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Frogs and Branching Random Walks

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Contents

Introduction

There are about 4800 known species of frogs that can be found all over the world. There is also at least one species in mathematics. It can be found in probability theory. Here, frogs appear as particles in a particle system known as the frog model that belongs to the family of interacting random walks. This family also includes branching random walks. We will study these two mathematical models in this thesis.

Let us first explain what interacting random walks are. A random walk on a graph is one of the most fundamental objects in probability theory. A particle moves from vertex to vertex, each time choosing its next destination randomly among the neighbouring vertices. To have interaction, we need more than one particle. An interacting random walk is indeed a system of finitely or infinitely many particles that simultaneously perform random walks on a graph and obey rules that govern their coexistence. For example, it might be forbidden for a vertex to be occupied by more than one particle at a time.

Both models, the frog model and the branching random walks, feature a growing set of particles that perform nearest neighbour random walks on the d-dimensional lattice $\mathbb{Z}^d$. Interaction between the particles arises whenever a new particle comes into existence. The time and place of the birth of the new particle depends on the behaviour of the other particles. Otherwise, the particles move independently in both models.

Though it does not have anything to do with the amphibians, in the frog model the particles are thought of as frogs. They can be in one of two states, active or sleeping. The model can informally be described as follows. Consider the d-dimensional lattice $\mathbb{Z}^d$ and place one active frog at the origin 0 and one sleeping frog at every other vertex. The active frog jumps from vertex to vertex according to a nearest-neighbour random walk. Every time it visits a vertex occupied by a sleeping frog, that frog wakes up by the loud croaking and becomes active itself. It starts a random walk as well and can also activate sleeping frogs. Thus, we get increasingly many active frogs leaping around the lattice.

The frog model was first studied in 1999 by Telcs and Wormald in [52] who called it "egg model". The term frog model was, according to [43], coined by Durrett. As mentioned, the frog model is not suitable to describe the behaviour of natural frogs that rarely move in the described fashion. However, it can be used to model the spread of information or of a disease. Every active particle carries some information or illness, moves freely around, and passes it to inactive particles upon meeting them. It can also be interpreted as a thermal reaction process taking place during the combustion of a solid. Here, the active particles play the role of diffusing heat, and the sleeping particles represent molecules that are not yet ignited. See the introduction of [44] and references therein for more information.

The branching random walk that we study in the second part of this thesis also starts with one particle at 0. At any point in time, every particle first moves randomly to a neighbouring vertex and then generates offspring at that vertex according to some fixed reproduction distribution before it dies. We assume that all particles move and reproduce independently and independent of everything that happened in the past. A priori, it is possible that there are eventually no more particles and the process dies out. However, we shall always assume that particles reproduce in such a way that the process survives with positive probability. Branching random walks are a natural extension of Galton-Watson processes. These only take the evolution of the number of particles into account, not their locations.

As mentioned, in both models we have a growing set of particles. The aim of this thesis is to describe the speed of the growth and movement of the cloud of particles. These questions have so far mainly been studied if the particles perform simple random walks. In this case, they move in every direction with equal probability. Yet it is also natural to assume that the system has a drift in one direction. The frogs might, for example, spot a swarm of juicy flies in the distance and prefer to leap towards it. The particles then move as biased random walks, choosing to go in drift direction with higher probability than to go in any other direction. The question how the drift effects the behaviour of the particle system will accompany us throughout this thesis.

To describe the evolution of the cloud of particles in the frog model mathematically, we use the concept of recurrence and transience. A particle system is called recurrent, if the origin is visited infinitely many times by particles with probability one, otherwise it is called transient. It is known that in the latter case the origin is in fact visited only finitely many times with probability one. Thus, the cloud of particles moves away from the origin. If the system is recurrent, then eventually every vertex in the lattice is visited by a particle. Our results show that in dimension $d \geq 2$ both cases can occur depending on the drift. In contrast, it is shown in [22] that this is not true in dimension $d = 1$. In that case, the frog model is transient if the frogs are exposed to a drift, no matter how small the drift may be. We discuss the model and our findings in detail in Section 1.3.1. The results are stated in Theorems 1.14, 1.15, 1.16, and 1.17. They also appear in the preprint [16] that is submitted for publication. This article was written in collaboration with Döbler, Gantert, Höfelsauer and Popov.

Further, we consider both models in the dimension $d = 1$ and discuss how fast the set of occupied sites moves. For the frog model we assume without loss of generality that the particles have a drift to the right. We derive a formula for the speed of the leftmost occupied vertex that depends on the drift parameter. We also show that the speed of the rightmost occupied vertex is a monotone function of the drift parameter and that it is strictly smaller than 1. These results can be found in Section 1.3.2 in Theorems 1.21, 1.22, and 1.23, respectively. The distribution of active frogs is of interest in this context as well. It turns out that the frogs are spread out uniformly in between the leftmost and rightmost occupied site. See Theorem 1.25. All these results are joint work with Höfelsauer and are published in [27].

For the branching random walk we study the speed of the rightmost occupied vertex in two cases. In the first case, we assume that the particles are moving in an i.i.d. random environment. This means that the drift experienced by the particles is not the same at every vertex. In the second case, the particles are restricted to the non-negative integers, have a bias, and are reflected at 0. In both cases, we show an explicit formula for the speed of the rightmost occupied vertex involving the large deviation rate function of the underlying random walk. These models are explained and discussed in Section 3. The results are given in Theorems 3.5 and 3.7. Lastly, as a technical preparation, a large deviation principle for random walk with drift and reflection on the non-negative integers is derived in Theorem 3.6.
Before we discuss the two models in more detail in the following chapters, let us briefly compare them and remark on some of their properties. Pick a particle in the branching random walk and observe it. At any point in time the position of this singled out particle has the same distribution as the position of a random walk moving in the same environment. In the frog model on the other hand, this is not true. Consider the trajectory of one frog. We can extend it backwards in time by recursively attaching the trajectories of the frogs that are responsible for activating our chosen frog. Thus, we end up with a trajectory starting at the origin at time 0. The part that describes the actual trajectory of the active frog looks like the path of a random walk, but the beginning has a much stronger bias away from the origin. Therefore, one singled out frog tends to be further away from the origin than a random walk started at the origin at time 0. This difference can also be seen in Figure 0.1. The picture shows the trajectories of the frog model and of a branching random walk in dimension $d = 1$. In each case, one particle is singled out and its trajectory is highlighted. The trajectory of the particle in the branching random walk looks like a simple random walk path while the one of the frog does not.
Another difference between the two models becomes apparent when considering the genealogical trees. The genealogical tree of the branching random walk is the family tree of the underlying Galton-Watson process decorated with positions of the various particles. The frog model can be viewed as a modification of a branching random walk, in which particles generate two children whenever they are at a vertex that has never been occupied by a particle before. Using this approach we can construct a family tree for the frog model as well. The genealogical tree of the branching random walk exhibits self-similarity. Every sub-tree has the same distribution as the full tree, shifted by the position of the root of the sub-tree. Once again, the family tree of the frog model does not have this property.
These two properties are repeatedly used in the proofs concerning branching random walks. They enable us to perform many explicit calculations. Unfortunately, we do not have such strong tools when dealing with frogs. We can, however, couple the frog model with a more approachable system. We will e.g. use this method in the proofs concerning recurrence and transience of the frog model. To show transience, we carefully smuggle more frogs into the system until we end up with a branching random walk. Transience of this branching random walk then implies transience of the frog model. To show recurrence, we couple the frog model with independent site percolation. Roughly speaking, the open cluster containing the origin

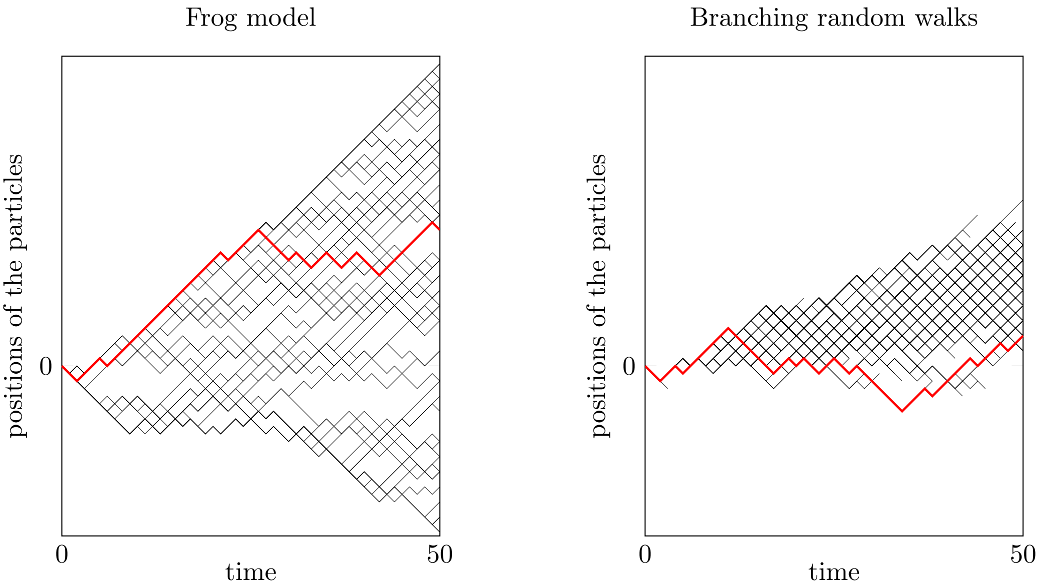

Figure 0.1: The figure shows a simulation of the frog model on the left hand side and one of a branching random walk on the right hand side. Each picture shows the first 50 steps of the trajectories of the active, respectively alive particles. The particles are exposed to the same drift in both models. They go to the right with probability 0.6 and to the left with probability 0.4. In each case one arbitrary particle is picked and its trajectory is highlighted as a thick red line.

in the percolation will be a subset of the set of vertices ever visited by active frogs. The theory of percolation provides us with information on the structure of this open cluster, and thus on the behaviour of the frogs.

This discussion explains why the results for branching random walks are much more refined than those for the frog model. As mentioned above, we can derive precise formulas for the speed of a branching random walk, yet we can only prove qualitative statements about the speed of the rightmost particle in the frog model. While there are precise criteria for distinguishing recurrence and transience for branching random walks, we can only prove the existence of recurrent and transient regimes for the frogs. We believe that indeed a phase transition between recurrence and transience takes place, but this problem remains open.

1 The Frog Model

1.1 Description of the model

We briefly review the construction of the frog model in a slightly more general setting than in the introduction. Consider a graph with a fixed root vertex. Initially, there is one active particle at the root, and at every other vertex x there is a (possibly zero) number η_x of inactive particles. All active particles perform nearest neighbour random walks in discrete time. Whenever an inactive particle is visited by an active particle, the inactive particle becomes active and and also starts a random walk, independently of the others, and according to the same dynamics. The active particles are also called awake frogs, and the inactive particles are referred to as sleeping frogs.

We focus on the frog model on the d-dimensional lattice $\mathbb{Z}^d$ with $d \geq 1$. By the d-dimensional lattice we mean, with the usual abuse of notation, the graph with vertex set $\mathbb{Z}^d$ and edge set $\{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \colon \|x - y\|_1 = 1\}$. Here, $\|x\|_1 = \sum_{i=1}^{d} |x_i|$ denotes the 1-norm. Let us introduce the model for this case in a more formal way. Set $\mathcal{E}_d = \{\pm e_i \colon 1 \leq i \leq d\}$, where e_i denotes the i-th standard basis vector in $\mathbb{Z}^d$, $i = 1, \ldots, d$, and let $\pi \colon \mathcal{E}_d \to [0, 1]$ be a function with $\sum_{e \in \mathcal{E}_d} \pi(e) = 1$. The function π provides the transition probabilities for the random walks performed by the active particles. A particle at a vertex x jumps to $x + e$ with probability $\pi(e)$. The transition probabilities for $d = 2$ are depicted in Figure 1.1. Further, let η be a random variable taking values in $\mathbb{N}_0$ with $\mathbb{P}(\eta \geq 1) > 0$. Now, let $\{\eta_x \colon x \in \mathbb{Z}^d \setminus \{0\}\}$ and $\{(S_n^x(i))_{n \in \mathbb{N}_0} \colon i \in \mathbb{N}, x \in \mathbb{Z}^d\}$ be independent families of i.i.d. random variables as follows: For every $x \in \mathbb{Z}^d$ the random variable η_x has the same distribution as η and describes the initial number of sleeping frogs at vertex x. As we assume that there is one active frog at 0 in the beginning we further set $\eta_0 = 1$. We call η the initial distribution of particles. For every $x \in \mathbb{Z}^d$ and $i \in \mathbb{N}$ the process $(S_n^x(i))_{n \in \mathbb{N}_0}$ is a discrete time nearest neighbour random walk with transition function π and starting point x. If $i \leq \eta_x$, it describes the trajectory of the i-th particle that is initially at vertex x.

For $x, y \in \mathbb{Z}^d$, $x \neq y$, we define $t(x, y) = \min\{n \in \mathbb{N} \colon S_n^x(i) = y, i \leq \eta_x\}$, the first time a particle initially at x reaches vertex y. Here, as in the next definition, we use the convention $\min \emptyset = \infty$. Then

$$T_x = \inf_{\substack{k \in \mathbb{N},\, x_0, x_1, \ldots, x_k \in \mathbb{Z}^d, \\ x_0 = 0,\, x_k = x}} \big\{ t(x_0, x_1) + t(x_1, x_2) + \ldots + t(x_{k-1}, x_k) \big\}$$

is the first time vertex x is visited by an active frog. We call T_x the activation time of x. At that time all frogs at x are activated and start to follow their respective trajectories. The location $Z_n^x(i)$ of the i-th frog initially at vertex $x \in \mathbb{Z}^d$ at time $n \in \mathbb{N}_0$ is then given by

$$Z_n^x(i) = \begin{cases} x & \text{for } n < T_x, \\ S_{n-T_x}^x(i) & \text{for } n \geq T_x. \end{cases}$$

We denote the frog model on the d-dimensional lattice $\mathbb{Z}^d$ with initial distribution η and underlying transition probabilities π by $\mathrm{FM}(d, \pi, \eta)$.

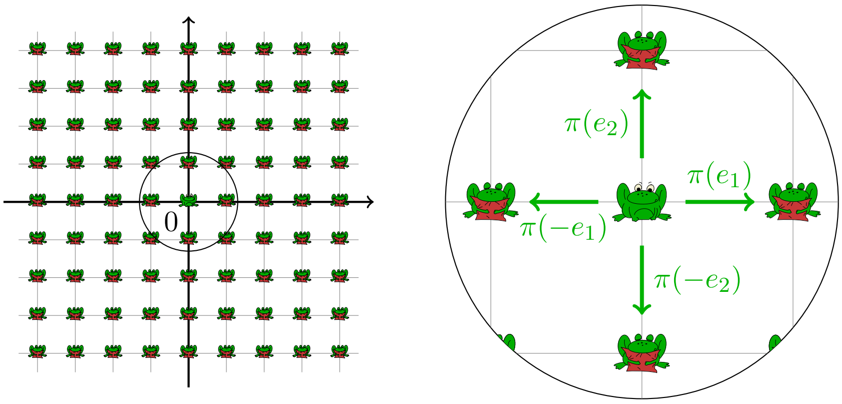

Figure 1.1: The picture on the left shows the set-up at time 0 of the frog model on the lattice $\mathbb{Z}^2$ with initial configuration $\eta \equiv 1$. There is one awake frog at 0 and one sleeping frog at every other vertex. In every step the active frogs independently move to a neighbouring vertex that is chosen according to the transition function π as depicted in the right picture.

The first question usually asked about particle systems deals with recurrence and transience. Is the origin visited infinitely often or does the cloud of particles move away from 0? In the first case we call the frog model recurrent, in the second transient. More precisely, we use the following definition.

Definition 1.1. *The frog model* $\mathrm{FM}(d, \pi, \eta)$ *is called recurrent, if*

$$\mathbb{P}(0 \textit{ is visited infinitely often}) = 1$$

Otherwise, it is called transient.

If the model is recurrent, then all vertices are visited and all particles get activated. The frog model satisfies a zero-one law. This has recently been shown by Kosygina and Zerner in [33]. See also Appendix A in [33] for a comment on the slightly different definition of recurrence used there.

Theorem 1.2 ([33, Theorem 1])**.** *In* $\mathrm{FM}(d, \pi, \eta)$ *the probability that the origin is visited infinitely many times by active frogs is either* 0 *or* 1*.*

In fact, Kosygina and Zerner show that this zero-one law holds in greater generality. It e.g. also applies to frog models that live on countably infinite state spaces and have frog trajectories given by a common transitive and irreducible Markov chain.

As mentioned in the introduction, one of the main topics of this thesis is recurrence and transience of frogs with drift. We restrict ourselves to the most natural initial distribution $\eta \equiv 1$, i.e. there is one particle at every vertex at time 0. This initial set-up is shown in Figure 1.1 for $d = 2$. According to the zero-one law we can classify the transition laws into a recurrent and a transient class. Our results show that both classes exist. We also present more quantitative statements. Further, in dimension $d = 1$, we study the speed of the spread of the frogs as a function of the drift and discuss the distribution of active frogs. The results are presented in Section 1.3.

1.2 A short survey on symmetric frogs

Most papers published on the frog model so far assume that the particles perform simple random walks. In this chapter we shortly summarise known results on this symmetric frog model which are either relevant for the topic of this thesis or of interest in the context. Some of them apply to slightly different versions of the frog model. In these cases we clearly specify the changes.
We denote the transition function corresponding to this symmetric evolution by π_{sym}, i.e. we define the function $\pi_{\text{sym}} \colon \mathcal{E}_d \to [0,1], \pi_{\text{sym}}(e) = \frac{1}{2d}$.

1.2.1 Recurrence and transience

For dimension $d = 1$ and $d = 2$ simple random walk on $\mathbb{Z}^d$ is recurrent. Thus, in the symmetric frog model, already the frog that starts its life at 0 visits the origin infinitely many times almost surely, and hence the frog model is recurrent. In $d = 3$ simple random walk is transient. Therefore, the first frog activates at least one frog at every distance n from 0. A frog activated at such a distance from 0 has probability of order $\frac{1}{n}$ to visit the origin. The Borel-Cantelli Lemma then shows that in $d = 3$ the symmetric frog model is recurrent. In higher dimensions it is not obvious whether the symmetric frog model is recurrent or transient. This question is solved by Telcs and Wormald in [52]. They prove that the frog model is indeed recurrent in any dimension $d \geq 1$ when the underlying random walk is symmetric and the frog model is started with one particle per site. Popov refines this result in [42] by considering a special random initial configuration of frogs. He considered the situation where there is for each $x \in \mathbb{Z}^d \backslash \{0\}$ originally one sleeping frog at x with probability $p(x)$ and no frog with probability $1 - p(x)$, independently of all other vertices, and found the exact rate of decay for the function $p(x)$ to separate transience from recurrence.

Theorem 1.3 ([42, Theorem 1.1]). *For $d \geq 3$ let $p \colon \mathbb{Z}^d \setminus \{0\} \to [0,1]$. Let $\{\eta_x \colon x \in \mathbb{Z}^d\}$ be a collection of independent random variables with $\mathbb{P}(\eta_x = 1) = 1 - \mathbb{P}(\eta_x = 0) = p(x)$ for all $x \in \mathbb{Z}^d \setminus \{0\}$ as well as $\mathbb{P}(\eta_0 = 1) = 1$. Consider the frog model with initial configuration $\{\eta_x \colon x \in \mathbb{Z}^d\}$ and transition function π_{sym}. There exists $\alpha_c = \alpha_c(d) \in (0,\infty)$ such that*

(i) if $\alpha < \alpha_c$ and $p(x) \leq \alpha \|x\|^{-2}$ for all x large enough, then the frog model is transient,

(ii) if $\alpha > \alpha_c$ and $p(x) \geq \alpha \|x\|^{-2}$ for all x large enough, then the frog model is recurrent.

The question of recurrence and transience of the frog model on the d-ary tree has been open for a long time. It was posed in [43] and has recently been mostly answered by Hoffman et al. in [28]. They considered the frog model on the infinite rooted d-ary tree $\mathbb{T}_d$, which denotes the regular tree in which every vertex has $d+1$ neighbours, with initially one sleeping frog at every vertex, i.e. with $\eta \equiv 1$. The frogs all perform simple random walks. In accordance with our notation we denote this model by $\mathrm{FM}(\mathbb{T}_d, \pi_{\mathrm{sym}}, 1)$.

Theorem 1.4 ([28, Theorem 1])**.** *Consider the frog model* $\mathrm{FM}(\mathbb{T}_d, \pi_{sym}, 1)$.

(i) For $d = 2$ *the frog model on* $\mathbb{T}_d$ *is recurrent.*

(ii) For $d \geq 5$ *the frog model on* $\mathbb{T}_d$ *is transient.*

The transience proof relies on a comparison with branching random walks. To prove recurrence, Hoffman et al. derive a recursion relation for the probability generating function of the number of visits to the root. They further conjecture that the frog model is recurrent for $d = 3$ and transient for $d = 4$. In [29] and [30] Hoffman, Johnson and Junge study recurrence and transience of the frog model on the regular tree with the initial number of frogs being Poisson distributed. Interesting in this context is also the article [31]. In this paper Johnson and Junge show that certain frog model statistics are monotone in the initial configuration. Their prime example for a suitable statistic is the number of returns of active particles to the origin. In [47] Rosenberg proves that the frog model on the $3,2$-alternating tree is recurrent. The $3,2$-alternating tree is the rooted tree in which nodes have 2, respectively 3 children, depending on whether they are in an even or odd generation.

1.2.2 Frogs with death

One modification of the model is the frog model with death. Here, activated particles are allowed to disappear after a random, e.g. geometric, lifetime. Let $s \in [0,1]$. After being activated, every active frog dies at every step with probability $1-s$ independently of everything else. The parameter s is called the survival probability. Otherwise, the frog model is defined as usual. We denote this frog model on $\mathbb{Z}^d$ by $\mathrm{FM}^*(d, \pi, \eta, s)$ if the underlying random walk has transition function π and random initial configuration distributed according to η.

In the symmetric case, i.e. with the underlying random walk being simple, this model was intensely analysed in [1] by Alves et al. Furthermore, it is discussed in [18] and [35]. Also, results on the frog model on regular trees are presented in [1]. We summarise a selection of the results in [1]. Some of them are relevant for the proofs of the results of this thesis.

The first question that comes to one's mind is whether or not this particle systems survives.

Definition 1.5. *The frog model with death survives if at any time there is at least one active frog. Otherwise, we say that it dies out.*

As the probability $\mathbb{P}(\mathrm{FM}^*(d, \pi, \eta, s) \text{ survives})$ is an increasing function in s, we can further define

$$s_c(d, \pi, \eta) = \inf\{s \colon \mathbb{P}(\mathrm{FM}^*(d, \pi, \eta, s) \text{ survives}) > 0\}.$$

First, it is established in [1] that, under weak conditions on η, the one-dimensional frog model $\mathrm{FM}^*(1, \pi_{\mathrm{sym}}, \eta, s)$ becomes extinct almost surely for every $s < 1$.

Theorem 1.6 ([1, Theorem 1.1])**.** *Consider the frog model with death for* $d = 1$ *and transition function* π_{sym}*. If* $\mathbb{E}[\ln^+(\eta)] < \infty$*, then* $s_c(1, \pi_{sym}, \eta) = 1$*.*

For $d \geq 2$ the frogs have a chance to survive if s is sufficiently close to 1. Further, Alves et al. describe sufficient conditions for extinction for small s.

Theorem 1.7 ([1, Theorem 1.4],[1, Theorem 1.3])**.** *Consider the frog model with death for* $d \geq 2$ *and transition function* π_{sym}*.*

(i) If $\mathbb{P}(\eta \geq 1) > 0$*, then* $s_c(d, \pi_{sym}, \eta) < 1$*.*

(ii) If $\mathbb{E}[(\log^+(\eta))^d] < \infty$*, then* $s_c(d, \pi_{sym}, \eta) > 0$*.*

Alves et al. also prove asymptotics for the critical survival parameter.

Theorem 1.8 ([1, Theorem 1.8])**.** *In the symmetric frog model with death, if* $\mathbb{E}[\eta] < \infty$*, then*

$$\lim_{d \to \infty} s_c(d, \pi_{sym}, \eta) = \frac{1}{1 + \mathbb{E}[\eta]}.$$

This result in particular implies that for all $s > \frac{1}{1+\mathbb{E}[\eta]}$ the frog model $\mathrm{FM}^*(d, \pi_{\mathrm{sym}}, s)$ survives with positive probability for d large enough. Dominating the frog model with a branching random walk it is also easy to see that $s_c(d, \pi_{\mathrm{sym}}, \eta) > \frac{1}{1+\mathbb{E}[\eta]}$ for all $d \geq 1$ and η with $\mathbb{E}[\eta] < \infty$. For details see [1, Proposition 1.1]. This suggests that $d \mapsto s_c(d, \pi_{\mathrm{sym}}, \eta)$ is monotone decreasing. Whether or not this is true, is also asked in [1].
Further, Alves et al. study recurrence and transience of the frog model with death in [2].

Theorem 1.9 ([1, Theorem 1.10], [1, Theorem 1.12])**.** *For* $d \geq 1$ *consider the frog model* $\mathrm{FM}^*(d, \pi_{sym}, \eta, s)$*.*

(i) If $\mathbb{E}[(\ln^+(\eta))^d] < \infty$*, then the probability that the origin* 0 *is hit infinitely many times is* 0 *for all* $s \in [0, 1)$*.*

(ii) If there is $\beta < d$ *such that* $\mathbb{P}(\eta \geq n) \geq \frac{1}{(\ln n)^\beta}$ *for all* n *large enough, then the probability that* 0 *is hit infinitely many times is positive for all* $s \in (0, 1]$*.*

For the frog model with death that starts with one sleeping particle at every vertex, i.e. $\eta \equiv 1$, these theorems have the following consequences: In dimension $d = 1$ the process dies out for all $s < 1$. For $d \geq 2$ the model exhibits a phase transition in the survival parameter s, i.e. $0 < s_c(d, \pi_{\mathrm{sym}}, 1) < 1$. Furthermore, in any dimension $d \geq 1$, the frog model with death is transient for any $s < 1$.

1.2.3 A shape theorem

In the symmetric frog model the set of vertices visited by active frogs, rescaled by time, converges to a convex set. This shape theorem is proven by Alves et al. in [2] for the frog model with initially one frog per vertex. It is generalised in [3] to the frog model with an i.i.d. random initial configuration. Recall that T_x denotes the activation time of vertex x. Let ξ_n be the set of all sites visited by active frogs by time n, i.e.

$$\xi_n(\eta) = \{x \in \mathbb{Z}^d \colon T_x \leq n\}.$$

Further, we define

$$\bar{\xi}_n(\eta) := \Big\{x + \Big(-\frac{1}{2}, \frac{1}{2}\Big]^d \colon x \in \xi_n(\eta)\Big\} \subseteq \mathbb{R}^d.$$

Theorem 1.10 ([3, Theorem 1.1])**.** *For any dimension $d \geq 1$ there is a non-empty convex symmetric set $\mathcal{A} = \mathcal{A}(d, \eta) \subseteq \mathbb{R}^d$, $\mathcal{A} \neq \{0\}$, such that for almost all initial configurations η and for any $0 < \varepsilon < 1$ we have*

$$(1-\varepsilon)\mathcal{A} \subseteq \frac{\bar{\xi}_n}{n} \subseteq (1+\varepsilon)\mathcal{A}$$

for all n large enough $\mathbb{P}_\eta$-almost surely. Here $\mathbb{P}_\eta = \mathbb{P}(\cdot|\eta)$ is the conditional measure for a given initial configuration η.

The proof of this shape theorem relies on Liggett's subadditive ergodic theorem. The authors of [3] and [2] also make some comments on the shape of $\mathcal{A}$, but this question is harder than the existence of $\mathcal{A}$. Of course, $\mathcal{A} \subseteq \{x \in \mathbb{R}^d \colon \|x\|_1 \leq 1\}$. Equality holds here, if the initial distribution η is heavy-tailed enough. See [3, Theorem 1.2] for details.

1.2.4 Further results

In [8] Junge et al. consider the symmetric frog model on the complete finite graph with n vertices and study the asymptotics in n of the wake-up time. The wake-up time is the expected time it takes until every vertex is visited.
An overview and a collection of problems up to the year 2003 can also be found in [43].

1.3 Frogs with drift

What happens if a drift is imposed on the frog model on $\mathbb{Z}^d$? As mentioned in the introduction, this is the leading questions of this thesis. In order to avoid technical problems and to be able to give more quantitative statements, we focus on a model in which the particles perform nearest neighbour random walks which are balanced in all but one direction. We expect that our methods also work for a frog model with a more general drift. More precisely, we assume that all particles move according to the following transition probabilities which depend on two parameters $w \in [0,1]$ and $\alpha \in [0,1]$:

$$\pi_{w,\alpha}(e) = \begin{cases} \frac{w(1+\alpha)}{2} & \text{for } e = e_1 \\ \frac{w(1-\alpha)}{2} & \text{for } e = -e_1 \\ \frac{1-w}{2(d-1)} & \text{for } e \in \{\pm e_2, \dots \pm e_d\} \end{cases} \tag{1.1}$$

The parameter w is the weight of the drift direction $\pm e_1$, i.e. the random walk chooses to go in direction $\pm e_1$ with probability w. The parameter α describes the strength of the drift. If the random walk has chosen to move in drift direction, it takes a step in direction e_1 with probability $\frac{1+\alpha}{2}$ and in direction $-e_1$ with probability $\frac{1-\alpha}{2}$. All other directions are balanced. As $\alpha \in [0,1]$, the model has a drift to the right.
In dimension $d = 1$ we have to demand $w = 1$. Thus, the transition probabilities on $\mathbb{Z}$ are defined by

$$\pi_\alpha(e_1) = 1 - \pi_\alpha(-e_1) = \frac{1+\alpha}{2}. \tag{1.2}$$

Figure 1.2 illustrates the behaviour of the frog model with transition function $\pi_{w,\alpha}$. We simulated the frog model $\mathrm{FM}(2, \pi_{w,\alpha}, 1)$ for various parameters (α, w) as indicated in the figure. Each of the pictures in the panel belongs to one pair (α, w) of parameters and shows the subset $\{x \in \mathbb{Z}^2 \colon \|x\|_\infty \leq 500\}$ of $\mathbb{Z}^2$. Vertices that are occupied by active frogs at time 500 in the simulation are coloured green. In the leftmost column the shape of the cloud of frogs for $\alpha = 0$ and various weight parameters can be seen. In this case the transition probabilities are balanced and the shape of the cloud of frogs is symmetric. Looking at the pictures in one row from left to right shows how the shape changes when the drift parameter is increased. The higher the drift, the more arrow-like the shape looks and the more to the right the cloud of particles has moved. The columns show the impact of the weight parameter. The higher the weight on the drift direction is, the stronger the effect of the drift becomes. If the cloud of frogs moves away from the origin that is located in the middle of each of the pictures, then the frog model is transient. This behaviour can be seen in the pictures at the top right corner of the panel. If the cloud of frogs grows in all directions and stays located around the origin, then the model is recurrent. This can be most clearly seen in the leftmost column showing the balanced case, but also in the pictures corresponding to small drift α or small weight w. In the next section we will describe these phenomena mathematically.

1.3.1 Recurrence and transience

Let us again first summarise known results, to put our results into context. For dimension $d = 1$ the question of recurrence and transience has been settled by Gantert and Schmidt in [22]. They consider both fixed and i.i.d. random initial configurations $(\eta_x)_{x \in \mathbb{Z} \setminus \{0\}}$ of sleeping frogs and derive a criterion separating transience from recurrence.

Theorem 1.11 ([22, Theorem 2.2]). *For $d = 1$ and $\alpha \in (0,1)$ the frog model $\mathrm{FM}(1, \pi_{1,\alpha}, \eta)$ is recurrent if and only if $\mathbb{E}[\ln^+(\eta)] = \infty$.*

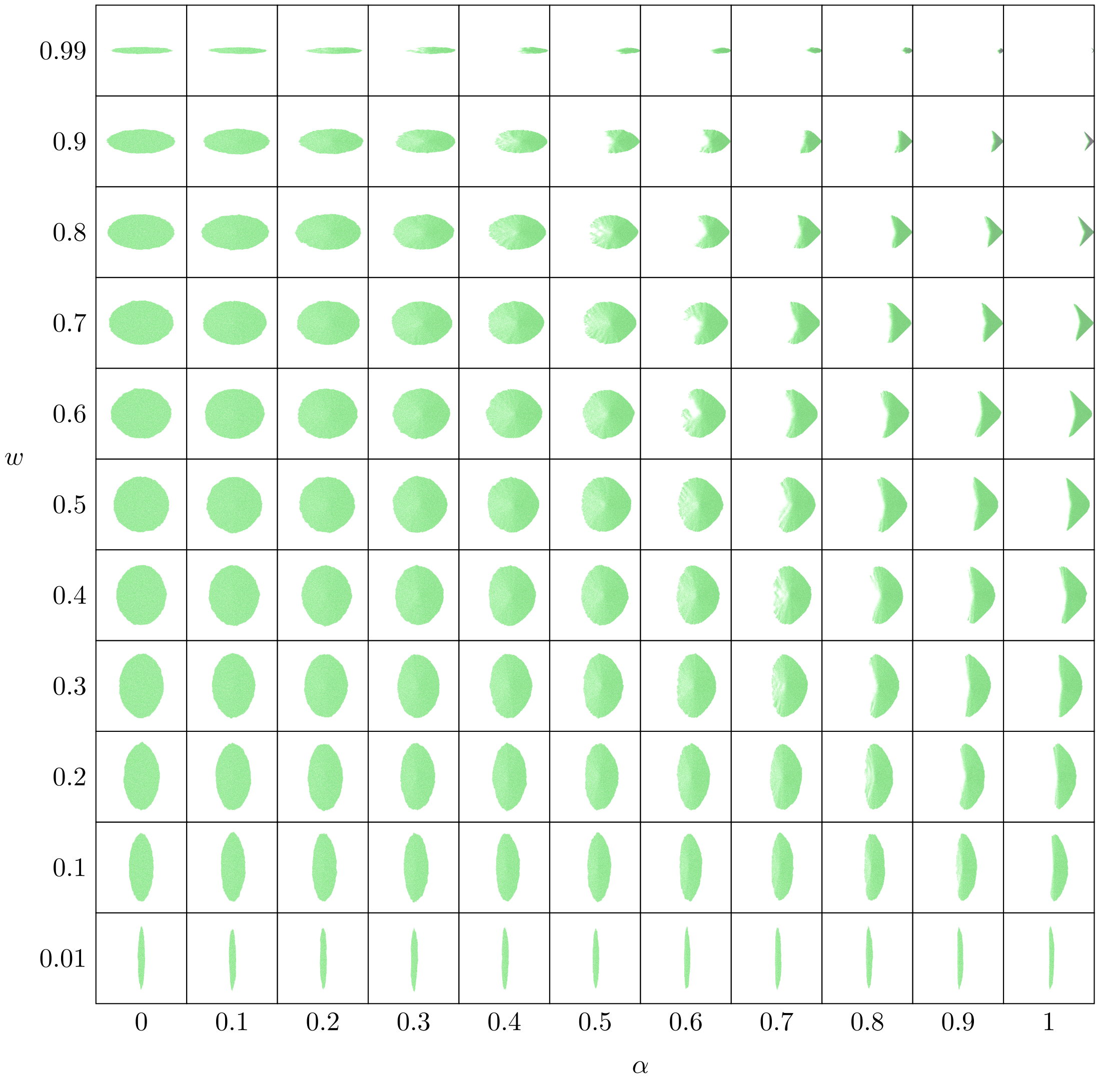

Figure 1.2: We have simulated the frog model $\mathrm{FM}(2, \pi_{w,\alpha}, 1)$ for parameters (α, w) as indicated above. Each of the pictures contained in the figure shows the cloud of active frogs after 500 steps in time. In each picture the origin is located in the middle.

In the case of one frog per site initially, the system is transient for any positive drift. This result is generalised to a nonhomogeneous version of the frog model by Rosenberg in [46]. In that paper the transition probabilities of the frogs may depend on their starting points. Also, the initial numbers η_x do not need to be identically distributed.

The recurrence part of Theorem 1.11 is generalised to any dimension d by Döbler and Pfeifroth in [17].

Theorem 1.12 ([17, Theorem 2.1])**.** *Let* $d \geq 1$*. If* $\mathbb{E}\left[\ln^+(\eta)^{(d+1)/2}\right] = \infty$*, then the frog model* $\mathrm{FM}(d, \pi_{w,\alpha}, \eta)$ *is recurrent.*

In [33] Kosygina and Zerner also show a sufficient recurrence criterion involving the tail behaviour of η. They derive it for a rather large class of frog models on $\mathbb{Z}^d$. They only assume that the nearest neighbour transition probabilities are bounded from below. Restricted to our set-up, the theorem can be stated as follows.

Theorem 1.13 ([33, Theorem 5]). *Let $d \geq 1$ and $\varepsilon > 0$. Assume $\pi(e) \geq \varepsilon$ for all $e \in \mathcal{E}_d$. Then there is a constant $c = c(d, \varepsilon) > 0$ such that if*

$$\mathbb{P}(\eta(0) > s) \geq \frac{c}{(\ln s)^d}$$

for all large s, then with probability 1 the frog model $\mathrm{FM}(d, \pi, \eta)$ is recurrent.

Kosygina and Zerner further show in [33, Proposition 23] that, if $\mathbb{E}\big[\ln^+(\eta)^d\big] < \infty$ and all frogs have a drift away from the origin, then the frog model is transient. Note that in this scenario the underlying transition matrix is not translation invariant.

Zerner also shortly presents a result on the frog model in [56] as an application of his results on autoregressive processes. He considers a one-dimensional frog model with death, initially i.i.d. many frogs on every non-negative site and a drift to the left. This model is almost surely transient if and only if there are not too many frogs initially, depending on the death and drift parameters.

Remarkably, the sharp criterion of Theorem 1.11 for the one-dimensional frog model only depends on the distribution of η and does, in particular, not depend on the value of the drift. The results below show that such a sharp criterion separating transience from recurrence that depends only on the distribution of η cannot exist in higher dimensions. Indeed, we show that for $\eta \equiv 1$ and $d \geq 2$ the frog model can be recurrent as well as transient depending on the drift.

Let us from now on assume that at each vertex in $\mathbb{Z}^d \setminus \{0\}$ there is exactly one sleeping frog at time 0. Further, we assume for the rest of this section that $d \geq 2$, and we concentrate on transition probabilities given by (1.1).

First, we discuss the extreme cases. For $w = 1$ the frog model is one-dimensional. This can also be seen in the topmost row in Figure 1.2. Thus, it is transient for any $\alpha \in (0, 1]$ and recurrent for $\alpha = 0$ by Theorem 1.11.

For $\alpha = 1$ one easily checks that it is transient for any $w \in (0, 1]$. Indeed, note that no vertex in the half-space $\{x \in \mathbb{Z}^d \colon x_1 < 0\}$ is ever visited. In Figure 1.2 in the rightmost column there are therefore no particles in the left half of the pictures. Further, no particle that is initially in the half-space $\{x \in \mathbb{Z}^d \colon x_1 > 0\}$ can ever reach 0. The probability that a particle originating in the hyperplane $\{x \in \mathbb{Z}^d \colon x_1 = 0\}$ ever reaches 0 decays exponentially with the distance of its starting point from 0. Therefore, only finitely many particles starting in the hyperplane $\{x \in \mathbb{Z}^d \colon x_1 = 0\}$ visit the origin.

If $w = 0$, then $\mathrm{FM}(d, \pi_{0,\alpha})$ is equivalent to the symmetric frog model in $d - 1$ dimensions and hence recurrent by Theorem 1.3. If $\alpha = 0$ we are back in the balanced case and the model is recurrent. This follows from Theorem 1.14 (i) and Theorem 1.16 below.

In dimension $d = 2$ the frog model is recurrent whenever α or w are sufficiently small, i.e. if the underlying transition mechanism is almost balanced. The frog model is transient for α or w close to 1.

Theorem 1.14. *Let $d = 2$ and $w \in (0, 1)$.*

(i) There exists $\alpha_r = \alpha_r(w) > 0$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is recurrent for all $0 \leq \alpha \leq \alpha_r$.

(ii) There exists $\alpha_t = \alpha_t(w) < 1$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is transient for all $\alpha_t \leq \alpha \leq 1$.

Theorem 1.15. *Let $d = 2$ and $\alpha \in (0, 1)$.*

(i) There exists $w_r = w_r(\alpha) > 0$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is recurrent for all $0 \leq w \leq w_r$.

(ii) There exists $w_t = w_t(\alpha) < 1$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is transient for all $w_t \leq w \leq 1$.

In dimension $d \geq 3$ the frog model is also recurrent if the transition probabilities are almost balanced. Further, for any fixed drift parameter $\alpha \in (0, 1]$ it is transient if the weight w is close to 1. However, in contrast to $d = 2$, for fixed $w \in [0, 1)$ there is not necessarily a transient regime. This follows from Theorem 1.17 (i) below.

Theorem 1.16. *Let $d \geq 3$ and $w \in (0, 1)$. There exists $\alpha_r = \alpha_r(d, w) > 0$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is recurrent for all $0 \leq \alpha \leq \alpha_r$.*

Theorem 1.17. *Let $d \geq 3$ and $\alpha \in (0, 1)$.*

(i) There exists $w_r > 0$, independent of d and α, such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is recurrent for all $0 \leq w \leq w_r$.

(ii) There exists $w_t = w_t(\alpha) < 1$, independent of d, such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is transient for all $w_t \leq w \leq 1$.

The results are graphically summarised in Figure 1.3. Note that the above theorems only make statements about the existence of recurrent, respectively transient regimes. We do not describe their shapes, as might be suggested by the curves depicted in Figure 1.3. However, we believe that there is a monotone curve separating the transient from the recurrent regime in the phase diagram shown in Figure 1.3.

Conjecture 1.18. *For any $d \geq 2$ there is a decreasing function $f_d\colon [0, 1] \to [0, 1]$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$ is recurrent for all $w, \alpha \in [0, 1]$ with $w < f_d(\alpha)$, and transient for all $w, \alpha \in [0, 1]$ with $w > f_d(\alpha)$.*

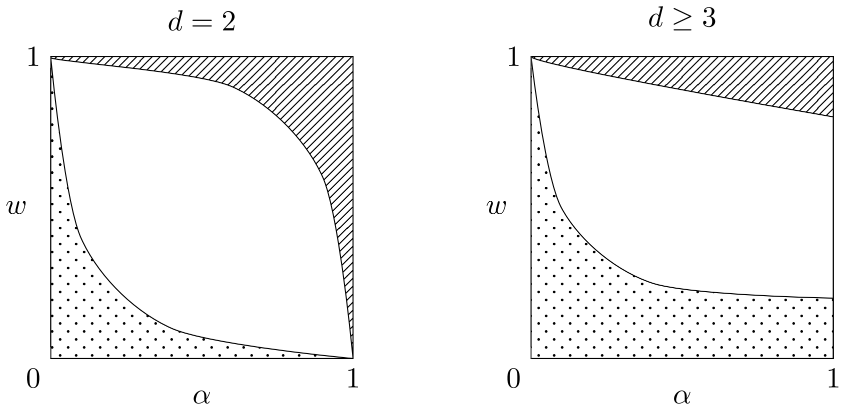

Figure 1.3: Phase diagram for the frog model $\mathrm{FM}(d, \pi_{w,\alpha}, 1)$: the recurrent regime is marked by ⬚, the transient one by ▨.

This conjecture is also supported by simulations. Figure 1.4 gives an idea of the form of the two regimes. We simulated 500 steps of the frog model in $d = 2$ and $d = 3$ with initially one frog per site, and counted the number of visits of active frogs to the origin. Besides showing the monotone curve separating the two regimes as described in Conjecture 1.18, the figure shows monotonicity in the dimension d. In $d = 3$, the transient regime seems to be smaller than in $d = 2$. We can also give an intuition for this effect. Intuitively, the frog model approximates a binary branching random walk for $d \to \infty$ from below, as each frog activates a new frog in every step if there are “infinitely” many directions to choose from. This leads to the following conjecture.

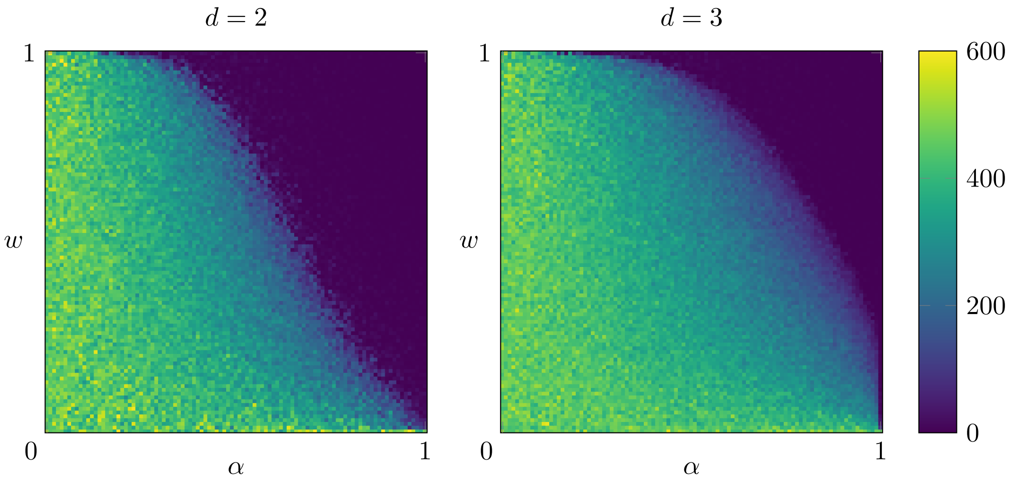

Figure 1.4: This figure shows the results of simulations of the frog model in $d = 2$ and $d = 3$ for 500 steps in time and various pairs of parameters (α, w). It colour codes the number of visits of active particles to the origin.

Conjecture 1.19. *The sequence of functions $(f_d)_{d \geq 2}$ is increasing in d.*

We will describe some further intuition on why we believe in this conjecture at the end of Section 2.2. For the binary branching random walk the recurrent and transient regimes are known precisely. Let

$$g\colon [0,1] \to [0,1],\ g(\alpha) = \min\left\{1, \left(2\left(1-\sqrt{1-\alpha^2}\right)\right)^{-1}\right\}.$$

A binary branching random walk on $\mathbb{Z}^d$ with transition probabilities as in (1.1) is recurrent iff $w < g(\alpha)$, see [20, Section 4]. The phase diagram is shown in Figure 1.5. The comparison of the frog model and a binary branching random walk also raises the following question.

Question 1.20. *Does the sequence of functions $(f_d)_{d\geq 2}$ converge pointwise to g as $d \to \infty$?*

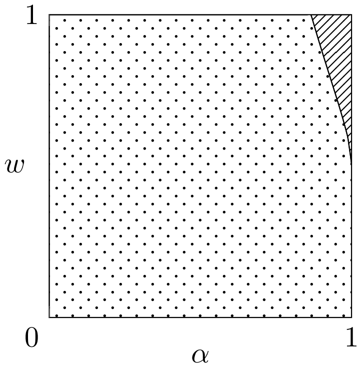

Figure 1.5: Phase diagram for the binary branching random walk when the underlying transition probabilities are given by $\pi_{w,\alpha}$: the recurrent regime is marked by ⬚, the transient one by ▨.

1.3.2 Speed of the fastest frog

Consider the one-dimensional frog model $\mathrm{FM}(1, \pi_\alpha, 1)$. How does the set of sites occupied by active frogs evolve over time? Theorem 1.11 implies that this frog model is transient for any $\alpha > 0$. Thus, the set of occupied sites moves to the right. In this section we study how fast it moves. We therefore consider the leftmost and rightmost occupied vertex, i.e. the positions of the “slowest” and “fastest” frogs, and discuss their speed. In particular, we are interested in the dependence of the speed on the drift parameter α.

Figure 1.6 shows the paths of all active frogs in a simulation of the frog model $\mathrm{FM}(1, 0.2, 1)$. For a particle initially at $x \in \mathbb{Z}$ the function $n \mapsto Z_n^x$ is plotted for $n \geq T_x$. The speed of the minimal and maximal occupied site corresponds to the asymptotic slope of the lower, respectively upper boundary of the coloured region containing all frog paths.

Before we start, let us mention two papers that deal with related questions. In [23] Ghosh et al. study a frog model on $\mathbb{Z}$ with drift and calculate the moments of the leftmost visited site. For a continuous time version of the frog model the behaviour of the right front is studied in [13] and [5].

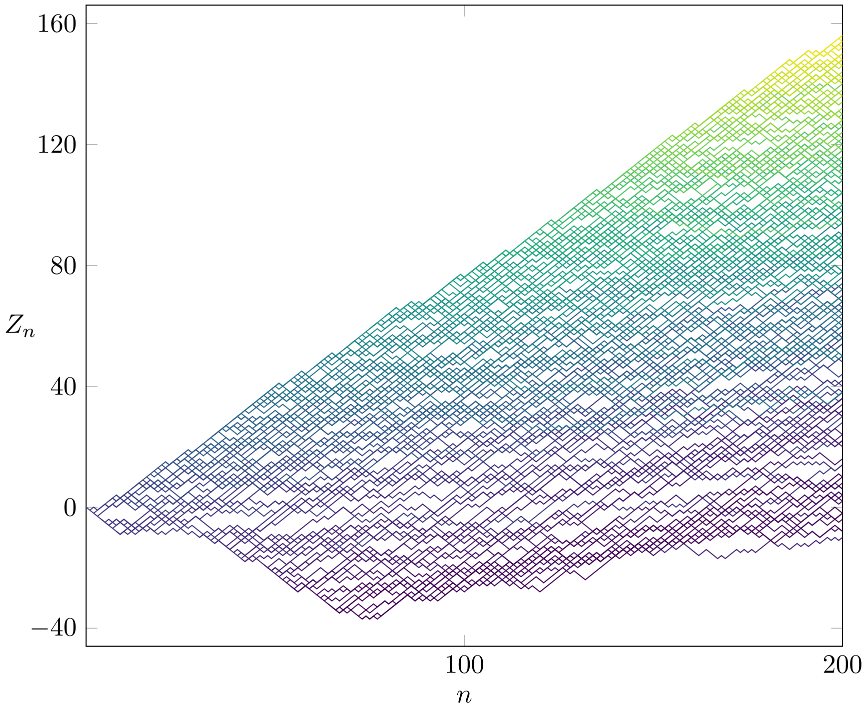

Figure 1.6: A simulation of all frog paths in $\mathrm{FM}(1, 0.2, 1)$ for 200 steps in time. Each path is coloured according to the starting point of the frog. Whenever paths overlap, the one belonging to the frog that was activated last is displayed.

Let A_n denote the set of active frogs at time n, i.e. $A_n = \{x \in \mathbb{Z} \colon T_x \leq n\}$. We define

$$M_n = \max_{x \in A_n} Z_n^x \quad \text{and} \quad m_n = \min_{x \in A_n} Z_n^x.$$

M_n describes the maximum and m_n the minimum of the locations of the active frogs at time n. We refer to M_n and m_n as the maximum and the minimum. One can show that there are non-zero constants $v_{\max}$ and $v_{\min}$ such that

$$v_{\max} = \lim_{n \to \infty} \frac{M_n}{n} \quad \text{a.s.}$$
$$v_{\min} = \lim_{n \to \infty} \frac{m_n}{n} \quad \text{a.s.}$$

The existence of $v_{\max}$ is well known. If follows from the shape theorem, i.e. Theorem 1.10 and is also discussed in Section 2.4. The existence of $v_{\min}$ is part of Theorem 1.21 below. We call $v_{\max}$ the speed of the maximum and $v_{\min}$ the speed of the minimum. We study $v_{\max}$ and $v_{\min}$ as functions of the drift parameter α. First, we show that the speed of the minimum equals the speed of a single frog.

Theorem 1.21. *For $\alpha > 0$ the speed of the minimum in the frog model* $\mathrm{FM}(1, \pi_\alpha, 1)$ *exists and is given by*

$$v_{\min} = \alpha.$$

We unfortunately cannot give a precise formula for the speed of the maximum. The following two theorems describe some properties of the speed.

Theorem 1.22. *The speed of the maximum in the frog model* $\mathrm{FM}(1, \pi_\alpha, 1)$ *is an increasing function of* α.

Theorem 1.23. *For the frog model* $\mathrm{FM}(1, \pi_\alpha, 1)$ *with* $\alpha < 1$ *it holds that* $v_{\max} < 1$.

In comparison to the last result note that for branching random walk on $\mathbb{Z}$ with binary branching the speed of the maximum equals 1 for every $\alpha \in [0, 1]$, see for example [41, Theorem 18.3] or Chapter 3.
Simulations suggest that the speed of the maximum is a concave function in the drift parameter α. See Figure 1.7.

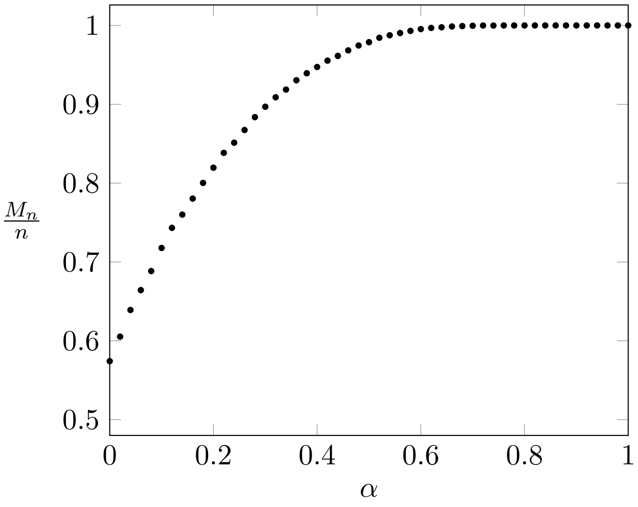

Figure 1.7: The figure shows results of simulation of $\frac{M_n}{n}$ for $n = 100000$ and various $\alpha \in [0, 1]$.

A heuristic argument might be the following. We expect that the number of frogs in the maximum converges to a stationary distribution τ_α for $\alpha < 1$. Therefore, the speed of the maximum should equal

$$\sum_{x \in \mathbb{N}} \tau_\alpha(x) \Big(1 - 2\Big(\frac{1-\alpha}{2}\Big)^x\Big).$$

If τ_α was independent of α, this would be a concave function. However, we believe that the dependence on α does not destroy the concavity.

Conjecture 1.24. *The speed of the maximum is a concave function of* α.

In addition to studying the behaviour of the minimum and the maximum we investigate the distribution of the active frogs. In the limit they are distributed uniformly in between the minimum and the maximum. To make this statement precise we rescale the positions of all active frogs at time n roughly to the interval $[0, 1]$ and then consider the empirical distribution μ_n, which is defined for $\alpha < 1$ by

$$\mu_n(B) = \frac{1}{|A_n|} \sum_{x \in A_n} \mathbb{1}_{\left\{ \frac{Z_n^x - v_{\mathrm{min}} n}{(v_{\mathrm{max}} - v_{\mathrm{min}}) n} \in B \right\}}$$

for every Borel set $B \subseteq [0,1]$. Recall that A_n denotes the set of active frogs at time n. Note that μ_n is a random measure.

Theorem 1.25. *Consider the frog model* $\mathrm{FM}(1, \pi_\alpha, 1)$ *with* $\alpha \in [0,1]$. *Almost surely, as* $n \to \infty$, *the empirical distribution* μ_n *converges weakly to the Lebesgue measure* λ *on* $[0,1]$.

Our interest in the distribution of active particles was motivated by the following idea. Assume we knew that at every vertex occupied by frogs there were approximately the same number k of frogs. Then we could count the number of active frogs in two ways. The largest occupied vertex is approximately nv_{max}, the smallest nv_{min}. For parity reasons only every second vertex is occupied at a time. Hence, the number of active frogs should approximately be $k \cdot n(v_{\mathrm{max}} - v_{\mathrm{min}}) \cdot \frac{1}{2}$. On the other hand, it should also be close to nv_{max}, as only finitely many frogs on negative sites are activated due to transience. This argument, which is illustrated in Figure 1.8, gives $k \cdot n(v_{\mathrm{max}} - v_{\mathrm{min}}) \cdot \frac{1}{2} \approx nv_{\mathrm{max}}$. Thus, if we knew k, we had a formula for v_{max}. In average $\frac{1+\alpha}{2}k$ of the k frogs located at the maximum go to the right in the next step and activate one new frog. Thus, in equilibrium the number k of frogs should satisfy $\frac{1+\alpha}{2}k + 1 \approx k$, which would give k. Unfortunately, the behaviour at the front is more complicated. We conjecture that the number of frogs at the front is actually larger.

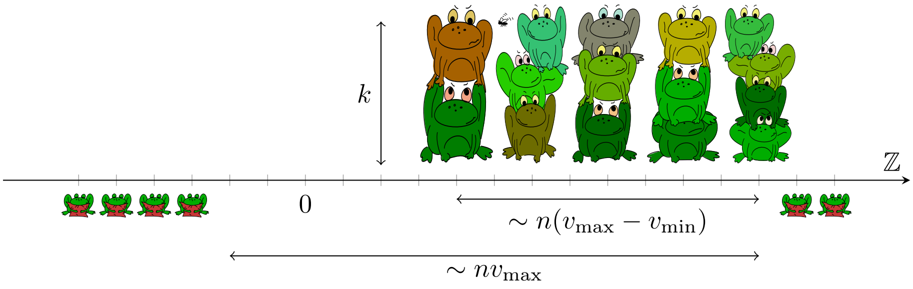

Figure 1.8: At time n, the number of active frogs is approximately nv_{max}. They are spread out uniformly in the interval $[nv_{\mathrm{min}}, nv_{\mathrm{max}}]$. Suppose we knew that there are approximately k frogs at every occupied vertex and suppose we knew k, then we could calculate the number of active frogs in a second way, multiplying k and the number of occupied sites. This would lead to a formula for the speed of the maximum v_{max}. This idea motivated our study of the distribution of active frogs.

2 Proofs for the Frog Model

Before we prove the theorems stated in Chapter 1, we need to introduce more notation. Further, we collect some basic facts about random walks on $\mathbb{Z}^d$ that are used in the proofs. This is done in Section 2.1. In Section 2.2 all statements concerning recurrence are proved, i.e. Theorems 1.14 (i), 1.15 (i), 1.16, and 1.17 (i). In Section 2.2 we show the claims made about transience of the frog model, i.e. Theorems 1.14 (ii), 1.15 (ii), and 1.17 (ii). Finally, Section 2.4 deals with the statements on the speed of the minimum and maximum of the cloud of frogs in the one-dimensional frog model, which are presented in Section 1.3.2.

2.1 Preliminaries

2.1.1 Notation

Recall that we refer to the frog model on $\mathbb{Z}^d$ with initial distribution η and transition function π as $\mathrm{FM}(d, \pi, \eta)$. As we only work with $\eta \equiv 1$ here, we abbreviate this notation and write $\mathrm{FM}(d, \pi)$ instead of $\mathrm{FM}(d, \pi, 1)$. Similarly, we drop the reference to η in the other notation introduced in the Chapter 1: We write $(S_n^x)_{n \in \mathbb{N}_0}$ instead of $(S_n^x(1))_{n \in \mathbb{N}_0}$ for the trajectory of the first and only frog initially at vertex $x \in \mathbb{Z}^d$. Further, we denote the position at time $n \in \mathbb{N}_0$ of the frog initially at $x \in \mathbb{Z}^d$ by Z_n^x instead of $Z_n^x(1)$.

To keep the sentence structure simple we from now on refer to the frog that is initially at vertex $x \in \mathbb{Z}^d$ as "frog x".

For $x, y \in \mathbb{Z}^d$ we write $x \to y$ if frog x (potentially) ever visits y, i.e. $y \in \{S_n^x \colon n \in \mathbb{N}_0\}$. For $x, y \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$ we say that there exists a frog path from x to y in A and write $x \overset{A}{\rightsquigarrow} y$ if there exist $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in A$ such that $x \to z_1$, $z_i \to z_{i+1}$ for all $1 \leq i < n$ and $z_n \to y$, or if $x \to y$ directly. Note that x and y are not necessarily in A. Also the trajectories of the frogs z_i, $1 \leq i \leq n$, do not need to be in A.

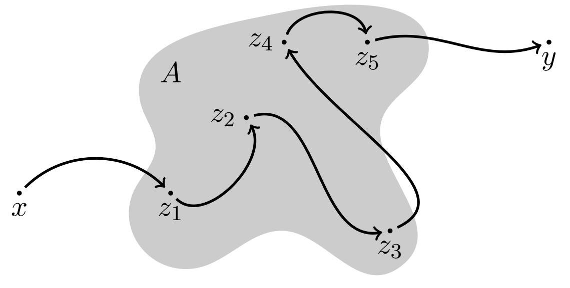

Figure 2.1: A frog path from x to y in a set A.

For $x \in \mathbb{Z}^d$ we call the set

$$FC_x = \left\{ y \in \mathbb{Z}^d \colon x \overset{\mathbb{Z}^d}{\rightsquigarrow} y \right\} \tag{2.1}$$

the frog cluster of x. Note that, if frog x ever becomes active, then every frog $y \in FC_x$ is also activated. Observe that to decide whether the system is transient or recurrent, we do not need to know the exact activation times of the frogs. Only the information whether or not a frog is activated is important. Thus, this notation comes in handy in the proofs concerning recurrence and transience. The activation times of the particles become relevant for the proofs concerning the speed of the frogs.

In the recurrence and transience proofs we often use the $(d-1)$-dimensional hyperplane H_n in $\mathbb{Z}^d$ defined by

$$H_n := \{x \in \mathbb{Z}^d \colon x_1 = n\} \tag{2.2}$$

for $n \in \mathbb{Z}$.

2.1.2 Hitting probabilities of random walks

We need to deal with hitting probabilities of random walks on $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$ recall that $\{x \to y\}$ denotes the event that the random walk started at x ever visits the vertex y. Analogously, for $A \subseteq \mathbb{Z}^d$ we write $\{x \to A\}$ for the event that the random walk started at x ever visits a vertex in A.

Lemma 2.1. *For $d \geq 3$ consider a random walk on $\mathbb{Z}^d$ with transition function $\pi_{w,0}$. There exists a constant $c = c(d, w) > 0$ such that for all $x \in \mathbb{Z}^d$*

$$\mathbb{P}(0 \to x) \geq c \|x\|_2^{-(d-2)},$$

where $\|x\|_2 = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$ is the Euclidean norm.

A proof of the lemma for simple random walk, i.e. with transition function π_{sym}, can e.g. be found in [2, Theorem 2.4] and [1, Lemma 2.4]. It relies on well known estimates of the Green function. The proof can immediately be generalised to our set-up using the Green function estimates provided by [34, Theorem 2.1.3].

Lemma 2.2. *For $d \geq 1$ consider a random walk on $\mathbb{Z}^d$ with transition function $\pi_{w,\alpha}$. Then for each $\gamma > 0$ there is a constant $c = c(d, \gamma, w, \alpha) > 0$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ with $x_1 = -n$ and $|x_i| \leq \gamma\sqrt{n}$, $2 \leq i \leq d$, it holds that*

$$\mathbb{P}(x \to 0) \geq c n^{-(d-1)/2}.$$

For a proof see e.g. [17, Lemma 3.1].

Lemma 2.3. *For $d \geq 1$ consider a random walk on $\mathbb{Z}^d$ with transition function $\pi_{w,\alpha}$. Then for every $n \in \mathbb{N}$ and H_{-n} as defined in* (2.2)

$$\mathbb{P}(0 \to H_{-n}) = \left(\frac{1-\alpha}{1+\alpha}\right)^n.$$

Proof. As $\mathbb{P}(0 \to H_{-n}) = \mathbb{P}(0 \to H_{-1})^n$ for $n \in \mathbb{N}$, it suffices to prove the lemma for $n = 1$. By the Markov property

$$\mathbb{P}(0 \to H_{-1}) = \frac{1-\alpha}{2} + \frac{1+\alpha}{2}\mathbb{P}(0 \to H_{-2}).$$

The results follows after a straightforward calculation.

2.2 Recurrence

To prove recurrence we make use of the theory of independent site percolation on $\mathbb{Z}^d$. The basic idea is to dominate the frog cluster of the origin by a supercritical percolation cluster. Then we can employ the knowledge about the structure of such a percolation cluster. We therefore give a brief introduction to percolation here.

2.2.1 Some facts about percolation

Let $p \in [0,1]$. Every site in $\mathbb{Z}^d$ is independently of the other sites declared open with probability p and closed with probability $1-p$. An open cluster is a connected component of the subgraph induced by all open sites. It is well known that for $d \geq 2$ there is a critical parameter $p_c = p_c(d) \in (0,1)$ such that for all $p > p_c$ (supercritical phase) there is a unique infinite open cluster C almost surely, and for $p < p_c$ (subcritical phase) there is no infinite open cluster almost surely. Furthermore, denoting the open cluster containing the site $x \in \mathbb{Z}^d$ by C_x, it holds that $\mathbb{P}(|C_x| = \infty) > 0$ for $p > p_c$, and $\mathbb{P}(|C_x| = \infty) = 0$ for $p < p_c$ and all $x \in \mathbb{Z}^d$. The following lemma states that the critical probability p_c is small for d large.

Lemma 2.4. *For independent site percolation on* $\mathbb{Z}^d$*,*

$$\lim_{d\to\infty} p_c(d) = 0.$$

Indeed, $p_c(d) = O\big(d^{-1}\big)$ holds. A proof of this result can e.g. be found in [7, Chapter 1, Theorem 7]. Further, in the recurrence proofs we use the fact that an infinite open cluster is "dense" in $\mathbb{Z}^d$. The following weak version of denseness suffices.

Lemma 2.5. *Consider supercritical independent site percolation on the lattice* $\mathbb{Z}^d$ *with* $d \geq 2$*. Let* $B_n = \{-n\} \times [-\sqrt{n}, \sqrt{n}]^{d-1}$ *and* $B'_n = [-\sqrt{n}, \sqrt{n}]^d$ *for* $n \in \mathbb{N}$*. Then, there are constants* $a, b > 0$ *and* $N \in \mathbb{N}$ *such that we have for all* $n \geq N$ *and* $x \in \mathbb{Z}^d$

$$\begin{aligned} \mathbb{P}\big(|B_n \cap C_x| \geq a n^{(d-1)/2}\big) &> b, \\ \mathbb{P}\big(|B'_n \cap C_x| \geq a n^{d/2}\big) &> b. \end{aligned}$$

Proof. For $y \in \mathbb{Z}^d$ consider the event $A_y = \{y \in C\}$. The process $(\mathbb{1}_{A_y})_{y \in \mathbb{Z}^d}$ is stationary and ergodic. By the spatial ergodic theorem

$$\lim_{n\to\infty} \frac{|B'_n \cap C|}{|B'_n|} = \lim_{n\to\infty} \frac{1}{|B'_n|} \sum_{y \in B'_n} \mathbb{1}_{A_y} = \mathbb{E}[\mathbb{1}_{A_0}] = \mathbb{P}(A_0)$$

almost surely. Note that $\mathbb{P}(A_0) > 0$ since the percolation is supercritical. Hence, there are constants $a, c > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$

$$\mathbb{P}\big(|B_n' \cap C| \geq a n^{d/2}\big) > c.$$

By the FKG inequality and the uniqueness of the infinite cluster

$$\begin{aligned}\mathbb{P}\big(|B_n' \cap C_x| \geq a n^{d/2}\big) &\geq \mathbb{P}\big(|B_n' \cap C| \geq a n^{d/2}, |C_x| = \infty\big)\\ &\geq \mathbb{P}\big(|B_n' \cap C| \geq a n^{d/2}\big)\mathbb{P}(|C_x| = \infty)\\ &\geq c \cdot \mathbb{P}(|C_x| = \infty).\end{aligned}$$

As $\mathbb{P}(|C_x| = \infty) = \mathbb{P}(|C_0| = \infty) > 0$ this shows the second inequality of the lemma.
For the first inequality consider the boxes $B_n(m) = \{-m\} \times [-\sqrt{n}, \sqrt{n}]^{d-1}$ for fixed $m \in \mathbb{N}$. Since $(\mathbb{1}_{A_y})_{y \in \{-m\} \times \mathbb{Z}^{d-1}}$ is stationary and ergodic, we get for every $m \in \mathbb{N}$

$$\lim_{n \to \infty} \frac{|B_n(m) \cap C|}{|B_n(m)|} = \mathbb{P}(A_0)$$

almost surely. Analogously to the proof of the second inequality, there are constants $a', c' > 0$ and $N' \in \mathbb{N}$, all independent of m, such that for all $n \geq N'$ and $m \in \mathbb{N}$

$$\mathbb{P}\big(|B_n(m) \cap C_x| \geq a' n^{(d-1)/2}\big) > c' \cdot \mathbb{P}(|C_x| = \infty). \tag{2.3}$$

Setting $m = n$ in (2.3) yields the claim.

2.2.2 A lemma on Bernoulli random variables

In the recurrence proofs we repeatedly use the following simple lemma. Note that the random variables in this lemma do not have to be independent.

Lemma 2.6. *For $i \in \mathbb{N}$ let X_i be a Bernoulli(p_i) random variable with $\inf_{i \in \mathbb{N}} p_i =: p > 0$. Then for every $a > 0$ and $n \in \mathbb{N}$*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) \geq p - a.$$

Proof. Since $\mathbb{E}[X_i] \geq p$ and $\frac{1}{n}\sum_{i=1}^{n} X_i \leq 1$, we have

$$p \leq \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \leq \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) + a,$$

which yields the claim.

2.2.3 Recurrence for $d \geq 2$ and arbitrary weight

In this section we prove Theorem 1.14 (i) and Theorem 1.16. Throughout this section assume that $w < 1$ is fixed. To illustrate the basic idea of the proof we first sketch it for $d = 2$. We call a site x in $\mathbb{Z}^2$ open if the trajectory $(S_n^x)_{n\in\mathbb{N}_0}$ of frog x includes the four neighbouring vertices $x \pm e_1, x \pm e_2$ of x, i.e. if $x \to x \pm e_1$ and $x \to x \pm e_2$. Note that for this definition it does not matter whether frog x is ever activated or not. All sites are open independently of each other due to the independence of the trajectories of the frogs. Furthermore, the probability of a site to be open is the same for all sites. Consider the percolation cluster C_0 that consists of all sites that can be reached from 0 by open paths, i.e. paths containing only open sites. Note that all frogs in C_0 are activated as frog 0 is active in the beginning. In this sense the frog model dominates the percolation. As we are in $d = 2$, the probability of a site x being open equals 1 for $\alpha = 0$ and by continuity is close to 1 if α is close to 0. Thus, if α is close enough to 0 the percolation is supercritical. Hence, with positive probability the cluster C_0 containing the origin is infinite. By Lemma 2.5 this infinite cluster contains many sites close to the negative e_1-axis. This shows that many frogs that are initially close to this axis get activated. Each of them travels in the direction of the e_1-axis and has a decent chance of visiting 0 on its way. Hence, this will happen infinitely many times. This argument shows that the origin is visited by infinitely many frogs with positive probability. Using the zero-one law stated in Theorem 1.2 yields the claim.

In higher dimensions the probability of a frog to visit all its neighbours is not close to 1 however small the drift may be. We can still make the argument work by using a renormalization type argument. To make this argument precise let K be a non-negative integer that will be chosen later. We tessellate $\mathbb{Z}^d$ for $d \geq 2$ with cubes $(Q_x)_{x\in\mathbb{Z}^d}$ of size $(2K+1)^d$. For every $x \in \mathbb{Z}^d$ we define

$$\begin{aligned} q_x &= q_x(K) = (2K+1)x, \\ Q_x &= Q_x(K) = \{y \in \mathbb{Z}^d \colon \|y - q_x\|_\infty \leq K\}, \end{aligned} \tag{2.4}$$

where $\|x\|_\infty = \max_{1\leq i\leq d} |x_i|$ is the maximum norm. A tessellation for $d = 2$ is shown in Figure 2.2. For $d = 2$ actually no tessellation is necessary, but the picture illustrates the construction.

We call a site $x \in \mathbb{Z}^d$ open if for every $e \in \mathcal{E}_d$ there exists a frog path from q_x to q_{x+e} in Q_x. Otherwise, x is said to be closed. The probability of a site x to be open does not depend on x, but only on the drift parameter α and the cube size K. We denote it by $p(K, \alpha)$. This defines an independent site percolation on $\mathbb{Z}^d$, which, as mentioned before, is dominated by the frog model in the following sense: For any $x \in C_0$ the frog at q_x will be activated in the frog model, i.e. $q_x \in FC_0$ with FC_0 as defined in (2.1).

In the next two lemmas we show that the probability $p(K, \alpha)$ of a site to be open is close to 1 if the drift parameter α is small and the cube size K is large. We first show this claim for the symmetric case $\alpha = 0$.

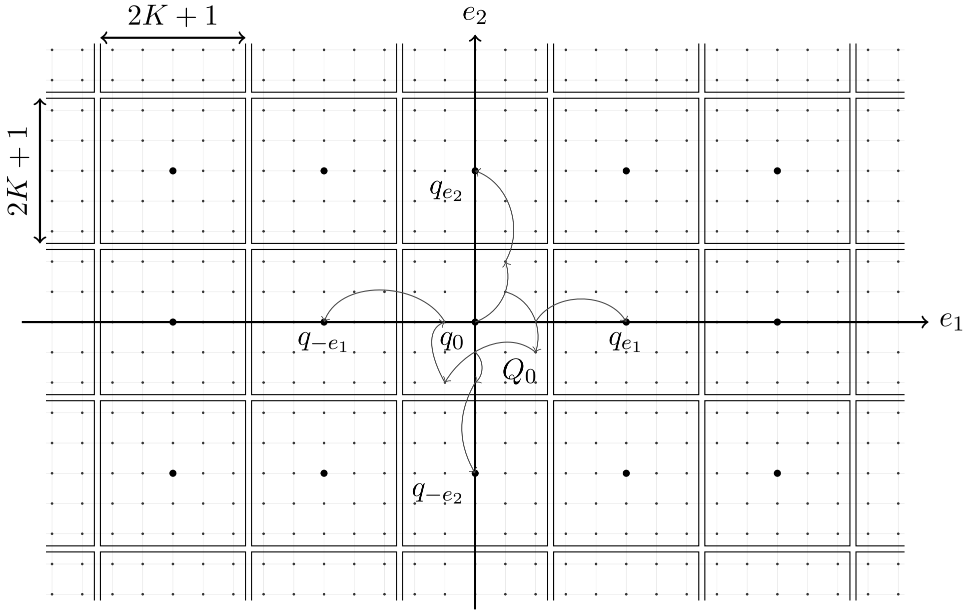

Figure 2.2: Tessellation of $\mathbb{Z}^2$ for $K = 2$; The site 0 is open if there a frog paths in Q_0 to q_e for all $e \in \mathcal{E}_d$.

Lemma 2.7. *For every $w < 1$ in the frog model $\mathrm{FM}(d, \pi_{w,0})$ we have*

$$\lim_{K\to\infty} p(K, 0) = 1.$$

Proof. For $d = 2$ we obviously have $p(K, 0) = 1$ for all $K \in \mathbb{N}_0$ as balanced random walk on $\mathbb{Z}^2$ is recurrent. Therefore, we can assume $d \geq 3$. The proof of the lemma relies on the shape theorem (Theorem 1.10) for the frog model. This theorem assumes equal weights on all directions. As in our model the e_1-direction has a different weight, we need a workaround. We couple our model with a modified frog model on $\mathbb{Z}^{d-1}$ in which the frogs in every step stay where they are with probability w and move according to a simple random walk otherwise. A direct coupling shows that, up to any fixed time, in the modified frog model on $\mathbb{Z}^{d-1}$ there are at most as many frogs activated as in the frog model $\mathrm{FM}(d, \pi_{w,0})$. Note that Theorem 1.10 holds true for the modified frog model on $\mathbb{Z}^{d-1}$ as the process is only slowed down by a constant depending on w. Let ξ_K, respectively ξ_K^{mod}, be the set of all sites visited by active frogs by time K in the frog model $\mathrm{FM}(d, \pi_{w,0})$, respectively the modified frog model on $\mathbb{Z}^{d-1}$. Further, let $\overline{\xi_K^{\mathrm{mod}}} := \{x + (-\frac{1}{2}, \frac{1}{2}]^{d-1} \colon x \in \xi_K^{\mathrm{mod}}\}$. By Theorem 1.10 there exists a non-trivial convex symmetric set $\mathcal{A} = \mathcal{A}(d) \subseteq \mathbb{R}^{d-1}$ and an almost surely finite random variable $\mathcal{K}$ such that

$$\mathcal{A} \subseteq \frac{\overline{\xi_K^{\mathrm{mod}}}}{K}$$

for all $K \geq \mathcal{K}$. This implies that there is a constant $c_1 = c_1(d) > 0$ such that $|\xi_K^{\mathrm{mod}}| \geq c_1 K^{d-1}$ for all $K \geq \mathcal{K}$. By the coupling the same statement holds true for ξ_K. As $\xi_K \subseteq Q_0(K)$ and any vertex in ξ_K can be reached by a frog path from 0 in Q_0, this implies

$$\Big|\Big\{y \in Q_0 \colon 0 \overset{Q_0}{\rightsquigarrow} y\Big\}\Big| \geq |\xi_K| \geq c_1 K^{d-1}$$

for all $K \geq \mathcal{K}$. Thus we have at least $c_1 K^{d-1}$ vertices in the box Q_0 that can be reached by frog paths from 0. Each frog in Q_0 has a chance to reach the centre q_e of a neighbouring box. More precisely, by Lemma 2.1 there is a constant $c_2 = c_2(d) > 0$ such that

$$\mathbb{P}\big(y \to q_e\big) \geq \frac{c_2}{K^{d-2}}$$

for any vertex $y \in Q_0$ and $e \in \mathcal{E}_d$. Hence, for any $e \in \mathcal{E}_d$

$$\begin{aligned}\mathbb{P}\big((0 \overset{Q_0}{\rightsquigarrow} q_e)^c \mid K \geq \mathcal{K}\big) &\leq \mathbb{P}\Big(\big\{y \not\to q_e \text{ for all } y \in Q_0 \text{ with } 0 \overset{Q_0}{\rightsquigarrow} y\big\} \bigm| K \geq \mathcal{K}\Big) \\ &\leq \Big(1 - \frac{c_2}{K^{d-2}}\Big)^{c_1 K^{d-1}} \\ &\leq \mathrm{e}^{-c_1 c_2 K}.\end{aligned}$$

Therefore,

$$\begin{aligned}p(K,0) &\geq \mathbb{P}\Big(\bigcap_{e \in \mathcal{E}_d} \{0 \overset{Q_0}{\rightsquigarrow} q_e\} \Bigm| K \geq \mathcal{K}\Big)\mathbb{P}_0(K \geq \mathcal{K}) \\ &\geq \Big[1 - 2d\,\mathrm{e}^{-c_1 c_2 K}\Big]\mathbb{P}(K \geq \mathcal{K}). \qquad (2.5)\end{aligned}$$

Since $\mathcal{K}$ is almost surely finite, we have $\lim_{K\to\infty} \mathbb{P}_0(K \geq \mathcal{K}) = 1$. Thus, the right hand side of (2.5) tends to 1 in the limit $K \to \infty$.

Lemma 2.8. *For fixed $w < 1$, in the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ we have for all $K \in \mathbb{N}_0$*

$$\lim_{\alpha \to 0} p(K,\alpha) = p(K,0).$$

Proof. Let $L(a,b,c,K)$ be the number of possible realisations such that all $q_{\pm e}$, $e \in \mathcal{E}_d$, are visited by frogs in Q_0 for the first time after in total (of all frogs) exactly a steps in e_1-direction, b steps in $-e_1$-direction and c steps in all other directions. Note that $L(a,b,c,K)$ is independent of α. We have

$$\begin{aligned}p(K,\alpha) &= \sum_{a,b,c=1}^{\infty} L(a,b,c,K)\Big(\frac{w(1+\alpha)}{2}\Big)^a\Big(\frac{w(1-\alpha)}{2}\Big)^b\Big(\frac{1-w}{2(d-1)}\Big)^c \\ &\xrightarrow[\alpha\to 0]{} \sum_{a,b,c=1}^{\infty} L(a,b,c,K)\Big(\frac{w}{2}\Big)^{a+b}\Big(\frac{1-w}{2(d-1)}\Big)^c = p(K,0).\end{aligned}$$

Proof of Theorem 1.14 (i) and Theorem 1.16. By Lemma 2.7 and Lemma 2.8 we can assume that K is big enough and $\alpha > 0$ small enough such that $p(K,\alpha) > p_c(d)$, i.e. the percolation with parameter $p(K,\alpha)$ on $\mathbb{Z}^d$ constructed at the beginning of this section is supercritical.

Consider boxes $B_n = \{-n\} \times [-\sqrt{n}, \sqrt{n}]^{d-1}$ for $n \in \mathbb{N}$. By Lemma 2.5 there are constants $a, b > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$

$$\mathbb{P}(|B_n \cap C_0| \geq an^{(d-1)/2}) > b.$$

After rescaling, the boxes B_n correspond to the boxes

$$FB_n = \{y \in \mathbb{Z}^d \colon |y_1 + (2K+1)n| \leq K,\ |y_i| \leq (2K+1)\sqrt{n} + K,\ 2 \leq i \leq d\}.$$

Recall that FC_0 consists of all vertices reachable by frog paths from 0 as defined in (2.1), and note that $x \in B_n \cap C_0$ implies $q_x \in FB_n \cap FC_0$. This shows

$$\mathbb{P}(|FB_n \cap FC_0| \geq an^{(d-1)/2}) > b \tag{2.6}$$

for n large enough. By Lemma 2.2 and (2.6) the probability that at least one frog in FB_n is activated and reaches 0 is at least

$$\Big(1 - (1 - cn^{-(d-1)/2})^{an^{(d-1)/2}}\Big)b \geq \big(1 - \mathrm{e}^{-ac}\big)b,$$

where $c = c(K, d, w) > 0$ is a constant. Altogether we get by Lemma 2.6

$$\begin{aligned}\mathbb{P}(0 \text{ visited infinitely often}) &= \lim_{n\to\infty} \mathbb{P}(0 \text{ is visited } \varepsilon n \text{ many times })\\ &\geq \liminf_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{1}_{\{\exists x \in FB_i \cap FC_0 \colon x \to 0\}} \geq \varepsilon n\right)\\ &\geq \big(1 - \mathrm{e}^{-ac}\big)b - \varepsilon > 0\end{aligned}$$

for ε sufficiently small. The claim now follows from Theorem 1.2.

2.2.4 Recurrence for $d = 2$ and arbitrary drift

In this section we prove Theorem 1.15 (i). Throughout the section let $\alpha < 1$ be fixed. The proof is similar to the one presented in the previous section. We couple the frog model with independent site percolation on $\mathbb{Z}^2$. Let K be an integer that will be chosen later. We tessellate $\mathbb{Z}^2$ with segments $(Q_x)_{x \in \mathbb{Z}^2}$ of size $2K+1$ as illustrated in Figure 2.3. For every $x = (x_1, x_2) \in \mathbb{Z}^2$ we define

$$\begin{aligned} q_x &= q_x(K) = \big(x_1, (2K+1)x_2\big),\\ Q_x &= Q_x(K) = \{y \in \mathbb{Z}^2 \colon y_1 = x_1, |y_2 - (2K+1)x_2| \leq K\}.\end{aligned}$$

We call the site $x \in \mathbb{Z}^2$ open if there are frog paths from q_x to q_{x+e} in Q_x for all $e \in \mathcal{E}_2$. As before, we denote the probability of a site to be open by $p(K, w)$. Note that this probability does not depend on x.

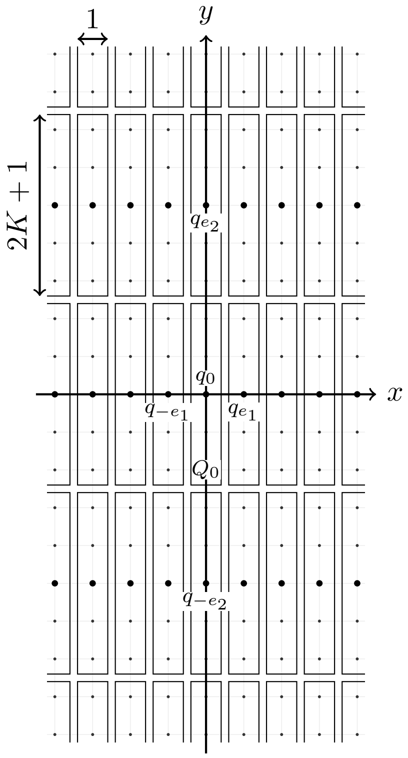

Figure 2.3: Tessellation of $\mathbb{Z}^2$ for $K = 2$.

Lemma 2.9. *For $\alpha < 1$, in the frog model* $\mathrm{FM}(2, \pi_{w,\alpha})$ *we have*

$$\lim_{K\to\infty} \liminf_{w\to 0} p(K, w) = 1.$$

Proof. We claim that there is a constant $c = c(\alpha) > 0$ such that for any $K \in \mathbb{N}_0$ and $x \in Q_0$

$$\liminf_{w\to 0} \mathbb{P}\Big(\bigcap_{e\in\mathcal{E}_2} \{x \to q_e\}\Big) \geq c. \tag{2.7}$$

We can estimate the probability in (2.7) by

$$\mathbb{P}\Big(\bigcap_{e\in\mathcal{E}_2} \{x \to q_e\}\Big) \geq \mathbb{P}\big(x \to q_{-e_2}\big)\mathbb{P}\big(q_{-e_2} \to q_{-e_1}\big)\mathbb{P}\big(q_{-e_1} \to q_{e_2}\big)\mathbb{P}\big(q_{e_2} \to q_{e_1}\big).$$

The probability of moving in $\pm e_2$-direction for $\lceil w^{-1}\rceil$ steps is $(1-w)^{\lceil w^{-1}\rceil}$. Conditioning on moving in this way, we just deal with a simple random walk on $\mathbb{Z}$. Therefore, there exists a constant $c_1 > 0$ such that for w close to 0

$$\mathbb{P}\big(x \to q_{-e_2}\big) \geq c_1(1-w)^{\lceil w^{-1}\rceil} \geq \frac{c_1}{4}.$$

The probability of moving exactly once in $-e_1$-direction and otherwise in $\pm e_2$-direction within $\lceil w^{-1} \rceil + 1$ steps is

$$\big(\lceil w^{-1} \rceil + 1\big)\frac{(1-\alpha)w}{2}(1-w)^{\lceil w^{-1} \rceil} \geq \frac{1-\alpha}{8}$$

for w close to 0. Therefore, there exists a constant $c_2 > 0$ such that

$$\mathbb{P}\big(q_{-e_2} \to q_{-e_1}\big) \geq \frac{c_2(1-\alpha)}{8}$$

for w sufficiently close to 0. The two remaining probabilities $\mathbb{P}\big(q_{-e_1} \to q_{e_2}\big)$ and $\mathbb{P}\big(q_{e_2} \to q_{e_1}\big)$ can be estimated analogously, which implies (2.7).
If frog 0 activates all frogs in Q_0 and any of these $2K$ frogs manages to visit the centres of all neighbouring segments, then 0 is open. By independence of the trajectories of the individual particles in Q_0 this implies

$$p(K,w) \geq \mathbb{P}\Big(\bigcap_{x \in Q_0} \{0 \to x\}\Big)\bigg(1 - \Big(1 - \mathbb{P}\Big(\bigcap_{1 \leq i \leq 4} \{x \to q_{e_i}\}\Big)\Big)^{2K}\bigg). \tag{2.8}$$

As in the proof of Lemma 2.8 one can show that for $w \to 0$ the first factor in (2.8) converges to 1. Therefore, taking limits in (2.8) and using (2.7) yields the claim.

Proof of Theorem 1.15 (i). By Lemma 2.9 we can choose K big and w small enough such that $p(K,w) > p_c(2)$, where $p_c(2)$ is the critical parameter for independent site percolation on $\mathbb{Z}^2$. As in the proof of Theorem 1.14 (i) and Theorem 1.16 the coupling with supercritical percolation now yields recurrence of the frog model. As we rescaled the lattice $\mathbb{Z}^2$ slightly different this time, the box B_n defined in the proof of Theorem 1.14 (i) and Theorem 1.16 now corresponds to the box

$$FB_n = \{y \in \mathbb{Z}^2 \colon y_1 = -n,\ |y_2| \leq (2K+1)\sqrt{n} + K\}.$$

Since only asymptotics in n matter for the proof, it otherwise works unchanged.

2.2.5 Recurrence for $d \geq 3$ and arbitrary drift

The proof of Theorem 1.17 (i) again relies on the idea of comparing the frog model with percolation. But instead of looking at the whole space $\mathbb{Z}^d$ as in the previous proofs, we consider a sequence of $(d-1)$-dimensional hyperplanes $(H_{-n})_{n \in \mathbb{N}_0}$ with H_{-n} as defined in (2.2) and depicted in Figure 2.4. We compare the frogs in each hyperplane with supercritical percolation, ignoring the frogs once they have left their hyperplane and all the frogs from other hyperplanes. Within a hyperplane we now deal with a frog model without drift, but allow the frogs to die in each step with probability w by leaving their hyperplane, i.e. we are interested in $\mathrm{FM}^*(d-1, \pi_{\mathrm{sym}}, 1-w)$ as defined in Section 1.2.2. Hence, the argument does not depend on the value of the drift parameter $\alpha < 1$.
We start with one active particle in the hyperplane H_0. With positive probability this particle initiates an infinite frog cluster in H_0 if w and therefore the probability to leave the

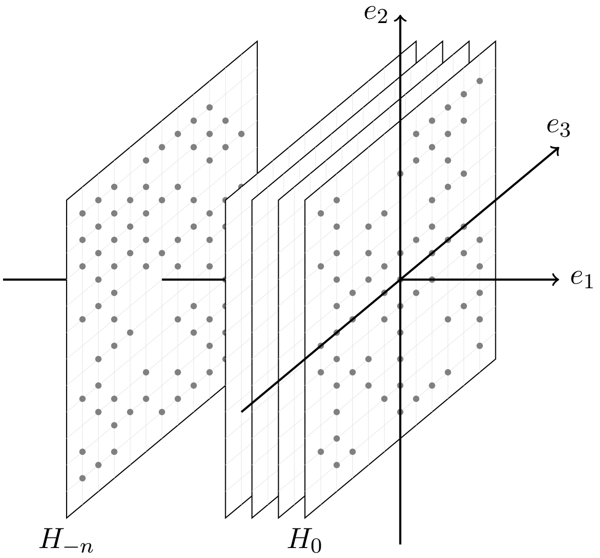

Figure 2.4: Hyperplanes $(H_{-n})_{n\in\mathbb{N}_0}$ in $\mathbb{Z}^3$; We will show that every hyperplane contains an infinite frog cluster, indicated by gray dots.

hyperplane is sufficiently small. Every frog eventually leaves H_0 and has for every $n \in \mathbb{N}$ a positive chance of activating a frog in the hyperplane H_{-n}, which might start an infinite cluster there. This is the only time where we need $\alpha < 1$ in the proof of Theorem 1.17 (i). Using the denseness of such clusters we can then proceed as before. We split the proof of Theorem 1.17 (i) into two parts:

Proposition 2.10. *There is $d_0 \in \mathbb{N}$ and $w_r > 0$, independent of d and α, such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is recurrent for all $0 \le w \le w_r$, $0 \le \alpha < 1$ and $d \ge d_0$.*

Proposition 2.11. *For every $d \ge 3$ there is $w_r = w_r(d) > 0$, independent of α, such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is recurrent for all $0 \le w \le w_r$ and all $0 \le \alpha < 1$.*

We first prove Proposition 2.10. As indicated above we need to study the frog model with death and no drift in $\mathbb{Z}^{d-1}$. To increase the readability let us first work in dimension d instead of $d-1$ and with a general survival parameter s, i.e. we investigate $\mathrm{FM}^*(d, \pi_{\mathrm{sym}}, s)$ for $d \ge 2$.
We tessellate $\mathbb{Z}^d$ with cubes $(Q'_x)_{x\in\mathbb{Z}^d}$ of size 3^d. More precisely, define for $x \in \mathbb{Z}^d$

$$Q'_x = \{y \in \mathbb{Z}^d \colon \|y - 3x\|_\infty \le 1\}.$$

Further, for technical reasons, for $a \in (\frac{2}{3}, 1)$ we define

$$W_x = \{y \in Q'_x \colon \|y - 3x\|_1 \le ad\},$$

where $\|z\|_1 = \sum_{i=1}^{2d} |z_i|$ is the graph distance from $z \in \mathbb{Z}^d$ to 0. Informally, W_x is the set of all vertices in Q'_x which are “sufficiently close” to the centre of the cube. Consider the box Q'_x for some $x \in \mathbb{Z}^d$ and let $o \in W_x$. If there are frog paths in Q'_x from o to vertices close to the centres of all neighbouring boxes, i.e. if the event

$$\bigcap_{e \in \mathcal{E}_d} \bigcup_{y \in W_{x+e}} \{o \overset{Q'_x}{\rightsquigarrow} y\}$$

occurs, we call the vertex o good. Note that this event only depends on the trajectories of all the frogs originating in the cube Q'_x and the choice of o. If o is good and is activated, then also the neighbouring cubes are visited. We show that the probability of a vertex being good is bounded from below uniformly in d and this bound does not depend on the choice of o.

Lemma 2.12. *Consider the frog model* $\mathrm{FM}^*(d, \pi_{sym}, s)$. *There are constants* $\beta > 0$ *and* $d_0 \in \mathbb{N}$ *such that for all* $d \geq d_0$, $s > \frac{3}{4}$, $\frac{2}{3} < a < 2 - \frac{1}{s}$, $x \in \mathbb{Z}^d$ *and* $o \in W_x$

$$\mathbb{P}(o \textit{ is good}) > \beta.$$

To show this we first need to prove that many frogs in the cube are activated. In the proof of Theorem 1.14 (i) and Theorem 1.16 this is done by means of Lemma 2.7 using the shape theorem. Here, we use a lemma that is analogous to Lemma 2.5 in [1].

Lemma 2.13. *Consider the frog model* $\mathrm{FM}^*(d, \pi_{sym}, s)$. *There exist constants* $\gamma > 0$, $\mu > 1$ *and* $d_0 \in \mathbb{N}$ *such that for all* $d \geq d_0$, $s > \frac{3}{4}$, $\frac{2}{3} < a < 2 - \frac{1}{s}$ *and* $o \in W_0$ *we have*

$$\mathbb{P}\Big(\big|\{y \in W_0 \colon o \overset{Q'_0}{\rightsquigarrow} y\}\big| \geq \mu^{\sqrt{d}}\Big) \geq \gamma.$$

Proof of Lemma 2.13. The proof consists of two parts. In the first part we show that with positive probability there are exponentially many vertices in Q'_0 reached from o by frog paths in Q'_0, and in the second part we prove that many of these vertices are indeed in W_0. For the first part we closely follow the proof of Lemma 2.5 in [1] and rewrite the details for the convenience of the reader.

We examine the frog model with initially one active frog at the vertex o and one sleeping frog at every other vertex in Q'_0 for $\sqrt{d}$ steps in time. Consider the sets $\mathcal{S}_0 = \{o\}$ and $\mathcal{S}_k = \{x \in Q'_0 \colon \|x - o\|_1 = k, \|x - o\|_\infty = 1\}$ for $k \geq 1$ and let ξ_k denote the set of active frogs which are in $\mathcal{S}_k$ at time k. We will show that, conditioned on an event to be defined later, the process $(\xi_k)_{k \in \mathbb{N}_0}$ dominates a process $(\tilde{\xi}_k)_{k \in \mathbb{N}_0}$, which again itself dominates a supercritical branching process. The process $(\tilde{\xi}_k)_{k \in \mathbb{N}_0}$ is defined as follows. Initially, there is one particle at o. Assume that the process has been constructed up to time $k \in \mathbb{N}_0$. In the next step each particle in $\tilde{\xi}_k$ survives with probability s. If it survives, it chooses one of the neighbouring vertices uniformly at random. If that vertex belongs to $\mathcal{S}_{k+1}$ and no other particle in $\tilde{\xi}_k$ intends to jump to this vertex, the particle moves there, activates the sleeping particle, and both particles enter $\tilde{\xi}_{k+1}$. Otherwise, the particle is deleted. In particular, if two or more particles attempt to jump to the same vertex, all of them will be deleted. Obviously, $\tilde{\xi}_k \subseteq \xi_k$ for all $k \in \mathbb{N}_0$.

First, we show that for d large it is unlikely that two particles in $\tilde{\xi}_k$ attempt to jump to the same vertex. To make this argument precise we need to introduce some notation. For $x \in \mathcal{S}_k$ and $y \in \mathcal{S}_{k+1}$ with $\|x-y\|_1 = 1$ define

$$\begin{aligned} \mathcal{D}_x &= \{z \in \mathcal{S}_{k+1} \colon \|x-z\|_1 = 1\}, \\ \mathcal{A}_y &= \{z \in \mathcal{S}_k \colon \|z-y\|_1 = 1\}, \\ \mathcal{E}_x &= \{z \in \mathcal{S}_k \colon \mathcal{D}_x \cap \mathcal{D}_z \neq \emptyset\}. \end{aligned}$$

$\mathcal{D}_x$ denotes the set of possible descendants of x, $\mathcal{A}_y$ the set of ancestors of y and $\mathcal{E}_x$ the set of enemies of x. Note that $\mathcal{E}_x = \bigcup_{y \in \mathcal{D}_x} (\mathcal{A}_y \setminus \{x\})$ is a disjoint union. Further, let $n_x = \sum_{i=1}^d \mathbb{1}_{\{o_i = 0,\, x_i \neq 0\}}$. Then one can check that

$$\begin{aligned} |\mathcal{D}_x| &= 2(d - \|o\|_1 - n_x) + \|o\|_1 - (k - n_x) = 2d - \|o\|_1 - k - n_x, \qquad (2.9) \\ |\mathcal{A}_y| &= k+1. \end{aligned}$$

For $x \in \mathcal{S}_k$ let $\chi(x)$ denote the number of particles of $\tilde{\xi}_k$ in x. Note that $\chi(x) \in \{0, 2\}$ for any $x \in \mathcal{S}_k$ with $k \in \mathbb{N}$.

Let ζ_{xy}^k denote the indicator function of the event that there is $z \in \mathcal{E}_x$ with $\chi(z) \geq 1$ such that one of the particles at z intends to jump to y at time $k+1$. If $\zeta_{xy}^k = 1$, then a particle on x cannot move to y at time $k+1$.

Further, we introduce the event $U_x = \{\chi(z) = 2 \text{ for all } z \in \mathcal{E}_x\}$. This event describes the worst case for x, when it is most likely that particles at x will not be able to jump. For $k \leq \sqrt{d}$ we have

$$\mathbb{P}(\zeta_{xy}^k = 1) \leq \mathbb{P}(\zeta_{xy}^k = 1 \mid U_x) \leq \sum_{z \in \mathcal{A}_y \setminus \{x\}} \frac{2s}{2d} = \frac{ks}{d} \leq \frac{1}{\sqrt{d}}.$$

Given $\sigma > 0$ we can choose d large such that $\mathbb{P}(\zeta_{xy}^k = 1) < \sigma$ for all $k \leq \sqrt{d}$. Now, consider the set of all descendants y of x such that there is a particle at some vertex $z \in \mathcal{E}_x$ that tries to jump to y at time $k+1$. This set contains $\sum_{y \in \mathcal{D}_x} \zeta_{xy}^k$ elements. Let ζ_x^k denote the indicator function of the event $\left\{\sum_{y \in \mathcal{D}_x} \zeta_{xy}^k > 2\sigma d\right\}$. If $\zeta_x^k = 1$, then more than $2\sigma d$ of the $2d$ neighbours of x are blocked to a particle at x.

The random variables $\{\zeta_{xy}^k \colon y \in \mathcal{D}_x\}$ are independent with respect to the measure $\mathbb{P}(\cdot \mid U_x)$ as $\mathcal{E}_x = \bigcup_{y \in \mathcal{D}_x} (\mathcal{A}_y \setminus \{x\})$ is a disjoint union. Using $2d - ad - 2k \leq |\mathcal{D}_x| \leq 2d$ and a standard large deviation estimate we get for $k \leq \sqrt{d}$

$$\begin{aligned} \mathbb{P}(\zeta_x^k = 1) &\leq \mathbb{P}\left(\sum_{y \in \mathcal{D}_x} \zeta_{xy}^k > 2\sigma d \;\middle|\; U_x \right) \\ &\leq \mathbb{P}\left(\frac{1}{|\mathcal{D}_x|} \sum_{y \in \mathcal{D}_x} \zeta_{xy}^k > \sigma \;\middle|\; U_x \right) \\ &\leq \mathrm{e}^{-c_1 |\mathcal{D}_x|} \\ &\leq \mathrm{e}^{-c_2 d} \end{aligned}$$

with constants $c_1, c_2 > 0$. Next, let us consider the bad event

$$B = \bigcup_{k=1}^{\sqrt{d}} \bigcup_{x \in \tilde{\xi}_k} \{\zeta_x^k = 1\}.$$

Then with $|\tilde{\xi}_k| \leq 2^k \leq 2^{\sqrt{d}}$ we get

$$\mathbb{P}(B) \leq \sqrt{d} \cdot 2^{\sqrt{d}} \cdot \mathrm{e}^{-c_2 d}.$$

In particular $\mathbb{P}(B)$ can be made arbitrarily small for d large. Conditioned on B^c, in each step for every particle there are at least

$$|\mathcal{D}_x| - 2\sigma d - 1 \geq (2 - a - 2\sigma)d - 3\sqrt{d}$$

available vertices in $\mathcal{S}_{k+1}$, i.e. vertices a particle at x can jump to in the next step. Thus, conditioned on B^c, the process $\tilde{\xi}_k$ dominates a branching process with mean offspring at least

$$\frac{\big((2 - a - 2\sigma)d - 3\sqrt{d}\big) \cdot 2 \cdot s}{2d}.$$

For σ small and d large the mean offspring is bigger than 1 as we assumed $a < 2 - \frac{1}{s}$. Since a supercritical branching process grows exponentially with positive probability, there are constants $c_3 > 1$, $q \in (0,1)$ that do not depend on d such that

$$\mathbb{P}\big(|\tilde{\xi}_{\sqrt{d}}| \geq c_3^{\sqrt{d}}\big) \geq q. \tag{2.10}$$

For the second part of the proof we condition on the event $\big\{|\tilde{\xi}_{\sqrt{d}}| \geq c_3^{\sqrt{d}}\big\}$ and choose $0 < \varepsilon < a - \frac{2}{3}$. If $\|o\|_1 \leq (a-\varepsilon)d$, all particles of $\tilde{\xi}_{\sqrt{d}}$ are in W_0 for d large. This immediately implies the claim of the lemma. Otherwise, let $n = |\tilde{\xi}_{\sqrt{d}}|$, enumerate the particles in $\tilde{\xi}_{\sqrt{d}}$ and let $\tilde{S}^i$, $1 \leq i \leq n$, denote the position of the i-th particle. Further, we define for $1 \leq i \leq n$

$$X_i = \begin{cases} 1 & \text{if } \|\tilde{S}^i\|_1 \leq \|o\|_1, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that $\mathbb{P}(X_1 = 1) > 0$. Then Lemma 2.6 applied to the random variables $X_1, \dots, X_n$ implies that with positive probability a positive proportion of the particles in $\tilde{\xi}_{\sqrt{d}}$ indeed have L_1-norm smaller than o, and are thus in W_0. Together with (2.10) this finishes the proof.

For the proof of the claim let $\tilde{S}_k^1$ denote the position of the ancestor of $\tilde{S}^1$ in $\mathcal{S}_k$, where $0 \leq k \leq \sqrt{d}$. Note that $\tilde{S}_0^1 = o$ and $\tilde{S}_{\sqrt{d}}^1 = \tilde{S}^1$.

We are interested in the process $(\|\tilde{S}_k^1\|_1)_{1 \leq k \leq \sqrt{d}}$. By the construction of the process $(\tilde{\xi}_k)_{k \in \mathbb{N}_0}$ it either increases or decreases by 1 in every step. The positions $\tilde{S}_k^1$ and $\tilde{S}_{k+1}^1$ differ in exactly one coordinate. If this coordinate is changed from 0 to ± 1, then $\|\tilde{S}_{k+1}^1\|_1 = \|\tilde{S}_k^1\|_1 + 1$. If it is changed from ± 1 to 0, then we have $\|\tilde{S}_{k+1}^1\|_1 = \|\tilde{S}_k^1\|_1 - 1$. There are at least $(a-\varepsilon)d - \sqrt{d}$

many ± 1-coordinates in $\tilde{S}^1_k$ that can be changed to 0. As we also know that $\tilde{S}^1_{k+1} \in \mathcal{D}_{\tilde{S}^1_k}$, we have for all $k \le \sqrt{d}$ by (2.9) and the choice of ε

$$\mathbb{P}\big(\|\tilde{S}^1_{k+1}\|_1 = \|\tilde{S}^1_k\|_1 - 1\big) \ge \frac{(a-\varepsilon)d - \sqrt{d}}{|\mathcal{D}_{\tilde{S}^1_k}|} \ge \frac{(a-\varepsilon)d - \sqrt{d}}{2d - (a-\varepsilon)d} > \frac{1}{2}$$

for d large. Hence, $\|\tilde{S}^1_k\|_1$ dominates a random walk with drift on $\mathbb{Z}$ started in $\|o\|_1$. Therefore,

$$\mathbb{P}(X_1 = 1) = \mathbb{P}\big(\|\tilde{S}^1_{\sqrt{d}}\|_1 \le \|o\|_1\big) \ge \frac{1}{2},$$

which finishes the proof. ∎

Proof of Lemma 2.12. By Lemma 2.13, with probability at least γ there are frog paths in Q'_x from o to at least $\mu^{\sqrt{d}}$ vertices in W_x for d large. We divide the frogs on these vertices into $2d$ groups of size at least $\mu^{\sqrt{d}}/2d$ and assign each group the task of visiting one of the neighbouring boxes W_{x+e}, $e \in \mathcal{E}_d$. Notice that this job is done if at least one of the frogs in the group visits at least one vertex in the neighbouring box. If all groups succeed, o is good. Any frog in any group is just three steps away from its respective neighbouring box W_{x+e}, $e \in \mathcal{E}_d$, and thus has probability at least $(\frac{s}{2d})^3$ of achieving its group's goal. Hence,

$$\mathbb{P}(o \text{ is good}) \ge \left(1 - \left(1 - \left(\frac{s}{2d}\right)^3\right)^{\mu^{\sqrt{d}}/2d}\right)^{2d} \gamma \ge \frac{\gamma}{2}$$

for d large. ∎

In the other recurrence proofs we couple the frog model with percolation by calling a cube open if its centre is good. Here, the choice of a “starting” vertex, like the centre, is not independent of the other cubes. Therefore, we cannot directly couple the frog model with independent percolation. However, the following lemma allows us to compare the distributions of a frog cluster and a percolation cluster.

Lemma 2.14. *Consider the frog model* $\mathrm{FM}^*(d, \pi_{sym}, s)$. *Let* $\beta > 0$ *and assume that for all* $o \in W_x$, $x \in \mathbb{Z}^d$ *we have* $\mathbb{P}(o \text{ is good}) > \beta$. *Further, consider independent site percolation on* $\mathbb{Z}^d$ *with parameter* β. *Then for all sets* $A \subseteq \mathbb{Z}^d$, $v \in \mathbb{Z}^d$ *and for all* $k \ge 0$

$$\mathbb{P}(|A \cap C_v| \ge k) \le \mathbb{P}\Big(\Big|\bigcup_{x \in A} Q'_x \cap FC^*_{3v}\Big| \ge k\Big).$$

Proof. For technical reasons we introduce a family of independent Bernoulli random variables $(X_o)_{o \in \mathbb{Z}^d}$ which are also independent of the choice of all the trajectories of the frogs and satisfy $\mathbb{P}(X_o = 1) = \mathbb{P}(o \text{ is good})^{-1}\beta$. Their job will be justified soon. Further, we fix an ordering of all vertices in $\mathbb{Z}^d$.

Now we are ready to describe a process that explores a subset of the frog cluster FC^*_{3v}. Its distribution can be related to the cluster C_v in independent site percolation with parameter β. The process is a random sequence $(R_t, D_t, U_t)_{t \in \mathbb{N}_0}$ of tripartitions of $\mathbb{Z}^d$. As the letters indicate, R_t will contain all sites reached by time t, D_t all those declared dead by time t, and U_t the unexplored sites. We construct the process in such a way that for all $t \in \mathbb{N}_0$, $x \in R_t$ and $e \in \mathcal{E}_d$ there is $y \in W_{x+e}$ such that there is a frog path from $3v$ to y in $\bigcup_{z \in R_t} Q'_z$. We start with $R_0 = D_0 = \emptyset$ and $U_0 = \mathbb{Z}^d$. If $3v$ is good and $X_{3v} = 1$, set $U_1 = \mathbb{Z}^d \setminus \{v\}$, $R_1 = \{v\}$, and $D_1 = \emptyset$. Otherwise, stop the algorithm. If the process is stopped at time t, let $U_s = U_{t-1}$, $R_s = R_{t-1}$ and $D_s = D_{t-1}$ for all $s \geq t$. Assume we have constructed the process up to time t. Consider the set of all sites in U_t that have a neighbour in R_t. If it is empty, stop the process. Otherwise, pick the site x in this set with the smallest number in our ordering. By the choice of x there is $y \in W_x$ such that there is a frog path from $3v$ to y in $\bigcup_{z \in R_t} Q'_z$. Choose any vertex y with this property. If y is good and $X_y = 1$, set

$$R_{t+1} = R_t \cup \{x\},\ D_{t+1} = D_t,\ U_{t+1} = U_t \setminus \{x\}.$$

Otherwise, update the sets as follows:

$$R_{t+1} = R_t,\ D_{t+1} = D_t \cup \{x\},\ U_{t+1} = U_t \setminus \{x\}$$

In every step t the algorithm picks an unexplored site x and declares it to be reached or dead, i.e. added to the set R_t or D_t. The probability that x is added to R_t equals β. This event is (stochastically) independent of everything that happened before time t in the algorithm. Note that every unexplored neighbour of a reached site will eventually be explored due to the fixed ordering of all sites.

In the same way we can explore independent site percolation on $\mathbb{Z}^d$ with parameter β. Construct a sequence $(R'_t, D'_t, U'_t)_{t \in \mathbb{N}_0}$ of tripartitions of $\mathbb{Z}^d$ as above, but whenever the algorithm evaluates whether a site x is declared reached or dead we toss a coin independently of everything else. Note that $\bigcup_{t \in \mathbb{N}_0} R'_t = C_v$, where C_v is the cluster containing v. This exploration process is well known for percolation, see e.g. [7, Proof of Theorem 4, Chapter 1]. By construction, $\bigcup_{t \in \mathbb{N}_0} R_t$ equals the percolation cluster C_v in distribution. The claim follows since for every $x \in \bigcup_{t \in \mathbb{N}_0} R_t$ there is a $y \in W_x$ such that there is a frog path from $3v$ to y, i.e. $y \in FC^*_{3v}$. □

Now we can show Proposition 2.10. Note that we are again working with the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ (without death).

Proof of Proposition 2.10. Throughout this proof we assume that d is so large that we can apply Lemma 2.12 for $d-1$ and such that $p_c(d-1) < \beta$, where β is the constant introduced in the statement of Lemma 2.12. This is possible because of Lemma 2.4. These assumptions in particular imply that we can use Lemma 2.14 and that the percolation introduced there is supercritical.

Consider the sequence of hyperplanes $(H_{-n})_{n \in \mathbb{N}_0}$ defined in (2.2) and let A denote the event that there is at least one frog v_n activated in every hyperplane H_{-n}. For technical reasons we want v_n of the form $v_n = (-n, 3w_n)$ for some $w_n \in \mathbb{Z}^{d-1}$. We first show that A occurs with positive probability. To see this consider the first hyperplane H_0 and couple the frogs in this

hyperplane with $\mathrm{FM}^*(d-1, \pi_{\mathrm{sym}}, 1-w)$ in the following way: Whenever a frog takes a step in $\pm e_1$-direction, i.e. leaves its hyperplane, it dies instead. By Theorem 1.8 or Lemma 2.14 this process survives with positive probability if w is sufficiently small (independent of the dimension d). This means that infinitely many frogs are activated in H_0. Obviously, this implies the claim.
From now on we condition on the event A. Note that $FC_{v_n} \subseteq FC_0$ for $n \in \mathbb{N}$. Analogously to the proofs in the last sections we introduce boxes

$$FB'_n = \{-n\} \times [-(3\sqrt{n}+1), 3\sqrt{n}+1]^{d-1}$$

for $n \in \mathbb{N}$. We claim that analogously to Lemma 2.5 there are constants $a, b > 0$ and $N \in \mathbb{N}$ such that for $n \geq N$

$$\mathbb{P}\big(|FB'_n \cap FC_0| \geq an^{(d-1)/2}\big) \geq b. \tag{2.11}$$

To prove this claim let $a, b > 0$ and $N \in \mathbb{N}$ be the constants provided by Lemma 2.5 for percolation with parameter β. For $n \geq N$ couple the frog model with $\mathrm{FM}^*(d-1, \pi_{\mathrm{sym}}, 1-w)$ in the hyperplane H_n as above. Let $B'_n = [-\sqrt{n}, \sqrt{n}]^{d-1}$ and note that B'_n corresponds to FB'_n restricted to H_n after rescaling. Then by Lemma 2.14 and Lemma 2.5

$$\begin{aligned}\mathbb{P}\big(|FB'_n \cap FC_{v_n}| \geq an^{(d-1)/2} | A\big) &\geq \mathbb{P}\big(|FB'_n \cap (\{-n\} \times FC^*_{3w_n})| \geq an^{(d-1)/2)} | A\big)\\ &\geq \mathbb{P}\big(|B'_n \cap C_{w_n}| \geq an^{(d-1)/2)} | A\big)\\ &\geq b.\end{aligned}$$

Here, C_{w_n} is the open cluster containing w_n in a percolation model with parameter β in $\mathbb{Z}^{d-1}$, independently of the frogs. As $FC_{v_n} \subseteq FC_0$, this implies inequality (2.11).
By Lemma 2.2 and (2.11), the probability that there is at least one activated frog in FB'_n that reaches 0 is at least

$$\Big(1 - (1 - c'n^{-(d-1)/2})^{an^{(d-1)/2}}\Big)b \geq \big(1 - \mathrm{e}^{-ac'}\big)b,$$

where $c' > 0$ is a constant. Altogether we get by Lemma 2.6

$$\begin{aligned}\mathbb{P}(0 \text{ visited infinitely often}) &= \lim_{n\to\infty} \mathbb{P}(0 \text{ is visited } \varepsilon n \text{ many times })\\ &\geq \lim_{n\to\infty} \mathbb{P}\bigg(\sum_{i=1}^{n} \mathbb{1}_{\{\exists x \in FB'_n \cap FC_0 \colon x \to 0\}} \geq \varepsilon n\bigg)\\ &\geq \Big(\big(1 - \mathrm{e}^{-ac'}\big)b - \varepsilon\Big) > 0\end{aligned}$$

for ε sufficiently small. The claim now follows from Theorem 1.2.

To prove Proposition 2.11 we again first study the frog model with death $\mathrm{FM}^*(d, \pi_{\mathrm{sym}}, s)$ in the hyperplanes and couple it with percolation. This time we use cubes of size $(2K+1)^d$ for some $K \in \mathbb{N}_0$. By choosing K large we increase the number of frogs in the cubes. In the proof of the previous proposition this was done by increasing the dimension d. For $x \in \mathbb{Z}^d$

and $K \in \mathbb{N}_0$ we define

$$\begin{aligned} q_x &= q_x(K) = (2K+1)x, \\ Q_x &= Q_x(K) = \{y \in \mathbb{Z}^d \colon \|y - q_x\|_\infty \leq K\}. \end{aligned}$$

Note that this definition coincides with (2.4). In analogy to Lemma 2.14 the frog cluster dominates a percolation cluster.

Lemma 2.15. *For $d \geq 2$ there are constants $s_r(d) < 1$ and $K \in \mathbb{N}_0$ such that for any $s \geq s_r(d)$ the frog model* $\mathrm{FM}^*(d, \pi_{sym}, s)$ *can be coupled with supercritical site percolation on $\mathbb{Z}^d$ such that for all sets $A \subseteq \mathbb{Z}^d$, $v \in \mathbb{Z}^d$ and for all $k \geq 0$*

$$\mathbb{P}(|A \cap C_v| \geq k) \leq \mathbb{P}\Big(\Big|\bigcup_{x \in A} Q_x \cap FC^*_{q_v}\Big| \geq k\Big).$$

Proof. We couple the frog model with percolation as follows: A site $x \in \mathbb{Z}^d$ is called open if for every $e \in \mathcal{E}_d$ there exists a frog path from q_x to q_{x+e} in Q_x. We denote the probability of a site x to be open by $p(K, s)$. By Lemma 2.7 $p(K, 1)$ is close to 1 for K large. As in the proof of Lemma 2.8 one can show that $\lim_{s \to 1} p(K, s) = p(K, 1)$. Thus, we can choose $K \in \mathbb{N}$ and $s_r > 0$ such that $p(K, s) > p_c(d)$ for all $s > s_r$, i.e. the percolation is supercritical. Now it remains to note that by the construction of the percolation $x \in C_v$ implies $q_x \in FC^*_{q_v}$ for any $v \in \mathbb{Z}^d$. 🐸

Proof of Proposition 2.11. Using Lemma 2.15 instead of Lemma 2.14 and boxes Q_x instead of Q'_x, the proof is analogous to the proof of Proposition 2.10. 🐸

Proof of Theorem 1.17 (i). Theorem 1.17 (i) follows from Proposition 2.10 and Proposition 2.11. 🐸

Before we conclude this section, let us shortly reflect on what we did. In the proof of Theorem 1.17 (i) we use Lemma 2.12 to show that in the frog model with death a frog cluster is dense with positive probability if the survival probability is larger than $\frac{3}{4}$ and d is large. Indeed, we believe that every infinite frog cluster is dense. Hence, $\mathrm{FM}(d, \pi_{w,\alpha})$ would be recurrent for all $\alpha < 1$ if $\mathrm{FM}^*(d-1, \pi_{\mathrm{sym}}, 1-w)$ has a positive survival probability. Further, as mentioned in Section 1.2.2, we believe that the critical survival probability is decreasing in d. See also the discussion in [1, Chapter 1.2]. Recall that we also suppose that there is a curve f_d separating the transient from the recurrent regime in the phase diagram. See Conjecture 1.18. All together, this argument would imply that $f_d(1^-)$ is increasing in d and thus support Conjecture 1.19.

2.3 Transience

To show the transience of the frog model we couple the frog model with branching random walks. For the proofs we need the following lemma concerning the one-dimensional frog model. It enables us to estimate the number of activated frogs.

2.3.1 One result about one-dimensional frogs

Recall that $\mathrm{FM}^*(d, \pi_{sym}, s)$ refers to the frog model on $\mathbb{Z}^d$ with survival parameter s and the underlying random walk being simple.

Lemma 2.16. *For* $\mathrm{FM}(1, \pi_{1,\alpha})$ *with* $\alpha > 0$ *and* $\mathrm{FM}^*(1, \pi_{sym}, s)$ *with* $s < 1$ *there is* $c > 0$ *such that* $\mathbb{P}(0 \overset{\mathbb{Z}}{\rightsquigarrow} -n) \le \mathrm{e}^{-cn}$ *for all* $n \in \mathbb{N}$.

Proof. Let p be the probability that a frog starting from 0 ever hits the vertex -1. In both models we have $p < 1$. Obviously, as $s < 1$, this is true for $\mathrm{FM}^*(d, \pi_{\mathrm{sym}}, s)$. For $\mathrm{FM}(1, \pi_{1,\alpha})$ it follows from Lemma 2.3.

For $n \in \mathbb{N}$ define $Y_n = |\{m > -n\colon m \to -n\}|$ if $-n \in FC_0$, respectively $-n \in FC_0^*$. Otherwise set $Y_n = 0$. If $-n$ is visited by active frogs, then Y_n counts the number of frogs to the right of $-n$ that potentially ever reach $-n$. The process $(Y_n)_{n\in\mathbb{N}}$ is a Markov chain on $\mathbb{N}_0$ with

$$Y_{n+1} = \begin{cases} 0 & \text{if } Y_n = 0, \\ \mathrm{Binomial}(Y_n + 1, p) & \text{if } Y_n > 0. \end{cases}$$

Note that $\mathbb{P}(0 \overset{\mathbb{Z}}{\rightsquigarrow} -n) = \mathbb{P}(Y_n > 0)$ by definition. A straightforward calculation shows that there is $k_0 \in \mathbb{N}$ such that $\mathbb{P}(Y_{n+1} < Y_n \mid Y_n = k) > \frac{2}{3}$ for all $k \ge k_0$. Hence, we can dominate the Markov chain $(Y_n)_{n\in\mathbb{N}}$ by the Markov chain $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ on $\{0, k_0, k_0+1, \ldots\}$ with transition probabilities

$$\mathbb{P}(\widetilde{Y}_{n+1} = l \mid \widetilde{Y}_n = k) = \begin{cases} \frac{1}{3} & \text{if } l = k+1,\ k > k_0, \\ \frac{2}{3} & \text{if } l = k-1,\ k > k_0, \\ (1-p)^{k_0+1} & \text{if } l = 0,\ k = k_0, \\ 1-(1-p)^{k_0+1} & \text{if } l = k+1,\ k = k_0, \\ 1 & \text{if } l = k = 0 \end{cases}$$

for all $n \in \mathbb{N}$ and starting point $\widetilde{Y}_1 = \max\{Y_1, k_0\}$. Obviously, we have the inequality $\mathbb{P}(Y_n > 0) \le \mathbb{P}(\widetilde{Y}_n > 0)$ for all $n \in \mathbb{N}$. Let $T_k = \min\{n \in \mathbb{N}\colon \widetilde{Y}_n = k\}$ and $T_{k,l} = T_l - T_k$. Note that $\mathbb{P}(\widetilde{Y}_n > 0) = \mathbb{P}(T_0 > n)$. For $t > 0$ the Markov inequality implies

$$\begin{aligned} \mathbb{P}(T_0 > n) &= \mathbb{P}\Bigg(\sum_{k=k_0}^{\widetilde{Y}_1 - 1} T_{k+1,k} + T_{k_0,0} > n\Bigg) \\ &\le \mathrm{e}^{-tn}\mathbb{E}\Bigg[\exp\Bigg(t\sum_{k=k_0}^{\widetilde{Y}_1-1} T_{k+1,k} + tT_{k_0,0}\Bigg)\Bigg] \\ &= \mathrm{e}^{-tn}\sum_{l=k_0}^{\infty}\prod_{k=k_0}^{l-1}\mathbb{E}\big[\exp(tT_{k+1,k})\big]\mathbb{E}\big[\exp(tT_{k_0,0})\big]\mathbb{P}(\widetilde{Y}_1 = l) \\ &= \mathrm{e}^{-tn}\sum_{l=0}^{\infty}\mathbb{E}\big[\exp(tT_{k_0+1,k_0})\big]^l\mathbb{E}\big[\exp(tT_{k_0,0})\big]\mathbb{P}(\widetilde{Y}_1 = l + k_0). \end{aligned} \tag{2.12}$$

$\widetilde{Y}_1$ can only be equal to $l + k_0$ if at least one frog to the right of $l - 1$ reaches -1. Thus,

$$\mathbb{P}(\widetilde{Y}_1 = l + k_0) \leq \sum_{i=l}^{\infty} p^{i+1} = p^l \frac{p}{1-p}. \tag{2.13}$$

Now, we choose $t > 0$ small enough such that $\mathbb{E}\big[\exp(tT_{k_0+1,k_0})\big] < p^{-1}$. Then (2.13) shows that the sum in (2.12) is finite, which yields the claim.

2.3.2 Transience for $d \geq 2$ and arbitrary drift

Proof of Theorem 1.15 (ii) and Theorem 1.17 (ii). Let the parameters $\alpha > 0$ and $d \geq 2$ be fixed throughout the proof. For $x \in \mathbb{Z}^d$ we define

$$L_x = \{y \in \mathbb{Z}^d \colon y_i = x_i \text{ for all } 2 \leq i \leq d\}. \tag{2.14}$$

L_x consists of all vertices which agree in all coordinates with x except the e_1-coordinate. The key observation used in the proof is that all particles mainly move along these lines if the weight w is large.

We dominate the frog model by a branching random walk on $\mathbb{Z}^d$. At time $n = 0$ the branching random walk starts with one particle at the origin. At every step in time every particle produces offspring as follows: For every particle located at $x \in \mathbb{Z}^d$ consider an independent copy of the frog model. At any vertex $z \in \mathbb{Z}^d \setminus L_x$ the particle produces $|\{y \in L_x \colon x \overset{L_x}{\rightsquigarrow} y, y \to z\}|$ many children. Notice that this number might be 0 or infinite. The particle does not produce any offspring at a vertex in L_x. Further, note that the particles reproduce independently of each other as we use independent copies of the frog model to generate the offspring.

One can couple this branching random walk with the original frog model. To explain the coupling, let us briefly describe how to go from the original frog model to the branching random walk. Recall that the frog model is entirely determined by a set of trajectories $(S_n^x)_{n \in \mathbb{N}_0, x \in \mathbb{Z}^d}$ of random walks. We use this set of trajectories to produce the particles in the first generation of the branching random walk, i.e. the children of the particle initially at 0, as explained above. Now, assume that the first n generations of the branching random walk have been created. Enumerate the particles in the n-th generation. When generating the offspring of the i-th particle in this generation, delete all trajectories of the frog model used for generating the offspring of a particle j with $j < i$ or a particle in an earlier generation, and replace them by independent trajectories. Otherwise, use the original trajectories.

One can check that the branching random walk dominates the frog model in the following sense: For every frog in $\mathbb{Z}^d \setminus L_0$ that is activated and visits 0 there is a particle at 0 in the branching random walk. Thus, the number of visits to the origin by particles in the branching random walk is at least as big as the number of visits to 0 by frogs in the frog model, not counting those visits to 0 made by frogs initially in L_0. Note that, if the frog model was recurrent, then almost surely there would be infinitely many frogs in $\mathbb{Z}^d \setminus L_0$ activated that return to 0. In particular, also in the branching random walk infinitely many particles would return to 0. Therefore, to prove transience of the frog model it suffices to show that in the branching random walk only finitely many particles return to 0 almost surely.

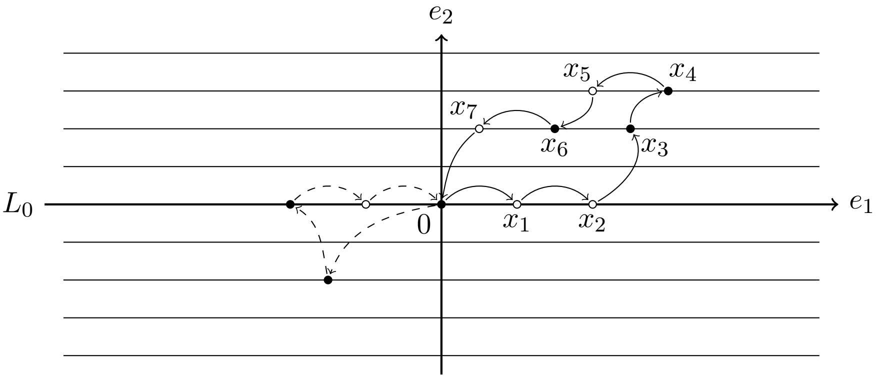

Figure 2.5: The figure shows the lines L_x, $x \in \mathbb{Z}^2$. Assume $x_1, \dots, x_7$ is a frog path from 0 to 0. In the branching process there is one descendant of 0 at x_3. This particle produces one child at x_4, this child has one descendant at x_6. The particle at x_6 produces one descendant at 0. The dashed frog path, on the other hand, does not result in a particle being placed at 0 in the branching random walk.

Let D_n denote the set of descendants in the n-th generation of the branching random walk. Further, for $i \in D_n$ let X_n^i be the e_1-coordinate of the location of particle i. Define for $\theta > 0$ and $n \in \mathbb{N}_0$

$$\mu = \mathbb{E}\Big[\sum_{i \in D_1} \mathrm{e}^{-\theta X_1^i}\Big] \qquad \text{and} \qquad M_n = \frac{1}{\mu^n} \sum_{i \in D_n} \mathrm{e}^{-\theta X_n^i}. \tag{2.15}$$

We claim that $\mu < 1$ for w close to 1 and θ small, which, in particular, implies that $(M_n)_{n\in\mathbb{N}_0}$ is well-defined. We show this claim in the end of the proof. We next show that $(M_n)_{n\in\mathbb{N}_0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ with

$$\mathcal{F}_n = \sigma\big(D_1, \dots, D_n, (X_1^i)_{i\in D_1}, \dots, (X_n^i)_{i \in D_n}\big).$$

Obviously, M_n is $\mathcal{F}_n$-measurable. For a particle $i \in D_n$ denote its descendants in generation $n+1$ by D_{n+1}^i. Since particles branch independently, we get

$$\begin{aligned}\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\Big[\frac{1}{\mu^{n+1}} \sum_{i \in D_{n+1}} \mathrm{e}^{-\theta X_{n+1}^i} \mid \mathcal{F}_n\Big] \\ &= \frac{1}{\mu^n} \sum_{i\in D_n} \mathrm{e}^{-\theta X_n^i} \cdot \frac{1}{\mu} \mathbb{E}\Big[\sum_{j \in D_{n+1}^i} \mathrm{e}^{-\theta\left(X_{n+1}^j - X_n^i\right)}\Big].\end{aligned}$$

Note that the expectation on the right hand side is independent of i and n and therefore, by the definition of μ, we conclude

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n.$$

This calculation also yields $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mathbb{E}[M_0] = 1$, and therefore $M_n \in \mathcal{L}^1$. This in particular implies that M_n is finite almost surely for every $n \in \mathbb{N}_0$. Thus, $X_n^i = 0$ can only occur for finitely many $i \in D_n$ almost surely for every $n \in \mathbb{N}_0$, i.e. in every generation only finitely many particles can be at 0. By the martingale convergence theorem, there exists an almost surely finite random variable M_∞, such that $\lim_{n\to\infty} M_n = M_\infty$ almost surely. Combining this with $\mu < 1$, we get $\lim_{n\to\infty} \sum_{i\in D_n} \mathrm{e}^{-\theta X_n^i} = 0$ almost surely. Hence, $X_n^i = 0$ for some $i \in D_n$ occurs only for finitely many times n. Overall, this shows that the branching random walk is transient.
It remains to show $\mu < 1$. Note that the particles in D_1 are located at vertices in the set $\{y \in \mathbb{Z}^d \setminus L_0 \colon 0 \overset{L_0}{\rightsquigarrow} y\}$. Therefore, for the calculation of μ we first need to consider all sites in L_0 that are reached from 0 by frog paths in L_0. The idea is to control the number of frogs activated on the negative e_1-axis using Lemma 2.16 and estimating the number of frogs activated on the positive e_1-axis by assuming the worst case scenario that all of them will be activated. Then, for every $k \in \mathbb{Z}$ we have to estimate the number of vertices with e_1-coordinate k visited by each of these active frogs on the e_1-axis. Due to the definition of μ, the sites visited by frogs on the positive e_1-axis do not contribute much to μ. Recall that H_k denotes the hyperplane that consists of all vertices with e_1-coordinate equal to $k \in \mathbb{Z}$, see (2.2). For $k, i \in \mathbb{Z}$ define

$$N_{k,i} = |\{x \in H_k \setminus L_0 \colon (i, 0, \ldots, 0) \to x\}|.$$

As $N_{k,i}$ equals $N_{k-i,0}$ in distribution for all $i, k \in \mathbb{Z}$, we get

$$\begin{aligned} \mu &= \mathbb{E}\Big[\sum_{i\in D_1} \mathrm{e}^{-\theta X_1^i}\Big] \\ &= \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i,0,\ldots,0)\big)\mathbb{E}[N_{k,i}]\mathrm{e}^{-\theta k} \\ &= \sum_{k=-\infty}^{\infty} \mathbb{E}[N_{k,0}]\mathrm{e}^{-\theta k} \sum_{i=-\infty}^{\infty} \mathrm{e}^{-\theta i}\mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i,0,\ldots,0)\big). \end{aligned} \tag{2.16}$$

Note that $\mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i,0,\ldots,0)\big)$ is smaller or equal than the probability of the event $\{0 \overset{\mathbb{Z}}{\rightsquigarrow} i\}$ in the frog model $\mathrm{FM}(1,1,\alpha)$. Hence, by Lemma 2.16, there is a constant $c_1 > 0$ such that $\mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i,0,\ldots,0)\big) \le \mathrm{e}^{c_1 i}$ for all $i \le 0$. Thus, (2.16) implies that for $\theta < c_1$ there is a constant $c_2 = c_2(\theta) < \infty$ such that

$$\mu \le c_2 \sum_{k=-\infty}^{\infty} \mathbb{E}[N_{k,0}]\mathrm{e}^{-\theta k}. \tag{2.17}$$

Next we estimate $\mathbb{E}[N_{k,0}]$, the expected number of vertices in $H_k \setminus L_0$ visited by a single particle starting at 0. Recall that the trajectory of frog 0 is denoted by $(S_n^0)_{n\in\mathbb{N}_0}$. We define $T_k = \min\{n \in \mathbb{N}_0 \colon S_n^0 \in H_k\}$, the entrance time of the hyperplane H_k, and further $T_k' = \max\{n \in \mathbb{N}_0 \colon S_n^0 \in H_k\}$, the last time frog 0 is in the hyperplane H_k. Obviously,

$N_{k,0} = 0$ on the event $\{T_k = \infty\}$. Hence, assume we are on $\{T_k < \infty\}$. The particle can only visit a vertex in $H_k \setminus L_0$ at time T_k if the random walk took at least one step in non-e_1-direction up to time T_k. This happens with probability $\mathbb{E}[1 - w^{T_k}]$. Furthermore, the number of vertices visited in H_k after time T_k can be estimated by the number of steps in non-e_1-direction taken between times T_k and T_k'. This number is binomially distributed and, thus, its expectation equals $(1-w)\mathbb{E}[T_k' - T_k]$. Overall, this implies

$$\mathbb{E}[N_{k,0}] \le \mathbb{P}(T_k < \infty)\big(\mathbb{E}\big[1 - w^{T_k} \mid T_k < \infty\big] + (1-w)\mathbb{E}\big[T_k' - T_k \mid T_k < \infty\big]\big).$$

For $k < 0$ the probability $\mathbb{P}(T_k < \infty)$ decays exponentially in k by Lemma 2.3. Therefore, we can choose θ small such that $\mathbb{P}(T_k < \infty)\mathrm{e}^{-\theta k} \le \mathrm{e}^{-\theta|k|}$ for all $k \in \mathbb{Z}$. Thus, (2.17) implies

$$\mu \le c_2 \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\theta|k|}\Big(\mathbb{E}\big[1 - w^{T_k} \mid T_k < \infty\big] + (1-w)\mathbb{E}\big[T_k' - T_k \mid T_k < \infty\big]\Big). \tag{2.18}$$

Note that the sum in (2.18) is finite as $\mathbb{E}\big[T_k' - T_k \mid T_k < \infty\big]$ is independent of k. By monotone convergence $\lim_{w\to 1} \mu = 0$ and the right hand side of (2.18) is continuous in w. Therefore, we can choose w close to 1 such that $\mu < 1$, as claimed. ∎

2.3.3 Transience for $d = 2$ and arbitrary weight

Proof of Theorem 1.14 (ii). Let $w > 0$ be fixed throughout the proof. As in the proof of Theorem 1.15 (ii) and Theorem 1.17 (ii) we dominate the frog model by a branching random walk. This time we use a one-dimensional branching random walk on $\mathbb{Z}$. For the construction of the process, let ξ be the number of activated frogs in an independent one-dimensional frog model $\mathrm{FM}^*(1, \pi_{\mathrm{sym}}, 1-w)$ with two active frogs at 0 initially. At time $n = 0$, the branching random walk starts with one particle in the origin. At every time $n \in \mathbb{N}$, the process repeats the following two steps. First, every particle produces offspring independently of all other particles with the number of offspring being distributed as ξ. Then, each particle jumps to the right with probability $\frac{1+\alpha}{2}$ and to the left with probability $\frac{1-\alpha}{2}$.

As an intermediate step to understand the relation between the frog model and this branching random walk on $\mathbb{Z}$, we first couple the frog model with a branching random walk on $\mathbb{Z}^2$ with initially one particle at 0. Partition the lattice $\mathbb{Z}^2$ into hyperplanes $(H_n)_{n\in\mathbb{Z}}$ as defined in (2.2). Let the frog model $\mathrm{FM}(2, \pi_{w,\alpha})$ with initially two active frogs at $0 \in H_0$ evolve and stop every frog when it first enters H_1 or H_{-1}. Every frog leaves its hyperplane in every step with probability w. Thus, the number of stopped frogs is distributed according to ξ. A stopped frog is in H_1 with probability $\frac{1+\alpha}{2}$ and in H_{-1} with probability $\frac{1-\alpha}{2}$. The stopped particles form the offspring of the particle at 0 in the branching random walk. We repeat this procedure to generate the offspring of an arbitrary particle in the branching random walk. Introduce an ordering of all particles in the branching random walk and let the particles branch one after another. Before generating the offspring of the i-th particle, refill every vertex which is no longer occupied by a sleeping frog with an extra independent sleeping frog. Unstop frog i and let it continue its work as usual, ignoring all other stopped frogs. Note that there is a sleeping frog at the starting vertex of frog i that is immediately activated. This explains our definition of ξ. Again stop every frog once it enters one of the neighbouring hyperplanes. These newly stopped frogs form the offspring of the i-th particle. This procedure creates a branching random walk with independent identically distributed offspring. Every vertex visited in the frog model is obviously also visited by the branching random walk.

Now, project all particles in the intermediate two-dimensional branching random walk onto $\mathbb{Z}$. This creates a branching random walk on $\mathbb{Z}$ distributed as the one described above. The construction shows that transience of this one-dimensional branching random walk implies transience of the frog model.

To prove that the one-dimensional branching random walk is transient for α close to 1, we proceed as in the proof of Theorem 1.15 (ii) and Theorem 1.17 (ii). The proof only differs in the calculation of the parameter μ defined by

$$\mu = \mathbb{E}\Big[\sum_{i \in D_1} \mathrm{e}^{-\theta X_1^i}\Big]$$

for $\theta > 0$ with D_1 denoting the set of descendants in the first generation of the branching random walk and X_1^i the e_1-coordinate of the location of particle $i \in D_1$. Here, we immediately get

$$\mu = \frac{1}{2}\big((1-\alpha)\mathrm{e}^{\theta} + (1+\alpha)\mathrm{e}^{-\theta}\big)\mathbb{E}[\xi].$$

Lemma 2.16 implies $\mathbb{E}[\xi] < \infty$. Thus, we can choose $\theta = \log\big(2\mathbb{E}[\xi]\big)$. Then $\lim_{\alpha \to 1} \mu = \frac{1}{2}$ and by continuity $\mu < 1$ for α close to 1, as required. □

2.4 About the maximum and minimum

In this section we present the proofs of Theorems 1.21–1.25 presented in Section 1.3.2. They concern the one-dimensional frog model $\mathrm{FM}(1, \pi_\alpha)$ for $\alpha \in [0,1]$ with π_α as defined in (1.2).

2.4.1 Existence of the speed of the maximum

The existence of the speed of the maximum follows from Liggett's Subadditive Ergodic Theorem. Indeed, this theorem yields more information which we use throughout this section. We summarize it in the following lemma.

Lemma 2.17. *For each $\alpha \in [0,1]$ there exists a positive constant $v_{\max}$ such that in the frog model* $\mathrm{FM}(1, \pi_\alpha)$

$$v_{\max} = \lim_{n \to \infty} \frac{M_n}{n} \quad a.s.$$

Furthermore,

$$v_{\max}^{-1} = \lim_{x \to \infty} \frac{T_x}{x} = \lim_{x \to \infty} \frac{\mathbb{E}[T_x]}{x} = \inf_{x \in \mathbb{N}} \frac{\mathbb{E}[T_x]}{x} \quad a.s. \tag{2.19}$$

Proof. Let $T_{x,y}$ denote the activation time of the frog at site y when initially there is one active frog at site x and one sleeping frog at every other site. An application of Liggett's Subadditive Ergodic Theorem (see e.g. [38]) to the times $(T_{x,y})_{x,y\in\mathbb{Z}}$ shows the existence of a positive constant $v_{\max}$ such that (2.19) holds. For $\alpha = 0$ this is proved for a more general model by Alves et al. in [2]. In our setting their argument immediately applies to $\alpha > 0$ as well.

By a standard argument it now follows that $\lim_{n\to\infty} \frac{M_n}{n}$ exists almost surely: There exists a unique random sequence $(x_n)_{n\in\mathbb{N}}$ with values in $\mathbb{N}_0$ such that $T_{x_n} \le n < T_{x_n+1}$. Note that $\lim_{n\to\infty} x_n = \infty$. Hence,

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} \frac{T_{x_n}}{x_n} = \lim_{n\to\infty} \frac{n}{x_n} \quad \text{a.s.}$$

Obviously, $x_n - (n - T_{x_n}) \le M_n \le x_n$. This implies

$$\frac{x_n}{n} - \left(1 - \frac{T_{x_n}}{x_n} \cdot \frac{x_n}{n}\right) \le \frac{M_n}{n} \le \frac{x_n}{n}.$$

Taking limits yields the claim.

2.4.2 Speed of the minimum

In order to prove Theorem 1.21 we compare the frogs initially on non-negative sites with independent random walks. The speed of the minimum of independent random walks can be computed explicitly which is done in the first of the following lemmas. Then it remains to deal with the frogs initially on negative sites. Luckily, they can be ignored due to the transience of the frog model. See Theorem 1.11.

We often need to talk about the frogs initially on negative sites and refer to them as the "negative frogs". Analogously we speak of "non-negative" and "positive frogs".

Lemma 2.18. *Let $\{(\widetilde{S}^x_n)_{n\in\mathbb{N}_0} : x \in \mathbb{Z}\}$ be a collection of independent random walks with start in* 0 *and transition probabilities π_α for $\alpha \in [0,1]$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \min_{x\in\{-n,\dots,n\}} \widetilde{S}^x_n = \alpha \quad \textit{a.s.}$$

Proof. We only need to prove $\liminf_{n\to\infty} \frac{1}{n} \min_{x\in\{-n,\dots,n\}} \widetilde{S}^x_n \ge \alpha$. The converse immediately follows from the law of large numbers. For all $\varepsilon > 0$ we have

$$\mathbb{P}\left(\frac{1}{n} \min_{x\in\{-n,\dots,n\}} \widetilde{S}^x_n \le \alpha - \varepsilon\right) = \mathbb{P}\left(\bigcup_{x=-n}^{n} \left\{\frac{\widetilde{S}^x_n}{n} \le \alpha - \varepsilon\right\}\right) \le (2n+1)\mathbb{P}\left(\frac{\widetilde{S}^0_n}{n} \le \alpha - \varepsilon\right).$$

By Cramér's Theorem the probability in the last term of this calculation decays exponentially fast in n. Hence, it is summable. An application of the Borel-Cantelli Lemma and letting $\varepsilon \to 0$ completes the proof.

This result now enables us to prove a formula for the speed of the minimum of the non-negative frogs.

Lemma 2.19. *Consider* $\mathrm{FM}(1, \pi_\alpha)$ *for* $\alpha \in [0,1]$ *and let* $A_n^+ = \{x \geq 0 \colon T_x \leq n\}$*. Then*

$$\lim_{n\to\infty} \frac{1}{n} \min_{x \in A_n^+} Z_n^x = \alpha \quad a.s.$$

Before proving Lemma 2.19 we make another short observation. Obviously $v_{\max}$ is at least as big as the speed of a single frog, i.e. $v_{\max} \geq \alpha$. In fact, this inequality is strict for all $\alpha \in [0,1)$. For $\alpha = 0$ this fact is known from [2] or Theorem 1.10.

Lemma 2.20. *Consider* $\mathrm{FM}(1, \pi_\alpha)$ *for* $\alpha \in (0,1)$*. Then it holds that* $v_{\max} > \alpha$*.*

Proof. The key point is to notice that $\mathbb{E}[T_1] < \mathbb{E}[T_1^s]$ holds, where $T_1^s = \inf\{n \in \mathbb{N} \colon S_n^0 = 1\}$ denotes the hitting time of the vertex 1 of a single simple random walk with drift α and start in 0. Hence, by Lemma 2.17

$$v_{\max}^{-1} = \inf_{x \in \mathbb{N}} \frac{\mathbb{E}[T_x]}{x} \leq \mathbb{E}[T_1] < \mathbb{E}[T_1^s] = \frac{1}{\alpha}.$$

One can of course find better lower bounds for the speed of the maximum by estimating $\mathbb{E}[T_x]$ for $x \geq 1$, but this is not done or needed here.

Proof of Lemma 2.19. It is enough to show $\liminf_{n\to\infty} \frac{1}{n} \min_{x \in A_n^+} Z_n^x \geq \alpha$ almost surely. In this proof we use a different but equivalent way of defining the movement of the frogs. Let $\{(\widetilde{S}_n^x)_{n\in\mathbb{N}_0} \colon x \in \mathbb{Z}\}$ be a collection of independent random walks with start in 0 and transition probabilities π_α for $\alpha \in [0,1]$. We assume that, once it is activated, frog x follows the trajectory $(\widetilde{S}_n^x - \widetilde{S}_{\widetilde{T}_x}^x)_{n\in\mathbb{N}_0}$, where $\widetilde{T}_x$ denotes the activation time of frog x in this set-up. More formally, for every $x \in \mathbb{Z}$ the position of frog x at time $n \in \mathbb{N}_0$ is given by

$$\widetilde{Z}_n^x = \begin{cases} x & \text{for } n < \widetilde{T}_x, \\ x + \widetilde{S}_n^x - \widetilde{S}_{\widetilde{T}_x}^x & \text{for } n \geq \widetilde{T}_x. \end{cases}$$

Note that $(\widetilde{Z}_n^x)$ equals (Z_n^x) in distribution. We now want to compare the trajectory $(\widetilde{Z}_n^x)_{n\in\mathbb{N}_0}$ of each frog with the trajectory $(\widetilde{S}_n^x)_{n\in\mathbb{N}_0}$ of the corresponding random walk started at 0. From time $\widetilde{T}_x$ onwards they move synchronously by definition. Therefore, we only need to compare their locations at time $\widetilde{T}_x$. Note that $\widetilde{Z}_{\widetilde{T}_x}^x = x$ and define $G = \{x \geq 0 \colon \widetilde{S}_{\widetilde{T}_x}^x \leq x\}$ to be the set of good frogs. Further, let $\tilde{A}_n^+ = \{x \geq 0 \colon \widetilde{T}_x \leq n\}$. Now, $x \in \tilde{A}_n^+ \cap G$ implies $\widetilde{S}_n^x \leq \widetilde{Z}_n^x$ for all $n \in \mathbb{N}$, i.e. all good frogs stay to the right of their corresponding random walk. Hence,

$$\min_{x\in\tilde{A}_n^+} \widetilde{Z}_n^x \geq \min_{x\in\tilde{A}_n^+} \widetilde{S}_n^x - \sum_{x \in G^c \cap \tilde{A}_n^+} \big(\widetilde{S}_n^x - \widetilde{Z}_n^x\big) \geq \min_{x\in\tilde{A}_n^+} \widetilde{S}_n^x - \sum_{x\in G^c} \big(\widetilde{S}_{\widetilde{T}_x}^x - x\big). \tag{2.20}$$

We claim that the set G^c is finite almost surely. For $\alpha = 1$ this is obviously true. For $\alpha < 1$ it is enough to show that

$$\lim_{x\to\infty} \frac{\widetilde{S}^x_{\widetilde{T}_x} - x}{\widetilde{T}_x} = \alpha - v_{\max} \quad \text{a.s.} \tag{2.21}$$

since by Lemma 2.20 the last term is strictly negative and hence $\widetilde{S}^x_{\widetilde{T}_x} - x > 0$ can occur only for finitely many $x \geq 0$ almost surely.

Note that $(\widetilde{S}^x_n)_{n \leq \widetilde{T}_x}$ is independent of the movement of the frogs up to time $\widetilde{T}_x$. Thus, $\widetilde{S}^x_{\widetilde{T}_x}$ equals $\widetilde{S}^0_{\widetilde{T}_x}$ in distribution. Using a standard large deviation estimate we get for every $\varepsilon > 0$

$$\mathbb{P}\left(\frac{\widetilde{S}^x_{\widetilde{T}_x}}{\widetilde{T}_x} \leq \alpha - \varepsilon \right) = \mathbb{P}\left(\frac{\widetilde{S}^0_{\widetilde{T}_x}}{\widetilde{T}_x} \leq \alpha - \varepsilon \right) \leq \mathbb{E}\left[\mathrm{e}^{-c\widetilde{T}_x}\right] \leq \mathrm{e}^{-cx}$$

where $c = c(\varepsilon, \alpha) > 0$ is a constant. Analogously $\mathbb{P}\left(\frac{\widetilde{S}^x_{\widetilde{T}_x}}{\widetilde{T}_x} \geq \alpha + \varepsilon \right)$ decays exponentially fast in x. An application of the Borel-Cantelli Lemma and letting $\varepsilon \to 0$ therefore shows

$$\lim_{n\to\infty} \frac{\widetilde{S}^x_{\widetilde{T}_x}}{\widetilde{T}_x} = \alpha \quad \text{a.s.}$$

Further, we know from Lemma 2.17 that $\lim_{x\to\infty} \frac{x}{\widetilde{T}_x} = v_{\max}$ almost surely. This proves equation (2.21) which implies that G^c is finite almost surely.

Therefore, the second term on the right side in inequality (2.20) is finite almost surely. Also note that it does not depend on n. Thus,

$$\liminf_{n\to\infty} \frac{1}{n} \min_{x \in \tilde{A}^+_n} \widetilde{Z}^x_n \geq \liminf_{n\to\infty} \frac{1}{n} \min_{x \in \tilde{A}^+_n} \widetilde{S}^x_n \quad \text{a.s.}$$

As $\tilde{A}^+_n \subseteq \{-n, \dots, n\}$ an application of Lemma 2.18 finishes the proof. ∎

Proof of Theorem 1.21. As shown in [22, Theorem 2.3] and cited in Theorem 1.11 the frog model with drift, as considered here for $\alpha \neq 0$, is transient almost surely. This means that the origin is visited by only finitely many frogs almost surely. Therefore only finitely many negative frogs are ever activated. Hence, Theorem 1.21 follows from Lemma 2.19. ∎

2.4.3 Monotonicity of the speed of the maximum

Next we prove that the speed of the maximum is an increasing function in the drift parameter α. Though this statement might seem obvious at first, no direct coupling of the frog models for different drift parameters seems possible, since for smaller values of α more negative frogs will eventually be woken up, which might help in pushing the front forward. But we can ignore all these frogs without changing the speed of the maximum, similar to the proof of Theorem 1.21. This is shown in the next lemma. We therefore consider the frog model without negative frogs and initially one sleeping frog at every positive integer, i.e. the frog model $\mathrm{FM}(1, \pi_\alpha, \eta)$ with $\eta_x = 1$ for $x > 0$ and $\eta_x = 0$ for $x < 0$.

Lemma 2.21. *Let* $v_{\max}$ *be the speed of the maximum in the frog model* $\mathrm{FM}(1,\pi_\alpha)$ *with* $\alpha \in [0,1]$*. Further, let* η *be the initial configuration given by* $\eta_x = 0$ *for* $x < 0$ *and* $\eta_x = 1$ *for* $x > 0$*. Let* T_x^+ *denote the activation time of* $x \in \mathbb{N}$ *in the frog model* $\mathrm{FM}(1,\pi_\alpha,\eta)$*. Then*

$$v_{\max}^{-1} = \lim_{x\to\infty} \frac{T_x^+}{x} \quad \textit{a.s.}$$

Proof. We only need to prove $\limsup_{x\to\infty} \frac{T_x^+}{x} \le v_{\max}^{-1}$ almost surely. We couple the two frog models by assuming that the corresponding frogs in both models follow the same paths. They might be activated at different times, but otherwise move in the same way. All variables in this proof, apart from the activation times T_x^+, refer to the frog model $\mathrm{FM}(1,\pi_\alpha)$.
First, we consider the frog model $\mathrm{FM}(1,\pi_\alpha)$ and show that the speed of the maximum of all negative frogs equals α almost surely. I.e. setting $A_n^- = \{x < 0\colon T_x \le n\}$ we claim that

$$\lim_{n\to\infty} \frac{1}{n} \max_{x\in A_n^-} Z_n^x = \alpha \quad \text{a.s.} \tag{2.22}$$

For $\alpha > 0$ only finitely many negative frogs will ever be activated almost surely by Theorem 1.11. In this case equation (2.22) follows immediately. If $\alpha = 0$, then by symmetry the claim follows from Lemma 2.19.
Let E be the set of all positive frogs which are activated by negative frogs, meaning that at the time of their activation at least one negative frog is present. Since $v_{\max} > \alpha$ as proved in Lemma 2.20 and by equation (2.22) the set E is finite almost surely.
Hence, $T = \sup_{x\in E}(T_x^+ - T_x)$ is an almost surely finite random variable. For all $x \in E$ we the definition of T yields $T_x^+ \le T_x + T$. Actually, we claim that this inequality is true for all $x \in \mathbb{N}_0$, which immediately implies the claim of the lemma.
The inequality can e.g. be proven inductively. For $x = 0$ the inequality is obviously true as $T_0^+ = T_0 = 0$. Now assume that $x \in \mathbb{N}$ and $T_y^+ \le T_y + T$ holds for all $0 \le y \le x-1$. If $x \in E$, there is nothing to show. Otherwise, let $0 \le z \le x-1$ be the (random) frog that activates the frog x in the version $\mathrm{FM}(1,\pi_\alpha)$. Then we have

$$T_x^+ \le T_z^+ + (T_x - T_z) \le T_x + T,$$

as claimed. ∎

Proof of Theorem 1.22. Using a standard coupling we can achieve that $T_i^+(\alpha)$ is monotone decreasing in α. As $v_{\max}(\alpha) = \lim_{x\to\infty} \frac{x}{T_x^+}$ almost surely by Lemma 2.21, we conclude that $v_{\max}(\alpha)$ is increasing in α. ∎

2.4.4 Upper bound for the speed of the maximum

In order to bound the speed of the maximum from above we prove an upper bound for the number of frogs in the maximum. We do this for a slightly modified frog model: Each time the maximum moves to the left we put a sleeping frog at the site that has just been left by the maximum. Hence, in this new model there is one sleeping frog at every site to the right of the maximum at any time. Further notice that, except at time 0, there are always at least two frogs in the maximum. We use the same notation as in the usual frog model, but add an index "mod" when referring to the modified model. Further, let a_n denote the number of frogs in the maximum at time $n \in \mathbb{N}_0$ in the modified frog model.

Lemma 2.22. *In the modified frog model as explained above, we have for* $\alpha \in (0,1)$ *and all* $n \in \mathbb{N}_0$

$$\mathbb{E}[a_n] \leq \frac{(3-\alpha)(1+\alpha)}{2(1-\alpha)\alpha}.$$

Proof. We define $p = \frac{1+\alpha}{2}$, i.e. we use p to denote the probability of a jump to the right by a particle. This notation slightly improves the readability of the following calculations.

We prove bounds not only for the number of frogs in the maximum, but for every other site as well. Therefore, let $a_n(k)$ be the number of frogs at location $M_n^{\text{mod}} - 2k$ at time n for $k, n \in \mathbb{N}_0$. We prove by induction on n that for all $n, k \in \mathbb{N}_0$

$$\mathbb{E}[a_n(k)] \leq \frac{(2-p)p}{(1-p)(2p-1)p^k}. \tag{2.23}$$

For $n = 0$ and $n = 1$ one easily checks that the claim is true. Assume that the claim holds for some integer $n \in \mathbb{N}$.

First we show inequality (2.23) for $k = 0$. Distinguishing whether all a_n particles in the maximum at time n move to the left or not in the next step one calculates

$$\begin{aligned}\mathbb{E}[a_{n+1}] &= \mathbb{E}\big[(1-p)^{a_n}\big(a_n + pa_n(1)\big)\big] + \mathbb{E}\Big[\big(1-(1-p)^{a_n}\big)\Big(\frac{pa_n}{1-(1-p)^{a_n}} + 1\Big)\Big] \\ &= \mathbb{E}\big[(1-p)^{a_n}\big(a_n + pa_n(1) - 1\big) + pa_n + 1\big].\end{aligned}$$

Note here that the expectation of a binomial random variable with parameters $p > 0$ and $k \in \mathbb{N}$ conditioned on being at least 1 is given by $\frac{pk}{1-(1-p)^k}$. Using $a_n \geq 2$ yields

$$\mathbb{E}[a_{n+1}] \leq (1-p)^2\mathbb{E}\big[a_n + pa_n(1) - 1\big] + p\mathbb{E}[a_n] + 1. \tag{2.24}$$

Inserting the induction hypothesis (2.23) in (2.24) the claim follows after a straightforward calculation.

For $k = 1$ an analogous calculation yields

$$\begin{aligned}\mathbb{E}[a_{n+1}(1)] &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(2) + (1-p)a_n(1)\big)\big] \\ &\quad + \mathbb{E}\Big[\big(1-(1-p)^{a_n}\big)\Big(a_n - \frac{pa_n}{1-(1-p)^{a_n}} + pa_n(1)\Big)\Big] \\ &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(2) - (2p-1)a_n(1) - a_n\big)\big] \\ &\quad + \mathbb{E}\big[(1-p)a_n + pa_n(1)\big].\end{aligned} \tag{2.25}$$

For $k \geq 2$ one gets

$$\begin{aligned}\mathbb{E}[a_{n+1}(k)] &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(k+1)+(1-p)a_n(k)\big)\big]\\ &\quad+\mathbb{E}\big[(1-(1-p)^{a_n})\big(pa_n(k)+(1-p)a_n(k-1)\big)\big]\\ &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(k+1)-(2p-1)a_n(k)-(1-p)a_n(k-1)\big)\big]\\ &\quad+\mathbb{E}\big[(1-p)a_n(k-1)+pa_n(k)\big]. \end{aligned}\tag{2.26}$$

Thus, for $k \geq 1$ equations (2.25) and (2.26) imply

$$\mathbb{E}[a_{n+1}(k)] \leq p(1-p)^2\mathbb{E}[a_n(k+1)]+p\mathbb{E}[a_n(k)]+(1-p)\mathbb{E}[a_n(k-1)]. \tag{2.27}$$

As before, inserting the induction hypothesis (2.23) into inequality (2.27) completes the proof. □

Proof of Theorem 1.23. Consider the event that in the modified frog model at time n all a_n frogs sitting in the maximum move to the left. Using Jensen's inequality and Lemma 2.22, we conclude that the probability of this event is bounded from below by

$$\mathbb{E}\Big[\Big(\frac{1-\alpha}{2}\Big)^{a_n}\Big] \geq \Big(\frac{1-\alpha}{2}\Big)^{\mathbb{E}[a_n]} \geq \Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2(1-\alpha)\alpha}}.$$

Therefore, for all $n \in \mathbb{N}_0$

$$\mathbb{E}\big[T_{n+1}^{\mathrm{mod}}-T_n^{\mathrm{mod}}\big] \geq 1+2\,\mathbb{E}\Big[\Big(\frac{1-\alpha}{2}\Big)^{a_{T_n^{\mathrm{mod}}}}\Big] \geq 1+2\Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2(1-\alpha)\alpha}}.$$

Clearly, in the modified model, frogs are activated no later than in the normal version of the frog model. Thus,

$$\mathbb{E}[T_n] \geq \mathbb{E}\big[T_n^{\mathrm{mod}}\big] = \sum_{k=1}^{n}\mathbb{E}\big[T_k^{\mathrm{mod}}-T_{k-1}^{\mathrm{mod}}\big] \geq \Big(1+2\Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2(1-\alpha)\alpha}}\Big)n.$$

By Lemma 2.17 we conclude

$$v_{\max}^{-1} = \inf_{n\in\mathbb{N}}\frac{\mathbb{E}[T_n]}{n} \geq 1+2\Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2(1-\alpha)\alpha}} > 1.$$ □

2.4.5 Equidistribution of frogs

It remains to prove Theorem 1.25. The idea of the proof is quite simple: From the point of view of the minimum the front moves with a positive speed, but all the frogs only fluctuate around their locations with $\sqrt{n}$, so basically they stay where they are.

First, we show that for large enough times n all active frogs do not deviate much from their expected locations. To formalize this statement we define for $n \in \mathbb{N}_0$ the set

$$G_n = \{x \in A_n\colon \big|Z_n^x - \mathbb{E}[Z_n^x]\big| < n^{3/4}\}.$$

Lemma 2.23. *In the frog model* $\mathrm{FM}(1,\pi_\alpha)$ *with* $\alpha \in [0,1]$, *we almost surely have* $G_n = A_n$ *for all* n *large enough.*

Proof. As $A_n \subseteq \{-n,\dots,n\}$ we have

$$\begin{aligned}\mathbb{P}\big(A_n \neq G_n\big) &= \mathbb{P}\Big(\bigcup_{x\in A_n} \big\{\big|Z_n^x - \mathbb{E}[Z_n^x]\big| \geq n^{3/4}\big\}\Big)\\ &\leq \sum_{x=-n}^{n} \mathbb{P}\big(\big|Z_n^x - \mathbb{E}[Z_n^x]\big| \geq n^{3/4}\big)\\ &= \sum_{x=-n}^{n}\sum_{k=0}^{n} \mathbb{P}\big(\big|Z_n^x - \mathbb{E}[Z_n^x]\big| \geq n^{3/4} \big| T_x = k\big)\cdot \mathbb{P}(T_x = k). \qquad (2.28)\end{aligned}$$

Further, for every $x \in \mathbb{Z}$ and $0 \leq k \leq n$ it holds that

$$\begin{aligned}\mathbb{P}\big(\big|Z_n^x - \mathbb{E}[Z_n^x]\big| \geq n^{3/4}\big|T_x = k\big) &= \mathbb{P}\big(\big|S_{n-k}^x - \mathbb{E}[S_{n-k}^x]\big| \geq n^{3/4}\big)\\ &\leq 2\exp\Big(-\frac{n^{3/2}}{4(n-k)}\Big)\\ &\leq 2\exp\Big(-\frac{n^{1/2}}{4}\Big).\end{aligned}$$

In the first inequality in the above estimate we use Höffding's inequality. Thus, (2.28) implies

$$\mathbb{P}\big(A_n \neq G_n\big) \leq 2\exp\Big(-\frac{n^{1/2}}{4}\Big)\sum_{x=-n}^{n}\sum_{k=0}^{n}\mathbb{P}(T_x = k) \leq 2(2n+1)\exp\Big(-\frac{n^{1/2}}{4}\Big)$$

which is summable. An application of the Borel-Cantelli Lemma completes the proof.

For $\varepsilon > 0$ and $b \in [0,1]$ define

$$L_n(b,\varepsilon) = \begin{cases}\big\{x \in \mathbb{Z}\colon -(v_{\max}-\varepsilon)n \leq x \leq \big((2b-1)v_{\max}-\varepsilon\big)n\big\} & \text{for } \alpha = 0,\\ \big\{x \in \mathbb{Z}\colon 0 \leq x \leq (bv_{\max}-\varepsilon)n\big\} & \text{for } \alpha > 0\end{cases}$$

and

$$R_n(b,\varepsilon) = \begin{cases}\big\{x \in \mathbb{Z}\colon \big((2b-1)v_{\max}+\varepsilon\big)n \leq x \leq (v_{\max}-\varepsilon)n\big\} & \text{for } \alpha = 0,\\ \big\{x \in \mathbb{Z}\colon (bv_{\max}+\varepsilon)n \leq x \leq (v_{\max}-\varepsilon)n\big\} & \text{for } \alpha > 0.\end{cases}$$

Lemma 2.24. *Consider the frog model* $\mathrm{FM}(1,\pi_\alpha)$ *with* $\alpha \in [0,1]$ *and let* $b \in [0,1]$. *For* n *large enough,* $x \in L_n(b,\varepsilon) \cap G_n$ *implies*

$$\frac{Z_n^x - v_{\min}n}{(v_{\max}-v_{\min})n} \leq b, \qquad (2.29)$$

whereas $x \in R_n(b,\varepsilon) \cap G_n$ *implies*

$$\frac{Z_n^x - v_{\min}n}{(v_{\max}-v_{\min})n} \geq b. \qquad (2.30)$$

Proof. For $\alpha = 0$ note that by symmetry $v_{\min} = -v_{\max}$. Thus, (2.29) holds if and only if $Z_n^x \leq (2b-1)v_{\max}n$. Assume $x \in L_n(b,\varepsilon) \cap G_n$. A straightforward calculation shows

$$Z_n^x \leq \mathbb{E}[Z_n^x] + n^{3/4} = x + n^{3/4} \leq (2b-1)v_{\max}n$$

for n big enough. Analogously, one shows (2.30) in this case.
For $\alpha > 0$ the proof works essentially in the same way as for $\alpha = 0$, but the estimation of $\mathbb{E}[Z_n^x]$ is less trivial. We have $\mathbb{E}[Z_n^x] = x + (n - \mathbb{E}[T_x])v_{\min}$. For $x \in L_n(b,\varepsilon) \cap G_n$ we thus get

$$Z_n^x \leq \mathbb{E}[Z_n^x] + n^{3/4} = v_{\min}n + \frac{x}{v_{\max}}\Big(v_{\max} - \frac{\mathbb{E}[T_x]}{x}v_{\min}v_{\max}\Big) + n^{3/4}.$$

Lemma 2.17 yields that $\frac{\mathbb{E}[T_x]}{x} \geq \inf_{x\in\mathbb{N}} \frac{\mathbb{E}[T_x]}{x} = v_{\max}^{-1}$. Hence, for n big enough

$$\begin{aligned} Z_n^x &\leq v_{\min}n + \frac{x}{v_{\max}}(v_{\max} - v_{\min}) + n^{3/4} \\ &\leq v_{\min}n + b(v_{\max} - v_{\min})n, \end{aligned}$$

as claimed in (2.29). On the other hand, $x \in R_n(b,\varepsilon) \cap G_n$ analogously implies

$$Z_n^x \geq v_{\min}n + \frac{x}{v_{\max}}\Big(v_{\max} - \frac{\mathbb{E}[T_x]}{x}v_{\min}v_{\max}\Big) - n^{3/4}.$$

Since $\lim_{x\to\infty} \frac{\mathbb{E}[T_x]}{x} = v_{\max}^{-1}$, and x tends to infinity whenever n does by the definition of $R_n(x,\varepsilon)$, we know that $\frac{\mathbb{E}[T_x]}{x} \leq v_{\max}^{-1} + \delta\varepsilon$ for n big enough and a small constant δ. Therefore,

$$Z_n^x \geq v_{\min}n + \frac{x}{v_{\max}}(v_{\max} - v_{\min} - \varepsilon\delta v_{\min}v_{\max}) - n^{3/4}.$$

Using $x \geq (bv_{\max} + \varepsilon)n$ and choosing δ small enough finishes the proof.

Proof of Theorem 1.25. We need to show that $\lim_{n\to\infty}\mu_n([0,b]) = \lambda([0,b])$ holds for every $b \in [0,1]$ almost surely.
Take a realisation of the frog model such that $A_n = G_n$ holds for sufficiently large n, such that $\lim_{n\to\infty}\frac{M_n}{n} = v_{\max}$ and $\lim_{n\to\infty}\frac{m_n}{n} = v_{\min}$, and finally such that $A_n \cap \mathbb{Z}^-$ is finite. This happens with probability 1 as we have seen in Lemma 2.23, Lemma 2.17, Theorem 1.21 and previous discussions about the transience of the frog model. Now fix $b \in [0,1]$ and $\varepsilon > 0$ small. Lemma 2.24 yields that, for n large enough,

$$\mu_n([0,b]) \geq \frac{1}{|A_n|}|G_n \cap L_n(b,\varepsilon)| = \frac{n}{|A_n|} \cdot \frac{|L_n(b,\varepsilon)|}{n}. \tag{2.31}$$

For the last equation we used $L_n(b,\varepsilon) \subseteq A_n$ for sufficiently large n as $\lim_{n\to\infty}\frac{M_n}{n} = v_{\max}$.
The definition of $L_n(b,\varepsilon)$ implies

$$|L_n(b,\varepsilon)| \geq \begin{cases} 2(bv_{\max} - \varepsilon)n & \text{for } \alpha = 0, \\ (bv_{\max} - \varepsilon)n & \text{for } \alpha > 0. \end{cases}$$

Further, $\lim_{n\to\infty} \frac{n}{|A_n|} = \frac{1}{2} v_{\max}^{-1}$ for $\alpha = 0$, respectively $\lim_{n\to\infty} \frac{n}{|A_n|} = v_{\max}^{-1}$ for $\alpha > 0$. Thus, the limit inferior of the last term in (2.31) as $n \to \infty$ is bounded from below by $b - \varepsilon v_{\max}^{-1}$. Since $\varepsilon > 0$ was chosen arbitrarily we conclude

$$\liminf_{n\to\infty} \mu_n([0,b]) \geq b.$$

On the other hand, Lemma 2.24 shows that, for n large enough,

$$\mu_n([0,b]) \leq \frac{1}{|A_n|} \big|A_n \setminus \big(G_n \cap R_n(b,\varepsilon)\big)\big| = 1 - \frac{n}{|A_n|} \cdot \frac{|R_n(b,\varepsilon)|}{n} \tag{2.32}$$

since $A_n = G_n$ and $R_n(b,\varepsilon) \subseteq A_n$ for n big enough. By the definition of $R_n(b,\varepsilon)$ we have

$$|R_n(b,\varepsilon)| \geq \begin{cases} 2\big((1-b)v_{\max} - \varepsilon\big)n & \text{for } \alpha = 0, \\ \big((1-b)v_{\max} - 2\varepsilon\big)n & \text{for } \alpha > 0. \end{cases}$$

Analogous to the above estimation this yields that the limit superior of the right hand side of (2.32) is bounded from above by $b + 2\varepsilon v_{\max}^{-1}$. As before we get, since $\varepsilon > 0$ is arbitrary,

$$\limsup_{n\to\infty} \mu_n([0,b]) \leq b,$$

which finishes the proof.

3 Branching Random Walks

3.1 Introduction

In this chapter we study branching random walks in discrete time on the one-dimensional lattice $\mathbb{Z}$. In general, such a branching random walk is a system of particles living on $\mathbb{Z}$ that evolves as follows: Initially, there is one particle at 0. The particle produces offspring that are placed on $\mathbb{Z}$ according to some point process. The offspring of this initial particle form the first generation of the branching random walk. Each particle in the n-th generation itself has children whose positions are distributed according the same point process (from the point of view of the parent particle). All the children of the particles in the n-th generation form the $(n+1)$-th generation. Usually, one also refers to the particles of the n-th generation as the particles present at time n. We assume that all particles reproduce independently of each other and independently of the past of the process. The point process governing the reproduction may in general depend on time and the location of the parent particle. An introduction to branching random walks and overview of classical results can e.g. be found in [48], [55], and [53].

There is an abundance of variants of branching random walks studied in the literature. We will take a look at one of them, in which the production of offspring consists of two parts. First, every particle moves according to a nearest neighbour random walk. The transition probabilities may depend on the location of the particle. Then it reproduces according to a fixed distribution. More precisely, let the sequence $(\omega_x)_{x\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$ describe some (for the moment arbitrary) environment on $\mathbb{Z}$ that is responsible for the motion of the particles. Further, let $(p_k)_{k\in\mathbb{N}_0}$ with $p_k \geq 0$, $\sum_{k=0}^{\infty} p_k = 1$ denote the reproduction distribution. At time 0 the process starts with one particle at 0. At time n all particles move one step according to the environment $(\omega_x)_{x\in\mathbb{Z}}$ and then reproduce according to $(p_k)_{k\in\mathbb{N}_0}$, i.e. a particle at a vertex $x \in \mathbb{Z}$ jumps to $x+1$ with probability ω_x and to $x-1$ with probability $1-\omega_x$, and is then replaced by k particles with probability p_k. Let $m = \sum_{k=0}^{\infty} k p_k$ denote the expected number of offspring of one particle. Throughout, we assume $m > 1$ and $\sum_{k=0}^{\infty} k^2 p_k < \infty$. In particular, this implies that the branching process survives with positive probability. Let

$$M_n = \max\{x \in \mathbb{Z}\colon x \text{ occupied at time } n\}$$

denote the rightmost position occupied by a particle at time n. We are interested in the limit $\lim_{n\to\infty} \frac{M_n}{n}$.

We consider this particle process in a random environment as well as in a homogeneous environment on the non-negative integers $\mathbb{N}$ with reflection at 0. In the first case we assume that $(\omega_x)_{x\in\mathbb{Z}}$ are i.i.d. random variables, in the latter that $\omega_0 = 1$ and $\omega_x = \alpha$ for all $x > 0$

and some $\alpha \in (0,1)$. In both cases the limit exists almost surely and can be determined via the mean m of the offspring distribution and the large deviation rate function I of a single particle moving according to the respective environment. Indeed, in both cases, one can define b as the largest solution of $\log m = I(b)$ if $\log m < I(1)$, or let $b = 1$ otherwise. Then, conditioned on the survival of the branching process, we almost surely have that

$$\lim_{n\to\infty} \frac{M_n}{n} = b. \tag{3.1}$$

This result (and often much more) is known for many variations of branching random walks. Let us just mention, that the classical version, in which the particles move in a homogeneous environment on $\mathbb{Z}$, is studied e.g. in [25], [32] and [6].
Our study of the problem was motivated by the question whether the speed of the maximum can in general be characterised by the rate function I and the reproduction mean m, or whether there is a natural counterexample. There is a simple and robust standard technique for showing that b is indeed an upper bound for the limit in (3.1). The proof of a lower bound is more involved and requires information on the environment. For the branching random walk in random environment we use the method of constructing an embedded Galton-Watson process. This method uses translation invariance of the environment. The proof in the case of the branching random walk on $\mathbb{N}$ therefore uses another method.

3.2 In random environment

The aim of this section is to discuss a branching random walk in an i.i.d. random environment. Before we delve into the discussion of the movement of a whole cloud of particles, let us deal with the motion of a single particle. In the following section, we therefore very briefly summarise some classical results about random walk in random environment.

3.2.1 Random walk in random environment

Let $\omega = (\omega_x)_{x\in\mathbb{Z}}$ be a collection of i.i.d. random variables with values in $(0,1)$. We denote the distribution of $\omega = (\omega_x)_{x\in\mathbb{Z}}$ by η. The process ω plays the role of the random environment. Given an environment ω let $(S_n)_{n\in\mathbb{N}_0}$ be a Markov chain on $\mathbb{Z}$ with $S_0 = 0$ and transition probabilities

$$\begin{aligned} P_\omega(S_{n+1} = x+1 | S_n = x) &= \omega_x, \\ P_\omega(S_{n+1} = x-1 | S_n = x) &= 1-\omega_x \end{aligned}$$

for all $x \in \mathbb{Z}$ as shown in Figure 3.1. P_ω is called the quenched measure. Further, let

$$\mathbb{P}(\cdot) = \int P_\omega(\cdot)\eta(d\omega)$$

define the annealed measure. The corresponding expectations are denoted by E_ω and $\mathbb{E}$, respectively.

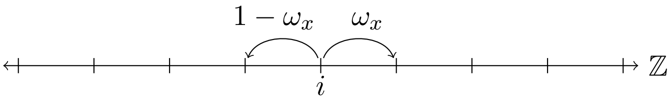

Figure 3.1: Transition probabilities in environment ω; Being at a vertex $x \in \mathbb{Z}$, the particle jumps to the right with probability ω_x and to the left with probability $1-\omega_x$.

A key role is played by the ratio $\rho_x = \frac{1-\omega_x}{\omega_x}$ of the probability of a jump to the left and the probability of a jump to the right. Indeed, transience and recurrence of random walk in the random environment depend on $\mathbb{E}[\log \rho_0]$. This is shown in [50] by Solomon. Also see [54, Theorem 2.1.2].

Theorem 3.1 ([50, Theorem 1.7]). *Assume* $\mathbb{E}[\log \rho_0] \in [-\infty, \infty]$ *is well-defined. Then it holds* $\mathbb{P}$*-a.s. that*

$$\begin{aligned}
&\lim_{n\to\infty} S_n = +\infty && \textit{if } \mathbb{E}[\log \rho_0] < 0,\\
&\lim_{n\to\infty} S_n = -\infty && \textit{if } \mathbb{E}[\log \rho_0] > 0,\\
&-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = +\infty && \textit{if } \mathbb{E}[\log \rho_0] = 0.
\end{aligned}$$

In particular, the model is $\mathbb{P}$*-almost surely recurrent if* $\mathbb{E}[\log \rho_0] = 0$*, and transient otherwise.*

Further, the following law of large numbers is given in [50]. Also see [54, Theorem 2.1.9].

Theorem 3.2 ([50, Theorem 1.16]). *There exists* $v \in [-1,1]$ *such that* $\lim_{n\to\infty} \frac{S_n}{n} = v$ $\mathbb{P}$*-almost surely, where*

$$v = \begin{cases} \frac{1-\mathbb{E}[\rho_0]}{1+\mathbb{E}[\rho_0]} & \textit{if } \mathbb{E}[\rho_0] < 1,\\ -\frac{1-\mathbb{E}[\rho_0^{-1}]}{1+\mathbb{E}[\rho_0^{-1}]} & \textit{if } \mathbb{E}[\rho_0^{-1}] < 1,\\ 0 & \textit{otherwise.}\end{cases}$$

Jensen's inequality implies $\mathbb{E}[\log \rho_0] \leq \log \mathbb{E}[\rho_0]$. Thus, if the random walk is, say, transient to the left, then $v \leq 0$. One can find examples of transient environments with $v = 0$.
We also make use of the following quenched large deviation principle shown by Greven and Hollander in [24, Theorem 1] and Comets et al. in [9, Theorem 1]. See also [54, Theorem 2.3.12].

Theorem 3.3 ([24, Theorem 1], [9, Theorem 1]). *Assume that there is* $\delta > 0$ *such that* $\omega_0 \in (\delta, 1-\delta)$ η*-almost surely. There exists a deterministic convex rate function* I_η^q *such that* η*-almost surely for any measurable set* A

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega\left(\frac{S_n}{n} \in A\right) \geq -\inf_{x\in A^\circ} I_\eta^q(x), \tag{3.2}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega\left(\frac{S_n}{n} \in A\right) \leq -\inf_{x\in \overline{A}} I_\eta^q(x). \tag{3.3}$$

Here, A° *denotes the interior of the set* A*, and* $\overline{A}$ *the closure.*

The rate function I_η^q is increasing on $[v,1]$, and, as it is convex, it is strictly increasing on $[v,1]\cap\{I_\eta^q\neq 0\}$. Further, $I_\eta^q(v)=0$ and $I_\eta^q(1)>0$. More details on the shape of the rate function can be found in [9, Section 5] and [24, Section 0.3].

For more detailed information and an excellent introduction to random walk in random environment see e.g. Zeitouni's lecture notes [54].

3.2.2 Speed of the maximum

Now we are ready to study a branching random walk in the random environment introduced in the previous section. Let us first present a more precise construction of the branching random walk which is taken from [15]. Recall that $\omega=(\omega_x)_{x\in\mathbb{Z}}$ denotes the random environment. Further, $(p_k)_{k\in\mathbb{N}_0}$ gives the reproduction distribution, and $m=\sum_{k=0}^{\infty}kp_k$ denotes the branching mean. Throughout this section we make the following assumptions:

$$\omega_0\in(\delta,1-\delta)\ \eta\text{-almost surely for some } \delta>0 \text{ and} \tag{3.4}$$

$$m>1 \text{ and } \sum_{k=0}^{\infty}k^2p_k<\infty. \tag{3.5}$$

The assumption on the environment ω ensures the applicability of all theorems introduced in Section 3.2.1. In particular, the large deviation principle stated in Theorem 3.3 holds. The assumption on the reproduction distribution $(p_k)_{k\in\mathbb{N}_0}$ implies that the branching random walk survives with positive probability and that the number of particles grows exponentially. The process survives if at any time there exists at least one particle. We will always condition on the event that the branching random walk survives.

Let $\{Z(x,n,i)\colon x\in\mathbb{Z},n\in\mathbb{N}_0,i\in\mathbb{N}\}$ be a collection of i.i.d. random variables that are also independent of ω such that $\mathbb{P}(Z(0,0,1)=k)=p_k$ for all $k\in\mathbb{N}_0$. The random variable $Z(x,n,i)$ describes the number of descendants of the i-th particle that is at $x\in\mathbb{Z}$ at time $n\in\mathbb{N}_0$. Further, let $\{X(x,n,i)\colon x\in\mathbb{Z},n\in\mathbb{N}_0,i\in\mathbb{N}\}$ be another collection of i.i.d. random variables, which are uniformly distributed on $[0,1]$. Assume that they are independent of the random variables $\{Z(x,n,i)\colon x\in\mathbb{Z},n\in\mathbb{N}_0,i\in\mathbb{N}\}$ and $(\omega_x)_{x\in\mathbb{Z}}$. These random variables are responsible for the motion of the particles in a given random environment ω. If $X(x,n,i)\leq\omega_x$, then the i-th particle that is at x at time n goes to the right in its next step. Otherwise, it goes to the left. Given the environment ω, we can now inductively define a process $(\lambda(x,n))_{x\in\mathbb{Z},\,n\in\mathbb{N}_0}$ by setting $\lambda(0,0)=1$, $\lambda(x,0)=0$ for all $x\neq 0$, and for $n\geq 1$ and $x\in\mathbb{Z}$

$$\begin{aligned}\lambda(x,n+1)=&\sum_{i=1}^{\lambda(x-1,n)}\mathbb{1}_{\{X(x-1,n,i)\leq\omega_{x-1}\}}Z(x-1,n,i)\\&+\sum_{i=1}^{\lambda(x+1,n)}\mathbb{1}_{\{X(x+1,n,i)>\omega_{x+1}\}}Z(x+1,n,i).\end{aligned}\tag{3.6}$$

$\lambda(x,n)$ denotes the number of particles at the vertex $x\in\mathbb{Z}$ at time $n\in\mathbb{N}_0$. Then the process $(M_n)_{n\in\mathbb{N}_0}$, that describes the maximal position occupied by a particle, satisfies

$$M_n = \max\{x \in \mathbb{Z} \colon \lambda(x,n) > 0\}.$$

Of course, this definition only makes sense on the event that the branching random walk survives. In [15] A. Devulder shows a phase transition result for the speed of the maximum.

Theorem 3.4 ([15, Theorem 1.1 and Proposition 1.2])**.** *Suppose that the Assumptions* (3.4) *and* (3.5) *hold. Further, assume* $\mathbb{E}[\log \rho_0] \geq 0$ *and define* $m_c = \exp(I_\eta^q(0))$*. Then,* $\mathbb{P}$*-almost surely, conditioned on survival,*

$$\begin{aligned} \limsup_{n\to\infty} \frac{M_n}{n} &< 0 && \text{if } 1 < m < m_c, \\ \limsup_{n\to\infty} \frac{M_n}{n} &\leq 0 && \text{if } m = m_c, \\ \liminf_{n\to\infty} \frac{M_n}{n} &> 0 && \text{if } m > m_c. \end{aligned}$$

The assumption $\mathbb{E}[\log \rho_0] \geq 0$ implies that every particle moving in the random environment is recurrent or transient to the left by Theorem 3.1, and that $v \leq 0$ by Theorem 3.2. On the other hand, the number of particles in the branching random walk grows exponentially in n. A larger reproduction mean m results in a higher probability of particles being far to the right of 0. The theorem shows that for $m > m_c$ this huge mass of particles wins against the drift of the environment and pushes the rightmost front to $+\infty$. For $m < m_c$, on the contrary, the effect of the environment prevails and the rightmost front goes to $-\infty$.

Results by Gantert and Müller in [20] imply the following statement: Consider the setting of Theorem 3.4 with $\mathbb{E}[\log \rho_0] > 0$ and assume that every particle has at least one offspring. Then, in the critical case $m = m_c$, we have $\lim_{n\to\infty} M_n = -\infty$ $\mathbb{P}$-almost surely.

Using the description of the shape of the rate function in [9, Section 5] and [24, Section 0.3] one can calculate m_c more explicitly. Without loss of generality, we continue to assume $\mathbb{E}[\log \rho_0] \geq 0$ and define $\omega_{\max} = \sup\{q \colon q \in \operatorname{supp} \omega_0\}$. If $\omega_{\max} < \frac{1}{2}$, the local drift is negative at any vertex. In this case we have $m_c = \frac{1}{2}\big(\omega_{\max}(1-\omega_{\max})\big)^{-1/2} > 1$. If $\omega_{\max} \geq \frac{1}{2}$, then the local drift can be positive as well as negative. Particles might be "trapped" in areas with positive drift. In this case $m_c = 1$ and the maximum of our supercritical branching random walk has positive speed.

We give an explicit formula for the speed of the right front and thus improve Theorem 3.4. Let $b \in (v,1)$ be the unique solution of $\log m = I_\eta^q(b)$ if $\log m < I_\eta^q(1)$, and let $b = 1$ otherwise. Note that the function I_η^q is continuous.

Theorem 3.5. *Assume that Assumptions* (3.4) *and* (3.5) *are given. Conditioned on the survival of the process,* $\mathbb{P}$*-almost surely it holds that*

$$\lim_{n\to\infty} \frac{M_n}{n} = b.$$

In [37] Liu et al. consider a more general version of this branching random walk, allowing the offspring distribution to vary randomly in time. Their results show that b is indeed an upper limit for the speed. They also prove an analogous statement to Theorem 3.5 for the case of a random offspring distribution and a deterministic homogeneous environment. In [39, Theorem 2.20] Liu claims to have a proof of Theorem 3.5, but this proof, to the best of our knowledge, is not published.

3.3 On the non-negative integers

This section is devoted to branching random walks on the non-negative integers $\mathbb{N}_0$ with reflection at 0. As before, we start with a short discussion of the motion of a single particle on $\mathbb{N}_0$. In particular, we need to derive a large deviation principle for random walk on $\mathbb{N}$.

3.3.1 Large deviations

Let $(S^r_n)_{n\in\mathbb{N}_0}$ denote the simple random walk with drift on $\mathbb{N}_0$ with $S^r_0 = 0$ and transition probabilities given by

$$\mathbb{P}(S^r_{n+1} = y \mid S^r_n = x) = \begin{cases} 1 & \text{if } x = 0, y = 1, \\ \alpha & \text{if } x > 0, y = x + 1, \\ 1 - \alpha & \text{if } x > 0, y = x - 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.7}$$

where $\alpha \in (0, 1)$ is a drift parameter. The transition probabilities are depicted in Figure 3.2. If $\alpha \le \frac{1}{2}$, the random walk is recurrent, otherwise it is transient.

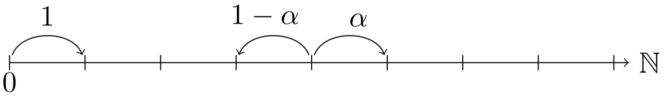

Figure 3.2: Transition probabilities for reflected random walk on $\mathbb{N}_0$

Further, let $(S_n)_{n\in\mathbb{N}_0}$ denote the corresponding homogeneous random walk on $\mathbb{Z}$ with start in 0 and transition probabilities given by

$$\begin{aligned} \mathbb{P}(S_{n+1} = x + 1 | S_n = x) &= \alpha, \\ \mathbb{P}(S_{n+1} = x - 1 | S_n = x) &= 1 - \alpha \end{aligned} \tag{3.8}$$

for all $x \in \mathbb{Z}$. It is well known that the random walk $(S_n)_{n\in\mathbb{N}_0}$ satisfies a large deviation principle with rate function I_α given by

$$I_\alpha(x) = \frac{1}{2}\bigg((1 - x)\log\Big(\frac{1 - x}{2(1 - \alpha)}\Big) + (1 + x)\log\Big(\frac{1 + x}{2\alpha}\Big)\bigg) \tag{3.9}$$

for $x \in [-1, 1]$. We get a corresponding result for the reflected random walk on the non-negative integers.

Theorem 3.6. *For* $\alpha \in (0, \frac{1}{2}]$ *and* $x \in [0,1]$ *define*

$$I^r_\alpha(x) = \begin{cases} x \log \frac{1-\alpha}{\alpha} & \text{if } 0 \le x \le 1-2\alpha, \\ I_\alpha(x) & \text{if } 1-2\alpha < x \le 1. \end{cases} \tag{3.10}$$

For $\alpha \in (\frac{1}{2}, 1)$ *let* $I^r_\alpha(x) = I_\alpha(x)$ *for all* $x \in [0,1]$*. Then the process* $(S^r_n)_{n\in\mathbb{N}_0}$ *satisfies a large deviation principle with rate function* I^r_α*. This means that for any measurable set* $A \subseteq [0,1]$ *we have*

$$-\inf_{x\in A^\circ} I^r_\alpha(x) \le \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{S^r_n}{n} \in A\right) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{S^r_n}{n} \in A\right) \le -\inf_{x\in\overline{A}} I^r_\alpha(x).$$

As before, A° *denotes the interior of the set* A*, and* $\overline{A}$ *the closure.*

The rate functions for the random walk on the integers and for the reflected random walk on the non-negative integers are shown in Figure 3.3.

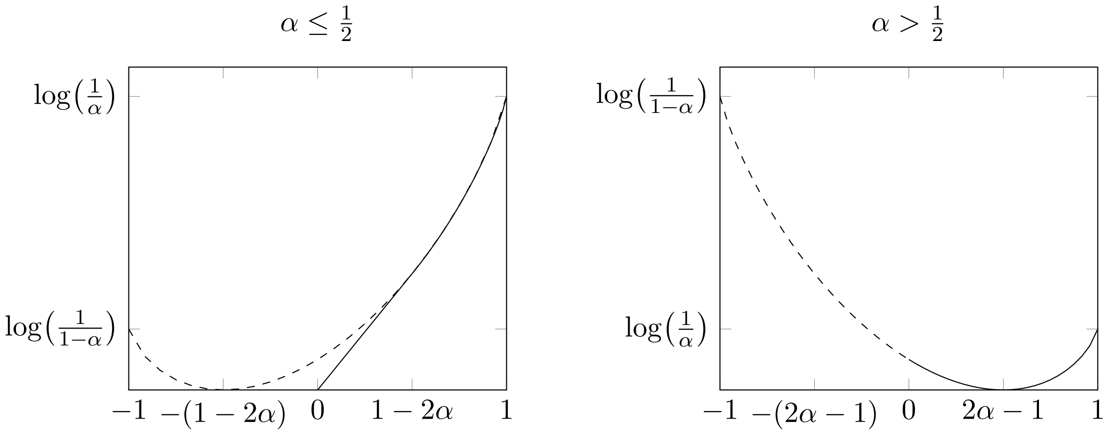

Figure 3.3: The figure shows the rate functions for $\alpha \le \frac{1}{2}$ on the left and for $\alpha > \frac{1}{2}$ on the right. The rate function I_α for the random walk on $\mathbb{Z}$ is marked by - - -. The rate function I^r_α for the reflected random walk on $\mathbb{N}_0$ by ——.

If the random walk has a drift away from 0, then the rate function of the random walk on $\mathbb{N}_0$ agrees with the one of the random walk on $\mathbb{Z}$. Otherwise, they differ. Let us briefly give an intuition why this happens. Let $x > 0$. The best strategy for a random walk on the integers $\mathbb{Z}$ to end up to the right of nx after n steps is to go right with constant speed all the way. The same strategy works for the random walk on the non-negative integers $\mathbb{N}_0$ as long as x is big. For small x a better strategy consists in spending some time close to 0 at no cost, and then go to the right with constant speed.

3.3.2 Speed of the maximum

Consider a branching random walk on $\mathbb{N}_0$ with reflection at 0. Assume that the transition probabilities of the underlying random walk are given by (3.7) for $\alpha \in (0,1)$. The branching random walk can be constructed as the branching random walk in random environment in Section 3.2.2, using the deterministic environment ω given by $\omega_0 = 1$ and $\omega_x = \alpha$ for $x \neq 0$. We denote the position of the maximum at time n by M_n^r.
A single particle moving in this environment has speed $v = 2\alpha - 1$ if $\alpha > \frac{1}{2}$, and $v = 0$ if $\alpha \leq \frac{1}{2}$. Further, recall that m denotes the reproduction mean. Analogous to Section 3.2.2 let $b^r \in (v, 1)$ be the unique solution of the equation $\log m = I_\alpha^r(b^r)$ if $\log m < I_\alpha^r(1)$, and let $b^r = 1$ otherwise. We have the following law of large numbers for the process $(M_n^r)_{n \in \mathbb{N}_0}$.

Theorem 3.7. *Let $\alpha \in (0,1)$ and assume that the branching random walk on $\mathbb{N}_0$ satisfies the assumption (3.5). Conditioned on survival of the process, almost surely*

$$\lim_{n \to \infty} \frac{M_n^r}{n} = b^r.$$

As we assume that the branching process is supercritical, the reproduction mean m is strictly bigger than 1. Therefore, the maximum always has a strictly positive speed, no matter how big the drift towards the origin is. The speed can be strictly bigger than the speed of the maximum of the branching random walk on $\mathbb{Z}$ with the same drift.

4 Proofs for Branching Random Walks

The aim of this chapter is to prove that the maximum of the branching random walk in random environment as well as on the non-negative integers has linear speed as claimed in Theorems 3.5 and 3.7. The proofs are presented in Sections 4.2 and 4.3, respectively. In Section 4.1, we collect a few facts about Galton-Watson processes and introduce some notation.

4.1 Preliminaries

4.1.1 Branching processes in a nutshell

If we consider a branching random walk and only pay attention to evolution of the number of particles, then we end up with a Galton-Watson branching process. Several times in this chapter we will make use of well known basic results about this process. For an introduction to branching processes we refer the reader e.g. to [26]. In the proof of the lower bound of Theorem 3.5 we further use a result about branching processes in random environment. Here, the offspring distribution varies randomly in time. As this process is less well known, we briefly introduce it here, following the presentation in [4] and [51].

Let $\mathcal{M}$ denote the set of offspring distributions $(p_k)_{k\in\mathbb{N}_0}$ satisfying $\sum_{k=1}^{\infty} kp_k < \infty$. Further, let $(\ell^\infty, \mathcal{B}_\infty)$ denote the probability space of all bounded sequences of real numbers equipped with the Borel-σ-algebra generated by the product topology. $\mathcal{M}$ is a Borel subset of ℓ^∞.

Let $(\zeta_n)_{n\in\mathbb{N}_0}$ be a stochastic process taking values in $\mathcal{M}$. This process plays the role of the random environment. Given an environment process $(\zeta_n)_{n\in\mathbb{N}_0}$, we can construct a non-homogeneous branching process $(Z_n)_{n\in\mathbb{N}_0}$. The number Z_n of particles in generation $n \in \mathbb{N}_0$ is recursively defined by $Z_0 = 1$, and for $n \in \mathbb{N}_0$ by

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i},$$

where $(X_{n,i})_i$ are independent and identically distributed according to the distribution ζ_n, i.e. $\mathbb{P}(X_{n,i} = k) = p_k(\zeta_n)$. The random variable $X_{n,i}$ describes the number of descendants of the i-th particle in generation n.

This model was first studied by Smith and Wilkinson in [49] for the case where the environment process $(\zeta_n)_{n\in\mathbb{N}_0}$ consists of independent and identically distributed random variables. Athreya and Karlin in [4], as well as Tanny in [51] extended the model to a stationary and ergodic environment process.

If the reproduction distribution $(p_k)_{k\in\mathbb{N}_0}$ is deterministic and does not depend on time, then it is well known that the process $(Z_n)_{n\in\mathbb{N}_0}$ survives with positive probability if the reproduction mean $m = \sum_{k\in\mathbb{N}_0} kp_k$ is strictly bigger than 1. As with branching random walks, we say that the process survives if $Z_n \neq 0$ for all $n \in \mathbb{N}_0$. If $m \leq 1$, then the branching process dies out almost surely. A branching process with $m > 1$ is called supercritical, one with $m = 1$ critical, and one with $m \leq 1$ subcritical. We also use this terminology for branching random walks.

In [4] and [51], conditions for a positive survival probability of the branching process in random environment are derived. Given an environment ζ_0, let $m(\zeta_0)$ denote the expected number of descendants of a particle. It is defined by $m(\zeta_0) = \sum_{k=1}^{\infty} kp_k(\zeta_0)$. Further, let $q(\zeta)$ denote the extinction probability given the environment process $\zeta = (\zeta_n)_{n\in\mathbb{N}_0}$, i.e. $q(\zeta) = \mathbb{P}(Z_n = 0 \text{ for some } n|\zeta)$.

Theorem 4.1 ([4, Theorem 3], [51, Corollary 6.3])**.** *Let $(Z_n)_{n\in\mathbb{N}_0}$ be a branching process in random environment with stationary and ergodic environment process $\zeta = (\zeta_n)_{n\in\mathbb{N}_0}$. If $\mathbb{E}[-\log(1-p_0(\zeta_0))] < \infty$ and $\mathbb{E}[\log m(\zeta_0)] > 0$, then $\mathbb{P}(q(\zeta) < 1) = 1$.*

As mentioned in the Chapter 3, the number of particles in a supercritical branching process with finite reproduction variance grows exponentially. We only need this result for branching processes with deterministic offspring distribution.

Theorem 4.2. *[26, Chapter 1, Theorem 8.1] Let $(Z_n)_{n\in\mathbb{N}_0}$ be a branching process with reproduction distribution $(p_k)_{k\in\mathbb{N}_0}$. Assume $m = \sum_{k\in\mathbb{N}_0} kp_k > 1$ and $\sum_{k\in\mathbb{N}_0} k^2 p_k < \infty$. There is a non-negative random variable W such that $\lim_{n\to\infty} \frac{Z_n}{m^n} = W$ almost surely and in L^2. Furthermore, $\mathbb{E}[W] = 1$ and $W > 0$ on the event of survival.*

4.1.2 Notation and one basic lemma

In addition to the notation introduced in Section 3.2.2 we use the following notation when dealing with branching random walks. For a set $A \subseteq \mathbb{Z}$ and $n \in \mathbb{N}_0$, the number of particles in generation n located in the set A is given by $\lambda(A,n) = \sum_{x\in A} \lambda(x,n)$. According to this notation, $\lambda(\mathbb{Z},n)$ counts the total number of particles in generation $n \in \mathbb{N}_0$. To abbreviate notation, we define $Z_n = \lambda(\mathbb{Z},n)$ in accordance with the notation used for branching processes. As we have just discussed, $(Z_n)_{n\in\mathbb{N}_0}$ is a Galton-Watson branching process with reproduction law $(p_k)_{k\in\mathbb{N}_0}$ and $Z_0 = 1$. The probability that the process is extinct by time n is given by $q_n = \mathbb{P}(Z_n = 0)$. Recall that we always assume $m = \sum_{k\in\mathbb{N}_0} kp_k > 1$ and $\sum_{k\in\mathbb{N}_0} kp_k < \infty$. Thus, the extinction probability $q = \lim_{n\to\infty} q_n$ is strictly smaller than 1. Further, for an environment $\omega = (\omega_x)_{x\in\mathbb{Z}}$ and time $N \in \mathbb{N}$ we define the σ-algebra

$$\mathcal{F}_\omega(N) = \sigma(\lambda(x,n), \omega \colon x \in \mathbb{Z}, 0 \leq n \leq N).$$

The σ-algebra $\mathcal{F}_\omega(N)$ contains all information about the distribution of particles up to time N.

We often reduce problems we encounter with branching random walks to problems concerning the movement of one single particle. We then use $(S_n)_{n\in\mathbb{N}_0}$ to denote a random walk with start in 0 moving in the same environment as the branching random walk..
Several times we will use the following standard lemma. The proof is very short, so we present it here for the sake of completeness.

Lemma 4.3 ([15, Lemma 2.2], [37, Lemma 4]). *For $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}$ we have for any environment ω*

$$E_\omega[\lambda(x, n)] = m^n P_\omega(S_n = x).$$

Proof. At time $n \in \mathbb{N}_0$ there are Z_n particles. We enumerate them and denote the position of the i-th particle by S_n^i. As the distribution of the position of any particle equals the distribution of the position of a normal random walk in the respect environment, we have

$$E_\omega\big[\lambda(x, N+1)\big] = E_\omega\left[E_\omega\left[\sum_{i=1}^{Z_n} \mathbb{1}_{\{S_n^i = x\}}\middle| Z_n\right]\right] = E_\omega\big[Z_n P_\omega(S_n = x)\big] = m^n P_\omega(S_n = x). \quad \square$$

4.2 In random environment

4.2.1 Upper bound on the speed

As mentioned in Section 3.2.2, the upper bound on the speed is shown in [37]. It also follows from the proof of Theorem 3.4 presented in [15, Section 3]. We rewrite it here for the sake of completeness.

Proposition 4.4 ([37, Theorem 1]). *$\mathbb{P}$-almost surely, conditioned on survival,*

$$\limsup_{n\to\infty} \frac{M_n}{n} \le b.$$

The proof relies on the Markov inequality and the large deviation principle given by Theorem 3.3.

Proof. Without loss of generality assume $b < 1$ and let $b < \beta < 1$. By the definition of b, and since I_η^q is strictly increasing on $[v, 1] \cap \{I_\eta^q \neq 0\}$, there is $\varepsilon > 0$ such that

$$\log m < I_\eta^q(\beta) - \varepsilon. \tag{4.1}$$

By Theorem 3.3 for almost all ω and n large

$$P_\omega(S_n \ge \beta n) \le \mathrm{e}^{-(I_\eta^q(\beta)-\varepsilon)n}.$$

Hence, for almost all ω, by the Markov inequality and Lemma 4.3

$$\begin{aligned} P_\omega\big(\lambda\big([\beta n,\infty),n\big)\geq 1\big) &\leq E_\omega\left[\sum_{x\geq\beta n}\lambda(x,n)\right] \\ &= m^n P_\omega(S_n\geq\beta n) \\ &\leq \mathrm{e}^{(\log m - I_\eta^q(\beta)+\varepsilon)n}. \end{aligned}$$

By inequality (4.1) the exponent in the last term is negative. Therefore, the sequence $\big(P_\omega\big(\lambda\big([\beta n,\infty),n\big)\geq 1\big)\big)_{n\in\mathbb{N}_0}$ is summable in n. Thus, by the Borel-Cantelli Lemma, P_ω-almost surely, at large times n there is no particle in $[n\beta,\infty)$. Thus, $\frac{M_n}{n}\leq\beta$ for n large, if $Z_n>0$. Hence, conditioned on the survival of the process, P_ω-almost surely $\limsup_{n\to\infty}\frac{M_n}{n}\leq\beta$. As $\beta>b$ was chosen arbitrarily, this yields the claim. □

4.2.2 Lower bound on the speed

Next we show the lower bound of the speed of the maximum by constructing an embedded branching process. The idea is the following. Fix $\beta<b$ and an integer $k>0$. For every $n\in\mathbb{N}$ consider the branching random walk at time nk. Delete all particles that are at vertices smaller than $nk\beta$ and all their descendants. For suitably chosen k the remaining branching process is supercritical and survives with positive probability. Hence, with positive probability, the speed of the maximum is bigger than β. It then remains to invoke a zero-one-law. The random environment complicates the proof a little. To ensure that at time nk, $n\in\mathbb{N}$, the surviving particles see the same environment, we relocate them all to $nk\beta$. The proof follows along the lines of the proof of Theorem 3.4 given in [15]. We construct the embedded branching process in a slightly different way to get the stronger result. This also simplifies some estimates needed in the proof.

Proposition 4.5. *$\mathbb{P}$-almost surely, conditioned on survival, it holds that*

$$\liminf_{n\to\infty}\frac{M_n}{n}\geq b.$$

Proof. The proof consists of three parts. In the first part we choose a suitable integer k. In the second part we construct the embedded branching process and show that it is supercritical. The last part shows that the event $\{\liminf_{n\to\infty}\frac{M_n}{n}\geq b\}$ indeed occurs almost surely.

Fix $\beta\in(v,b)\cap\mathbb{Q}$. By the definition of b there is $\varepsilon>0$ such that

$$\log m > I_\eta^q(\beta)+\varepsilon \tag{4.2}$$

as I_η^q is strictly increasing on $[v,1]\cap\{I_\eta^q\neq 0\}$. By Theorem 3.3, for η-almost every environment ω, there is $N(\omega)<\infty$ such that for all $n\geq N(\omega)$

$$P_\omega(S_n\geq n\beta)\geq \mathrm{e}^{-(I_\eta^q(\beta)+\frac{\varepsilon}{2})n}. \tag{4.3}$$

As $\lim_{N\to\infty}\eta(N(\omega)\le N)=1$, we can choose $N<\infty$ such that

$$\eta(N(\omega)>N)<\frac{\varepsilon}{2|\log\delta|}. \tag{4.4}$$

Recall that δ was introduced in the assumption (3.4), which demands $\eta(\omega_0\in(\delta,1-\delta))=1$. By Lemma 4.3 we have for $k\ge N$

$$\begin{aligned}\int\log E_\omega\Bigg[\sum_{x\ge k\beta}\lambda(x,k)\Bigg]\,\eta(\mathrm{d}\omega)&=\int\log\big(m^kP_\omega(S_k\ge k\beta)\big)\,\eta(\mathrm{d}\omega)\\&=k\log m+\int\log P_\omega(S_k\ge k\beta)\,\eta(\mathrm{d}\omega).\end{aligned}\tag{4.5}$$

Using (4.3) and the assumption $\omega_x\in(\delta,1-\delta)$ η-almost surely for $x\in\mathbb{Z}$, we get for $k\ge N$

$$\begin{aligned}\log P_\omega(S_k\ge k\beta)&\ge-\Big(I_\eta^q(\beta)+\frac{\varepsilon}{2}\Big)k\mathbb{1}_{\{N\ge N(\omega)\}}+\log P_\omega(S_k\ge k)\mathbb{1}_{\{N<N(\omega)\}}\\&\ge-\Big(I_\eta^q(\beta)+\frac{\varepsilon}{2}\Big)k+k\log\delta\,\mathbb{1}_{\{N<N(\omega)\}}.\end{aligned}$$

Inserting this estimate into (4.5) and using (4.4) and (4.2) yields for $k\ge N$

$$\begin{aligned}\int\log E_\omega\Bigg[\sum_{x\ge k\beta}\lambda(x,k)\Bigg]\,\eta(\mathrm{d}\omega)&\ge k\Big(\log m-I_\eta^q(\beta)-\frac{\varepsilon}{2}+\log\delta\,\eta(N<N(\omega))\Big)\\&\ge k(\log m-I_\eta^q(\beta)-\varepsilon)>0.\end{aligned}\tag{4.6}$$

From now on we fix an even integer $k\ge N$ such that $k\beta$ is also an even integer. We construct a sequence $Y=(Y_n)_{n\in\mathbb{N}_0}$ of random variables as follows. Set $Y_0=1$. This corresponds to the fact that the branching random walk starts with one particle at 0. Let the branching random walk evolve and let Y_1 denote the number of particles located in $[k\beta,\infty)$ at time k. Relocate all those Y_1 particles to the vertex $k\beta$, delete all other particles, and let the system run for another k steps of time. Now, let Y_2 be the number of particles located in $[2k\beta,\infty)$ at time $2k$. Repeat this procedure. Suppose the process is constructed up to time nk. There are Y_n particles in $[nk\beta,\infty)$. Relocate them to $nk\beta$, delete the others, and let the system evolve. Define Y_{n+1} to be the number of particles in $[(n+1)k\beta,\infty)$ at time $(n+1)k$. Recall that the branching random walk is entirely determined by the families $\{Z(x,n,i)\colon x\in\mathbb{Z},n\in\mathbb{N}_0,i\in\mathbb{N}\}$ and $\{X(x,n,i)\colon x\in\mathbb{Z},n\in\mathbb{N}_0,i\in\mathbb{N}\}$ of random variables as described at the beginning of Section 3.2.2. The first set of random variables determines the number of descendants, the second the movement of the particles. Every particle in the modified process corresponds to one particle in the original branching random walk. To determine the movement and number of descendants of the particle use the corresponding Z and X of the particle in the original process. This coupling ensures that a particle in the modified process never overtakes its counterpart in the original process, thus ending up on a bigger vertex. Otherwise, the particle and its counterpart would meet somewhere and afterwards be forced to move together due to the coupling. As we chose k such that k and $k\beta$

are even, the corresponding particles are on vertices with an even distance and thus cannot bypass each other without meeting. This argument shows that for all $n \in \mathbb{N}_0$

$$Y_n \leq \lambda([nk\beta, \infty), nk). \tag{4.7}$$

The sequence $(Y_n)_{n\in\mathbb{N}_0}$ forms a branching process in random environment. Indeed, for all $n \in \mathbb{N}_0$ we can write

$$Y_{n+1} = \sum_{i=1}^{Y_n} X_{n,i},$$

where $X_{n,i}$ counts how many of the descendants of the i-th particle placed at $nk\beta$ at time nk are in $[(n+1)k\beta, \infty)$ at time $(n+1)k$. Given an environment ω, for every $n \in \mathbb{N}_0$ the random variables $(X_{n,i})_i$ are independent and identically distributed with respect to P_ω. Thus, the process $(Y_n)_{n\in\mathbb{N}_0}$ fits into the setting of the branching process in random environment introduced in Section 4.1.1. We want to apply Theorem 4.1 and next check whether all conditions are satisfied. The sequence $(\mathcal{L}(X_{n,1}))_{n\in\mathbb{N}_0}$ of offspring distributions is stationary and ergodic. Furthermore, the choice of k and (4.6) imply

$$\int \log E_\omega[X_{0,1}]\,\eta(\mathrm{d}\omega) = \int \log E_\omega\left[\sum_{x\geq k\beta} \lambda(x,k)\right]\eta(\mathrm{d}\omega) > 0.$$

Recall that q_k denotes the probability that the branching process is extinct by time k. Using $\beta < 1$, for η-almost every environment ω we have

$$\begin{aligned}
P_\omega\big(\lambda\big([k\beta,\infty),k\big) = 0\big) &\leq P_\omega\big(\lambda\big([k,\infty),k\big) = 0\big) \\
&\leq P_\omega(Z_k = 0) + P_\omega\big(\lambda\big([k,\infty),k\big) = 0 \mid Z_k \neq 0\big)P_\omega(Z_k \neq 0) \\
&\leq q_k + (1-q_k)P_\omega(S_k \notin [k,\infty)) \\
&\leq q_k + (1-q_k)(1-\delta^k) = 1 - \delta^k(1-q_k).
\end{aligned}$$

As $q_k < 1$, the last estimate implies

$$\begin{aligned}
\int -\log(1 - P_\omega(X_{0,1} = 0))\,\eta(\mathrm{d}\omega) &= \int -\log\Big(1 - P_\omega\big(\lambda\big([k\beta,\infty),k\big) = 0\big)\Big)\,\eta(\mathrm{d}\omega) \\
&\leq \int -\log\big(\delta^k(1-q_k)\big)\,\eta(\mathrm{d}\omega) \\
&= -\log\big(\delta^k(1-q_k)\big) < \infty.
\end{aligned}$$

Hence, we can apply Theorem 4.1 and get for η-almost all ω

$$P_\omega(Y \text{ survives}) > 0.$$

By (4.7), survival of the process $(Y_n)_{n\in\mathbb{N}_0}$ implies $\frac{M_{nk}}{nk} \geq \beta$ for all $n \in \mathbb{N}$, and thus

$\liminf_{n\to\infty} \frac{M_{nk}}{nk} \geq \beta$. This leads to

$$P_\omega\Big(\liminf_{n\to\infty} \frac{M_n}{n} \geq \beta\Big) > 0 \tag{4.8}$$

for η-almost all ω. It remains to show that the event $\{\liminf_{n\to\infty} \frac{M_n}{n} \geq \beta\}$ has indeed probability 1, conditioned on the survival of the process.
To abbreviate notation let $A = \{\liminf_{n\to\infty} \frac{M_n}{n} \geq \beta\}$. Further, let θ denote the shift-operator given by $(\theta\omega)_x = \omega_{x+1}$ for all $x \in \mathbb{Z}$. The sequence $\big(P_{\theta^{2x}\omega}(A)\big)_{x\in\mathbb{Z}}$ is stationary. A coupling argument similar to the one used previously in this proof further shows that the sequence is non-decreasing. Thus, by stationarity, it has to be constant, i.e. for η-almost every ω we have

$$P_\omega(A) = P_{\theta^{2x}\omega}(A) \tag{4.9}$$

for all $x \in \mathbb{Z}$. Choose $1 < r < m$ and $N \in \mathbb{N}$. Consider the genealogical tree of the branching random walk. In generation $2N$, there are $Z_{2N} = \sum_{x\in\mathbb{Z}} \lambda(2x, 2N)$ particles. The descendants of each of them form Z_{2N} subtrees that are independent given an environment ω. If the maximum of the entire tree has speed smaller than β, then the speed of the maximum of each subtree has to be smaller than β as well. The law of a subtree started by a particle at a vertex $2x$ equals the one of the entire tree shifted by $2x$, i.e. $P_{\theta^{2x}\omega}$. Using (4.9), we therefore have for η-almost every environment ω

$$\begin{aligned} P_\omega\Big(\liminf_{n\to\infty} \frac{M_n}{n} < \beta, Z_{2N} \geq r^{2N} \Big| \mathcal{F}_\omega(2N)\Big) &\leq \prod_{x\in\mathbb{Z}} P_{\theta^{2x}\omega}\Big(\liminf_{n\to\infty} \frac{M_n}{n} < \beta\Big)^{\lambda(2x,2N)} \mathbb{1}_{\{Z_{2N}\geq r^{2N}\}} \\ &= \prod_{x\in\mathbb{Z}} \big(1 - P_{\theta^{2x}\omega}(A)\big)^{\lambda(2x,2N)} \mathbb{1}_{\{Z_{2N}\geq r^{2N}\}} \\ &\leq \big(1 - P_\omega(A)\big)^{r^{2N}}. \end{aligned}$$

We know that $P_\omega(A) > 0$ for η-almost every ω by (4.8). Therefore, the last estimate yields

$$\lim_{N\to\infty} P_\omega\Big(\liminf_{n\to\infty} \frac{M_n}{n} < \beta, Z_{2N} \geq r^{2N}\Big) = 0.$$

The branching process $(Z_n)_{n\in\mathbb{N}_0}$ has reproduction mean $m > 1$ and reproduction variance $\sum_{k=0}^{\infty} k^2 p_k < \infty$. For such a branching process the event that $(Z_n)_{n\in\mathbb{N}_0}$ survives equals the event $\liminf_{N\to\infty}\{Z_N \geq r^N\}$ by Theorem 4.2. Therefore, we conclude

$$P_\omega\Big(\liminf_{n\to\infty} \frac{M_n}{n} < \beta, (Z_n)_{n\in\mathbb{N}_0} \text{ survives}\Big) = 0,$$

which finishes the proof. □

Theorem 3.5 now follows immediately from Proposition 4.4 and Proposition 4.5.

4.3 On the non-negative integers

In this last section of the thesis we turn to branching random walk on the non-negative integers. Before we can prove Theorem 3.7, we derive the large deviation rate function for random walk on $\mathbb{N}_0$.

4.3.1 Large deviations

We split the proof into the cases $\alpha \leq \frac{1}{2}$ and $\alpha > \frac{1}{2}$.

Proof of Theorem 3.6 for $\alpha \leq \frac{1}{2}$. It is enough to prove

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) = -I_\alpha^r(x) \tag{4.10}$$

for $0 \leq x \leq 1$ with I_α^r as defined in the statement of the theorem. The proof is split into proofs of lower and upper bounds on $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\frac{S_n^r}{n} \geq x\big)$ for $x \in [0, 1-2\alpha]$ and $x \in (1-2\alpha, 1]$, respectively.

Lower bound A direct coupling of the random walks $(S_n^r)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$ immediately shows the following lower bound that holds for all $0 \leq x \leq 1$:

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) \geq \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \geq x\Big) = -I_\alpha(x) \tag{4.11}$$

Lower bound on $[0, 1-2\alpha]$ For this part we can without loss of generality assume $\alpha < \frac{1}{2}$. We need to find a good strategy for the random walk to end up to the right of nx after n steps. The best strategy for small x is to spend some time close to 0 and only then go to the right. Indeed, assume $0 \leq x \leq 1-2\alpha$ and let $x < \gamma < 1$. Using the Markov property and the coupling with the random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ as before we get

$$\begin{aligned}
\mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) &\geq \mathbb{P}\Big(S^r_{2\lceil \frac{n}{2}(1-\gamma)\rceil} = 0\Big)\mathbb{P}\Bigg(\frac{S^r_{n-2\lceil \frac{n}{2}(1-\gamma)\rceil}}{n} \geq x\Bigg) \\
&\geq \mathbb{P}\Big(S^r_{2\lceil \frac{n}{2}(1-\gamma)\rceil} = 0\Big)\mathbb{P}\Bigg(\frac{S_{n-2\lceil \frac{n}{2}(1-\gamma)\rceil}}{n} \geq x\Bigg).
\end{aligned}$$

The random walk $(S_n^r)_{n\in\mathbb{N}_0}$ is a 2-periodic positive recurrent Markov chain starting at 0. Therefore, $\lim_{n\to\infty} \mathbb{P}\big(S^r_{2\lceil \frac{n}{2}(1-\gamma)\rceil} = 0\big) = 2\pi(0) > 0$, where $(\pi(i))_{i\in\mathbb{N}_0}$ is the stationary distribution. Thus,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) &\geq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Bigg(\frac{S_{n-2\lceil \frac{n}{2}(1-\gamma)\rceil}}{n\gamma} \geq \frac{x}{\gamma}\Bigg) \\
&\geq \liminf_{n\to\infty} \gamma \frac{1}{n\gamma} \log \mathbb{P}\Bigg(\frac{S_{n-2\lceil \frac{n}{2}(1-\gamma)\rceil}}{n - 2\lceil \frac{n}{2}(1-\gamma)\rceil} \geq \frac{x}{\gamma}\Bigg) \\
&= -\gamma I_\alpha\Big(\frac{x}{\gamma}\Big).
\end{aligned}$$

For $\gamma = \frac{x}{1-2\alpha}$ a straightforward calculation yields $\gamma I_\alpha\big(\frac{x}{\gamma}\big) = x\log\big(\frac{1-\alpha}{\alpha}\big)$. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n}\log \mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) \geq -x\log\Big(\frac{1-\alpha}{\alpha}\Big). \tag{4.12}$$

Upper bound on $[0, 1-2\alpha]$ Again, we can without loss of generality assume $\alpha < \frac{1}{2}$. Using the exponential Markov inequality we get for all $\lambda \geq 0$ and $0 \leq x \leq 1$

$$\frac{1}{n}\log \mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) \leq \frac{1}{n}\log\Big(\mathbb{E}\big[\mathrm{e}^{\lambda S_n^r}\big]\mathrm{e}^{-\lambda n x}\Big) = \frac{1}{n}\log \mathbb{E}\big[\mathrm{e}^{\lambda S_n^r}\big] - \lambda x. \tag{4.13}$$

One can check by induction that $\mathbb{P}(S_n^r = z) \leq \big(\frac{1-\alpha}{\alpha}\big)\big(\frac{\alpha}{1-\alpha}\big)^z$ for all $n, z \in \mathbb{N}_0$. Therefore,

$$\limsup_{n\to\infty} \mathbb{E}\big[\mathrm{e}^{\lambda S_n^r}\big] = \limsup_{n\to\infty} \sum_{z=0}^{\infty} \mathbb{P}(S_n^r = z)\mathrm{e}^{\lambda z} \leq \sum_{z=0}^{\infty}\Big(\frac{1-\alpha}{\alpha}\Big)\Big(\frac{\alpha}{1-\alpha}\Big)^z \mathrm{e}^{\lambda z}. \tag{4.14}$$

The last sum is finite if $\lambda < \log\frac{1-\alpha}{\alpha}$. In this case (4.13) and (4.14) imply

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) \leq -\lambda x.$$

Thus, optimizing the parameter λ,

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}\Big(\frac{S_n^r}{n} \geq x\Big) \leq -x\log\Big(\frac{1-\alpha}{\alpha}\Big). \tag{4.15}$$

Upper bound on $(1-2\alpha, 1]$ We want to use inequality (4.13) and claim that for $\lambda > \log\frac{1-\alpha}{\alpha}$

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{E}\big[\mathrm{e}^{\lambda S_n^r}\big] = \log\big(\alpha\mathrm{e}^{\lambda} + (1-\alpha)\mathrm{e}^{-\lambda}\big). \tag{4.16}$$

Indeed, for $n \geq 1$

$$\mathbb{E}\big[\mathrm{e}^{\lambda S_n^r}\big] = \mathbb{E}\big[\mathbb{1}_{\{S_{n-1}^r \neq 0\}}\mathrm{e}^{\lambda S_{n-1}^r}(\alpha\mathrm{e}^{\lambda} + (1-\alpha)\mathrm{e}^{-\lambda})\big] + \mathbb{E}\big[\mathbb{1}_{\{S_{n-1}^r = 0\}}\mathrm{e}^{\lambda}\big].$$

Abbreviating $a = \alpha\mathrm{e}^{\lambda} + (1-\alpha)\mathrm{e}^{-\lambda}$, this recursion formula simplifies to

$$\mathbb{E}\big[\mathrm{e}^{\lambda S_n^r}\big] = a\mathbb{E}\big[\mathrm{e}^{\lambda S_{n-1}^r}\big] + \mathbb{P}(S_{n-1}^r = 0)(\mathrm{e}^{\lambda} - a).$$

Hence, since we assumed $S_0^r = 0$,

$$\mathbb{E}\big[\mathrm{e}^{\lambda S_n^r}\big] = a^n\Big(1 + (\mathrm{e}^{\lambda} - a)\sum_{k=0}^{n-1}\mathbb{P}(S_k^r = 0)a^{-(k+1)}\Big) \leq a^n\Big(1 + (\mathrm{e}^{\lambda} - a)\sum_{k=0}^{n-1} a^{-(k+1)}\Big). \tag{4.17}$$

Note that we used $\mathrm{e}^{\lambda} - a \geq 0$ in the last estimate. A straightforward calculation using $\alpha \leq \frac{1}{2}$ and $\lambda > \log\frac{1-\alpha}{\alpha}$ shows $a > 1$. Thus, the second factor on the right hand side of (4.17) is bounded and therefore (4.17) implies (4.16). Now, inserting (4.16) in (4.13) yields

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_n^r}{n}\geq x\Big)\leq\log\big(\alpha\mathrm{e}^{\lambda}+(1-\alpha)\mathrm{e}^{-\lambda}\big)-\lambda x. \tag{4.18}$$

Set $\lambda=\frac{1}{2}\log\frac{(1-\alpha)(1+x)}{\alpha(1-x)}$. This satisfies $\lambda>\log\frac{1-\alpha}{\alpha}$ since we assumed $x>1-2\alpha$. Then another straightforward calculation shows that the last inequality implies, for $x>1-2\alpha$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_n^r}{n}\geq x\Big)\leq -I_\alpha(x). \tag{4.19}$$

(4.10) now follows from (4.11), (4.12), (4.15) and (4.19). □

Proof of Theorem 3.6 for $\alpha>\frac{1}{2}$. We need to show

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_n^r}{n}\geq x\Big)&=-I_\alpha(x)\quad\text{for } x\geq 2\alpha-1 \text{ and}\\ \lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_n^r}{n}\leq x\Big)&=-I_\alpha(x)\quad\text{for } x\leq 2\alpha-1.\end{aligned}$$

A coupling with the random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ shows $\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(\frac{S_n^r}{n}\geq x\big)\geq -I_\alpha(x)$ as well as $\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(\frac{S_n^r}{n}\leq x\big)\leq -I_\alpha(x)$ for all $x\in[0,1]$. It remains to prove an upper bound for $x\in[2\alpha-1,1]$ and a lower bound for $x\in[0,2\alpha-1]$.

Upper bound on $[2\alpha-1,1]$ The proof follows along the lines of the proof of the upper bound on $(1-2\alpha,1]$ in the case $\alpha\leq\frac{1}{2}$. Let $\lambda\geq 0$ and set $a=\alpha\mathrm{e}^{\lambda}+(1-\alpha)\mathrm{e}^{-\lambda}$. A straightforward calculation shows $a>1$ and $\mathrm{e}^{\lambda}-a\geq 0$. Thus, we also get Estimate (4.18),

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_n^r}{n}\geq x\Big)\leq\log\big(\alpha\mathrm{e}^{\lambda}+(1-\alpha)\mathrm{e}^{-\lambda}\big)-\lambda x.$$

As before, setting $\lambda=\frac{1}{2}\log\frac{(1-\alpha)(1+x)}{\alpha(1-x)}$ yields the claim. This time we have $\lambda\geq 0$ as $x\geq 2\alpha-1$.

Lower bound on $[0,2\alpha-1]$ Let $x\in[0,2\alpha-1]$, $n\in\mathbb{N}$ and $\varepsilon>0$. For technical reasons we define the event $A_\varepsilon(n)=\{S_k^r=k\colon 0\leq k\leq\lceil\varepsilon n\rceil\}$. The Markov property implies

$$\begin{aligned}\mathbb{P}\Big(\frac{S_n^r}{n}\leq x\Big)&\geq\mathbb{P}\big(A_\varepsilon(n)\big)\mathbb{P}\big(S_n^r\leq xn\,\big|\,S_{\lceil\varepsilon n\rceil}^r=\lceil\varepsilon n\rceil\big)\\ &\geq\mathbb{P}\big(A_\varepsilon(n)\big)\mathbb{P}\Big(0<S_k^r\leq xn\text{ for all }\lceil\varepsilon n\rceil\leq k\leq n\,\Big|\,S_{\lceil\varepsilon n\rceil}^r=\lceil\varepsilon n\rceil\Big).\end{aligned} \tag{4.20}$$

As

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(A_\varepsilon(n)\big)=0, \tag{4.21}$$

we only need to estimate the second factor on the right hand side of (4.20). The probability to never hit 0 and to stay below xn when starting in $\lceil\varepsilon n\rceil$ is the same for random walk on $\mathbb{N}_0$ and random walk on $\mathbb{Z}$. By shifting everything by $\lceil\varepsilon n\rceil$, it therefore remains to show that

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(-\lceil\varepsilon n\rceil<S_k\leq xn-\lceil\varepsilon n\rceil\text{ for all }0\leq k\leq n-\lceil\varepsilon n\rceil\Big)\geq -I_\alpha(x). \tag{4.22}$$

Then, the lower bound follows from (4.20), (4.21), and (4.22). Estimate (4.22), however, is a consequence of large deviation results for paths of random walks that can e.g. be found in [14, Chapter 5.1]. To show this, we define the process $(Z_n(t))_{t\in[0,1]}$ for $n \in \mathbb{N}_0$ by

$$Z_n(t) = \frac{S_{\lfloor\lfloor(1-\varepsilon)n\rfloor t\rfloor}}{\lfloor(1-\varepsilon)n\rfloor}.$$

The paths of this process are elements in the space $L_\infty[0,1]$ of bounded functions $[0,1] \to \mathbb{R}$ equipped with the uniform norm. Further, let

$$B_\varepsilon = \Big\{ f \in L_\infty[0,1] \colon f(t) \in \Big(-\frac{\varepsilon}{1-\varepsilon}, \frac{x-2\varepsilon}{1-\varepsilon}\Big) \text{ for all } t \in [0,1] \Big\}.$$

Note that $Z_n \in B_\varepsilon$ implies $S_k \le xn - \lceil \varepsilon n \rceil$ as well as $S_k > -\lceil \varepsilon n \rceil$ for all $0 \le k \le \lfloor(1-\varepsilon)n\rfloor$ and n sufficiently large. Hence,

$$\mathbb{P}\Big(-\lceil \varepsilon n\rceil < S_k \le xn - \lceil \varepsilon n\rceil \text{ for all } 0 \le k \le n - \lfloor \varepsilon n \rfloor\Big) \ge \mathbb{P}(Z_n \in B_\varepsilon).$$

Using [14, Theorem 5.1.2] we get

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n \in B_\varepsilon) = - \inf_{f \in B_\varepsilon^\circ} J(f), \tag{4.23}$$

where for $f \in L_\infty([0,1])$ absolutely continuous with $f(0) = 0$ we have

$$J(f) = \int_0^1 I_\alpha(\dot{f}(t))\,\mathrm{d}t.$$

For $h \in \mathbb{R}$ let $f_h(t) = ht$ for $t \in [0,1]$. Observe that $f_h \in B_\varepsilon^\circ$ for all $h \in \big(0, \frac{x-2\varepsilon}{1-\varepsilon}\big)$. Thus, as I_α is decreasing on $[0, 2\alpha - 1]$,

$$\inf_{f\in B_\varepsilon^\circ} J(f) \le \inf_{h\in(0,\frac{x-2\varepsilon}{1-\varepsilon})} J(f_h) = \inf_{h\in(0,\frac{x-2\varepsilon}{1-\varepsilon})} I_\alpha(h) = I_\alpha\Big(\frac{x-2\varepsilon}{1-\varepsilon}\Big) \xrightarrow{\varepsilon\to 0} I_\alpha(x).$$

Together with (4.23) this shows the claimed Estimate (4.22). □

4.3.2 The speed of the maximum

For the proof of the lower bound of Theorem 3.7 we compare the branching random walk on the non-negative integers $\mathbb{N}$ with the corresponding branching random walk on $\mathbb{Z}$. This branching random walk has the same reproduction law and the underlying transition probabilities are given by (3.8). To distinguish the processes, we decorate all notation used for the branching random walk on $\mathbb{N}$ with reflection at 0 with a superscript r.

In particular, we use a large deviation result for the maximum process $(M_n)_{n\in\mathbb{N}_0}$ of the branching random walk on $\mathbb{Z}$. It can be found in a paper by Gantert and Höfelsauer. Simplified to the scenario and case we need, it can be stated as follows:

Lemma 4.6 ([19])**.** *Consider a branching random walk on* $\mathbb{Z}$ *that satisfies Condition* (3.5) *and has transition probabilities given by* (3.8) *with* $\alpha \in (0,1)$*. Let* $(M_n)_{n\in\mathbb{N}_0}$ *denote the maximum process. Assume that is has speed* b*, i.e.* $\lim_{n\to\infty}\frac{M_n}{n} = b$ *almost surely. Then we have for* $x \geq b$

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{M_n}{n}\geq x\Big) \geq -I_\alpha(x)+\log m.$$

See (3.9) *for the definition of* I_α*.*

Recall that b exists e.g. by Theorem 3.5. It satisfies $\log m = I_\alpha(b)$ if $\log m < I_\alpha(1)$, and $b = 1$ otherwise. In [19] the authors discuss large deviation principles for branching random walks with reproduction laws that vary randomly in time. As this paper is not yet published, we present the proof of Lemma 4.6 here.

Proof of Lemma 4.6. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We enumerate the $Z_{\lfloor\varepsilon n\rfloor}$ particles in the $\lfloor\varepsilon n\rfloor$-th generation of the branching process and denote the position of the i-th of them by $S^i_{\lfloor\varepsilon n\rfloor}$. The descendants of each of these $Z_{\lfloor\varepsilon n\rfloor}$ particles form independent branching random walks. For every $1 \leq i \leq Z_{\lfloor\varepsilon n\rfloor}$ let M^i_m denote the maximum of the positions of the descendants of the i-th particle at time $\lfloor\varepsilon n\rfloor + m$, relative to their parent particle i. The processes $(M^i_m)_{m\in\mathbb{N}_0}$ are independent and have the same distribution as $(M_n)_{n\in\mathbb{N}_0}$. For $x > b$ we have

$$\begin{aligned}\mathbb{P}\Big(\frac{M_n}{n}\geq x\Big) &= \mathbb{P}\Big(\max_{1\leq i\leq Z_{\lfloor\varepsilon n\rfloor}}\frac{M^i_{n-\lfloor\varepsilon n\rfloor}}{(1-\varepsilon)n}+\frac{S^i_{\lfloor\varepsilon n\rfloor}}{(1-\varepsilon)n}\geq\frac{x}{1-\varepsilon}\Big)\\ &\geq \mathbb{P}\Big(\max_{1\leq i\leq Z_{\lfloor\varepsilon n\rfloor}}\frac{M^i_{n-\lfloor\varepsilon n\rfloor}}{n-\lfloor\varepsilon n\rfloor}+\frac{S^i_{\lfloor\varepsilon n\rfloor}}{(1-\varepsilon)n}\geq\frac{x}{1-\varepsilon}\Big). \qquad (4.24)\end{aligned}$$

Let $(S_n)_{n\in\mathbb{N}_0}$ be an independent random walk with start in 0 that moves in the same environment as the branching random walk. Note that $S^i_{\lfloor\varepsilon n\rfloor}$ has the same distribution as the random variable $S_{n\varepsilon}$ for every $1 \leq i \leq Z_{\lfloor\varepsilon n\rfloor}$. One can show that (4.24) implies

$$\begin{aligned}\mathbb{P}\Big(\frac{M_n}{n}\geq x\Big) &\geq \mathbb{P}\Big(\max_{1\leq i\leq Z_{\lfloor\varepsilon n\rfloor}}\frac{M^i_{n-\lfloor\varepsilon n\rfloor}}{n-\lfloor\varepsilon n\rfloor}+\frac{S_{\lfloor\varepsilon n\rfloor}}{(1-\varepsilon)n}\geq\frac{x}{1-\varepsilon}\Big)\\ &\geq \mathbb{P}\Big(\max_{1\leq i\leq Z_{\lfloor\varepsilon n\rfloor}}\frac{M^i_{n-\lfloor\varepsilon n\rfloor}}{n-\lfloor\varepsilon n\rfloor}\geq x\Big)\mathbb{P}\Big(\frac{S_{\lfloor\varepsilon n\rfloor}}{(1-\varepsilon)n}\geq\frac{\varepsilon x}{1-\varepsilon}\Big). \qquad (4.25)\end{aligned}$$

As $(S_n)_{n\in\mathbb{N}_0}$ satisfies a large deviation principle with rate function I_α, we have

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_{\lfloor\varepsilon n\rfloor}}{(1-\varepsilon)n}\geq\frac{\varepsilon x}{(1-\varepsilon)}\Big) = \lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_{\lfloor\varepsilon n\rfloor}}{\varepsilon n}\geq x\Big) = -\varepsilon I_\alpha(x). \qquad (4.26)$$

To estimate the first factor in (4.25) we use the independence of the processes $(M^i_m)_{m\in\mathbb{N}_0}$. Recall that m denotes the reproduction mean of the branching random walk. For $c > 0$ it holds that

$$\begin{aligned}\mathbb{P}\Big(\max_{1\le i\le Z_{\lfloor \varepsilon n\rfloor}} \frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big) &\ge \mathbb{P}\Big(\max_{1\le i\le Z_{\lfloor \varepsilon n\rfloor}} \frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x \Big| Z_{\lfloor \varepsilon n\rfloor} \ge cm^{\lfloor \varepsilon n\rfloor}\Big)\mathbb{P}\big(Z_{\lfloor \varepsilon n\rfloor} \ge cm^{\lfloor \varepsilon n\rfloor}\big)\\ &\ge \left(1-\left(1-\mathbb{P}\Big(\frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big)\right)^{cm^{\lfloor \varepsilon n\rfloor}}\right)\mathbb{P}\big(Z_{\lfloor \varepsilon n\rfloor} \ge cm^{\lfloor \varepsilon n\rfloor}\big).\end{aligned}$$

Using the inequality $1-(1-x)^y \ge xy(1-xy)$, that holds for $x \in [0,1]$ and $y \ge 0$, this implies

$$\begin{aligned}\mathbb{P}\Big(\max_{1\le i\le Z_{\lfloor \varepsilon n\rfloor}} \frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big) &\ge \mathbb{P}\Big(\frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big)cm^{\lfloor \varepsilon n\rfloor}\\ &\cdot\left(1-cm^{\lfloor \varepsilon n\rfloor}\mathbb{P}\Big(\frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big)\right)\mathbb{P}\big(Z_{\lfloor \varepsilon n\rfloor} \ge cm^{\lfloor \varepsilon n\rfloor}\big). \qquad (4.27)\end{aligned}$$

By Theorem 4.2 and Assumption (3.5) the sequence $\big(\frac{Z_n}{m^n}\big)_{n\in\mathbb{N}_0}$ converges almost surely towards a non-negative random variable W. We further know that W is positive if the process survives. Thus, we can choose $c>0$ such that $\lim_{n\to\infty}\mathbb{P}\big(Z_{\lfloor \varepsilon n\rfloor} \ge cm^{\lfloor \varepsilon n\rfloor}\big) = \mathbb{P}(W \ge c) > 0$, and therefore

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(Z_{\lfloor \varepsilon n\rfloor} \ge cm^{\lfloor \varepsilon n\rfloor}\big) = 0. \qquad (4.28)$$

If at time $n-\lfloor \varepsilon n\rfloor$ the maximum is bigger or equal than x, then at least one of the $Z_{n-\lfloor \varepsilon n\rfloor}$ particles present at time $n-\lfloor \varepsilon n\rfloor$ has to be at a position bigger or equal than x. Thus, using $\mathbb{E}\big[Z_{n-\lfloor \varepsilon n\rfloor}\big] = m^{n-\lfloor \varepsilon n\rfloor}$, we get

$$cm^{\lfloor \varepsilon n\rfloor}\mathbb{P}\Big(\frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big) \le cm^{\lfloor \varepsilon n\rfloor}\mathbb{E}\big[Z_{n-\lfloor \varepsilon n\rfloor}\big]\mathbb{P}\Big(\frac{S_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big) = cm^n\mathbb{P}\Big(\frac{S_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big).$$

The choice of x implies $\log m < I_\alpha(x)$. If we choose $\varepsilon > 0$ small enough, the last term tends to 0 in the limit $n \to \infty$ by the large deviation principle for $(S_n)_{n\in\mathbb{N}_0}$. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\log\left(1-cm^{\lfloor \varepsilon n\rfloor}\mathbb{P}\Big(\frac{M^i_{n-\lfloor \varepsilon n\rfloor}}{n-\lfloor \varepsilon n\rfloor} \ge x\Big)\right) = 0. \qquad (4.29)$$

Now, (4.25), (4.26), (4.27), (4.28), and (4.29) imply

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{M_n}{n} \ge x\Big) \ge (1-\varepsilon)\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{M_n}{n} \ge x\Big) + \varepsilon\log m - \varepsilon I_\alpha(x).$$

This immediately yields the claim of the lemma. □

Proof of Theorem 3.7. Upper bound We first show that

$$\limsup_{n\to\infty}\frac{M^r_n}{n} \le b^r \qquad (4.30)$$

holds almost surely. The proof of this upper bound is identical to the proof of Lemma 4.4.

Without loss of generality assume $b^r < 1$. Let $b^r < \beta < 1$. As I_α^r is strictly increasing on the interval $[v, 1]$ there is $\varepsilon > 0$ such that

$$\log m < I_\alpha^r(\beta) - \varepsilon. \tag{4.31}$$

By Theorem 3.6 for n large

$$\mathbb{P}(S_n^r \geq \beta n) \leq \mathrm{e}^{-(I_\alpha^r(\beta)-\varepsilon)n}.$$

Hence, the Markov inequality implies

$$\begin{aligned}\mathbb{P}\big(\lambda\big([\beta n, \infty), n\big) \geq 1\big) &\leq \mathbb{E}\left[\sum_{x \geq \beta n} \lambda(x, n)\right] \\ &= m^n \mathbb{P}(S_n^r \geq n\beta) \\ &\leq \mathrm{e}^{(\log m - I_\alpha^r(\beta)+\varepsilon)n}.\end{aligned}$$

By (4.31) the exponent in the last term is negative. The probability $\mathbb{P}\big(\lambda\big([n\beta, \infty), n\big) \geq 1\big)$ is hence summable in n. Thus, by the Borel-Cantelli Lemma, almost surely, for n large there is no particle in $[\beta n, \infty)$. Therefore, $\frac{M_n^r}{n} \leq \beta$ for n large, if $Z_n > 0$. Hence, on $\{Z_n \to \infty\}$ almost surely $\limsup_{n\to\infty} \frac{M_n^r}{n} \leq \beta$. As $\beta > b^r$ was chosen arbitrary, this shows (4.30).

Lower bound: A direct coupling with the branching random walk on $\mathbb{Z}$ yields $M_n^r \geq M_n$ for every $n \in \mathbb{N}_0$. Thus, $\liminf_{n\to\infty} \frac{M_n^r}{n} \geq b$. If $\alpha \leq \frac{1}{2}$ and $b \geq 1 - 2\alpha$, then $b^r = b$ by Theorem 3.6. The same holds whenever $\alpha > \frac{1}{2}$. Hence, we can assume $\alpha \leq \frac{1}{2}$ and $b < 1 - 2\alpha$. The last condition is equivalent to $\log m < I_\alpha^r(1 - 2\alpha)$ and $b^r < 1 - 2\alpha$.

Let $0 < \beta < b^r$ and define $\gamma = \frac{\beta}{1-2\alpha}$. Note that $\gamma \in (0, 1)$. As the rate function is strictly increasing, there is $\varepsilon > 0$ such that

$$\log m > I_\alpha^r(\beta) + \varepsilon. \tag{4.32}$$

Let us first motivate the following calculations. We want to estimate from above the probability of the event that the maximum M_n^r is at most βn, show that this probability is summable in n, and then apply the Borel-Cantelli Lemma. In the proof of Theorem 3.6 we have seen that the best strategy for a single particle to get to a vertex to the right of βn within n steps is to spend $(1-\gamma)n$ steps close to 0 and only then go to the right with positive speed. Here, we also let the branching random walk run for $(1-\gamma)n$ steps in time, and then relocate all $Z_{(1-\gamma)n}$ particles to 0. The descendants of these $Z_{(1-\gamma)n}$ particles form independent branching random walks. By the usual coupling, if $M_n^r \leq \beta n$, then the maximum of each of these branching random walks after the remaining γn steps has to be at most at βn as well.

For the sake of correctness we have to round γn and $(1-\gamma)n$. To abbreviate notation let $n(\gamma) = n - 2\lfloor \frac{n}{2}(1-\gamma)\rfloor$ and note that $n(\gamma) \sim \gamma n$. By the above argument we have for

$1 < r < m$ and $n \in \mathbb{N}_0$

$$\begin{aligned}\mathbb{P}\Big(\frac{M_n^r}{n} \le \beta, Z_{n-n(\gamma)} \ge r^{n-n(\gamma)}\Big) &= \mathbb{E}\Big[\mathbb{P}\Big(\frac{M_n^r}{n} \le \beta \Big| \mathcal{F}(n-n(\gamma))\Big)\mathbb{1}_{\{Z_{n-n(\gamma)} \ge r^{n-n(\gamma)}\}}\Big] \\ &\le \mathbb{E}\Big[\mathbb{P}\Big(\frac{M_{n(\gamma)}^r}{n} \le \beta\Big)^{Z_{n-n(\gamma)}}\mathbb{1}_{\{Z_{n-n(\gamma)} \ge r^{n-n(\gamma)}\}}\Big] \\ &\le \Big(1 - \mathbb{P}\Big(\frac{M_{n(\gamma)}^r}{n(\gamma)} \ge \frac{\beta}{\gamma}\Big)\Big)^{r^{n-n(\gamma)}} \\ &\le \Big(1 - \mathbb{P}\Big(\frac{M_{n(\gamma)}}{n(\gamma)} \ge \frac{\beta}{\gamma}\Big)\Big)^{r^{(1-\gamma)n-2}}. \end{aligned} \tag{4.33}$$

By the above assumption we have $\frac{\beta}{\gamma} = 1 - 2\alpha > b$ and can apply Lemma 4.6. Therefore,

$$\mathbb{P}\Big(\frac{M_{n(\gamma)}}{n(\gamma)} \ge \frac{\beta}{\gamma}\Big) \ge \mathrm{e}^{-n(\gamma)(I_\alpha(\frac{\beta}{\gamma}) - \log m + \frac{\varepsilon}{\gamma})}$$

for n large enough. Inserting this estimate into (4.33) and using $\gamma I_\alpha(\frac{\beta}{\gamma}) = I_\alpha^r(\beta)$ shows that for n large

$$\begin{aligned}\mathbb{P}\Big(\frac{M_n^r}{n} \le \beta, Z_{n-n(\gamma)} \ge r^{n-n(\gamma)}\Big) &\le \Big(1 - \mathrm{e}^{-\frac{n(\gamma)}{\gamma}(I_\alpha^r(\beta) - \gamma\log m + \varepsilon)}\Big)^{r^{(1-\gamma)n-2}} \\ &\le \exp\big(-\mathrm{e}^{-\frac{n(\gamma)}{\gamma}(I_\alpha^r(\beta) - \gamma\log m + \varepsilon - (1-\gamma)\log r)} \cdot r^{-2}\big).\end{aligned}$$

As $\lim_{r\to m} I_\alpha^r(\beta) - \gamma\log m + \varepsilon - (1-\gamma)\log r = I_\alpha^r(\beta) - \log m + \varepsilon < 0$ by (4.32), we can choose r close enough to m such that

$$\sum_{n\in\mathbb{N}_0} \mathbb{P}\Big(\frac{M_n^r}{n} \le \beta, Z_{n-n(\gamma)} \ge r^{n-n(\gamma)}\Big) < \infty.$$

By the Borel-Cantelli Lemma the event $\{\frac{M_n^r}{n} \le \beta, Z_{(1-\gamma)n} \ge r^{(1-\gamma)n}\}$ thus occurs only for finitely many $n \in \mathbb{N}_0$ almost surely. Conditioned on the survival of the process $(Z_n)_{n\in\mathbb{N}_0}$, we know that $Z_{(1-\gamma)n} \ge r^{(1-\gamma)n}$ for all but finitely many $n \in \mathbb{N}_0$ almost surely. Therefore, conditioned on survival, we conclude

$$\liminf_{n\to\infty} \frac{M_n^r}{n} \ge \beta$$

almost surely. As $\beta < b^r$ was chosen arbitrarily, this finishes the proof. □

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