac{1+o_P(1)}{\mu(A)^2}\left[\underbrace{\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\hat{p}_m(A)-p_m(A)\right)}_{=o_P(1)\text{ compare equation (22)}}\tau_{AB}(h_i)\right]\\
&\qquad-\frac{1+o_P(1)}{\mu(A)^2}\left[\underbrace{\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\tau_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\right)}_{=o(1)\text{ by assumption (19)}}p_m(A)\right]\\
&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{1+o_P(1)}{\mu(A)}\left(\hat{\tau}_{AB,m}(h_i)-\mathbb{E}\left[\hat{\tau}_{AB,m}(h_i)\right]\right)+o_P(1).
\end{aligned}
$$

This then gives by (25)

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}} \begin{pmatrix} \hat{\rho}_{AB,m}(h_1) - \rho_{AB,m}(h_1) \\ \vdots \\ \hat{\rho}_{AB,m}(h_p) - \rho_{AB,m}(h_p) \end{pmatrix} = \sqrt{\frac{|S_n|\lambda_n^2}{m_n}} \frac{1 + o_P(1)}{\mu(A)} \begin{pmatrix} \hat{\tau}_{AB,m}(h_1) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h_1)\right] \\ \vdots \\ \hat{\tau}_{AB,m}(h_p) - \mathbb{E}\left[\hat{\tau}_{AB,m}(h_p)\right] \end{pmatrix} + o_P(1)$$

$$\xrightarrow{d} N\left(0, \mu(A)^{-2}V\right), \quad n \to \infty$$

such that we can define the asymptotic covariance matrix Σ by $\Sigma \coloneqq \mu(A)^{-2}V$. $\square$

4 The extremogram for the Brown-Resnick process

4.1 The Brown-Resnick process

In this section we consider the *strictly stationary isotropic Brown-Resnick process* $\{\eta_s : s \in \mathbb{R}^2\}$, introduced in Brown and Resnick [3] in a time setting and in a spatial setting in Kabluchko et al. [12], with

$$\eta_s = \bigvee_{j=1}^{\infty} \xi_j^{-1} \exp(W_j(s) - \delta(|s|), \qquad s \in \mathbb{R}^2 \tag{26}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$, $(\xi_j)_{j\in\mathbb{N}}$ are points of a unit rate Poisson process on $[0,\infty)$, the *dependence function* $\delta(\cdot)$ is nonnegative and conditionally negative definite. $\{W_j(s) : s \in \mathbb{R}^2\}_{j\in\mathbb{N}}$ are independent samples of the Gaussian process $\{W(s) : s \in \mathbb{R}^2\}$ with stationary increments $W(0) = 0,\ \mathbb{E}[W(s)] = 0$ and covariance function

$$\mathbb{C}\mathrm{ov}\,(W(s_1), W(s_2)) = \delta(|s_1|) + \delta(|s_2|) - \delta(|s_1 - s_2|).$$

All finite dimensional distributions are multivariate extreme value distributions with standard unit Fréchet margins. This implies, applying a Taylor expansion, for the sequence a_n according to Definition 2.1

$$\frac{1}{n} \sim \mathbb{P}(\eta_0 > a_n) = 1 - e^{-\frac{1}{a_n}} = \frac{1}{a_n} + O\left(\frac{1}{a_n^2}\right) \Rightarrow a_n \sim n.$$

In our case, let the dependence function be given by

$$\delta(u) = 2\theta_1 u^{\alpha_1}, \qquad u \geq 0. \tag{27}$$

According to equation (2.6) in Davis et al. [10], the bivariate distribution function $F(x_1, x_2)$ of (η_0, η_s) is given for $x_1, x_2 > 0$ by

$$F(x_1, x_2) = \exp\left[-\frac{1}{x_1}\Phi\left(\frac{\log(x_2/x_1)}{\sqrt{2\delta(|s|)}} + \sqrt{\frac{1}{2}\delta(|s|)}\right) - \frac{1}{x_2}\Phi\left(\frac{\log(x_1/x_2)}{\sqrt{2\delta(|s|)}} + \sqrt{\frac{1}{2}\delta(|s|)}\right)\right]. \tag{28}$$

The following Lemma, coming from equation (3.1) in Davis et al. [10], gives the explicit form of the tail dependence coefficient of a Brown-Resnick process.

Lemma 4.1. *Let $\{\eta_s : s \in \mathbb{R}^2\}$ be the strictly stationary Brown-Resnick process in $\mathbb{R}^2$ as defined in* (26) *with dependence function given by* (27). *Then the tail dependence coefficient is given by*

$$\chi(u) = 2\left(1 - \Phi\left(\sqrt{\frac{1}{2}\delta(u)}\right)\right) = 2\left(1 - \Phi\left(\sqrt{\theta_1 u^{\alpha_1}}\right)\right), \qquad u \in \mathbb{R}, \tag{29}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

Thus, applying a Taylor expansion, we show that for $x > 0$

$$\begin{aligned}\mathbb{P}(\eta_0 > x, \eta_s > x) &= 2[1 - F(x)] - 1 + F(x,x) \\ &= 1 - 2\exp\left(-\frac{1}{x}\right) + \exp\left(-\frac{2}{x}\Phi\left(\sqrt{\frac{1}{2}\delta(|s|)}\right)\right) \\ &= 1 - 2\exp\left(-\frac{1}{x}\right) + \exp\left(\frac{1}{x}\left(\chi(|s|) - 2\right)\right) \\ &= 1 - 2 + \frac{2}{x} + O\left(\frac{1}{x^2}\right) + 1 + \frac{\chi(|s|) - 2}{x} + O\left(\frac{1}{x^2}\right) \\ &= \frac{\chi(|s|)}{x} + O\left(\frac{1}{x^2}\right), \qquad \text{as } x \to \infty, \end{aligned} \tag{30}$$

where, by stationarity, $F(\cdot)$ denotes the univariate distribution function of η_s for every $s \in \mathbb{R}^2$.

4.2 The central limit theorem for the tail dependence coefficient in the Brown-Resnick case

Since we want to apply Theorem 3.6 to the tail dependence coefficient of the Brown-Resnick process, we need to show that η satisfies the corresponding regularity conditions. We will see that this is the case if the rates from **(M1)** and **(M2)** satisfy the conditions

$$\sup_n \frac{\lambda_n^2 n^{2\gamma}}{m_n} < \infty \tag{31}$$

$$\sup_n \frac{m_n}{\lambda_n^2 n^{2\gamma}} < \infty \tag{32}$$

$$\text{and } \sup_n m_n^3 \lambda_n^6 n^{2\gamma} < \infty. \tag{33}$$

Lemma 4.2. *Let the rates from **(M1)** and **(M2)** satisfy* (31)*,* (32) *and* (33)*. Then the Brown-Resnick process* $\left\{\eta_s : s \in \mathbb{R}^2\right\}$ *as defined in* (26) *satisfies the regularity conditions **M(1)**, **M(2)** and the LUNC when considering* $A = B = (1, \infty)$.

Proof. In the following proof, we denote appropriate constants by C, where C might vary between the lines and different equations but is always a positive and finite constant.
We start with showing (1). Then, since $a_m \sim m_n$, we may infer for $\theta_1, \alpha_1 > 0$ and every $\epsilon > 0$

$$\begin{aligned}&\lim_{k\to\infty} \limsup_{n\to\infty} m_n \int_{B[k,r_n]} \mathbb{P}\left(|X_y| > \epsilon a_m, |X_0| > \epsilon a_m\right) dy \\ &\quad= \lim_{k\to\infty} \limsup_{n\to\infty} m_n \int_{B[k,r_n]} \frac{1}{\epsilon m_n}\chi(|y|) + O\left(\frac{1}{m_n^2}\right) dy, \qquad \text{by (30)} \\ &\quad= \lim_{k\to\infty} \limsup_{n\to\infty} \int_{B[k,r_n]} \frac{2}{\epsilon}\left(1 - \Phi\left(\sqrt{\theta_1 |y|^{\alpha_1}}\right)\right) + O\left(\frac{1}{m_n}\right) dy, \qquad \text{by (29)}\end{aligned}$$

$$= \lim_{k\to\infty} \limsup_{n\to\infty} \left(\int_k^{r_n} \int_0^{2\pi} \frac{2R}{\epsilon} \left(1 - \Phi\left(\sqrt{\theta_1 R^{\alpha_1}}\right)\right) d\theta dR + O\left(\frac{r_n^2}{m_n}\right)\right), \quad \text{transformation to polar coordinates}$$

$$= \lim_{k\to\infty} \limsup_{n\to\infty} \left(\int_k^{r_n} \frac{4\pi R}{\epsilon} \left(1 - \Phi\left(\sqrt{\theta_1 R^{\alpha_1}}\right)\right) dR + O\left(\frac{r_n^2}{m_n}\right)\right)$$

$$\leq \lim_{k\to\infty} \limsup_{n\to\infty} \left(\frac{4\pi}{\epsilon} \int_k^{r_n} R e^{-\theta_1 R^{\alpha_1}/2} dR + O\left(\frac{r_n^2}{m_n}\right)\right), \quad \text{since } 1 - \Phi(x) \leq e^{-x^2/2}$$

$$\leq \lim_{k\to\infty} \limsup_{n\to\infty} \left(Ck^2 e^{-\theta_1 k^{\alpha_1}/2} + O\left(\frac{r_n^2}{m_n}\right)\right), \quad \text{by Lemma } A.5$$

$$= \lim_{k\to\infty} Ck^2 e^{-\theta_1 k^{\alpha_1}/2} + \underbrace{\limsup_{n\to\infty} O\left(\frac{r_n^2}{m_n}\right)}_{=0, \text{ since } r_n^2 = o(m_n) \text{ by } \textbf{(M1)}} = 0.$$

We proceed with showing (2) and obtain for every $k, l \in \mathbb{N}$ the more general result

$$\lim_{n\to\infty} \int_{\mathbb{R}^2 \setminus B[0,r_n)} m_n \alpha_{k,l}(|y|) dy$$

$$= \lim_{n\to\infty} m_n \int_{r_n}^{\infty} \int_0^{2\pi} R \alpha_{k,l}(R) d\theta dR, \quad \text{transformation to polar coordinates}$$

$$= \lim_{n\to\infty} m_n \int_{r_n}^{\infty} 2\pi R \alpha_{k,l}(R) dR$$

$$\leq \lim_{n\to\infty} 8kl\pi m_n \int_{r_n}^{\infty} R e^{-\theta_1 R^{\alpha_1}/2} dR, \quad \text{by Lemma } A.4$$

$$\leq \lim_{n\to\infty} C m_n r_n^2 e^{-\theta_1 r_n^{\alpha_1}/2}, \quad \text{by Lemma } A.5$$

$$= 0, \quad \text{since } \theta_1, \alpha_1 > 0 \text{ and } r_n \xrightarrow{n\to\infty} \infty.$$

In the next step, we prove (3) and observe

$$\int_{\mathbb{R}^2} \tau_{AA}(y) dy = \int_{\mathbb{R}^2} \lim_{n\to\infty} m_n \mathbb{P}\left(X_0 > a_m, X_y > a_m\right) dy$$

$$= \int_{\mathbb{R}^2} \lim_{n\to\infty} \left(\chi(|y|) + \underbrace{O\left(\frac{1}{m_n}\right)}_{\to 0} \right) dy, \quad \text{by (30)}$$

$$= \int_{\mathbb{R}^2} 2\left(1 - \Phi\left(\sqrt{\theta_1 |y|^{\alpha_1}}\right)\right) dy, \quad \text{by (29)}$$

$$= \int_0^{\infty} \int_0^{2\pi} 2R\left(1 - \Phi\left(\sqrt{\theta_1 R^{\alpha_1}}\right)\right) d\theta dR, \quad \text{transformation to polar coordinates}$$

$$\le 4\pi \int_0^\infty Re^{-\theta_1 R^{\alpha_1}/2} dR, \qquad \text{since } 1-\Phi(x) \le e^{-x^2/2}$$

$$\le 4\pi + 4\pi \int_1^\infty Re^{-\theta_1 R^{\alpha_1}/2} dR$$

$$\le 4\pi + Ce^{-\theta/2} < \infty, \qquad \text{by Lemma A.5.}$$

Note that in our case $A = B = (1, \infty)$. Hence, (5) and (3) are equivalent.
We continue with the proof of (6) and find for all $k, l \in \mathbb{N}$ by the same techniques as before

$$\begin{aligned}
\int_{\mathbb{R}^2} \alpha_{k,l}(|y|)dy &= \int_0^\infty \int_0^{2\pi} R\alpha_{k,l}(R) d\theta dR, && \text{transformation to polar coordinates} \\
&\le 2\pi \int_0^\infty klRe^{-\theta_1 R^{\alpha_1}/2} dR, && \text{by Lemma A.4} \\
&\le 2kl\pi + 2kl\pi \int_1^\infty Re^{-\theta_1 R^{\alpha_1}/2} dR \\
&\le 2kl\pi + Ce^{-\theta_1/2} < \infty && \text{by Lemma A.5.}
\end{aligned} \tag{34}$$

Turning to (7), we calculate for every $h \in \mathbb{R}^2$ by Lemma A.4

$$\sup_l \frac{\alpha_{l,l}(|h|)}{l^2} \le \sup_l 4e^{-\theta_1 |h|^{\alpha_1}/2} = 4e^{-\theta_1 |h|^{\alpha_1}/2} = O(|h|^{-\epsilon}), \qquad \text{for every } \epsilon > 0,$$

which is even stronger than (7) since this holds for arbitrary $\epsilon > 0$.
In the following, we show that the Brown-Resnick process satisfies the LUNC from Definition 2.5. For this purpose, we consider $\eta_s = U_s^1 \vee U_s^2$, where $U_s^1 := \xi_1^{-1} Y_s^1$, $U_s^2 := \bigvee_{j \ge 2} \xi{-1}_j Y_s^j$ and $Y_s^j := \exp(W_j(s) - \delta(|s|))$, where $\vee$ denotes the maximum operator. First, we show that for arbitrary $\delta > 0$

$$\left\{ |U_s^1 \vee U_s^2 - U_0^1 \vee U_0^2| > \delta a_n \right\} \subset \left\{ |U_s^1 - U_0^1| > \delta a_n \right\} \cup \left\{ |U_s^2 - U_0^2| > \delta a_n \right\}. \tag{35}$$

This can be verified by considering the following cases

$$|U_s^1 \vee U_s^2 - U_0^1 \vee U_0^2| > \delta a_n \Rightarrow \begin{cases} |U_s^1 - U_0^1| > \delta a_n, & \text{if } U_s^1 \ge U_s^2 \text{ and } U_0^1 \ge U_0^2 \\ |U_s^1 - U_0^1| > \delta a_n, & \text{if } U_s^1 \ge U_s^2,\ U_0^1 < U_0^2 \text{ and } U_s^1 \ge U_0^2 \\ |U_s^2 - U_0^2| > \delta a_n, & \text{if } U_s^1 \ge U_s^2,\ U_0^1 < U_0^2 \text{ and } U_s^1 < U_0^2 \\ |U_s^2 - U_0^2| > \delta a_n, & \text{if } U_s^1 < U_s^2 \text{ and } U_0^1 \le U_0^2 \\ |U_s^2 - U_0^2| > \delta a_n, & \text{if } U_s^1 < U_s^2,\ U_0^1 > U_0^2 \text{ and } U_s^2 \ge U_0^1 \\ |U_s^1 - U_0^1| > \delta a_n, & \text{if } U_s^1 < U_s^2,\ U_0^1 > U_0^2 \text{ and } U_s^2 < U_0^2. \end{cases}$$

Hence, we obtain for all $\delta > 0$, $\delta' > 0$

$$
\begin{aligned}
n\mathbb{P}\left(\sup_{|s|<\delta'} |\eta_s - \eta_0| > a_n\delta\right) &= n\mathbb{P}\left(\sup_{|s|<\delta'} |U_s^1 \vee U_s^2 - U_0^1 \vee U_0^2| > a_n\delta\right) \\
&\le n\mathbb{P}\left(\left\{\sup_{|s|<\delta'} |U_s^1 - U_0^1| > a_n\delta\right\} \cup \left\{\sup_{|s|<\delta'} |U_s^2 - U_0^2| > a_n\delta\right\}\right), \qquad \text{by (35)} \\
&\le n\mathbb{P}\left(\sup_{|s|<\delta'} |U_s^1 - U_0^1| > a_n\delta\right) + n\mathbb{P}\left(\sup_{|s|<\delta'} |U_s^2 - U_0^2| > a_n\delta\right) =: A_1 + A_2.
\end{aligned}
$$

Note that Y_s has continuous sample paths, since it is a continuous function of a continuous Gaussian process. Therefore, every path of Y_s is bounded on a compact set such that $\mathbb{E}\left[\sup_{|s|<\delta'} |Y_s|\right] < \infty$. We define $Z := \sup_{|s|<\delta'} |Y_s^1 - Y_0^1|$ and obtain

$$
\begin{aligned}
A_1 &= n\mathbb{P}\left(\sup_{|s|<\delta'} |U_s^1 - U_0^1| > a_n\delta\right) \\
&= n\mathbb{P}\left(\xi_1 < \frac{Z}{a_n\delta}\right) \\
&= n\mathbb{E}\left[\mathbb{1}\left\{\xi_1 < \frac{Z}{a_n\delta}\right\}\right] \\
&= n\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}\left\{\xi_1 < \frac{Z}{a_n\delta}\right\}\middle| Z\right]\right] \\
&= n\mathbb{E}\left[\mathbb{P}\left(\xi_1 < \frac{Z}{a_n\delta}\middle| Z\right)\right] \\
&= n\int_0^\infty 1 - e^{-\frac{z}{a_n\delta}} F(dz), \qquad \text{since } \xi_1 \sim \text{Exp}(1) \text{ as the first time point of a Poisson process} \\
&= n\int_0^\infty \frac{z}{a_n\delta} + O\left(\frac{1}{a_n^2}\right) F(dz), \qquad \text{applying a Taylor expansion,} \qquad (36)
\end{aligned}
$$

where $F(\cdot)$ denotes the distribution function of Z. In particular, the continuity of Y_s implies for every $\delta' > 0$ that $\mathbb{E}\,[Z] = \mathbb{E}\left[\sup_{|s|<\delta'} |Y_s - Y_0|\right] < \infty$. Since $a_n \sim n$, we have

$$
A_1 = \frac{n}{a_n}\int_0^\infty \frac{z}{\delta} F(dz) + O\left(\frac{1}{a_n^2}\right) \xrightarrow{n\to\infty} \frac{1}{\delta}\mathbb{E}[Z] \xrightarrow{\delta\to\infty} 0.
$$

Turning to A_2, we follow the arguments of Remark 3.6 in Davis and Mikosch [8] and denote the distribution function of $\sup_{|s|<\delta'} |Y_s^j|$ by $G(\cdot)$ such that

$$
\begin{aligned}
A_2 &= n\mathbb{P}\left(\sup_{|s|<\delta'} \bigvee_{j=2}^{\infty} \xi_j^{-1}|Y_s^j - Y_0^j| > a_n\delta\right) \\
&\le n\mathbb{P}\left(\sup_{|s|<\delta'} \bigvee_{j=2}^{\infty} 2\xi_j^{-1}|Y_s^j| > a_n\delta\right) \\
&= n\mathbb{P}\left(\bigcup_{j=2}^{\infty} \left\{\sup_{|s|<\delta'} 2\xi_j^{-1}|Y_s^j| > a_n\delta\right\}\right) \\
&\le n\sum_{j=2}^{\infty} \mathbb{P}\left(2\sup_{|s|<\delta'} |Y_s^j| > \xi_j\delta a_n\right) \\
&= n\int_0^{\infty} \sum_{j=2}^{\infty} \mathbb{P}\left(\xi_j < \frac{2y}{\delta a_n}\right) G(dy), \qquad \text{by the law of total probability and Fubini} \\
&= n\int_0^{\infty} \sum_{j=1}^{\infty} \mathbb{P}\left(\xi_j < \frac{2y}{\delta a_n}\right) - \mathbb{P}\left(\xi_1 < \frac{2y}{\delta a_n}\right) G(dy) \\
&= n\int_0^{\infty} \frac{2y}{\delta a_n} - \left(1 - e^{-\frac{2y}{\delta a_n}}\right) G(dy).
\end{aligned}
$$

The last equality holds since $\xi_1 \sim$Exp(1) and $\sum_{j=1}^{\infty} \mathbb{P}\left(\xi_j < \frac{2y}{\delta a_n}\right) = \mathbb{E}\left[N\left(0, \frac{2y}{\delta a_n}\right)\right] = \frac{2y}{\delta a_n}$, where $N[0,t) = \sum_{j=1}^{\infty} \mathbb{1}\{\xi_j < t\}$, $t > 0$, is a homogeneous unit-rate Poisson point process on $\mathbb{R}$.
Define $f_n(y) \coloneqq n\left(\frac{2y}{\delta a_n} - \left(1 - e^{-\frac{2y}{\delta a_n}}\right)\right)$ and note that since $a_n \sim n$ as $n \to \infty$ and $e^{-\frac{2y}{\gamma a_n}} \le 1$, we obtain $f_n(y) \le \frac{2yn}{\delta a_n} \le Cy$ for some appropriate constant $C > 0$. Hence, again due to $\mathbb{E}\left[\sup_{|s|<\delta'} |Y_s^j|\right] < \infty$ which implies that $\int CyG(dy) < \infty$, we may apply dominated convergence and calculate via a Taylor expansion

$$
\lim_{n\to\infty} A_2 = \lim_{n\to\infty} n\int_0^{\infty} \frac{2y}{\delta a_n} - \left(1 - e^{-\frac{2y}{\delta a_n}}\right) G(dy) = \int_0^{\infty} \lim_{n\to\infty} O\left(\frac{ny^2}{a_n^2}\right) G(dy) = 0, \quad \text{since } a_n \sim n.
$$

Altogether, we showed $\lim_{\delta\to\infty} \limsup_{n\to\infty} n\mathbb{P}\left(\sup_{|s|<\delta'} |\eta_s - \eta_0| > a_n\delta\right) = 0$, for all $\delta' > 0$, which implies the LUNC for the process $\{\eta_s : s \in \mathbb{R}^2\}$.

Next, we check condition (4) for $\delta = 1$ by showing that $\sup_n \frac{|B_n|^{\frac{3}{2}}\lambda_n^3}{m_n^{\frac{3}{2}}}\mathbb{E}\left[\hat{\tau}_{AB,m}(h:B_n)^3\right] < \infty$ for every $h \in \mathbb{R}^2$ when $|B_n| = O(n^{2\gamma})$. This means, similar to the proof of Proposition 3.3 ii), we have to compute

$$\frac{1}{\nu^6}\frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}}\lambda_n^3}\int_{B_n^6} w\left(\frac{h+s_1-s_2}{\lambda_n}\right) w\left(\frac{h+s_3-s_4}{\lambda_n}\right) w\left(\frac{h+s_5-s_6}{\lambda_n}\right)$$
$$\times\,\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A, \frac{X_{s_2}}{a_m}\in B, \frac{X_{s_3}}{a_m}\in A, \frac{X_{s_4}}{a_m}\in B, \frac{X_{s_5}}{a_m}\in A, \frac{X_{s_6}}{a_m}\in B\right)\mathbb{E}\left[N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)N^{(2)}(ds_5,ds_6)\right]. \tag{37}$$

Thus, we consider terms that are integrated with respect to

$$\mathbb{E}\left[N^{(2)}(ds_1,ds_2)N^{(2)}(ds_3,ds_4)N^{(2)}(ds_5,ds_6)\right] = \sum_{k=2}^{6}\nu^k \sum_{I_k^j\in\bar{I}_k} I_k^j,$$

where

$$\bar{I}_k := \{ds_1ds_2ds_3ds_4ds_5ds_6 : s_1\notin ds_2, s_3\notin ds_4, s_5\notin ds_6 \text{ and } |\{s_i\in ds_j : i,j=1,\dots,6\}| = 6-k\}.$$

Instead of showing the result for every single integrand, we show it for representative terms. By the same arguments, it can easily be checked that the remaining terms give analogous results.

i) We start with the integral with respect to $\nu^2 ds_1ds_2\mathbb{1}\{s_1\in ds_3\}\mathbb{1}\{s_2\in ds_4\}\mathbb{1}\{s_2\in ds_5\}\mathbb{1}\{s_1\in ds_6\}$. Then the integral (37) reads

$$\frac{1}{\nu^4}\frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}}\lambda_n^3}\int_{B_n^2} w\left(\frac{h+s_1-s_2}{\lambda_n}\right)^2 w\left(\frac{h+s_2-s_1}{\lambda_n}\right)\underbrace{\mathbb{P}\left(\frac{X_{s_1}}{a_m}\in A\cap B, \frac{X_{s_2}}{a_m}\in A\cap B\right)}_{=\frac{1}{m_n}\tau_{A\cap BA\cap B,m}(s_2-s_1)} ds_1ds_2$$

and we substitute with $y = \frac{h+s_1-s_2}{\lambda_n}$ and $u = s_2$ such that above term equals

$$= \frac{1}{\nu^4}\frac{m_n^{\frac{1}{2}}}{|B_n|^{\frac{3}{2}}\lambda_n}\int_{\frac{h+(B_n-B_n)}{\lambda_n}}\int_{B_n\cap(B_n+h-\lambda_n y)} w(y)^2 w\left(\frac{2h}{\lambda_n}-y\right)\tau_{A\cap BA\cap B,m}(h-\lambda_n y)du dy$$
$$\le \frac{1}{\nu^4}\frac{m_n^{\frac{1}{2}}}{|B_n|^{\frac{1}{2}}\lambda_n}\int_{\frac{h+(B_n-B_n)}{\lambda_n}} w(y)^2 w\left(\frac{2h}{\lambda_n}-y\right)\tau_{A\cap BA\cap B,m}(h-\lambda_n y)dy,$$

where we used that the integrand does not depend on u and $\frac{|B_n\cap(B_n+h-\lambda_n y)|}{|B_n|}\le 1$. Since $w(\cdot)$ is a bounded probability density on $\mathbb{R}^2$, and $\tau_{A\cap BA\cap B,m}(h-\lambda_n y)\le p_m(A\cap B)$, we apply dominated convergence and obtain by $\lambda_n \xrightarrow{n\to\infty} 0$, condition (32) and Proposition 3.2

$$\le \frac{1}{\nu^4}\int_{\mathbb{R}^2}\underbrace{\frac{m_n^{\frac{1}{2}}}{|B_n|^{\frac{1}{2}}\lambda_n}}_{\le C<\infty} w(y)^2 \underbrace{w\left(\frac{2h}{\lambda_n}-y\right)}_{\to 0}\underbrace{p_m(A\cap B)}_{\to\mu(A\cap B)<\infty} dy \xrightarrow{n\to\infty} 0<\infty.$$

ii) We proceed with the integral with respect to $\nu^2 ds_1 ds_2 \mathbb{1}\{s_1 \in ds_3\} \mathbb{1}\{s_2 \in ds_4\} \mathbb{1}\{s_1 \in ds_5\} \mathbb{1}\{s_2 \in ds_6\}$ such that the integral (37) becomes

$$\frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \frac{1}{\nu^4} \int_{B_n^2} w\left(\frac{h+s_1-s_2}{\lambda_n}\right)^3 \underbrace{\mathbb{P}\left(\frac{X_{s_1}}{a_m} \in A, \frac{X_{s_2}}{a_m} \in B\right)}_{=\frac{1}{m_n}\tau_{AB,m}(s_2-s_1)} ds_1 ds_2$$

and again by substituting with $y = \frac{h+s_1-s_2}{\lambda_n}$ and $u = s_2$, we have by boundedness of $w(\cdot)$, Proposition 3.1 and condition (32)

$$\limsup_{n\to\infty} \frac{m_n^{\frac{1}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n} \frac{1}{\nu^4} \int_{\frac{h+(B_n-B_n)}{\lambda_n}} \int_{B_n \cap (B_n + h - \lambda_n y)} w(y)^3 \tau_{AB,m}(h - \lambda_n y) du dy$$

$$\leq \limsup_{n\to\infty} \frac{1}{\nu^4} \int_{\mathbb{R}^2} \underbrace{\frac{m_n^{\frac{1}{2}}}{|B_n|^{\frac{1}{2}} \lambda_n}}_{\leq C < \infty} \underbrace{\tau_{AB,m}(h - \lambda_n y)}_{\to \tau_{AB}(h) < \infty} w(y)^3 dy < \infty,$$

where we used dominated convergence in the last step.

iii) Next, we consider the integral (37) with respect to $\nu^3 ds_1 ds_2 \mathbb{1}\{s_1 \in ds_3\} ds_4 \mathbb{1}\{s_1 \in ds_5\} \mathbb{1}\{s_2 \in ds_6\}$ and obtain

$$\frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \frac{1}{\nu^3} \int_{B_n^3} w\left(\frac{h+s_1-s_2}{\lambda_n}\right)^2 w\left(\frac{h+s_1-s_4}{\lambda_n}\right) \underbrace{\mathbb{P}\left(\frac{X_{s_1}}{a_m} \in A, \frac{X_{s_2}}{a_m} \in B, \frac{X_{s_1}}{a_m} \in A, \frac{X_{s_4}}{a_m} \in B\right)}_{=\frac{1}{m_n}\tau_m^*(0, s_2-s_1, 0, s_4-s_1)} ds_1 ds_2 ds_4$$

$$= \frac{m_n^{\frac{1}{2}} \lambda_n}{|B_n|^{\frac{3}{2}}} \frac{1}{\nu^3} \int_{\left(\frac{h+(B_n-B_n)}{\lambda_n}\right)^2} \int_{B_n \cap (B_n - \lambda_n y + h)} w(x)^2 w(y) \tau_m^*(0, h - \lambda_n x, 0, h - \lambda_n y) du dx dy,$$

where we substituted $x = \frac{h+s_1-s_2}{\lambda_n}$, $y = \frac{h+s_1-s_4}{\lambda_n}$ and $u = s_4$ and $\tau_m^*(\cdot,\cdot,\cdot,\cdot)$ is defined in part ii) in the proof of Proposition 3.3. We observe that the integrand is independent of u and $\tau_m^*(0, h-\lambda_n x, 0, h-\lambda_n y) \leq p_m(A)$. Thus, we may bound the above term by

$$\leq \frac{1}{\nu^3} \frac{m_n^{\frac{1}{2}} \lambda_n}{|B_n|^{\frac{1}{2}}} p_m(A) \int_{\left(\frac{h+(B_n-B_n)}{\lambda_n}\right)^2} w(x)^2 w(y) dx dy$$

$$\leq \frac{1}{\nu^3} \underbrace{\frac{m_n^{\frac{1}{2}} \lambda_n}{|B_n|^{\frac{1}{2}}}}_{\to 0 \text{ by } \textbf{(M2)}} \underbrace{p_m(A)}_{\to \mu(A) < \infty} \underbrace{\int_{\mathbb{R}^2} w(x)^2 dx}_{< \infty} \underbrace{\int_{\mathbb{R}^2} w(y) dy}_{=1} \xrightarrow{n\to\infty} 0.$$

iv) We continue with the integral (37) with respect to $\nu^3 ds_1 ds_2 \mathbb{1}\{s_2 \in ds_3\}\mathbb{1}\{s_1 \in ds_4\} ds_5 \mathbb{1}\{s_1 \in ds_6\}$ and find

$$\frac{1}{\nu^3}\frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}}\lambda_n^3}\int_{B_n^3} w\left(\frac{h+s_1-s_2}{\lambda_n}\right) w\left(\frac{h+s_2-s_1}{\lambda_n}\right) w\left(\frac{h+s_5-s_1}{\lambda_n}\right)$$
$$\times \mathbb{P}\left(\frac{X_{s_1}}{a_m} \in A\cap B, \frac{X_{s_2}}{a_m} \in A\cap B, \frac{X_{s_5}}{a_m} \in A\right) ds_1 ds_2 ds_5$$

$$\leq \frac{1}{\nu^3}\frac{m_n^{\frac{1}{2}}}{|B_n|^{\frac{3}{2}}\lambda_n^3}\int_{B_n^3} w\left(\frac{h+s_1-s_2}{\lambda_n}\right) w\left(\frac{h+s_2-s_1}{\lambda_n}\right) w\left(\frac{h+s_5-s_1}{\lambda_n}\right) \underbrace{\tau_{A\cap BA\cap B,m}(s_2-s_1)}_{\leq p_m(A)} ds_1 ds_2 ds_5$$

$$\leq \frac{1}{\nu^3}\frac{m_n^{\frac{1}{2}}\lambda_n}{|B_n|^{\frac{3}{2}}} p_m(A)\int_{\left(\frac{h+(B_n-B_n)}{\lambda_n}\right)^2}\int_{B_n\cap(B_n-\lambda_n x+h)} w(x)\, w\left(\frac{2h}{\lambda_n}-x\right) w(y)\, du dx dy,$$

where we substituted $x = \frac{h+s_1-s_2}{\lambda_n}$, $y = \frac{h+s_5-s_1}{\lambda_n}$ and $u = s_2$. Then the above term is bounded by

$$\leq \frac{1}{\nu^3}\frac{m_n^{\frac{1}{2}}\lambda_n}{|B_n|^{\frac{1}{2}}} p_m(A)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(x)\, w\left(\frac{2h}{\lambda_n}-x\right) w(y)\, dx dy$$

$$= \frac{1}{\nu^3}\frac{m_n^{\frac{1}{2}}\lambda_n}{|B_n|^{\frac{1}{2}}} p_m(A)\int_{\mathbb{R}^2} w(x)\, w\left(\frac{2h}{\lambda_n}-x\right) dx$$

$$= \frac{1}{\nu^3}\underbrace{\frac{m_n^{\frac{1}{2}}\lambda_n}{|B_n|^{\frac{1}{2}}}}_{\to 0,\text{ by }\mathbf{(M2)}}\underbrace{p_m(A)}_{\to\mu(A)}\int_{\mathbb{R}^2} w(x)\underbrace{w\left(\frac{2h}{\lambda_n}-x\right)}_{\to 0} dx \xrightarrow{n\to\infty} 0, \qquad \text{by dominated convergence.}$$

v) In the following, we investigate the integral (37) with respect to $\nu^4 ds_1 ds_2 \mathbb{1}\{s_1 \in ds_3\} ds_4 ds_5 \mathbb{1}\{s_2 \in ds_6\}$ and compute

$$\frac{1}{\nu^2}\frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}}\lambda_n^3}\int_{B_n^4} w\left(\frac{h+s_1-s_2}{\lambda_n}\right) w\left(\frac{h+s_1-s_4}{\lambda_n}\right) w\left(\frac{h+s_5-s_2}{\lambda_n}\right)$$
$$\times \underbrace{\mathbb{P}\left(\frac{X_{s_1}}{a_m} \in A, \frac{X_{s_2}}{a_m} \in B, \frac{X_{s_4}}{a_m} \in B, \frac{X_{s_5}}{a_m} \in A\right)}_{\leq \frac{1}{m_n}p_m(A)} ds_1 ds_2 ds_4 ds_5$$

$$= \frac{1}{\nu^2}\frac{m_n^{\frac{1}{2}}\lambda_n^3}{|B_n|^{\frac{3}{2}}} p_m(A)\int_{\left(\frac{h+(B_n-B_n)}{\lambda_n}\right)^3}\int_{B_n\cap(B_n-\lambda_n y_2+h)} w(y_1)\, w(y_2)\, w(y_3)\, du dy_1 dy_2 dy_3,$$

where we substituted $y_1 = \frac{h+s_1-s_2}{\lambda_n}$, $y_2 = \frac{h+s_1-s_4}{\lambda_n}$, $y_3 = \frac{h+s_5-s_2}{\lambda_n}$ and $u = s_4$. Hence, the integrand does not depend on u and is bounded by

$$\leq \frac{1}{\nu^2} \frac{m_n^{\frac{1}{2}} \lambda_n^3}{|B_n|^{\frac{1}{2}}} p_m(A) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} w(y_1)\, w(y_2)\, w(y_3)\, dy_1 dy_2 dy_3$$

$$\leq \frac{1}{\nu^2} \underbrace{\frac{m_n^{\frac{1}{2}} \lambda_n^3}{|B_n|^{\frac{1}{2}}}}_{\to 0,\text{ by } \mathbf{(M2)}} \underbrace{p_m(A)}_{\to \mu(A) < \infty} \xrightarrow{n \to \infty} 0.$$

vi) Next, we integrate (37) with respect to $\nu^4 ds_1 ds_2 \mathbb{1}\{s_2 \in ds_3\} \mathbb{1}\{s_1 \in ds_4\} ds_5 ds_6$. Then the supremum of (37) over n becomes

$$\sup_n \frac{1}{\nu^2} \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \int_{B_n^4} w\left(\frac{h + s_1 - s_2}{\lambda_n}\right) w\left(\frac{h + s_2 - s_1}{\lambda_n}\right) w\left(\frac{h + s_5 - s_6}{\lambda_n}\right)$$
$$\times \mathbb{P}\left(\frac{X_{s_1}}{a_m} \in A \cap B, \frac{X_{s_2}}{a_m} \in A \cap B, \frac{X_{s_5}}{a_m} \in A, \frac{X_{s_6}}{a_m} \in B\right) ds_1 ds_2 ds_5 ds_6.$$

We make use of $A = B = (1, \infty)$ and stationarity. This gives

$$= \sup_n \frac{1}{\nu^2} \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \int_{B_n^4} w\left(\frac{h + s_1 - s_2}{\lambda_n}\right) w\left(\frac{h + s_2 - s_1}{\lambda_n}\right) w\left(\frac{h + s_5 - s_6}{\lambda_n}\right)$$
$$\times \mathbb{P}\left(X_0 > a_m, X_{s_2 - s_1} > a_m, X_{s_5 - s_1} > a_m, X_{s_6 - s_1} > a_m\right) ds_1 ds_2 ds_5 ds_6$$

$$= \sup_n \frac{1}{\nu^2} \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \int_{(B_n - B_n)^3} \int_{B_n \cap (B_n - v_1) \cap (B_n - v_2) \cap (B_n - v_3)} w\left(\frac{h - v_1}{\lambda_n}\right) w\left(\frac{h + v_1}{\lambda_n}\right) w\left(\frac{h - v_3}{\lambda_n}\right)$$
$$\times \mathbb{P}\left(X_0 > a_m, X_{v_1} > a_m, X_{v_2} > a_m, X_{v_3} > a_m\right) du\, dv_1 dv_2 dv_3,$$

where we substituted $v_1 = s_2 - s_1$, $v_2 = s_5 - s_1$, $v_3 = s_6 - s_1$ and $u = s_1$. We use the bivariate distribution function (30) of the Brown-Resnick process and note that the integral does not depend on u such that the above term is bounded by

$$\leq \sup_n \frac{1}{\nu^2} \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{1}{2}} \lambda_n^3} \int_{(B_n - B_n)^3} w\left(\frac{h - v_1}{\lambda_n}\right) w\left(\frac{h + v_1}{\lambda_n}\right) w\left(\frac{h - v_3}{\lambda_n}\right)$$
$$\times \mathbb{P}\left(X_0 > a_m, X_{v_1} > a_m, X_{v_2} > a_m, X_{v_3} > a_m\right) dv_1 dv_2 dv_3$$

$$= \sup_n \frac{1}{\nu^2} \frac{m_n^{\frac{3}{2}} \lambda_n}{|B_n|^{\frac{1}{2}}} \int_{\left(\frac{h - (B_n - B_n)}{\lambda_n}\right)^2} \int_{(B_n - B_n) \cap ((B_n - B_n) + \lambda_n y_2 - h)} w(y_1)\, w\left(\frac{2h}{\lambda_n} - y_1\right) w(y_2)$$
$$\times \mathbb{P}\left(X_0 > a_m, X_{h - \lambda_n y_1} > a_m, X_z > a_m, X_{z + h - \lambda_n y_2} > a_m\right) dz\, dy_1 dy_2$$

$$\leq \sup_n \frac{1}{\nu^2} \frac{m_n^{\frac{3}{2}} \lambda_n}{|B_n|^{\frac{1}{2}}} \underbrace{\int_{\mathbb{R}^2} w(y_1) w\left(\frac{2h}{\lambda_n} - y_1\right) dy_1}_{<\infty \text{, proved in the proof of Proposition 3.3}} \underbrace{\int_{\mathbb{R}^2} w(y_2)\, dy_2}_{=1}$$
$$\times \int_{(B_n - B_n) \cap ((B_n - B_n) + \lambda_n y_2 - h)} \mathbb{P}(X_0 > a_m, X_z > a_m)\, dz$$
$$\leq \sup_n \frac{C}{\nu^2} \frac{m_n^{\frac{3}{2}} \lambda_n}{|B_n|^{\frac{1}{2}}} \int_{(B_n - B_n) \cap ((B_n - B_n) + \lambda_n y_2 - h)} \left(\frac{\chi(z)}{a_m} + O\left(\frac{1}{a_m^2}\right)\right) dz, \qquad \text{by (30)}$$
$$\leq \sup_n C \left(\frac{m_n^{\frac{3}{2}} \lambda_n}{|B_n|^{\frac{1}{2}} a_m} \underbrace{\int_{\mathbb{R}^2} e^{-\theta_1 |z|^{\alpha_1}/2} dz}_{<\infty \text{, proved when showing (3)}} + \frac{|(B_n - B_n) \cap ((B_n - B_n) + \lambda_n y_2 - h)|}{|B_n|^{\frac{1}{2}}} \frac{m_n^{\frac{3}{2}} \lambda_n}{a_m^2} \right)$$
$$\sim C \left(\underbrace{\sup_n \frac{m_n^{\frac{1}{2}} \lambda_n}{|B_n|^{\frac{1}{2}}}}_{<\infty \text{, by } \mathbf{(M2)}} + \underbrace{\sup_n \frac{|B_n|^{\frac{1}{2}} \lambda_n}{m_n^{\frac{1}{2}}}}_{<\infty \text{, by (31)}} \right) < \infty, \qquad \text{since } a_m \sim m_n \text{ and } O(|B_n - B_n|) = O(|B_n|).$$

Here we applied an integral substitution with $y_1 = \frac{h - v_1}{\lambda_n}$, $y_2 = \frac{h - (v_3 - v_2)}{\lambda_n}$ and $u = v_2$ for the first equality.

vii) In the next case, we integrate (37) with respect to $\nu^5 ds_1 ds_2 ds_3 ds_4 ds_5 \mathbb{1}\{s_1 \in ds_6\}$ and obtain for its supremum over n with $A = B = (1, \infty)$

$$\sup_n \frac{1}{\nu} \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \int_{B_n^5} w\left(\frac{h + s_1 - s_2}{\lambda_n}\right) w\left(\frac{h + s_3 - s_4}{\lambda_n}\right) w\left(\frac{h + s_5 - s_1}{\lambda_n}\right)$$
$$\times \mathbb{P}(X_{s_1} > a_m, X_{s_2} > a_m, X_{s_3} > a_m, X_{s_4} > a_m, X_{s_5} > a_m)\, ds_1 ds_2 ds_3 ds_4 ds_5$$
$$= \sup_n \frac{1}{\nu} \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \int_{(B_n - B_n)^4} \int_{B_n \cap (B_n - v_1) \cap (B_n - v_2) \cap (B_n - v_3) \cap (B_n - v_4)} w\left(\frac{h - v_1}{\lambda_n}\right) w\left(\frac{h - (v_3 - v_2)}{\lambda_n}\right) w\left(\frac{h - v_4}{\lambda_n}\right)$$
$$\times \underbrace{\mathbb{P}(X_0 > a_m, X_{v_1} > a_m, X_{v_2} > a_m, X_{v_3} > a_m, X_{v_4} > a_m)}_{\leq \mathbb{P}(X_0 > a_m, X_{v_2} > a_m)} du dv_1 dv_2 dv_3 dv_4,$$

where we substituted $v_1 = s_2 - s_1$, $v_1 = s_2 - s_1$, $v_1 = s_2 - s_1$, $v_1 = s_2 - s_1$ and $u = s_1$. The integral does not depend on u and $|B_n \cap (B_n - v_1) \cap (B_n - v_2) \cap (B_n - v_3) \cap (B_n - v_4)|/|B_n| \leq 1$ such that we may bound the above term by

$$\leq \sup_n \frac{1}{\nu} \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{1}{2}} \lambda_n^3} \int_{(B_n - B_n)^4} w\left(\frac{h - v_1}{\lambda_n}\right) w\left(\frac{h - (v_3 - v_2)}{\lambda_n}\right) w\left(\frac{h - v_4}{\lambda_n}\right) \mathbb{P}(X_0 > a_m, X_{v_2} > a_m)\, dv_1 dv_2 dv_3 dv_4$$

$$= \sup_n \frac{1}{\nu} \frac{m_n^{\frac{3}{2}} \lambda_n^3}{|B_n|^{\frac{1}{2}}} \int_{\left(\frac{h-(B_n-B_n)}{\lambda_n}\right)^2} \int_{\frac{h-(B_n-B_n)+(B_n-B_n)}{\lambda_n}} \int_{(B_n-B_n)\cap((B_n-B_n)+\lambda_n y_2 - h)} w(y_1)\, w(y_2)\, w(y_3)$$
$$\times \mathbb{P}\left(X_0 > a_m, X_u > a_m\right) du dy_2 dy_1 dy_3.$$

The equality holds due to substituting $y_1 = \frac{h-v_1}{\lambda_n}$, $y_2 = \frac{h-(v_3-v_2)}{\lambda_n}$, $y_3 = \frac{h-v_4}{\lambda_n}$ and $u = v_2$. Since the integrand is non-negative, the integral can be bounded by

$$= \sup_n \frac{1}{\nu} \frac{m_n^{\frac{3}{2}} \lambda_n^3}{|B_n|^{\frac{1}{2}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{(B_n-B_n)\cap((B_n-B_n)+\lambda_n y_2 - h)} w(y_1)\, w(y_2)\, w(y_3) \underbrace{\mathbb{P}\left(X_0 > a_m, X_u > a_m\right)}_{=\frac{\chi(u)}{a_m}+O\left(\frac{1}{a_m^2}\right),\text{ by (30)}} du dy_1 dy_2 dy_3$$

$$\leq \sup_n \frac{1}{\nu} \frac{m_n^{\frac{3}{2}} \lambda_n^3}{|B_n|^{\frac{1}{2}}} \left(\frac{2}{a_m} \underbrace{\int_{\mathbb{R}^2} e^{-\theta_1 |u|^{\alpha_1}/2} du}_{<\infty} + C \frac{|(B_n - B_n) \cap ((B_n - B_n) + \lambda_n y_2 - h)|}{a_m^2} \right), \qquad \text{by Lemma A.4}$$

$$\sim C \left(\underbrace{\sup_n \frac{m_n^{\frac{1}{2}} \lambda_n^3}{|B_n|^{\frac{1}{2}}}}_{<\infty,\text{ by } \mathbf{(M2)}} + \underbrace{\sup_n \frac{|B_n|^{\frac{1}{2}} \lambda_n}{m_n^{\frac{1}{2}}}}_{<\infty,\text{ by (31)}} \underbrace{\lambda_n^2}_{\to 0} \right) < \infty, \qquad \text{since } a_m \sim m_n \text{ and } O(|B_n - B_n|) = O(|B_n|).$$

vii) Finally, we show that the supremum of (37) is bounded when integrating with respect to $\nu^6 ds_1 ds_2 ds_3 ds_4 ds_5 ds_6$. This term then reads with $A = B = (1, \infty)$

$$I := \frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}} \lambda_n^3} \int_{B_n^6} w\left(\frac{h + s_1 - s_2}{\lambda_n}\right) w\left(\frac{h + s_3 - s_4}{\lambda_n}\right) w\left(\frac{h + s_5 - s_6}{\lambda_n}\right) \times$$
$$\times \mathbb{P}\left(X_{s_1} > a_m, X_{s_2} > a_m, X_{s_3} > a_m, X_{s_4} > a_m, X_{s_5} > a_m, X_{s_6} > a_m\right) ds_1 ds_2 ds_3 ds_4 ds_5 ds_6.$$

We will show

$$\Bigg| I - \frac{m_n^{\frac{3}{2}} \lambda_n^3}{|B_n|^{\frac{3}{2}}} \int_{B_n^6} w_n(h + s_1 - s_2) w_n(h + s_3 - s_4) w_n(h + s_5 - s_6)$$
$$\times \mathbb{P}\left(X_{s_1} > a_m, X_{s_2} > a_m, X_{s_3} > a_m, X_{s_4} > a_m\right) \mathbb{P}\left(X_{s_5} > a_m, X_{s_6} > a_m\right) ds_1 ds_2 ds_3 ds_4 ds_5 ds_6 \Bigg|$$
$$\xrightarrow{n\to\infty} 0. \tag{38}$$

By substituting $v_1 = s_2 - s_1, v_2 = s_3 - s_1, v_3 = s_4 - s_1, v_4 = s_5 - s_1, v_5 = s_6 - s_1$ and $u = s_1$ and employing stationarity, we find that the left side of (38) is bounded by

$$\begin{aligned}
&\frac{m_n^{\frac{3}{2}}}{|B_n|^{\frac{3}{2}}\lambda_n^3} \int_{(B_n-B_n)^5} \int_{B_n\cap(B_n-v_1)\cap(B_n-v_1)\cap(B_n-v_2)\cap(B_n-v_3)\cap(B_n-v_4)\cap(B_n-v_5)} \\
&\qquad w\left(\frac{h-v_1}{\lambda_n}\right) w\left(\frac{h-(v_3-v_2)}{\lambda_n}\right) w\left(\frac{h-(v_5-v_4)}{\lambda_n}\right) \\
&\qquad \times \Big|\mathbb{P}\left(X_0 > a_m, X_{v_1} > a_m, X_{v_2} > a_m, X_{v_3} > a_m, X_{v_4} > a_m, X_{v_5} > a_m\right) \\
&\qquad - \mathbb{P}\left(X_0 > a_m, X_{v_1} > a_m, X_{v_2} > a_m, X_{v_3} > a_m\right) \mathbb{P}\left(X_{v_4} > a_m, X_{v_5} > a_m\right)\Big| du dv_1 dv_2 dv_3 dv_4 dv_5 \\
&\leq \frac{m_n^{\frac{3}{2}}\lambda_n^3}{|B_n|^{\frac{1}{2}}} \int_{\frac{h-(B_n-B_n)}{\lambda_n}} \int_{\left(\frac{h-(B_n-B_n)+(B_n-B_n)}{\lambda_n}\right)^2} \int_{(B_n-B_n)\cap((B_n-B_n)+\lambda_n y_3-h)} \int_{(B_n-B_n)\cap((B_n-B_n)+\lambda_n y_2-h)} \\
&\qquad w(y_1)\, w(y_2)\, w(y_3) \\
&\qquad \times \Big|\mathbb{P}\left(X_0 > a_m, X_{h-\lambda_n y_1} > a_m, X_{u_1+\lambda_n y_2-h} > a_m, X_{u_1} > a_m, X_{u_2+\lambda_n y_3-h} > a_m, X_{u_2} > a_m\right) \\
&\qquad - \mathbb{P}\left(X_0 > a_m, X_{h-\lambda_n y_1} > a_m, X_{u_1+\lambda_n y_2-h} > a_m, X_{u_1} > a_m\right) \mathbb{P}\left(X_{u_2+\lambda_n y_3-h} > a_m, X_{u_2} > a_m\right)\Big| \\
&\qquad du_1 du_2 dy_1 dy_2 dy_3,
\end{aligned} \tag{39}$$

where we substituted $y_1 = \frac{h-v_1}{\lambda_n}, y_2 = \frac{h-(v_3-v_2)}{\lambda_n}, y_3 = \frac{h-(v_5-v_4)}{\lambda_n}, u_1 = v_3$ and $u_2 = v_5$ and exploited the fact that the first integral does not depend on u. By the same arguments that we employed in the proof of Propositon 3.3 for showing convergence of I_1, we find that the absolute value above is bounded by the mixing coefficient $\alpha_{44}(k)$ with

$$\begin{aligned}
k = \min\{&|u_2|, |u_2 - \lambda_n y_3 + h|, |u_2 - h + \lambda_n y_1|, |u_2 - \lambda_n(y_3 - y_1)|, |u_2 - u_1 - \lambda_n y_2 + h|, |u_2 - \lambda_n(y_3 - y_2)|, \\
&|u_2 - u_1|, |u_2 + h - \lambda_n y_3 - u_1|\}
\end{aligned}$$

such that

$$\begin{aligned}
\alpha_{44}(k) \leq\, &\alpha_{44}(|u_2|) + \alpha_{44}(|u_2 - \lambda_n y_3 + h|) + \alpha_{44}(|u_2 - h + \lambda_n y_1|) + \alpha_{44}(|u_2 - \lambda_n(y_3 - y_1)|) \\
&+ \alpha_{44}(|u_2 - u_1 - \lambda_n y_2 + h|) + \alpha_{44}(|u_2 - \lambda_n(y_3 - y_2)|) + \alpha_{44}(|u_2 - u_1|) + \alpha_{44}(|u_2 + h - \lambda_n y_3 - u_1|).
\end{aligned} \tag{40}$$

Then (39) is bounded by the sum of 8 integrals $A_i, i \in \{1, \ldots, 8\}$, where A_i corresponds to (39) with the absolute value replaced by the i-th summand in (40). Recall that, by (34), $\int_{\mathbb{R}^2} \alpha_{kl}(|u|)du < \infty$ for the Brown-Resnick process for every $k, l \in \mathbb{N}$. This gives with $|B_n| = O(n^{2\gamma})$

$$\begin{aligned}
A_1 &\leq \frac{m_n^{\frac{3}{2}}\lambda_n^3}{|B_n|^{\frac{1}{2}}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{(B_n-B_n)\cap((B_n-B_n)+\lambda_n y_2-h)} w(y_1)\, w(y_2)\, w(y_3)\, \alpha_{44}(|u_2|) du_1 du_2 dy_1 dy_2 dy_3 \\
&\leq m_n^{\frac{3}{2}}\lambda_n^3 |B_n|^{\frac{1}{2}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \underbrace{\frac{|(B_n - B_n) \cap ((B_n - B_n) + \lambda_n y_2 - h)|}{|B_n|}}_{\leq C < \infty, \text{ since } |B_n| \sim |B_n - B_n|} w(y_2)\, \alpha_{44}(|u_2|) du_1 dy_2.
\end{aligned}$$

This means that by (34) and (33), we can conclude

$$A_1 \sim C m_n^{\frac{3}{2}} \lambda_n^3 |B_n|^{\frac{1}{2}} < \infty$$

and by analogous substitution techniques as in the proof of Proposition 3.3, when considering I_1, we find $A_j < \infty$, for all $j \in \{1, \ldots, 8\}$. Hence, we have to show that the limit of I is finite which follows from condition (31), Proposition 3.3 i) and iii), since

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{m_n^{\frac{3}{2}} \lambda_n^3}{|B_n|^{\frac{3}{2}}} \int_{B_n^6} w_n(h + s_1 - s_2) w_n(h + s_3 - s_4) w_n(h + s_5 - s_6) \\
&\qquad \times \mathbb{P}\left(X_{s_1} > a_m, X_{s_2} > a_m, X_{s_3} > a_m, X_{s_4} > a_m\right) \mathbb{P}\left(X_{s_5} > a_m, X_{s_6} > a_m\right) ds_1 ds_2 ds_3 ds_4 ds_5 ds_6 \\
&\leq \limsup_{n\to\infty} \underbrace{\frac{|B_n|^{\frac{1}{2}} \lambda_n}{m_n^{\frac{1}{2}}}}_{<\infty} \underbrace{\frac{|B_n| \lambda_n^2}{m_n} \mathbb{E}\left[\hat{\tau}_{AB,m}(h : B_n)^2\right]}_{\to C \leq \infty} \underbrace{\mathbb{E}\left[\hat{\tau}_{AB,m}(h : B_n)\right]}_{\to C \leq \infty} < \infty.
\end{aligned}$$

Altogether, this proves the boundedness of the supremum of (37) over all n.

□

By Lemma 4.2, the Brown-Resnick $\left\{\eta_s : s \in \mathbb{R}^2\right\}$ satisfies the conditions of Theorem 3.6 if $A = B = (1, \infty)$, so that we can derive the following CLT as $n \to \infty$

$$\sqrt{\frac{|S_n| \lambda_n^2}{m_n}} \left(\hat{\rho}_{AB,m}(h) - \rho_{AB,m}(h)\right)_{h \in H} \xrightarrow{d} N(0, \Sigma).$$

We require (31), (32) and (33) whereas in contrast Cho et al. [7] assume the first two conditions to hold in addition to $\log m_n = o(r_n^\gamma)$.
However, as we will see in the following chapter, we are required to impose some further restrictions on the rates as we want to center the empirical extremogram with its theoretical version. For this purpose, we will consider the tail dependence coefficient from Buhl et al. [4] as seen in Definition 2.2.

4.3 The bias corrected empirical extremogram in the Brown-Resnick case

We follow the idea of the bias correction presented in Remark 3.4 of Buhl et al. [4] and consider the tail dependence coefficient denoted by $\chi(h) := \rho_{(1,\infty)(1,\infty)}(h)$ and its pre-asymptotic version $\chi_m(h) := \rho_{(1,\infty)(1,\infty),m}(h)$. We introduce the rate coefficients $\beta_1 > 0$ and $\beta_2 > 0$ for the rates m_n and λ_n such that the conditions imposed by **(M1)**, **(M2)** and (33) read

$$
\begin{aligned}
|S_n| &= O(n^2) && \\
m_n &= n^{\beta_1} = o(n^\gamma) && \Rightarrow \beta_1 < \gamma < 1 \\
\lambda_n &= n^{-\beta_2} \to 0 && \Rightarrow \beta_2 > 0 \\
\lambda_n^2 |S_n| &\to \infty && \Rightarrow \beta_2 < 1 \\
\lambda_n^2 m_n &= n^{\beta_1 - 2\beta_2} \to 0 && \Rightarrow \beta_2 > \frac{1}{2}\beta_1 \\
\sup_n m_n^3 \lambda_n^6 n^{2\gamma} &= \sup_n n^{3\beta_1 - 6\beta_2 + 2\gamma} < \infty && \Rightarrow \beta_2 > \frac{1}{3}\gamma + \frac{1}{2}\beta_1 > \frac{1}{2}\beta_1.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\gamma \in (0,1) \\
&\beta_1 \in (0,\gamma) \\
&\beta_2 \in \left(\frac{1}{3}\gamma + \frac{1}{2}\beta_1, 1\right).
\end{aligned}
$$

If the conditions from Lemma 4.2 hold in addition to Theorem 3.6, we conclude that the pre-asymptotic extremogram in the central limit theorem can be replaced by the theoretical one if

$$
\sqrt{\frac{|S_n|\lambda_n^2}{m_n}} \left(\chi_m(h) - \chi(h)\right) \xrightarrow{n\to\infty} 0
$$

holds for all lags $h \in H$. For the Brown-Resnick process we obtain from Lemma 3.1 in Buhl et al. [4] and the fact that $a_m \sim m_n$

$$
\begin{aligned}
\sqrt{\frac{|S_n|\lambda_n^2}{m_n}} \left(\chi_m(h) - \chi(h)\right) &= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}} \left(\frac{\mathbb{P}\left(X_0 > a_m, X_h > a_m\right)}{\mathbb{P}\left(X_0 > a_m\right)} - \chi(h)\right) \\
&= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}} \frac{1}{2a_m} \left(\chi(h) - 2\right)\left(\chi(h) - 1\right)\left(1 + o(1)\right) \\
&\sim \frac{1}{2}\sqrt{\frac{|S_n|\lambda_n^2}{m_n^3}} \left(\chi(h) - 2\right)\left(\chi(h) - 1\right)\left(1 + o(1)\right) \xrightarrow{n\to\infty} 0,
\end{aligned}
$$

if and only if, also compare Remark A8 in the supplement of Cho et al. [7],

$$
\frac{|S_n|\lambda_n^2}{m_n^3} \sim n^2 n^{-2\beta_2} n^{-3\beta_1} = n^{2-2\beta_2-3\beta_1} \xrightarrow{n\to\infty} 0 \;\Leftrightarrow\; 2 - 2\beta_2 - 3\beta_1 < 0 \;\Leftrightarrow\; \beta_1 > \frac{2}{3}\left(1 - \beta_2\right).
$$

This implies that $\beta_1 \in \left(\frac{2}{3}(1-\beta_2), \gamma\right)$ and, additionally, to ensure existence of such a β_1, we require

$$\frac{2}{3}(1-\beta_2) < \gamma \Leftrightarrow \beta_2 > 1 - \frac{3}{2}\gamma. \tag{41}$$

(I) Therefore, if $\beta_1 \in \left(\frac{2}{3}(1-\beta_2), \gamma\right)$ and (41) is met, i.e. $\beta_2 \in \left(\max\left\{1-\frac{3}{2}\gamma, \frac{1}{3}\gamma + \frac{1}{2}\beta_1\right\}, 1\right)$, we obtain

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\widehat{\chi}_m(h) - \chi(h)\right)_{h\in H} \xrightarrow{d} N(0,\Sigma), \quad n\to\infty.$$

(II) On the other hand, consider $\beta_1 \in \left(0, \frac{2}{3}(1-\beta_2)\right]$, requiring $\beta_2 \in \left(\frac{1}{3}\gamma + \frac{1}{2}\beta_1, 1\right)$, we need a bias correction. For this virtue, recall that by Lemma 3.1 in Buhl et al. [4] for $n \to \infty$

$$\chi_m(h) = (1+o(1))\left(\chi(h) + \frac{1}{2m_n}\underbrace{(\chi(h)-2)(\chi(h)-1)}_{=:v(h)}\right). \tag{42}$$

Therefore, in line with Buhl et al. [4], we introduce the bias corrected empirical extremogram

$$\widehat{\chi}_m(h) - \frac{1}{2m_n}\widehat{v}_m(h)$$

where $\widehat{v}_m(\cdot)$ is the empirical version of $v(\cdot)$. We define the bias corrected estimator as

$$\tilde{\chi}_m(h) := \begin{cases} \widehat{\chi}_m(h) - \frac{1}{2m_n}\widehat{v}_m(h), & \text{if } \beta_1 \in \left(\frac{2}{5}(1-\beta_2), \frac{2}{3}(1-\beta_2)\right] \\ \widehat{\chi}_m(h), & \text{if } \beta_1 \in \left(\frac{2}{3}(1-\beta_2), \gamma\right). \end{cases}$$

We prove asymptotic normality for the bias corrected version of the estimator. In particular, we will see why we have to postulate $\beta_1 > \frac{2}{5}(1-\beta_2)$.

Theorem 4.3. *Let $\{\eta_s : s \in \mathbb{R}^2\}$ be the Brown-Resnick prcoess with dependence function* (27)*. If the rate coefficients β_1 and β_2 satisfy conditions* (31) *and* (32) *as well as $\beta_1 \in \left(\frac{2}{5}(1-\beta_2), \frac{2}{3}(1-\beta_2)\right]$ and $\beta_2 \in \left(\frac{1}{3}\gamma + \frac{1}{2}\beta_1, 1\right)$, then the bias corrected estimator of the tail dependence coefficient satisfies*

$$\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\tilde{\chi}_m(h) - \chi(h)\right)_{h\in H} \xrightarrow{d} N(0,\Sigma), \quad n\to\infty,$$

where Σ is the covariance matrix from Theorem 3.6.

Proof. Following the arguments of Buhl et al. [4], our equation (42) and the definition of the bias corrected extremogram we get

$$\begin{aligned}\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\tilde{\chi}_m(h) - \chi(h)\right) &= \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\widehat{\chi}_m(h) - \chi(h)\right) - \frac{1}{2}\sqrt{\frac{|S_n|\lambda_n^2}{m_n^3}}\widehat{v}_m(h) \\ &\sim \sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\widehat{\chi}_m(h) - \chi_m(h)\right) - \sqrt{\frac{|S_n|\lambda_n^2}{4m_n^3}}\left(\widehat{v}_m(h) - v(h)\right), \quad n\to\infty.\end{aligned}$$

By Theorem 3.6, it suffices to show that $\sqrt{\frac{|S_n|\lambda_n^2}{4m_n^3}}\left(\widehat{v}_m(h)-v(h)\right)\xrightarrow{P}0$.
We define the pre-asymptotic version of $v(\cdot)$ by $v_m(h):=(\chi_m(h)-2)(\chi_m(h)-1)$. This then gives

$$\begin{aligned}\sqrt{\frac{|S_n|\lambda_n^2}{4m_n^3}}\left(\widehat{v}_m(h)-v(h)\right)&=\sqrt{\frac{|S_n|\lambda_n^2}{4m_n^3}}\left(\widehat{v}_m(h)-v_m(h)\right)+\sqrt{\frac{|S_n|\lambda_n^2}{4m_n^3}}\left(v_m(h)-v(h)\right)\\&=:A_1+A_2.\end{aligned}$$

In the next step, we calculate

$$\begin{aligned}\frac{2m_n}{2\chi(h)-3}A_1&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{1}{2\chi(h)-3}\Big(\widehat{v}_m(h)-v_m(h)\Big)\\&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{1}{2\chi(h)-3}\Big(\widehat{\chi}_m(h)^2-3\widehat{\chi}_m(h)-\big(\chi_m(h)^2-3\chi_m(h)\big)\Big)\\&=\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\frac{1}{2\chi(h)-3}\Big(\left(\widehat{\chi}_m(h)-\chi_m(h)\right)\left(\widehat{\chi}_m(h)+\chi_m(h)\right)-3\left(\widehat{\chi}_m(h)-\chi_m(h)\right)\Big)\\&=\underbrace{\sqrt{\frac{|S_n|\lambda_n^2}{m_n}}\left(\widehat{\chi}_m(h)-\chi_m(h)\right)}_{\xrightarrow{d}N(0,\sigma^2)}\frac{\widehat{\chi}_m(h)+\chi_m(h)-3}{2\chi(h)-3}\end{aligned}$$

and with $A=B=(1,\infty)$, we obtain from Proposition 3.3, Slutsky's theorem and the continuous mapping theorem

$$\begin{aligned}\widehat{\chi}_m(h)+\frac{\tau_{AB,m}(h)}{\widehat{p}_m(A)}&=\underbrace{\frac{1}{\widehat{p}_m(A)}}_{\xrightarrow{P}\frac{1}{\mu(A)}}\Bigg(\underbrace{\widehat{\tau}_{AB,m}(h)-\mathbb{E}\left[\widehat{\tau}_{AB,m}(h)\right]}_{\xrightarrow{P}0,\text{ as }n\to\infty\text{ by Proposition 3.5}}+\underbrace{\mathbb{E}\left[\widehat{\tau}_{AB,m}(h)\right]}_{\xrightarrow{n\to\infty}\tau_{AB}(h)}+\underbrace{\tau_{AB,m}(h)}_{\xrightarrow{n\to\infty}\tau_{AB}(h)}\Bigg)\\&\xrightarrow{P}\frac{2\tau_{AB}(h)}{\mu(A)}=2\chi(h),\quad n\to\infty.\end{aligned}$$

Hence by Proposition 3.5, Slutsky's theorem and the continuous mapping theorem, we may infer that

$$\begin{aligned}\widehat{\chi}_m(h)+\chi_m(h)&=\widehat{\chi}_m(h)+\frac{\tau_{AB,m}(h)}{\widehat{p}_m(A)}-\frac{\tau_{AB,m}(h)}{\widehat{p}_m(A)}+\frac{\tau_{AB,m}(h)}{p_m(A)}\\&=\widehat{\chi}_m(h)+\frac{\tau_{AB,m}(h)}{\widehat{p}_m(A)}+\frac{\tau_{AB,m}(h)\left(\widehat{p}_m(A)-p_m(A)\right)}{\widehat{p}_m(A)p_m(A)}\\&=\widehat{\chi}_m(h)+\frac{\tau_{AB,m}(h)}{\widehat{p}_m(A)}+\frac{\tau_{AB}(h)o_P(1)}{\mu(A)^2(1+o_P(1))}\xrightarrow{P}2\chi(h),\quad n\to\infty.\end{aligned}$$

This then implies that

$$\frac{\widehat{\chi}_m(h)+\chi_m(h)-3}{2\chi(h)-3}\xrightarrow{P}1,\quad n\to\infty.$$

Thus, we see

$$\frac{2m_n}{2\chi(h)-3}A_1 \xrightarrow{d} N(0,\sigma^2), \quad n\to\infty$$

such that $A_1 \xrightarrow{P} 0$ for $n\to\infty$ results from the fact that $m_n\to\infty$ as $n\to\infty$. Next, we consider A_2 and employ (42) to find

$$\begin{aligned}
v_m(h) &= \chi_m^2(h) - 3\chi_m(h) + 2 \\
&= \left(\chi(h) + \frac{1}{2m_n}v(h)\right)^2 (1+o_P(1)) - 3\left(\chi(h) + \frac{1}{2m_n}v(h)\right)(1+o_P(1)) + 2 \\
&= \left(\chi^2(h) - 3\chi(h) + 2 + \frac{1}{m_n}\chi(h)v(h) + \frac{1}{4m_n^2}v(h)^2 - \frac{3}{2m_n}v(h)\right)(1+o_P(1)) \\
&= \left((\chi(h)-2)(\chi(h)-1) + \frac{1}{m_n}\chi(h)v(h) + \frac{1}{4m_n^2}v(h)^2 - \frac{3}{2m_n}v(h)\right)(1+o_P(1)) \\
&= \left[v(h) + \frac{v(h)}{m_n}\left(\chi(h) + \frac{1}{4m_n}v(h) - \frac{3}{2}\right)\right](1+o_P(1)).
\end{aligned}$$

This implies that

$$\begin{aligned}
\sqrt{\frac{4m_n^3}{|S_n|\lambda_n^2}}A_2 &= \frac{v(h)}{m_n}\left(\chi(h) + \frac{1}{4m_n}v(h) - \frac{3}{2}\right)(1+o_P(1)) \\
\Leftrightarrow A_2 &= \frac{1}{2}\sqrt{\frac{|S_n|\lambda_n^2}{m_n^5}}v(h)\left(\chi(h) + \frac{1}{4m_n}v(h) - \frac{3}{2}\right)(1+o_P(1)).
\end{aligned}$$

So, A_2 converges to 0 in probability if

$$\begin{aligned}
&\frac{|S_n|\lambda_n^2}{m_n^5} \to 0, \quad n\to\infty \\
\Leftrightarrow\ &\frac{n^2 n^{-2\beta_2}}{n^{5\beta_1}} \to 0, \quad n\to\infty \\
\Leftrightarrow\ &n^{2(1-\beta_2)-5\beta_1} \to 0, \quad n\to\infty \\
\Leftrightarrow\ &2(1-\beta_2) - 5\beta_1 < 0 \\
\Leftrightarrow\ &\beta_1 > \frac{2}{5}(1-\beta_2).
\end{aligned}$$

Note that $\beta_1 \in \left(\frac{2}{5}(1-\beta_2), \frac{2}{3}(1-\beta_2)\right]$ requires

$$\frac{2}{5}(1-\beta_2) < \frac{2}{3}(1-\beta_2) \Leftrightarrow \beta_2 < 1,$$

which does not impose a new condition on β_2. Thus, the suitable sets for β_1 and β_2 are

$$\beta_1 \in \left(\frac{2}{5}(1-\beta_2), \frac{2}{3}(1-\beta_2)\right]$$
$$\beta_2 \in \left(\frac{1}{2}\beta_1, 1\right).$$

□

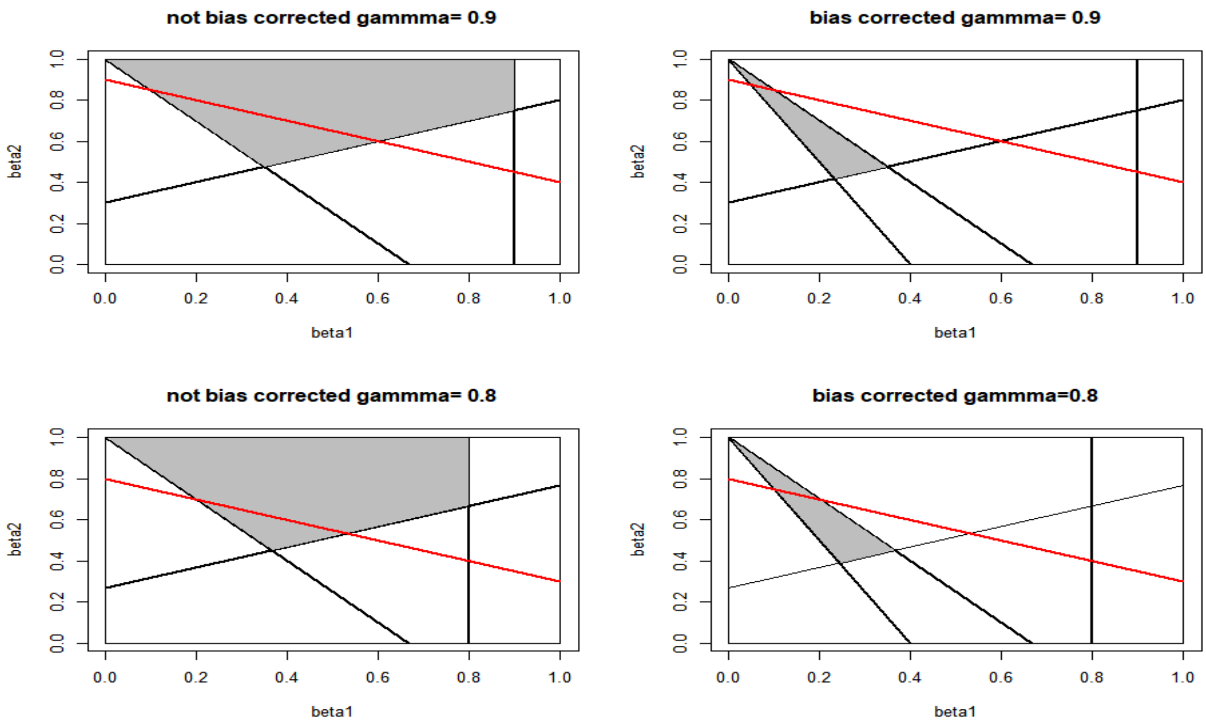

Figure 1: Comparison of feasible sets for rates of $\gamma = 0.9$ and $\gamma = 0.8$ in the case of no bias correction versus the bias corrected one.

We visualize the feasible sets of rates for β_1 and β_2 in Figure 1 for fixed $\gamma = 0.9$ and $\gamma = 0.8$. The conditions (31) and (32) translate into $\beta_2 = \gamma - \frac{1}{2}\beta_1$, which corresponds to the red line in the diagrams. Furthermore, the grey areas are restrictions imposed by $\beta_2 > \frac{1}{3}\gamma + \frac{1}{2}\beta_1$ in addition to $\beta_1 \in \left(\frac{2}{3}(1-\beta_2), \gamma\right)$ in the not bias corrected case and $\beta_1 \in \left(\frac{2}{5}(1-\beta_2), \frac{2}{3}(1-\beta_2)\right)$ for the bias corrected extremogram, respectively. Then, the overall feasible set to apply the CLT centered by the true extremogram is the intersection of the red line with the grey polygon. In particular, we may infer from Figure 1 that there exist rates β_1 and β_2 for which the bias correction in Theorem 4.3 is justified.

5 Data example: German rainfall

In this section, we apply the theory of Brown-Resnick process and estimate the empirical extremogram on real rainfall data measurements in Germany. We remark in passing that real data usually does not follow a Brown-Resnick process precisely. This is why Buhl et al. [4] generalized their Lemma 3.1, which we used in chapter 4.3, to a Brown-Resnick process disturbed by some noise variable, Lemma 5.1 in [4]. Thus, all other results from that section are still applicable.
The dataset is provided by the German Meteorological Service (Deutscher Wetterdienst) and was collected by a total of 5556 measuring stations between 01/01/1781 and 31/12/2016. It includes their location (in longitude and latitude), the height above sea level at which the station is located as well as the amount of rainfall per day (in millimeter). Since we focus on a 2-dimensional Poisson Process, we eliminate the influence of height in the measurements by taking only stations into account that are located between 0 and 521 meters above sea level where 521 meters correspond to the 85%-quantile of the height of all stations such that 4711 stations are left. Furthemore, we only consider stations that recorded data in the period from 01/01/1971 to 31/12/2010. Thus, 1272 stations remain, visualized in Figure 2.

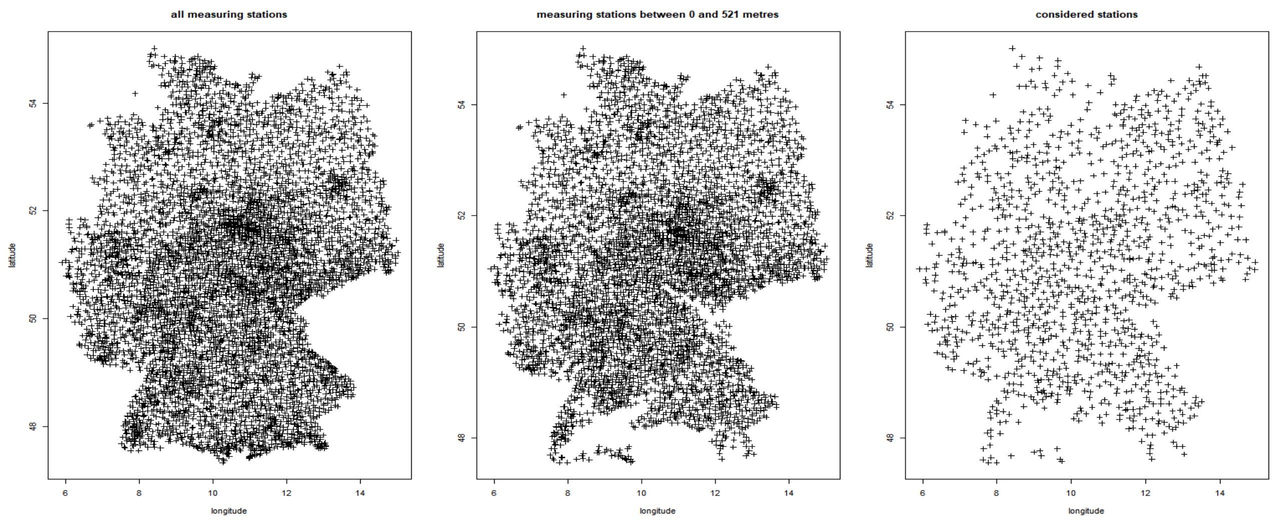

Figure 2: All measuring stations that are available in the dataset (left), the stations located below 521 meters above sea level (center) and those that are active between 01/01/1971 and 31/12/2010 and are located below 521 meters above sea level (right).

Since our model does not take time-dependence into account, we split the data into 731 periods of 20 days and consider the first observation of each such period to obtain (nearly) independent samples. Figure 3 illustrates the daily rainfall data at the considered time points within the period of 01/01/1971 until 31/12/2010 at the two measuring stations in Augsburg (coordinates: $(10.9351, 48.3474)$) and Munich-City (coordinates: $(11.5429, 48.1631)$). This corresponds to a spatial lag of approximately $(0.6078, -0.1843)$.
In the next step, we transform the data to standard Fréchet margins. This is done by setting a threshold at the 90%-quantile for each time point to model the tails by Generalized Pareto distributions, where the parameter estimation is done by maximum likelihood. Missing values in the dataset are treated in such way that they

are removed before conducting the computations. Figure 4 compares the untransformed with the transformed data for the particular measuring date 01/01/1971. Note that, as expected, after the transformation to Fréchet margins the extreme values become considerably larger.

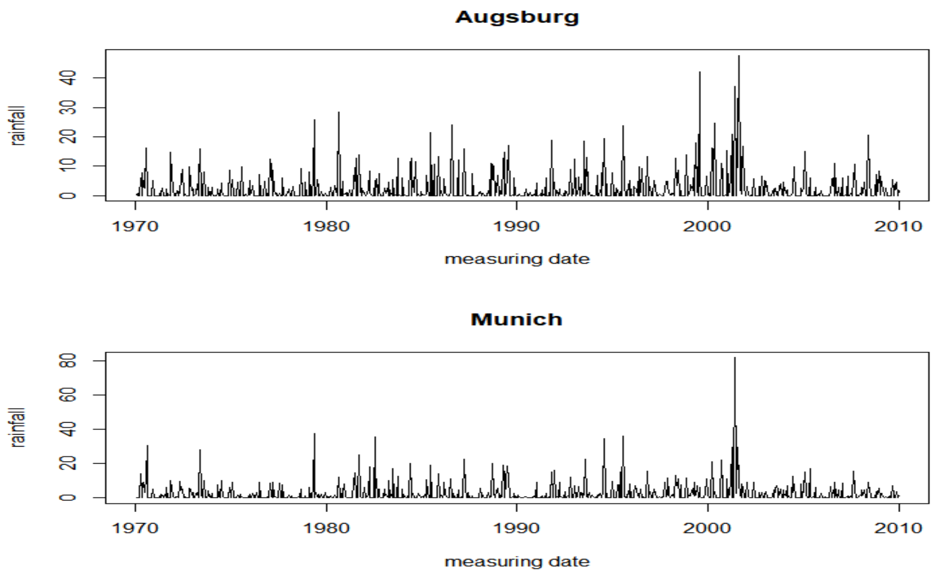

Figure 3: Comparison of the rainfall (in millimeter) in Munich and Augsburg during the considered time period.

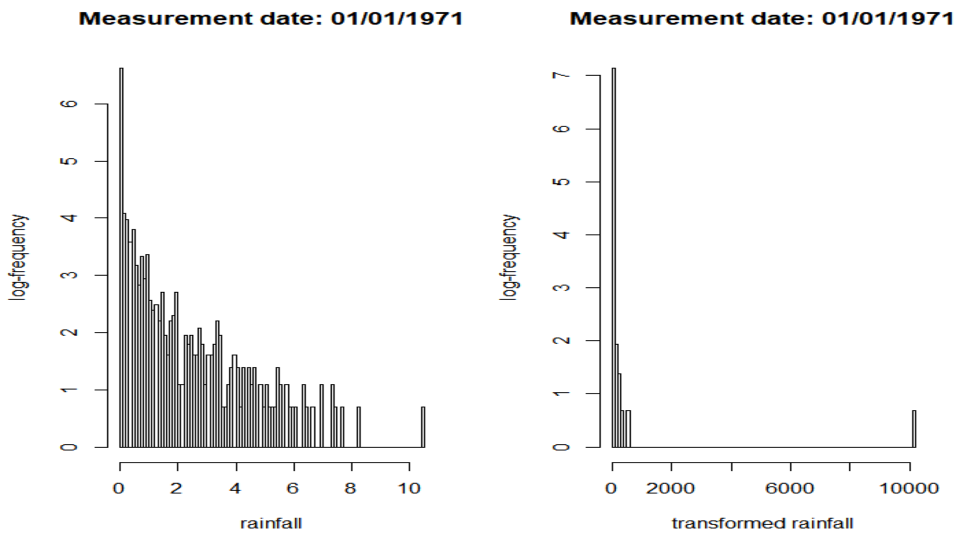

Figure 4: Rainfall data, measured in millimeter, on 01/01/1971 before (left) and after (right) the transformation to standard Fréchet-margins with frequencies in log-scale.

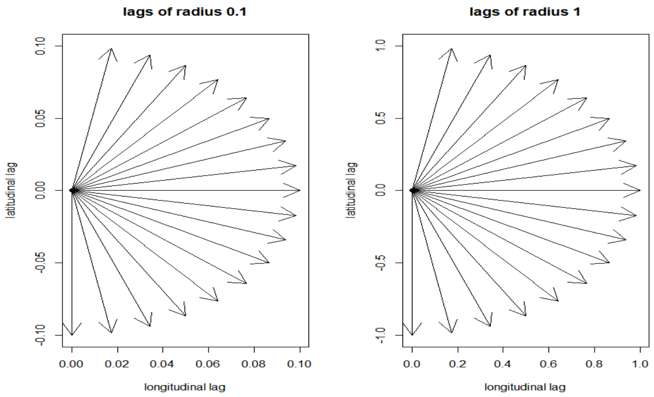

Figure 5: Radial lags of total length 0.1 and 1. For estimation we use the bivariate density of independent normal random variables, the centralized beta density of independent random variables and the uniform density on a centered 6×6 square for the weight function $w_n(\cdot)$. Then by symmetry of the weight function $w_n(\cdot)$ the above lags are sufficient to consider.

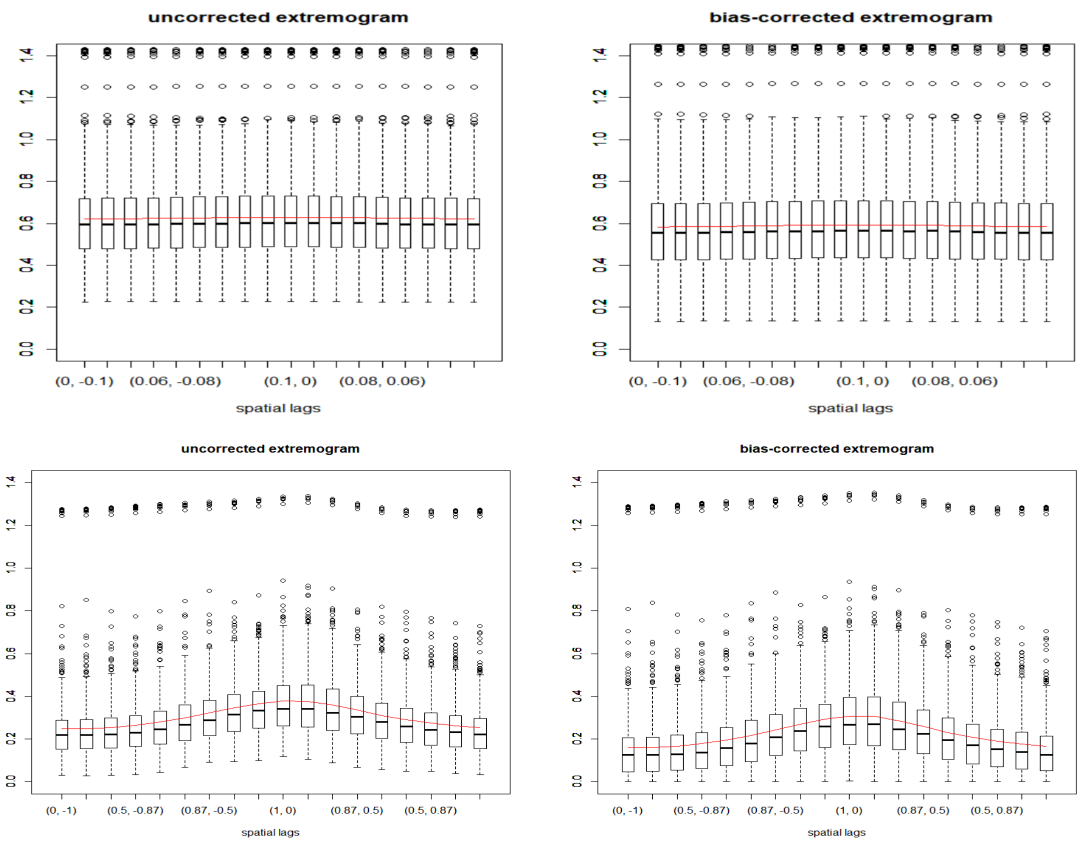

Figure 6: Boxplots of the uncorrected and bias corrected empirical extremogram for radial lags of 0.1 (upper row) and 1 (lower row) employing a normal kernel. The red lines correspond to the mean of the extremograms.

We then compare the results for the bias corrected version of the tail dependence coefficient from section 4.3 with the original version as introduced in section 2. Since we modelled the tails on the basis of a 90%-quantile, we choose the threshold $a_m = 90\%$.

First, we analyze the data for radial dependence, i.e. we estimate the extremogram for lags (in langitude and latitude) with a total length of 0.1 and 1, respectively, but different angles, compare Figure 5. This way, we obtain lags of the same length but different directions, allowing for detection of directional dependence. Figure 6 shows the boxplots of both the uncorrected and bias corrected empirical extremogram for those lags using a normal kernel function. We conclude that the direction does not have an impact for small distances whereas with growing total lags it becomes clear that stations lagging primarily in the longitude, i.e. in east-west direction, are more likely to assess extreme events simultaneously. This does not fit our assumption of an isotropic Brown-Resnick process with dependence function (27). Moreover, considering lags with a length of 1 in Figure 6, we observe that there are estimates that take values larger than 1 and are far off the mean and the median. Note that in contrast to the theoretical extremogram the empirical one may take values greater than 1, where one might think about introducing a restricted version of the estimator. We observe that for all lags the estimates with values greater than 1 always correspond to the same 9 dates. On these days the threshold was exceeded at more than 1200 stations, i.e. nearly all measuring stations recorded an extreme event, corresponding to total extremal dependence. Figure 7 shows the number of exceedances from 1970 to 2010, the marked points symbolize the discussed observations with empirical extremograms larger than 1.

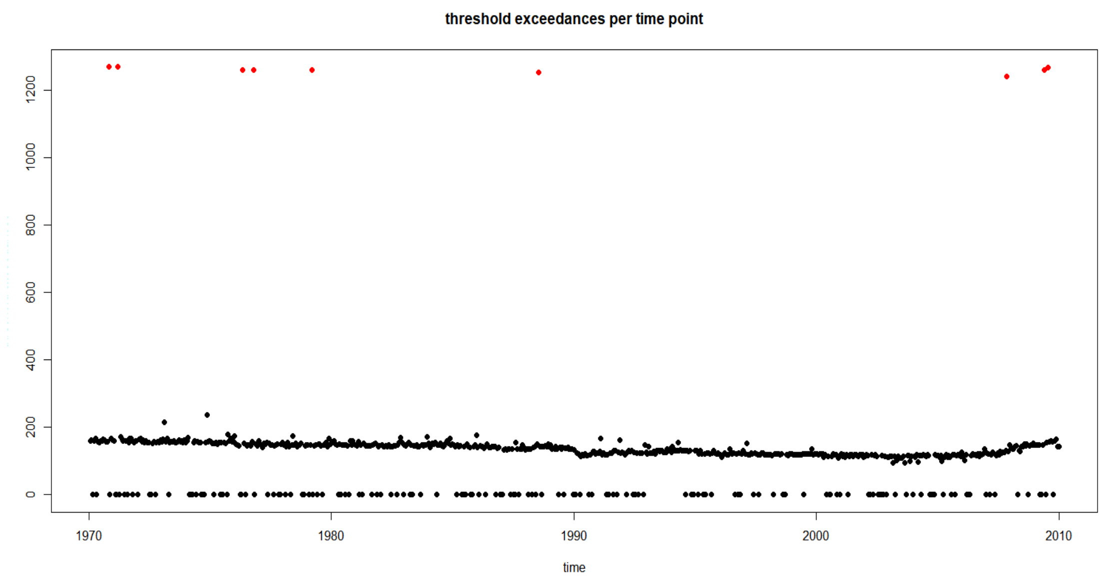

Figure 7: Illustration of the number of threshold exceedances per day. The red marks correspond to the estimates of the empirical extremograms in Figure 6 that are larger than 1.3 for lags of length 0.1 or larger than 1 for lags of length 1, respectively.

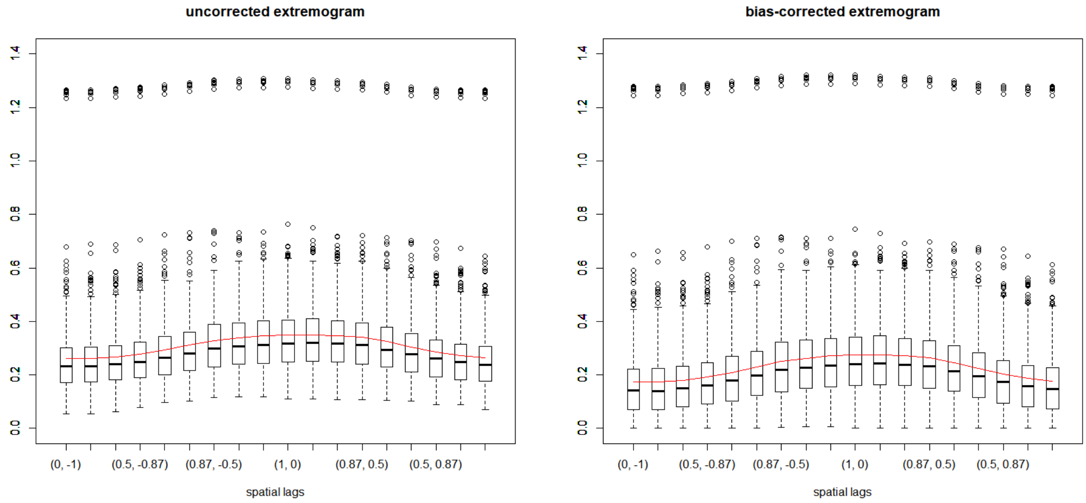

Figure 8: Boxplots of the uncorrected and bias corrected empirical extremogram for radial lags of 1 employing a uniform 6×6 kernel density $w(\cdot)$. The red lines correspond to the mean of the empirical extremograms.

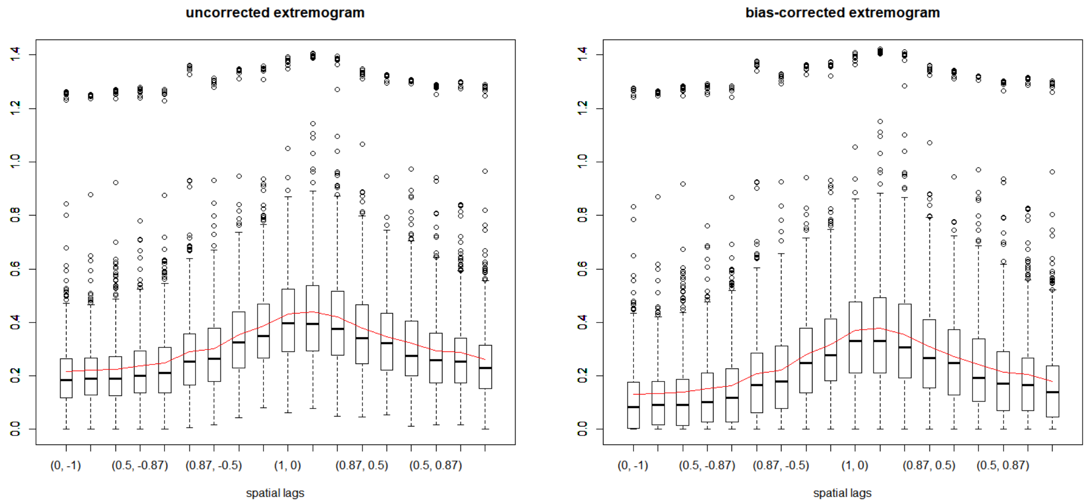

Figure 9: Boxplots of the uncorrected and bias corrected empirical extremogram for radial lags of 1 employing a centered beta(2,2) kernel density $w(\cdot)$. The red lines correspond to the mean of the estimates.

To compensate for the non-isotropy we try different kernel functions, namely a uniform density on a square of 6×6 as well as the centered product measure of two independent beta distributed random variables with both shape parameters set to 2. Recall that the beta distribution is only symmetric and bell-shaped when both parameters take the same value larger than 1. The resulting extremograms are depicted in Figure 8 and Figure 9, respectively. We recognize that the uniform density smoothens the effect of directional extremal dependence. On the other hand, the issue still occurs under the usage of the beta kernel. Therefore, we compare the results

of the normal with the uniform kernel in the following.
Blanchet and Davison [1] investigate non-isotropic models, Schlather's model and Smith's storm model, in the context of snow depth. They present two modelling approaches, one by directly transforming the data and another by transforming the space. Since in our case the first approach would require to know the value of α_1 in the dependence function (27), we transformed the observation space. The $2-$dimensional space S is given in terms of longitude and latitude of 1272 stations, i.e. $S \in \mathbb{R}^{1272\times 2}$. We calculate the 2×2 covariance matrix V of these coordinates and compute the Cholesky decomposition of its inverse $V^{-1} = U^\top U$, where U is a 2×2 upper triangular matrix. This allows us to standardize the covariance of the data space and to obtain the standardized space $\tilde{S} = SU^\top$. Figure 10 shows the original space and the transformed space. We notice that the overall shape of the map does not change dramatically. However, the map rescales to a more quadratic shape, i.e. approximately 4×4 in longitude and latitude in contrast to 7×9 in the original space. When investigating directional dependence on this transformed space, we see that the effect declines distinctly using a bivariate standard normal kernel and disappears as we employ a uniform 6×6 kernel density, see Figures 11 and 12. Additionally, considering the quartiles of the estimates, the variation of the empirical extremogram becomes significantly smaller in comparison to the variation of the estimates on the original space.

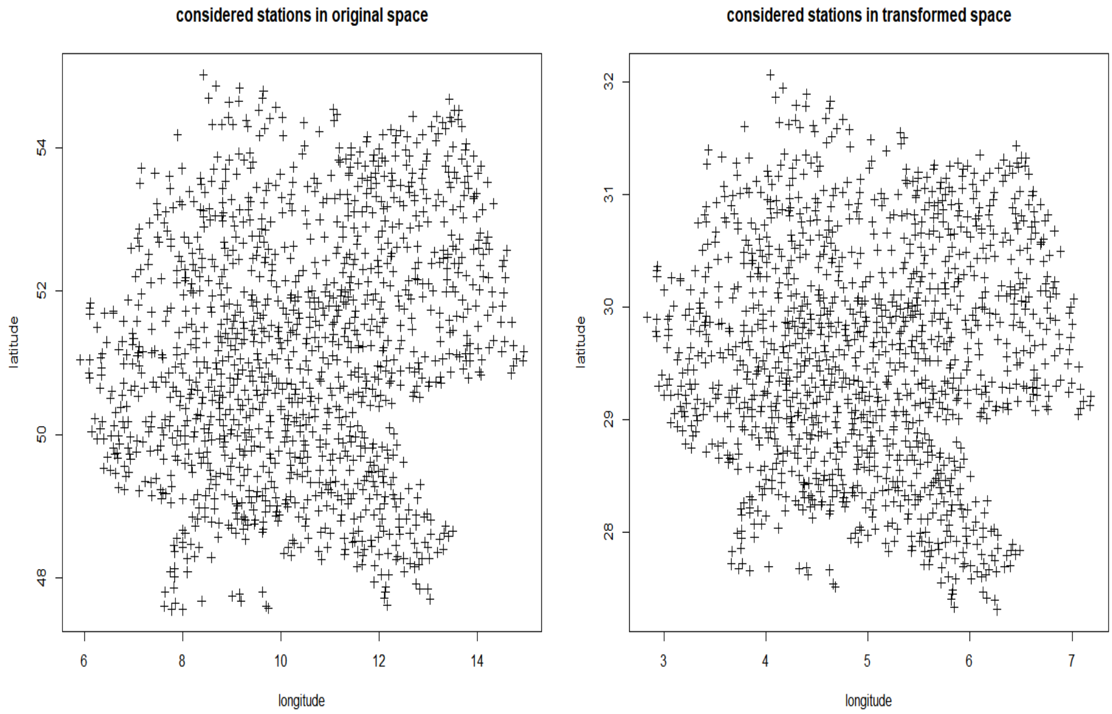

Figure 10: Comparison of the considered measuring stations before (left) and after (right) the transformation of the observation space.

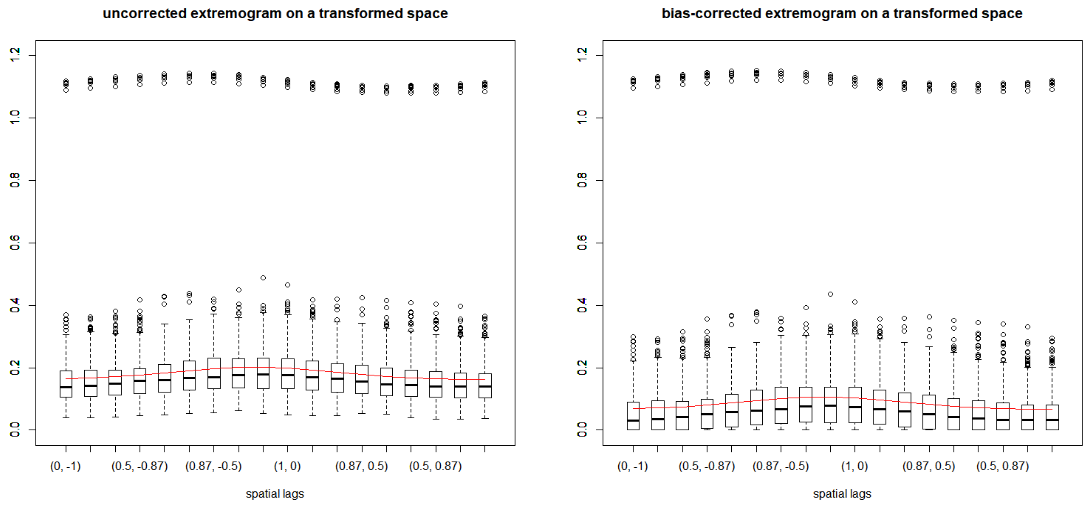

Figure 11: Boxplots of the uncorrected and bias corrected empirical extremogram on the transformed space for radial lags of 1 employing the bivariate density of two independent standard normally distributed random variables. The red lines correspond to the mean of the estimates.

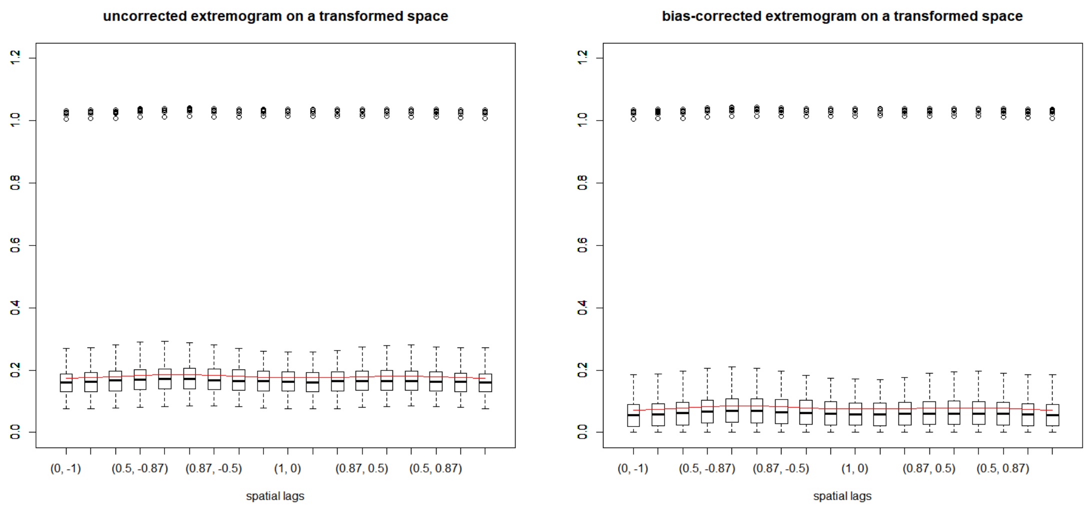

Figure 12: Boxplots of the uncorrected and bias corrected empirical extremogram on the transformed space for radial lags of 1 using a uniform kernel. The red lines correspond to the mean of the estimates.

Figures 13, 14, 15 and 16 compare the empirical extremogram with its bias corrected version in the original space and the transformed space, where we use a normal and a uniform kernel. For example, when estimating with a normal kernel on the original space (Figure 13) we find that for a lag of 0.01 in longitude both the uncorrected and the bias corrected extremogram take values of approximately 0.6, while for measuring stations that are in distance of 5 degrees in lattitude and 1 degree in longitude, extreme dependence declines distinctly. Note that for small values of $\hat{\chi}_m$ the bias corrected $\tilde{\chi}_m$ could attain values smaller than 0. Since this is not possible for its theoretical equivalent, we restrict $\tilde{\chi}_m$ to 0 if the correction would lead to negative values. This can be seen in the lags $(3, 1)$, $(3, 3)$, $(1, 5)$ and $(1, 7)$ for which the bias corrected extremogram takes the value 0 more frequently than $\hat{\chi}_m$. Furthermore, we test for extremal independence by employing a permutation test. In more detail, for every lag and each of the 731 time points we permute the transformed data randomly and compute the extremogram with a threshold of 90%. Repeating this procedure three times gives a total of 2193 estimates for the extremogram in an independent setting for each lag and we may compute the $95\%-$ quantile of these estimates. Then a value below the 95%-quantile is an indicator for extremal independence. The quantile is visualized by the blue lines in the corresponding figures. For example, in Figure 13, regarding the uncorrected extremogram, we may conclude that there is extremal independence of the lag $(1, 5)$, because the mean (and median) are smaller than the quantile. On the other hand, there is no sign of extremal independence for the smaller lag $(0.5, 0.5)$.

Figure 14 shows the results of the extremogram and the bias corrected version for a normal kernel on the transformed space $\tilde{S}$. We note that the general magnitude of the estimates is smaller when estimating on the transformed space, e.g. for the lag $(0.01, 0)$ the estimate of the uncorrected extremogram declines by approximately 0.2. This means that the transformation of space eliminates some extremal dependence in the data. Moreover, regarding the uncorrected empirical extremogram, the independence test gives no indication of extremal dependence for the lag $(1.1, 1)$ anymore. The results for the bias corrected extremogram show no sign of extremal dependence for the lag $(3, 1)$ contrasting the findings from Figure 13.

In Figure 15 we consider a uniform kernel on the original space and find that the results are smiliar to those for the normal kernel in Figure 13 except for a general decrease in extreme dependence for lags of small total length $(0.01, 0)$ and $(0, 0.1)$. The results of the independence test are identical for all lags when considering the uncorrected extremogram as well as its bias corrected version.

When employing a uniform kernel on the transformed space, see Figure 16, we observe the same decline in general extremal dependence as in the transition from Figure 13 to Figure 16. In opposition to our findings for the transformed space with a normal kernel, the independence test still indicates extremal dependence for the lag $(1.1, 1)$ when considering the uncorrected extremogram and shows no sign of extremal independence for lags of $(3, 1)$ and longer.

Moreover, we find that the transformation of the observation space reduces the variation in the estimates, this can be seen in the smaller distance between the quartiles.

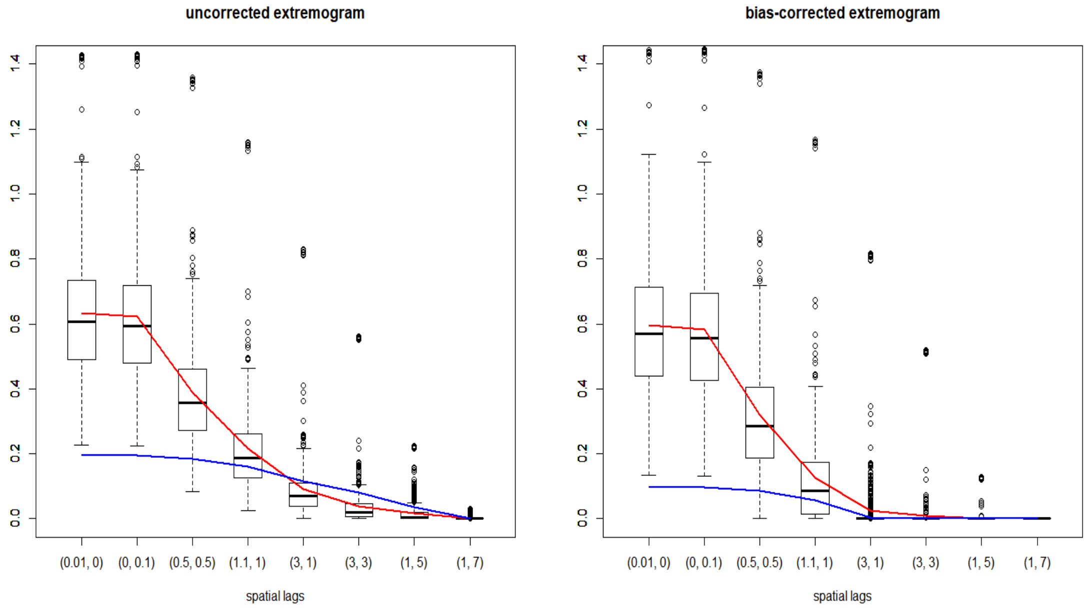

Figure 13: Comparison of the empirical extremogram with the bias corrected version for different lags in longitude and latitude on the original space using a normal kernel density. The red lines correspond to the mean of the extremograms. The blue lines represent the 95% quantile of the extremograms for randomly permuted data.

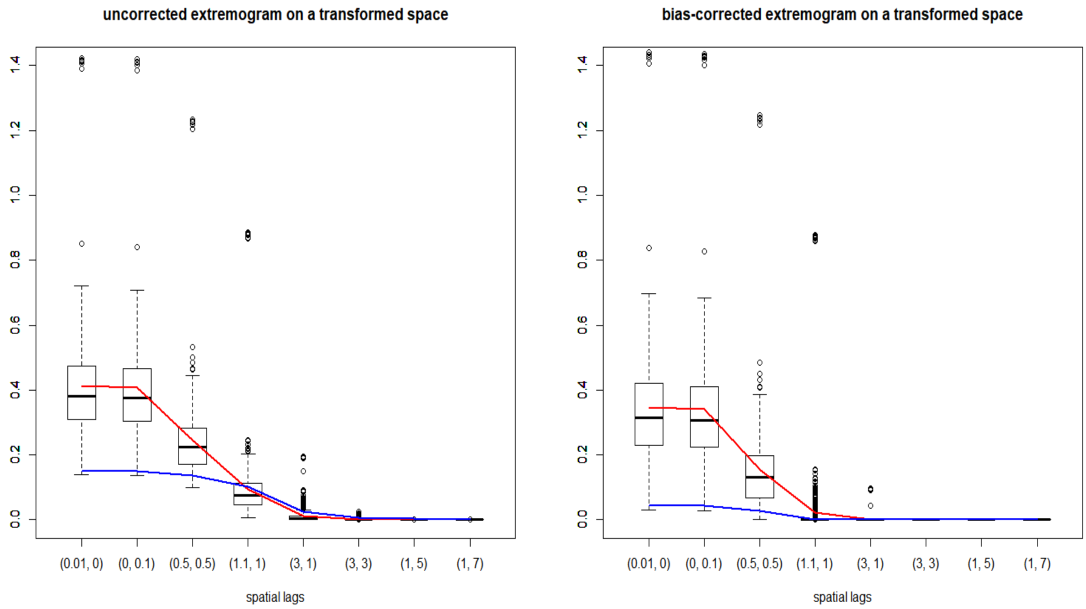

Figure 14: Comparison of the empirical extremogram with the bias corrected version for different lags in longitude and latitude after transforming the observation space and employing a normal kernel density. The red lines correspond to the mean of the extremograms. The blue lines represent the 95% quantile of the extremograms for randomly permuted data on the transformed space.

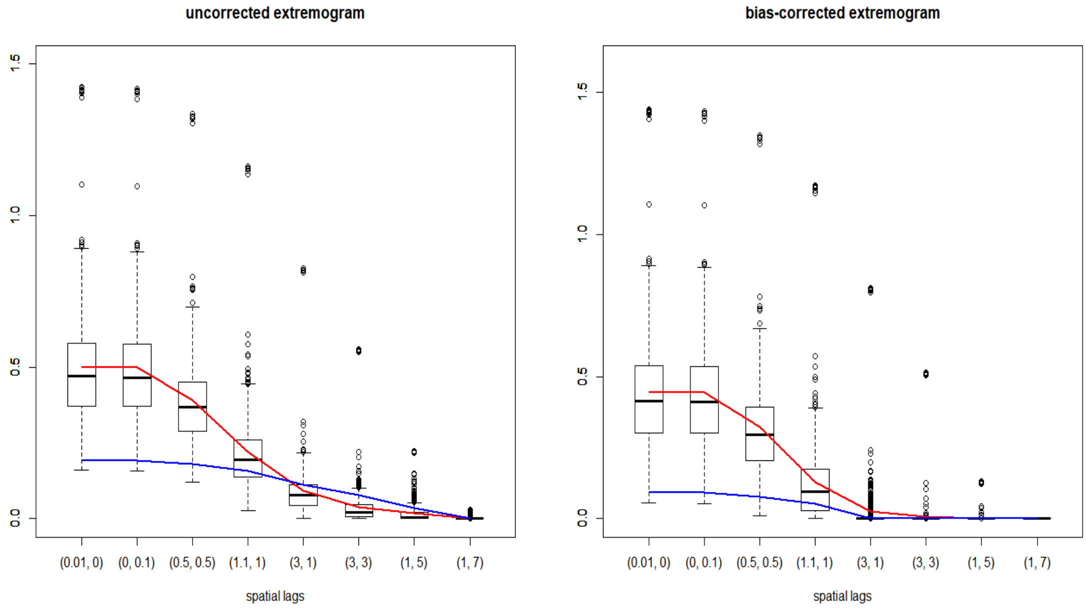

Figure 15: Comparison of the empirical extremogram with the bias corrected version for different lags in longitude and latitude on the original space employing a uniform 6×6 kernel density. The red lines correspond to the mean of the extremograms. The blue lines represent the 95% quantile of the extremograms for randomly permuted data on the transformed space.

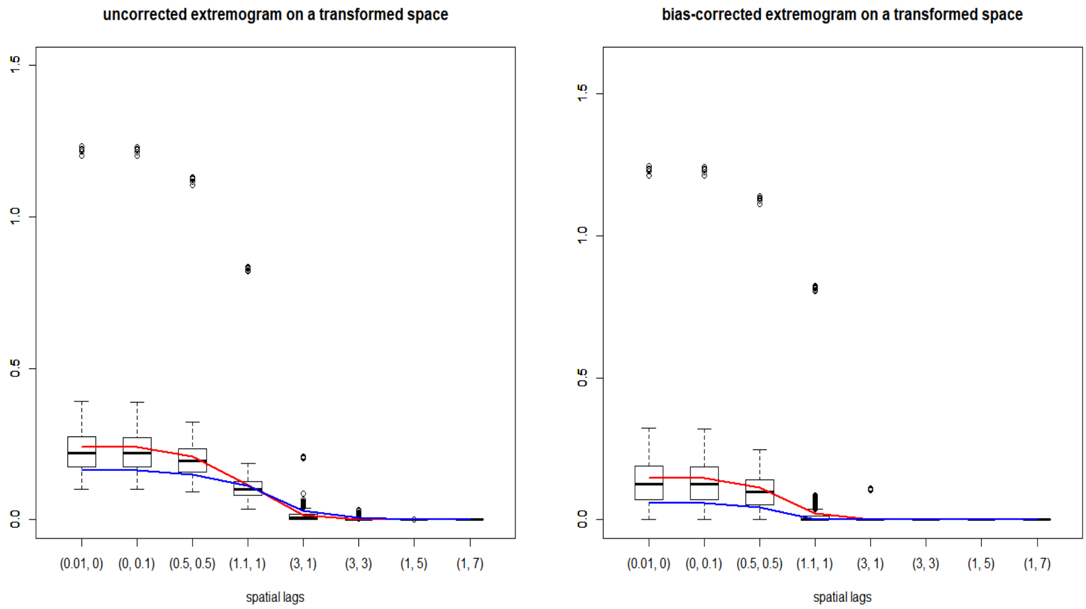

Figure 16: Comparison of the empirical extremogram with the bias corrected version for different lags in longitude and latitude after transforming the observation space employing a uniform 6×6 kernel density. The red lines correspond to the mean of the extremograms. The blue lines represent the 95% quantile of the extremograms for randomly permuted data on the transformed space.

6 Conclusion and outlook

Extremal dependence in space is a topic of high interest in the ongoing research on extreme value theroy. In this thesis we presented a new spatial central limit theorem, Theorem 3.6 for the empirical extremogram, when observing a random process $\{X_s : s \in \mathbb{R}^2\}$ on a space where we assume that the underlying distribution of the locations is inherited by a 2-dimensional Poisson process N with intensity measure ν. The results are shown for processes in two dimensions, because this gives the widest range of applications since the locations of measuring stations are expected to follow a 2-dimensional Poisson process. We remark in passing that this results can be generalized to d-dimensional spaces in an analogous fashion as presented in this work. The multivariate CLT applies in terms of spatial lags in the random process and was already proved in Cho [7]. We developed the proof in more detail and fixed obscurities.

Then we focused on the Brown-Resnick process and showed that the assumptions of our CLT in particular hold for this process, where we considered an isotropic dependence function (27). Additionally, we were able to introduce a new bias corrected version of the extremogram that allowed us to generalize the previous CLT to an advanced result with better rates, Theorem 4.3. This bias correction was useful as we aimed for centering the empirical extremogram by its theoretical and not only its pre-asymptotic version.

Furthermore, we estimated the empirical extremogram to rainfall data in Germany from 1971 to 2010. We found that the extremal dependence is larger when considering lags in east-west direction so that the direction of the lag plays a significant role, which is consistent with Germany being located in prevailing westerlies. However, the effect of the length of the considered lag dominates the impact of the direction. Furthermore, employing a permutation test, we showed that for lags of certain length (and direction) there are signs of extremal independence.

For future research, it might be of further interest to investigate the asymptotic behaviour for an empirical extremogram in a space-time setting with observations on an irregular space, where the observation locations follow the law of a Poisson-process but are fixed over time. Buhl et al. [4] looked into such a framework on a regular grid. Note that for real life problems, one usually has to deal with a restricted space and only the time component can be treated as going to infinity, i.e. a CLT for fixed space and a growing amount of measurements of time would be a highly useful result.

We proved Theorem 3.6 for an isotropic kernel $w(\cdot)$, when considering an isotropic Brown-Resnick process. Thus, we suggest that there is a CLT in terms of absolut spatial lags, as done in Buhl et al. [4] on regular grids. However, note that their proof is not transferrable in an analagous manner as they employ the fact that on a regular grid for a certain distance there is only a finite number of spatial lags, whereas this does not hold true in our case. For every specific absolute lag the Poisson process may generate an unbounded number of observations with the corresponding absolute lags.

When having completed the proof for such a CLT in terms of absolute lags, the empirical extremogram and its bias corrected version may be used to conduct parameter estimation for the Brown-Resnick process, as done in Buhl et al. [4] on regular grids. Their arguments, regarding asymptotic normality of the resulting parameter estimates, will then work in our setting as well.

Due to time restrictions, we were not able to conduct a simulation study on the empirical extremogram. Since we are interested in extreme events, a large amount of simulations is necessary to obtain a significant number of extremes that allow for consistent estimation of the extremogram, making such computations time-consuming. If being provided with sufficient computational resources, simulating a Brown-Resnick process on an irregular grid and comparing the empirical extremogram with the bias corrected version from Theorem 4.3, will be an interesting study to compare the performance of the estimators with regard to their rate of convergence and variation with respect to the theoretical value. Oesting et al. [17] give an algorithm to simulate the Brown-Resnick process.

In chapter 4.5 of her dissertation Steinkohl [21] developed a pairwise maximum likelihood estimator. Hence, analyzing the performances of her approach with parameter estimates based on the empirical extremogram and the bias corrected empirical extremogram are of particular interest. Additionally, one might consider comparing the accuracy of these estimators on a confined space with a growing number of observations in time.

A Auxiliary results

We compute the expected value of the product measure $N^{(2)}(ds_1, ds_2) = N(ds_1)N(ds_2)\mathbb{1}\{s_1 \neq s_2\}$.

Proposition A.1. *Let $N(\cdot)$ be a 2-dimensional Poisson process with intensity rate $\nu > 0$. If $s_1 \neq s_2$, $s_1, s_2 \in \mathbb{R}^2$, we have*

$$\mathbb{E}\left[N^{(2)}(ds_1, ds_2)\right] = \nu^2 ds_1 ds_2. \tag{43}$$

Proof. For $s_1 \neq s_2$, we note that there are disjoint, open, convex and non-empty environments around s_1 and s_2. Then $N(ds_1)$ and $N(ds_2)$ are independent random variables by the properties of the Poisson process. This gives

$$\begin{aligned}
\mathbb{E}\left[N^{(2)}(ds_1, ds_2)\right] &= \mathbb{E}\left[N(ds_1)N(ds_2)\right] \\
&= \mathbb{E}\left[N(ds_1)\right]\mathbb{E}\left[N(ds_2)\right], \quad && \text{by independence} \\
&= \nu^2 ds_1 ds_2, && \text{since } \mathbb{E}\left[N(ds_1)\right] = \nu ds_1, \text{ since } N \text{ is a Poisson process.}
\end{aligned}$$

□

The subsequent result is stated in Karr [14] in Lemma 10.20.

Lemma A.2. *Let $(S_n)_{\{n\in\mathbb{N}\}}$ be a sequence of convex, compact sets with $S_1 \subset S_2 \subset \cdots \subset \mathbb{R}^d$ such that $|S_n| = O(n^2)$ and $y \in \mathbb{R}^d$, $d \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{|S_n \cap (S_n - y)|}{|S_n|} = 1,$$

where $S_n - y := \{z - y : z \in S_n\}$.

In the course of the proof for the CLT of the empirical extremogram, we need an upper bound for the covariance of two random variables in terms of the mixing coefficient. The following Lemma can be found in the proof of Lemma 1 in Politis et al. [18].

Lemma A.3. *Let N be a Poisson process on $\mathbb{R}^d$ independent of the random process $\{X_s : s \in \mathbb{R}^d\}$. Let U, V be two closed sets in $\mathbb{R}^d$ such that $M := \max\{|U|, |V|\}$ exists and $d(U, V) \geq r > 0$. Also let Y and Z be two random variables that are functions of X observed on points of N that fall into U and V, respectively. Assume that, given N, $Y \leq C_1$ and $Z \leq C_2$ almost surely. Then it holds that*

$$\mathbb{C}\text{ov}(Y, Z|N) \leq 4C_1C_2\alpha_{MM}(r).$$

The following Lemma is a result shown in the proofs of Proposition 1 and 2 in Buhl and Klüppelberg [5].

Lemma A.4. *The mixing coefficient $\alpha_{kl}(\cdot)$, $k, l \in \mathbb{N}$, of the Brown-Resnick process $\{\eta_s : s \in \mathbb{R}^2\}$ satisfies the inequality*

$$\alpha_{kl}(r) \leq 2kl \sup_{s \in \mathbb{R}^2,\ |s| \geq r} \chi(s) \leq 4kl e^{-\theta_1 r^{\alpha_1}/2}, \quad r \geq 0.$$

The proof of the next result can be found in the appendix of Buhl et al. [4].

Lemma A.5. *Let $k \in \mathbb{N}$. Then for $\theta, \alpha > 0$*

$$\int_r^\infty y^k e^{-\theta y^\alpha/2} dy \leq C r^{k+1} e^{-\theta r^\alpha/2}, \qquad r > 0$$

for some constant $C = C(k) > 0$.

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