Technische Universität München
Zentrum Mathematik
Lehrstuhl für Mathematische Optimierung

# Analysis of optimal control problems for the optical flow equation under mild regularity assumptions

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Vollständiger Abdruck der von der Fakultät der Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

**Doktors der Naturwissenschaften (Dr.rer.nat.)**

genehmigten Dissertation.

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Die Dissertation wurde am 16.11.2017 bei der Technischen Universität München eingereicht und durch die Fakultät der Mathematik am 06.06.2018 angenommen.

**Abstract**

In many fields of image and video processing like image stabilization and video compression information about movements of image values are required. These are frequently given as an optical flow. The optical flow of an image sequence is identified as the velocity field of apparent points of movements of objects projected to the image plane. The determination of the optical flow is normally carried out by solving an optimization problem. In this thesis, we investigate an optimal control problem for a given image sequence with the transport equation as a side constraint which yields a time-continuous optical flow field of the image sequence as the optimal control. The corresponding optimal state then represents a time-continuous image interpolation of the sequence. For the transport equation we use results about well-posedness of Ambrosio for $BV$-regular vector fields. Furthermore, we improve existing stability results of solutions in the setting of spatial $BV$-regularity of the vector fields. With the aid of these results we show the existence of minima of the objective function under various regularization terms. In the second part of the thesis, we attend to differentiability of the problem. In a first step we show Fréchet differentiability of the control-to-state operator with $BV$-regular initial values of the transport equation. By smoothing this operator, Fréchet differentiability of the composition of the control-to-state operator with the tracking term of the objective function can be immediately proven. In a further proof, we show directly Fréchet differentiability of this composition under the requirement that the image sequence satisfies a certain condition. In the case that the condition is not fulfilled we are able to prove that the composition still possesses a one-sided directional derivative. At the end, we show two duality relations which are based on the adjoint equation of the transport equation on the one hand and on the backward transport equation on the other hand. With the aid of these results we find two different representations of the gradient of the composition. The thesis finally ends with necessary optimality conditions of first order.

**Zusammenfassung**

In vielen Bereichen der Bild- und Videoverarbeitung wie Bildstabilisierung und Videokomprimierung werden Bewegungsinformationen von Bildwerten benötigt. Diese sind häufig als optischer Fluss gegeben. Der optische Fluss einer Bildfolge bezeichnet dabei das Geschwindigkeitsfeld sichtbarer Punkte der in die Bildebene projizierten Bewegungen von Objekten. Die Bestimmung des optischen Flusses erfolgt gewöhnlicherweise durch das Lösen eines Optimierungsproblems. In dieser Arbeit untersuchen wir zu einer gegebenen Bildsequenz ein Optimalsteuerungsproblem mit der Transportgleichung als Nebenbedingung, das als optimale Steuerung ein zeit-kontinuierliches optisches Flussfeld dieser Bildfolge liefert. Der zugehörige optimale Zustand stellt dann eine zeit-kontinuierliche Bildinterpolation der Folge dar. Für die Transportgleichung benutzen wir Ergebnisse von Ambrosio zur Wohlgestelltheit dieser Gleichung bei $BV$-regulären Vektorfeldern. Darüber hinaus verbessern wir vorhandene Stabilitätsaussagen von Lösungen im Rahmen räumlicher $BV$-Regularität der Vektorfelder. Mithilfe dieser Resultate zeigen wir die Existenz von Minima der Zielfunktion unter verschiedenen Regularisierunsgtermen. In der zweiten Hälfte der Arbeit widmen wir uns der Differenzierbarkeit des Problems. In einem ersten Schritt zeigen wir die Fréchet-Differenzierbarkeit des Steuerungs-Zustands-Operators bei $BV$-regulären Anfangsdaten der Transportgleichung. Durch eine Glättung dieses Operators lässt sich dann unmittelbar die Fréchet-Differenzierbarkeit der Komposition des Steuerungs-Zustands-Operators mit dem Trackingterm des Zielfunktionals zeigen. In einem weiteren Nachweis zeigen wir direkt die Fréchet-Differenzierbarkeit dieser Komposition unter der Voraussetzung, dass die Bildfolge eine bestimmte Bedingung erfüllt. Im Falle des Nichterfüllens dieser Bedingung können wir nachweisen, dass die Komposition immer noch eine einseitige Richtungsableitung besitzt. Zum Schluss zeigen wir zwei Dualitätsrelationen, die zum einen auf der adjungierten Gleichung der Transportgleichung und zum anderen auf der rückwärts gerichteten Transportgleichung basieren. Mithilfe dieser finden wir zwei verschiedene Darstellungen des Gradienten der Komposition. Die Arbeit endet schließlich mit notwendigen Optimalitätsbedingungen erster Ordnung.

# Contents

# 1. Introduction

In many fields of image processing, a main task is to process information about motion of pixel values and objects appearing in sequences of images. In medical applications, motion information is used for comparing and matching ultrasound images and radiograms. For compressing videos, knowledge about motion is applied to reduce the information which has to be stored. In video decompression, motion information is used to generate intermediate images. In these and many other applications, the generation of motion information is often based on the so-called optical flow.

Optical flow basically describes the vector field of velocities of apparent points in the 2D image plane. In general, it differs from the projection of velocities of moving object points in the 3D space to the 2D image plane. In the 2D image plane, this projection of 3D movements is normally not observable in total, but the changing of intensity values of visible points. The vector field of velocities corresponding to these changes is then called optical flow ([AK06]). In applications, a continuous projection in time of 3D motion is usually not available, but a sequence of projections of 3D scenes onto the 2D image plane at consecutive time points, i.e. a sequence of images located at subsequent time points. In this situation, the optical flow is not a vector field of velocities anymore but a vector field of displacements of apparent points from one image to its subsequent image.

For a given image sequence, the computation of the displacement field was investigated by numerous researchers in the last decades. Most of their works are based on the groundbreaking work [HS81] of Horn and Schunck in 1981. In their work, Horn and Schunck developed the computation of the optical flow based on the assumption of constant intensity values. More precisely, Horn and Schunck assumed that the intensity values $Y$ of some image do not change, but their locations in the subsequent images can be different, i.e.

$$Y(t, x_1, x_2) - Y(t + \Delta t, x_1 + \Delta x_1, x_2 + \Delta x_2) = 0.$$

Assuming in addition that the changes in location are small, the first order term of a Taylor expansion represents a good approximation which leads to the so-called optical flow equation

$$\partial_t Y + \partial_{x_1} Y \Delta x_1 + \partial_{x_2} Y \Delta x_2 = 0.$$

In this case, the optical flow field is given by $(b_1, b_2) = (\Delta x_1, \Delta x_2)$. For the computation, additional constraints are needed since components of $(b_1, b_2)$ perpendicular to the gradient of the intensity values $(\partial_{x_1} Y, \partial_{x_2} Y)$ cannot be determined. Therefore, Horn and Schunck introduced some smoothness constraint which penalizes discontinuities of the displacement field. This constraint is based on the assumption that neighboring points have similar deviations and thus, the displacement field is smooth in general. As a consequence, Horn and Schunck minimized the objective function

$$J(b_1, b_2) = \int_{\Omega} (\partial_t Y + b_1 \partial_{x_1} Y + b_2 \partial_{x_2} Y)^2 + \lambda(|\nabla b_1|^2 + |\nabla b_2|^2) \; dx_1 dx_2$$

for computing the optical flow field, where $\Omega$ describes the image domain. The partial derivatives $\partial_t Y$, $\partial_{x_1} Y$ and $\partial_{x_2} Y$ were determined by using the image sequence. This approach of Horn and Schunck and other approaches (e.g. [LK81]) were further developed in numerous works in the subsequent decades. An overview of these works is given in [BSL$^+$11].

Beside these time independent approaches, optimal control based formulations ([HS01, BIK03]) of the form

$$\begin{aligned} \min_{u,b} J(u,b) &= \sum_{k=2}^{K} \|u(t_k,\cdot) - Y_k\|_{L^2(\Omega)} + R(b), \\ \text{s.t. } \partial_t u + \nabla u \cdot b &= 0 \qquad \text{in } (0,T)\times\Omega, \\ u(0,\cdot) &= Y_1 \qquad \text{in } \Omega \end{aligned} \tag{P}$$

came up at the beginning of 2000 leading to time dependent optical flow fields. In [HS01, BIK03], an image variable $u$ for the intensity values is introduced in addition to the optical flow variable $b = (b_1, b_2)$. These variables are defined on a spatio-temporal domain $(0,T)\times\Omega$ and images $Y_k$, $k \in \{1,\dots,K\}$ of a given image sequence are placed at specific time points $t_k$ in the time interval $[0,T]$. The assumption that $u$ and $b$ satisfy the optical flow equation in the domain $(0,T)\times\Omega$ leads to the transport equation which appears as a side constraint of the problems. The objective functions then consist of some regularization terms $R$ and a tracking term measuring the $L^2$-distance between the given images $Y_k$ and $u$ at specific time points. In this case, a solution $u$ of the transport equation can be seen as a continuous interpolation in time of the image sequence and the corresponding optical flow field is then a vector field of velocities in contraction to the vector fields of deviations in previously developed approaches.

In this thesis, our focus lies in the investigation of optimal control problems of the form (P). In this connection, our interest is directed towards the determination of a setting for the optimal control problems allowing preferably weak regularity and constraints for the optical flow fields $b$ and solutions $u$. In real problems, motion with discontinuities in the spatial domain naturally appear. Thus, one main task is to find conditions under which vector fields $b$ and solutions $u$ with discontinuities in the spatial variable $x$ are allowed and which still yields well-posedness of the transport equation.
In [BIK03], the authors presented some statements about well-posedness of the transport equation in a setting with Sobolev regularity. However, theoretical results concerning the existence of some minimizing points are absent. In 2011, Chen in [Che11] and Chen and Lorenz in [CL11] investigated a version of these optimal control problems and gave some further theoretical results. For vector fields $b$ with $C^1$-regularity in space and vanishing divergence, they could prove that $BV$-regularity is preserved in time for solutions of the transport equation. Furthermore, they showed existence of minimizing points for their optimal control problem under Sobolev regularity assumptions on the optical flow fields. Their theoretical results are based on results of DiPerna and Lions ([DL89]) about well-posedness of solutions for the transport equation with Sobolev regular vector fields.
Unfortunately, Sobolev regularity of the vector fields in the spatial variable is too strict to describe discontinuities. Therefore, we need new concepts for well-posedness of solutions allowing discontinuities in the spatial argument of the vector fields and the solutions. In the literature, different such concepts exist. One condition satisfying these requirements is the so-called one-sided Lipschitz condition for vector fields. A vector field $b$ satisfies this condition

if there exists some positive function $\alpha \in L^1((0,T))$ such that

$$\langle b(t,x) - b(t,y), x - y \rangle \leq \alpha |x - y|^2$$

holds for almost all $(t,x),(t,y)$ of the spatio-temporal domain. For the one-sided Lipschitz condition Bouchut, James and Mancini introduced in [Fra05] the concept of duality solutions for the forward multidimensional transport equation. A duality solution is a function satisfying a certain duality relation with all solutions of the corresponding dual backward equation. For this dual backward equation, Bouchut, James and Mancini generalized a solution concept, the so-called reversible solutions, developed in [Fra98] for the one-dimensional case to the multidimensional case. A crucial drawback of this approach is that the one-sided Lipschitz condition only allows to show uniqueness of reversible solutions of the backward continuity equation. Thus, a solution concept via duality solutions can only be established for the forward transport equation.

In this thesis, we will use a different concept, which builds on the so-called renormalized solutions and does not possess the previous drawback. It was developed by DiPerna and Lions in 1989 in [DL89]. A function $u$ is called a renormalized solution if it satisfies the weak formulation of the transport equation and every composition $\beta(u)$ of $u$ and some $C^1$ function $\beta$ is again a weak solution of the transport equation. DiPerna and Lions proved that any weak solution of the transport equation with Sobolev regular vector fields is a renormalized solution. This renormalization property then yields uniqueness of weak solutions for the transport equation. In 2004, Ambrosio could extend this theory in [Amb04] to vector fields with $BV$-regularity in space and absolutely continuous divergence. Some refinements and extensions to this theory were given in some later works by Ambrosio, Crippa, De Lellis and others ([Cri07, Lel07, CDS14b, CDS14a]). Since $BV$ functions can have discontinuities, the concept of renormalized solutions with $BV$-regular vector fields enables us to investigate the optimal control problems in our favored setting. As pointed out in [Fra05], these two solution concepts are in some sense orthogonal to each other, i.e. vector fields satisfying the one-sided Lipschitz condition behave in an orthogonal way at its discontinuity points than vector fields with $BV$-regularity and absolutely continuous divergence do.

A crucial step in the theory of renormalized solutions is the proof of convergence to zero of the so-called commutator

$$r_\varepsilon = b \cdot \nabla(u * \rho_\varepsilon) - (b \cdot \nabla u) * \rho_\varepsilon$$

as $\varepsilon \to 0$, where $b$ denotes some vector field, $u$ the corresponding solution and $\rho_\varepsilon$ some mollifier. In contrast to $L^1$-convergence to zero of the commutator in the Sobolev regular case, the commutator only converge weakly* to some measure $\sigma$ for general $BV$-regular vector fields. Therefore, Ambrosio had to develop various new techniques to give an upper bound for $\sigma$ which then turns out to be zero. For our purposes, the existing stability results for the solution operator $b \mapsto u_b$ of the transport equation provided in [DL89, Amb04, Cri07] were too weak. Therefore, we proved essentially stronger stability results. In these proofs, a similar term as the commutator appears and we used the same techniques Ambrosio had developed to prove convergence to zero of this term as $\varepsilon \to 0$. As a consequence we could show existence of minimizing points of the optimal control problem in a quite general setting.

A further considered aspect in this thesis is the investigation of differentiability of the control-to-state operator as well as its composition with the regarded objective function. Linearizing the transport equation formally shows that derivatives of the control-to-state operator involve derivatives of solutions with respect to the spatial variable. Thus, solutions of the transport equation need at least $BV$-regularity in space to derive the control-to-state operator at the

corresponding vector field in a meaningful way. Unfortunately, Colombini et al. showed in [CLR04] that in general $BV$-regularity is not propagated in time for vector fields with less than Lipschitz regularity in the spatial domain. Therefore, we change our assumptions on the vector fields to Lipschitz regularity in the investigation of differentiability. Beside the regularity for the spatial variable, the integrability in the time variable plays an important role in the investigation of differentiability. As we will see, the derivatives of the control-to-state operator and its composition with the tracking term can be represented as integrals of vector valued functions. Therefore, regarding the vector fields and the solutions as time dependent vector valued functions, we demand Gelfand integrability in time for these functions.
Results about differentiability of the solution operator in several dimensions with regularities assumed in this thesis are not known to exist in the literature. However, there are some results for nonlinear one dimensional conservation equations treating differentiability (e.g. [BJ99, Ulb01]). In these works, differentiability is proved for the solution operator depending on the initial values of the partial differential equation. The idea of the proofs are based on representing the difference quotient as the difference of two unique solutions of two linear partial differential equations and to show that this difference tends to zero as the difference of the initial values vanishes. We adopt this idea to prove Fréchet differentiability of our control-to-state operator in $C([0,T], \mathcal{M}(\Omega) - w^*)$. Smoothing of the solution operator then leads to Fréchet differentiability of the composition with the tracking term of the objective function. Under some refinements on the assumptions of the initial values and the functions $Y_k$ Fréchet differentiability in the non-smoothed case can be shown if the jump sets of the initial value and $Y_k$ satisfy some condition. If this condition is not fulfilled, we then prove that the composition is still one-sided directional differentiable. Finally, using some kind of duality relations, we present two different gradient representations at the end.

We divided the thesis into seven chapters. In the *second chapter*, we present mathematical basics and function spaces playing an essential role in this thesis. Beside the space of functions of bounded variation and some of its subspaces we give a short overview about time dependent function spaces. Here, we distinguish between Gelfand and Bochner measurability and integrability for time dependent vector valued functions. Furthermore, we introduce differentiability concepts and lists some theorems frequently appearing in the subsequent chapters.
In the *third chapter*, our focus lies on the transport equation. The first section treats existence and uniqueness of solutions for vector fields with spatial $BV$-regularity. In this part, we first present results of Ambrosio and others ([Amb04, Lel07, Cri07]) on existence and uniqueness of solutions for spatial $BV$-regular vector fields with the whole $\mathbb{R}^N$ as the spatial domain. We then use results on trace distributions ([CDS14b, CDS14a]) to conclude uniqueness of solutions on bounded spatio-temporal domains via extension from the results on unbounded domains. In [CDS14b, CDS14a], uniqueness of solutions on general domains is already shown but we have chosen this way to be able to extend any solution on bounded domains to general domains in the succeeding parts of the thesis. In the second part of chapter 3, we give two essential improvements on stability of the solution operator. These results improve existing stability statements given in [Amb04, Lel07, Cri07, DL89]. For the proofs, a compensated compactness result, given for Sobolev regular vector fields in [Mou16] is generalized to $BV$-regular vector fields.
In the *fourth chapter*, we consider optimal control problems with objective functions consisting of some tracking term and some regularization terms. The transport equation appears as one

of the side constraints. For the regularization part, various combinations of regularization terms are investigated. In the first section, our focus lies on time dependent vector valued functions with $BV(\Omega)$ as the codomain. We investigate the predual of $BV(\Omega)$ to clarify the relation between the weak*-topology naturally appearing in dual spaces and the weak*-topology usually used in $BV(\Omega)$. Furthermore, we show closedness of some set of time dependent vector fields with respect to some quite weak topology. With these results we are then able to show existence of minimizing points for our optimal control problems with different regularization terms in the second section.

In the *fifth chapter*, we turn to vector fields with spatial Lipschitz regularity. The chapter serves as a supporting chapter, providing us with (established) results being necessary in the successive chapters. In the first section, we present established theory about flows for Lipschitz regular vector fields. Furthermore, we extend results of [Che11, CL11] for solutions of the transport equation with $BV$-regular initial values. In the second section, our focus lies on general measure solutions of the inhomogeneous continuity equation. We present known existence and uniqueness results for this equation. Furthermore, we prove existence of solutions for the continuity equation in the case that the vector fields have less than Lipschitz regularity. Finally, we show a stability result for these measure solutions

In the *sixth chapter*, we investigate differentiability properties of the control-to-state operator $L$ as well as its composition $G$ with the tracking term of the objective functions of chapter 4. In the first section, we prove continuous Fréchet differentiability of $L$ as an operator mapping to $C([0,T],\mathcal{M}(\Omega))$. In the second section, we first smooth $L$ and prove that the smoothed control-to-state operator is again continuously Fréchet differentiable. In a second step, we then show that the composition of the smoothed $L$ with the tracking term is continuously Fréchet differentiable by using the chain rule for Fréchet differentiable functions. In the last section of this chapter, we introduce some further assumptions on the initial values $u_0$ and the functions $Y_k$. Then, we directly prove Fréchet differentiability of $G$ if some condition on the initial value and the given functions $Y_k$ is satisfied. If this condition does not hold, we finally prove that $G$ is still one-sided directional differentiable.

In the *seventh chapter*, we first apply results of the fifth chapter to obtain existence and uniqueness of solutions for the adjoint equation as well as for the backward transport equation. Then, we prove two relations depending on solutions of these equations. These relations will give us two representations of the gradient of $G$: one based on solutions of the adjoint equation and the other one based on solutions of the backward transport equation. In the second part of this chapter, we apply results of the previous chapters to obtain optimality conditions of first order for the optimal control problems in chapter 4. The setting will be stricter than in chapter 4 in order to ensure Fréchet differentiability of $G$. Due to the gradient representations of the first part, we obtain specific representations of these conditions.

# 2. Mathematical basics and function spaces

## 2.1. Function spaces

### 2.1.1. Spaces of continuous functions and approximation by mollifiers

Let $\mathcal{O} \subset \mathbb{R}^N$ be a subset. We set for $m \in \mathbb{N}_0$ the Banach space $C^m(\mathcal{O})$ as the set of functions

$$f : \mathcal{O} \to \mathbb{R},$$

such that $f$ is continuous, $D^\alpha f$ exists and is continuous for all multi-indices $\alpha \in \mathbb{N}_0^N$ with $|\alpha|_1 \leq m$ and is bounded with respect to $\|\cdot\|_{C^m(\mathcal{O})}$. The norm is given by

$$\|f\|_{C^m(\mathcal{O})} := \sum_{|\alpha| \leq m} \|D^\alpha f\|_\infty .$$

Here the sup-norm $\|\cdot\|_\infty$ is defined for a function $g : \mathcal{O} \to \mathbb{R}^k$ with $k \in \mathbb{N}$ as

$$\|g\|_\infty := \sup_{x \in \mathcal{O}} |g(x)| ,$$

where $|\cdot|$ denotes a fixed norm in $\mathbb{R}^k$. We will use the Euclidean norm in this thesis if we do not specify differently at some point. For $m = 0$ we write $C(\mathcal{O})$ instead of $C^0(\mathcal{O})$. Furthermore,

$$C^\infty(\mathcal{O}) := \bigcap_{m=0}^{\infty} C^m(\mathcal{O})$$

is the space of infinitely often continuously differentiable functions and $C_c^\infty(\mathcal{O})$ denotes the functions $f \in C^\infty(\mathcal{O})$ with compact support in $\mathcal{O}$. Analogously, for $m \in \mathbb{N}_0$ $C_c^m(\mathcal{O})$ denotes the functions in $C^m(\mathcal{O})$ with compact support in $\mathcal{O}$ and $C_0^m(\mathcal{O})$ is the closure of $C_c^\infty(\mathcal{O})$ with respect to $\|\cdot\|_{C^m(\mathcal{O})}$. Moreover, a function $f : \mathcal{O} \to \mathbb{R}$ is called Hölder continuous with Hölder exponent $\beta \in (0, 1]$ if there exists some $c > 0$ such that

$$|f(x) - f(y)| \leq c\,|x - y|^\beta$$

is satisfied for all $x, y \in \mathcal{O}$. We then set

$$|f|_{C^\beta(\mathcal{O})} := \sup_{\substack{x,y \in \mathcal{O}, \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta}$$

and define Hölder spaces as

$$C^{m,\beta}(\mathcal{O}) := \{ f \in C^m(\mathcal{O}) |\ D^\alpha f \text{ is Hölder continuous with exponent } \beta \text{ for all } |\alpha| \leq m \} ,$$

which are Banach spaces with norm

$$\|f\|_{C^{m,\beta}(\mathcal{O})} := \|f\|_{C^m(\mathcal{O})} + \max_{|\alpha| = m} |D^\alpha f|_{C^\beta(\mathcal{O})} .$$

In the case $\beta = 1$ the functions $f \in C^{0,1}(\mathcal{O})$ are called Lipschitz continuous and we will use the term $Lip(\mathcal{O})$ for the space $C^{0,1}(\mathcal{O})$.
In this thesis we use mollifiers a number of times. A mollifier is a function $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$ such that

$$\rho(x) \geq 0 \text{ for all } x \in \mathbb{R}^N, \quad \operatorname{supp}(\rho) \subset \mathbb{R}^N \text{ is compact and } \int\limits_{\mathbb{R}^N} \rho(x)\ dx = 1.$$

The standard mollifier we will use is given by

$$\rho(x) = \begin{cases} \rho_0 e^{\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $\rho_0 > 0$ is chosen such that

$$\int\limits_{\mathbb{R}^N} \rho(x)\ dx = 1.$$

For $\varepsilon > 0$ and some mollifier $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$ we set

$$\rho_\varepsilon(x) = \varepsilon^{-N} \rho(\varepsilon^{-1} x) \qquad \text{for } x \in \mathbb{R}^N.$$

Then $\rho_\varepsilon$ is a mollifier again and in the case of the standard mollifier $\operatorname{supp}(\rho_\varepsilon) = \overline{B_\varepsilon(0)}$. Now, for a function $f \in L^1(\mathcal{O})$ the convolution with some mollifier $\rho_\varepsilon$ is defined by

$$(\rho_\varepsilon * f)(x) := \int\limits_{\mathbb{R}^N} \rho_\varepsilon(x-y) f(y)\ dy,$$

where $f$ outside of $\mathcal{O}$ is set to zero. In this case, the following statement about approximations holds.

**Theorem 2.1.1** *Let $f : \mathcal{O} \to \mathbb{R}$ be a function and $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$. We set $f \equiv 0$ in $\mathbb{R}^N \backslash \mathcal{O}$. Then, the following holds:*

*(i) If $f \in L^p(\mathcal{O})$ with $1 \leq p < \infty$, then $(\rho_\varepsilon * f)|_{\mathcal{O}} \in L^p(\mathcal{O})$ with*

$$\|(\rho_\varepsilon * f)|_{\mathcal{O}}\|_{L^p(\mathcal{O})} \leq \|f\|_{L^p(\mathcal{O})} \quad \text{and} \quad (\rho_\varepsilon * f)|_{\mathcal{O}} \to f \quad \text{in } L^p(\mathcal{O}) \text{ as } \varepsilon \to 0.$$

*ii) If $f \in C(\mathcal{O})$, then for every compact set $K \subset \mathcal{O}$*

$$(\rho_\varepsilon * f)|_K \to f|_K \quad \text{in } C(K) \text{ as } \varepsilon \to 0.$$

**Proof:** The statement is proven in Lemma 4.22 in [Dob10].

□

### 2.1.2. Space of Radon measures and subspaces of the space of $BV$-functions

We start this subsection with a brief summary about Radon measures and their role as elements of a dual space. The results and definitions can be found in [AFP00].

Let $m \in \mathbb{N}$, $\mathcal{O} \subset \mathbb{R}^N$ be a set and let $\mathcal{B}(\mathcal{O})$ be the Borel $\sigma$-Algebra. We call a set function

$$\mu : \mathcal{B}(\mathcal{O}) \to \mathbb{R}^m$$

a measure if $\mu(\emptyset) = 0$ and if

$$\mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \mu(B_n)$$

is satisfied for any sequence of pairwise disjoint sets $(B_n) \subset \mathcal{B}(\mathcal{O})$. For a measure $\mu$ its total variation is then defined as

$$|\mu|(B) := \sup\left\{\sum_{n\in\mathbb{N}} |\mu(B_n)| \mid B_n \in \mathcal{B}(\mathcal{O}) \text{ pairwise disjoint, } B = \bigcup_{n\in\mathbb{N}} B_n\right\}.$$

Now, we call a set function $\mu : \mathcal{B}(\mathcal{O}) \to \mathbb{R}^m$ a Radon measure if $\mu$ is a measure on $(K, \mathcal{B}(K))$ for every compact set $K \subset \mathcal{O}$. If $\mu$ is a measure on $(\mathcal{O}, \mathcal{B}(\mathcal{O}))$, then we say that it is a finite Radon measure. The term $\mathcal{M}_{loc}(\mathcal{O})^m$ (resp. $\mathcal{M}(\mathcal{O})^m$) denotes the space of $\mathbb{R}^m$-valued Radon (resp. finite Radon) measures on $\mathcal{O}$. For these spaces we have the following Riesz representation: for every additive and bounded functional $L : C_0(\mathcal{O})^m \to \mathbb{R}$, there exists a unique finite Radon measure $\mu_L \in \mathcal{M}(\mathcal{O})^m$ such that

$$L(f) = \sum_{i=1}^{m} \int_{\mathcal{O}} f_i(x)\; d\mu_{L_i}(x) \quad \forall\; f \in C_0(\mathcal{O})^m.$$

In this case, we have that

$$\|L\| := \sup\left\{|L(f)| \mid f \in C_0(\mathcal{O})^m, \|f\|_{C(\mathcal{O})^m} \leq 1\right\} = |\mu_L|(\mathcal{O}) = \|\mu_L\|_{\mathcal{M}(\mathcal{O})^m}.$$

Analogously, for every linear and bounded functional $L : C_c(\mathcal{O})^m \to \mathbb{R}^m$, there exists a unique Radon measure $\mu_L \in \mathcal{M}_{loc}(\mathcal{O})^m$ such that

$$L(f) = \sum_{i=1}^{m} \int_{\mathcal{O}} f_i(x)\; d\mu_{L_i}(x) \quad \forall\; f \in C_c(\mathcal{O})^m.$$

The above result just states that the dual of the Banach space $C_0(\mathcal{O})^m$ can be identified with $\mathcal{M}(\mathcal{O})^m$ and the dual of the locally convex space $C_c(\mathcal{O})^m$ with $\mathcal{M}_{loc}(\mathcal{O})^m$. Now, let $\mu$ be a finite, $\mathbb{R}^m$-valued Radon measure and $(\mu_n) \subset \mathcal{M}(\mathcal{O})^m$ a sequence. We say that $(\mu_n)$ converges weakly* to $\mu$ in $\mathcal{M}(\mathcal{O})^m$ if

$$\int_{\mathcal{O}} f(x)\; d\mu_n(x) \to \int_{\mathcal{O}} f(x)\; d\mu(x) \quad \forall\; f \in C_0(\mathcal{O})^m.$$

In the same way, we say that a sequence $(\mu_n) \subset \mathcal{M}_{loc}(\mathcal{O})^m$ converges locally weakly* to $\mu \in \mathcal{M}_{loc}(\mathcal{O})^m$ if

$$\int_{\mathcal{O}} f(x)\; d\mu_n(x) \to \int_{\mathcal{O}} f(x)\; d\mu(x) \quad \forall\; f \in C_c(\mathcal{O})^m.$$

Finally, a bounded sequence $(\mu_n) \subset \mathcal{M}(\mathcal{O})^m$ has a weakly* convergent subsequence and the map $\mu \to |\mu|(\mathcal{O})$ is lower semi-continuous with respect to the weak* convergence.

In the second part of this subsection we introduce the Banach space of $BV$ functions on a set $\mathcal{O}$ and some of its subspaces. As above, the results and definitions can be found in [AFP00].

Let $\mathcal{O} \subset \mathbb{R}^N$ be an open, connected and bounded subset with Lipschitz boundary $\partial\mathcal{O}$. A function $f \in L^1(\mathcal{O})$ is a function of bounded variation if the distributional derivative $Df$ is given by a finite Radon measure in $\mathcal{O}$, i.e. if there exists $(\mu_1, \ldots, \mu_N)^\top \in \mathcal{M}(\mathcal{O})^N$ such that

$$\int_{\mathcal{O}} f(x)\partial_{x_i}\varphi(x)\ dx = -\int_{\mathcal{O}} \varphi(x)\ d\mu_i(x) \quad \forall\ \varphi \in C_c^\infty(\mathcal{O})$$

and for all $i = 1, \ldots, N$. We then set $\partial_{x_i} f := \mu_i$ for $i = 1, \ldots, N$. The vector space of all functions of bounded variation in $\mathcal{O}$ is denoted by $BV(\mathcal{O})$. For $f \in L^1(\mathcal{O})^m$ the variation $\mathrm{V}(f, \mathcal{O})$ is given by

$$\mathrm{V}(f, \mathcal{O}) := \left\{ \sum_{i=1}^m \int_{\mathcal{O}} f_i(x) \operatorname{div} \varphi_i(x)\ dx |\ \varphi \in C_c^1(\mathcal{O})^{m\times N}, \|\varphi\|_{C(\mathcal{O})^{m\times N}} \leq 1 \right\}$$

and the following properties hold:

(i) $f \in BV(\mathcal{O})^m$ if and only if $\mathrm{V}(f, \mathcal{O}) < \infty$,

(ii) $\mathrm{V}(f, \mathcal{O}) = |Df|(\mathcal{O})$ for any $f \in BV(\mathcal{O})^m$ and

(iii) $f \mapsto |Df|(\mathcal{O})$ is lower semi-continuous in $BV(\mathcal{O})^m$ with respect to the $L^1(\mathcal{O})^m$-topology.

Together with the norm

$$\|f\|_{BV(\mathcal{O})} = \|f\|_{L^1(\mathcal{O})^m} + |Df|(\mathcal{O})$$

$BV(\mathcal{O})^m$ is a Banach space. The derivative of a $BV$ function can be split into three components according to Lebesgue's decomposition theorem, i.e.

$$Df = D^a f + D^c f + D^j f,$$

where $D^a f$ denotes the absolutely continuous part of $Df$ with respect to the Lebesgue measure $\mathcal{L}^N$, $D^c f$ denotes the Cantor part and $D^j f$ denotes the jump part of $Df$. Then, a function $f \in BV(\mathcal{O})^m$ is a special function of bounded variation if $D^c f = 0$. The set of special functions of bounded variation is denoted by $SBV(\mathcal{O})^m$ and is a subspace of $BV(\mathcal{O})^m$.
For our purposes the norm topology is too strong and thus, as in [AFP00], we introduce two weaker topologies leading to weaker terms of convergence.

**Definition 2.1.2 (Weak* convergence)** *A sequence $(f_n) \subset BV(\mathcal{O})^m$ converges weakly* to some $f \in BV(\mathcal{O})^m$ if $(f_n)$ converges to $f$ in $L^1(\mathcal{O})^m$ and $(Df_n)$ converges weakly* to $Df$ in $\mathcal{M}(\mathcal{O})^{m\times N}$, i.e.*

$$\lim_{n\to\infty} \sum_{j=1}^m \int_{\mathcal{O}} \varphi_j(x)\ d\partial_{x_i} f_{n,j}(x) = \sum_{j=1}^m \int_{\mathcal{O}} \varphi_j(x)\ d\partial_{x_i} f_j(x) \qquad \text{for all } \varphi \in C_0(\Omega)^m$$

*and for all $i = 1, \ldots, N$.*

In the literature the term weak* convergence for this kind of convergence is commonly used but in general it is not equal to the usual weak* topology in functional analysis if we consider $BV(\mathcal{O})$ as a dual space (see Remark 3.12 in [AFP00]). However, for sufficiently regular domains $\mathcal{O}$ these two topologies coincides and we will have a closer look in Chapter 4 for which regularity of the domain this case occurs.

**Definition 2.1.3 (Strict convergence)** *A sequence $(f_n) \subset BV(\mathcal{O})^m$ converges strictly to some $f \in BV(\mathcal{O})^m$ if $(f_n)$ converges to $f$ in $L^1(\mathcal{O})^m$ and $(|Df_n|(\mathcal{O}))$ converges to $|Df|(\mathcal{O})$ as $n \to \infty$.*

Due to Proposition 3.13 in [AFP00] strict convergence implies weak* convergence. Now, Theorem 3.87 and Theorem 3.88 in [AFP00] yield that for $m \in \mathbb{N}$, there exists a linear mapping

$$T : BV(\mathcal{O})^m \to L^1\left(\partial\mathcal{O}, \mathcal{H}^{N-1} \llcorner \partial\mathcal{O}\right)^m$$

which is continuous with respect to the topology induced by strict convergence. Next, we consider the set

$$BV_0(\mathcal{O})^m := \{f \in BV(\mathcal{O})^m |\ T(f) = 0 \text{ on } \partial\mathcal{O}\}.$$

Obviously, this set is a vector space and it is closed with respect to the strict topology since it is the preimage of $\{0\}$. As the strict topology is weaker than the norm topology, $BV_0(\mathcal{O})^m$ is closed in the norm topology and hence a Banach space. Now, for a function $f \in BV(\mathcal{O})^N$ we use the notation

$$\operatorname{Div} f := \sum_{k=1}^{N} \frac{\partial}{\partial x_k} f_k.$$

If $\operatorname{Div} f$ is absolutely continuous with respect to some measure $\mu \in \mathcal{M}(\mathcal{O})$, i.e. $\operatorname{Div} f \ll \mu$, we denote its density function by $\operatorname{div} f$, i.e.

$$\operatorname{Div} f = \operatorname{div} f \mu.$$

For $BV$-functions, we consider the following subspace

$$BV_{\mathrm{div}}(\mathcal{O})^N := \left\{f \in BV(\mathcal{O})^N |\ \operatorname{Div} f \ll \mathcal{L}^N\right\}$$

and we obtain the result:

**Theorem 2.1.4** *$BV_{\mathrm{div}}(\mathcal{O})^N$ is a closed subspace of $BV(\mathcal{O})^N$ with respect to the norm topology and thus it is a Banach space.*

**Proof:** Obviously, $BV_{\mathrm{div}}(\mathcal{O})^N$ is a subspace of $BV(\mathcal{O})^N$. Now, let $(f_n) \subset BV_{\mathrm{div}}(\mathcal{O})^N$ be a sequence, converging to $f \in BV(\mathcal{O})^N$, i.e.

$$|Df_n - Df|(\mathcal{O}) \to 0 \qquad \text{as } n \to \infty.$$

Then, the total variation is given by

$$|Df_n - Df|(\mathcal{O}) := \sup\left\{\sum_j \|Df_n(B_j) - Df(B_j)\|_M \ |\ B_j \subset \mathcal{O} \text{ is a countable partition of } \mathcal{O}\right\},$$

where $\|\cdot\|_M$ denotes a fixed matrix norm in $\mathbb{R}^{N\times N}$. Hence, since for all $B \in \mathcal{B}(\mathcal{O})$, $B\cup B^c = \mathcal{O}$ is a partition of $\mathcal{O}$, we obtain that for all $i = 1, \dots, N$

$$|(\partial_{x_i} f_{n,i} - \partial_{x_i} f_i)(B)| \to 0 \qquad \forall\ B \in \mathcal{B}(\mathcal{O}).$$

Thus,

$$|(\operatorname{Div} f_n - \operatorname{Div} f)(B)| \to 0 \qquad \forall\ B \in \mathcal{B}(\mathcal{O}).$$

As $\operatorname{Div} f_n \ll \mathcal{L}^N$, there are functions $\operatorname{div} f_n \in L^1(\mathcal{O})$ such that

$$\int_B \operatorname{div} f_n(x)\ dx \to (\operatorname{Div} f)(B) \qquad \forall\ B \in \mathcal{B}(\mathcal{O}).$$

Using Theorem 10 in Chapter 5 of [AS85] yields that there exists some function $g \in L^1(\mathcal{O})$ such that

$$\int_B \operatorname{div} f_n(x)\ dx \to \int_B g(x)\ dx \qquad \forall\ B \in \mathcal{B}(\mathcal{O}).$$

Hence, $\operatorname{Div} f$ is absolutely continuous with respect to the Lebesgue measure with density function $g$. Thus, $f \in BV_{\mathrm{div}}(\mathcal{O})^N$ and $BV_{\mathrm{div}}(\mathcal{O})^N$ is a Banach space.

□

### 2.1.3. Time dependent function spaces

In this thesis, we work with functions defined on bounded time intervals $I \subset \mathbb{R}$ with values in some Banach space $X$. For these functions we use the term $L^p(I, X)$ with $p \in [1, \infty]$ to describe the function space wherein those lie. Depending on the kind of Banach space $X$, the term has different meanings:

(i) In the case $X$ is separable, the term $L^p(I, X)$ means the space of Bochner integrable functions with values in $X$. Typical Banach spaces which will appear are

$$X = C^m(\mathcal{O}),\ L^p(\mathcal{O}),\ W^{1,p}(\mathcal{O}), \dots \text{ for } m \in \mathbb{N}_0 \text{ and } p < \infty.$$

(ii) In the case $X$ is a non-separable dual space, the term $L^p(I, X)$ denotes the space of Gelfand integrable functions with values in $X$. This case will appear for the Banach spaces $X = BV(\mathcal{O})$, $X = \mathcal{M}(\mathcal{O})$ and $X = W^{1,\infty}(\mathcal{O})$.

In the following, we give a brief summary about Bochner and Gelfand integrability of time dependent functions. These results can be found in [AB06, Sch13, Emm04, Sus08].
Let $X$ be a Banach space, $I \subset \mathbb{R}$ a bounded interval and let $f : I \to X$ be a function. Then $f$ is called Bochner measurable if there exists a sequence of simple functions $(f_n)$ such that

$$f_n(t) \to f(t) \quad \text{in } X \quad \text{for } \mathcal{L}^1\text{-almost all } t \in I$$

as $n \to \infty$. A simple function is a function $g : I \to X$ with finitely many different values in $X$, which are defined on Lebesgue measurable subsets of $I$. Furthermore, $f$ is called weak measurable if for any $x' \in X'$ the function

$$t \mapsto \langle x', f(t) \rangle_{X',X}$$

is Lebesgue measurable. If $X$ is separable, these two definitions are equivalent. Now, a Bochner measurable function $f$ is called Bochner integrable if for some sequence of simple functions $(f_n)$ being almost everywhere pointwise convergent to $f$ and if for every $\varepsilon > 0$ there exists some $N(\varepsilon) \in \mathbb{N}$ such that

$$\left( \int\limits_I \|f_n(t) - f_m(t)\|_X \; dt \right) \leq \varepsilon$$

for all $m, n \geq N(\varepsilon)$. In this case, the integral on a Lebesgue measurable set $B \subset I$ is defined as

$$\int\limits_B f(t) \; dt := \lim_{n\to\infty} \int\limits_I f_n(t)\chi_B(t) \; dt = \lim_{n\to\infty} \sum_{k=1}^{K(n)} f_{n,k}\mathcal{L}^1(\{t| \; t \in B \cap B_{n,k}\}),$$

where $B_{n,k} \subset I$ denotes the $K(n) \in \mathbb{N}$ disjoint, Lebesgue measurable sets where $f_n$ is constant with value $f_{n,k} \in X$. Then, a Bochner measurable function $f$ is Bochner integrable if and only if $t \mapsto \|f(t)\|_X$ is Lebesgue integrable and we have

$$\left\| \int_B f(t) \; dt \right\|_X \leq \int\limits_B \|f(t)\|_X \; dt$$

for any measurable subset $B \subset I$. For $p \in [1, \infty)$ we denote by $L^p(I, X)$ the space of equivalence classes of Bochner integrable functions $f : I \to X$ such that

$$\int\limits_I \|f(t)\|_X^p \; dt < \infty.$$

For $p = \infty$ we define the space $L^\infty(I, X)$ as the space of Bochner measurable functions $f : I \to X$ such that $t \mapsto \|f(t)\|_X$ is essentially bounded in $I$. Now, let $X$ be a non-separable dual Banach space. Then, a function $f : I \to X$ is called weak* measurable if for any $p \in P$ with predual Banach space $P$ of $X$ the function

$$t \mapsto \langle f(t), p\rangle_{X,P}$$

is Lebesgue measurable. A weak* measurable function $f$ is Gelfand integrable over some measurable set $B \subset I$ if there exists some $x_B \in X$ such that

$$\langle x_B, p\rangle = \int\limits_B \langle f(t), p\rangle \; dt$$

is satisfied for all $p \in P$. The unique vector $x_B$ is called the Gelfand integral of $f$ over the set $B$ and is denoted by $x_B = \int_B f dt$. We call the function $f$ Gelfand integrable if the integral exists for all measurable subsets $B \subset I$. If a function $f : I \to X$ has the property that $t \mapsto \langle f(t), p\rangle$ is Lebesgue integrable for any $p \in P$, then $f$ is Gelfand integrable. In this case, we obtain that

$$t \mapsto \sup_{\substack{p\in P,\\ \|p\|\leq 1}} \langle f(t), p\rangle = \|f(t)\|_X$$

is measurable and if $t \mapsto \|f(t)\|_X$ is additionally integrable, then we have that

$$\|x_B\|_X \leq \int\limits_B \|f(t)\|_X \ dt.$$

For non-separable dual Banach spaces $X$ we set for $p \in [1, \infty)$ the space $L^p(I, X)$ as the space of Gelfand integrable functions $f : I \to X$ such that

$$\int\limits_I \|f(t)\|_X^p \ dt < \infty.$$

Analogously as before, we define the space $L^\infty(I, X)$ as the space of weak* measurable functions $f : I \to X$ such that $t \mapsto \|f(t)\|_X$ is essentially bounded in $I$. Finally, for a reflexive and separable Banach space $Y$ with $X \hookrightarrow Y$ we have that

$$L^p(I, X) \hookrightarrow L^p(I, Y)$$

for any $1 \leq p < \infty$ if the identity map

$$\mathrm{id} : X \to Y \simeq (Y')'$$

is weak*-weak* continuous, i.e. a Gelfand integrable function is Bochner integrable if it is considered as a mapping with codomain $Y$: due to Lemma 7.37 in [Sus08] any function $f$ in $L^p(I, X)$ represents a Gelfand integrable function from $I$ into $(Y')'$. Consequently, for any $y' \in Y'$ we obtain that

$$t \mapsto \left\langle f(t), y' \right\rangle_{(Y')', Y'} = \left\langle y', f(t) \right\rangle_{Y', Y}$$

is measurable, i.e $f$ is weak Bochner measurable. Since $Y$ is separable and $t \mapsto \|f(t)\|_X$ is integrable, we obtain that $f \in L^p(I, Y)$. In the thesis, this relation will be applied in the following cases, if $\mathcal{O} \subset \mathbb{R}^N$ is a bounded set:

(i) $BV(\mathcal{O}) \hookrightarrow L^p(\mathcal{O})$ with $1 < p \leq \frac{N}{N-1}$,

(ii) $W^{1,\infty}(\mathcal{O}) \hookrightarrow W^{1,p}(\mathcal{O})$ for $1 \leq p < \infty$,

(iii) $L^\infty(\mathcal{O}) \hookrightarrow L^p(\mathcal{O})$ for $1 \leq p < \infty$.

In all three cases, the identity map is obviously weak*-weak* continuous.

## 2.2. Theorems and concepts for vector space valued functions

### 2.2.1. Differentiability of vector space valued functions

Main statements in this thesis investigate various differentiability properties of vector space valued functions. We give a short overview of differentiability concepts used in this work. The definitions and results can be found in chapter 1, §3 in [Lan95] and in chapter 40, §4.10 in [Zei85].

**Definition 2.2.1** *Let $X$ and $Y$ be two topological vector spaces, $U \subset X$ a neighborhood of zero and $\varphi : U \to Y$ some mapping. The function $\varphi$ is called tangent to zero if for each neighborhood $W \subset Y$ of $0$ there exists a neighborhood $V \subset X$ of zero such that*

$$\varphi(tV) \subset o(t)W$$

*for some function $o : (-a, a) \to [0, \infty)$ with some $a > 0$ and $\frac{o(t)}{t} \to 0$ as $t \to 0$.*

If $X$ and $Y$ are normed spaces, then the above definition is equivalent to

$$|\varphi(x)| \leq |x|\psi(x),$$

where $\psi : X \to \mathbb{R}$ is a function with $\psi(x) \to 0$ as $|x| \to 0$.

**Definition 2.2.2 (Fréchet differentiability in topological vector spaces)** *Let $X$ and $Y$ be two topological vector spaces and $U \subset X$ an open set. Furthermore, let $f : U \to Y$ be some continuous map. We say that $f$ is Fréchet differentiable at $x_0 \in U$ if there exists a continuous linear map $T : X \to Y$ such that*

$$f(x_0 + y) = f(x_0) + Ty + \varphi(y)$$

*holds for some neighborhood $V \subset X$ of zero with $x_0 + V \subset U$ and $\varphi : V \to Y$ is tangent to zero. In this case we set $Df(x_0) = T$. If $f$ is Fréchet differentiable at every point $x \in U$, then we say that $f$ is Fréchet differentiable.*

For compositions $f \circ g$ of functions $g : U \subset X \to Y$ and $f : V \subset Y \to Z$ with $g(U) \subset V$ and topological vector spaces $X, Y$ and $Z$, the chain rule holds: if $g$ is Fréchet differentiable in $x_0 \in U$ and $f$ is Fréchet differentiable in $g(x_0)$, then $f(g(x_0))$ is Fréchet differentiable in $x_0$ with

$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0).$$

Beside Fréchet differentiability, one-sided directional differentiability will play some role.

**Definition 2.2.3 (One-sided directional differentiability)** *Let $X$ be some normed space, $U \subset X$ an open set and $f : U \to \mathbb{R}$ some continuous function. We say that $f$ is one-sided right directional differentiable at some $x_0 \in U$ in direction $\tilde{x} \in X$ if*

$$\delta_+ f(x_0, \tilde{x}) = \lim_{t \to 0^+} \frac{f(x_0 + t\tilde{x}) - f(x_0)}{t}$$

*exists. In the same way, we say that $f$ is one-sided left directional differentiable at some $x_0 \in U$ in direction $\tilde{x} \in X$ if*

$$\delta_- f(x_0, \tilde{x}) = \lim_{t \to 0^-} \frac{f(x_0 + t\tilde{x}) - f(x_0)}{t}$$

*exists.*

### 2.2.2. General theorems

In this subsection we repeat some well-known results which will be used several times in this thesis. We start with Grönwall's lemma.

**Lemma 2.2.4 (Grönwall's lemma)** *Let $a, b \in [0, \infty]$ with $a < b$ and let $f$ and $g$ be elements of $L^\infty((a, b))$ as well as $h \in L^1((a, b))$ such that $h(t) \geq 0$ for almost all $t \in (a, b)$. Assume that the inequality*

$$f(t) \leq g(t) + \int_a^t h(s)f(s)\, ds$$

*holds for almost all $t \in (a,b)$. Then*

$$f(t) \leq g(t) + \int_a^t h(s)g(s)e^{\int_s^t h(r)dr}\, ds$$

*for almost all $t \in (a,b)$. In addition, if $g$ is monotonically increasing and continuous in $(a,b)$, then*

$$f(t) \leq e^{\int_a^t h(s)ds} g(t).$$

**Proof:** The statement is proven in Lemma 7.3.1 in [Emm04].

□

Our next theorem is Lebesgue's dominated convergence theorem which we present for general Bochner spaces. The classical statement for functions with values in finite dimensional Banach spaces is a special case of this statement.

**Theorem 2.2.5 (Lebesgue's dominated convergence theorem)** *Let $1 \leq p < \infty$, $X$ be a separable Banach space and denote $I \subset \mathbb{R}$ an open, bounded interval. If addition, let $(f_n) \subset L^p(I,X)$ and $(g_n) \subset L^p(I,\mathbb{R})$ be sequences and $f : I \to X$ as well as $g : I \to \mathbb{R}$ two functions. Then, if the following holds*

*(i)* $f_n(t) \to f(t)$ *in $X$ for almost all $t \in I$,*

*(ii)* $\|f_n(t)\|_X \leq g_n(t)$ *for almost all $t \in I$ and*

*(iii)* $g_n \to g$ *in $L^p(I,\mathbb{R})$,*

*we have that $f \in L^p(I,X)$ and*

$$f_n \to f \quad \text{in } L^p(I,X).$$

**Proof:** The proof can be found in Theorem 10.4 in [Sch13].

□

**Theorem 2.2.6 (Arzelà-Ascoli for locally convex Hausdorff spaces)** *Let $X$ be a compact space, $Y$ be a locally convex Hausdorff space. Then a closed subset $F$ of $C(X,Y)$ with respect to the compact-open topology is compact if and only if $F$ is an equicontinuous family of mappings and the set $\{f(x)|\ f \in F\} \subset Y$ has a compact closure for every $x \in X$.*

**Proof:** The theorem is a consequence of Theorem 3.4.20 in [Eng89].

□

In the thesis we will apply this theorem to the case where $X = [0,T]$ and $Y$ is a locally convex space of the form

$$Y = (Z, \mathcal{P}_{Z'}) \quad \text{or} \quad Y = (Z', \mathcal{P}_Z)$$

where $Z$ denotes some Banach space and

(i) $\mathcal{P}_{Z'} = \{p_{z'}|\ p_{z'}(z) = |\langle z', z\rangle|, z \in Z, z' \in Z'\}$

(ii) $\mathcal{P}_Z = \{p_z|\ p_z(z') = |\langle z', z\rangle|, z \in Z, z' \in Z'\}$

denote the sets of seminorms defining the locally convex spaces. These locally convex spaces are obviously Hausdorff spaces. Furthermore, we know that in both cases a subset $F \subset C([0,T],Y)$ is equicontinuous if the sets $\{p \circ f |\ f \in F\}$ are equicontinuous for each $p \in \mathcal{P}_Z$ or $p \in \mathcal{P}_{Z'}$. Finally, if a sequence $(f_n) \subset C([0,T],Y)$ converges to $f \in C([0,T],Y)$ in the compact-open topology, then

$$p \circ f_n \to p \circ f \quad \text{in } C([0,T]) \quad \text{as } n \to \infty$$

for each $p \in \mathcal{P}_Z$ or $p \in \mathcal{P}_{Z'}$.

# 3. Well-posedness of transport equation

In this chapter, we consider the transport equation

$$\begin{aligned} \partial_t u + b \cdot \nabla u &= 0 \qquad \text{in } (0,T) \times \mathcal{O}, \\ u(0,\cdot) &= u_0 \qquad \text{in } \mathcal{O}, \end{aligned}$$

on the spatio-temporal domain $(0,T) \times \mathcal{O} \subset \mathbb{R} \times \mathbb{R}^N$ for some given functions $b$ and $u_0$. In the first section we present the uniqueness theory for solutions of the transport equations with vector fields having spatial $BV$-regularity. In this generality, the theory was first shown by Ambrosio in his groundbreaking work [Amb04]. It is based on the concept of renormalized solutions saying that any composition of a solution with some $C^1(\mathbb{R})$ function is again a solution of the same transport equation. This concept was developed by DiPerna and Lions in [DL89] where they proved uniqueness of solutions of the transport equation for vector fields with spatial Sobolev regularity. In the following, we give a detailed summary of this theory where we also use a slightly different definition of renormalization as Ambrosio and others ([Amb04, Cri07, CDS14b]) did. Our definition, which is used by De Lellis in [Lel07] and originally by DiPerna and Lions in [DL89] also includes the demand that the composition $\beta(u)$ of any solution $u$ with any $C^1(\mathbb{R})$ function $\beta$ has to be equal to some composed initial value at $t = 0$, i.e. if $u_0$ is the initial value for $u$ then $\beta(u)$ has to be equal to $\beta(u_0)$ at $t = 0$. Ambrosio as well as DiPerna and Lions developed their theories for domains where the spatial component is the whole $\mathbb{R}^N$. Our requirement is to have a corresponding theory for domains with general spatial subsets $\mathcal{O} \subset \mathbb{R}^N$. This theory is provided by Crippa et al. in [CDS14b, CDS14a] where they showed the results directly on general domains. Our approach here is different. We show that, under our assumptions, solutions on general domains can be extended to the domain with spatial component $\mathbb{R}^N$. Then, we use the existing theory of Ambrosio for this case to prove uniqueness of solutions which leads to uniqueness of solutions on general domains. For extending solutions to $\mathbb{R}^N$ in the spatial variable we apply results about trace distributions of specific functions developed by Ambrosio and others in [ACM07] and also appearing in [CDS14b]. Beside uniqueness, the existing theory provides us with a first stability result. In the second part, we improve this stability result with two further stability theorems. In the proof of the first stability theorem we use Arzelà-Ascoli and the renormalization property to show strong convergence of solutions in $C([0,T], L^p(\Omega))$ for any $p < \infty$ under weak convergence of the vector fields in $L^1((0,T) \times \Omega)^N$ and some further assumptions on the divergence as well as on the initial data. This result can already be found in [DL89] for vector fields satisfying some condition on uniform translation in the spatial argument. However, this result does not appear in [Amb04, Cri07] and we will show that our additional assumptions on the vector fields yield that the condition of DiPerna and Lions is fulfilled. In the second stability theorem we further weaken the assumptions of the first theorem using an idea of DiPerna and Lions in [DL89] to prove the statement. In this proof the statement of the first stability theorem will be needed. Again, the result of the second statement already appears in [DL89] under Sobolev regularity assumptions but does not exist for vector fields with spatial $BV$-regularity.

## 3.1. Existence and uniqueness of solutions of the transport equation

### 3.1.1. Existence of solutions

Let $T > 0$ and $\mathcal{O} \subset \mathbb{R}^N$ be an open set. In the beginning, we will consider the following problem before we concentrate on a bounded spatial domain $\Omega \subset \mathbb{R}^N$:

$$\begin{aligned} \partial_t u + b \cdot \nabla u &= 0 \qquad \text{in } (0,T) \times \mathcal{O}, \\ u(0,\cdot) &= u_0 \qquad \text{in } \mathcal{O}. \end{aligned} \tag{3.1}$$

We look for solutions solving the above partial differential equation in a distributional sense. Before we can start investigating the above problem we have to clarify what is meant by $b \cdot \nabla u$ when the vector field $b$ is not smooth: if $u \in L^\infty((0,T) \times \mathcal{O})$ with

$$b \in L^1((0,T) \times \mathcal{O})^N \quad \text{and} \quad \operatorname{div} b \in L^1((0,T) \times \mathcal{O}),$$

we define the distribution $b \cdot \nabla u \in \mathcal{D}'(\mathbb{R} \times \mathcal{O})$ by

$$\langle b \cdot \nabla u, \varphi \rangle = -\langle bu, \nabla \varphi \rangle - \langle u \operatorname{div} b, \varphi \rangle \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R} \times \mathcal{O})$$

where we extend the involved functions in the temporal domain by zero. Thus, we have the following general definition of weak solution of (3.1):

**Definition 3.1.1 (Weak solution)** *Let $b \in L^1((0,T) \times \mathcal{O})^N$, $u_0 \in L^\infty(\mathcal{O})$ and let the distributional divergence* $\operatorname{div} b \in L^1((0,T) \times \mathcal{O})$. *Then, we call a function*

$$u \in C([0,T], L^\infty(\mathcal{O}) - w^*)$$

*a weak solution of* (3.1)*, if the following equation is satisfied*

$$\int_0^T \int_{\mathcal{O}} u(t,x) \left( \varphi_t(t,x) + b(t,x) \cdot \nabla \varphi(t,x) + \varphi(t,x) \operatorname{div} b(t,x) \right) \, dxdt = -\int_{\mathcal{O}} u_0(x) \varphi(0,x) \, dx$$

*for all $\varphi \in C_c^\infty([0,T) \times \mathcal{O})$.*

We start with the existence of weak solutions for (3.1) in the case $\mathcal{O} = \mathbb{R}^N$.

**Theorem 3.1.2 (Existence of weak solutions in $\mathbb{R}^N$)** *Let $b \in L^1\left((0,T) \times \mathbb{R}^N\right)^N$ with $\operatorname{div} b \in L^1\left((0,T) \times \mathbb{R}^N\right)$ and let $u_0 \in L^\infty\left(\mathbb{R}^N\right)$. Then there exists a weak solution $u \in C([0,T], L^\infty\left(\mathbb{R}^N\right) - w^*)$ to* (3.1) *in the case $\mathcal{O} = \mathbb{R}^N$ with*

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)}$$

*for all $t \in [0,T]$.*

**Proof:** The theorem is proven in [Cri07]: the existence of a solution $u \in L^\infty\left((0,T) \times \mathbb{R}^N\right)$ is shown in Theorem 2.2.3, the bound on the $L^\infty$-norm of $u$ is a consequence of the maximum principle and Remark 2.2.2 shows that $u$ can be seen as an element of $C([0,T], L^\infty(\mathbb{R}^N) - w^*)$. □

Now we consider an open and bounded subset $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. Our next aim is to show existence of solutions on the domain $(0,T) \times \Omega$. Extending the involved functions by zero to the entire $\mathbb{R}^N$ and using the above existence result is not possible without further ado since the divergence of the vector field must not be an element of $L^1\left((0,T) \times \mathbb{R}^N\right)$ anymore. But if we restrict us to vector fields $b \in L^1((0,T,BV_0(\Omega))^N$, then this kind of extension yields Lebesgue integrable divergence. For this case we have the following result.

**Lemma 3.1.3** *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary $\partial\mathcal{O}$ and let $f \in BV_0(\mathcal{O})^N$ with $\operatorname{div} f \in L^1(\mathcal{O})$. Then $f$ can be extended to a function in $BV\left(\mathbb{R}^N\right)^N$ with $\operatorname{div} f \in L^1\left(\mathbb{R}^N\right)$.*

**Proof:** Since the zero function in $\mathbb{R}^N \backslash \overline{\mathcal{O}}$ belongs to $BV\left(\mathbb{R}^N \backslash \overline{\mathcal{O}}\right)$, Theorem 1 in chapter 5.4 in [EG92] yields that $\bar{f}$, given by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{O}, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash \mathcal{O}, \end{cases}$$

lies in $BV\left(\mathbb{R}^N\right)^N$. If we define the function

$$h(x) = \begin{cases} \operatorname{div} f(x) & \text{if } x \in \mathcal{O}, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash \mathcal{O} \end{cases}$$

then $h \in L^1\left(\mathbb{R}^N\right)$. Hence, by using Theorem I in subsection 5.3 of [EG92], we obtain the following: for $\varphi \in C_c^\infty\left(\mathbb{R}^N\right)$ and $i \in \{1, \ldots, N\}$

$$\int_{\mathbb{R}^N} \bar{f}_i(x) \partial_{x_i} \varphi(x)\ dx = \int_{\mathcal{O}} f_i(x) \partial_{x_i} \varphi(x)\ dx = -\int_{\mathcal{O}} \varphi(x)\ d(\partial_{x_i} f_i)(x),$$

and thus

$$\begin{aligned} \int_{\mathbb{R}^N} \bar{f}(x) \cdot \nabla \varphi(x)\ dx &= -\int_{\mathcal{O}} \varphi(x)\ d(\operatorname{Div} f)(x) = -\int_{\mathcal{O}} \varphi(x) \operatorname{div} f(x)\ dx \\ &= -\int_{\mathbb{R}^N} \varphi(x) h(x)\ dx. \end{aligned}$$

Hence $\operatorname{Div} \bar{f} = h\mathcal{L}^1$ and thus is absolutely continuous with density $h \in L^1\left(\mathbb{R}^N\right)$.

□

With the above lemma, we are now able to show the existence of solutions on $(0,T) \times \Omega)$.

**Theorem 3.1.4 (Existence of weak solutions on bounded domains)** *Let $u_0 \in L^\infty(\Omega)$ and let $b \in L^1((0,T), BV_0(\Omega))^N$ with $\operatorname{div} b \in L^1((0,T) \times \Omega)$. Then there exists a weak solution $u \in C([0,T], L^\infty(\Omega) - w^*)$ of (3.1) in the case $\mathcal{O} = \Omega$.*

**Proof:** We use Lemma 3.1.3 to extend $b(t,\cdot)$ in the spatial variable by 0 to the entire $\mathbb{R}^N$ for almost all $t \in (0,T)$. Analogously, we extend $u_0$ to $\mathbb{R}^N$ by 0. Obviously, $b \in L^1\left((0,T,BV(\mathbb{R}^N)\right)^N \subset L^1\left((0,T) \times \mathbb{R}^N\right)^N$ with $\operatorname{div} b \in L^1\left((0,T) \times \mathbb{R}^N\right)$. Thus, Theorem 3.1.2 yields a weak solution $u \in C([0,T], L^\infty\left(\mathbb{R}^N\right) - w^*)$ of the transport equation with vector field $b$ and initial data $u_0 \in L^\infty\left(\mathbb{R}^N\right)$. As

$$C_c^\infty([0,T) \times \Omega) \subset C_c^\infty\left([0,T) \times \mathbb{R}^N\right),$$

the restriction of $u$ on $(0,T)\times\Omega$ is a weak solution of (3.1) in the case $\mathcal{O}=\Omega$.

□

**Remark 3.1.5** *In the above case, the solution $u \in C\left([0,T], L^\infty\left(\mathbb{R}^N\right) - w^*\right)$ is equal to zero on $(0,T)\times\mathbb{R}^N\backslash\overline{\Omega}$: let $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^N\backslash\overline{\Omega})$. Then*

$$0 = \int_0^T \int_{\mathbb{R}^N\backslash\overline{\Omega}} u(t,x)(\partial_t\varphi(t,x) + b(t,x)\cdot\nabla\varphi(t,x) + \varphi(t,x)\operatorname{div} b(t,x))\ dxdt + \int_{\mathbb{R}^N\backslash\overline{\Omega}} u_0(x)\varphi(0,x)\ dx$$

$$= \int_0^T \int_{\mathbb{R}^N\backslash\overline{\Omega}} u(t,x)\partial_t\varphi(t,x)\ dxdt.$$

*Thus, the weak derivative of $u|_{(0,T)\times\mathbb{R}^N\backslash\overline{\Omega}}$ in $t$ exists and is equal to zero. Hence, $u|_{(0,T)\times\mathbb{R}^N\backslash\overline{\Omega}}$ must be constant with respect to the time $t$, i.e.*

$$u|_{(0,T)\times\mathbb{R}^N\backslash\overline{\Omega}}(t,\cdot) = v \in L^\infty\left(\mathbb{R}^N\backslash\overline{\Omega}\right) \qquad \textit{for all } t\in(0,T).$$

*Since $u \in C\left([0,T], L^\infty\left(\mathbb{R}^N\right) - w^*\right)$, we obtain for all $\psi \in L^1\left(\mathbb{R}^N\backslash\overline{\Omega}\right)$:*

$$\int_{\mathbb{R}^N\backslash\overline{\Omega}} v(x)\psi(x)\ dx = \int_{\mathbb{R}^N\backslash\overline{\Omega}} u(0,x)\psi(x)\ dx = \int_{\mathbb{R}^N\backslash\overline{\Omega}} u_0(x)\psi(x)\ dx = 0.$$

*Hence $v\equiv 0$ and thus $u|_{(0,T)\times\mathbb{R}^N\backslash\overline{\Omega}} \equiv 0$.*

**Remark 3.1.6** *Using Remark* (3.1.5)*, we obtain for a solution $u \in C([0,T], L^\infty(\Omega) - w^*)$ with initial value $u_0 \in L^\infty(\Omega)$:*

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} = \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)} = \|u_0\|_{L^\infty(\Omega)}$$

*for all $t\in[0,T]$.*

### 3.1.2. Renormalization property and uniqueness

Existence results can usually be obtained with quite weak assumptions. To gain uniqueness of solutions, special concepts are needed which solutions of the PDE have to satisfy. For the transport equation, DiPerna and Lions developed in [DL89] the concept of renormalization for proving uniqueness of weak solutions. This concept is based on a feature of smooth solutions: a smooth solution $u$ is constant on specific characteristics $X(t,0,x)$ given by the flow $X$ of the vector field $b$, i.e. the value $u_0(x)$ of the initial function $u_0$ at a spatial point $x$ is just transported along its characteristic $X(t,0,x)$ and thus it does not change. Therefore,

$$u(t,\cdot) = u_0(X(0,t,\cdot)) \quad \text{for all } t\ge 0.$$

In chapter 5, we will give a precise definition of the flow together with some of its properties. Now, the composition with any $\beta\in C^1(\mathbb{R})$ yields that

$$\beta(u(t,\cdot)) = \beta(u_0(X(0,t,\cdot))) = (\beta(u_0))(X(0,t,\cdot)) \quad \text{for all } t\ge 0$$

and thus, the composition is again a solution of the transport equation with initial value $\beta(u_0)$. This property is used in the concept of renormalization for proving uniqueness.

**Definition 3.1.7 (Renormalized solution)** *Let* $u_0 \in L^\infty(\Omega)$*,* $b \in L^1((0,T)\times\Omega)^N$ *with* $\operatorname{div} b \in L^1((0,T)\times\Omega)$ *and let* $u \in C([0,T], L^\infty(\Omega) - w^*)$ *be a solution of* (3.1)*. We call* $u$ *a renormalized solution if for any* $C^1$*-function* $\beta : \mathbb{R} \to \mathbb{R}$ *the function* $\beta(u)$ *is a weak solution of*

$$\begin{aligned} \partial_t \beta(u) + b \cdot \nabla \beta(u) &= 0 \qquad && \text{in } (0,T)\times\Omega, \\ \beta(u(0,\cdot)) &= \beta(u_0) && \text{in } \Omega. \end{aligned} \tag{3.2}$$

We are not only interested in solutions which can be renormalized but in vector fields leading to transport equations having only renormalized solutions. This property is called renormalization property.

**Definition 3.1.8 (Renormalization property)** *Let* $b$ *be a vector field in* $L^1((0,T)\times\Omega)^N$ *with* $\operatorname{div} b \in L^1((0,T)\times\Omega)$*. We say that* $b$ *has the renormalization property if for every* $u_0 \in L^\infty(\Omega)$*, every bounded solution of the transport equation with vector field* $b$ *and initial data* $u_0$ *is a renormalized solution.*

Now, if a vector field $b$ has the renormalization property, then uniqueness of solutions of the corresponding transport equation can be concluded. Furthermore, a first stability result can be deduced for convergent sequences of vector fields.

**Theorem 3.1.9 (Uniqueness and stability of solutions on bounded domains)** *Let the vector field* $b \in L^1((0,T), BV_0(\Omega))^N$ *with* $\operatorname{div} b \in L^1((0,T), L^\infty(\Omega))$ *has the renormalization property. Then, for every* $u_0 \in L^\infty(\Omega)$*, the solution to the transport equation* (3.1) *is unique. Furthermore, the solution depends continuously on the vector field* $b$ *and the initial data* $u_0$ *in the following sense: let* $(b_n) \subset L^1((0,T), BV_0(\Omega))^N$ *and* $(u_{0,n}) \subset L^\infty(\Omega)$ *be sequences satisfying*

*(i)* $b_n$ *has the renormalization property for all* $n \in \mathbb{N}$ *and* $(\operatorname{div} b_n) \subset L^1((0,T), L^\infty(\Omega))$*,*

*(ii)* $b_n \to b$ *in* $L^1((0,T)\times\Omega)^N$*,* $\quad \operatorname{div} b_n \to \operatorname{div} b$ *in* $L^1((0,T)\times\Omega)$*,*

*(iii)* $\sup_{n\in\mathbb{N}} \|u_{0,n}\|_{L^\infty(\Omega)} < \infty$ *and* $u_{0,n} \to u_0$ *in* $L^1(\Omega)$*.*

*Then the sequence of solutions* $(u_n)$ *of the corresponding transport equations converges strongly in* $L^p(0,T)\times\Omega)$ *to the solution* $u$ *of* (3.1) *for any* $p < \infty$*.*

**Proof:** The result and its proof can be found for domains with spatial component $\mathbb{R}^N$ in [Lel07, Cri07] and the uniqueness part for spatial component $\Omega$ in [CDS14b]. Therefore, we only show the stability statement for functions with domain $(0,T)\times\Omega$.
Let $(b_n)$, $(\operatorname{div} b_n)$ and $(u_{0_n})$ be sequences with properties as assumed in the theorem and let $(u_n) \subset C([0,T], L^\infty(\Omega) - w^*) \subset L^\infty((0,T)\times\Omega)$ be the sequence of unique solutions. Due to Remark (3.1.6), $(u_n)$ is uniformly bounded and thus there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ weakly$^*$ converging to some function $\tilde{u} \in L^\infty((0,T)\times\Omega)$. Obviously, $\tilde{u}$ is a weak solution of the transport equation with vector field $b$ and initial value $u_0$. By the uniqueness part of the proof, this solution is unique (i.e. $u = \tilde{u}$) and we conclude that the whole sequence converges weakly$^*$ to $u$ in $L^\infty((0,T)\times\Omega)$ by a proof by contradiction. As $b_n$ has the renormalization

property for all $n \in \mathbb{N}$, $u_n^2$ solves the corresponding transport equation with initial data $u_{0_n}^2$. As $(u_{0_n}^2)$ is bounded in $L^\infty(\Omega)$ and $u_{0_n} \to u_0$ in $L^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega \left|u_{0_n}^2(x) - u_0^2(x)\right| \, dx &\le \int_\Omega |u_{0_n}(x) - u_0(x)| \, |u_{0_n}(x) + u_0(x)| \, dx \\ &\le \|u_{0_n} + u_0\|_{L^\infty(\Omega)} \|u_{0_n} - u_0\|_{L^1(\Omega)} \\ &\le C \|u_{0_n} - u_0\|_{L^1(\Omega)} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

With the same argument as above $u_n^2$ converges weakly* to the unique weak solution of the transport equation with vector field $b$ and initial data $u_0^2$. Using the renormalization property this solution is given by $u^2$. Now, as

$$u_n^2 \overset{*}{\rightharpoonup} u^2 \quad \text{and} \quad u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty((0,T) \times \Omega),$$

and the weak as well as the weak* topology is identical in $L^2((0,T) \times \Omega)$, we have that

$$u_n^2 \rightharpoonup u^2 \quad \text{and} \quad u_n \rightharpoonup u \quad \text{in } L^2((0,T) \times \Omega).$$

Using the constant 1-function on $(0,T) \times \Omega$, the first convergence yields that

$$\|u_n\|_{L^2((0,T)\times\Omega)} \to \|u\|_{L^2((0,T)\times\Omega)} .$$

But weak convergence and norm convergence yields strong convergence in $L^2((0,T) \times \Omega)$. It remains to show strong convergence for general $p < \infty$. For $p \le 2$ it is obviously true due to the continuous embedding of $L^2((0,T) \times \Omega)$ into $L^q((0,T) \times \Omega)$ for $q \le 2$. Thus, we can restrict to the case $2 < p < \infty$. Since $(u_n)$ is bounded in $L^\infty((0,T) \times \Omega)$ we estimate

$$\begin{aligned}\|u_n - u\|_{L^p((0,T)\times\Omega)}^p &= \int_0^T \int_\Omega |u_n(t,x) - u(t,x)|^{p-2} \, |u_n(t,x) - u(t,x)|^2 \, dxdt \\ &\le \|u_n - u\|_{L^\infty((0,T)\times\Omega)}^{p-2} \|u_n - u\|_{L^2((0,T)\times\Omega)}^2 \\ &\le C \|u_n - u\|_{L^2((0,T)\times\Omega)}^2 \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

□

### 3.1.3. Renormalization property for vector fields on $\mathbb{R}^N$

So far, we know that for vector fields having the renormalization property, solutions to the transport equation are unique. It remains to clarify under which conditions some vector field $b$ has the renormalization property. Ambrosio showed in [Amb04] that any bounded vector field $b \in L^1\left((0,T), BV(\mathbb{R}^N)\right)^N$ with absolute continuous divergence possesses this property.

**Theorem 3.1.10 (Renormalization property on entire $\mathbb{R}^N$)** *Let $b$ be a bounded vector field belonging to $L^1\left((0,T), BV(\mathbb{R}^N)\right)^N$ such that $\operatorname{div} b \in L^1\left((0,T) \times \mathbb{R}^N\right)$. Then $b$ has the renormalization property.*

In this subsection, we give a proof of this theorem with the aid of several lemmas. Our intention is to present the strategy of Ambrosio for handling the commutator in the $BV$ regular case. We do this since in section 3.2 we will be confronted with a term similar to the commutator and we will apply the same steps to handle this term. The presentation of the proof is a mixture of the ones presented by De Lellis in [Lel07] and by Crippa in his PhD thesis [Cri07]. The proofs are adjusted to our situation with domain $(0,T) \times \mathbb{R}^N$ and definition of renormalized solution with included initial value. We start with the following general lemma of De Lellis in [Lel07]. In [Lel07], a quite brief and inexact proof is given with different assumptions on the time regularity than in this lemma. Since this lemma plays an important role in the proof of the second stability theorem in the next subsection we give a detailed proof. Furthermore, we have added some further statements to the lemma which do not appear in the original one.

**Lemma 3.1.11** *Let $1 \leq q < \infty$, let $g \in L^q\left((0,T), BV(\mathbb{R}^N)\right)^N$ and let $z, w \in \mathbb{R}^N$. Then, the difference quotient*

$$\frac{w^\top (g(t, x + \delta z) - g(t,x))}{\delta}$$

*can be written as $w^\top g_{1,\delta,z} + w^\top g_{2,\delta,z}$, where*

*(i) $w^\top g_{1,\delta,z} \to w^\top J_g z$ in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$ as $\delta \to 0$, where $J_g$ denotes the Radon-Nikodym derivative of the absolute continuous part $D^a g$ of $Dg$ with respect to $\mathcal{L}^N$.*

*(ii) For any compact set $K \subset \mathbb{R}^N$ and for almost all $t \in (0,T)$ we have*

$$\limsup_{\delta \to 0} \int_K \left| w^\top g_{2,\delta,z}(t,x) \right| \, dx \leq \left| (w^\top D^s g z)(t, \cdot) \right| (K)$$

*where $D^s g$ denotes the singular part of the measure $Dg$ with respect to $\mathcal{L}^N$. Furthermore, for any measurable set $I \subset (0,T)$ we have*

$$\limsup_{\delta \to 0} \int_I \left( \int_K \left| w^\top g_{2,\delta,z}(t,x) \right| \, dx \right)^q \, dt \leq \int_I \left( \left| (w^\top D^s g z)(t, \cdot) \right| (K) \right)^q \, dt.$$

*(iii) For every compact set $K \subset \mathbb{R}^N$, for almost all $t \in (0,T)$ and $\varepsilon > 0$ we have*

$$\sup_{\delta \in (0,\varepsilon)} \int_K \left( \left| w^\top g_{1,\delta,z}(t,x) \right| + \left| w^\top g_{2,\delta,z}(t,x) \right| \right) \, dx \leq |w||z||Dg(t,\cdot)|(K_\varepsilon),$$

*where $K_\varepsilon = \{x \in \mathbb{R}^N \,|\, \operatorname{dist}(x,K) \leq \varepsilon\}$. Furthermore, for any measurable set $I \subset (0,T)$ we have*

$$\sup_{\delta \in (0,\varepsilon)} \int_I \left( \int_K \left( \left| w^\top g_{1,\delta,z}(t,x) \right| + \left| w^\top g_{2,\delta,z}(t,x) \right| \right) \, dx \right)^q \, dt \leq \int_I \left( |w||z||Dg(t,\cdot)|(K_\varepsilon) \right)^q \, dt.$$

Before we prove the lemma we present the following auxiliary corollary which can be found in section F of chapter 10 in [Jon01]:

**Corollary 3.1.12** *Let $1 \leq q < \infty$ and let $f \in L^q\left(\mathbb{R}^N\right)$. Then for $h \in \mathbb{R}^N$, the mapping*

$$T_h : [0,T] \to L^q\left(\mathbb{R}^N\right), \qquad r \mapsto f(\cdot + rh)$$

*is continuous, i.e. $T_h \in C\left([0,T], L^q(\mathbb{R}^N)\right)$.*

Now, we turn to the proof of Lemma 3.1.11.

**Proof:** We start with some basic facts: for $\delta > 0$ and a Radon measure $\mu$ on $\mathbb{R}$ we have

$$\mu_\delta(\tau) := \mu * \frac{\mathbb{1}_{[-\delta,0]}}{\delta}(\tau) = \frac{1}{\delta}\int\limits_{\mathbb{R}} \mathbb{1}_{[-\delta,0]}(\tau - r)\ d\mu(r) = \frac{1}{\delta}\int\limits_{[\tau,\tau+\delta]} d\mu(r) = \frac{\mu([\tau,\tau+\delta])}{\delta}$$

with $\tau \in \mathbb{R}$. Furthermore, for a compact set $K \subset \mathbb{R}$ we set

$$K_\delta := \{x \in \mathbb{R}|\ \operatorname{dist}(x, K) \leq \delta\}.$$

Then, we estimate

$$\begin{aligned}\int\limits_K |\mu_\delta(\tau)|\ d\tau &\leq \frac{1}{\delta}\int\limits_K\int\limits_{\mathbb{R}} \mathbb{1}_{[-\delta,0]}(\tau - r)\ d|\mu|(r)d\tau = \frac{1}{\delta}\int\limits_{\mathbb{R}}\int\limits_K \mathbb{1}_{[-\delta,0]}(\tau - r)\ d\tau d|\mu|(r)\\ &= \int\limits_{\mathbb{R}}\int\limits_K \frac{1}{\delta}\mathbb{1}_{[r-\delta,r]}(\tau)\ d\tau d|\mu|(r). \end{aligned} \tag{3.3}$$

For $\tau \in K$ and $r \in \mathbb{R}$ we have

$$r - \delta \leq \tau \leq r \quad \Leftrightarrow \quad \tau \leq r \leq \tau + \delta.$$

Thus, for $r \notin K_\delta$ the inner integral in (3.3) is equal to zero and for $r \in K_\delta$ we have the estimate

$$\int\limits_K \frac{1}{\delta}\mathbb{1}_{[r-\delta,r]}(\tau)\ d\tau \leq \int\limits_{\mathbb{R}} \frac{1}{\delta}\mathbb{1}_{[r-\delta,r]}(\tau)\ d\tau \leq 1.$$

Hence, we obtain

$$\int\limits_K |\mu_\delta(\tau)|\ d\tau \leq \int\limits_{\mathbb{R}} \mathbb{1}_{K_\delta}(r)\ d|\mu|(r) = |\mu|(K_\delta).$$

In the following, we denote the orthonormal basis vectors in $\mathbb{R}^N$ $e_1, \ldots, e_N$ and we define for a vector $x \in \mathbb{R}$

$$x = (x_1, \ldots, x_{N-1}, x_N) = (x', x_N).$$

We will show the result for the case $z = e_N$. The general case $z \in \mathbb{R}^N$ can be traced back to the case $z = e_N$ via a change of the coordinate system: take an orthonormal matrix $A : \mathbb{R}^N \to \mathbb{R}^N$ such that $Az = e_N$ and set $y = Ax$. Then, due to Theorem 3.16 in [AFP00]

$$\tilde{g}(t, y) = g(t, A^{-1}y) \quad \text{for all } y \in \mathbb{R}^N$$

is an element of $L^q\left((0,T), BV(\mathbb{R}^N)\right)$ and the general case corresponds to the case $z = e_N$ with function $\tilde{g}$ in the new coordinate system.
Lebesgue's decomposition theorem yields that the measure $Dg$ can be split into the measures $D^a g$ and $D^s g$ where the former is absolutely continuous and the latter is singular with respect to the Lebesgue measure. We set $J_g \in L^q\left((0,T), L^1(\mathbb{R}^N)\right)^{N\times N}$ as the Radon-Nikodym derivative of $D^a g$. For the measure $Dge_N$ we immediately obtain that $J_g e_N$ then is the Radon-Nikodym derivative of $D^a g e_N$ with respect to the Lebesgue measure and we set the singular measure $D^s g e_N = Dge_N - J_g e_N \mathcal{L}^N$. Furthermore, we define

$$w^\top g_{1,\delta,e_N}(t, x', x_N) = \frac{1}{\delta}\int\limits_{x_N}^{x_N+\delta} (w^\top J_g e_N)(t, x', r)\ dr.$$

It holds that $w^\top g_{1,\delta,e_N} \in L^q\left((0,T), L^1(\mathbb{R}^N)\right)$: for $t \in (0,T)$ we have

$$\begin{aligned} \left\| w^\top g_{1,\delta,e_N}(t,\cdot) \right\|_{L^1(\mathbb{R}^N)} &\leq \frac{1}{\delta} \sum_{i=1}^N \int_{\mathbb{R}^N} \int_{x_N}^{x_N+\delta} |w_i| \left| \nabla g_i(t,x',r) \right| |e_N| \, dr dx \\ &\leq \frac{1}{\delta} \sum_{i=1}^N \delta |w_i| \left\| \nabla g_i(t,\cdot) \right\|_{L^1(\mathbb{R}^N)} = \sum_{i=1}^N |w_i| \left\| \nabla g_i(t,\cdot) \right\|_{L^1(\mathbb{R}^N)} \end{aligned} \tag{3.4}$$

and thus

$$\left\| w^\top g_{1,\delta,e_N} \right\|_{L^q((0,T),L^1(\mathbb{R}^N))} \leq \sum_{i=1}^N |w_i| \left\| \nabla g_i \right\|_{L^q((0,T),L^1(\mathbb{R}^N))}.$$

Furthermore, for almost all $t \in (0,T)$ we conclude

$$\begin{aligned} &\int_{\mathbb{R}^N} \left| w^\top g_{1,\delta,e_N}(t,x) - (w^\top J_g e_N)(t,x) \right| \, dx \\ &= \int_{\mathbb{R}^N} \left| \frac{1}{\delta} \int_{x_n}^{x_n+\delta} (w^\top J_g e_N)(t,x',r) \, dr - (w^\top J_g e_N)(t,x) \right| \, dx \\ &\leq \int_{\mathbb{R}^N} \frac{1}{\delta} \int_0^\delta \left| (w^\top J_g e_N)(t,x',r+x_N) - (w^\top J_g e_N)(t,x) \right| \, dr dx \\ &= \frac{1}{\delta} \int_0^\delta \int_{\mathbb{R}^N} \left| (w^\top J_g e_N)(t,x',r+x_N) - (w^\top J_g e_N)(t,x) \right| \, dx dr. \end{aligned}$$

If we set $f_t := (w^\top J_g e_N)(t,\cdot)$ for $t \in (0,T)$, then $f_t \in L^1\left(\mathbb{R}^N\right)$ and we obtain using Corollary 3.1.12 with $h = e_N$:

$$\begin{aligned} &\frac{1}{\delta} \int_0^\delta \int_{\mathbb{R}^N} \left| (w^\top J_g e_N)(t,x',r+x_N) - (w^\top J_g e_N)(t,x) \right| \, dx dr \\ &\qquad \leq \frac{1}{\delta} \int_0^\delta \int_{\mathbb{R}^N} \left| f_t(x+rh) - f_t(x) \right| \, dx dr \\ &\qquad \leq \sup_{r \in [0,\delta]} \left\| f_t(\cdot + rh) - f_t \right\|_{L^1(\mathbb{R}^N)} \to 0 \end{aligned}$$

as $\delta \to 0$. Thus, we apply Lebesgue's dominated convergence theorem since

$$w^\top g_{1,\delta,e_N}(t,\cdot) \to (w^\top J_g e_N)(t,\cdot) \qquad \text{in } L^1\left(\mathbb{R}^N\right)$$

for almost all $t \in (0,T)$ as $\delta \to 0$ and since it is pointwise uniformly bounded by some function in $L^q((0,T))$ due to estimate (3.4). This gives us point (i). Now, we set

$$w^\top g_{2,\delta,e_N}(t,x',x_N) = \frac{w^\top(g(t,x',x_N+\delta) - g(t,x',x_N))}{\delta} - w^\top g_{1,\delta,e_N}(t,x',x_N),$$

for almost all $x \in \mathbb{R}^N$ and $t \in (0,T)$. In addition, we define for $\mathcal{L}^{N-1}$-a.e. $y \in \mathbb{R}^{N-1}$ and $\mathcal{L}^1$-a.e. $t \in (0,T)$

$$g_{t,y} : \mathbb{R} \to \mathbb{R}, \quad r \mapsto g(t,y,r).$$

Then, due to Remark 3.104 in [AFP00] we have that $g_{t,y} \in BV(\mathbb{R})$ for almost all $y \in \mathbb{R}^{N-1}$ and $t \in (0,T)$. In addition, if we use Lebesgue's decomposition

$$Dg_{t,y} = D^a g_{t,y} + D^s g_{t,y} = g'_{t,y}\mathcal{L}^1 + D^s g_{t,y},$$

then Theorem 3.107 in [AFP00] yields that

$$J_g(t,y,r)e_N = g'_{t,y}(r)$$

for $\mathcal{L}^{N-1}$-a.e. $y \in \mathbb{R}^{N-1}$, for $\mathcal{L}^1$-a.e. $t \in (0,T)$ and for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ as well as

$$|(D^s g e_N)(t,\cdot)|(A) = \int\limits_{\mathbb{R}^{N-1}} |D^s g_{t,y}|(\{r \in \mathbb{R} |\ (y,r) \in A\})\ dy$$

for all Borel sets $A \subset \mathbb{R}^N$. Moreover, for almost all $y \in \mathbb{R}^{N-1}$ and $t \in (0,T)$, using Theorem 3.28 and Theorem 3.108 in [AFP00], we obtain that for $r \in \mathbb{R} \backslash B$ the function $g^*_{t,y}$ is continuous in $r$, where $B$ denotes the set of atoms of $Dg_{t,y}$ (i.e. $r \in B$ if and only if $Dg_{t,y}(\{r\}) \neq 0$). Since the set of atoms is at most countable we have that it is a null set with respect to the Lebesgue measure and thus we get for the continuity points $r, r+\delta \in \mathbb{R}$ (i.e. for almost all $r$):

$$g^*_{t,y}(r+\delta) - g^*_{t,y}(r) = g^r_{t,y}(r+\delta) - g^l_{t,y}(r) = Dg_{t,y}([r, r+\delta])$$

with $\delta > 0$. Therefore, we conclude for any $\delta > 0$ and for $\mathcal{L}^{N-1}$-a.e. $y \in \mathbb{R}^{N-1}$, $\mathcal{L}^1$-a.e. $t \in (0,T)$ and $\mathcal{L}^1$-a.e. $x_N \in \mathbb{R}$

$$\begin{aligned}
&w^\top g_{1,\delta,e_N}(t,y,x_N) + w^\top g_{2,\delta,e_N}(t,y,x_N) \\
&\quad = \frac{w^\top (g(t,y,x_N+\delta) - g(t,y,x_N))}{\delta} \\
&\quad = \frac{w^\top (g^*_{t,y}(x_N+\delta) - g^*_{t,y}(x_N))}{\delta} = \frac{w^\top Dg_{t,y}([x_N, x_N+\delta])}{\delta} \\
&\quad = \frac{w^\top D^a g_{t,y}([x_N, x_N+\delta])}{\delta} + \frac{w^\top D^s g_{t,y}([x_N, x_N+\delta])}{\delta} \\
&\quad = \frac{1}{\delta} \int\limits_{x_n}^{x_n+\delta} (w^\top J_g e_N)(t,y,r)\ dr + \frac{w^\top D^s g_{t,y}([x_N, x_N+\delta])}{\delta} \\
&\quad = w^\top g_{1,\delta,e_N}(t,y,x_N) + \frac{w^\top D^s g_{t,y}([x_N, x_N+\delta])}{\delta}.
\end{aligned}$$

Thus, for any compact set $K \subset \mathbb{R}^N$ we set

$$K^y := \{x_N \in \mathbb{R} |\ (y, x_N) \in K\} \qquad \text{with } y \in \mathbb{R}^{N-1}.$$

Then, $K^y$ is a closed, bounded set and therefore compact and we obtain for $t \in (0,T)$

$$\int\limits_K \left| w^\top g_{2,\delta,e_N}(t,x) \right|\ dx \le \int\limits_{\mathbb{R}^{N-1}} \int\limits_{K^y} \frac{\left| w^\top D^s g_{t,y}([r, r+\delta]) \right|}{\delta}\ drdy$$

$$= \int_{\mathbb{R}^{N-1}} \int_{K^y} \left|(w^\top D^s g_{t,y})_\delta(r)\right| \, drdy \leq \int_{\mathbb{R}^{N-1}} |w^\top D^s g_{t,y}|((K^y)_\delta) \, dy.$$

The set given by

$$S^y = \{(y,r)| \ r \in (K^y)_\delta\}$$

is a subset of $K_\delta$ since for $(y,r) \in S^y$ there exists some $s \in K^y$ such that $|s-r| \leq \delta$. Then $|(y,r)-(y,s)| \leq \delta$ and since $(y,s) \in K$ we get that $(y,r) \in K_\delta$. Thus, we have for $t \in (0,T)$

$$\begin{aligned} \int_K \left|w^\top g_{2,\delta,e_N}(t,x)\right| \, dx &\leq \int_{\mathbb{R}^{N-1}} |w^\top D^s g_{t,y}|((K^y)_\delta) \, dy \\ &\leq \int_{\mathbb{R}^{N-1}} |w^\top D^s g_{t,y}|(\{r \in \mathbb{R}| \ (y,r) \in K_\delta\}) \, dy \\ &= |(w^\top D^s g e_N)(t,\cdot)|(K_\delta). \end{aligned} \tag{3.5}$$

Taking the limes superior over $\delta$ yields the first part of point (ii). Moreover, as above, we have for any compact $K \subset \mathbb{R}^N$ and $t \in (0,T)$

$$\begin{aligned} \int_K \left|w^\top g_{1,\delta,e_N}(t,x)\right| \, dx &\leq \int_{\mathbb{R}^{N-1}} \int_{K^y} \left|(w^\top g'_{t,y}\mathcal{L}^1)_\delta(r)\right| \, drdy \\ &\leq \int_{K_\delta} |(w^\top J_g e_N)(t,\cdot)|(x) \, dx \\ &= \left|(w^\top D^a g e_N)(t,\cdot)\right|(K_\delta). \end{aligned} \tag{3.6}$$

Finally, for any measurable set $I \subset (0,T)$ we obtain

$$\int_I \left(\int_K \left|w^\top g_{2,\delta,e_N}(t,x)\right| \, dx\right)^q dt \leq \int_I \left(|(w^\top D^s g e_N)(t,\cdot)|(K_\delta)\right)^q \, dt$$

and taking the limes superior over $\delta$ yields the second part of point (ii). Combining the estimates (3.5) and (3.6) yield both parts of point (iii):

$$\begin{aligned} \int_K \left|w^\top g_{1,\delta,e_N}(t,x)\right| + \left|w^\top g_{2,\delta,e_N}(t,x)\right| \, dx &\leq \sup_{\delta\in(0,\varepsilon)} \left|(w^\top D g e_N)(t,\cdot)\right|(K_\delta) \\ &\leq |e_N| \, |w| \, |Dg(t,\cdot)| \, (K_\varepsilon). \end{aligned}$$

Then, taking the supremum on the left side yields the first part and integrating with subsequently taking the supremum on the left side yields the second part.

□

The next lemma states that the composition of a solution with some $C^1(\mathbb{R})$ function satisfies a transport equation whose right-hand side is given by a Radon measure. Later, it will be proven that this Radon measure is the zero measure.

**Lemma 3.1.13** *Let $I \subset \mathbb{R}$ be an open interval in $\mathbb{R}$ and let $b \in L^1\left(I, BV(\mathbb{R}^N)\right)^N$ with $\operatorname{div} b \in L^1\left(I \times \mathbb{R}^N\right)$. Let $u \in L^\infty\left(I \times \mathbb{R}^N\right)$ be a distributional solution of the transport equation, i.e.*

$$\partial_t u + \operatorname{div}(ub) - u \operatorname{div} b = 0 \qquad \text{in } \mathcal{D}'\left(I \times \mathbb{R}^N\right).$$

*Then, for every* $\beta \in C^1(\mathbb{R})$,

$$\partial_t(\beta(u)) + \operatorname{div}(\beta(u)b) - \beta(u)\operatorname{div} b = \mu_\beta, \tag{3.7}$$

*where* $\mu_\beta \in \mathcal{M}_{loc}\left(I \times \mathbb{R}^N\right)$.

**Proof:** Let $\rho$ be an even mollifier in $\mathbb{R}^N$ and let $u \in L^\infty\left(I \times \mathbb{R}^N\right)$ be a solution of the transport equation with vector field $b \in L^1\left(I, BV(\mathbb{R}^N)\right)^N$. Convolving the transport equation in the distributional form yields

$$\partial_t(u * \rho_\varepsilon) + b \cdot \nabla(u * \rho_\varepsilon) = r_\varepsilon,$$

with commutator

$$r_\varepsilon = -\underbrace{\left[(\operatorname{div}(bu)) * \rho_\varepsilon - \operatorname{div}(b(u * \rho_\varepsilon))\right]}_{=:R_\varepsilon} + \left[(u \operatorname{div} b) * \rho_\varepsilon - u * \rho_\varepsilon \operatorname{div} b\right]. \tag{3.8}$$

As a convolved function, $u_\varepsilon := u * \rho_\varepsilon$ is smooth with respect to the spatial variable and since $\partial_t u_\varepsilon = -(b \cdot \nabla u) * \rho_\varepsilon$, $u_\varepsilon$ has Sobolev regularity in its spatio-temporal domain. Hence, for $\beta \in C^1(\mathbb{R})$, we can use the chain rule for Sobolev functions (Theorem 4, section 4.2.2 in [EG92]) and we obtain

$$\partial_t\left[\beta(u_\varepsilon)\right] + b \cdot \nabla\left[\beta(u_\varepsilon)\right] = \beta'(u_\varepsilon)\left[-R_\varepsilon + \left[(u \operatorname{div} b) * \rho_\varepsilon - u_\varepsilon \operatorname{div} b\right]\right]. \tag{3.9}$$

The left part of (3.9) converges distributionally to

$$\partial_t\beta(u) + b \cdot \nabla\beta(u).$$

Furthermore,

$$\beta'(u_\varepsilon)\left[(u \operatorname{div} b) * \rho_\varepsilon - u_\varepsilon \operatorname{div} b\right] \to 0 \qquad \text{in } L^1_{loc}\left(I \times \mathbb{R}^N\right)$$

and hence in the distributional sense. It remains term $R_\varepsilon = \left[(\operatorname{div}(bu)) * \rho_\varepsilon - \operatorname{div}(b(u_\varepsilon))\right]$. By using the elementary identity

$$R_\varepsilon = \sum_{i=1}^N (ub_i) * \partial_{x_i}\rho_\varepsilon - \sum_{i=1}^N b_i(u * \partial_{x_i}\rho_\varepsilon) - u_\varepsilon \operatorname{div} b,$$

we conclude performing the change of variables $z = (y - x)/\varepsilon$:

$$R_\varepsilon(t,x) = -\int_{\mathbb{R}^N} u(t, x + \varepsilon z)\frac{b(t, x + \varepsilon z) - b(t,x)}{\varepsilon} \cdot \nabla\rho(z)\ dz - [u_\varepsilon \operatorname{div} b](t,x). \tag{3.10}$$

In the remaining part, our aim is to show that $R_\varepsilon$ is bounded in $L^1_{loc}\left(I \times \mathbb{R}^N\right)$. Thus, we obtain for any compact set $K \subset I \times \mathbb{R}^N$:

$$\begin{aligned}\int_K |R_\varepsilon(t,x)|\ dxdt \leq{}& \int_K \int_{\mathbb{R}^N} \left| u(t, x + \varepsilon z)\frac{b(t, x + \varepsilon z) - b(t,x)}{\varepsilon} \cdot \nabla\rho(z)\right|\ dzdxdt \\ &+ \int_K \left|[u_\varepsilon \operatorname{div} b](t,x)\right|\ dxdt\end{aligned}$$

The second term is bounded by $\|u\|_{L^\infty(I\times\mathbb{R}^N)} \|\operatorname{div} b\|_{L^1(I\times\mathbb{R}^N)}$ and hence uniformly bounded. Since $K$ is compact, we find a cuboid $\prod_{i=1}^N [a_i, b_i]$ with $a_i, b_i \in \mathbb{R}$ for $i = 1, \dots, N$ such that $K \subset I \times \prod_{i=1}^N [a_i, b_i]$. Using Lemma 3.1.11, we conclude for the first term:

$$\int_K \int_{\mathbb{R}^N} \left| u(t, x + \varepsilon z) \frac{b(t, x + \varepsilon z) - b(t, x)}{\varepsilon} \cdot \nabla \rho(z) \right| \, dz dx dt$$
$$\leq \|u\|_{L^\infty(I\times\mathbb{R}^N)} \int_I |Db(t, \cdot)| \left( \prod_{i=1}^N [a_i - \varepsilon, b_i + \varepsilon] \right) \, dt \int_{\mathbb{R}^N} |z||\nabla \rho(z)| \, dz.$$

Now, taking $\varepsilon \in (0, \delta)$ for some $\delta > 0$, we obtain

$$\sup_{\varepsilon \in (0,\delta)} \int_K \int_{\mathbb{R}^N} \left| u(t, x + \varepsilon z) \frac{b(t, x + \varepsilon z) - b(t, x)}{\varepsilon} \cdot \nabla \rho(z) \right| \, dz dx dt$$
$$\leq \|u\|_{L^\infty(I\times\mathbb{R}^N)} \int_I |Db(t, \cdot)| \left( \prod_{i=1}^N [a_i - \delta, b_i + \delta] \right) \, dt \int_{\mathbb{R}^N} |z||\nabla \rho(z)| \, dz.$$

Hence, $R_\varepsilon$ is bounded in $L^1_{loc}\left(I \times \mathbb{R}^N\right)$ and thus, there exists a subsequence of $(-\beta'(u_\varepsilon)R_\varepsilon)$ which converges locally weakly* to some Radon measure $\mu_\beta \in \mathcal{M}_{loc}\left(I \times \mathbb{R}^N\right)$. The limit $\mu_\beta$ is unique and is given by

$$\mu_\beta = \partial_t \beta(u) + b \cdot \nabla \beta(u)$$

due to the following argument: the left side of equation (3.9) converges distributionally to $\partial_t \beta(u) + b \cdot \nabla \beta(u)$, whereas the right side converges weakly* to $\mu_\beta \in \mathcal{M}_{loc}\left(I \times \mathbb{R}^N\right)$. Hence, the distribution $\partial_t \beta(u) + b \cdot \nabla \beta(u)$ can be represented by a measure and every weak*-limit must be equal to it and thus the weak*-limit of $(-\beta'(u_\varepsilon)R_\varepsilon)$ is unique.

□

As a next step, we first give a definition and then show that the previous measure is dominated by some specific measure.

**Definition 3.1.14** *For any $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$ and any $N \times N$-matrix $M$ we define*

$$\Lambda(M, \rho) = \int_{\mathbb{R}^N} |(\nabla \rho(z))^\top M z| \, dz.$$

**Lemma 3.1.15** *Let $I \subset \mathbb{R}$ be an open interval in $\mathbb{R}$ and let $b \in L^1\left(I, BV(\mathbb{R}^N)\right)^N$ with $\operatorname{div} b \in L^1\left(I \times \mathbb{R}^N\right)$. Let $u \in L^\infty\left(I \times \mathbb{R}^N\right)$ be a distributional solution of the transport equation and $\beta \in C^1(\mathbb{R})$. Denote $M_b$ the matrix-valued Borel function such that $D^s b = M_b |D^s b|$ and let $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$ be an even mollifier. Then, the Radon measure $\mu_\beta$ in (3.7) satisfies*

$$|\mu_\beta| \leq C \Lambda(M_b, \rho) |D^s b| \tag{3.11}$$

*for some $C > 0$.*

**Proof:** We take $\varphi \in C_c\left(I \times \mathbb{R}^N\right)$ and conclude by using Lemma 3.1.11

$$\int_{I\times\mathbb{R}^N} \varphi \; d\mu_\beta = \int_{I\times\mathbb{R}^N} \varphi(t,x)[\beta'(u)u \operatorname{div} b](t,x) \; dxdt$$

$$+ \lim_{\varepsilon\to 0} \int_{I\times\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(t,x)\beta'(u_\varepsilon)(t,x)u(t,x+\varepsilon z) b_{1,\varepsilon,z}(t,x)^\top \nabla\rho(z) \; dzdxdt \tag{3.12}$$

$$+ \lim_{\varepsilon\to 0} \int_{I\times\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(t,x)\beta'(u_\varepsilon)(t,x)u(t,x+\varepsilon z) b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z) \; dzdxdt. \tag{3.13}$$

We first start to show that (3.12) vanishes. Combining Property (i) and (iii) of Lemma 3.1.11 as well as the boundedness of $\beta'(u_\varepsilon)$ and $u$ in $L^\infty\left(I \times \mathbb{R}^N\right)$, we deduce that the second integral in (3.12) converges to

$$\int_{I\times\mathbb{R}^N} \varphi(t,x)\beta'(u(t,x))u(t,x) \sum_{i,j=1}^{N} e_j^\top J_b(t,x)e_i \int_{\mathbb{R}^N} z_i \partial_{z_j}\rho(z) \; dzdxdt \tag{3.14}$$

The above term (3.14) is equal to

$$-\int_{I\times\mathbb{R}^N} \varphi(t,x)u(t,x)\beta'(u(t,x)) \operatorname{div} b(t,x) \; dxdt,$$

since $\int_{\mathbb{R}^N} z_i \partial_{z_j}\rho(z) \; dz = -\delta_{ij}$. Thus (3.12) vanishes. We now investigate the term in (3.13). Since $u$ and thus also $\beta'(u_\varepsilon)$ are bounded, we estimate (3.13) by

$$C \limsup_{\varepsilon\to 0} \int_{I\times\mathbb{R}^N} |\varphi(t,x)| \int_{\mathbb{R}^N} |b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z)| \; dzdxdt. \tag{3.15}$$

Next, let $S = \|\varphi\|_\infty$ and let $K_\sigma$ be the closure of $\{(t,x) : |\varphi(t,x)| > \sigma\}$. Then we can rewrite (3.15) as

$$C \limsup_{\varepsilon\to 0} \int_0^S \int_{K_\sigma} \int_{\mathbb{R}^N} |b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z)| \; dzdxdtd\sigma. \tag{3.16}$$

Using Property (ii) in Lemma 3.1.11, we obtain that

$$C \limsup_{\varepsilon\to 0} \int_{K_\sigma} |b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z)| \; dxdt \leq C \left|\nabla\rho(z)\right| |z| \left|D^s b\right| (K_\sigma). \tag{3.17}$$

Since for $z$ outside the support of $\rho$ the term in (3.17) vanishes, the map

$$(\sigma, z) \mapsto \int_{K_\sigma} |b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z)| \; dxdt$$

is bounded. Therefore, we first integrate in $(t,x)$ in (3.16) and obtain

$$C \limsup_{\varepsilon\to 0} \int_0^S \int_{K_\sigma} \int_{\mathbb{R}^N} |b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z)| \; dzdxdtd\sigma \leq C \int_0^S \int_{\mathbb{R}^N} |(\nabla\rho(z))^\top D^s b z|(K_\sigma) \; dzd\sigma. \tag{3.18}$$

Let $\nu_z$ be the measure $|(\nabla\rho(z))^\top D^s b z| = |(\nabla\rho(z))^\top M_b z||D^s b|$. Then (3.18) (without $C$) is simply

$$\begin{aligned}\int_0^S \int_{\mathbb{R}^N} \nu_z(K_\sigma)\, dz d\sigma &= \int_{\mathbb{R}^N} \int_{I\times\mathbb{R}^N} |\varphi(t,x)||(\nabla\rho(z))^\top M_b(t,x)z|\; d|D^s b|(t,x)dz \\ &= \int_{I\times\mathbb{R}^N} |\varphi(t,x)| \left[\int_{\mathbb{R}^N} |(\nabla\rho(z))^\top M_b(t,x)z|\; dz\right]\; d|D^s b|(t,x) \\ &= \int_{I\times\mathbb{R}^N} |\varphi(t,x)|\Lambda(M_b(t,x),\rho)\; d|D^s b|(t,x).\end{aligned}$$

Summarizing, we obtain that

$$\int_{I\times\mathbb{R}^N} \varphi\; d\mu_\beta \leq C \int_{I\times\mathbb{R}^N} |\varphi(t,x)|\Lambda(M_b(t,x),\rho)\; d|D^s b|(t,x)$$

for any $\varphi \in C_c\left(I\times\mathbb{R}^N\right)$.

□

In (3.11), the measure $\mu_\beta$ and the constant $C$ are independent of the mollifier $\rho$ and hence we can optimize over $\rho$. We define the following set

$$\mathcal{K} := \left\{\rho \in C_c^\infty(B_1(0)) \text{ such that } \rho \geq 0 \text{ is even, and } \int_{B_1(0)} \rho(x)\; dx = 1\right\}.$$

The next lemma states that the inequality of the previous lemma is still valid if the infimum over $\mathcal{K}$ is taken on the right side of the inequality.

**Lemma 3.1.16** *Let $I \subset \mathbb{R}$ be an open interval in $\mathbb{R}$ and let $b \in L^1\left(I, BV(\mathbb{R}^N)\right)^N$ with $\operatorname{div} b \in L^1(I\times\mathbb{R}^N)$. Let $u \in L^\infty\left(I\times\mathbb{R}^N\right)$ be a distributional solution of the transport equation and $\beta \in C^1(\mathbb{R})$. Denote $M_b$ the matrix-valued Borel function such that $D^s b = M_b|D^s b|$. Then*

$$|f_\beta(t,x)| \leq C \inf_{\rho\in\mathcal{K}} \Lambda(M_b(t,x),\rho) \qquad \textit{for } |D^s b| - a.e.\ (t,x) \in I\times\mathbb{R}^N,$$

*where $f_\beta$ is a Borel function satisfying $\mu_\beta = f_\beta|D^s b|$.*

**Proof:** The argumentation of the proof can be found in subsection 2.6.4 in [Cri07] or in the proof of Theorem 3.6 in [Lel07].

□

Now, the Lemma of Alberti gives an expression for the above infimum which then turns out to be zero.

**Lemma 3.1.17 (Alberti)** *For any $N\times N$-matrix $M$ we have*

$$\inf_{\rho\in\mathcal{K}} \Lambda(M,\rho) = |\operatorname{trace}(M)| \tag{3.19}$$

**Proof:** See proof of Lemma 2.6.2 in [Cri07]

□

Finally, we are able to prove Theorem 3.1.10:

**Proof:** Let $b$ be as assumed in Theorem 3.1.10 and let $u \in L^\infty\left((0,T)\times\mathbb{R}^N\right)$ be a weak solution of the transport equation with initial data $u_0 \in L^\infty\left(\mathbb{R}^N\right)$. We extend the solution by $u_0$ and the vector field by zero to the negative time axis. We denote the extensions $\bar{u}$ and $\bar{b}$. Then, $\bar{u}$ is a distributional solution of the transport equation with vector field $\bar{b}$ and using Lemma 3.1.13 we obtain that for $\beta \in C^1(\mathbb{R})$

$$\partial_t(\beta(\bar{u})) + \operatorname{div}(\beta(\bar{u})\bar{b}) - \beta(\bar{u})\operatorname{div}\bar{b} = \mu_\beta$$

for some Radon measure $\mu_\beta \in \mathcal{M}_{loc}\left((-\infty,T)\times\mathbb{R}^N\right)$. Now, Lemma 3.1.15 yields that for $\mu_\beta$ there exists a Borel function $f_\beta$ such that $\mu_\beta = f_\beta|D^s\bar{b}|$. For this function, Lemma 3.1.16 and Lemma 3.1.17 yield the estimate

$$|f_\beta(t,x)| \leq C \inf_{\rho\in\mathcal{K}} \Lambda(M_{\bar{b}}(t,x),\rho) = C|\operatorname{trace}(M_{\bar{b}}(t,x))| \qquad \text{for } |D^s\bar{b}| - \text{a.e. } (t,x).$$

Since $\operatorname{Div}\bar{b}$ is absolutely continuous with respect to the Lebesgue measure, the singular part of $\operatorname{Div}\bar{b}$ is zero, i.e.

$$0 = \operatorname{trace}(M_{\bar{b}})|D^s\bar{b}|.$$

Hence, the right side in the above inequality is zero and thus $\mu_\beta = 0$. It remains to show that $\beta(u(0,\cdot)) = \beta(u_0)$. For $\varphi \in C_c^\infty\left((-\infty,T)\times\mathbb{R}^N\right)$ we obtain

$$\begin{aligned} 0 = \int_{-\infty}^{T}\int_{\mathbb{R}^N} \beta(\bar{u})(\partial_t\varphi + \bar{b}\nabla\varphi + \varphi\operatorname{div}\bar{b})\; dxdt &= \int_0^T\int_{\mathbb{R}^N} \beta(u)(\partial_t\varphi + b\nabla\varphi + \varphi\operatorname{div} b)\; dxdt \\ &+ \int_{-\infty}^{0}\int_{\mathbb{R}^N} \beta(\bar{u})\partial_t\varphi\; dxdt. \end{aligned}$$

If we integrate by parts in $t$ in the second integral on the right side we obtain

$$0 = \int_0^T\int_{\mathbb{R}^N} \beta(u)(\partial_t\varphi + b\nabla\varphi + \varphi\operatorname{div} b)\; dxdt + \int_{\mathbb{R}^N} \beta(u_0)\varphi(0,\cdot)\; dx.$$

□

### 3.1.4. Renormalization property for vector fields on bounded spatial domains

In this last subsection, our aim is to show the statement of Theorem 3.1.10 for bounded space domains $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. The idea is to extend weak solutions and the corresponding vector fields on bounded space domains to the spatio-temporal domain $(0,T)\times\mathbb{R}^N$ and to use Theorem 3.1.10. $BV$-functions on bounded domains can be extended to the entire space in a meaningful way due to the extension theorem for $BV$-functions (see for example Theorem 1 in section 5.4 in [EG92]). In our case, we are faced with the situation that we also have to extend a solution $u \in C([0,T], L^\infty(\Omega) - w^*)$ to the domain $(0,T) \times \mathbb{R}^N$ in a meaningful way, i.e. that the extension of $u$ remains a weak solution in $(0,T) \times \mathbb{R}^N$. This problem leads to the problem of assigning the product $ub$ of a solution $u \in C([0,T], L^\infty(\Omega) - w^*)$ with its vector field $b \in L^1((0,T), BV_0(\Omega))^N$ a meaningful trace on the boundary $(0,T) \times \partial\Omega$. Therefore, we introduce the concept of normal traces as it is presented in [CDS14b]. We start with two definitions adjusted to bounded sets.

**Definition 3.1.18** *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open bounded set. Then, $\mathcal{M}_\infty(\mathcal{O})$ denotes the set of functions $f \in L^\infty(\mathcal{O})^N$ such that the distributional divergence $\operatorname{div} f$ is a finite Radon measure on $\mathcal{O}$.*

**Definition 3.1.19** *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary $\partial\mathcal{O}$. Let $f \in \mathcal{M}_\infty(\mathcal{O})$. Then the normal trace of $f$ on $\partial\mathcal{O}$ is defined as the following distribution:*

$$\langle Tr(f, \partial\mathcal{O}), \varphi \rangle = \int\limits_{\mathcal{O}} \nabla\varphi(x) \cdot f(x)\ dx + \int\limits_{\mathcal{O}} \varphi(x)\ d(\operatorname{Div} f)(x) \qquad \forall\ \varphi \in C_c^\infty\left(\mathbb{R}^N\right).$$

Then, we quote Lemma 2.2 in [CDS14b]:

**Lemma 3.1.20** *For vector fields $f \in \mathcal{M}_\infty(\mathcal{O})$ the normal trace distribution is induced by some $L^\infty(\partial\mathcal{O}, \mathcal{H}^{N-1})$ function which we still call $Tr(f, \partial\mathcal{O})$. For this function we have*

$$\|Tr(f, \partial\mathcal{O})\|_{L^\infty(\partial\mathcal{O}, \mathcal{H}^{N-1})} \leq \|f\|_{L^\infty(\mathcal{O})^N}$$

*Furthermore, if $\Sigma$ is a Borel set contained in $\mathcal{O}_1 \cap \mathcal{O}_2$ and if $\vec{n}_1 = \vec{n}_2$ on $\Sigma$, then*

$$Tr(f, \partial\mathcal{O}_1) = Tr(f, \partial\mathcal{O}_2) \qquad \text{for } \mathcal{H}^{N-1}\text{-almost every } x \in \Sigma.$$

**Remark 3.1.21** *If $f \in L^\infty(\mathcal{O})^N \cap BV(\mathcal{O})^N$ with $\mathcal{O} \subset \mathbb{R}^N$ an open, bounded set with Lipschitz boundary, then the normal trace of $f$ and the trace of $f$ (see for example Theorem 1 in section 5.3 in [EG92]) are equal $\mathcal{H}^{N-1}$-a.e. on $\partial\mathcal{O}$.*

Next, we present Lemma 3.3 in [CDS14b] applied to bounded sets $\mathcal{O} \subset \mathbb{R}^N$. The assumptions of the presented Lemma are slightly weaker than in Lemma 3.3 in [CDS14b] since we assume $g, c \in L^1((0,T) \times \mathcal{O})$ instead of being elements of $L^\infty((0,T) \times \mathcal{O})$. The proof remains the same for these assumptions.

**Lemma 3.1.22** *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary $\partial\mathcal{O}$. Furthermore, let $b \in L^\infty((0,T) \times \mathcal{O})^N$ be a vector field such that $\operatorname{div} b$ is a finite Radon measure on $(0,T) \times \mathcal{O}$. Then, there is a unique function $Tr(b)$, that belongs to $L^\infty((0,T) \times \partial\mathcal{O}, \mathcal{L}^1 \otimes \mathcal{H}^{N-1})$ and satisfies*

$$\int\limits_0^T \int\limits_{\partial\mathcal{O}} Tr(b)\varphi\ d\mathcal{H}^{N-1}(x) dt = \int\limits_0^T \int\limits_{\mathcal{O}} b \cdot \nabla\varphi\ dxdt + \int\limits_0^T \int\limits_{\mathcal{O}} \varphi\ d(\operatorname{div} b)(t,x)$$

*for all $\varphi \in C_c^\infty\left([0,T) \times \mathbb{R}^N\right)$. Also, if $w \in L^\infty((0,T) \times \mathcal{O})$, $c \in L^1((0,T) \times \mathcal{O})$ and $g \in L^1((0,T) \times \mathcal{O})$ satisfy*

$$\int\limits_0^T \int\limits_{\mathcal{O}} w(\partial_t\eta + b \cdot \nabla\eta) + g\eta + cw\eta\ dxdt = 0$$

*for all* $\eta \in C_c^\infty((0,T) \times \mathcal{O})$*, then there are two uniquely determined functions, which in the following are denoted by* $Tr(bw) \in L^\infty((0,T) \times \partial\mathcal{O}, \mathcal{L}^1 \otimes \mathcal{H}^{N-1})$ *and* $w_0 \in L^\infty(\mathcal{O})$*, that satisfy*

$$\int_0^T \int_{\partial\mathcal{O}} Tr(bw)\varphi \, d\mathcal{H}^{N-1}(x)dt - \int_{\mathcal{O}} \varphi(0,\cdot)w_0 \, dx = \int_0^T \int_{\mathcal{O}} w(\partial_t\varphi + b \cdot \nabla\varphi) + g\varphi + wc\varphi \, dxdt$$

*for all* $\varphi \in C_c^\infty\left([0,T) \times \mathbb{R}^N\right)$.

This result in [CDS14b] contains almost all statements which we need for extending a weak solution $u$ on $(0,T)\times\Omega$ to a weak solution on $(0,T)\times\mathbb{R}^N$. The last ingredient is a result about the form of the function $Tr(bw)$. In our case it would be useful if $Tr(bw) \equiv 0$. Fortunately, the following lemma yields this result for our case.

**Lemma 3.1.23** *Let the assumptions of Lemma 3.1.22 holds. Then* $Tr(bw) \equiv 0$ *if* $Tr(b) \equiv 0$.

**Proof:** The statement is a consequence of Theorem 4.2 in [ACM07].

□

Now, we are able to show the main lemma needed for extending weak solutions.

**Lemma 3.1.24** *Let* $b \in L^1((0,T), BV_0(\Omega))^N \cap L^\infty((0,T)\times\Omega)^N$ *such that* $\operatorname{div} b \in L^1((0,T)\times\Omega)$*. Let* $u \in C([0,T], L^\infty(\Omega) - w^*)$ *be a weak solution of* (3.1) *with initial data* $u_0 \in L^\infty(\Omega)$*. Then, for all* $\psi \in C_c^\infty\left([0,T) \times \mathbb{R}^N\right)$ *we obtain:*

$$\begin{aligned}\int_0^T \int_\Omega u(t,x)\left(\partial_t\psi(t,x) + b(t,x)\cdot\nabla\psi(t,x) + \psi(t,x)\operatorname{div} b(t,x)\right) \, dxdt \\ = -\int_\Omega u_0(x)\psi(0,x) \, dx\end{aligned} \tag{3.20}$$

**Proof:** We apply Lemma 3.1.22 and obtain, that there are functions $w_0 \in L^\infty(\Omega)$ and $Tr(bu) \in L^\infty((0,T)\times\partial\Omega, \mathcal{L}^1 \otimes \mathcal{H}^{N-1})$ such that

$$\int_0^T \int_{\partial\mathcal{O}} Tr(bu)\varphi \, d\mathcal{H}^{N-1}(x)dt - \int_{\mathcal{O}} \varphi(0,\cdot)w_0 \, dx = \int_0^T \int_{\mathcal{O}} u(\partial_t\varphi + b\cdot\nabla\varphi) + u \operatorname{div} b\varphi \, dxdt$$

for all $\varphi \in C_c^\infty\left([0,T)\times\mathbb{R}^N\right)$. Obviously, if we take $\psi \in C_c^\infty(\Omega)$ and $\eta \in C_c^\infty([0,T))$ with $\eta(0) = 1$, we obtain for $\varphi = \psi\eta$

$$\int_\Omega \psi(x)w_0(x) \, dx = \int_\Omega \psi(x)u_0(x) \, dx.$$

Since $\psi$ can be chosen arbitrarily, we get that $w_0 = u_0$. It remains to show that $Tr(bu) \equiv 0$. We show that $Tr(b) \equiv 0$. Then Lemma 3.1.23 yields that $Tr(bu) \equiv 0$. We define the vector field

$$B(t,x) = \begin{cases} (1, b(t,x)) & \text{if } (t,x) \in (0,T)\times\Omega, \\ 0 & \text{if } (t,x) \notin (0,T)\times\Omega. \end{cases}$$

Then we have $\mathrm{Div}_{t,x} B|_{(0,T)\times\Omega} = \operatorname{div} b \left(\mathcal{L}^1 \otimes \mathcal{L}^N\right)$ and we can apply Lemma 3.2 in [CDS14b]. The definition of normal trace now yields

$$\langle Tr(B, \partial((0,T)\times\Omega)), \varphi\rangle = \int_0^T \int_\Omega \nabla_{t,x}\varphi \cdot B \; dxdt + \int_0^T \int_\Omega \varphi \operatorname{div} b(t,x)) \; dxdt$$

for all $\varphi \in C_c^\infty\left(R^{N+1}\right)$. Obviously, we have that

$$Tr(B, \partial((0,T)\times\Omega))|_{\{0\}\times\Omega} = -1 \quad \text{and} \quad Tr(B, \partial((0,T)\times\Omega))|_{\{T\}\times\Omega} = 1.$$

Since due to the proof of Lemma 3.1.22, $Tr(b) = Tr(B, \partial((0,T)\times\Omega))|_{(0,T)\times\partial\Omega}$, we deduce

$$\langle Tr(B, \partial((0,T)\times\Omega))|_{(0,T)\times\partial\Omega}, \varphi\rangle = \int_0^T \int_\Omega \nabla_x\varphi(t,x)\cdot b(t,x) \; dx + \int_\Omega \varphi(t,x) \operatorname{div} b(t,x) \; dxdt$$

for all $\varphi \in C_c^\infty\left(R^{N+1}\right)$. Using Theorem 1 in chapter 5.3 in [EG92], we obtain that

$$\int_\Omega \nabla\varphi(t,x)\cdot b(t,x) \; dx + \int_\Omega \varphi(t,x) \operatorname{div} b(t,x) \; dx = 0$$

for almost all $t \in (0,T)$, since $b(t,\cdot) \in BV_0(\Omega)^N$ for almost all $t \in (0,T)$. Thus, we have

$$Tr(b) = Tr(B, \partial((0,T)\times\Omega))|_{(0,T)\times\partial\Omega} = 0.$$

□

Finally, we are prepared for the proof of the main results of this subsection. For

$$b \in L^1((0,T), BV_0(\Omega))^N \quad \text{with} \quad \operatorname{div} b \in L^1((0,T)\times\Omega),$$

using Lemma 3.1.3, we denote by $\bar{b} \in L^1\left((0,T), BV\left(\mathbb{R}^N\right)\right)$ the extension of $b$ to the entire $\mathbb{R}^N$ in the spatial variable. In addition we denote by $\bar{u}_0 \in L^\infty\left(\mathbb{R}^N\right)$ the extension by zero of $u_0 \in L^\infty(\Omega)$ to the entire $\mathbb{R}^N$.

**Theorem 3.1.25 (Extension of weak solutions)** *Let $b \in L^1((0,T), BV_0(\Omega))^N$ be a bounded vector field such that $\operatorname{div} b \in L^1((0,T)\times\Omega)$. Furthermore, let $u_0 \in L^\infty(\Omega)$ and denote $u \in C([0,T], L^\infty(\Omega) - w^*)$ some weak solution of (3.1) with initial data $u_0$. Then*

$$\bar{u}(t,x) = \begin{cases} u(t,x) & \text{if } (t,x) \in (0,T)\times\Omega \\ 0 & \text{if } (t,x) \in (0,T)\times\mathbb{R}^N\setminus\Omega \end{cases}$$

*is a weak solution of the transport equation with vector field $\bar{b}$ and initial data $\bar{u}_0$.*

**Proof:** Obviously, $\bar{u} \in C\left([0,T], L^\infty\left(\mathbb{R}^N\right) - w^*\right)$. Let $\psi \in C_c^\infty\left([0,T)\times\mathbb{R}^N\right)$. Then, Lemma 3.1.24 yields

$$\int_0^T \int_{\mathbb{R}^N} \bar{u}(\partial_t\psi + \bar{b}\cdot\nabla\psi + \psi \operatorname{div} b) \; dxdt = \int_0^T \int_\Omega u(\partial_t\psi + b\cdot\nabla\psi + \psi \operatorname{div} b) \; dxdt$$

$$= -\int_{\Omega} u_0 \psi(0,\cdot)\ dx = -\int_{\mathbb{R}^N} \bar{u}_0 \psi(0,\cdot)\ dx.$$

□

**Theorem 3.1.26 (Renormalization property on bounded domains)** *Let $b$ be a bounded vector field belonging to $L^1((0,T), BV_0(\Omega))^N$, such that $\operatorname{div} b \in L^1((0,T)\times\Omega)$. Then $b$ has the renormalization property.*

**Proof:** Let $u \in L^\infty((0,T)\times\Omega)$ be a weak solution of the transport equation with vector field $b$ and initial data $u_0 \in L^\infty(\Omega)\cap BV(\Omega)$. Then Theorem 3.1.25 yields that there exists a solution extension $\bar{u}$ of $u$ to the entire $\mathbb{R}^N$ in the space variable. Hence, since $\bar{u}$ is a weak solution of the transport equation with vector field $\bar{b}$ and initial data $\bar{u}_0$, we obtain that $\bar{u}$ is a renormalized solution since $\bar{b}$ has the renormalization property due to Theorem 3.1.10, i.e. for $\beta \in C^1(\mathbb{R})$ and $\varphi \in C_c^\infty([0,T)\times\Omega)$ we obtain

$$\begin{aligned}\int_0^T\int_{\Omega} \beta(u)(\partial_t\varphi + b\cdot\nabla\varphi + \varphi \operatorname{div} b)\ dxdt &= \int_0^T\int_{\mathbb{R}^N} \beta(\bar{u})(\partial_t\varphi + \bar{b}\cdot\nabla\varphi + \varphi \operatorname{div}\bar{b})\ dxdt \\ &= -\int_{\mathbb{R}^N} \beta(\bar{u}_0)\varphi(0,\cdot)\ dx = -\int_{\Omega} \beta(u_0)\varphi(0,\cdot)\ dx.\end{aligned}$$

□

The above theorem finishes the theory about well-posedness of solutions of the transport equation on the domain $(0,T)\times\Omega$. Therefore we can define the solution operator $S$: we set

$$\mathrm{VF} := \left\{ b \in L^1((0,T), BV(\Omega))^N \cap L^\infty((0,T)\times\Omega)^N \middle|\ \operatorname{div} b \in L^1((0,T), L^\infty(\Omega)) \right\} \tag{3.21}$$

and

$$\mathrm{VF}_0 := \left\{ b \in \mathrm{VF} \middle|\ b \in L^1((0,T), BV_0(\Omega))^N \right\}.$$

Then, the solution operator $S$ is given by

$$\begin{aligned} S : L^\infty(\Omega)\times \mathrm{VF}_0 &\to C([0,T], L^\infty(\Omega) - w^*), \\ (u_0, b) &\mapsto S(u_0,b) = u, \end{aligned} \tag{3.22}$$

where $u$ denotes the unique weak solution of the transport equation (3.1) in the case $\mathcal{O} = \Omega$. In Theorem 3.1.9, a first stability statement was shown. In the following section, we highly improve this statement.

## 3.2. Stability results for the transport equation

### 3.2.1. A compensated compactness result for weakly convergent sequences

In this first subsection, we prove a result which reminds one of the compensated compactness results of Tartar ([Tar79]) and Murat ([Mur81]): the product of two weakly convergent sequences converges to the product of their weak limits if the sequences satisfy some regularity assumptions. The theorem we present is a generalization of Proposition 1 in [Mou16] to the case that one of the sequences has codomain $BV(\Omega)$ instead of Sobolev regularity as in [Mou16]. We will use this statement in the proofs for the stability theorems where we will be faced with the situation that we have to specify the limit of the product of weakly convergent vector fields with their weakly convergent solutions. We start with the main statement.

**Theorem 3.2.1** *Let $q \in (1,\infty]$. Furthermore, let $(f_n) \subset L^q((0,T), BV_0(\Omega))$ and $(g_n) \subset L^{q'}((0,T), L^\infty(\Omega))$ be bounded sequences such that*

$$f_n \rightharpoonup f \quad \text{in } L^1((0,T)\times\Omega) \quad \text{and} \quad g_n \rightharpoonup g \quad \text{in } L^{q'}((0,T)\times\Omega),$$

*where $f \in L^q((0,T), BV_0(\Omega))$ and $g \in L^{q'}((0,T), L^\infty(\Omega))$. If $(\partial_t g_n)$ is a bounded sequence in $L^1((0,T), (W^{m,2}(\Omega))')$ for some $m \in \mathbb{N}$, then*

$$f_n g_n \overset{*}{\rightharpoonup} fg \quad \text{in } \mathcal{M}((0,T)\times\Omega).$$

Before we prove this theorem we need two auxiliary lemmas.

**Lemma 3.2.2** *Let $q \in [1,\infty]$ and let $(f_n) \subset L^q((0,T), BV_0(\Omega))$ be a bounded sequence. Then*

$$f_n(\cdot,\cdot+h) - f_n \to 0 \quad \text{in } L^q((0,T), L^1(\Omega)) \quad \text{as } h \to 0$$

*uniformly in $n \in \mathbb{N}$.*

**Proof:** We extend the functions $f_n$ by zero to the entire $\mathbb{R}^N$ in the spatial variable and convolve those with the standard mollifier in the spatial domain:

$$g_{n,k} := f_n * \rho_{1/k}.$$

Then, for $g_{n,k}$, for almost all $t \in (0,T)$ and for $h \in \mathbb{R}^N$ we obtain the estimate

$$\begin{aligned}
\int_{\mathbb{R}^N} |g_{n,k}(t,x+h) - g_{n,k}(t,x)| \; dx &= \int_{\mathbb{R}^N} \left| \int_0^1 \nabla g_{n,k}(t,x+rh)^\top h \; dr \right| \; dx \\
&\leq |h|_\infty \int_0^1 \int_{\mathbb{R}^N} |\nabla g_{n,k}(t,x)|_1 \; dxdr \\
&\leq |h|_\infty \left\|\nabla f_n(t,\cdot)\right\|_{\mathcal{M}(\Omega)^N}.
\end{aligned}$$

Integrating over $(0,T)$ yields

$$\left( \int_0^T \|g_{n,k}(t,\cdot+h) - g_{n,k}(t,\cdot)\|_{L^1(\Omega)}^q \; dt \right)^{1/q} \leq |h|_\infty \|f_n\|_{L^q((0,T),BV(\Omega))} \leq C\,|h|_\infty,$$

where $C > 0$ denotes an upper bound for the sequence $(f_n)$. With the following estimate

$$\|f_n(\cdot,\cdot+h) - f_n\|_{L^q((0,T,L^1(\Omega))} = \|f_n(\cdot,\cdot+h) - f_n\|_{L^q((0,T,L^1(\mathbb{R}^N))}$$

$$\leq 2 \|f_n - g_{n,k}\|_{L^q((0,T,L^1(\mathbb{R}^N))} + \|g_{n,k}(t, \cdot + h) - g_{n,k}\|_{L^q((0,T),L^1(\mathbb{R}^N))}$$

we conclude: for every $\varepsilon > 0$ we choose for each $n \in \mathbb{N}$ $k(n) \in \mathbb{N}$ such that

$$\|f_n - g_{n,k}\|_{L^q((0,T,L^1(\mathbb{R}^N))} \leq \frac{\varepsilon}{4}$$

for all $k \geq k(n)$ and $\delta = \varepsilon/2C$. Then for $|h|_\infty \leq \delta$

$$\|f_n(\cdot, \cdot + h) - f_n\|_{L^q((0,T,L^1(\Omega))} \leq 2\frac{\varepsilon}{4} + C\,|h|_\infty \leq \varepsilon.$$

□

**Lemma 3.2.3** *Let $q \in [1, \infty]$, $\rho \in C_c^\infty\left(\mathbb{R}^N\right)$ a mollifier for the spatial variable and let $(f_n) \subset L^q((0,T), BV_0(\Omega))$ and $(g_n) \subset L^{q'}((0,T), L^\infty(\Omega))$ be bounded sequences. Then, the commutator*

$$S_{n,k} := f_n(g_n * \rho_{1/k}) - (f_n g_n) * \rho_{1/k}$$

*converges uniformly in $n \in \mathbb{N}$ to zero in $L^1((0,T) \times \Omega)$ as $k \to \infty$.*

**Proof:** For $t \in (0,T)$ and $x \in \Omega$ we have

$$S_{n,k}(t,x) = \int\limits_{\mathbb{R}^N} \left(f_n(t,x) - f_n(t, x-y)\right) g_n(t, x-y)\rho_{1/k}(y)\ dy$$

and thus, integrating over $(0,T) \times \Omega$ yields

$$\begin{aligned}\int\limits_0^T \int\limits_\Omega |S_{n,k}(t,x)|\ dxdt &\leq \|g_n\|_{L^{q'}((0,T),L^\infty(\Omega))} \int\limits_{\mathbb{R}^N} \rho_{1/k}(y)\,\|f_n - f_n(\cdot, \cdot - y)\|_{L^q((0,T),L^1(\Omega))}\ dy \\ &\leq C \int\limits_{\{y|\ |y| \leq 1/k\}} \rho_{1/k}(y)\,\|f_n - f_n(\cdot, \cdot - y)\|_{L^q((0,T),L^1(\Omega))}\ dy,\end{aligned}$$

where $C > 0$ denotes an upper bound for $(g_n)$ in $L^{q'}((0,T), L^\infty(\Omega))$. Then, Lemma 3.2.2 yields the statement.

□

Now, we can prove Theorem 3.2.1. The proof is a reproduction of the proof of Proposition 1 in [Mou16] adjusted and extended to functions $f_n, f \in L^q((0,T), BV_0(\Omega))$ and weak convergence in $L^1((0,T) \times \Omega)$.

**Proof:** We do the same steps as in the previously mentioned proof. Obviously, using Lebesgue's dominated convergence theorem, we obtain

$$f(g * \rho_{1/k}) \to fg \quad \text{in } L^1((0,T) \times \Omega) \quad \text{as } k \to \infty. \tag{3.23}$$

Furthermore, for a fixed $k \in \mathbb{N}$, since $(g_n) \subset L^{q'}((0,T), L^\infty(\Omega))$ is bounded we obtain that

$$(g_n * \rho_{1/k})_n \quad \text{and} \quad \left(\nabla(g_n * \rho_{1/k})\right)_n = \left(g_n * \nabla\rho_{1/k}\right)_n$$

are bounded in $L^1((0,T)\times\Omega)$ and $L^1((0,T)\times\Omega)^N$, respectively. In addition, if we consider for almost all $t\in(0,T)$ $\partial_t g_n(t,\cdot)$ as a distribution on $\mathbb{R}^N$, i.e. if we define its application on $\varphi\in C_c^\infty(R^N)$ as

$$\partial_t g_n(t,\cdot)(\varphi|_\Omega)$$

then, the convolution is defined as

$$(\partial_t g_n(t,\cdot)*\rho_{1/k})(x)=\partial_t g_n(t,\cdot)(\rho_{1/k}(x-\cdot)|_\Omega).$$

Hence, we obtain for $\varphi\in C_0((0,T)\times\Omega)$

$$\begin{aligned}
&\left|\int_0^T\int_\Omega(\partial_t g_n(t,\cdot)*\rho_{1/k})(x)\varphi(t,x)\ dxdt\right|\\
&\quad=\left|\int_0^T\int_\Omega \partial_t g_n(t,\cdot)(\rho_{1/k}(x-\cdot)|_\Omega)\varphi(t,x)\ dxdt\right|\\
&\quad\le\|\varphi\|_{C((0,T)\times\Omega)}\int_0^T\int_\Omega\left\|\rho_{1/k}(x-\cdot)\right\|_{W^{m,2}(\Omega)}\|\partial_t g_n(t,\cdot)\|_{(W^{m,2}(\Omega))'}\ dxdt\\
&\quad\le|\Omega|\,\|\varphi\|_{C((0,T)\times\Omega)}\left\|\rho_{1/k}\right\|_{W^{m,2}(\mathbb{R}^N)}\|\partial_t g_n\|_{L^1((0,T),(W^{m,2}(\Omega))')}\\
&\quad\le C_k\,\|\varphi\|_{C((0,T)\times\Omega)}\,,
\end{aligned}$$

where $C_k>0$ denotes a bound depending on $k\in\mathbb{N}$. Thus, the principle of uniform boundedness yields that $\big(\partial_t(g_n*\rho_{1/k})\big)$ is a bounded sequence in $\mathcal{M}((0,T)\times\Omega)$. Summing up, we have that $(g_n*\rho_{1/k})_n$ is a bounded sequence in $BV((0,T)\times\Omega)$ for any $k\in\mathbb{N}$. Thus, for a fixed $k\in\mathbb{N}$, there exists a subsequence $(g_{n_l}*\rho_{1/k})_l$ being convergent to some $h_k$ in $L^1((0,T)\times\Omega)$. Since $g_n\rightharpoonup g$ in $L^{q'}((0,T)\times\Omega)$ we easily obtain that $g_n*\rho_{1/k}\rightharpoonup g*\rho_{1/k}$ in $L^1((0,T)\times\Omega)$ as $n\to\infty$ and thus $h_k=g*\rho_{1/k}$. With a proof by contradiction we deduce that the whole sequence $g_n*\rho_{1/k}\to g*\rho_{1/k}$ in $L^1((0,T)\times\Omega)$ as $n\to\infty$. Now, using a standard diagonal argument, we can find a subsequence (labeled by $n$ again) such that

$$g_n*\rho_{1/k}(t,x)\to g*\rho_{1/k}(t,x)\quad\text{for almost all }(t,x)\in(0,T)\times\Omega\text{ and for all }k\in\mathbb{N}$$

as $n\to\infty$. In addition, we have that $(g_n*\rho_{1/k})_n$ is a bounded subset of $L^\infty((0,T)\times\Omega)$ for each $k\in\mathbb{N}$ which is a consequence of the boundedness of $(g_n)$ in $L^{q'}((0,T),L^\infty(\Omega))$ and of $(\partial_t g_n)$ in $L^1((0,T),(W^{m,2}(\Omega))')$. Thus, $g_n*\rho_{1/k}\to g*\rho_{1/k}$ in $L^p((0,T)\times\Omega)$ for any $p<\infty$. Furthermore, $(f_n)$ is bounded in $L^r((0,T)\times\Omega)$ for $r=\min(q,N/(N-1))$ and we obtain for any $\varphi\in L^\infty((0,T)\times\Omega)$ and $k\in\mathbb{N}$

$$\begin{aligned}
\left|\langle f_n(g_n*\rho_{1/k})-f(g*\rho_{1/k}),\varphi\rangle\right|&\le\|\varphi\|_{L^\infty((0,T)\times\Omega)}\|f_n\|_{L^r((0,T)\times\Omega)}\\
&\quad\cdot\left\|g_n*\rho_{1/k}-g*\rho_{1/k})\right\|_{L^{r'}((0,T)\times\Omega)}\\
&\quad+\left|\langle f_n-f,(g*\rho_{1/k})\varphi\rangle\right|\to0
\end{aligned}\tag{3.24}$$

as $n\to\infty$, i.e. $f_n(g_n*\rho_{1/k})\rightharpoonup f(g*\rho_{1/k})$ in $L^1((0,T)\times\Omega)$. Finally, we deduce that for any fixed $\varphi\in C_0((0,T)\times\Omega)$

$$\begin{aligned}
\left|\langle(f_ng_n)*\rho_{1/k}-f_ng_n,\varphi\rangle\right|&=\left|\langle f_ng_n,\varphi*\rho_{1/k}-\varphi\rangle\right|\\
&\le\|f_ng_n\|_{L^1((0,T)\times\Omega)}\left\|\varphi*\rho_{1/k}-\varphi\right\|_{C((0,T)\times\Omega)}\\
&\le C\left\|\varphi*\rho_{1/k}-\varphi\right\|_{C((0,T)\times\Omega)}\to0
\end{aligned}\tag{3.25}$$

since $\varphi$ is uniformly continuous in $(0,T)\times\Omega$. Summing up, we conclude for any $\varphi \in C_0((0,T)\times\Omega)$:

$$\begin{aligned}|\langle fg - f_n g_n, \varphi\rangle| \leq &\left|\langle fg - f(g * \rho_{1/k}), \varphi\rangle\right| \\ &+ \left|\langle f(g * \rho_{1/k}) - f_n(g_n * \rho_{1/k}), \varphi\rangle\right| \\ &+ \left|\langle f_n(g_n * \rho_{1/k}) - (f_n g_n) * \rho_{1/k}, \varphi\rangle\right| \\ &+ \left|\langle (f_n g_n) * \rho_{1/k} - f_n g_n, \varphi\rangle\right|.\end{aligned}$$

Then, the first, third and fourth term on the right side converge uniformly in $n \in \mathbb{N}$ as $k \to \infty$ due to Lemma 3.2.3 and estimates (3.23) and (3.25). Therefore, for any $\varepsilon$ we choose $k(\varepsilon) \in \mathbb{N}$ such that the sum of the first, third and fourth term is smaller than $\varepsilon$ for any $k \geq k(\varepsilon)$. Then for $k(\varepsilon)$ fixed, we can choose $n(\varepsilon) \in \mathbb{N}$ such that the second term is smaller than $\varepsilon$ for all $n \geq n(\varepsilon)$ due to estimate (3.24). Consequently,

$$|\langle fg - f_n g_n, \varphi\rangle| \leq 2\varepsilon \quad \forall\, n \geq n(\varepsilon)$$

which proves the statement.

□

### 3.2.2. Stability of solution operator: first improvement

In the works [Cri07, DL89] of Crippa, DiPerna and Lions, it is mentioned (and proven) that solutions of the transport equation are elements of $C\left([0,T], L^p_{loc}\left(\mathbb{R}^N\right)\right)$ for any $p \in [1,\infty)$. This can be easily deduced from the renormalization property of solutions. In [DL89] it is additionally shown that sequences of solutions are strongly convergent in $C\left([0,T], L^p_{loc}\left(\mathbb{R}^N\right)\right)$ if the sequences of vector fields and initial data satisfy some convergence assumptions. For the proof, arguments of Arzelà-Ascoli type are used. Arzelà-Ascoli is also used by Crippa in [Cri07], but it is just shown that sequences of solutions are convergent in $C\left([0,T], L^p\left(\mathbb{R}^N\right) - w\right)$. In the first stability theorem we present the proof for convergence in $C([0,T], L^p(\Omega) - w)$ based on the theorem of Arzelà-Ascoli in locally convex spaces. In contrast to Crippa where strong convergence of the vector fields is required, our assumptions only demand weak convergence of the vector fields in $L^1((0,T)\times\Omega)^N$. In [DL89], it is shown that weak convergence of the vector fields is sufficient if the uniform convergence of the translation relation appearing in Lemma 3.2.2 is satisfied by the sequence of vector fields. In addition, it is also mentioned that this condition is fulfilled if the vector fields are a bounded sequence in $L^q((0,T), X)^N$, where $X$ is a Banach space embedding compactly into $L^1(\Omega)$. In Lemma 3.2.2, we have shown this for the special case $X = BV_0(\Omega)$. These results were sufficient for DiPerna and Lions to prove weak convergence of $b_n u_n$ to $bu$ in $L^1((0,T)\times\Omega)^N$ which we summed up to the compensated compactness result in the previous subsection. With the aid of some auxiliary statements building on renormalization arguments we additionally show strong convergence of solutions in $C([0,T], L^p(\Omega))$ for any $p \in [1,\infty)$. We start with the main statement of this subsection.

**Theorem 3.2.4 (First stability theorem)** *Let $b \in \mathrm{VF}_0$ and let the initial value $u_0 \in L^\infty(\Omega)$. Furthermore, let $(b_n) \subset \mathrm{VF}_0$ and $(u_{0,n}) \subset L^\infty(\Omega)$ be two sequences with the following properties:*

*(i) $(u_{0,n})$ is bounded in $L^\infty(\Omega)$ and converges to $u_0$ in $L^1(\Omega)$,*

*(ii)* *(a)* $(b_n)$ *converges strongly to* $b$ *in* $L^1((0,T)\times\Omega)^N$ *or*

*(b)* $(b_n)$ *is bounded in* $L^q((0,T),BV_0(\Omega))^N$ *for some* $q>1$ *and* $b_n \rightharpoonup b$ *in* $L^1((0,T)\times\Omega)^N$.

*(iii)* $(\operatorname{div} b_n)$ *converges strongly to* $\operatorname{div} b$ *in* $L^1((0,T)\times\Omega)$.

*Then, for any* $1\le p<\infty$*, the sequence of unique solutions* $(u_n)\subset C([0,T],L^\infty(\Omega)-w^*)$ *of* (3.1) *with vector fields* $b_n$ *and initial data* $u_{0,n}$ *is a subset of* $C([0,T],L^p(\Omega))$ *and converges in* $C([0,T],L^p(\Omega))$ *to the unique solution* $u\in C([0,T],L^p(\Omega))$ *of* (3.1) *with vector field* $b$ *and initial value* $u_0$.

Before we are able to prove this statement we need two auxiliary lemmas which we introduce in the following.

**Lemma 3.2.5** *Let* $g,g^2\in C([0,T],L^2(\Omega)-w)$. *Then* $g\in C([0,T],L^2(\Omega))$.

**Proof:** For $\varphi\equiv 1\in L^2(\Omega)$ we have

$$\int_\Omega \left(g^2(t,x)-g^2(s,x)\right)\varphi\,dx\to 0 \qquad \text{as } t\to s \text{ in } [0,T],$$

i.e.

$$\|g(t,\cdot)\|_{L^2(\Omega)}\to\|g(s,\cdot)\|_{L^2(\Omega)} \qquad \text{as } t\to s \text{ in } [0,T].$$

In addition we have $g(t,\cdot)\rightharpoonup g(s,\cdot)$ in $L^2(\Omega)$ as $t\to s$. Thus, using Theorem 1.37 in [AFP00], we obtain that

$$\|g(t,\cdot)-g(s,\cdot)\|_{L^2(\Omega)}\to 0 \qquad \text{as } t\to s \text{ in } [0,T].$$

□

**Lemma 3.2.6** *Let* $(g_n),(g_n^2)\subset C([0,T],L^2(\Omega)-w)$ *be two sequences such that*

$$g_n\to g \quad \text{and} \quad g_n^2\to g^2 \quad \text{in } C([0,T],L^2(\Omega)-w),$$

*where* $g,g^2\in C([0,T],L^2(\Omega)-w)$. *Then* $g_n,g\in C([0,T],L^2(\Omega))$ *for all* $n\in\mathbb{N}$ *and* $g_n\to g$ *in* $C([0,T],L^2(\Omega))$.

**Proof:** The previous Lemma 3.2.5 yields that $g_n,g\in C([0,T],L^2(\Omega))$ for all $n\in\mathbb{N}$. Furthermore for all $\varphi\in L^2(\Omega)$ we have

$$\sup_{t\in[0,T]}\left|\int_\Omega \varphi(x)(g_n(t,x)^2-g(t,x)^2)\,dx\right|\to 0 \qquad \text{as } n\to\infty,$$

and thus $\|g_n(t,\cdot)\|^2_{L^2(\Omega)}\to\|g(t,\cdot)\|^2_{L^2(\Omega)}$ in $C([0,T])$. In addition, we estimate

$$\sup_{t\in[0,T]}\left|\int_\Omega (g_n(t,x)-g(t,x))^2\,dx\right|\le\sup_{t\in[0,T]}\left|\int_\Omega \left(g_n(t,x)^2-g(t,x)^2\right)\,dx\right| \tag{3.26}$$

$$+ 2 \sup_{t\in[0,T]} \left| \int_\Omega g(t,x)(g(t,x) - g_n(t,x))\, dx \right| \tag{3.27}$$

Obviously, for $n \to \infty$ term (3.26) tends to zero. For the second term (3.27) we introduce the operators

$$L_n : L^2(\Omega) \to \mathbb{R}, \qquad \varphi \mapsto \sup_{t\in[0,T]} \left| \int_\Omega \varphi(x)(g(t,x) - g_n(t,x))\, dx \right|.$$

These operators are Lipschitz continuous: if we set

$$h_\varphi(t) := \int_\Omega \varphi(x)(g(t,x) - g_n(t,x))\, dx,$$

then $h_\varphi \in C([0,T])$ for $\varphi \in L^2(\Omega)$ and we estimate

$$\begin{aligned} |L_n(\varphi) - L_n(\psi)| &= \left| \|h_\varphi\|_{C([0,T])} - \|h_\psi\|_{C([0,T])} \right| \le \|h_\varphi - h_\psi\|_{C([0,T])} \\ &\le \sup_{t\in[0,T]} \|g(t,\cdot) - g_n(t,\cdot)\|_{L^2(\Omega)} \|\varphi - \psi\|_{L^2(\Omega)} \le C \|\varphi - \psi\|_{L^2(\Omega)} \end{aligned}$$

for some $C > 0$, independent of $n \in \mathbb{N}$ since

$$\begin{aligned} \|g_n(t,\cdot)\|^2_{L^2(\Omega)} &\le \left| \|g_n(t,\cdot)\|^2_{L^2(\Omega)} - \|g(t,\cdot)\|^2_{L^2(\Omega)} \right| + \|g(t,\cdot)\|^2_{L^2(\Omega)} \\ &\le \sup_{t\in[0,T]} \left| \|g_n(t,\cdot)\|^2_{L^2(\Omega)} - \|g(t,\cdot)\|^2_{L^2(\Omega)} \right| + \sup_{t\in[0,T]} \|g(t,\cdot)\|^2_{L^2(\Omega)} \le C. \end{aligned}$$

Now we have the set

$$A := \{g(t,\cdot) | t \in [0,T]\} \subset L^2(\Omega).$$

This set is compact since $g \in C([0,T], L^2(\Omega))$ and thus it is the image of a compact set under a continuous function. Hence, for each operator $L_n$ there exists an element $\varphi_n \in A$ such that

$$L_n(\varphi_n) = \max_{\psi\in A} L_n(\psi).$$

Since $(\varphi_n) \subset A$, there exists a subsequence $(\varphi_{n_k})$ converging to some $\varphi \in A$ in $L^2(\Omega)$. Furthermore we have the estimate for any $n \in \mathbb{N}$

$$\left| \int_\Omega g(t,x)(g(t,x) - g_n(t,x))\, dx \right| \le \sup_{s\in[0,T]} \left| \int_\Omega g(t,x)(g(s,x) - g_n(s,x))\, dx \right| \le L_n(\varphi_n).$$

Thus, we conclude

$$\begin{aligned} \sup_{t\in[0,T]} \left| \int_\Omega g(t,x)(g(t,x) - g_{n_k}(t,x))\, dx \right| &\le \sup_{t\in[0,T]} \left| \int_\Omega (\varphi_{n_k}(x) - \varphi(x))(g(t,x) - g_{n_k}(t,x))\, dx \right| \\ &+ \sup_{t\in[0,T]} \left| \int_\Omega \varphi(x)(g(t,x) - g_{n_k}(t,x))\, dx \right| \end{aligned}$$

$$\leq C \left\|\varphi_{n_k} - \varphi\right\|_{L^2(\Omega)} + \sup_{t\in[0,T]} \left| \int_\Omega \varphi(x)(g(t,x) - g_{n_k}(t,x))\, dx \right|.$$

Both terms tend to zero as $k \to \infty$. Summing up, the term in (3.27) converges to 0 for $n = n_k$, $k \to \infty$ and therefore, $g_{n_k} \to g$ in $C([0,T], L^2(\Omega))$. Now a standard proof by contradiction yields that the whole sequence $(g_n)$ converges to $g$ in $C([0,T], L^2(\Omega))$.

□

With the above lemma we have collected all required statements and we can prove Theorem 3.2.4.

**Proof:** We divide the proof in two parts. In the first part we prove the result for the case $p = 2$ and in the second part we prove the general result.

*First part:* Let $(b_n) \subset \mathrm{VF}_0$ be a sequence such that $(b_n)$ satisfies case (a) or (b) as well as $(\operatorname{div} b_n)$ converges to $\operatorname{div} b$ in $L^1((0,T) \times \Omega)$. In addition, let $(u_{0,n})$ be a bounded sequence in $L^\infty(\Omega)$ and being convergent to $u_0$ in $L^1(\Omega)$. Then Remark 3.1.6 yields that for $t \in [0,T]$ and any $n \in \mathbb{N}$

$$\left\|u_n(t,\cdot)\right\|_{L^2(\Omega)} \leq C \left\|u_n(t,\cdot)\right\|_{L^\infty(\Omega)} \leq C \left\|u_{0,n}\right\|_{L^\infty(\Omega)} \leq CC_1$$

for some $C, C_1 > 0$. Therefore, $(u_n(t,\cdot)) \subset L^2(\Omega)$ is a relatively compact subset with respect to the weak topology in $L^2(\Omega)$ for all $t \in [0,T]$. In addition, we have for $\varphi \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty((0,T))$

$$\begin{aligned}\int_0^T \psi(t) \frac{d}{dt} \langle u_n(t,\cdot), \varphi\rangle\, dt &= -\int_0^T \psi'(t) \langle u_n(t,\cdot), \varphi\rangle\, dt \\ &= \int_0^T \psi(t)\langle u_n(t,\cdot) b_n(t,\cdot), \nabla\varphi\rangle + \psi(t)\langle u_n(t,\cdot) \operatorname{div} b_n(t,\cdot), \varphi\rangle\, dt,\end{aligned}$$

i.e. $t \mapsto \langle u_n(t,\cdot), \varphi\rangle$ is an element of $W^{1,1}((0,T))$ with weak derivative

$$t \mapsto \langle u_n(t,\cdot) b_n(t,\cdot), \nabla\varphi\rangle + \langle u_n(t,\cdot) \operatorname{div} b_n(t,\cdot), \varphi\rangle.$$

We estimate for $t, s \in [0,T]$ with $s < t$

$$\begin{aligned}\int_s^t \left| \frac{d}{dr} \langle u_n(r,\cdot), \varphi\rangle \right| dr &\leq \int_s^t \left\|u_n(r,\cdot)\right\|_{L^\infty(\Omega)} \left\|\nabla\varphi\right\|_{C(\Omega)^N} \left\|b_n(r,\cdot)\right\|_{L^1(\Omega)^N}\, dr \\ &\quad + \left\|\varphi\right\|_{C(\Omega)} \int_s^t \left\|u_n(r,\cdot)\right\|_{L^\infty(\Omega)} \left\|\operatorname{div} b_n(r,\cdot)\right\|_{L^1(\Omega)}\, dr \\ &\leq C_1 C \cdot C(\varphi) \int_s^t \left\|b_n(r,\cdot)\right\|_{L^1(\Omega)^N}\, dr + C_1 C \left\|\varphi\right\|_{C(\Omega)} \int_s^t \left\|\operatorname{div} b_n(r,\cdot)\right\|_{L^1(\Omega)}\, dr,\end{aligned}$$

where $C(\varphi) > 0$ is a bound depending on $\varphi$. Now, the functions given by

$$h_n : (0,T) \to \mathbb{R}, \quad t \mapsto C_1 C \cdot C(\varphi) \left\| b_n(t,\cdot) \right\|_{L^1(\Omega)^N} + C_1 C \left\| \varphi \right\|_{C(\Omega)} \left\| \operatorname{div} b_n(t,\cdot) \right\|_{L^1(\Omega)}$$

form a uniformly integrable set in both cases: in case (a) due to the strong convergence of $(b_n)$ to $b$ in $L^1((0,T)\times\Omega)^N$ and in case (b) we obtain the estimate

$$\int_s^t \left\| b_n(r,\cdot) \right\|_{L^1(\Omega)^N} \, dr \le \left\| b_n \right\|_{L^q((0,T),L^1(\Omega))^N} |t-s|^{1/q'} \le \tilde{C} \, |t-s|^{1/q'}$$

for some constant $\tilde{C} > 0$. Hence, the set of functions $t \mapsto \left| \frac{d}{dt} \langle u_n(t,\cdot), \varphi \rangle \right|$ is also uniformly integrable for fixed $\varphi \in C_c^\infty(\Omega)$ and we deduce equicontinuity of $t \mapsto \langle u_n(t,\cdot), \varphi \rangle$ for any $\varphi \in L^2(\Omega)$: let $(\varphi_k) \subset C_c^\infty(\Omega)$ be a sequence converging to $\varphi$ in $L^2(\Omega)$ and let $0 \le s < t \le T$. Then, we obtain

$$\begin{aligned} |\langle u_n(t,\cdot) - u_n(s,\cdot), \varphi \rangle| \le & \left( \left\| u_n(t,\cdot) \right\|_{L^2(\Omega)} + \left\| u_n(s,\cdot) \right\|_{L^2(\Omega)} \right) \left\| \varphi_k - \varphi \right\|_{L^2(\Omega)} \\ & + \int_s^t \left| \frac{d}{dt} \langle u_n(r,\cdot), \varphi_k \rangle \right| \, dr. \end{aligned}$$

Now for $\varepsilon > 0$, we find some $k(\varepsilon) \in \mathbb{N}$ and some $\delta(\varepsilon) > 0$ such that

$$\left\| \varphi_k - \varphi \right\|_{L^2(\Omega)} \le \varepsilon \quad \text{and} \quad \int_s^t \left| \frac{d}{dt} \langle u_n(r,\cdot,\varphi_{k(\varepsilon)} \rangle \right| \, dr \le \varepsilon$$

for all $k \ge k(\varepsilon)$ and $|t-s| \le \delta(\varepsilon)$. Then, for $|t-s| \le \delta(\varepsilon)$, we obtain

$$|\langle u_n(t,\cdot) - u_n(s,\cdot), \varphi \rangle| \le C\varepsilon + \varepsilon = (C+1)\varepsilon$$

where

$$C := 2 \sup_{n \in \mathbb{N}, t \in [0,T]} \left\| u_n(t,\cdot) \right\|_{L^2(\Omega)}.$$

Using Arzelà-Ascoli, we deduce that there exists a subsequence $(u_{n_k})$ and some

$$w \in C([0,T], L^2(\Omega) - w)$$

such that $u_{n_k} \to w$ in $C([0,T], L^2(\Omega) - w)$. Some simple calculations yield in case (a) that $w$ satisfies the weak formulation with vector field $b$ and initial data $u_0$. Hence, $w$ is a weak solution of the transport equation with vector field $b$ and initial value $u_0$ and thus unique, i.e. $u = w$. In case (b), the same calculations yield that for any $\varphi \in C_c^\infty([0,T)\times\Omega)$

$$\begin{aligned} & \int_\Omega u_{0,n}(x)\varphi(0,x) \, dx + \int_0^T \int_\Omega u_n(t,x)\partial_t\varphi(t,x) + u_n(t,x)\varphi(t,x) \operatorname{div} b_n(t,x) \, dxdt \\ & \quad \to \int_\Omega u_0(x)\varphi(0,x) \, dx \int_0^T \int_\Omega w(t,x)\partial_t\varphi(t,x) + w(t,x)\varphi(t,x) \operatorname{div} b(t,x) \, dxdt. \end{aligned}$$

It remains to show that

$$\int_0^T \int_\Omega u_n(t,x) b_n(t,x) \cdot \nabla\varphi(t,x)\ dxdt \to \int_0^T \int_\Omega w(t,x) b(t,x) \cdot \nabla\varphi(t,x)\ dxdt$$

is satisfied. Our aim is to use Theorem 3.2.1 of the previous subsection. Therefore, we have to show that $(\partial_t u_n)$ is a bounded subset of $L^1((0,T),(W^{m,2}(\Omega))')$. We choose $m$ so large that $W^{m,2}(\Omega) \hookrightarrow C^1(\Omega)$. Then, as a consequence of Lemma 3.1.24, we have for any $\varphi \in W^{m,2}(\Omega)$ and for almost all $t \in (0,T)$

$$\langle \partial_t u_n(t,\cdot), \varphi \rangle = \langle u_n(t,\cdot) b_n(t,\cdot), \nabla\varphi \rangle + \langle u_n(t,\cdot) \operatorname{div} b_n(t,\cdot), \varphi \rangle,$$

i.e. $\partial_t u_n(t,\cdot) \in (W^{m,2}(\Omega))'$ and thus, we estimate for $\psi \in L^\infty((0,T),W^{m,2}(\Omega))$

$$\begin{aligned} |\langle \partial_t u_n, \psi \rangle| &\le \|u_n b_n\|_{L^1((0,T)\times\Omega)^N} \|\nabla\psi\|_{L^\infty((0,T),C(\Omega))^N} + \|u_n \operatorname{div} b_n\|_{L^1((0,T\times\Omega))} \|\psi\|_{L^\infty((0,T),C(\Omega))} \\ &\le C \|\psi\|_{L^\infty((0,T),W^{m,2}(\Omega))} \end{aligned}$$

for some $C > 0$ independent of $n \in \mathbb{N}$. The principle of uniform boundedness now yields that $(\partial_t u_n)$ is a bounded sequence in $L^1((0,T),(W^{m,2}(\Omega))')$ and we can apply Theorem 3.2.1 leading to

$$\int_0^T \int_\Omega u_n(t,x) b_n(t,x) \cdot \nabla\varphi(t,x)\ dxdt \to \int_0^T \int_\Omega w(t,x) b(t,x) \cdot \nabla\varphi(t,x)\ dxdt$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega)$. The general case, i.e. for test functions in $C_c^\infty([0,T)\times\Omega)$ can be deduced using smooth cut-off functions in time $(\eta_k) \subset C_c^\infty((0,T))$ with

$$0 \le \eta_k(t) \le 1, \quad \eta_k(t) \to \chi_{(0,T)} \quad \text{and } \eta_k' \overset{*}{\rightharpoonup} \delta_0 - \delta_T$$

for all $t \in (0,T)$, $k \in \mathbb{N}$ and as $k \to \infty$. Thus, $w$ satisfies the weak formulation and as above we deduce that $w$ is the unique solution of the transport equation with vector field $b$ and initial value $u_0$ which we denote $u$, i.e. $w = u$. Finally, by a standard proof of contradiction, we obtain that the whole sequence $(u_n)$ converges to $u$ in $C([0,T],L^2(\Omega)-w)$. Furthermore, following the previous argumentation, we obtain that $(u_n)^2$ converges to $u^2$ in $C([0,T],L^2(\Omega)-w)$. Then Lemma 3.2.6 yields that $u_n, u \in C([0,T],L^2(\Omega))$ for all $n \in \mathbb{N}$ and $u_n \to u$ in $C([0,T],L^2(\Omega))$. This ends the first part of the proof.

*Second part:* It remains to show the result for general $p < \infty$. The case $1 \le p \le 2$ is obviously satisfied due to the continuous embedding of $C([0,T],L^2(\Omega))$ into $C([0,T],L^p(\Omega))$ for $p \le 2$. Therefore, it remains to show the statement for the case $2 < p < \infty$. So, let $2 < p < \infty$ and let $t, s \in [0,T]$. Then, we estimate

$$\begin{aligned} \|u_n(t,\cdot) - u_n(s,\cdot)\|_{L^p(\Omega)}^p &\le \|u_n(t,\cdot) - u_n(s,\cdot)\|_{L^\infty(\Omega)}^{p-2} \|u_n(t,\cdot) - u_n(s,\cdot)\|_{L^2(\Omega)}^2 \\ &\le C^{p-2} \|u_n(t,\cdot) - u_n(s,\cdot)\|_{L^2(\Omega)}^2 \to 0 \quad \text{as } t \to s \end{aligned}$$

where $C > 0$ is a bound for $2\|u_{0,n}\|_{L^\infty(\Omega)}$. Obviously, the estimate also works for $u$. In the same way we estimate for $t \in [0,T]$

$$\|u_n(t,\cdot) - u(t,\cdot)\|_{L^p(\Omega)}^p \le C^{p-2} \|u_n(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)}^2$$

where $C > 0$ again is a suitable upper bound. Taking the supremum over $[0,T]$ yields

$$\|u_n - u\|^p_{C([0,T],L^p(\Omega))} \leq C^{p-2} \|u_n - u\|^2_{C([0,T],L^2(\Omega))} \to 0 \quad \text{as } n \to \infty.$$

□

**Remark 3.2.7** *In the proof, we only show that the unique solutions $u_n \in C([0,T], L^\infty(\Omega) - w^*)$ can be seen as elements of $C([0,T], L^p(\Omega))$ and converge to the unique solution $u \in C([0,T], L^\infty(\Omega) - w^*)$ in $C([0,T], L^p(\Omega))$ under the assumptions of Theorem 3.2.4. Uniqueness of solutions in $C([0,T], L^p(\Omega))$ is not shown here. Crippa shows in his thesis [Cri07] some results (Theorem 3.1.1 and Theorem 3.2.1) yielding uniqueness in $C([0,T], L^2(\Omega))$ if $b$ satisfies some specific requirements.*

### 3.2.3. Stability of solution operator: second improvement

In this subsection, we improve the previous stability result. The improvement consists of the point that we replace the strong convergence of $(\operatorname{div} b_n)$ to some $\operatorname{div} b$ in $L^1((0,T) \times \Omega)$ with boundedness of $(\operatorname{div} b_n)$ in $L^1((0,T), L^\infty(\Omega))$. This improvement will be needed in the proof for existence of a minimizing point for optimal control problems in the next chapter. In [DL89] this result is shown in Theorem II.5 for vector fields with spatial Sobolev regularity under stronger assumptions on the convergence of the vector fields than we will require. The idea of DiPerna and Lions's proof is the following: they convolve the unique solution $u$ corresponding to the vector field $b$ with some mollifier $\rho_\varepsilon$ and obtain $u_\varepsilon := u * \rho_\varepsilon$. Then, they show that the function $u_\varepsilon$ satisfies the same transport equation but with some inhomogeneity $r_\varepsilon$. This inhomogeneity converges strongly to zero in some Lebesgue space as $\varepsilon \to 0$ (Theorem II.1 in [DL89]). As a next step they consider the difference $u_n - u_\varepsilon$ of unique weak solutions $u_n$ corresponding to the vector fields $b_n$ and the smoothed $u_\varepsilon$. For this difference they can show that it is uniformly bounded in $n$ by two terms: by the $L^1$-norm of the difference $u - u_\varepsilon$ and by the Lebesgue norm of $r_\varepsilon$. Taking the limit in $\varepsilon$ yields their statement in the end. We take the same way to show our results for vector fields with spatial $BV$-regularity. Unfortunately, the proof is much more complicated and we are confronted with the same problem as Ambrosio had with the commutator (3.8): DiPerna and Lions had the case that their commutator converged strongly to zero in some Lebesgue space as $\varepsilon \to 0$ whereas Ambrosio's commutator can only be split into a strongly convergent part $r_{1,\varepsilon}$ and some weakly*-convergent part $r_{2,\varepsilon}$. Then, Ambrosio had to show carefully that this second term also vanishes as $\varepsilon \to 0$. The same problem appears here with the inhomogeneity $r_\varepsilon$ appearing in the transport equation satisfied by the convolved solution $u_\varepsilon$. This inhomogeneity can only be split into a „good" part $r_{1,\varepsilon}$ being convergent in some Lebesgue space and a „bad" part for which we have some estimate for the limit as $\varepsilon \to 0$. Therefore, most parts of this subsection resemble the approach of subsection 3.1.3 and we use the same techniques to tackle the problems. We start with a theorem whose statements remind one of Lemma 3.1.11. Before we start we want to remind one of Theorem 3.1.25 which enables us to extend (by zero) any solution $u$ on $(0,T) \times \Omega$ to a solution $u$ on $(0,T) \times \mathbb{R}^N$.

**Theorem 3.2.8** *Let $1 \leq q < \infty$ and $b \in L^q((0,T), BV_0(\Omega))^N$ with $\operatorname{div} b \in L^q((0,T), L^\infty(\Omega))$ and denote $u$ the unique weak solution of the transport equation with initial data $u_0 \in L^\infty(\Omega)$.*

*We set $u_\varepsilon := u * \rho_\varepsilon$, where $\rho$ denotes an even mollifier with $\operatorname{supp}(\rho) \subset \overline{B_1(0)}$ and $u$ denotes the extension (by zero) to $(0,T) \times \mathbb{R}^N$. Then $u_\varepsilon$ satisfies*

$$\begin{aligned} \partial_t u_\varepsilon + \operatorname{div}(b u_\varepsilon) - u_\varepsilon \operatorname{div} b = r_\varepsilon \qquad &\text{in } (0,T) \times \mathbb{R}^N, \\ u_\varepsilon(0,\cdot) = u_0 * \rho_\varepsilon \qquad &\text{on } \mathbb{R}^N, \end{aligned}$$

*where*

$$r_\varepsilon = r_{1,\varepsilon} + r_{2,\varepsilon} \qquad \text{with } r_{1,\varepsilon}, r_{2,\varepsilon} \in L^q\left((0,T), L^1(\mathbb{R}^N)\right)$$

*and $r_{1,\varepsilon}, r_{2,\varepsilon}$ having the following properties:*

*(i) There exists some compact set $K \subset \mathbb{R}^N$ independent of $\rho$ such that*

$$r_{1,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0 \quad \text{and} \quad r_{2,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0$$

*for any $1 \geq \varepsilon > 0$.*

*(ii) $r_{1,\varepsilon} \to 0$ in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$ as $\varepsilon \to 0$ and*

*(iii) for any measurable set $I \subset (0,T)$ and any compact set $W \subset \mathbb{R}^N$ we have*

$$\limsup_{\varepsilon\to 0} \int\limits_I \left(\int_W |r_{2,\varepsilon}(t,x)| \; dx\right)^q \; dt \leq C \int\limits_I \left(\int_W \Lambda(M_b(t,x),\rho) \; d\left|D^s b(t,\cdot)\right|(x)\right)^q \; dt.$$

*Here, $M_b$ denotes the matrix valued Borel function such that $D^s b = M_b\,|D^s b|$ and $C > 0$ is a constant depending only on $u$.*

**Proof:** We have

$$\begin{aligned} 0 &= [\partial_t u + \operatorname{div}(bu) - u \operatorname{div} b] * \rho_\varepsilon \\ &= \partial_t(u * \rho_\varepsilon) + \operatorname{div}(b(u*\rho_\varepsilon)) - u * \rho_\varepsilon \operatorname{div} b + \operatorname{div}(bu) * \rho_\varepsilon \\ &- (u \operatorname{div} b) * \rho_\varepsilon - \operatorname{div}(b(u*\rho_\varepsilon)) + u * \rho_\varepsilon \operatorname{div} b \end{aligned}$$

and thus

$$\partial_t(u_\varepsilon) + \operatorname{div}(b(u_\varepsilon)) - u_\varepsilon \operatorname{div} b = r_\varepsilon,$$

where $r_\varepsilon$ is given by

$$r_\varepsilon = (u \operatorname{div} b) * \rho_\varepsilon - u * \rho_\varepsilon \operatorname{div} b + \operatorname{div}(b(u*\rho_\varepsilon)) - \operatorname{div}(bu) * \rho_\varepsilon.$$

Obviously, the term $(u \operatorname{div} b) * \rho_\varepsilon - u * \rho_\varepsilon \operatorname{div} b$ converges to zero in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$. Thus, we have a closer look on the commutator

$$R_\varepsilon := \operatorname{div}(bu) * \rho_\varepsilon - \operatorname{div}(b(u*\rho_\varepsilon)).$$

As in the proof of Lemma 3.1.13 we can rewrite $R_\varepsilon$ using Lemma 3.1.11 as

$$R_\varepsilon(t,x) = -\int\limits_{\mathbb{R}^N} u(t,x+\varepsilon z) b_{1,\varepsilon,z}(t,x)^\top \nabla\rho(z) \; dz - (u*\rho_\varepsilon)(t,x) \operatorname{div} b(t,x) \tag{3.28}$$

$$- \int\limits_{\mathbb{R}^N} u(t,x+\varepsilon z) b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z) \; dz. \tag{3.29}$$

Then we define $s_{1,\varepsilon}$ as the function given in (3.28) and $s_{2,\varepsilon}$ as the function given in (3.29). We set

$$K := \left\{x \in \mathbb{R}^N \middle|\ \operatorname{dist}(x,\Omega) \leq 2\right\}.$$

Then, since $u$ is zero outside of $\Omega$ we immediately obtain that

$$r_{1,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0 \quad \text{and} \quad r_{2,\varepsilon}|_{(0,T)\times(\mathbb{R}^N\setminus K)} \equiv 0,$$

where we define $r_{1,\varepsilon} := (u \operatorname{div} b) * \rho_\varepsilon - u * \rho_\varepsilon \operatorname{div} b - s_{1,\varepsilon}$ and $r_{2,\varepsilon} = -s_{2,\varepsilon}$. The functions $s_{1,\varepsilon}$ and $s_{2,\varepsilon}$ are elements of $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$ due to the following reason: we set $i = 1, 2$ and estimate

$$\begin{aligned}
&\int_0^T \left(\int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} u(t,x+\varepsilon z) b_{i,\varepsilon,z}(t,x)^\top \nabla\rho(z)\ dz\right|\ dx\right)^q\ dt \\
&\qquad \leq \|u\|_{L^\infty((0,T)\times\Omega)} \int_0^T \left(\int_{B_1(0)} \int_K \left|b_{i,\varepsilon,z}(t,x)^\top \nabla\rho(z)\right|\ dxdz\right)^q\ dt \\
&\qquad \leq \|u\|_{L^\infty((0,T)\times\Omega)} |B_1(0)|^{q-1} \int_{B_1(0)} \int_0^T \left(\int_K \left|b_{i,\varepsilon,z}(t,x)^\top \nabla\rho(z)\right|\ dx\right)^q\ dtdz \\
&\qquad \leq \|u\|_{L^\infty((0,T)\times\Omega)} |B_1(0)|^{q-1} \int_{B_1(0)} \int_0^T \left(|\nabla\rho(z)|\,|z|\,|Db(t,\cdot)|\,(K_\varepsilon)\right)^q\ dtdz < \infty,
\end{aligned}$$

where we used point (iii) of Lemma 3.1.11. To finish the proof of point (ii) it remains to show that $s_{1,\varepsilon} \to 0$ in $L^q\left((0,T), L^1(\mathbb{R}^N)\right)$. For almost all $t \in (0,T)$ we deduce as in the proof of Lemma 3.1.15 that

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u(t,x+\varepsilon z) b_{1,\varepsilon,z}(t,x)^\top \nabla\rho(z)\ dzdx &\to \int_{\mathbb{R}^N} u(t,x) \sum_{i,j=1}^N e_i^\top J_b(t,x) e_j \int_{\mathbb{R}^N} z_j \partial_{z_i}\rho(z)\ dzdx \\
&= -\int_{\mathbb{R}^N} u(t,x) \operatorname{div} b(t,x)\ dx
\end{aligned}$$

as $\varepsilon \to 0$. Using Lebesgue's dominated convergence theorem and point (iii) of Lemma 3.1.11 then yields that

$$s_{1,\varepsilon} \to 0 \quad \text{in } L^q\left((0,T), L^1(R^N)\right)$$

as $\varepsilon \to 0$. It remains to show the property of $s_{2,\varepsilon}$. A similar statement is already proven in the proof of Lemma 3.1.15. Therefore, we transfer the proof to our situation:
Due to point (ii) in Lemma 3.1.11 we know that for almost all $t \in (0,T)$ and for any compact set $W \subset \mathbb{R}^N$

$$\limsup_{\varepsilon\to 0} \int_W \left|b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z)\right|\ dx \leq \left|(\nabla\rho(z))^\top D^s b(t,\cdot) z\right|(W).$$

Moreover, since the support of $\rho$ is a subset of $\overline{B_1(0)}$ we obtain with Fubini for a measurable set $I \subset (0,T)$

$$\limsup_{\varepsilon\to 0} \int_I \left(\int_{\mathbb{R}^N} \int_W \left|b_{2,\varepsilon,z}(t,x)^\top \nabla\rho(z)\right|\ dxdz\right)^q dt$$

$$\leq \int_I \left( \int_{\mathbb{R}^N} \limsup_{\varepsilon \to 0} \int_W \left| b_{2,\varepsilon,z}(t,x)^\top \nabla \rho(z) \right| \, dxdz \right)^q \, dt$$
$$\leq \int_I \left( \int_{\mathbb{R}^N} \left| (\nabla \rho(z))^\top D^s b(t,\cdot) z \right| (W) \, dz \right)^q \, dt.$$

The last term can be rewritten as

$$\int_I \left( \int_{\mathbb{R}^N} \left| (\nabla \rho(z))^\top D^s b(t,\cdot) z \right| (W) \, dz \right)^q \, dt$$
$$= \int_I \left( \int_{\mathbb{R}^N} \int_W \, d \left| (\nabla \rho(z))^\top D^s b(t,\cdot) z \right| (x) dz \right)^q \, dt$$
$$= \int_I \left( \int_W \int_{\mathbb{R}^N} \left| (\nabla \rho(z))^\top M_b(t,x) z \right| \, dzd \left| D^s b(t,\cdot) \right| (x) \right)^q \, dt$$
$$= \int_I \left( \int_W \Lambda(M_b(t,x),\rho) \, d \left| D^s b(t,\cdot) \right| (x) \right)^q \, dt.$$

Thus, we conclude

$$\limsup_{\varepsilon \to 0} \int_I \left( \int_W \left| s_{2,\varepsilon}(t,x) \right| \, dx \right)^q \, dt$$
$$\leq \limsup_{\varepsilon \to 0} \int_I \left( \int_W \int_{\mathbb{R}^N} \left| u(t,x+\varepsilon z) b_{2,\varepsilon,z}(t,x)^\top \nabla \rho(z) \right| \, dzdx \right)^q \, dt$$
$$\leq \|u\|^q_{L^\infty((0,T)\times\mathbb{R}^N)} \limsup_{\varepsilon \to 0} \int_I \left( \int_W \int_{\mathbb{R}^N} \left| b_{2,\varepsilon,z}(t,x)^\top \nabla \rho(z) \right| \, dzdx \right)^q \, dt$$
$$\leq \|u\|^q_{L^\infty((0,T)\times\mathbb{R}^N)} \int_I \left( \int_W \Lambda(M_b(t,x),\rho) \, d \left| D^s b(t,\cdot) \right| (x) \right)^q \, dt.$$

□

Now, we are prepared for the main result of this subsection which is a generalization of Theorem II.5 in [DL89] to vector fields with spatial $BV$-regularity.

**Theorem 3.2.9 (Second stability theorem)** *Let $q \in (1,\infty)$, $u_0 \in L^\infty(\Omega)$ and let $b \in L^\infty((0,T) \times \Omega)^N \cap L^q((0,T), BV_0(\Omega))^N$ with $\operatorname{div} b \in L^q((0,T), L^\infty(\Omega))$. Furthermore, let $(b_n) \subset \mathrm{VF}_0$ and $(u_{0,n}) \subset L^\infty(\Omega)$ be two sequences with the following properties:*

*(i) $(u_{0,n})$ is bounded in $L^\infty(\Omega)$ and converges to $u_0$ in $L^1(\Omega)$,*

*(ii) $(b_n) \subset L^q((0,T), BV_0(\Omega))^N$ is bounded and converges weakly to $b$ in $L^1((0,T) \times \Omega)^N$,*

*(iii) $(\operatorname{div} b_n) \subset L^q((0,T), L^\infty(\Omega))$ and is bounded in $L^1((0,T), L^\infty(\Omega))$.*

*Then, for any $1 \leq p < \infty$, the sequence of unique solutions $(u_n) \subset C([0,T], L^\infty(\Omega) - w^*)$ of (3.1) with vector fields $b_n$ and initial data $u_{0,n}$ is a subset of $C([0,T], L^p(\Omega))$ and converges in $C([0,T], L^p(\Omega))$ to the unique solution $u \in C([0,T], L^p(\Omega))$ of (3.1) with vector field $b$ and initial value $u_0$.*

In the following, if some Lebesgue function is just defined on a proper subset of $\mathbb{R}^N$ in the spatial variable, then we extend this function by zero to the whole $\mathbb{R}^N$ if we consider the function as some function defined on $\mathbb{R}^N$ in our calculations.

We take some even mollifier $\rho \in C_c^\infty(B_1(0))$ and we set $u_\varepsilon := u * \rho_\varepsilon$ for the unique solution $u$ of the transport equation with vector field $b$ and initial value $u_0$. We will prove the theorem in several consecutive lemmas. In the first lemma we obtain an expression for the difference of $u_n - u_\varepsilon$.

**Lemma 3.2.10** *Under the assumptions of Theorem 3.2.9 the following expression for the difference $u_n - u_\varepsilon$ holds:*

$$\begin{aligned}
&\partial_t \int_K (u_n(t,x) - u_\varepsilon(t,x))^2 \, dx - \int_K (u_n(t,x) - u_\varepsilon(t,x))^2 \operatorname{div} b_n(t,x) \, dx \\
&= 2 \int_K (u_n(t,x) - u_\varepsilon(t,x)) \left(-r_{1,\varepsilon}(t,x) - r_{2,\varepsilon}(t,x) + (b(t,x) - b_n(t,x)) \cdot \nabla u_\varepsilon(t,x)\right) \, dx,
\end{aligned} \tag{3.30}$$

*where $K \subset \mathbb{R}^N$ denotes the compact set of Theorem 3.2.8.*

**Proof:** Due to Theorem 3.2.8 we obtain that $u_\varepsilon$ satisfies

$$\begin{aligned}
\partial_t u_\varepsilon + \operatorname{div}(b u_\varepsilon) - u_\varepsilon \operatorname{div} b = r_{1,\varepsilon} + r_{2,\varepsilon} \qquad & \text{in } (0,T) \times \mathbb{R}^N, \\
u_\varepsilon(0,\cdot) = u_0 * \rho_\varepsilon \qquad & \text{on } \mathbb{R}^N.
\end{aligned}$$

We assume first that $u_{0,l} \in C_c^\infty(\Omega)$ and $b_l$ is smooth in $(0,T) \times \Omega$ with zero spatial boundary value. Then, the corresponding solution $u_l$ of the transport equation is also smooth with zero spatial boundary value. These functions can be obviously extended in a smooth way to $\mathbb{R}^N$ in the spatial domain. We take $\beta \in C^1(\mathbb{R})$ such that $\beta(0) = 0$. Then, we write

$$\begin{aligned}
&\partial_t \beta(u_l - u_\varepsilon) + \operatorname{div}(b_l \beta(u_l - u_\varepsilon)) - \beta(u_l - u_\varepsilon) \operatorname{div} b_l \qquad (3.31)\\
&= \beta'(u_l - u_\varepsilon) \left(\partial_t (u_l - u_\varepsilon) + \operatorname{div}(b_l (u_l - u_\varepsilon)) - (u_l - u_\varepsilon) \operatorname{div} b_l\right) \\
&= \beta'(u_l - u_\varepsilon) \left(-r_{1,\varepsilon} - r_{2,\varepsilon} + (b - b_l) \cdot \nabla u_\varepsilon\right). \qquad (3.32)
\end{aligned}$$

For the initial value we have that $\beta(u_l(0,\cdot) - u_\varepsilon(0,\cdot)) = \beta(u_{0,l} - u_0 * \rho_\varepsilon)$. In the following we denote by $K$ the compact set given in point (i) in Theorem 3.2.8 and we know that $\Omega \subset K$. Now, integrating over $K$ yields

$$\begin{aligned}
&\partial_t \int_K \beta(u_l(t,x) - u_\varepsilon(t,x)) \, dx - \int_K \beta(u_l(t,x) - u_\varepsilon(t,x)) \operatorname{div} b_l(t,x) \, dx \\
&= \partial_t \int_K \beta(u_l(t,x) - u_\varepsilon(t,x)) \, dx + \int_K \nabla \beta(u_l(t,x) - u_\varepsilon(t,x)) \cdot b_l(t,x) \, dx \\
&= \int_K \beta'(u_l(t,x) - u_\varepsilon(t,x)) \left(-r_{1,\varepsilon}(t,x) - r_{2,\varepsilon}(t,x) + (b(t,x) - b_l(t,x)) \cdot \nabla u_\varepsilon(t,x)\right) \, dx.
\end{aligned}$$

Choosing $\beta(t) = t^2$ for $t \in \mathbb{R}$ we obtain that

$$\partial_t \int_K (u_l(t,x) - u_\varepsilon(t,x))^2 \, dx - \int_K (u_l(t,x) - u_\varepsilon(t,x))^2 \operatorname{div} b_l(t,x) \, dx$$
$$= 2 \int_K (u_l(t,x) - u_\varepsilon(t,x)) \left(-r_{1,\varepsilon}(t,x) - r_{2,\varepsilon}(t,x) + (b(t,x) - b_l(t,x)) \cdot \nabla u_\varepsilon(t,x)\right) \, dx.$$

Our first assumption was that $u_l$, $b_l$ and $u_{0,l}$ are smooth functions. Therefore, we take a sequence of smooth functions $(b_{n,k})_k$ such that

$$b_{n,k} \to b_n \quad \text{in } L^1((0,T) \times \Omega)^N \quad \text{and} \quad \operatorname{div} b_{n,k} \to \operatorname{div} b_n \quad \text{in } L^1((0,T) \times \Omega) \quad \text{as } k \to \infty.$$

In addition, we take a sequence of smooth and bounded functions $(u_{0,n,k})_k \subset C_c^\infty(\Omega)$ converging to $u_{0,n}$ in $L^1(\Omega)$. Then, the above equation is valid for $b_{n,k}$ and $u_{n,k}$ and taking the limit, Theorem 3.2.4 yields

$$\partial_t \int_K (u_n(t,x) - u_\varepsilon(t,x))^2 \, dx - \int_K (u_n(t,x) - u_\varepsilon(t,x))^2 \operatorname{div} b_n(t,x) \, dx$$
$$= 2 \int_K (u_n(t,x) - u_\varepsilon(t,x)) \left(-r_{1,\varepsilon}(t,x) - r_{2,\varepsilon}(t,x) + (b(t,x) - b_n(t,x)) \cdot \nabla u_\varepsilon(t,x)\right) \, dx.$$

□

In the second lemma we get an upper estimate for the difference $u_n - u_\varepsilon$.

**Lemma 3.2.11** *Under the assumptions of Theorem 3.2.9 the following estimate holds:*

$$\begin{aligned}
&\int_K ((u_n(t,x) - u_\varepsilon(t,x)))^2 \, dx \\
&\quad \le (C_1 + 1) \cdot \left( C_0 \int_0^T \int_K |r_{1,\varepsilon}(s,x)| \, dxds + \int_K ((u_{0,n}(x) - u_{0,\varepsilon}(x))^2 \, dx \right) \\
&\quad + 2 \left| \int_0^t \int_K (u_n(s,x) - u_\varepsilon(s,x)) r_{2,\varepsilon}(s,x) \, dxds \right| \\
&\quad + 2C_1 \max_{s \in [0,T]} \left| \int_0^s \int_K (u_n(r,x) - u_\varepsilon(r,x)) r_{2,\varepsilon}(r,x) \, dxdr \right| \\
&\quad + 2C_2 \int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} \left| \int_0^s \int_K (u_n(r,x) - u_\varepsilon(r,x))(b(r,x) - b_n(r,x)) \cdot \nabla u_\varepsilon(r,x) \, dxdr \right| ds \\
&\quad + 2 \left| \int_0^t \int_K (u_n(s,x) - u_\varepsilon(s,x))(b(s,x) - b_n(s,x)) \cdot \nabla u_\varepsilon(t,x) \, dxds \right|
\end{aligned} \tag{3.33}$$

*for some constants $C_2, C_1, C_0 > 0$ and any $t \in [0,T]$.*

**Proof:** We use expression (3.30) of Lemma 3.2.10 and estimate:

$$\begin{aligned}
\partial_t \int_K ((u_n(t,x) - u_\varepsilon(t,x)))^2 \; dx \leq{}& \|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)} \int_K (u_n(t,x) - u_\varepsilon(t,x))^2 \; dx \\
&+ C_0 \int_K |r_{1,\varepsilon}(t,x)| \; dx - 2 \int_K (u_n(t,x) - u_\varepsilon(t,x)) r_{2,\varepsilon}(t,x) \; dx \\
&+ 2 \int_K (u_n(t,x) - u_\varepsilon(t,x))(b(t,x) - b_n(t,x)) \cdot \nabla u_\varepsilon(t,x) \; dx
\end{aligned}$$

where $C_0 > 0$ can be chosen as $C_0 := 2 \sup_n \|u_{0,n}\|_{L^\infty(\Omega)} + 2\|u_0\|_{L^\infty(\Omega)}$. Integrating in time yields

$$\begin{aligned}
&\int_K ((u_n(t,x) - u_\varepsilon(t,x)))^2 \; dx \\
&\quad \leq \int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} \int_K ((u_n(s,x) - u_\varepsilon(s,x)))^2 \; dxds \\
&\quad + C_0 \int_0^T \int_K |r_{1,\varepsilon}(s,x)| \; dxds + 2 \left| \int_0^t \int_K (u_n(s,x) - u_\varepsilon(s,x)) r_{2,\varepsilon}(s,x) \; dxds \right| \\
&\quad + 2 \left| \int_0^t \int_K (u_n(s,x) - u_\varepsilon(s,x))(b(s,x) - b_n(s,x)) \cdot \nabla u_\varepsilon(s,x) \; dxds \right| \\
&\quad + \int_K ((u_{0,n}(x) - u_{0,\varepsilon}(x))^2 \; dx.
\end{aligned}$$

Using Grönwall's Lemma 2.2.4, we obtain that

$$\begin{aligned}
\int_K ((u_n(t,x) - u_\varepsilon(t,x)))^2 \; dx \leq{}& \left( C_0 \int_0^T \int_K |r_{1,\varepsilon}(s,x)| \; dxds + \int_K ((u_{0,n}(x) - u_{0,\varepsilon}(x))^2 \; dx \right) \\
&\cdot \left( 1 + \int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} e^{\int_s^t \|\operatorname{div} b_n(r,\cdot)\|_{L^\infty(\Omega)} dr} \; ds \right) \\
&+ 2 \left| \int_0^t \int_K (u_n(s,x) - u_\varepsilon(s,x)) r_{2,\varepsilon}(s,x) \; dxds \right| \\
&+ 2 \left| \int_0^t \int_K (u_n(s,x) - u_\varepsilon(s,x))(b(s,x) - b_n(s,x)) \cdot \nabla u_\varepsilon(s,x) \; dxds \right| \\
&+ 2 \int_0^t \left| \int_0^s \int_K (u_n(r,x) - u_\varepsilon(r,x)) r_{2,\varepsilon}(r,x) \; dxdr \right|
\end{aligned}$$

$$\cdot \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} e^{\int_s^t \|\operatorname{div} b_n(r,\cdot)\|_{L^\infty(\Omega)} dr} \, ds$$
$$+ 2 \int_0^t \left| \int_0^s \int_K (u_n(r,x) - u_\varepsilon(r,x))(b(r,x) - b_n(r,x)) \cdot \nabla u_\varepsilon(r,x) \, dxdr \right|$$
$$\cdot \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} e^{\int_s^t \|\operatorname{div} b_n(r,\cdot)\|_{L^\infty(\Omega)} dr} \, ds.$$

Since $(\operatorname{div} b_n)$ is bounded in $L^1((0,T), L^\infty(\Omega))$, we set

$$C_1 := e^{\sup_n \int_0^T \|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)} dt} \sup_n \int_0^T \|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)} \, dt$$

and

$$C_2 := e^{\sup_n \int_0^T \|\operatorname{div} b_n(t,\cdot)\|_{L^\infty(\Omega)} dt}$$

and this yields the statement of the lemma.

□

In the third lemma we use estimate (3.33) to obtain an upper bound for the limes superior of $\left(\int_K |u_n(t,x) - u(t,x)| \, dx\right)^2$.

**Lemma 3.2.12** *Under the assumptions of Theorem 3.2.9 we have*

$$\begin{aligned} \limsup_{n\to\infty} \left( \int_K |u_n(t,x) - u(t,x)| \, dx \right)^2 & \\ \le C_0 \int_K |u_\varepsilon(t,x) - u(t,x)| \, dx + C \int_K (u_\varepsilon(t,x) - u(t,x))^2 \, dx + 2CC_1 R_\varepsilon(s^*) & \\ + CC_0(C_1+1) \int_0^T \int_K |r_{1,\varepsilon}(s,x)| \, dxds + C(C_1+1) \int_K ((u_0(x) - u_{0,\varepsilon}(x))^2 \, dx & \\ + 2C \left| \int_0^t \int_K (w_1(s,x) - u_\varepsilon(s,x)) r_{2,\varepsilon}(s,x) \, dxds \right| & \end{aligned} \tag{3.34}$$

*for some specific $w_1 \in L^\infty((0,T)\times\Omega)$, $s^* \in [0,T]$ and function $R_\varepsilon \in C([0,T])$.*

**Proof:** The proof of Theorem 3.2.4 shows that there are subsequences $(u_n), (u_n^2) \in C([0,T], L^\infty(\Omega) - w^*)$ and $(u_n \operatorname{div} b_n), (u_n^2 \operatorname{div} b_n) \in L^1((0,T), L^\infty(\Omega))$ (labeled by $n$ again) and $w_1, w_2 \in L^\infty((0,T)\times\Omega)$ and $w_3, w_4 \in L^1((0,T)\times\Omega)$ such that $u_n \overset{*}{\rightharpoonup} w_1$ in $L^\infty((0,T)\times\Omega)$ and

$$\begin{aligned} u_n \rightharpoonup w_1 \quad &\text{and} \quad u_n^2 \rightharpoonup w_2 \quad \text{in } C([0,T], L^2(\Omega) - w), \\ u_n \operatorname{div} b_n \rightharpoonup w_3 \quad &\text{and} \quad u_n^2 \operatorname{div} b_n \rightharpoonup w_4 \quad \text{in } L^1((0,T)\times\Omega). \end{aligned}$$

In particular, we have that $w_1(0,\cdot) = u_0$ and $w_2(0,\cdot) = u_0^2$. We restrict to these subsequences. Furthermore, the mappings $R_{n,\varepsilon} : [0,T] \to \mathbb{R}$ defined by

$$s \mapsto R_{n,\varepsilon}(s) := \left| \int_0^s \int_K (u_n(r,x) - u_\varepsilon(r,x)) r_{2,\varepsilon}(r,x) \, dxdr \right|$$

are equicontinuous in $n$: for $0 \leq s \leq t \leq T$ we obtain that

$$\begin{aligned}|R_{n,\varepsilon}(t) - R_{n,\varepsilon}(s)| &\leq \left| \int_s^t \int_K (u_n(r,x) - u_\varepsilon(r,x)) r_{2,\varepsilon}(r,x) \; dxdr \right| \\ &\leq C_0 \int_s^t \int_K |r_{2,\varepsilon}(r,x)| \; dxdr.\end{aligned}$$

We set

$$R_\varepsilon : [0,T] \to \mathbb{R}, \quad s \mapsto R_\varepsilon(s) := \left| \int_0^s \int_K (w_1(r,x) - u_\varepsilon(r,x)) r_{2,\varepsilon}(r,x) \; dxdr \right|$$

and get for all $s \in [0,T]$ that $R_{n,\varepsilon}(s) \to R_\varepsilon(s)$. As $R_{n,\varepsilon}$ are continuous functions for all $n \in \mathbb{N}$, we can find $s_n \in [0,T]$ such that $R_{n,\varepsilon}(s_n) := \max_{s \in [0,T]} R_{n,\varepsilon}(s)$. Then, $(s_n)$ represents a bounded sequence and there is a convergent subsequence $(s_n)$ (which is labeled by $n$ again) with limit $s^* \in [0,T]$. We restrict to this subsequence. For the subsequence we conclude

$$|R_{n,\varepsilon}(s_n) - R_\varepsilon(s^*)| \leq |R_{n,\varepsilon}(s_n) - R_{n,\varepsilon}(s^*)| + |R_{n,\varepsilon}(s^*) - R_\varepsilon(s^*)| \to 0 \tag{3.35}$$

as $n \to \infty$ since $R_{n,\varepsilon}$ are equicontinuous. Now, we estimate

$$\begin{aligned}\left( \int_K |u_n(t,x) - u(t,x)| \; dx \right)^2 &\leq \left( \int_K |u_n(t,x) - u_\varepsilon(t,x)| \; dx \right)^2 + \left( \int_K |u_\varepsilon(t,x) - u(t,x)| \; dx \right)^2 \\ &\quad + 2 \int_K |u_n(t,x) - u_\varepsilon(t,x)| \; dx \int_K |u_\varepsilon(t,x) - u(t,x)| \; dx \\ &\leq C \int_K (u_n(t,x) - u_\varepsilon(t,x))^2 \; dx + C \int_K (u_\varepsilon(t,x) - u(t,x))^2 \; dx \\ &\quad + C_0 \int_K |u_\varepsilon(t,x) - u(t,x)| \; dx,\end{aligned} \tag{3.36}$$

where $C = |K|^{1/2}$. As in the proof of Theorem 3.2.9 we obtain as a consequence of Theorem 3.2.1 that

$$u_n b_n \overset{*}{\rightharpoonup} w_1 b \quad \text{in } \mathcal{M}((0,T) \times \Omega)^N. \tag{3.37}$$

Since $(u_n)$ is bounded in $L^\infty((0,T) \times \Omega)$ and $(b_n)$ is bounded in $L^p((0,T) \times \Omega)^N$ for $p = \min(q, N/(N-1))$, we have that $(u_n b_n)$ is bounded in $L^p((0,T) \times \Omega)^N$ and thus with (3.37) we have that $u_n b_n \rightharpoonup w_1 b$ in $L^p((0,T) \times \Omega)^N$. Consequently, we obtain that

$$\left| \int_0^s \int_K (u_n(r,x) - u_\varepsilon(r,x))(b(r,x) - b_n(r,x)) \cdot \nabla u_\varepsilon(r,x) \; dxdr \right| \to 0 \quad \text{as } n \to 0$$

for any $s \in [0,T]$ and with Lebesgue's dominated convergence theorem that

$$\int_0^t \|\operatorname{div} b_n(s,\cdot)\|_{L^\infty(\Omega)} \left| \int_0^s \int_K (u_n(r,x) - u_\varepsilon(r,x))(b(r,x) - b_n(r,x)) \cdot \nabla u_\varepsilon(r,x) \; dxdr \right| \; ds \to 0$$

as $n \to \infty$ for any $t \in [0,T]$. Taking the limes superior over $n$ and using estimates (3.33), (3.36) as well as relation (3.35) yield

$$\begin{aligned}
&\limsup_{n\to\infty} \left( \int_K |u_n(t,x) - u(t,x)| \; dx \right)^2 \\
&\quad \le C_0 \int_K |u_\varepsilon(t,x) - u(t,x)| \; dx + C \int_K (u_\varepsilon(t,x) - u(t,x))^2 \; dx + 2CC_1 R_\varepsilon(s^*) \\
&\quad + CC_0(C_1+1) \int_0^T \int_K |r_{1,\varepsilon}(s,x)| \; dxds + C(C_1+1) \int_K ((u_0(x) - u_{0,\varepsilon}(x))^2 \; dx \\
&\quad + 2C \left| \int_0^t \int_K (w_1(s,x) - u_\varepsilon(s,x)) r_{2,\varepsilon}(s,x) \; dxds \right| .
\end{aligned}$$

□

In the next lemma we will show that the sequence of function $(w_1 - u_{\varepsilon_m}) r_{2,\varepsilon_m}$ converges weakly$^*$ to some measure $\sigma \in \mathcal{M}([0,T) \times K)$ which is independent of the mollifier $\rho$ for a sequence $(\varepsilon_m)$ with $0 < \varepsilon_m \le 1$ for all $m \in \mathbb{N}$ and $\varepsilon_m \to 0$ as $m \to \infty$.

**Lemma 3.2.13** *Under the assumptions of Theorem 3.2.9 there exists a sequence $(\varepsilon_m)$ with $0 < \varepsilon_m \le 1$ for all $m \in \mathbb{N}$ and $\varepsilon_m \to 0$ as $m \to \infty$ such that*

$$2(w_1 - u_{\varepsilon_m}) r_{2,\varepsilon_m} \overset{*}{\rightharpoonup} \sigma \quad \text{in } \mathcal{M}([0,T] \times K) \quad \text{as } m \to \infty.$$

*The measure $\sigma \in \mathcal{M}([0,T] \times K)$ is independent of the mollifier $\rho$.*

**Proof:** We know that

$$2 \sup_{0<\varepsilon\le 1} \int_0^T \int_K |w_1(t,x) - u_\varepsilon(t,x)| \, |r_{2,\varepsilon}(t,x)| \; dxdt < \infty$$

and thus, there exists a sequence $(\varepsilon_m)$ with $0 < \varepsilon_m \le 1$ for all $m \in \mathbb{N}$ and $\varepsilon_m \to 0$ such that $2(w_1 - u_{\varepsilon_m}) r_{2,\varepsilon_m}$ converges to some $\sigma_\rho \in \mathcal{M}([0,T] \times K)$. We show that the limit $\sigma_\rho$ is not depending on $\rho$:
for $t \in (0,T)$ we take the following sequence $(\eta_{t,k}) \subset C_c^\infty([0,T))$ such that

$$0 \le \eta_{t,k}(s) \le 1 \; \forall \; s \in (0,T), \quad \eta_{t,k}(s) \to \chi_{[0,t]}(s) \; \forall \; s \in [0,T) \quad \text{and} \quad \eta'_{t,k} \to \delta_0 - \delta_t$$

in the distributional sense. Lebesgue's dominated convergence theorem then yields that

$$\eta_{t,k} \to \chi_{[0,t]} \quad \text{in } L^r((0,T)) \text{ for all } 1 \le r < \infty$$

and for any $t \in [0,T)$. Hence, from the equation given by lines (3.31) and (3.32) we deduce, setting $\beta(t) = t^2$ for all $t \in \mathbb{R}$ and integrating over $[0,T] \times K$ with test functions $\varphi \in C_c^\infty([0,T] \times K)$ and fixed $s \in [0,T)$:

$$\begin{aligned} 0 = &\int_0^T \eta_{s,k}' \int_K (u_l - u_{\varepsilon_m})^2 \varphi \; dxdt + \int_K \eta_{s,k}(0)\varphi(0,\cdot)(u_l(0,\cdot) - u_{\varepsilon_m}(0,\cdot))^2 \; dx \\ &+ \int_0^T \int_K (u_l - u_{\varepsilon_m})^2 \eta_{s,k}(\partial_t \varphi + b_l \cdot \nabla \varphi + \varphi \operatorname{div} b_l) \; dxdt \\ &+ 2 \int_0^T \int_K (u_l - u_{\varepsilon_m})\varphi \eta_{s,k}(-r_{1,\varepsilon_m} - r_{2,\varepsilon_m} + (b - b_l) \cdot \nabla u_{\varepsilon_m}) \; dxdt. \end{aligned}$$

As above, this holds for smooth $b_l$ and $u_l$. Again, taking suitable sequences, we conclude that

$$\begin{aligned} 0 = &\int_0^T \eta_{s,k}' \int_K (u_n - u_{\varepsilon_m})^2 \varphi \; dxdt + \int_K \eta_{s,k}(0)\varphi(0,\cdot)(u_n(0,\cdot) - u_{\varepsilon_m}(0,\cdot))^2 \; dx \\ &+ \int_0^T \int_K (u_n - u_{\varepsilon_m})^2 \eta_{s,k}(\partial_t \varphi + b_n \cdot \nabla \varphi + \varphi \operatorname{div} b_n) \; dxdt \\ &+ 2 \int_0^T \int_K (u_n - u_{\varepsilon_m})\varphi \eta_{s,k}(-r_{1,\varepsilon_m} - r_{2,\varepsilon_m} + (b - b_n) \cdot \nabla u_{\varepsilon_m}) \; dxdt. \end{aligned}$$

where $u_n$ and $b_n$ denotes the above solutions and vector fields. Now, taking the limit in $n$ yields with the same argument as in the proof of the previous lemma for products of weakly convergent sequences

$$\begin{aligned} 0 = &\int_0^T \int_K (w_2 - 2w_1 u_{\varepsilon_m} + u_{\varepsilon_m}^2)(\varphi \eta_{s,k}' + \eta_{s,k}(\partial_t \varphi + b \cdot \nabla \varphi)) \; dxdt \\ &+ \int_0^T \int_K \varphi \eta_{s,k}(w_4 - 2w_3 u_{\varepsilon_m} + u_{\varepsilon_m}^2 \operatorname{div} b) \; dxdt \\ &+ \int_K \eta_{s,k}(0)\varphi(0,\cdot) \left(u_0^2 - 2u_{\varepsilon_m}(0,\cdot)u_0 + (u_{\varepsilon_m}(0,\cdot))^2\right) \; dx \\ &- 2 \int_0^T \int_K (w_1 - u_{\varepsilon_m})\varphi \eta_{s,k}(r_{1,\varepsilon_m} + r_{2,\varepsilon_m}) \; dxdt. \end{aligned} \tag{3.38}$$

For the last term in (3.38), we have

$$2 \left| \int_0^T \int_K (\eta_{s,k} - \chi_{[0,s]})(w_1 - u_{\varepsilon_m})\varphi(r_{1,\varepsilon_m} + r_{2,\varepsilon_m}) \; dxdt \right|$$

$$\leq 2\left(\int_0^T \left|\eta_{s,k}-\chi_{[0,s]}\right|^{q'}\ dt\right)^{1/q'}\left(\int_0^T\left(\int_K |(w_1-u_{\varepsilon_m})\varphi(r_{1,\varepsilon_m}+r_{2,\varepsilon_m})|\ dx\right)^q dt\right)^{1/q}$$
$$\leq 2C\left(\int_0^T \left|\eta_{s,k}-\chi_{[0,s]}\right|^{q'}\ dt\right)^{1/q'} \to 0 \quad \text{as } k\to\infty,$$

where $C>0$ is an upper bound for

$$\sup_{m\in\mathbb{N}}\left(\int_0^T\left(\int_K |(w_1-u_{\varepsilon_m})\varphi(r_{1,\varepsilon_m}+r_{2,\varepsilon_m})|\ dx\right)^q dt\right)^{1/q}.$$

Thus, we can switch the limiting processes of $k\to\infty$ and $m\to\infty$ and we obtain using $r_{1,\varepsilon_m}\to 0$ in $L^1((0,T)\times K)$ as $m\to\infty$

$$\begin{aligned}
\lim_{k\to\infty}\langle\sigma_\rho,\varphi\eta_{s,k}\rangle &= \lim_{m\to\infty}\lim_{k\to\infty} 2\int_0^T\int_K (w_1-u_{\varepsilon_m})r_{2,\varepsilon_m}\varphi\eta_{s,k}\ dxdt\\
&= \lim_{m\to\infty}\lim_{k\to\infty}\int_0^T\int_K (w_2-2w_1u_{\varepsilon_m}+u_{\varepsilon_m}^2)(\varphi\eta_{s,k}'+\eta_{s,k}(\partial_t\varphi+b\cdot\nabla\varphi))\ dxdt\\
&+\lim_{m\to\infty}\lim_{k\to\infty}\int_K \eta_{s,k}(0)\varphi(0,\cdot)\left(u_0^2-2u_{\varepsilon_m}(0,\cdot)u_0+(u_{\varepsilon_m}(0,\cdot))^2\right)\ dx\\
&+\lim_{m\to\infty}\lim_{k\to\infty}\int_0^T\eta_{s,k}\int_K\varphi(w_4-2w_3u_{\varepsilon_m}+u_{\varepsilon_m}^2\operatorname{div} b)\ dxdt\\
&= \lim_{m\to\infty}\Bigg[\int_K\varphi(0,\cdot)\left(u_0^2-2u_0u_{\varepsilon_m}(0,\cdot)+(u_{\varepsilon_m}(0,\cdot))^2+w_2(0,\cdot)\right.\\
&\qquad \left.-2w_1(0,\cdot)u_{\varepsilon_m}(0,\cdot)+(u_{\varepsilon_m}(0,\cdot))^2\right)\ dx\\
&\qquad -\int_K\varphi(s,\cdot)\left(w_2(s,\cdot)-2w_1(s,\cdot)u_{\varepsilon_m}(s,\cdot)+(u_{\varepsilon_m}(s,\cdot))^2\right)\ dx\Bigg]\\
&+\lim_{m\to\infty}\Bigg[\int_0^s\int_K (w_2-2w_1u_{\varepsilon_m}+u_{\varepsilon_m}^2)(\partial_t\varphi+b\cdot\nabla\varphi)\ dxdt\\
&\qquad +\varphi(w_4-2w_3u_{\varepsilon_m}+u_{\varepsilon_m}^2\operatorname{div} b)\ dxdt\Bigg]\\
&= \int_0^s\int_K (w_2-2w_1u+u^2)(\partial_t\varphi+b\cdot\nabla\varphi)+\varphi(w_4-2w_3u+u^2\operatorname{div} b)\ dxdt\\
&-\int_K\varphi(s,\cdot)\left(w_2(s,\cdot)-2w_1(s,\cdot)u+u(s,\cdot)^2\right)\ dx
\end{aligned}$$

since

$$w_2(0,\cdot) - 2w_1(0,\cdot)u_{\varepsilon_m}(0,\cdot) + (u_{\varepsilon_m}(0,\cdot))^2 = u_0^2 - 2u_0u_{\varepsilon_m}(0,\cdot) + (u_{\varepsilon_m}(0,\cdot))^2 \rightharpoonup 0 \quad \text{in } L^2(\Omega).$$

From the above equation and the preceding estimates and equations we get the following information: if we omit $\eta_{s,k}$ at the beginning and just test with $\varphi$, we see that the measure $\sigma_\rho$ is given by

$$\sigma_\rho = -\partial_t(w_2 - 2w_1u + u^2) - \operatorname{div}(b(w_2 - 2w_1u + u^2)) + (w_4 - 2w_3u + u^2 \operatorname{div} b)$$

and thus, independent of the mollifier $\rho$. Therefore, we call $\sigma_\rho$ just $\sigma$ in the following. Furthermore, if we restrict $\sigma$ to the set $[0,s] \times K$ and denote the restriction $\sigma_s$ we obtain from the above equation for any $\varphi \in C_c([0,T] \times K)$:

$$\begin{aligned}\int\limits_{[0,s]}\int\limits_K \varphi \, d\sigma_s &= \int\limits_{[0,T]}\int\limits_K \chi_{[0,s]}\varphi \, d\sigma = \lim_{k\to\infty} \int\limits_{[0,T]}\int\limits_K \varphi(\chi_{[0,s]} - \eta_{s,k}) \, d\sigma + \lim_{k\to\infty} \int\limits_{[0,T]}\int\limits_K \varphi\eta_{s,k} \, d\sigma \\ &= -\int\limits_K \varphi(s,\cdot)(w_2(s,\cdot) - 2w_1(s,\cdot)u + (u(s,\cdot))^2) \, dx \\ &\quad + \int\limits_0^s\int\limits_K (w_2 - 2w_1u + u^2)(\partial_t\varphi + b\cdot\nabla\varphi) + \varphi(w_4 - 2w_3u + u^2 \operatorname{div} b) \, dxdt,\end{aligned}$$

i.e. the restriction $2[(w_1 - u_{\varepsilon_m})r_{2,\varepsilon_m}]|_{[0,s]\times K}\mathcal{L}^1 \otimes \mathcal{L}^N$ converges weakly* to

$$\begin{aligned}\sigma_s = &-\partial_t\left((w_2 - 2w_1u + u^2)|_{[0,s]\times K}\right) - \operatorname{div}\left(b(w_2 - 2w_1u + u^2)|_{[0,s]\times K}\right) \\ &+ (w_4 - 2w_3u + u^2 \operatorname{div} b)|_{[0,s]\times K}.\end{aligned}$$

□

In the last lemma we use this measure to show that the right side of estimate (3.34) is zero. This gives us the statement of Theorem 3.2.9.

**Lemma 3.2.14** *Under the assumptions of Theorem 3.2.9 the statement of the theorem holds.*

**Proof:** So far, we have shown that our limits do not depend on the specific mollifier and we go back to estimate (3.34). Taking the supremum over $m \in \mathbb{N}$ with $t \in [0,T]$ and $\varphi \equiv 1$ on $[0, \max(t,s^*)] \times K$ yields:

$$\begin{aligned}&\limsup_{n\to\infty}\left(\int\limits_K |u_n(t,x) - u(t,x)| \, dx\right)^2 \\ &\leq 2C \sup_{m\in\mathbb{N}}\left|\int\limits_0^t\int\limits_K (w_1(s,x) - u_{\varepsilon_m}(s,x))r_{2,\varepsilon_m}(s,x) \, dxds\right| \\ &+ CC_1 \sup_{m\in\mathbb{N}} R_{\varepsilon_m}(s^*) \\ &= C\,|\sigma_t([0,t]\times K)| + CC_1\,|\sigma_{s^*}([0,s^*]\times K)|\,.\end{aligned}$$

Now, in the remaining part we show that $\sigma = 0$. This will work in the same way as it is shown that the limit measure of the commutator is zero in the previous chapter. The sequence $(|(w_1 - u_{\varepsilon_m})r_{2,\varepsilon_m}|)$ is bounded in $L^1((0,T)\times K)$ and thus, a subsequence converges weakly$^*$ to some measure $\lambda \in \mathcal{M}([0,T]\times K)$. Due to Proposition 1.62 in [AFP00] we have that $|\sigma| \leq \lambda$. Hence, restricting to this subsequence we obtain for $\varphi \in C_c([0,T]\times K)$

$$\begin{aligned}\int_{[0,T]}\int_K |\varphi(t,x)| \, d|\sigma|(t,x) &\leq \limsup_{m\to\infty}\int_0^T\int_K |\varphi(t,x)|\,|(w_1(t,x)-u_{\varepsilon_m}(t,x))r_{2,\varepsilon_m}(t,x)| \, dxdt \\ &\leq C\limsup_{m\to\infty}\int_0^T\int_K |\varphi(t,x)|\,|r_{2,\varepsilon_m}(t,x)| \, dxdt \\ &\leq C\limsup_{m\to\infty}\int_0^T\int_K |\varphi(t,x)|\int_{\mathbb{R}^N} |b_{2,\varepsilon_m,z}(t,x)\cdot\nabla\rho(z)| \, dzdxdt. \qquad (3.39)\end{aligned}$$

Now, setting $S := \|\varphi\|_{C([0,T]\times K)}$ and

$$W_{t,y} := \overline{\{x \in K| \ |\varphi|(t,x) > y\}}$$

we rewrite (3.39) and obtain

$$\begin{aligned} &C\limsup_{m\to\infty}\int_0^T\int_0^S\int_{W_{t,y}}\int_{\mathbb{R}^N} |b_{2,\varepsilon_m,z}(t,x)\cdot\nabla\rho(z)| \, dzdxdydt \\ &\leq C\int_0^T\int_0^S\int_{\mathbb{R}^N} \limsup_{m\to\infty}\int_{W_{t,y}} |b_{2,\varepsilon_m,z}(t,x)\cdot\nabla\rho(z)| \, dxdzdydt \\ &\leq C\int_0^T\int_0^S\int_{\mathbb{R}^N} \left|(\nabla\rho(z))^\top (D^s b)(t,\cdot)z\right|(W_{t,y}) \, dzdydt \\ &= C\int_0^T\int_0^S\int_{\mathbb{R}^N}\int_{W_{t,y}} d\left|(\nabla\rho(z))^\top (D^s b)(t,\cdot)z\right|(x)dzdydt \\ &= C\int_0^T\int_0^S\int_{W_{t,y}}\int_{\mathbb{R}^N} \left|(\nabla\rho(z))^\top M_b(t,x)z\right| \, dzd\,|D^s b(t,\cdot)|(x)dydt \\ &= C\int_0^T\int_K |\varphi(t,x)|\,\Lambda(M_b(t,x),\rho) \, d\,|D^s b(t,\cdot)|(x)dt. \end{aligned}$$

Thus, $|\sigma| \leq C\Lambda(M_b,\rho)\,|D^s b|$ and in the same way as in Lemma 3.1.16 we obtain that there exists a Borel function $f$ such that $|\sigma| = f\,|D^s b|$ and

$$|f(t,x)| \leq C\Lambda(M_b(t,x),\rho) \qquad \text{for } |D^s b|\text{-a.e. } (t,x).$$

Since $|\sigma|$ is not depending on the mollifier $\rho$, we deduce with the same argumentation as in the above mentioned proof that

$$|f(t,x)| \leq \inf_{\rho \in \mathcal{K}'} C\Lambda(M_b(t,x),\rho) = \inf_{\rho \in \mathcal{K}} C\Lambda(M_b(t,x),\rho) \qquad \text{for } |D^s b|\text{-a.e. } (t,x),$$

where $\mathcal{K}' \subset \mathcal{K}$ denotes a countable dense subset. Then, the Lemma of Alberti 3.1.17 yields that

$$|f(t,x)| \leq C\,|\mathrm{trace}(M_b(t,x))| = 0 \qquad \text{for } |D^s b|\text{-a.e. } (t,x),$$

since the singular part of $\mathrm{Div}\, b$ is zero. Therefore, we obtain that $\sigma = 0$ and thus for $t \in [0,T)$

$$\limsup_{n\to\infty} \left( \int_{\mathbb{R}^N} |u_n(t,x) - u(t,x)| \ dx \right)^2 = 0.$$

For the subsequence $(u_n)$ being convergent to $w_1$ in $C([0,T], L^2(\Omega) - w)$, we conclude that $w_1(t,\cdot) = u(t,\cdot)$ for all $t \in [0,T]$. Analogously, we obtain that $w_2(t,\cdot) = u^2(t,\cdot)$ for all $t \in [0,T]$. Using a proof by contradiction as in the proof of Theorem 3.2.4, we obtain that the whole sequence $(u_n)$ converges to $u$ in $C([0,T], L^2(\Omega))$ and using the boundedness of $(u_n)$ in $L^\infty((0,T)\times\Omega)$, we get that the convergence is valid in $C([0,T], L^p(\Omega))$ for any $1 \leq p < \infty$.

□

# 4. Optimal control problems

In this chapter, we consider optimal control problems of the form

$$\min_{u,b} J(u,b) = \frac{1}{2} \sum_{k=2}^{K} \Upsilon_k \left( \|u(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)} \right) + R(b)$$

with the transport equation of the previous chapter as one of the constraints. Here, the function $u$ denotes the solution of the transport equation with vector field $b$ and initial value $Y_1 \in L^\infty(\Omega)$, i.e. $u = S(Y_1, b)$. In the above objective function, the functions $Y_k \in L^\infty(\Omega)$ and $\Upsilon_k : \mathbb{R} \to \mathbb{R}$, $k = 2, \ldots, K$ are given and the mapping $R$ represents the regularization terms of the minimizing functional $J$. Our aim of this chapter is to show that $J$ attains a minimum on a bounded subset $S_{ad} \subset \mathrm{VF}_0$ with the transport equation as a side condition. In the optical flow framework this problem appears with various regularization terms in [HS01, BIK03]. Some of their considered regularization terms will be covered in this chapter, in particular the one appearing in [BIK03]. Furthermore, Chen and Lorenz examined this problem in [Che11, CL11] for some special regularization term. In all these works, results on existence of some minima require assumptions with much more regularity of the involved functions as we will need for our results in this chapter. The chapter is divided into two sections: in the first section we will have a closer look on time dependent functions with $BV(\Omega)$ as codomain. As mentioned in subsection 2.1.3 we consider time dependent functions whose codomain is given by a dual space as Gelfand integrable functions. For this purpose we need to clarify when the weak$^*$ topology, usually used in $BV(\Omega)$ coincides with the standard weak$^*$ topology in functional analysis when we consider $BV(\Omega)$ as the dual of a separable Banach space. In [AFP00] it is mentioned that these topologies are equal for sufficiently regular domains. As we will see Lipschitz regular domains are sufficient. In the second part of this subsection, we will use a generalization of Fatou's Lemma for unbounded Gelfand integrable functions ([CdR04]) to show that bounded sequences of time dependent vector fields contain subsequences being weakly$^*$ convergent with Gelfand integrable functions as limits. In the second section, we will use these results and the statements of the previous chapters to show existence of minima for the above optimal control problems with diverse regularization terms $R$. Beside the total variation of the vector field $b$, which will be a fixed part in all regularization terms, the following additional terms are considered:

$$R_2(b) = \frac{\beta}{2} \int_0^T \Gamma_2 \left( \|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)^N} \right) \, dt \quad \text{and} \quad R_3(b) = \frac{\gamma}{2} \int_0^T \Gamma_3 \left( \|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)} \right) \, dt,$$

where $\beta, \gamma > 0$ denote some regularization parameters and $\Gamma_2, \Gamma_3 : \mathbb{R} \to \mathbb{R}$ are given functions. For the theory about transport equations zero boundary of the vector fields in the spatial domain is needed. Unfortunately, the limit function of a convergent subsequence will not need to have zero boundary anymore, since the convergence is at best with respect to the weak$^*$ topology and the trace operator is not continuous with respect to this topology. Therefore, we will introduce some additional technical assumptions to tackle this problem and to enforce strict convergence. This will give us a limit with spatial zero boundary.

## 4.1. Time dependent vector fields with $BV(\Omega)$ as codomain

### 4.1.1. Predual of $BV(\Omega)$

In the space $BV(\Omega)$ an often used topology is the so-called weak* topology. The name of the topology is misleading since this topology is not the standard weak* topology in functional analysis if $BV(\Omega)$ is seen as a dual space of a separable Banach space. In Remark 3.12 in [AFP00] it is mentioned that these two topologies coincides if the domain is sufficiently regular. We will show that Lipschitz regularity for the domain is sufficiently enough. With this result we do not need to distinguish between these two topologies in the subsequent parts, in particular in the case when we consider vector fields as Gelfand integrable functions where $BV(\Omega)$ is regarded as a dual space with (dual) weak* topology. As in the previous parts of this thesis, we consider $\Omega \subset \mathbb{R}^N$ as an open, bounded domain with Lipschitz boundary in this subsection.

In Remark 3.12 in [AFP00], a sketch for constructing the predual of $BV(\Omega)$ is given. In the following, we call $\Gamma(\Omega)$ the predual of $BV(\Omega)$ and we give a precise construction of $\Gamma(\Omega)$: we set $X := C_0(\Omega)^{N+1}$ and

$$E := \left\{ \Phi = (\Phi_0, \dots, \Phi_N) \in X, \varphi = (\Phi_1, \dots, \Phi_N) \in C_c^\infty(\Omega)^N \text{ such that } \operatorname{div} \varphi = \Phi_0 \right\}.$$

Then $E$ is a subspace of $X$ and we set $Y$ as the closure of $E$ with respect to $\|\cdot\|_{C(\Omega)^{N+1}}$. Now Remark 3.12 in [AFP00] yields that the map $T$ given by

$$T : BV(\Omega) \to \mathcal{M}(\Omega)^{N+1}, \qquad u \mapsto (u\mathcal{L}^N, \partial_{x_1} u, \dots, \partial_{x_N} u)$$

is an isomorphism between $BV(\Omega)$ and $T(BV(\Omega))$ with

$$\|u\|_{BV(\Omega)} \leq 2 \|T(u)\|_{\mathcal{M}(\Omega)^{N+1}} \leq 2 \|u\|_{BV(\Omega)}.$$

Furthermore, for all $\Phi \in E$ and $u \in BV(\Omega)$ we have that

$$\begin{aligned} (T(u), \Phi)_{(\mathcal{M}(\Omega)^{N+1}, C_0(\Omega)^{N+1})} &= \left(u\mathcal{L}^N, \Phi_0\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} + \sum_{k=1}^N (\partial_{x_k} u, \Phi_k)_{(\mathcal{M}(\Omega), C_0(\Omega))} \\ &= \left(u\mathcal{L}^N, \operatorname{div} \varphi\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} + \sum_{k=1}^N (\partial_{x_k} u, \Phi_k)_{(\mathcal{M}(\Omega), C_0(\Omega))} \\ &= \left(u\mathcal{L}^N, \operatorname{div} \varphi\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} - \left(u\mathcal{L}^N, \operatorname{div} \varphi\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} = 0. \end{aligned} \tag{4.1}$$

Hence, we obtain that $(T(u), y) = 0$ for all $u \in BV(\Omega)$ and all $y \in Y$. This means that $T(BV(\Omega)) \subset Y^\circ$, the annihilator of $Y$, which is the set of linear functionals $L \in X'$ such that $Y$ lies in the kernel of $L$. By using the following result we conclude that $Y^\circ = T(BV(\Omega))$.

**Lemma 4.1.1** *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\mu, \nu_i \in \mathcal{M}(\Omega)$ for $i = 1, \dots, N$ such that*

$$\int_\Omega \partial_{x_i} \varphi(x) \, d\mu(x) = - \int_\Omega \varphi(x) \, d\nu_i(x) \qquad \forall \, \varphi \in C_c^1(\Omega), \; i = 1, \dots, N.$$

*Then, there exists a unique $u \in BV(\Omega)$ such that $\mu = u\mathcal{L}^N$.*

**Proof:** We smooth $\mu$ with some mollifier $\rho_\varepsilon$ and then, Theorem 2.2 in [AFP00] yields

$$\int\limits_{\Omega} |(\mu * \rho_\varepsilon)(x)| \ dx \leq \int\limits_{\mathbb{R}^N} |(\mu * \rho_\varepsilon)(x)| \ dx \leq |\mu| (\mathbb{R}^N) = \|\mu\|_{\mathcal{M}(\Omega)} . \tag{4.2}$$

In addition, we have

$$\int\limits_{\Omega} |D(\mu * \rho_\varepsilon)(x)| \ dx \leq \int\limits_{\mathbb{R}^N} |(D\mu * \rho_\varepsilon)(x)| \ dx \leq |\nu| (\mathbb{R}^N) = \|\nu\|_{\mathcal{M}(\Omega)^N} \tag{4.3}$$

with $\nu = (\nu_1, \ldots, \nu_N)$. Thus, $(\mu * \rho_\varepsilon) \subset BV(\Omega)$ is bounded and we conclude that there exists a subsequence $(\mu * \rho_\varepsilon)$ (labeled by $\varepsilon$ again) and some $u \in BV(\Omega)$ such that

$$\mu * \rho_\varepsilon \to u \quad \text{in } L^1(\Omega).$$

On the other hand, $\mu * \rho_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. Thus, $u\mathcal{L}^N = \mu$.

□

Hence, Theorem III.1.10 in [Wer11] yields that $Y^\circ \simeq (X/Y)'$ and an isomorphism is given by

$$T_1 : Y^\circ \to (X/Y)', \qquad y \mapsto T_1(y)$$

with

$$T_1(y) : X/Y \to \mathbb{R}, \quad [w] \mapsto \langle T_1(y), [w] \rangle_{((X/Y)', X/Y)} = \langle y, w \rangle_{(X', X)}$$

which is well-defined due to (4.1). Hence, $BV(\Omega)$ is isomorphic to $(X/Y)'$ via $T_1 \circ T$ and we can identify the predual $\Gamma(\Omega)$ with $X/Y$. Now, for some $u \in BV(\Omega)$, we define

$$\langle u, [w] \rangle_{(BV(\Omega), \Gamma(\Omega))} = \left(u\mathcal{L}^N, w_0\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} + \sum_{k=1}^{N} \left(\partial_{x_k} u, w_k\right)_{(\mathcal{M}(\Omega), C_0(\Omega))} \tag{4.4}$$

for all $[w] \in \Gamma(\Omega)$ with $w \in X$ and $w = (w_0, w_1, \ldots, w_N)$. Therefore, we conclude for a sequence $(u_n) \subset BV(\Omega)$ and some $u \in BV(\Omega)$ (we use the notation $\overset{*}{\rightharpoonup}$ for the standard weak$^*$ topology in functional analysis and $\overset{**}{\rightharpoonup}$ for the usually used weak$^*$ topology in $BV(\Omega)$):

$$\begin{aligned}
u_n \overset{*}{\rightharpoonup} u \Leftrightarrow \langle u_n - u, [w] \rangle_{(BV(\Omega), \Gamma(\Omega))} \quad & \forall\, [w] \in \Gamma(\Omega) \\
\Leftrightarrow u_n\mathcal{L}^N \overset{*}{\rightharpoonup} u\mathcal{L}^N \quad & \text{in } \mathcal{M}(\Omega) \text{ and} \\
\partial_{x_i} u_n \overset{*}{\rightharpoonup} \partial_{x_i} u \quad & \text{in } \mathcal{M}(\Omega) \quad \forall\, i \in \{1, \ldots, N\} \\
\Leftrightarrow u_n \to u \quad & \text{in } L^1(\Omega) \text{ and} \\
\partial_{x_i} u_n \overset{*}{\rightharpoonup} \partial_{x_i} u \quad & \text{in } \mathcal{M}(\Omega) \quad \forall\, i \in \{1, \ldots, N\} \\
\Leftrightarrow u_n \overset{**}{\rightharpoonup} u. \quad &
\end{aligned}$$

In the third equivalence relation we used the fact that for domains with compact Lipschitz boundary $BV(\Omega)$ is compactly imbedded in $L^1(\Omega)$ (see Proposition 3.21 and Corollary 3.49 in [AFP00]). Hence, for Lipschitz regular and bounded domains, these two topologies coincides and in the following we will use the term weak$^*$ and the notation $\overset{*}{\rightharpoonup}$ for both topologies.

### 4.1.2. Closedness of bounded sets of time dependent vector fields

In this subsection, we have a closer look on norm bounded sets of vector fields. In the main theorem we will prove that sequences $(b_n) \subset \mathrm{VF}$ which are bounded with respect to some norm contain subsequences which are convergent in a weak sense and whose limits are again vector fields with the same temporal and spatial regularities. The statement will play a crucial role in the next section: in the proof of existence of minima, the result of this subsection will give us a limit for which it can be shown that it represents a minimum. We start with some definition.

For $q \in (1, \infty)$ we define the set

$$\mathrm{VF}^q := \left\{ b \in \mathrm{VF} |\ b \in L^q((0,T), BV(\Omega))^N \text{ and } \operatorname{div} b \in L^q((0,T), L^\infty(\Omega)) \right\}.$$

Then, we can state the main result of this subsection.

**Theorem 4.1.2** *Let $q \in (1, \infty)$ and let $(b_n) \subset \mathrm{VF}^q$ be a sequence. If $(b_n)$ is bounded, i.e. for all $n \in \mathbb{N}$*

$$\|b_n\|_{L^q((0,T),BV(\Omega))^N} \leq C < \infty$$

*for some $C > 0$, then there exists a subsequence $(b_{n_k})$ and a function $b \in \mathrm{VF}^q$ such that the following properties are satisfied:*

*(i) for almost all $t \in (0,T)$ $b(t) \in \overline{\operatorname{conv}(\overline{\{b_n(t) | n \in \mathbb{N}\}}^{w^*})}^{w^*}$,*

*(ii) for any measurable set $B \in \mathcal{B}((0,T))$*

$$\int_B b_n(t,\cdot)\ dt \overset{*}{\rightharpoonup} \int_B b(t,\cdot)\ dt \quad \text{in } BV(\Omega)^N,$$

*(iii) for any measurable set $B \in \mathcal{B}((0,T))$ and any monotonically increasing and convex function $g : \mathbb{R} \to \mathbb{R}_0^+$ with $g \in \mathcal{O}(x)$*

$$\int_B g\left( \|Db(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}} \right)\ dt \leq \liminf_{n\to\infty} \int_B g\left( \|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}} \right)\ dt,$$

*(iv) $b_n \rightharpoonup b$ in $L^p((0,T) \times \Omega)^N$ as $n \to \infty$ for any $p \in [1, \min(q, N/(N-1)))$.*

**Proof:** We start to show that for any $[w] \in \Gamma(\Omega)^N$ the set of functions

$$t \mapsto \langle b_n(t,\cdot), [w] \rangle_{(BV(\Omega)^N, \Gamma(\Omega)^N)} \tag{4.5}$$

is uniformly integrable in $n \in \mathbb{N}$. Then, results from [CdR04] will yield most of our statements. Let $[w] \in \Gamma(\Omega)^N$. We take a fixed representative $w \in C_0(\Omega)^{N\times(N+1)}$ and estimate for any measurable set $B \subset (0,T)$

$$\int_B \left| \langle b_n(r,\cdot), [w] \rangle_{(BV(\Omega)^N, \Gamma(\Omega)^N)} \right|\ dr \leq \sum_{i=1}^N \int_B \left| \langle b_{i,n}(r,\cdot) \mathcal{L}^N, w_{i,1} \rangle \right|\ dr \tag{4.6}$$

$$+ \sum_{i=1}^N \sum_{j=1}^N \int_B \left| \langle \partial_{x_j} b_{i,n}(r,\cdot), w_{i,j+1} \rangle \right|\ dr. \tag{4.7}$$

Now, we have a closer look on the terms (4.6) and (4.7). For term (4.6) we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{B}\left|\langle b_{i,n}(r,\cdot)\mathcal{L}^N, w_{i,1}\rangle\right|\ dr &\leq |B|^{1/q'}\sum_{i=1}^{N}\|b_{i,n}\|_{L^q((0,T),L^1(\Omega))}\|w_{i,1}\|_{C(\Omega)}\\ &\leq |B|^{1/q'}C_1\sum_{i=1}^{N}\|w_{i,1}\|_{C(\Omega)}\end{aligned}\tag{4.8}$$

for some $C_1 > 0$ independent of $n \in \mathbb{N}$. For the second term (4.7) we estimate

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{B}\left|\langle \partial_{x_j}b_{i,n}(r,\cdot), \varphi_{i,j,k}\rangle\right|\ dr &\leq |B|^{1/q'}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|\partial_{x_j}b_{i,n}\right\|_{L^q((0,T),\mathcal{M}(\Omega))}\left\|w_{i,j+1}\right\|_{C(\Omega)}\\ &\leq |B|^{1/q'}C_2\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|w_{i,j+1}\right\|_{C(\Omega)}\end{aligned}\tag{4.9}$$

for some $C_2 > 0$ independent of $n \in \mathbb{N}$. The uniform integrability of the functions in (4.5) follows directly from estimates (4.6)-(4.9). Now, Theorem 3.1 (b) in [CdR04] yields that there exists a subsequence (labeled by $n$ again) and a Gelfand integrable function $b \in L^1((0,T), BV(\Omega))^N$ such that

$$\begin{aligned}\left\langle \int_B b(t,\cdot)\ dt, [w]\right\rangle &= \int_B \langle b(t,\cdot), [w]\rangle\ dt\\ &\leq \liminf_{n\to\infty}\int_B \langle b_n(t,\cdot), [w]\rangle\ dt = \liminf_{n\to\infty}\left\langle \int_B b_n(t,\cdot)\ dt, [w]\right\rangle\end{aligned}$$

for any $[w] \in \Gamma(\Omega)^N$ and for any measurable $B \in \mathcal{B}((0,T))$. Since the above inequality is satisfied both for $[w]$ and $-[w]$, we conclude that

$$\int_B b_n(t,\cdot)\ dt \overset{*}{\rightharpoonup} \int_B b(t,\cdot)\ dt \quad \text{in } BV(\Omega)^N \tag{4.10}$$

for any $B \in \mathcal{B}((0,T))$. Due to Proposition 3.1 in [CdR04] we can choose the subsequence $(b_n)$ such that it is K-convergent to $b$. Furthermore, part (c) of that theorem yield point (i). Since $BV(\Omega)$ is compactly imbedded in $L^p(\Omega)$ for any $p < N/(N-1)$, (4.10) yields that

$$\int_B b_n(t,\cdot)\ dt \to \int_B b(t,\cdot)\ dt \quad \text{in } L^p(\Omega)^N$$

for any $B \in \mathcal{B}((0,T))$ and any $p < N/(N-1)$. Hence, for $p \in (1, \min(q, N/(N-1)))$ and for $h \in L^{p'}((0,T)\times\Omega)^N$ Theorem 10.4 (i) in [Sch13] yields that there is a sequence $(h_k) \subset L^{p'}((0,T), L^{p'}((\Omega))^N$ of simple functions such that $h_k \to h$ in $L^{p'}((0,T), L^{p'}(\Omega))^N$. Denote $A_{k,i} \subset (0,T)$, $i = 1,\dots,K(k)$ the different measurable subsets where $h_k$ is constant with value $h_{k,i} \in L^{p'}(\Omega)$. Then, we conclude

$$|\langle h, b_n - b\rangle| \leq \sum_{i=1}^{K(k)}\left|\langle h_{k,i}, \int_{A_{k,i}} b_n(t,\cdot) - b(t,\cdot)\ dt\rangle\right|$$

$$
\begin{aligned}
&+ \|h_k - h\|_{L^{p'}((0,T),L^{p'}(\Omega))^N} \|b_n - b\|_{L^p((0,T),L^p(\Omega))^N} \\
&\le \sum_{i=1}^{K(k)} \left| \langle h_{k,i}, \int_{A_{k,i}} b_n(t,\cdot) - b(t,\cdot)\ dt \rangle \right| + C \|h_k - h\|_{L^{p'}((0,T),L^{p'}(\Omega))^N}
\end{aligned}
$$

for some $C > 0$ since $(b_n)$ is bounded in $L^p((0,T)\times\Omega)^N$. This yields that

$$
|\langle h, b_n - b\rangle| \to 0 \quad \text{as } n \to \infty.
$$

Thus, $b_n \rightharpoonup b$ in $L^p((0,T)\times\Omega)^N$ and hence in $L^1((0,T)\times\Omega)^N$. It remains to show that $b \in L^q((0,T), BV(\Omega))^N$ and point (iii). We consider the sequence $(Db_n) \subset L^q((0,T), \mathcal{M}(\Omega)^{N\times N})$. For this sequence we do the same steps as in the proof of Theorem 3.1 (a) in [CdR04] but with some differences: we choose a subsequence (labeled by $n$ again) such that

$$
\liminf_{n\to\infty} \int_0^T g\left(\|D_{b_n}(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)\ dt = \lim_{n\to\infty} \int_0^T g\left(\|D_{b_n}(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)\ dt
$$

since

$$
\sup_{n\in\mathbb{N}} \int_0^T g\left(\|D_{b_n}(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)\ dt < \infty
$$

due to the boundedness of $\left(\int_0^T \|D_{b_n}(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\ dt\right)$, $g \in \mathcal{O}(x)$ and the monotone increasing of $g$. Then, as in the above mentioned proof we can construct a subsequence $(Db_{n_k})$ being K-convergent to some $f \in L^1((0,T), \mathcal{M}(\Omega)^{N\times N})$. On the other hand, we already know that the whole sequence $(Db_n)$ is K-convergent to $Db$. Thus, we conclude $Db = f$ and we have as in [CdR04]

$$
\|Db(t,\cdot)\|_{\mathcal{M}(\Omega)^{N\times N}} \le \liminf_{n\to\infty} \left\| \frac{1}{n}\sum_{i=1}^n Db_i(t,\cdot) \right\|_{\mathcal{M}(\Omega)^{N\times N}} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \|Db_i(t,\cdot)\|_{\mathcal{M}(\Omega)^{N\times N}}
$$

for almost all $t \in (0,T)$. Thus, since $x \mapsto |x|^q$ is convex and $g$ is continuous as a convex function,

$$
g\left(\|Db(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right) \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n g\left(\|Db_i(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)
$$

for almost all $t \in (0,T)$. In addition, due to $g \in \mathcal{O}(x)$, the above expressions are integrable over measurable sets $B \subset (0,T)$. Fatou's lemma for positive functions then yields

$$
\begin{aligned}
\int_B g\left(\|Db(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)\ dt &\le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \int_B g\left(\|Db_i(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)\ dt \\
&= \liminf_{n\to\infty} \int_B g\left(\|Db_n(t,\cdot)\|^q_{\mathcal{M}(\Omega)^{N\times N}}\right)\ dt
\end{aligned}
$$

for any $B \in \mathcal{B}((0,T))$. The boundedness of $(b_n)$ in $L^q((0,T), BV(\Omega))^N$ and the choice $g(x) = x$ finally yields that $b \in L^q((0,T), BV(\Omega))^N$.

□

Beside this result for Gelfand integrable functions we need the following result for Bochner integrable functions in the subsequent section.

**Lemma 4.1.3** *Let $l \in \mathbb{N}$, $g : \mathbb{R} \to \mathbb{R}_0^+$ be a monotonically increasing and convex function with $g \in \mathcal{O}(x)$ and let $(f_n) \subset L^2((0,T), L^2(\Omega))^l$ be a bounded sequence. Then, there exists a subsequence $(f_{n_k})$ and some $f \in L^2((0,T), L^2(\Omega))^l$ such that*

$$\int_0^T g\left(\|f(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt \le \liminf_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt.$$

**Proof:** Due to the boundedness of $(f_n)$ in $L^2((0,T), L^2(\Omega))^l$, there exists a subsequence (labeled by $n$ again) and some $f \in L^2((0,T), L^2(\Omega))^l$ such that $f_n \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))^l$. Furthermore, due to the properties of $g$, we have

$$\sup_{n\in\mathbb{N}} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt < \infty$$

and thus, we can choose a subsequence $(f_n)$ (labeled by $n$ again) such that

$$\liminf_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt = \lim_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt$$

holds. Applying Theorem 2.1 in [DRS93], we then obtain that there is a sequence $(h_n) \subset L^2((0,T), L^2(\Omega))^l$ with $h_n \in \operatorname{conv}(\{f_k |\ k \ge n\})$ for $n \in \mathbb{N}$ such that $(h_n(t,\cdot))$ is convergent to some $h(t,\cdot) \in L^2(\Omega)^l$ for almost all $t \in (0,T)$, i.e.

$$h_n = \sum_{i=n}^{N(n)} \lambda_{n,i} f_i \quad \text{with } 0 \le \lambda_{n,i} \le 1 \quad \text{for } n \le i \le N(n) \in \mathbb{N} \quad \text{and } \sum_{i=n}^{N(n)} \lambda_{n,i} = 1$$

for all $n \in \mathbb{N}$. We assume that $h(t,\cdot) \ne f(t,\cdot)$ for $t \in B \subset (0,T)$ with $\mathcal{L}^1(B) > 0$. Then, we have for $\varphi \in L^2(\Omega)^l$

$$\begin{aligned}\int_0^T |\langle h_n(t,\cdot), \varphi\rangle|^2\, dt &\le \|\varphi\|^2_{L^2(\Omega)^l} \sup_{n\in\mathbb{N}} \int_0^T \|h_n(t,\cdot)\|^2_{L^2(\Omega)^l}\, dt \\ &\le \|\varphi\|^2_{L^2(\Omega)^l} \sup_{n\in\mathbb{N}} \int_0^T \|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\, dt < \infty.\end{aligned}$$

Due to Theorem 1.35 in [AFP00] we obtain that

$$[t \mapsto \langle h_n(t,\cdot), \varphi\rangle] \rightharpoonup [t \mapsto \langle h(t,\cdot), \varphi\rangle] \quad \text{in } L^2((0,T)).$$

Hence, we conclude for $\psi \in L^2(B)$

$$\int_B \int_\Omega \psi(t)\varphi(x)h(t,x)\, dxdt \leftarrow \int_B \int_\Omega \psi(t)h_n(t,x)\varphi(x)\, dxdt \to \int_B \int_\Omega \psi(t)\varphi(x)f(t,x)\, dxdt,$$

i.e. $\langle h(t,\cdot),\varphi\rangle = \langle f(t,\cdot),\varphi\rangle$ for almost all $t \in B$. Since $\varphi \in L^2(\Omega)^l$ can be arbitrarily chosen, we obtain that $h(t,\cdot) = f(t,\cdot)$ in $L^2(\Omega)^l$ for almost all $t \in B$. But this is a contradiction to our assumption and thus $h = f$ in $L^2((0,T),L^2(\Omega))^l$. Consequently, we obtain

$$g\left(\|f(t,\cdot)\|^2_{L^2(\Omega)^l}\right) = \lim_{n\to\infty} g\left(\|h_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right) \leq \liminf_{n\to\infty} \sum_{i=n}^{N(n)} \lambda_{n,i} g\left(\|f_i(t,\cdot)\|^2_{L^2(\Omega)^l}\right)$$

for almost all $t \in (0,T)$. Thus, Fatou's lemma finally yields

$$\begin{aligned}\int_0^T g\left(\|f(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt &\leq \liminf_{n\to\infty} \sum_{i=n}^{N(n)} \lambda_{n,i} \int_0^T g\left(\|f_i(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt \\ &= \liminf_{n\to\infty} \int_0^T g\left(\|f_n(t,\cdot)\|^2_{L^2(\Omega)^l}\right)\, dt.\end{aligned}$$

□

## 4.2. Existence of minima of optimal control problems

In this section, we have a closer look on the following type of optimal control problems

$$\min_{u,b} J(u,b) = \frac{1}{2}\sum_{k=2}^{K} \Upsilon_k\left(\|u(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)}\right) + \frac{\alpha}{2}\int_0^T \Gamma_1\left(\|Db(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right)\, dt \tag{4.11}$$

$$+ R(b) \tag{4.12}$$

with regularization parameter $\alpha > 0$, functions $\Upsilon_k, \Gamma_1 : \mathbb{R} \to \mathbb{R}$, $k = 2,\dots,K$ and constraints

$$u_t + \operatorname{div}(bu) - u\operatorname{div}(b) = 0 \qquad \text{in } (0,T] \times \Omega, \tag{4.13}$$

$$u(0,\cdot) = Y_1 \qquad \text{in } \Omega, \tag{4.14}$$

$$b = 0 \qquad \text{on } (0,T) \times \partial\Omega, \tag{4.15}$$

where $Y_k \in L^\infty(\Omega)$, $k = 1,\dots,K$ are given. The term $R$ denotes additional regularization terms and we will cover the following ones in our investigations:

(i)
$$R_1(b) \equiv 0,$$

(ii)
$$R_2(b) = \frac{\beta}{2}\int_0^T \Gamma_2\left(\|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)^N}\right)\, dt,$$

(iii)
$$R_3(b) = \frac{\gamma}{2}\int_0^T \Gamma_3\left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)}\right)\, dt,$$

(iv)

$$R_4(b) = \frac{\beta}{2} \int_0^T \Gamma_2 \left( \|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)^N} \right) \, dt + \frac{\gamma}{2} \int_0^T \Gamma_3 \left( \|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)} \right) \, dt$$

where $\beta, \gamma > 0$ are regularization parameters and $\Gamma_2, \Gamma_3 : \mathbb{R} \to \mathbb{R}$ are given. In the first two cases, we will additionally distinguish between two further cases: the set of constraints given by (4.13)-(4.15) and the same set plus the additional constraint

$$\operatorname{div} b = 0 \qquad \text{in } (0,T) \times \Omega. \tag{4.16}$$

For the functions $\Upsilon_k$, $k = 2, \ldots, K$ and $\Gamma_i$, $i = 1, 2, 3$ we assume the following:

(a) the functions $\Upsilon_k : \mathbb{R} \to \mathbb{R}_0^+$ are lower semi-continuous,

(b) the functions $\Gamma_i : \mathbb{R} \to \mathbb{R}_0^+$ are convex, monotonically increasing, in $\mathcal{O}(x)$ and

$$\lim_{x \to \infty} \Gamma_i(x) = \infty.$$

In this case, the regularization terms in (4.11) and in (ii)-(iv) are well-defined.

Before we can introduce a setting for an admissible set we have a closer look on the $BV$-regularity for our considered vector fields. So far, we have the obvious setting

$$b \in \mathrm{VF}^2 = \left\{ b \in L^\infty((0,T) \times \Omega)^N \cap L^2((0,T), BV(\Omega))^N \mid \operatorname{div} b \in L^2((0,T), L^\infty(\Omega)) \right\}.$$

For the existence and uniqueness of solutions we need vector fields $b$ which have zero trace at the boundary of the spatial domain. The demand $b \in L^2((0,T), BV_0(\Omega))$ would not be enough since the trace operator is not continuous with respect to the weak*-convergence but with respect to the strict convergence in $BV(\Omega)$. As we will get at best weak*-convergence for a subsequence of a minimizing sequence, the weak*-limit would not need to have zero trace at $\partial\Omega$ for almost all $t \in (0,T)$. The reason for this lack of continuity of the trace operator with respect to the weak*-topology is illustrated in the following simple example: it is based on the possible loss of measure of the derivative at the boundary in the weak*-limit. The derivatives of $BV$-functions with vanishing trace do not have nonzero measures on the boundary. This loss of measures is not possible if the derivatives converges narrowly in measure which corresponds to strict convergence of $BV$-functions. An example illustrating this problem in $BV((0,1))$ is given by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (0, 1/n) \\ 1 & \text{if } x \in [1/n, 1 - 1/n] \\ 0 & \text{if } x \in (1 - 1/n, 1) \end{cases}.$$

Then, $(f_n) \subset BV_0((0,1))$ and $f_n \to f \equiv 1$ in $L^1((0,1))$ and $Df_n = \delta_{1/n} - \delta_{1-1/n} \stackrel{*}{\rightharpoonup} 0$ in $\mathcal{M}((0,1))$, but $|Df_n|((0,1)) = 2 \nrightarrow 0$. The problem here is that the jumps keep constant and do not tend to zero when they approach the boundary. That means we have to control the behavior of our $BV$-functions close to the boundary to ensure that limits of weakly*-convergent sequences of $BV$-functions with zero boundary trace do have zero boundary trace. Therefore we introduce the following setting. Given an $\varepsilon > 0$ we define for an open bounded set $\mathcal{O} \subset \mathbb{R}^N$ with Lipschitz boundary

$$\mathcal{O}_\varepsilon = \{ x \in \mathcal{O} | \operatorname{dist}(x, \partial\mathcal{O}) \le \varepsilon \}.$$

Then, we can set for $\delta \geq 0$ and $\varepsilon > 0$

$$\mathrm{W}_{\varepsilon,\delta}(\mathcal{O}) := \left\{ w \in L^1(\mathcal{O}) | \, |w(x)| \leq \delta \operatorname{dist}(x, \partial\mathcal{O}) \text{ for almost all } x \in \mathcal{O}_\varepsilon \right\}. \tag{4.17}$$

and obtain the following result:

**Lemma 4.2.1** *Let $\mathcal{O} \subset \mathbb{R}^N$ be open and bounded with Lipschitz boundary $\partial\mathcal{O}$ and let $\varepsilon > 0$ and $\delta \geq 0$. Then, any $f \in BV(\mathcal{O})$ satisfying $f \in \mathrm{W}_{\varepsilon,\delta}(\mathcal{O})$ lies in $BV_0(\mathcal{O})$.*

**Proof:** For $\delta = 0$, this is obviously true since $f \equiv 0$ in $\mathcal{O}_\varepsilon$. Thus, let $\delta > 0$. Then, by Theorem 3.87 in [AFP00] there exists for $\mathcal{H}^{N-1}$-almost every $x \in \partial\mathcal{O}$ a unique $w_f(x) \in \mathbb{R}$ such that

$$\lim_{r \to 0} \frac{1}{r} \int_{\mathcal{O} \cap B_r(x)} |f(y) - w_f(x)| \, dy = 0.$$

Now, let $A \subset \partial\mathcal{O}$ be the set such that there is a unique $w_f(x)$. Then, we assume that there is some $x \in A$ such that $w_f(x) \neq 0$. Without loss of generality we can assume that $w_f(x) > 0$. For $0 < r \leq r_0 := \min(1/\delta \cdot w_f(x)/2, \varepsilon)$ we estimate

$$\frac{1}{r} \int_{\mathcal{O} \cap B_r(x)} |f(y) - w_f(x)| \, dy \geq \frac{1}{r} \int_{\mathcal{O} \cap B_r(x)} |w_f(x)/2 - w_f(x)| \, dy = \frac{w_f(x)}{2r} \mathcal{L}^N(\mathcal{O} \cap B_r(x)) \tag{4.18}$$

$$\geq C > 0 \qquad \forall \; 0 < r \leq r_0 \tag{4.19}$$

and for some $C > 0$. This is true due to the following argument: assume that

$$\frac{1}{r} \mathcal{L}^N(\mathcal{O} \cap B_r(x)) \to 0.$$

Then,

$$0 \leftarrow \frac{1}{r} \mathcal{L}^N(\mathcal{O} \cap B_r(x)) = \frac{1}{r} \int_{\mathcal{O} \cap B_r(x)} |1 - 0| \, dy,$$

i.e. the constant 1-function in $\mathcal{O}$, which is a $BV$ function, has zero boundary. This is a contradiction since the boundary trace of the constant 1-function is the 1-function in $L^1(\partial\mathcal{O}, \mathcal{H}^{N-1} \llcorner \partial\mathcal{O})$. Thus, our assumption is wrong and there must be a $C > 0$ such that

$$\frac{1}{r} \mathcal{L}^N(\mathcal{O} \cap B_r(x)) \geq C \qquad \forall \; r > 0.$$

Therefore, we conclude that

$$\frac{1}{r} \int_{\mathcal{O} \cap B_r(x)} |f(y) - w_f(x)| \, dy \nrightarrow 0$$

which is a contradiction. Thus, $w_f(x) = 0$ and $f \in BV_0(\Omega)$.

□

**Lemma 4.2.2** *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary $\partial\mathcal{O}$ and let $\varepsilon > 0$ and $\delta \geq 0$. Furthermore, let $(f_n) \subset L^1(\mathcal{O})$ be convergent to $f \in L^1(\mathcal{O})$ with $f_n \in \mathrm{W}_{\varepsilon,\delta}(\mathcal{O})$ for all $n \in \mathbb{N}$. Then $f \in \mathrm{W}_{\varepsilon,\delta}(\mathcal{O})$.*

**Proof:** Since $f_n \to f$ in $L^1(\mathcal{O})$, we have that

$$f_n|_{\mathcal{O}_\varepsilon} \to f|_{\mathcal{O}_\varepsilon} \qquad \text{in } L^1(\mathcal{O}_\varepsilon).$$

Thus, there exists a subsequence such that

$$[-\delta\,\mathrm{dist}(x,\partial\mathcal{O}), \delta\,\mathrm{dist}(x,\partial\mathcal{O})] \ni f_n|_{\mathcal{O}_\varepsilon}(x) \to f|_{\mathcal{O}_\varepsilon}(x) \qquad \text{for almost all } x \in \mathcal{O}_\varepsilon.$$

Hence, $f|_{\mathcal{O}_\varepsilon}(x) \in [-\delta\,\mathrm{dist}(x,\partial\mathcal{O}), \delta\,\mathrm{dist}(x,\partial\mathcal{O})]$ for almost all $x \in \mathcal{O}_\varepsilon$ and therefore $f \in \mathrm{W}_{\varepsilon,\delta}(\mathcal{O})$.

□

**Remark 4.2.3** *For the case $\delta = 0$ the set*

$$BV_{\varepsilon,0}(\mathcal{O}) := \{f \in BV(\mathcal{O}) |\ f \in \mathrm{W}_{\varepsilon,0}(\mathcal{O})\}$$

*is a closed subspace of $BV(\mathcal{O})$ and a subset of $BV_0(\mathcal{O})$.*

With this technical assumption we define the set of admissible vector fields $\mathrm{S}_{ad}$ for the various optimal control problems. We take fixed $M > 0$, $\delta \geq 0$ and $\varepsilon > 0$ and we consider vector fields $b : (0,T) \times \Omega \to \mathbb{R}^N$ with

$$b \in \mathrm{S}_{ad}^{\varepsilon,\delta} := \left\{b \in \mathrm{VF}^2 |\ b(t,\cdot) \in \mathrm{W}_{\varepsilon,\delta}(\Omega) \text{ for almost all } t \in (0,T)\right\}$$

and define the admissible set for $M$, $\varepsilon$ and $\delta$

$$\mathrm{S}_{ad}^{M,\varepsilon,\delta} := \left\{b \in \mathrm{S}_{ad}^{\varepsilon,\delta} \ |\ \|b\|_{L^\infty((0,T)\times\Omega))^N} + \|\mathrm{div}\, b\|_{L^2((0,T),L^\infty(\Omega))} \leq M\right\}. \tag{4.20}$$

Obviously, we have that

$$\mathrm{S}_{ad}^{\varepsilon,\delta} \subset \mathrm{VF}_0^2 := \left\{b \in \mathrm{VF}_0 |\ b \in \mathrm{VF}^2\right\}.$$

Furthermore, for the case of the additional constraint $\mathrm{div}\, b \equiv 0$ we define the set

$$\mathrm{S}_{ad,0}^{M,\varepsilon,\delta} := \left\{b \in \mathrm{S}_{ad}^{M,\varepsilon,\delta} |\ \mathrm{div}\, b \equiv 0\right\} \tag{4.21}$$

and in the case of time regularization

$$\mathrm{S}_{ad,\partial_t}^{M,\varepsilon,\delta} := \left\{b \in \mathrm{S}_{ad}^{M,\varepsilon,\delta} |\ \partial_t b \in L^2((0,T)\times\Omega)^N\right\}. \tag{4.22}$$

The previous chapter yields that there is a well-defined solution operator

$$S : L^\infty(\Omega) \times \mathrm{VF}_0 \to C([0,T], L^\infty(\Omega) - w^*), \quad (u_0, b) \mapsto S(u_0, b).$$

Based on this solution operator we define the control-to-state operator $L_{Y_1}$ as

$$L_{Y_1} : \mathrm{VF}_0 \to C([0,T], L^\infty(\Omega) - w^*), \quad b \mapsto L_{Y_1}(b) = S(Y_1, b) \tag{4.23}$$

and its restriction to $\mathrm{S}_{ad}^{M,\varepsilon,\delta}$ as $L_{Y_1,ad}$. We abbreviate the terms $\mathrm{S}_{ad}^{M,\varepsilon,\delta}$, $\mathrm{S}_{ad,0}^{M,\varepsilon,\delta}$ and $\mathrm{S}_{ad,\partial_t}^{M,\varepsilon,\delta}$ to $\mathrm{S}_{ad}$, $\mathrm{S}_{ad,0}$ and $\mathrm{S}_{ad,\partial_t}$, respectively, if it is clear which constants $M$, $\varepsilon$ and $\delta$ are used in the current setting. Incorporating these control-to-state mappings into the objective function $J$ leads to various reduced objective functions $F_i$ for our considered cases: we define

| in the case | the reduced objective function $J(L_{Y_1,ad}(\cdot),\cdot)$ as | with admissible set |
|---|---|---|
| $R = R_1$ | $F_1$ | $\mathrm{S}_{ad}$ |
| $R = R_1$ | $F_{1,0}$ | $\mathrm{S}_{ad,0}$ |
| $R = R_2$ | $F_2$ | $\mathrm{S}_{ad,\partial_t}$ |
| $R = R_2$ | $F_{2,0}$ | $\mathrm{S}_{ad,0} \cap \mathrm{S}_{ad,\partial_t}$ |
| $R = R_3$ | $F_3$ | $\mathrm{S}_{ad}$ |
| $R = R_4$ | $F_4$ | $\mathrm{S}_{ad,\partial_t}$ |

For these reduced objective functions we show in the subsequent theorem that they attain their infima on their admissible sets, i.e there are minima within the admissible sets for each optimal control problem.

**Theorem 4.2.4 (Existence of minima of optimal control problems)** *Let $M > 0$, $\varepsilon > 0$ and $\delta \geq 0$ be fixed chosen. Then, the reduced objective functions $F_i$, $i \in \{1, \dots, 4\}$ and $F_{j,0}$, $j = 1, 2$ attain their infima on their admissible sets.*

**Proof:** We just show the statement for the objective function $F_4$ since the proof works in the same way for the other problems.
The objective function $F_4$ has a finite infimum in $\mathrm{S}_{ad,\partial_t}$ since $F_4(b) \geq 0$ for all $b \in \mathrm{S}_{ad,\partial_t}$. Now, let $(b_n) \subset \mathrm{S}_{ad,\partial_t}$ be a minimizing sequence, i.e.

$$F_4(b_n) \geq F_4(b_{n+1}) \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} F_4(b_n) = \inf_{\tilde{b} \in \mathrm{S}_{ad,\partial_t}} F_4(\tilde{b}).$$

The sequence $(b_n)$ is bounded in $L^2((0,T), BV(\Omega))^N$:

$$\begin{aligned} F_4(b_1) \geq F_4(b_n) &\geq \frac{\alpha}{2} \int_0^T \Gamma_1\left(\|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right)\, dt \\ &\geq \frac{T\alpha}{2} \Gamma_1\left(\frac{1}{T}\int_0^T \|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\, dt\right) \qquad \forall\, n \in \mathbb{N} \end{aligned}$$

and thus,

$$\sup_{n\in\mathbb{N}} \int_0^T \|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\, dt < \infty$$

since $\Gamma_1(x) \to \infty$ if $x \to \infty$. In addition, $\|b_n\|_{L^\infty((0,T)\times\Omega)^N} \leq M$ for all $n \in \mathbb{N}$ and hence, $(b_n)$ is also bounded in $L^2((0,T), L^1(\Omega))^N$. Using Theorem 4.1.2, we obtain that there exists a subsequence $(b_n)$ (which is labeled by $n$ again) and some $b \in L^2((0,T), BV(\Omega))^N$ such that

$$\int_0^T \Gamma_1\left(\|Db(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right)\, dt \leq \liminf_{n\to\infty} \int_0^T \Gamma_1\left(\|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right)\, dt \tag{4.24}$$

and $b_n \rightharpoonup b$ in $L^1((0,T) \times \Omega)^N$. For the limit $b$ we have that $b(t,\cdot) \in \mathrm{W}_{\varepsilon,\gamma}(\Omega)$ for almost all $t \in (0,T)$: denote

$$\mathcal{N}_n := \{t \in (0,T), b_n(t,\cdot) \notin BV(\Omega)^N\} \cup \{t \in (0,T), b_n(t,\cdot) \notin \mathrm{W}_{\varepsilon,\delta}(\Omega)^N\}$$

and

$$\mathcal{N} := \{t \in (0,T), b(t,\cdot) \notin BV(\Omega)^N\}.$$

Then $\mathcal{N}_n$ and $\mathcal{N}$ are null sets and

$$\mathcal{W} = \mathcal{N} \cup \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$$

is also a null set as a countable union of null sets. Furthermore, due to Lemma 4.2.2 we conclude that for any $t \in (0,T)\backslash\mathcal{W}$

$$g \in \overline{\{b_n(t,\cdot)|\ n \in \mathbb{N}\}}^{w^*} \Rightarrow g \in \mathrm{W}_{\varepsilon,\delta}(\Omega)^N$$

is satisfied. Consequently, in the same way we conclude that for any $t \in (0,T)\backslash\mathcal{W}$

$$g \in \overline{\operatorname{conv}\left(\overline{\{b_n(t,\cdot)|\ n \in \mathbb{N}\}}^{w^*}\right)}^{w^*} \Rightarrow g \in \mathrm{W}_{\varepsilon,\delta}(\Omega)^N$$

is satisfied. Thus $b(t,\cdot) \in \mathrm{W}_{\varepsilon,\delta}(\Omega)^N$ for almost all $t \in (0,T)$. In addition, since $(b_n)$, $(\partial_t b_n)$ and $(\operatorname{div} b_n)$ are bounded sequences in $L^\infty((0,T)\times\Omega)^N$, in $L^2((0,T)\times\Omega)^N$ and in $L^2((0,T),L^\infty(\Omega))$, respectively, we conclude, using standard arguments, that $b_n \overset{*}{\rightharpoonup} b$ in $L^\infty((0,T)\times\Omega)^N$, $\partial_t b_n \rightharpoonup \partial_t b$ in $L^2((0,T)\times\Omega)^N$ and $\operatorname{div} b_n \rightharpoonup \operatorname{div} b$ in $L^2((0,T)\times\Omega)$ with $\operatorname{div} b \in L^2((0,T),L^\infty(\Omega))$ for some subsequences. Due to Lemma 4.1.3, we know that each of these subsequences contains a subsequence (labeled by $n$ again) such that

$$\int_0^T \Gamma_2\left(\|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)^N}\right)\,dt \le \liminf_{n\to\infty}\int_0^T \Gamma_2\left(\|\partial_t b_n(t,\cdot)\|^2_{L^2(\Omega)^N}\right)\,dt$$

and

$$\int_0^T \Gamma_3\left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)}\right)\,dt \le \liminf_{n\to\infty}\int_0^T \Gamma_3\left(\|\operatorname{div} b_n(t,\cdot)\|^2_{L^2(\Omega)}\right)\,dt$$

holds. We restrict to those subsequences. Summing up, we have shown that $b \in \mathrm{S}_{ad,\partial_t}$. Finally, using Theorem 3.2.9, we obtain that

$$L_{Y_1,ad}(b_n) \to L_{Y_1,ad}(b) \quad \text{in } C([0,T],L^r(\Omega)) \quad \text{for } 1 \le r < \infty$$

and thus we get for all $2 \le k \le K$

$$L_{Y_1,ad}(b_n)(t_k,\cdot) - Y_k \to L_{Y_1,ad}(b)(t_k,\cdot) - Y_k \qquad \text{in } L^2(\Omega) \quad \text{as } n \to \infty.$$

In total, we obtain with estimate (4.24):

$$\begin{aligned}F_4(b) = {} & \frac{1}{2}\sum_{k=2}^K \Upsilon_k\left(\|L_{Y_1,ad}(b)(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)}\right) + \frac{\alpha}{2}\int_0^T \Gamma_1\left(\|Db(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}}\right)\,dt \\ & + \frac{\beta}{2}\int_0^T \Gamma_2\left(\|\partial_t b(t,\cdot)\|^2_{L^2(\Omega)}\right)\,dt + \frac{\gamma}{2}\int_0^T \Gamma_3\left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)}\right)\,dt\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{n\to\infty} \frac{1}{2} \sum_{k=2}^{K} \Upsilon_k \left( \|L_{Y_1,ad}(b_n)(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)} \right) + \liminf_{n\to\infty} \frac{\alpha}{2} \int\limits_0^T \Gamma_1 \left( \|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}} \right) \, dt \\
&+ \liminf_{n\to\infty} \frac{\beta}{2} \int\limits_0^T \Gamma_2 \left( \|\partial_t b_n(t,\cdot)\|^2_{L^2(\Omega)} \right) \, dt + \liminf_{n\to\infty} \frac{\gamma}{2} \int\limits_0^T \Gamma_3 \left( \|\operatorname{div} b_n(t,\cdot)\|^2_{L^2(\Omega)} \right) \, dt \\
&\le \liminf_{n\to\infty} \Bigg[ \frac{1}{2} \sum_{k=2}^{K} \Upsilon_k \left( \|L_{Y_1,ad}(b_n)(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)} \right) + \frac{\alpha}{2} \int\limits_0^T \Gamma_1 \left( \|Db_n(t,\cdot)\|^2_{\mathcal{M}(\Omega)^{N\times N}} \right) \, dt \\
&\qquad + \frac{\beta}{2} \int\limits_0^T \Gamma_2 \left( \|\partial_t b_n(t,\cdot)\|^2_{L^2(\Omega)} \right) \, dt + \frac{\gamma}{2} \int\limits_0^T \Gamma_3 \left( \|\operatorname{div} b_n(t,\cdot)\|^2_{L^2(\Omega)} \right) \, dt \Bigg] \\
&= \liminf_{n\to\infty} F_4(b_n) = \inf_{\tilde{b}\in \mathrm{S}_{ad,\partial_t}} F_4(\tilde{b}).
\end{aligned}
$$

Thus, the infimum is attained and $F_4$ has a minimum in $\mathrm{S}_{ad,\partial_t}$.

□

# 5. Unique flow and measure solutions for Lipschitz regular vector fields

In the previous chapters, we have presented existence and uniqueness of solutions to the transport equation

$$\begin{aligned} \partial_t u + \operatorname{div}(bu) - u \operatorname{div}(b) &= 0 \qquad \text{in } (0,T] \times \Omega, \\ u(0,\cdot) &= u_0 \qquad \text{in } \Omega, \end{aligned} \tag{5.1}$$

with vector fields $b \in L^\infty((0,T) \times \Omega)^N \cap L^1((0,T), BV_0(\Omega))^N$, $\operatorname{div} b \in L^1((0,T), L^\infty(\Omega))$ and initial value $u_0 \in L^\infty(\Omega)$. In addition, we showed improved stability results for the solution operator

$$S : (u_0, b) \mapsto S(u_0, b) \in C([0,T], L^p(\Omega))$$

and the existence of some minima of the reduced objective functions $F_i$, $i \in \{1, \ldots, 4\}$ and $F_{j,0}$, $j = 1, 2$ with control-to-state operator $L_{Y_1}$ restricted to some admissible set. In the following we will abbreviate the notation for the control-to-state operator $L_{u_0}$ to $L$ if it is clear which initial value $u_0$ is used for the transport equation.

In this chapter, our focus lies on the unique flows and measure solutions for vector fields with Lipschitz regularity in the spatial domain. These concepts will be needed for proving differentiability properties of the control-to-state operator $L_{u_0}$ for some fixed initial value $u_0$ and the tracking part of the reduced objective functions in the successive chapter. We start with some considerations to substantiate the necessity of these concepts.

For two vector fields $\hat{b}$ and $\hat{b} + db$ with initial value $u_0$ the difference of the corresponding unique solutions $\hat{u} = L(\hat{b})$ and $u = L(\hat{b} + db)$, respectively, satisfies

$$\begin{aligned} 0 &= \partial_t(u - \hat{u}) + \operatorname{div}(\hat{b}(u - \hat{u})) - (u - \hat{u}) \operatorname{div}(\hat{b}) + \operatorname{div}(dbu) - u \operatorname{div}(db) \\ &= \partial_t(u - \hat{u}) + \operatorname{div}(\hat{b}(u - \hat{u})) - (u - \hat{u}) \operatorname{div}(\hat{b}) + \operatorname{div}(db\hat{u}) - \hat{u} \operatorname{div}(db) \\ &\quad + \operatorname{div}(db(u - \hat{u})) - (u - \hat{u}) \operatorname{div}(db). \end{aligned}$$

For differentiability, we need a linear approximation of the control-to-state operator at a given vector field $\hat{b}$. The above equation yields a hint that such a linear approximation has to be the solution operator of the following inhomogeneous transport equation:

$$\begin{aligned} \partial_t \tilde{u} + \operatorname{div}(\hat{b}(\tilde{u})) - \tilde{u} \operatorname{div}(\hat{b}) + \operatorname{div}(db\hat{u}) - \hat{u} \operatorname{div}(db) &= 0 \qquad \text{in } (0,T) \times \Omega \\ \tilde{u}(0,\cdot) &= 0 \qquad \text{in } \Omega, \end{aligned} \tag{5.2}$$

where $\tilde{u}$ denotes the solution to the vector field $db$. A crucial point in the equation is the term $\operatorname{div}(db\hat{u})$. Together with the term $\hat{u} \operatorname{div}(db)$ it forms the inhomogeneous part of the equation. If $\operatorname{div}(db\hat{u})$ would be a function of a Lebesgue space, we could extend the existence and uniqueness statements about weak solutions of the homogeneous transport equation to the inhomogeneous case. As we are interested in a preferably general setting, we assume that the term $\operatorname{div}(db\hat{u})$ represents a measure. Since this requirement must hold for arbitrary vector fields $db$, i.e. in particular for vector fields whose components are all zero except of one, we can conclude that $db\hat{u}(t,\cdot)$ and thus also $\hat{u}(t,\cdot)$ must be elements of $BV(\Omega)$ for almost all $t \in (0,T)$. As $\hat{u}$ represents a solution of the transport equation, we need requirements leading to solutions having $BV$-regularity in space. A first requirement is given by Colombini, Luo and Rauch in [CLR04] showing that $BV$-regularity is not propagated in general for vector fields having less than Lipschitz regularity in space. In [Che11, CL11], it is proven that initial $BV$-regular data is preserved under $C^1$-regular vector fields in the spatial domain. We will extend these results to vector fields with spatial Lipschitz regularity in the first section of this chapter. Under this assumption the term $\operatorname{div}(db\hat{u})$ will be a measure and since the term $\hat{u}\operatorname{div}(db)$ lies in $L^1((0,T),L^\infty(\Omega))$, the inhomogeneous part of (5.2) will be given by a measure. Consequently, the equations (5.2) describe an inhomogeneous continuity equation whose solutions have measure regularity in space. In the second section, we will have a closer look on such equations and we will present standard theory about existence and uniqueness of measure solutions. As we will see spatial Lipschitz regularity of the corresponding vector field is sufficient to guarantee uniqueness of solutions. Based on these sections we will be able to show some differentiability results of the control-to-state operator as well as of the tracking term of the objective functions in the subsequent chapter.

## 5.1. Transport equation with Lipschitz regular vector fields

In this section, we present the extension of some results given by Chen and Lorenz in [Che11, CL11]. In this and the following chapters, we consider bounded, convex, open subsets $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. For such domains Proposition 2.13 in [AFP00] yields that

$$W^{1,\infty}(\Omega) \simeq Lip(\Omega).$$

As a reminder, $Lip(\Omega)$ denotes the Banach space of Lipschitz functions with Lipschitz constant

$$\mathbb{L}(f) := \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} < \infty \quad \text{for } f \in Lip(\Omega)$$

and norm $\|f\|_{Lip(\Omega)} = \|f\|_{C(\Omega)} + \mathbb{L}(f)$. For the Lipschitz constant $\mathbb{L}(f)$ of $f \in Lip(\Omega)$, Proposition 2.13 in [AFP00] yields that $\mathbb{L}(f) = \|\nabla f\|_{L^\infty(\Omega)^N}$. Additionally, we define the subspace

$$Lip_0(\Omega) := \{f \in Lip(\Omega) |\ f \equiv 0 \text{ on } \partial\Omega\}.$$

In this section, we consider vector fields $b$ lying in $L^\infty((0,T)\times\Omega)^N \cap L^1((0,T),Lip_0(\Omega))^N$. These assumptions are weaker than the assumptions on vector fields required by Chen and Lorenz in [Che11, CL11], in particular with respect to two essential points:

(i) In [Che11, CL11] a fixed Lipschitz constant for almost all $t \in (0,T)$ is required, which is not the case in our assumptions.

(ii) Chen and Lorenz require $b(t,\cdot) \in H_0^{3,div}(\Omega)^2 \hookrightarrow C^1(\overline{\Omega})^2$ for almost all $t \in (0,T)$, which is obviously stronger than spatial Lipschitz regularity in our case.

We will show that all results in [Che11, CL11] about spatial $BV$-regularity of solutions to the transport equation with $BV$-regular initial value $u_0$ at time points $t \in (0,T)$ remain true under our weaker assumptions on the vector fields. In addition, the representation of solutions as the composition of initial values and unique flows of the vector fields will also hold true for our assumptions.

### 5.1.1. The unique flow of Lipschitz regular vector fields

Regular vector fields are strongly connected via a certain ordinary differential equation to the so-called flow of the vector field. The flow is a uniquely defined vector field given by the family of unique trajectories of spatial points $x \in \Omega$ satisfying an ODE with the vector fields as right hand sides. Therefore, we start with the forward Carathéodory equation for $s \in [0, T)$ and $x \in \Omega$

$$\frac{d}{dt}\gamma(t) = b(t, \gamma(t)) \quad \text{in } (s, T), \tag{5.3}$$

$$\gamma(s) = x, \tag{5.4}$$

and the corresponding backward Carathéodory equation for $s \in (0, T]$ and $x \in \Omega$

$$\frac{d}{dt}\gamma(t) = b(t, \gamma(t)) \quad \text{in } (0, s), \tag{5.5}$$

$$\gamma(s) = x, \tag{5.6}$$

yielding unique trajectories on which the definition of the flow is based. The utility of the flow will be revealed in the next subsection: any solution of the transport equation can be represented as the composition of the initial value with some backward flow.

**Definition 5.1.1 (Solutions of the Carathéodory equations)** *A function*

$$\gamma : [s, T] \to \mathbb{R}^N$$

*is called a solution of* (5.3)-(5.4) *if it is absolutely continuous on* $[s, T]$ *with* $\gamma(s) = x$ *and satisfies* (5.3) *for almost all* $t \in (s, T)$. *Analogously, a function*

$$\gamma : [0, s] \to \mathbb{R}^N$$

*is called a solution of* (5.5)-(5.6) *if it is absolutely continuous on* $[0, s]$ *with* $\gamma(s) = x$ *and satisfies* (5.5) *for almost all* $t \in (0, s)$.

For the Carathéodory equations (5.3)-(5.4) and (5.5)-(5.6) we have the following result, which is an extension of Theorem 3.1 in [Che11] (Theorem 1 in [CL11]).

**Theorem 5.1.2** *Let* $b \in L^1((0, T), Lip_0(\Omega))^N$. *Then for all* $s \in [0, T)$ *and* $x \in \Omega$, *there exists a unique solution* $\gamma \in C([s, T])$ *of* (5.3)-(5.4) *given by*

$$\gamma(t) = x + \int_s^t b(\tau, \gamma(\tau))\, d\tau. \tag{5.7}$$

*Furthermore,* $\gamma(t) \in \Omega$ *for all* $t \in [s, T]$. *Similarly, for all* $s \in (0, T]$ *and* $x \in \Omega$, *there exists a unique solution* $\gamma \in C([0, s])$ *of* (5.5)-(5.6) *given by*

$$\gamma(t) = x - \int_t^s b(\tau, \gamma(\tau))\, d\tau. \tag{5.8}$$

We will split the proof into several lemmas. Since the backward problem can be transformed into a forward problem via $t \mapsto s - t$, we only have to show the forward problem. In the following we will extend our vector field in the spatial domain to the entire $\mathbb{R}^N$:

$$\bar{b}(t,x) := \begin{cases} b(t,x) & \text{if } (t,x) \in (s,T) \times \Omega \\ 0 & \text{if } (t,x) \in (s,T) \times \mathbb{R}^N \backslash \Omega \end{cases}. \tag{5.9}$$

Then, $\bar{b}(t,\cdot) \in C(\mathbb{R}^N)$ for almost all $t \in (s,T)$ and $\bar{b} \in L^1((0,T), Lip(\mathbb{R}^N))$ with $\mathbb{L}(\bar{b}(t,\cdot)) = \mathbb{L}(b(t,\cdot))$ for almost all $t \in (s,T)$.

**Lemma 5.1.3** *For all $x \in \mathbb{R}^N$ the Carathéodory equation* (5.3) *has a unique solution $\gamma_x \in C([s,T])$ given by* (5.7)*.*

**Proof:** The vector field $\bar{b}$ satisfies the Carathéodory conditions in Chapter 1 in [Fil88]:

(i) the function $\bar{b}$ is defined and continuous in $x$ for almost all $t$,

(ii) the function $\bar{b}$ is measurable in $t$ for all $x$ and

(iii) there exists a function $\alpha \in L^1(0,T)$ such that

$$|\bar{b}(t,x)| \leq \alpha(t)$$

for all $x$ and for almost all $t$.

Point (iii) is obviously satisfied if we take $\alpha(t) = \left\|\bar{b}(t,\cdot)\right\|_{C(\mathbb{R}^N)}$. Hence, we can use Theorem 1 and Theorem 2 in Chapter 1 in [Fil88] to obtain the result.

□

**Lemma 5.1.4** *Let $\gamma_z$ be the solution of* (5.3)-(5.4) *with initial value $z \in \mathbb{R}^N$. Then, for $x, y \in \mathbb{R}^N$*

$$|\gamma_x(t) - \gamma_y(t)| \leq C\,|x-y|\,,$$

*where*

$$C := e^{\int_s^t \mathbb{L}(b(r,\cdot))dr}.$$

**Proof:** We have the estimate

$$|\gamma_x(t) - \gamma_y(t)| \leq |x-y| + \int_s^t \mathbb{L}(b(r,\cdot))\,|\gamma_x(r) - \gamma_y(r)|\;dr.$$

Then, Grönwall's lemma 2.2.4 yields the result.

□

**Lemma 5.1.5** *If $x \in \mathbb{R}^N \backslash \Omega$, then the unique solution $\gamma_x$ is given by*

$$\gamma_x(t) = x \qquad \forall\; t \in [s,T].$$

**Proof:** We first take $x \in \mathbb{R}^N \backslash \overline{\Omega}$. Then, there exists some $r_x > 0$ such that $B_{r_x}(x) \cap \overline{\Omega} = \emptyset$. Hence, since $\gamma_x$ is continuous, there exists a $t_x \in (s, T)$ such that $\gamma_x(t) \in B_{r_x}(x)$ for all $t \in [s, t_x]$. Since $\gamma_x^{-1}(\{x\}) \subset [s, T]$ is closed, there exists a maximal $t_x^* \in [s, T]$ such that $\gamma_x(t) = x$ for all $t \in [s, t_x^*]$. We assume that $t_x^* < T$. Then there exists a $\bar{t} > t_x^*$ such that

$$|\gamma_x(t) - \gamma_x(t_x^*)| < r_x \qquad \text{for all } t \leq \bar{t}.$$

Thus, $b(t, \gamma_x(t)) = 0$ for all $t \leq \bar{t}$. For $t \in (t_x^*, \bar{t}]$ we have

$$\gamma_x(t) = x + \int_{t_x^*}^{t} b(\tau, \gamma(\tau)) \, d\tau = x$$

since the integrand is zero. Hence, this is a contradiction to our assumption that $t_x^*$ is maximal. Thus, $t_x^* = T$. For $x \in \partial\Omega$ we take a sequence $(x_n) \subset \mathbb{R}^N \backslash \overline{\Omega}$ which converges to $x$. Then, Lemma 5.1.4 yields that for all $t \in [s, T]$ $\gamma_{x_n}(t) \to \gamma_x(t)$ as $n \to \infty$. Hence, we have

$$x \leftarrow x_n = \gamma_{x_n}(t) \to \gamma_x(t).$$

Thus, $\gamma_x$ is constant.

□

**Lemma 5.1.6** *For $x \in \Omega$ the solution $\gamma_x$ satisfies $\gamma_x(t) \in \Omega$ for all $t \in [s, T]$.*

**Proof:** Assume that there is some $t_0 \in (s, T]$ such that $\gamma_x(t_0) \in \mathbb{R}^N \backslash \Omega$. Then, we define the vector field $\tilde{b} \in L^1((0, t_0 - s), Lip(\mathbb{R}^N))$ in the following way:

$$\tilde{b}(t, x) := -\overline{b}(t_0 - t, x)$$

for almost all $t \in (0, t_0 - s)$ and for all $x \in \mathbb{R}^N$. Then Lemma 5.1.3 yields that the Carathéodory equation with vector field $\tilde{b}$ has a unique solution $\tilde{\gamma}$ for the point $\gamma_x(t_0) \in \mathbb{R}^N \backslash \Omega$. Furthermore, using Lemma 5.1.5, we obtain that $\tilde{\gamma}$ is constant. Now, we define

$$w(t) := \gamma_x(t_0 - t) = \gamma_x(s) + \int_{s}^{t_0 - t} b(r, \gamma_x(r)) \, dr, \qquad t \in [0, t_0 - s].$$

Then, $w \in C([0, t_0 - s])$ is absolutely continuous and $w(0) = \gamma_x(t_0)$ and $w(t_0 - s) = \gamma_x(s) \neq \gamma_x(t_0) = w(0)$. Furthermore, we obtain:

$$\frac{d}{dt} w(t) = -b(t_0 - t, \gamma_x(t_0 - t)) = -b(t_0 - t, w(t)) = \tilde{b}(t, w(t))$$

for almost all $t \in (0, t_0 - s)$. Thus, $w$ is also a solution of the Carathéodory equation with initial value $\gamma_x(t_0)$, but is not constant. Hence, this is a contradiction to the uniqueness of solutions and our assumption at the beginning of the proof is wrong.

□

The proof of Theorem 5.1.2 directly arises from Lemmas 5.1.3 - 5.1.6. Now, Theorem 5.1.2 enables us to define the flow of a vector field $b$.

**Definition 5.1.7 (Flow of vector field)** *Let $b \in L^1((0,T), Lip_0(\Omega))^N$. Then for $x \in \Omega$ and $s \in [0,T]$ we set*

$$X(t,s,x) := \gamma_x(t) \qquad \text{in } [0,T]$$

*where $\gamma_x$ denotes the unique solution of the Carathéodory equation*

(i) (5.3)-(5.4) *with initial value $x$ for $t \geq s$ and*

(ii) (5.5)-(5.6) *with final value $x$ for $t \leq s$.*

*We call $X$ the flow of the vector field $b$ in $[0,T]$.*

Obviously, the flow $X(\cdot, s, \cdot)$ satisfies

$$\begin{aligned} \partial_t X(t,s,x) = b(t, X(t,s,x)) \quad \text{on } (0,T)\times\Omega \\ X(s,s,x) = x, \quad \text{in } \Omega. \end{aligned} \tag{5.10}$$

and we obtain the following properties:

**Theorem 5.1.8** *(i) The map $X(t,s,\cdot): \Omega \to \Omega$ is bijective for all $t, s \in [0,T]$.*

*(ii) The flow $X(t,s,\cdot) \in Lip(\Omega)^N$ for all $t,s \in [0,T]$ and the derivative $DX(t,s,x)$ satisfies*

$$|DX(t,s,x)| \leq e^{\int_{\min(s,t)}^{\max(s,t)} \mathbb{L}(b(r,\cdot))dr}$$

*if it exists.*

*(iii) The map $X(\cdot,s,x)$ is absolutely continuous for all $x \in \Omega$ and $s \in [0,T]$.*

*(iv) The flow satisfies a semi-group property, i.e for $t_0, t_1, t_2 \in [0,T]$ and for all $x \in \Omega$ we have*

$$X(t_2, t_0, x) = X(t_2, t_1, X(t_1, t_0, x)).$$

*(v) The flow $X(t,\cdot,\cdot)$ satisfies the transport equation for all $t \in [0,T]$, i.e.*

$$\partial_s X(t,s,x) + DX(t,s,x)b(s,x) = 0 \qquad \text{in } (0,T)\times\Omega.$$

**Proof:**

(i) The map $X(t,s,\cdot)$ is surjective since for all $x \in \Omega$ there exists a solution

$$\gamma_x \in C([\min(t,s), \max(t,s)])$$

of the Carathéodory equations with initial/end value $x$ at time point $t$ and end/initial value $\gamma_x(s)$ at time point $s$ (in the cases $t \leq s$ and $t \geq s$). Then $X(t,s,\gamma_x(s)) = x$. It is injective due to the uniqueness of the forward equation if $t > s$ and of the backward equation if $t < s$.

(ii) The result is shown in Lemma 5.1.4.

(iii) Since $X(\cdot,s,x)$ is a solution of some Carathéodory equation, it is absolutely continuous.

(iv) This statement follows from the uniqueness and surjectivity of solutions of the forward and backward equations.

(v) Using (iv) and setting $y = X(t,s,x)$ as well as $x = X(s,t,y)$, we have that $y = X(t,s,X(s,t,y))$ for all $s \in [0,T]$. Hence, we obtain:

$$\begin{aligned} 0 = \frac{d}{ds}y &= \frac{\partial}{\partial_s}X(t,s,X(s,t,y)) + DX(t,s,X(s,t,y))\frac{\partial}{\partial_s}X(s,t,y) \\ &= \frac{\partial}{\partial_s}X(t,s,X(s,t,y)) + DX(t,s,X(s,t,y))b(s,X(s,t,y)) \\ &= \frac{\partial}{\partial_s}X(t,s,x) + DX(t,s,x)b(s,x). \end{aligned}$$

□

Before we finish this subsection, we show two further statements which will be helpful in chapter 6 for proving differentiability.

**Lemma 5.1.9** *Let $(b_n) \subset L^1((0,T), Lip_0(\Omega))^N$ such that*

$$\sup_{n\in\mathbb{N}} \int_0^T \mathbb{L}(b_n(t,\cdot))\ dt < \infty$$

*and*

$$b_n \to b \quad in\ L^1((0,T), C(\Omega))^N \quad as\ n \to \infty$$

*for some $b \in L^1((0,T), Lip_0(\Omega))^N$. Then for any $s \in [0,T]$,*

$$X_n(\cdot,s,\cdot) \to X(\cdot,s,\cdot) \quad in\ C([0,T]\times\Omega)^N \quad as\ n \to \infty.$$

**Proof:** We estimate for $s \le t$

$$\begin{aligned} |X_n(t,s,x) - X(t,s,x)| &\le \int_s^t |b_n(\tau, X_n(\tau,s,x)) - b_n(\tau, X(\tau,s,x))|\ d\tau \\ &+ \int_s^t |b_n(\tau, X(\tau,s,x)) - b(\tau, X(\tau,s,x))|\ d\tau \\ &\le \int_s^t \mathbb{L}(b_n(\tau,\cdot))\,|X_n(t,s,x) - X(\tau,s,x)|\ d\tau \\ &+ \int_0^T \|b_n(\tau,\cdot) - b(\tau,\cdot)\|_{C(\Omega)^N}\ d\tau. \end{aligned}$$

Using Grönwall's lemma, we have for $s \le t$

$$\begin{aligned} |X_n(t,s,x) - X(t,s,x)| &\le e^{\int_s^t \mathbb{L}(b_n(\tau,\cdot))d\tau} \int_0^T \|b_n(\tau,\cdot) - b(\tau,\cdot)\|_{C(\Omega)^N}\ d\tau \\ &\le C \int_0^T \|b_n(\tau,\cdot) - b(\tau,\cdot)\|_{C(\Omega)^N}\ d\tau \end{aligned}$$

for some $C > 0$. The same can be estimated for $t \leq s$. Thus, we have

$$\|X_n(\cdot, s, \cdot) - X(\cdot, s, \cdot)\|_{C([0,T]\times\Omega)^N} \leq C\,\|b_n - b\|_{L^1((0,T),C(\Omega))^N} \to 0 \quad \text{as } n \to \infty.$$

□

The last lemma in this subsection shows that the flows are continuously differentiable if the vector fields have $C^1$-regularity in the spatial variable. This statement will be needed to show some differentiability result in the successive chapter 6.

**Lemma 5.1.10** *Let $b \in L^1((0,T), C_0(\Omega) \cap C^1(\Omega))^N$. Then, the unique flow $X(\cdot, s, \cdot)$ is an element of $L^1((0,T), C^1(\Omega))^N$ for any $s \in [0,T]$ and the derivative is given by*

$$D_x X(t,s,x) = \mathrm{Id}_{N\times N} + \int_s^t D_x b(r, X(r,s,x)) D_x X(r,s,x)\; dr.$$

*In addition, if $(b_n) \subset L^1((0,T), C_0(\Omega) \cap C^1(\Omega))^N$ is a sequence such that*

$$b_n \to b \quad \text{in } L^1((0,T), C_0(\Omega) \cap C^1(\Omega))^N \quad \text{as } n \to \infty,$$

*then $D_x X_n(\cdot, s, \cdot) \to D_x X(\cdot, s, \cdot)$ in $L^1((0,T), C(\Omega))^{N\times N}$ for any $s \in [0,T]$.*

**Proof:** We consider for $x \in \Omega$, $s \in [0,T]$ and $e \in \mathbb{S}^{N-1}$ the ordinary differential equation

$$\begin{aligned} \partial_t w_e(t,s,x) &= D_x b(t, X(t,s,x)) \partial w_e(t,s,x) \qquad \text{in } (0,T), \\ w_e(s,s,x) &= e. \end{aligned} \tag{5.11}$$

Then, Theorems 5.1, 5.2 and 5.3 in [Hal80] yield that there exists a unique solution of (5.11) given by

$$w_e(t,s,x) = e + \int_s^t D_x b(r, X(r,s,x)) w_e(r,s,x)\; dr.$$

Now, in section 1.3 in [Cri07], it is proven that

$$\frac{X(t,s,x+he) - X(t,s,x)}{h} \to w_e(t,s,x) \quad \text{as } h \to 0,$$

i.e. $D_x X(t,s,x)e = w_e(t,s,x)$. Thus, we conclude

$$D_x X(t,s,x) = \mathrm{Id}_{N\times N} + \int_s^t D_x b(r, X(r,s,x)) D_x X(r,s,x)\; dr.$$

Furthermore, for all $s,t \in [0,T]$, $D_x X(t,s,\cdot) \in C(\Omega)$ due to the following argument: for $x, y \in \Omega$ we have

$$|D_x X(t,s,x) - D_x X(t,s,y)| \leq \int_s^t |D_x b(r, X(r,s,x)) - D_x b(r, X(r,s,y))|\,|D_x X(r,s,x)|\; dr$$

$$+ \int_s^t |D_x b(r, X(r,s,y))| \, |D_x X(r,s,x) - D_x X(r,s,y)| \; dr$$
$$\leq C \int_s^t |D_x b(r, X(r,s,x)) - D_x b(r, X(r,s,y))| \; dr$$
$$+ \int_s^t \|D_x b(r,\cdot)\|_{C(\Omega)^{N\times N}} \, |D_x X(r,s,x) - D_x X(r,s,y)| \; dr$$

since $D_x X(\cdot, s, x)$ is continuous in $[0,T]$ and thus, there exists a $C > 0$ such that

$$\max_{r\in[0,T]} |D_x X(r,s,x))| \leq C.$$

Applying Grönwall's lemma yields

$$\begin{aligned} &|D_x X(t,s,x) - D_x X(t,s,y)| \\ &\quad \leq C \int_s^t |D_x b(r, X(r,s,x)) - D_x b(r, X(r,s,y))| \; dr e^{\int_s^t \|D_x b(r,\cdot)\|_{C(\Omega)^{N\times N}} dr}. \end{aligned}$$

Since $D_x b$ and $X$ are continuous in the spatial variable, we immediately obtain that $D_x X(t,s,\cdot)$ is continuous in $\Omega$. In a similar way, we obtain that

$$\|D_x X(t,s,\cdot)\|_{C(\Omega)^{N\times N}} \leq C e^{\int_s^t \|D_x b(r, X(r,s,\cdot))\|_{C(\Omega)^{N\times N}} dr} \tag{5.12}$$

is valid for any $t \in [0,T]$, i.e. $D_x X(\cdot,s,\cdot) \in L^1((0,T), C(\Omega))^{N\times N}$. Now, for a sequence $b_n \subset L^1((0,T), C_0(\Omega) \cap C^1(\Omega))^N$ being convergent to $b$ we have

$$\begin{aligned} &\|D_x X_n(t,s,\cdot) - D_x X(t,s,\cdot)\|_{C(\Omega)^{N\times N}} \\ &\quad \leq C_1 \int_s^t \|D_x b_n(r,\cdot) - D_x b(r,\cdot)\|_{C(\Omega)^{N\times N}} \; dr \\ &\quad + \int_s^t \|D_x b_n(r,\cdot)\|_{C(\Omega)^{N\times N}} \, \|D_x X_n(r,s,\cdot) - D_x X(r,s,\cdot)\|_{C(\Omega)^{N\times N}} \; dr \end{aligned}$$

for some $C_1 > 0$. Grönwall's lemma then yields

$$\|D_x X_n(t,s,\cdot) - D_x X(t,s,\cdot)\|_{C(\Omega)^{N\times N}} \leq \tilde{C} \int_s^t \|D_x b_n(r,\cdot) - D_x b(r,\cdot)\|_{C(\Omega)^{N\times N}} \; dr$$

for some $\tilde{C} > 0$ due to the boundedness of $(b_n)$ in $L^1((0,T), C_0(\Omega) \cap C^1(\Omega))^N$. Thus, since $(b_n)$ converges to $b$, $D_x X_n(t,s,\cdot) \to D_x X(t,s,\cdot)$ in $C(\Omega)^{N\times N}$. Then, using estimate (5.12) and applying Lebesgue's dominated convergence theorem yields that $D_x X_n(\cdot,s,\cdot) \to D_x X(\cdot,s,\cdot)$ in $L^1((0,T), C(\Omega))^{N\times N}$.

□

### 5.1.2. Solutions of transport equations with Lipschitz regular vector fields

As mentioned at the beginning of the section, we consider weak solutions of the transport equation

$$\begin{aligned} \partial_t u + \operatorname{div}(bu) - u \operatorname{div}(b) &= 0 \quad \text{in } (0,T) \times \Omega \\ u(0,\cdot) &= u_0 \quad \text{in } \Omega \end{aligned} \tag{5.13}$$

in the case $b \in L^1((0,T), Lip_0(\Omega))^N \cap L^\infty((0,T) \times \Omega)^N$ and $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$. We will show that (5.13) possesses a unique solution $u \in C([0,T], BV(\Omega) - w^*)$, which is a slightly better result than the one in [Che11, CL11], since $C([0,T], BV(\Omega) - w^*) \subset L^\infty((0,T), BV(\Omega))$. Most of the proofs are based on the corresponding proofs in [Che11, CL11] and are extended by additional technical steps needed for the weaker assumptions. We start with a statement that Lipschitz regularity is preserved if the initial value owns this regularity. The result will be needed for the duality relations in chapter 7.

**Theorem 5.1.11** *Let $u_0 \in C(\Omega)$ and $b \in L^1((0,T), Lip_0(\Omega))^N \cap L^\infty((0,T) \times \Omega)^N$. Then the transport equation* (5.13) *has a unique solution $u \in C([0,T] \times \Omega)$ given by*

$$u(t,x) := u_0(X(0,t,x)) \qquad \text{for } (t,x) \in [0,T] \times \Omega, \tag{5.14}$$

*where $X$ denotes the flow of $b$. Furthermore, if $u_0 \in Lip(\Omega)$, then $u(t,\cdot) \in Lip(\Omega)$ for all $t \in [0,T]$ with Lipschitz constant*

$$\mathbb{L}(u(t,\cdot)) \le \mathbb{L}(u_0) e^{\int_0^t \mathbb{L}(b(r,\cdot)) dr}$$

*and $u(\cdot,x)$ is absolutely continuous for all $x \in \Omega$.*

**Proof:** Let $u_0 \in Lip(\Omega)$. By Definition 5.1.7 and Theorem 5.1.8, there exists a unique flow $X$ for the vector field $b$. Then, the composition $u = u_0(X(0,\cdot,\cdot))$ is Lipschitz continuous in $x$ for all $t \in [0,T]$ as a composition of Lipschitz functions and absolutely continuous in $t$ for all $x \in \Omega$ as a composition of a Lipschitz and an absolutely continuous function. The bound for the Lipschitz constant follows from point (ii) in Theorem 5.1.8. We obtain for $u$ and for almost all $t \in (0,T)$ and for almost all $x \in \Omega$:

$$\begin{aligned} &\partial_t u(t,x) + \operatorname{div}(b(t,x)u(t,x)) - u(t,x) \operatorname{div}(b(t,x)) = \partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) \\ &= \nabla u_0(X(0,t,x)) \cdot \partial_t X(0,t,x) + \nabla u_0(X(0,t,x)) \cdot (DX(0,t,x) b(t,x)) \\ &= \nabla u_0(X(0,t,x)) \cdot (\partial_t X(0,t,x) + DX(0,t,x) b(t,x)) \\ &= 0. \end{aligned}$$

Thus, $u$ is a solution of (5.13). Due to Theorem 3.1.26 and Theorem 3.1.9, the solution $u$ is unique. Now, let $u_0 \in C(\Omega)$ and let $(u_{0,n}) \subset Lip(\Omega)$ be a sequence being convergent to $u_0$ in $C(\Omega)$. Denote $u_n$ the unique solution of (5.13) with initial value $u_{0,n}$. Then, we have

$$\| u_{0,n}(X(0,\cdot,\cdot)) - u_0(X(0,\cdot,\cdot)) \|_{C([0,T] \times \Omega)} \to 0$$

as $n \to \infty$. Furthermore, we obtain

$$0 = \int_0^T \int_\Omega u_{0,n}(X(0,\cdot,\cdot)) \left( \partial_t \varphi + b \cdot \nabla \varphi + \varphi \operatorname{div} b \right) \, dxdt + \int_\Omega u_{0,n} \varphi(0,\cdot) \, dx$$

$$\to \int_0^T \int_\Omega u_0(X(0,\cdot,\cdot))\,(\partial_t\varphi + b\cdot\nabla\varphi + \varphi \operatorname{div} b)\ dxdt + \int_\Omega u_0\varphi(0,\cdot)\ dx$$

as $n \to \infty$. Thus, the function $u$ defined by $u(t,\cdot) := u_0(X(0,t,\cdot))$ for all $t \in [0,T]$ lies in $C([0,T]\times\Omega)$ and is a solution of (5.13). Since $b$ has the renormalization property, $u$ is unique. □

The next theorem is an extenuated version of Theorem 3.3 in [Che11] (Theorem 3 in [CL11]). The statement of the theorem will be needed in the proof of the main statement. In the theorem, the function $\rho$ is the standard mollifier.

**Theorem 5.1.12** *Let $u_0 \in BV(\Omega)$ and let $\varphi : \Omega \to \Omega$ be bijective and Lipschitz continuous with Lipschitz continuous inverse $\varphi^{-1}$. Then the composition $(u_0 * \rho_\varepsilon)\circ\varphi$ converges to $u_0\circ\varphi$ in the weak*-topology of $BV(\Omega)$ as $\varepsilon \to 0$.*

For the proof we will use the Hadamard inequality which is proven in Corollary 7.8.2 in [HJ90].

**Lemma 5.1.13 (Hadamard inequality)** *Let $A \in \mathbb{R}^{N\times N}$. Then*

$$|\det A| \le \prod_{i=1}^N \left(\sum_{j=1}^N |A_{ij}|^2\right)^{\frac{1}{2}} \quad \text{and} \quad |\det A| \le \prod_{j=1}^N \left(\sum_{i=1}^N |A_{ij}|^2\right)^{\frac{1}{2}}.$$

**Proof:** We start with the $L^1$-convergence:

$$\begin{aligned}\int_\Omega |(u_0*\rho_\varepsilon)(\varphi(x)) - u_0(\varphi(x))|\ dx &= \int_\Omega |u_0*\rho_\varepsilon(y) - u_0(y)|\,\left|\det(\nabla\varphi^{-1}(y))\right|\ dy\\ &\le \|u_0*\rho_\varepsilon - u_0\|_{L^1(\Omega)}\left\|\det(\nabla\varphi^{-1})\right\|_{L^\infty(\Omega)}.\end{aligned}$$

If $L = \left\|\nabla\varphi^{-1}\right\|_{L^\infty(\Omega)^{N\times N}}$, then using the Hadamard inequality, we obtain that

$$\left\|\det(\nabla\varphi^{-1})\right\|_{L^\infty(\Omega)} \le N^{\frac{N}{2}}L^N.$$

Thus, we have the convergence of $(u_0*\rho_\varepsilon)\circ\varphi$ to $u_0\circ\varphi$ in $L^1(\Omega)$. Due to Theorem 3.16 and Theorem 2.2 in [AFP00] we have

$$\begin{aligned}\left\|\nabla(u_0*\rho_\varepsilon\circ\varphi)\mathcal{L}^N\right\|_{\mathcal{M}(\Omega)^N} &= \left|\nabla((u_0*\rho_\varepsilon)(\varphi))\mathcal{L}^N\right|(\Omega) \le \mathbb{L}(\varphi^{-1})^{N-1}\left|\nabla(u_0*\rho_\varepsilon)\mathcal{L}^N\right|(\Omega)\\ &\le \mathbb{L}(\varphi^{-1})^{N-1}\left|\nabla u_0\right|(\Omega) = \mathbb{L}(\varphi^{-1})^{N-1}\left\|\nabla u_0\right\|_{\mathcal{M}(\Omega)^N}.\end{aligned}$$

Thus, $u_0*\rho_\varepsilon\circ\varphi$ is bounded in $BV(\Omega)$ and Proposition 3.13 in [AFP00] yields that

$$u_0*\rho_\varepsilon\circ\varphi \overset{*}{\rightharpoonup} u_0\circ\varphi \quad \text{in } BV(\Omega)$$

as $\varepsilon\to 0$. □

The following lemma resembles Lemma 3.2 in [Che11] (Lemma 1 in [CL11]) but the assumptions are slightly weaker and the statement is slightly different. Since in [Che11, CL11], there is no proof given, we present a short proof of the lemma.

**Lemma 5.1.14** *Let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$, $\varphi \in C([0,T] \times \Omega)$ such that $\varphi(t,\cdot)$ and $\varphi^{-1}(t,\cdot)$ are Lipschitz continuous and bijective functions in $\Omega$ for every $t \in [0,T]$ with*

$$\mathbb{L}(\varphi^{-1}(t,\cdot)) \leq C \qquad \text{for all } t \in [0,T]$$

*and for some $C > 0$. Then $u_\varepsilon$, defined by*

$$u_\varepsilon(t,x) = (u_0 * \rho_\varepsilon)(\varphi(t,x)) \quad \forall\ (t,x) \in [0,T] \times \Omega$$

*lies in $C([0,T], BV(\Omega) - w^*)$.*

**Proof:** We first show $L^1$-convergence. For $t, s \in [0,T]$, we have $\varphi(t,x) \to \varphi(s,x)$ for all $x \in \Omega$ as $t \to s$. Since $u_0 * \rho_\varepsilon \in C(\Omega)$, we have that

$$u_\varepsilon(t,x) = u_0 * \rho_\varepsilon(\varphi(t,x)) \to u_0 * \rho_\varepsilon(\varphi(s,x)) = u_\varepsilon(s,x) \quad \text{as } t \to s$$

for all $x \in \Omega$. Furthermore,

$$|u_0 * \rho_\varepsilon(\varphi(t,x))| \leq \|u_0 * \rho_\varepsilon\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$$

and hence, Lebesgue's dominated convergence theorem yields that $u_\varepsilon(t,\cdot) \to u_\varepsilon(s,\cdot)$ in $L^1(\Omega)$ as $t \to s$. By the statement before Corollary 3.19 in [AFP00] and using Theorem 3.16 in [AFP00] we have that

$$\left|\nabla(u_0 * \rho_\varepsilon(\varphi(t,\cdot)))\mathcal{L}^N\right| \leq \mathbb{L}(\varphi^{-1}(t,\cdot))^{N-1} \left|\nabla(u_0 * \rho_\varepsilon)\right|(\varphi(t,\cdot))$$

and thus

$$\begin{aligned}\left\|\nabla(u_0 * \rho_\varepsilon(\varphi(t,\cdot)))\mathcal{L}^N\right\|_{\mathcal{M}(\Omega)^N} &= \left|\nabla(u_0 * \rho_\varepsilon(\varphi(t,\cdot)))\mathcal{L}^N\right|(\Omega) \\ &\leq C^{N-1}\left|\nabla(u_0 * \rho_\varepsilon)\mathcal{L}^N\right|(\Omega) = C^{N-1}\left\|\nabla(u_0 * \rho_\varepsilon)\mathcal{L}^N\right\|_{\mathcal{M}(\Omega)^N}\end{aligned}$$

for all $t \in [0,T]$. Then, Proposition 3.13 in [AFP00] yields the statement of the lemma.

□

The following result is the main extended statement of the results (Theorem 3.4 or Theorem 4) given in [Che11, CL11] with stricter assumptions: it states that the composition of a unique flow $X$ with an initial function $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ is a weak solution and the $BV$-regularity in the spatial domain is preserved.

**Theorem 5.1.15** *Let $b \in L^1((0,T), Lip_0(\Omega))^N \cap L^\infty((0,T) \times \Omega)^N$ and let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$. Then, there exists a unique weak solution of* (5.13) *given by*

$$u(t,x) = u_0(X(0,t,x)),$$

*which lies in $C([0,T], BV(\Omega) - w^*)$. Furthermore, the sequence $(u_0 * \rho_\varepsilon(X(0,\cdot,\cdot))$ converges to $u$ in $C([0,T], BV(\Omega) - w^*)$ as $\varepsilon \to 0$.*

**Proof:** For $b \in L^1((0,T), Lip_0(\Omega))^N$, we have that $\int_0^T \mathbb{L}(b(t,\cdot))\, dt < \infty$ and using point (iii) in Theorem 5.1.8 we obtain that

$$\mathbb{L}(X(t,s,\cdot)) = \|DX(t,s,\cdot)\|_{L^\infty(\Omega)} \leq e^{\int_s^T \mathbb{L}(b(r,\cdot))dr} \leq e^{\int_0^T \mathbb{L}(b(r,\cdot))dr} < \infty.$$

Due to Theorem 5.1.12, we know that

$$u_0 * \rho_\varepsilon(X(0,t,\cdot)) \overset{*}{\rightharpoonup} u_0(X(0,t,\cdot)) \qquad \text{in } BV(\Omega)$$

for all $t \in [0,T]$ and thus $\|u_0 * \rho_\varepsilon(X(0,t,\cdot))\|_{BV(\Omega)}$ is bounded. The bound holds uniformly in $t$: as in the proof of the previous lemma, we have that

$$|u_0 * \rho_\varepsilon(X(0,t,x))| \leq \|u_0\|_{L^\infty(\Omega)} \tag{5.15}$$

for almost all $x \in \Omega$ and thus $\|u_0 * \rho_\varepsilon(X(0,t,\cdot))\|_{L^1(\Omega)} \leq c\,\|u_0\|_{L^\infty(\Omega)}$ for some $c > 0$. Furthermore, the previous proof yields that

$$\left\|\nabla(u_0 * \eta_\varepsilon(X(0,t,\cdot)))\mathcal{L}^N\right\|_{\mathcal{M}(\Omega)^N} \leq C^{N-1}\left\|\nabla(u_0 * \rho_\varepsilon)\mathcal{L}^N\right\|_{\mathcal{M}(\Omega)^N} \tag{5.16}$$

with

$$C := e^{\int_0^T \mathbb{L}(b(r,\cdot))dr}.$$

Hence, we obtain in total that

$$\begin{aligned}\|u_0 * \rho_\varepsilon(X(0,t,\cdot))\|_{BV(\Omega)} &\leq c\,\|u_0\|_{L^\infty(\Omega)} + C^{N-1}\left\|\nabla(u_0 * \rho_\varepsilon)\mathcal{L}^N\right\|_{\mathcal{M}(\Omega)} \\ &\leq c\,\|u_0\|_{L^\infty(\Omega)} + C^{N-1}\sup_{\varepsilon>0}\|u_0 * \rho_\varepsilon\|_{BV(\Omega)} =: D < \infty\end{aligned}$$

for all $t \in [0,T]$ and for all $\varepsilon > 0$. Now, as $b$ has the renormalization property, there exists a unique weak solution $\tilde{u} \in C([0,T], L^\infty(\Omega) - w^*)$ of the transport equation with vector field $b$ and initial value $u_0$ lying in $C([0,T], L^2(\Omega))$. Then, as $u_0 * \rho_\varepsilon(X(0,\cdot,\cdot))$ is the unique solution of the transport equation with initial data $u_0 * \rho_\varepsilon$ and vector field $b$, we apply Theorem 3.2.4 and obtain

$$\tilde{u}(t,\cdot) \leftarrow u * \rho_\varepsilon(X(0,t,\cdot)) \rightarrow u(t,\cdot) \qquad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0$$

for all $t \in [0,T]$. Thus, $\tilde{u} = u$. It remains to show that $u \in C([0,T], BV(\Omega) - w^*)$ and that $u * \rho_\varepsilon(X(0,\cdot,\cdot))$ converges to $u$ in $C([0,T], BV(\Omega) - w^*)$. If $(u_\varepsilon) \subset C([0,T], BV(\Omega) - w^*)$ is equicontinuous, then Arzelà-Ascoli yields the missing statements since $u_\varepsilon(t,\cdot)$ is uniformly bounded in $t \in [0,T]$ and $\varepsilon > 0$. Now, we have that

$$\begin{aligned}&\|u_0 * \rho_\varepsilon(X(0,t,\cdot)) - u_0 * \rho_\varepsilon(X(0,s,\cdot))\|_{L^1(\Omega)} \\ &\quad\leq \int_\Omega \int_{\mathbb{R}^N} |\rho_\varepsilon(z)(u_0(X(0,t,x) - z) - u_0(X(0,s,x) - z))|\ dzdx \\ &\quad\leq \int_{\mathbb{R}^N} \rho_\varepsilon(z) \int_\Omega |(u_0(X(0,t,x) - z) - u_0(X(0,s,x) - z))|\ dxdz \\ &\quad\leq \int_{\mathbb{R}^N} \rho_\varepsilon(z)\ dz\,\|(u_0(X(0,t,\cdot)) - u_0(X(0,s,\cdot)))\|_{L^1(\Omega)} \\ &\quad= \|u(t,\cdot) - u(s,\cdot)\|_{L^1(\Omega)} \to 0 \quad \text{as } t \to s.\end{aligned}$$

In addition, let $\varphi \in C_0(\Omega)$ and $(\varphi_n) \subset C_c^\infty(\Omega)$ such that $\varphi_n \to \varphi$ in $C_0(\Omega)$. Then, for all $\delta > 0$ there exists some $N(\delta) \in \mathbb{N}$ such that

$$\|\varphi_n - \varphi\|_{C(\Omega)}\, D \leq \frac{\delta}{3}$$

for all $n \geq N(\delta)$. Now, taking $N(\delta)$, we obtain that there exists some $\gamma(N(\delta)) > 0$ such that for all $t \in [0,T]$ with $|t-s| \leq \gamma(N(\delta))$

$$\left\|\operatorname{div}\varphi_{N(\delta)}\right\|_{L^\infty(\Omega)} \|u(t,\cdot) - u(s,\cdot)\|_{L^1(\Omega)} \leq \frac{\delta}{3}.$$

Finally, we conclude for $N(\delta)$ and $|t-s| \leq \gamma(N(\delta))$

$$\begin{aligned}
&|\langle \nabla(u_0 * \rho_\varepsilon(X(0,t,\cdot))) - \nabla(u_0 * \rho_\varepsilon(X(0,s,\cdot))), \varphi\rangle| \\
&\quad \leq \left|\left\langle \nabla(u_0 * \rho_\varepsilon(X(0,t,\cdot))), \varphi - \varphi_{N(\delta)}\right\rangle\right| + \left|\left\langle \nabla(u_0 * \rho_\varepsilon(X(0,s,\cdot))), \varphi_{N(\delta)} - \varphi\right\rangle\right| \\
&\quad + \left|\left\langle \nabla(u_0 * \rho_\varepsilon(X(0,t,\cdot))) - \nabla(u_0 * \rho_\varepsilon(X(0,s,\cdot))), \varphi_{N(\delta)}\right\rangle\right| \\
&\quad \leq \frac{\delta}{3} + \int_\Omega \left|(u_0 * \rho_\varepsilon(X(0,t,x)) - u_0 * \rho_\varepsilon(X(0,s,x)))\operatorname{div}\varphi_{N(\delta)}(x)\right| \, dx + \frac{\delta}{3} \\
&\quad \leq \frac{\delta}{3} + \left\|\operatorname{div}\varphi_{N(\delta)}\right\|_{L^\infty(\Omega)} \|u(t,\cdot) - u(s,\cdot)\|_{L^1(\Omega)} + \frac{\delta}{3} \leq \delta
\end{aligned}$$

Thus, $(u * \rho_\varepsilon)$ is equicontinuous in $C([0,T], BV(\Omega) - w^*)$ and Arzelà-Ascoli yields that there exists a subsequence converging to some $w \in C([0,T], BV(\Omega) - w^*)$. Obviously, $u = w$ and via an argument by contradiction we can show that the whole sequence converges to $u$ in $C([0,T], BV(\Omega) - w^*)$.

□

**Remark 5.1.16** *The estimates* (5.15) *and* (5.16) *in the above proof together with Theorem 2.2 in [AFP00] shows that a solution $u \in C([0,T], BV(\Omega) - w^*)$ of the transport equation is bounded at any time point $t \in [0,T]$ with respect to the $BV$-norm in the following way:*

$$\begin{aligned}
\|u(t,\cdot)\|_{BV(\Omega)} &\leq \liminf_{\varepsilon > 0} \|u_0 * \rho_\varepsilon(X(0,t,\cdot))\|_{BV(\Omega)} \\
&\leq c\,\|u_0\|_{L^\infty(\Omega)} + C^{N-1}\,\|u_0\|_{BV(\Omega)} < \infty,
\end{aligned}$$

*where $C := e^{\int_0^T \mathbb{L}(b(r,\cdot))dr}$ and $c := |\Omega|$.*

With this proof, the extension of results of Chen and Lorenz ends. So far we showed that spatial Lipschitz regularity of vector fields preserves initial $BV$-regularity for time points $t > 0$. Now we turn to the general inhomogeneous continuity equation since we indicated the solution operator of this kind of equation as the possible derivative of our control-to-state operator $L_{u_0}$.

## 5.2. Measure solutions of the inhomogeneous continuity equation

In this section, we have a closer look on the general inhomogeneous continuity equation

$$\begin{aligned}
\partial_t\mu + \operatorname{div}(b\mu) + g\mu + f = 0 \qquad &\text{in } (0,T) \times \Omega, \\
\mu(0,\cdot) = \mu_0 \qquad &\text{on } \Omega
\end{aligned}$$

and we present well-known results on existence and uniqueness in the setting of measure valued solutions. Furthermore, we give a stability result for the solution operator of this equation in the second subsection. In the subsequent chapter we will show that the Fréchet derivative of the control-to-state operator $L_{u_0}$ is given by the solution operator of some special case of the above continuity equation.

### 5.2.1. Existence and uniqueness of measure solutions

We start this subsection with some considerations and introductions. In the upcoming parts, we will consider vector fields

$$b \in \mathrm{V}^p := \left\{ b \in L^p((0,T), Lip_0(\Omega))^N \cap L^\infty((0,T)\times\Omega)^N \middle|\ \operatorname{div} b \in L^1((0,T), Lip(\Omega)) \right\}$$

for $p \geq 1$ or

$$b \in \mathrm{V}^{p,q} := \left\{ b \in L^p((0,T), W_0^{1,q}(\Omega))^N \middle|\ \operatorname{div} b \in L^1((0,T), C(\Omega)) \right\}.$$

for $p \geq 1$ and $q > N$. In this case,

$$W^{1,q}(\Omega) \hookrightarrow C(\Omega).$$

Furthermore, we will be confronted with products of continuous functions $g \in C(\Omega)$ with Radon measures $\omega \in \mathcal{M}(\Omega)$. These products can be seen as Radon measures in $\mathcal{M}(\Omega)$, defined as linear functionals on the space $C_0(\Omega)$:

$$g\omega : C_0(\Omega) \to \mathbb{R}, \qquad f \mapsto \int_\Omega f(x)g(x)\ d\omega(x).$$

For the measure norm of $g\omega$ we obtain:

$$|\langle g\omega, f\rangle| = |\langle \omega, gf\rangle| \leq \|gf\|_{C(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} \leq \|f\|_{C(\Omega)} \|g\|_{C(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)}$$

and hence, taking the supremum over the set $\{f \in C_0(\Omega) | \, \|f\|_{C(\Omega)} \leq 1\}$ yields

$$\|g\omega\|_{\mathcal{M}(\Omega)} \leq \|g\|_{C(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)}.$$

As the last consideration, we have a closer look on the Radon measures $b \cdot \nabla u$ and $\operatorname{Div}(bu) - u \operatorname{div} b$ for functions $u \in BV(\Omega)$ and $b \in W_0^{1,q}(\Omega)^N$. These measures are equal: for $\varphi \in C_c^\infty(\Omega)$, using the product rule for Sobolev functions (e.g. Theorem 5.18 in [Dob10]), we obtain

$$\begin{aligned}
\langle \operatorname{Div}(bu) - u \operatorname{div}(b)\mathcal{L}^N, \varphi\rangle &= \int_\Omega \varphi(x)\ d\operatorname{Div}(bu)(x) - \int_\Omega \varphi(x)u(x) \operatorname{div}(b(x))\ dx \\
&= -\int_\Omega u(x)(\nabla\varphi(x)\cdot b(x) + \varphi(x)\operatorname{div}(b(x))\ dx \\
&= -\int_\Omega u(x) \operatorname{div}(\varphi(x)b(x))\ dx = \int_\Omega \varphi(x)b(x)\cdot d(\nabla u)(x) \\
&= \langle b\cdot\nabla u, \varphi\rangle.
\end{aligned} \tag{5.17}$$

Hence, the measures coincide on a dense subset as linear operators on $C_0(\Omega)$ and thus are equal.
Now, we consider general inhomogeneous continuity equations of the form

$$\begin{aligned}
\partial_t\mu + \operatorname{div}(b\mu) + g\mu + f = 0 \qquad &\text{in } (0,T)\times\Omega, \\
\mu(0,\cdot) = \mu_0 \qquad &\text{on } \Omega,
\end{aligned} \tag{5.18}$$

where $\mu_0 \in \mathcal{M}(\Omega)$, $g \in L^1((0,T), Lip(\Omega))$ and $f \in L^1((0,T), \mathcal{M}(\Omega))$.

**Definition 5.2.1** *A function $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$ is a weak measure solution of* (5.18) *if the equation*

$$\int_0^T \int_\Omega (\partial_t \varphi + b \cdot \nabla \varphi - g\varphi) \; d\mu(t,\cdot)(x)dt - \int_0^T \int_\Omega \varphi \; df(t,\cdot)(x)dt = -\int_\Omega \varphi(0,\cdot) \; d\mu_0(x)$$

*holds for all $\varphi \in C_c^\infty([0,T) \times \Omega)$.*

We will start with the homogeneous case for vector fields $b \in \mathrm{V}^p$ with $p \geq 1$ and look for existence and uniqueness of weak measure solutions. For the uniqueness part in the proof we need the following auxiliary lemma.

**Lemma 5.2.2** *Let $m \in \mathbb{N}$, $1 \leq p < \infty$ and $g \in L^p((0,T), Lip_0(\Omega))$.*

*(i) There exists a sequence $(g_n) \subset C^\infty((0,T), C_0^m(\Omega))$ such that*

$$g_n \to g \quad \text{in } L^p((0,T), C(\Omega))$$

*and $\left(\int_0^T \mathbb{L}(g_n(t,\cdot))^p \; dt\right)$ is bounded.*

*(ii) If $g \in L^p((0,T), Lip_0(\Omega))^N$ with $\operatorname{div} g \in L^1((0,T), Lip(\Omega))$, then there exists a sequence $(g_n) \subset C^\infty((0,T), C_0^m(\Omega))^N$ such that $\left(\int_0^T \mathbb{L}(g_n(t,\cdot))^p \; dt\right)$ is bounded,*

$$g_n \to g \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad \operatorname{div} g_n \to \operatorname{div} g \quad \text{in } L^1((0,T), C(\Omega)).$$

**Proof:** The proof can be found in the appendix.

□

**Theorem 5.2.3 (Existence and uniqueness for the homogeneous continuity equation)** *Let $\mu_0 \in \mathcal{M}(\Omega)$, $g \in L^1((0,T), Lip(\Omega))$, $f = 0$ and $b \in \mathrm{V}^1$. Then there exists a unique weak measure solution $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$ of* (5.18)*, given by*

$$\mu(t,\cdot) = \mu_0(X(0,t,\cdot))e^{-\int_0^t g(s,X(s,t,\cdot))ds} \tag{5.19}$$

*in an explicit form as well as in implicit form*

$$\mu(t,\cdot) = \mu_0(X(0,t,\cdot)) - \int_0^t (g\mu)(s, X(s,t,\cdot)) \; ds \tag{5.20}$$

*for all $t \in [0,T]$.*

**Remark 5.2.4** *The form of the homogeneous solution in* (5.19) *is consistent with the form of the solutions in Theorem 5.1.15 and Theorem 5.1.11: for some vector field*

$$b \in L^1((0,T), Lip_0(\Omega))^N \cap L^\infty((0,T) \times \Omega)^N$$

*and some initial value $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$, Theorem 5.1.15 yields that the unique solution $u$ of (5.13) has the form*

$$u(t,x) = u_0(X(0,t,x)) \quad \textit{for } (t,x) \in (0,T) \times \Omega.$$

*If we additionally assume that $\operatorname{div} b \in L^1((0,T), Lip(\Omega))$ holds, i.e. $b \in \mathrm{V}^1$, Theorem 5.2.3 yields that $u\mathcal{L}^N$ is also unique among all measure solutions of (5.18) with $g = -\operatorname{div} b$, $f = 0$ and initial value $\mu_0 = u_0\mathcal{L}^N$. On the other hand, if we consider the initial measure $\mu_0 = u_0\mathcal{L}^N$, the above theorem yields that the unique measure solution of (5.18) with $g = -\operatorname{div} b$, $f = 0$ is given by*

$$\mu(t,\cdot) = \mu_0(X(0,t,\cdot))e^{\int_0^t \operatorname{div} b(s,X(s,t,\cdot))ds} = u_0(X(0,t,\cdot))\mathcal{L}^N(X(0,t,\cdot))e^{\int_0^t \operatorname{div} b(s,X(s,t,\cdot))ds}.$$

*That means,*

$$u_0(X(0,t,\cdot))\mathcal{L}^N = u(t,\cdot)\mathcal{L}^N = \mu(t,\cdot) = u_0(X(0,t,\cdot))\mathcal{L}^N(X(0,t,\cdot))e^{\int_0^t \operatorname{div} b(s,X(s,t,\cdot))ds}$$

*which leads to*

$$\mathcal{L}^N = \mathcal{L}^N(X(0,t,\cdot))e^{\int_0^t \operatorname{div} b(s,X(s,t,\cdot))ds} \tag{5.21}$$

*for all $t \in [0,T]$ due to the arbitrariness of $u_0$.*

The proof of Theorem 5.2.3 is based on the proof of Proposition 3.6 in [Man07] where the statement is proven for similar assumptions.

**Proof:** We start with the statement about existence. As test functions we choose $\psi \in C_c^\infty([0,T))$ and $\varphi \in C_c^\infty(\Omega)$. Any arbitrary test function $\xi \in C_c^\infty([0,T) \times \Omega)$ can be approximated by sums of products of such test functions. Therefore, it suffices to restrict to such test functions. As our first step we show that the map $t \mapsto \langle \mu(t), \varphi \rangle$ with $\mu$ given by the explicit formula is an element of $W^{1,1}((0,T))$, i.e. that the mapping is absolutely continuous. For disjoint subintervals $(s_i,t_i) \subset [0,T]$ with $i \in \{1,\dots,k\}$ and $k \in \mathbb{N}$ we obtain that

$$\begin{aligned}\sum_{i=1}^k |\varphi(X(t_i,0,x)) - \varphi(X(s_i,0,x))| &\leq \sum_{i=1}^k \int_{s_i}^{t_i} |\nabla\varphi(X(r,0,x)) \cdot \partial_r X(r,0,x)| \; dr \\ &\leq \|\nabla\varphi\|_{C(\Omega)^N} \int_{\bigcup_i (s_i,t_i)} \|b(r,\cdot)\|_{C(\Omega)^N} \; dr.\end{aligned} \tag{5.22}$$

Since

$$\left| \int_0^t g(s,X(s,t,x)) \; ds \right| \leq \int_0^T \|g(s,\cdot)\|_{C(\Omega)} \; ds =: A \tag{5.23}$$

for all $x \in \Omega$, $t \in [0,T]$ and $z \mapsto e^z$ is a Lipschitz continuous function on bounded, closed intervals, we get with some calculations

$$\sum_{i=1}^k |\langle \mu(t_i,\cdot) - \mu(s_i,\cdot), \varphi \rangle|$$

$$\leq e^{\int_0^T \|g(r,\cdot)\|_{C(\Omega)}dr} |\mu_0|(\Omega) \|\nabla\varphi\|_{C(\Omega)^N} \int\limits_{\bigcup_i (s_i,t_i)} \|b(r,\cdot)\|_{C(\Omega)^N} \, dr$$

$$+ |\mu_0|(\Omega) \|\varphi\|_{C(\Omega)} \mathbb{L}\left(e|_{[A,A]}\right) \int\limits_{\bigcup_i (s_i,t_i)} \|g(s,\cdot)\|_{C(\Omega)} \, ds.$$

Due to the absolute continuity of the integrals, we conclude the absolute continuity of the mapping $t \mapsto \langle \mu(t,\cdot), \varphi \rangle$. Hence $t \mapsto \langle \mu(t,\cdot), \varphi \rangle \in W^{1,1}((0,T))$. Then, the pointwise time derivative is given by

$$\begin{aligned}
\frac{d}{dt} \langle \mu(t,\cdot), \varphi \rangle &= \int\limits_{\Omega} e^{-\int_0^t g(s,X(s,0,x))ds} \frac{d}{dt} \varphi(X(t,0,x)) + \varphi(X(t,0,x)) \frac{d}{dt} e^{-\int_0^t g(s,X(s,0,x))ds} \, d\mu_0(x) \\
&= \int\limits_{\Omega} e^{-\int_0^t g(s,X(s,0,x))ds} (\nabla\varphi)(X(t,0,x)) \cdot b(t, X(t,0,x)) \, d\mu_0(x) \\
&- \int\limits_{\Omega} \varphi(X(t,0,x)) e^{-\int_0^t g(s,X(s,0,x))ds} g(t, X(t,0,x)) \, d\mu_0(x) \\
&= \langle \mu(t,\cdot), \nabla\varphi \cdot b(t,\cdot) - \varphi g(t,\cdot) \rangle
\end{aligned}$$

and thus equal to the weak derivative of $t \mapsto \langle \mu(t,\cdot), \varphi \rangle$ for almost all $t \in (0,T)$. Using that, we obtain

$$\begin{aligned}
\int\limits_0^T \int\limits_{\Omega} \partial_t(\psi(t)\varphi(x)) \, d\mu(t,\cdot)(x)dt = &- \int\limits_0^T \int\limits_{\Omega} \psi(t) \left[\nabla\varphi(x) \cdot b(t,x) - \varphi(x)g(t,x)\right] \, d\mu(t,\cdot)(x)dt \\
&- \int\limits_{\Omega} \psi(0)\varphi(x) \, d\mu_0(x).
\end{aligned}$$

Therefore, $\mu$, given in (5.19), is a solution of the homogeneous continuity equation with $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$. The explicit defined $\mu$ also satisfies the implicit formula (5.20). The proof is given in the proof of Proposition 3.6 in [Man07]. Now, it remains to show the uniqueness result. The proof is a standard procedure for showing uniqueness of measure solutions for the continuity equation adopted here to the more general situation that $g \neq 0$ (see e.g. Proposition 8.1.7 in [AGS08]):

Since the homogeneous equation is linear, it suffices to show that the equation with zero initial value has only the constant zero measure as a solution. So let $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$ solves (5.18) with $\mu_0 = 0$. Furthermore, let $\psi \in C_c^\infty((0,T) \times \Omega)$, $(b_n) \subset C^\infty((0,T), C_0^1(\Omega))^N$ be convergent to $b$ in $L^1((0,T), C_0(\Omega))^N$ such that $\left(\int_0^T \mathbb{L}(b_n(t,\cdot))dt\right)$ is bounded by some $C > 0$ and $(g_n) \subset C^\infty((0,T), C^\infty(\Omega))$ be convergent to $g$ in $L^1((0,T), C(\Omega))$ such that $\sup_n \int_0^T \mathbb{L}(g_n(t,\cdot)) \, dt < \infty$. Then, we consider the sequence of functions

$$\varphi_n(t,x) = -\int\limits_t^T \psi(s, X_n(s,t,x)) e^{-\int_t^s g_n(r, X_n(r,t,x))dr} \, ds.$$

Obviously, $\varphi_n \in C_0^1([0,T) \times \Omega)$: $\varphi_n$ is a composition of continuously differentiable functions and thus continuously differentiable. In addition, $X_n(s,t,x) = x$ for all $x \in \partial\Omega$ and for all

$t, s \in [0,T]$ due to Lemma 5.1.5 and thus

$$\varphi_n(t,x) = -\int_s^T \psi(s,x) e^{-\int_t^s g_n(r,X_n(r,t,x))dr} \, ds = 0 \quad \text{for all } x \in \partial\Omega \text{ and } t \in [0,T].$$

Finally, $\varphi_n(T,\cdot) = 0$. Now, $\varphi_n$ is a solution of

$$\begin{aligned} \partial_t \varphi_n + b_n \cdot \nabla \varphi - g_n \varphi_n = \psi \qquad & \text{in } (0,T) \times \Omega, \\ \varphi_n(T,\cdot) = 0 \qquad & \text{in } \Omega, \end{aligned}$$

which can be proved by direct calculation. Due to Lemma 5.1.10, $X_n(s,t,\cdot) \in C^1(\Omega)$ for $s,t \in [0,T]$ and we have the estimate

$$|DX_n(s,t,x)| \leq e^{\int_s^t \mathbb{L}(b_n(z,\cdot))dz} \, dr \leq e^{\int_0^T \mathbb{L}(b_n(z,\cdot))dz} \leq e^C.$$

Thus,

$$\begin{aligned} |\nabla \varphi_n(t,x)| \leq {} & e^{\int_0^T \|g_n(s,\cdot)\|_{C(\Omega)} ds} \int_t^T |\nabla \psi(s, X_n(s,t,x))| \, |DX_n(s,t,x)| \, ds \\ & + e^{\int_0^T \|g_n(s,\cdot)\|_{C(\Omega)} ds} \|\psi\|_{C((0,T)\times\Omega)} \int_t^T \int_t^s \left| \nabla g_n(t, X_n(r,t,x))^\top DX_n(r,t,x) \right| drds \\ & \leq e^{\int_0^T \|g_n(s,\cdot)\|_{C(\Omega)} ds} T e^C \left( \|\nabla \psi\|_{C((0,T)\times\Omega)^N} + \|\psi\|_{C((0,T)\times\Omega)} \int_0^T \mathbb{L}(g_n(t,\cdot)) \, dt \right) < \infty. \end{aligned}$$

Hence

$$\|\nabla \varphi_n\|_{C((0,T)\times\Omega)^N} \leq \tilde{C} < \infty$$

for all $n \in \mathbb{N}$ and we get

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \partial_t \varphi_n(t,x) + b(t,x) \cdot \nabla \varphi_n(t,x) - \varphi_n(t,x) g(t,x) \, d\mu(t,\cdot)(x)dt \\ &= \int_0^T \int_\Omega \psi(t,x) \, d\mu(t,\cdot)(x)dt + \int_0^T \int_\Omega (b(t,x) - b_n(t,x)) \cdot \nabla \varphi_n(t,x) \, d\mu(t,\cdot)(x)dt \\ &+ \int_0^T \int_\Omega (g_n(t,x) - g(t,x)) \varphi_n(t,x) \, d\mu(t,\cdot)(x)dt. \end{aligned} \tag{5.24}$$

For the second term in (5.24), we have the estimate:

$$\left| \int_0^T \int_\Omega (b(t,x) - b_n(t,x)) \cdot \nabla \varphi_n(t,x) \, d\mu(t,\cdot)(x)dt \right| \leq \tilde{C} \|b - b_n\|_{L^1((0,T),C(\Omega))^N} \sup_{t\in[0,T]} \|\mu(t,\cdot)\|_{\mathcal{M}(\Omega)}$$

for some $\tilde{C} > 0$. In a similar way, we have for the third term in (5.24)

$$\left| \int_0^T \int_\Omega (g_n(t,x) - g(t,x))\varphi_n(t,x)\ d\mu(t,\cdot)(x)dt \right| \leq C \left\| g - g_n \right\|_{L^1((0,T),C(\Omega))} \sup_{t\in[0,T]} \left\| \mu(t,\cdot) \right\|_{\mathcal{M}(\Omega)}$$

for some $C > 0$. Taking $n \to \infty$, we obtain that the second and third term vanish and we get that the first term in (5.24) is equal to zero. If we choose $\psi = \rho\eta$ with $\rho \in C_c^\infty((0,T))$ and $\eta \in C_c^\infty(\Omega)$, we obtain

$$0 = \int_0^T \rho(t)\langle \mu(t,\cdot), \eta\rangle\ dt.$$

Since $\rho$ can be chosen arbitrarily, we conclude that $t \mapsto \langle \mu(t,\cdot), \eta\rangle$ is equal to zero in $L^2((0,T))$ and therefore in $C([0,T])$. Analogously, as $\eta$ can be chosen arbitrarily, we obtain $\mu(t,\cdot) = 0$ for all $t \in [0,T]$. Hence $\mu = 0$ in $C([0,T], \mathcal{M}(\Omega) - w^*)$.

□

For the inhomogeneous case we will use Duhamel's principle to obtain a unique weak measure solution of the transport equation. Therefore, we will consider the inhomogeneous continuity equation with zero initial data

$$\begin{aligned} \partial_t\mu + \operatorname{div}(b\mu) + g\mu + f &= 0 \qquad \text{in } (0,T)\times\Omega, \\ \mu(0,\cdot) &= 0 \qquad \text{in } \Omega, \end{aligned} \tag{5.25}$$

where $f \in L^1((0,T), \mathcal{M}(\Omega))$ and $g \in L^1((0,T), Lip(\Omega))$.

**Theorem 5.2.5 (Existence of solutions for the inhomogeneous equation)** *Let $\mu_0$ be zero, $f \in L^1((0,T), \mathcal{M}(\Omega)), g \in L^1((0,T), Lip(\Omega))$ and $b \in \mathrm{V}^1$. Then the inhomogeneous continuity equation* (5.25) *has a unique weak measure solution $\mu \in C([0,T], \mathcal{M} - w^*)$, given by*

$$\mu(t,\cdot) = -\int_0^t f(s, X(s,t,\cdot))e^{\int_s^t -g(\tau, X(\tau,t,\cdot))d\tau}\ ds.$$

The proof works in the same way as the existence part of the previous proof.

**Proof:** The uniqueness is obvious, since the difference of two possible solutions satisfies the homogeneous continuity equation with zero initial value and Theorem 5.2.3 yields that the only solution to this equation is the constant zero measure. For the existence proof we follow the structure of the previous proof: we first take $\psi \in C_c^\infty([0,T))$ and $\varphi \in C_c^\infty(\Omega)$ and obtain for disjoint subintervals $(s_i,t_i) \subset [0,T]$ where $i \in \{1,\ldots,k\}$ with $k \in \mathbb{N}$ using estimate (5.22)

$$\begin{aligned} \sum_{i=1}^k |\langle \mu(t_i,\cdot) - \mu(s_i,\cdot), \varphi\rangle| \leq \|\varphi\|_{C(\Omega)}\, e^{\int_0^T \|g(s,\cdot)\|_{C(\Omega)}ds} \int_{\bigcup_i (s_i,t_i)} \|f(s,\cdot)\|_{\mathcal{M}(\Omega)}\ ds \\ + \|\nabla\varphi\|_{C(\Omega)^N}\, e^{\int_0^T \|g(s,\cdot)\|ds} \int_{\bigcup_i (s_i,t_i)} \|b(s,\cdot)\|_{C(\Omega)^N}\ ds \int_0^T \|f(s,\cdot)\|_{\mathcal{M}(\Omega))}\ ds \end{aligned}$$

$$+ \|\varphi\|_{C(\Omega)} \mathbb{L}\left(e|_{[-A,A]}\right) \int\limits_{\bigcup_i (s_i,t_i)} \|g(s,\cdot)\|_{C(\Omega)} \; ds \int\limits_0^T \|f(s,\cdot)\|_{\mathcal{M}(\Omega))} \; ds,$$

where we denote by $A$ the constant defined in (5.23). Hence, we get absolute continuity of $t \mapsto \langle \mu(t,\cdot), \varphi \rangle$ and we conclude for the weak derivative

$$\begin{aligned}
&\frac{d}{dt}\langle \mu(t,\cdot), \varphi \rangle \\
&= -\frac{d}{dt} \int\limits_0^t \int\limits_\Omega \varphi(X(t,s,x)) e^{-\int_s^t g(\tau, X(\tau,s,x))d\tau} \; df(s,\cdot)(x) ds \\
&= -\int\limits_\Omega \varphi(x) \; df(t,\cdot)(x) - \int\limits_\Omega \nabla\varphi(x) \cdot b(t,x) \; d\left( \int\limits_0^t e^{-\int_s^t g(\tau, X(\tau,t,\cdot))d\tau} f(s, X(s,t,\cdot)) \; ds \right)(x) \\
&+ \int\limits_\Omega \varphi(x) g(t,x) \; d\left( \int\limits_0^t e^{-\int_s^t g(\tau, X(\tau,t,\cdot))d\tau} f(s, X(s,t,\cdot)) \; ds \right)(x) \\
&= -\int\limits_\Omega \varphi(x) \; df(t,\cdot)(x) + \int\limits_\Omega \nabla\varphi(x) \cdot b(t,x) \; d\mu(t,\cdot)(x) - \int\limits_\Omega \varphi(x) g(t,x) \; d\mu(t,\cdot)(x)
\end{aligned}$$

for almost all $t \in [0,T]$. Using that, we conclude

$$\begin{aligned}
\int\limits_0^T \int\limits_\Omega \partial_t \psi(t) \varphi(x) \; d\mu(t,\cdot)(x) dt &= \int\limits_0^T \partial_t \psi(t) \langle \mu(t,\cdot), \varphi \rangle \; dt = -\int\limits_0^T \psi(t) \frac{d}{dt} \langle \mu(t,\cdot), \varphi \rangle \; dt \\
&= \int\limits_0^T \left( \int\limits_\Omega \psi(t)\varphi(x) \; df(t,\cdot)(x) + \int\limits_\Omega \psi(t)\varphi(x) g(t,x) \; d\mu(t,\cdot)(x) \right. \\
&\left. - \int\limits_\Omega \psi(t) \nabla\varphi(x) \cdot b(t,x) \; d\mu(t,\cdot)(x) \right) dt.
\end{aligned}$$

General test functions can be approximated by sums of products of functions of the above type. □

**Remark 5.2.6** *By combining the results of Theorem 5.2.3 and Theorem 5.2.5, we obtain a unique weak measure solution of the inhomogeneous transport equation with nonzero initial data: let* $\mu_H$ *be the unique weak measure solution of* (5.13) *with initial data* $\mu_0 \in \mathcal{M}(\Omega)$ *and* $\mu_I$ *be the unique weak measure solution of* (5.25). *Then the unique weak measure solution to* (5.18) *is given by*

$$\begin{aligned}
\mu(t,\cdot) &= \mu_H(t,\cdot) + \mu_I(t,\cdot) \\
&= \mu_0(X(0,t,\cdot)) e^{-\int_0^t g(s, X(s,t,\cdot))ds} - \int\limits_0^t f(s, X(s,t,\cdot)) e^{-\int_s^t g(\tau, X(\tau,t,\cdot))d\tau} \; ds
\end{aligned}$$

$$= \mu_0(X(0,t,\cdot)) - \int_0^t \left[(g\mu_H)(s,X(s,t,\cdot)) + f(s,X(s,t,\cdot))e^{-\int_s^t g(\tau,X(\tau,t,\cdot))d\tau}\right] ds.$$

*In particular, if* $\mu_0 = 0$ *we obtain that*

$$\int_0^t (g\mu_H)(s,X(s,t,\cdot))\ ds = 0$$

*for any* $t \in [0,T]$.

### 5.2.2. A generalized existence result about measure solutions and stability

In the subsequent chapter, we will apply these general results about measure solutions of inhomogeneous continuity equations to show continuous Fréchet differentiability of the control-to-state operator at Lipschitz regular vector fields. In this situation, the derivative will be given by the solution operator of some specific inhomogeneous continuity equation. For this purpose, we are interested in a generalized existence results of measure solutions for less spatial regularity, namely for vector fields with $W^{1,q}(\Omega)$-regularity in the spatial domain with $q > N$. In this case, the vector fields are still continuous with respect to the spatial variable, but the uniqueness of solutions cannot be guaranteed anymore. Nevertheless, these results will be helpful for showing the existence of derivatives. Furthermore, we need some stability result for sequences of measure solutions of the continuity equation with vector fields with spatial $W^{1,q}(\Omega)$-regularity. Therefore, we present a theorem stating convergence of solutions to some unique measure solution with respect to some topology if the corresponding sequence of vector fields converges to a Lipschitz regular vector field.
In this part, we consider vector fields $b \in \mathrm{V}^{p,q}$ for $p \geq 1$ and $q > N$. In this case, we have the embedding

$$W_0^{1,q}(\Omega) \hookrightarrow C_0(\Omega),$$

which can be found in Theorem 6.24 in [Dob10]. We begin the extension with existence of measure solutions for vector fields $b \in \mathrm{V}^{p,q}$.

**Theorem 5.2.7 (Existence of solutions for vector fields with Sobolev regularity)** *Let* $p \in [1,\infty)$ *and* $q > N$. *Furthermore, let* $b \in \mathrm{V}^{p,q}$, $\mu_0 \in \mathcal{M}(\Omega)$, $g \in L^1((0,T),C(\Omega))$ *and* $f \in L^1((0,T),\mathcal{M}(\Omega))$. *Then, there exists a weak measure solution* $\mu \in C([0,T],\mathcal{M}(\Omega) - w^*)$ *of* (5.18).

**Proof:** As $W_0^{1,q}(\Omega) \hookrightarrow C_0(\Omega)$ for $q > N$, we find sequences $(b_n) \subset L^p((0,T),C_0^m(\Omega))^N$ for some fixed $m \geq 2$ and $(g_n) \subset L^1((0,T),Lip(\Omega))$ such that

$$b_n \to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{and} \quad g_n \to g \quad \text{in } L^1((0,T),C(\Omega)).$$

Then, Remark 5.2.6 yields that the unique weak measure solutions $\mu_n \in C([0,T],\mathcal{M}(\Omega) - w^*)$ of (5.18) with functions $b_n$, $g_n$ and inhomogeneous term $f$ are given by

$$\mu_n(t,\cdot) = \mu_0(X_n(0,t,\cdot))e^{-\int_0^t g_n(s,X_n(s,t,\cdot))ds} - \int_0^t f(s,X_n(s,t,\cdot))e^{-\int_s^t g_n(\tau,X_n(\tau,t,\cdot))d\tau}\ ds,$$

where $X_n$ denotes the unique flow of $b_n$. We obtain the following estimates for $\varphi \in C_0(\Omega)$:

$$|\langle \mu_0(X_n(0,t,\cdot)), \varphi \rangle| = \left| \int_\Omega \varphi(X_n(t,0,x))\ d\mu_0(x) \right| \leq \|\mu_0\|_{\mathcal{M}(\Omega)} \|\varphi\|_{C(\Omega)}$$

and thus $\|\mu_0(X_n(0,t,\cdot))\|_{\mathcal{M}(\Omega)} \leq \|\mu_0\|_{\mathcal{M}(\Omega)}$. In the same way, we get that

$$\|f(s, X_n(s,t,\cdot))\|_{\mathcal{M}(\Omega)} \leq \|f(s,\cdot)\|_{\mathcal{M}(\Omega)}$$

for almost all $s \in (0,T)$. Hence, we have

$$\|\mu_n(t,\cdot)\|_{\mathcal{M}(\Omega)} \leq e^{\int_0^T \|g_n(s,\cdot)\|_{C(\Omega)} ds} \left( \|\mu_0\|_{\mathcal{M}(\Omega)} + \int_0^T \|f(s,\cdot)\|_{\mathcal{M}(\Omega)}\ ds \right) \leq C < \infty,$$

since $(g_n)$ is a bounded sequence in $L^1((0,T), C(\Omega))$. Furthermore, as in the proofs of Theorems 5.2.3 and 5.2.5 we obtain for $\varphi \in C_c^\infty(\Omega)$

$$\frac{d}{dt}\langle \mu_n(t,\cdot), \varphi \rangle = -\int_\Omega \varphi(x)\ df(t,\cdot)(x) + \int_\Omega [\nabla\varphi(x) \cdot b_n(t,x) - \varphi(x) g_n(t,x)]\ d\mu_n(t,\cdot)(x).$$

and thus we estimate

$$\begin{aligned} \left| \frac{d}{dt}\langle \mu_n(t,\cdot), \varphi \rangle \right| \leq\ & \|\varphi\|_{C(\Omega)} \|f(t,\cdot)\|_{\mathcal{M}(\Omega)} + \|\nabla\varphi\|_{C(\Omega)^N} \|b_n(t,\cdot)\|_{C(\Omega)^N} \|\mu_n(t,\cdot)\|_{\mathcal{M}(\Omega)} \\ & + \|\varphi\|_{C(\Omega)} \|g_n(t,\cdot)\|_{C(\Omega)} \|\mu_n(t,\cdot)\|_{\mathcal{M}(\Omega)} \end{aligned}$$

Consequently, we get for $0 \leq s < t \leq T$

$$\begin{aligned} |\langle \mu_n(t,\cdot) - \mu_n(s,\cdot), \varphi \rangle| \leq\ & \|\varphi\|_{C(\Omega)} \int_s^t \|f(z,\cdot)\|_{\mathcal{M}(\Omega)}\ dz \\ & + \|\nabla\varphi\|_{C(\Omega)^N} \int_s^t \|b_n(z,\cdot)\|_{C(\Omega)^N} \|\mu_n(z,\cdot)\|_{\mathcal{M}(\Omega)}\ dz \\ & + \|\varphi\|_{C(\Omega)} \int_s^t \|g_n(z,\cdot)\|_{C(\Omega)} \|\mu_n(z,\cdot)\|_{\mathcal{M}(\Omega)}\ dz \\ \leq\ & C(\varphi) \int_s^t \Big( \|f(z,\cdot)\|_{\mathcal{M}(\Omega)} + \|b_n(z,\cdot)\|_{C(\Omega)^N}\ dz + \|g_n(z,\cdot)\|_{C(\Omega)} \Big)\ dz \end{aligned} \tag{5.26}$$

for some $C(\varphi) > 0$ depending on $\varphi$. As $b_n \to b$ in $L^p((0,T), C_0(\Omega))^N$ and $g_n \to g$ in $L^1((0,T), C(\Omega))$, the sequence of mappings $(h_n)$ given by

$$h_n(t) := \|f(t,\cdot)\|_{\mathcal{M}(\Omega)} + \|b_n(t,\cdot)\|_{C(\Omega)^N} + \|g_n(t,\cdot)\|_{C(\Omega)}$$

converges to $h(t) := \|f(t,\cdot)\|_{\mathcal{M}(\Omega)} + \|b(t,\cdot)\|_{C(\Omega)^N} + \|g(t,\cdot)\|_{C(\Omega)}$ in $L^1((0,T))$. Hence, due to Theorem 1.38 in [AFP00], $(h_n)$ is uniformly integrable, i.e. for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $n \in \mathbb{N}$

$$\int_s^t h_n(z)\, dz \le \varepsilon \qquad \text{for all } t, s \text{ with } |t-s| \le \delta.$$

Now, let $\varphi \in C_0(\Omega)$. Then we take a sequence $(\varphi_k) \subset C_c^\infty(\Omega)$ such that $\varphi_k \to \varphi$ in $C_0(\Omega)$ as $k \to \infty$. We deduce

$$\begin{aligned}|\langle \mu_n(t,\cdot) - \mu_n(s,\cdot), \varphi\rangle| &\le |\langle \mu_n(t,\cdot) - \mu_n(s,\cdot), \varphi - \varphi_k\rangle| + |\langle \mu_n(t,\cdot) - \mu_n(s,\cdot), \varphi_k\rangle| \\ &\le C\, \|\varphi_k - \varphi\|_{C(\Omega)} \\ &\quad + C(\varphi_k) \int_s^t \Big( \|f(z,\cdot)\|_{\mathcal{M}(\Omega)} + \|b_n(z,\cdot)\|_{C(\Omega)^N}\ dz + \|g_n(z,\cdot)\|_{C(\Omega)} \Big)\ dz.\end{aligned}$$

Fo $\varepsilon > 0$ we find a $k(\varepsilon) \in \mathbb{N}$ such that

$$C\, \|\varphi_k - \varphi\|_{C(\Omega)} \le \varepsilon$$

for all $k \ge k(\varepsilon)$. We choose $\delta > 0$ such that

$$C\left(\varphi_{k(\varepsilon)}\right) \int_s^t h_n(z)\ dz \le \varepsilon \qquad \text{for all } n \in \mathbb{N} \text{ and } t, s \text{ with } |t-s| \le \delta.$$

Then,

$$|\langle \mu_n(t,\cdot) - \mu_n(s,\cdot), \varphi\rangle| \le 2\varepsilon$$

for all $t, s \in [0,T]$ with $|t-s| \le \delta$. Hence, the mappings $t \mapsto \langle \mu_n(t), \varphi\rangle$ are equicontinuous and thus $(\mu_n)$ is equicontinuous in $C([0,T], \mathcal{M}(\Omega) - w^*)$. In addition, since $(\mu_n)$ is pointwise bounded, the set $\{\mu_n(t,\cdot) | n \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}(\Omega)$ with respect to the weak*-topology. Consequently, the requirements of Arzelà-Ascoli are satisfied and we obtain that there exists some $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$ and a subsequence $(\mu_{n_k})$ of $(\mu_n)$ converging to $\mu$ in $C([0,T], \mathcal{M}(\Omega) - w^*)$. Lebesgue's dominated convergence theorem and some simple calculations yield that $\mu$ is a weak measure solution of (5.18) with vector field $b$, initial value $\mu_0$ and functions $g$ and $f$.

□

These weak measure solutions need not to be unique. But for vector fields with Lipschitz regularity in the spatial domain we have uniqueness due to Remark 5.2.6. Beside existence and uniqueness of solutions we need some stability results for sequences of vector fields and their corresponding solutions. Such a result will be shown in the following theorem. For vector fields in $\mathrm{V}^{p,q}$ with $p \ge 1$ and $q > N$, the lack of uniqueness of solutions is not a problem for our purposes, since later we will show Fréchet differentiability at vector fields $b \in \mathrm{V}^p$ with $p > 1$ which yield uniqueness of the corresponding weak solutions. Actually, we are interested in those solutions $\mu$ of vector fields $b \in \mathrm{V}^{p,q}$, which can be approximated by smooth solutions, i.e. for which a sequence of smooth vector fields $(b_n) \subset \mathrm{V}^p$ exists with corresponding sequence of unique solutions $(\mu_n)$ such that $b_n \to b$ and $\mu_n \to \mu$ in suitable Banach spaces. We introduce the following definition for such solutions.

**Definition 5.2.8 (Approximability of solutions)** *Let $p \geq 1$ and $q > N$. Then, we call a solution $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$ of (5.18) with vector field $b \in \mathrm{V}^{p,q}$, $g \in L^1((0,T), C(\Omega))$ and $f \in L^1((0,T), \mathcal{M}(\Omega))$ approximable if for some $m \geq 2$, there exist sequences $(b_n) \subset C^\infty((0,T), C_0^m(\Omega))^N$ and $(g_n) \subset L^1((0,T), Lip(\Omega))$ such that*

$$b_n \to b \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad g_n \to g \quad \text{in } L^1((0,T), C(\Omega))$$

*and the corresponding sequence of unique weak solutions $(\mu_n) \subset C([0,T], \mathcal{M}(\Omega) - w^*)$ with inhomogeneity $f$ converges to $\mu$ in $C([0,T], \mathcal{M}(\Omega) - w^*)$.*

**Remark 5.2.9** *The proof of Theorem 5.2.7 shows that each vector field $b \in \mathrm{V}^{p,q}$ has at least one approximable solution $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$ of (5.18) for any $f \in L^1((0,T), \mathcal{M}(\Omega))$, $g \in L^1((0,T), C(\Omega))$ and $\mu_0 \in \mathcal{M}(\Omega)$.*

With this definition we present a stability result for measure solutions in the extended setting.

**Theorem 5.2.10** *Let $p \geq 1$, $r > 1$ and $q > N$ be fixed. Furthermore, let $\mu_0 \in \mathcal{M}(\Omega)$, $(g_n) \subset L^1((0,T), C(\Omega))$ be a sequence such that*

$$g_n \to g \in L^1((0,T), Lip(\Omega)) \quad \text{in } L^1((0,T), C(\Omega))$$

*and $(f_n) \subset L^r((0,T), \mathcal{M}(\Omega))$ be a bounded sequence converging to some $f \in L^r((0,T), \mathcal{M}(\Omega))$ in the weak*-topology of $\mathcal{M}((0,T) \times \Omega)$. Then we have:*
*If $(b_n) \subset \mathrm{V}^{p,q}$ is a sequence being convergent to $b \in \mathrm{V}^p$ in $L^p((0,T), C(\Omega))^N$, then each sequence of approximable solutions $(\mu_n)$ of the inhomogeneous continuity equation (5.18) with vector fields $b_n$, initial value $\mu_0$, functions $g_n$ and $f_n$ converges in $C([0,T], \mathcal{M}(\Omega) - w^*)$ to the unique solution $\mu$ of (5.18) with vector field $b$, initial value $\mu_0$ and functions $g$ and $f$.*

**Proof:** For each $b_n$ and $g_n$, we find sequences $(b_{n,k}) \subset C^\infty((0,T), C_0^m(\Omega))^N$ and $(g_{n,k}) \subset L^1((0,T), C^m(\Omega))$ for some $m \geq 2$ such that

$$b_{n,k} \to b_n \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad g_{n,k} \to g_n \quad \text{in } L^1((0,T), C(\Omega))$$

with $\|g_{n,k}\|_{L^1((0,T),C(\Omega))} \leq \|g_n\|_{L^1((0,T),C(\Omega))}$ for all $k \in \mathbb{N}$. Let $\varphi \in C_0(\Omega)$. Then, we deduce for all $\varepsilon > 0$ that there exists some $k(\varepsilon) \in \mathbb{N}$ such that for all $k \geq k(\varepsilon)$ and for all $t \in [0,T]$

$$\begin{aligned}\left| \int_\Omega \varphi(x)\, d\mu_{n,k}(t,\cdot)(x) - \int_\Omega \varphi(x)\, d\mu_n(t,\cdot)(x) \right| &\leq \int_\Omega |\varphi(X_{n,k}(t,0,x)) - \varphi(X_n(t,0,x))|\, d|\mu_0|(x) \\ &\leq \varepsilon.\end{aligned}$$

This is a consequence of the uniform convergence of $(X_{n,k}(\cdot,0,\cdot))$ to $X_n(\cdot,0,\cdot)$ (see Lemma 5.1.9). Thus, $\mu_{n,k} \to \mu_n$ in $C([0,T], \mathcal{M}(\Omega) - w^*)$ as $k \to \infty$. As in the proof of Theorem 5.2.7, we obtain that

$$\begin{aligned}\|\mu_{n,k}(t,\cdot)\|_{\mathcal{M}(\Omega)} &\leq \|\mu_0\|_{\mathcal{M}(\Omega)}\, e^{\int_0^T \|g_{n,k}(s,\cdot)\|_{C(\Omega)} ds} + e^{\int_0^T \|g_{n,k}(s,\cdot)\|_{C(\Omega)} ds} \int_0^T \|f_n(s,\cdot)\|_{\mathcal{M}(\Omega)}\, ds \\ &\leq \|\mu_0\|_{\mathcal{M}(\Omega)}\, e^{\int_0^T \|g_n(s,\cdot)\|_{C(\Omega)} ds} + e^{\int_0^T \|g_n(s,\cdot)\|_{C(\Omega)} ds} \int_0^T \|f_n(s,\cdot)\|_{\mathcal{M}(\Omega)}\, ds \leq C\end{aligned}$$

for some $C > 0$ independent of $n \in \mathbb{N}$ due to the boundedness of the sequences $(g_n)$ and $(f_n)$. Now, for a fixed $\varphi \in C_0(\Omega)$ we choose $k(n,\varphi) \in \mathbb{N}$ such that:

$$\left\| b_{n,k(n,\varphi)} - b_n \right\|_{L^p((0,T),C(\Omega))^N} \leq \frac{1}{n} \quad \text{and} \quad \left\| g_{n,k(n,\varphi)} - g_n \right\|_{L^1((0,T),C(\Omega))} \leq \frac{1}{n}$$

as well as

$$\sup_{t \in [0,T]} \left| \langle \mu_{n,k(n,\varphi)}(t,\cdot) - \mu_n(t,\cdot), \varphi \rangle \right| \leq \frac{1}{n}.$$

Then, the sequence $(b_{n,k(n,\varphi)})$ converges to $b$ in $L^p((0,T),C(\Omega))^N$ and the sequence $(g_{n,k(n,\varphi)})$ to $g$ in $L^1((0,T),C(\Omega))$ as $n \to \infty$. Furthermore, as in the proof of Theorem 5.2.7 we obtain that for $\psi \in C_0(\Omega)$ the sequence of functions $t \mapsto \langle \mu_{n,k(n,\varphi)}(t,\cdot), \psi \rangle$ is absolutely continuous with derivative

$$\frac{d}{dt} \langle \mu_{n,k(n,\varphi)}(t,\cdot), \psi \rangle = \langle \mu_{n,k(n,\varphi)}(t,\cdot), \nabla\psi \cdot b_{n,k(n,\varphi)}(t,\cdot) - \psi g_{n,k(n,\varphi)}(t,\cdot) \rangle - \langle f_n(t,\cdot), \psi \rangle$$

for almost all $t \in (0,T)$. Estimating for $s,t \in [0,T]$

$$\int_s^t \|f_n(z,\cdot)\|_{\mathcal{M}(\Omega)} \, dz \leq |t-s|^{1/r'} \|f_n\|_{L^r((0,T)),\mathcal{M}(\Omega)} \leq C|t-s|^{1/r'},$$

where $C > 0$ is a bound for the bounded sequence $(f_n)$, yields equicontinuity of $(\mu_{n,k(n,\varphi)})$ in $C([0,T],\mathcal{M}(\Omega) - w^*)$ as in the proof of Theorem 5.2.7 and due to Arzelà-Ascoli to a subsequence $(\mu_{(n,k(n,\varphi))_l})$ converging to some $\omega \in C([0,T],\mathcal{M}(\Omega) - w^*)$. As in previous proofs, we deduce that $\omega$ is a weak solution of (5.18). Since weak measure solutions with vector fields in $\mathrm{V}^p$ are unique, $\omega = \mu$. Furthermore, the whole sequence $(\mu_{n,k(n,\varphi)})$ converges to $\mu$ in $C([0,T],\mathcal{M}(\Omega) - w^*)$. This can be proven via a proof by contradiction. Now, it remains to show that $\mu_n \to \mu$ in $C([0,T],\mathcal{M}(\Omega) - w^*)$: let $\varphi \in C_0(\Omega)$. Then, we have

$$\begin{aligned}
\sup_{t \in [0,T]} |\langle \mu_n(t,\cdot) - \mu(t,\cdot), \varphi \rangle| &\leq \sup_{t \in [0,T]} \left| \langle \mu_n(t,\cdot) - \mu_{n,k(n,\varphi)}(t,\cdot), \varphi \rangle \right| \\
&\quad + \sup_{t \in [0,T]} \left| \langle \mu_{n,k(n,\varphi)}(t,\cdot) - \mu(t,\cdot), \varphi \rangle \right| \\
&\leq \frac{1}{n} + \sup_{t \in [0,T]} \left| \langle \mu_{n,k(n,\varphi)}(t,\cdot) - \mu(t,\cdot), \varphi \rangle \right| \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

□

# 6. Differentiability properties of the control-to-state operator and the tracking term

The aim of this chapter is to present several results about continuous Fréchet differentiability of the control-to-state operator $L$ and its composition with the tracking term of the objective function $J$. In the introductive text of the previous chapter, we gave a motivation that the Fréchet derivative of $L$ is given by the solution operator of the following partial differential equation:

$$\begin{aligned} \partial_t\mu + \operatorname{div}(b\mu)) - \mu \operatorname{div}(b) + \operatorname{div}(\tilde{b}u) - u \operatorname{div}(\tilde{b}) &= 0 \qquad \text{in } (0,T)\times\Omega \\ \mu(0,\cdot) &= 0 \qquad \text{in } \Omega. \end{aligned} \tag{6.1}$$

In the first section, we substantiate this motivation: we prove that $L$ is continuously Fréchet differentiable in vector fields $b \in \mathrm{V}^p$ with the solution operator of (6.1) as the Fréchet derivative. In this setting, we need to consider $L$ as an operator mapping into a vector space with less spatial regularity, i.e. we regard $L$ as a mapping with codomain $C([0,T],\mathcal{M}(\Omega) - w^*)$ instead of $C([0,T],L^2(\Omega))$. In the second part of the chapter, we turn to the composition of $L$ with the tracking term of $J$. To be more precise, we consider

$$G : \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N \to \mathbb{R}, \quad b \mapsto \sum_{k=2}^{K} G_k(b) \tag{6.2}$$

with $G_k : \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N \to \mathbb{R}$, $k = 2,\dots,K$ given by

$$G_k(b) = \frac{1}{2}\left\| L(b)(t_k,\cdot) - Y_k \right\|^2_{L^2(\Omega)}. \tag{6.3}$$

We constrain us to this term, since any differentiability result of this chapter can be easily generalized to the more general situation, where $G_k$ is composed with the functions $\Upsilon_k$, $k \in \{2,\dots,K\}$, introduced in section 4.2 of chapter 4.
For the composition, the weakening of the regularity in the range of $L$ has a consequence for its Fréchet differentiability: a direct proof of Fréchet differentiability as a composition of Fréchet differentiable functions is not possible due to the incompatibility of $\mathcal{M}(\Omega)$-regularity with the $L^2(\Omega)$-norm. As a consequence, we smooth $L$ with some mollifier $\rho_\varepsilon$ in the second section and denote it $L_\varepsilon$. For the smoothed control-to-state operator $L_\varepsilon$, we easily deduce from the results of the first section that it is continuously Fréchet differentiable in spatial Lipschitz regular vector fields as a mapping from a subset of $\mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N$ into $C([0,T],L^p(\Omega))$ for any $p \geq 1$. Based on this result, we then conclude that the smoothed tracking term

$$G_\varepsilon : \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N \to \mathbb{R}, \quad b \mapsto \sum_{k=2}^{K} G_{\varepsilon,k}(b) \tag{6.4}$$

with components

$$G_{\varepsilon,k}(b) = \frac{1}{2} \left\| L_\varepsilon(b)(t_k, \cdot) - Y_k \right\|^2_{L^2(\Omega)}, \quad k \in \{2, \dots, K\} \tag{6.5}$$

is continuously Fréchet differentiable in spatial Lipschitz regular vector fields.
In the last section, we improve these results: we show directly under some further restrictions and assumptions that $G$ is Fréchet differentiable in spatial Lipschitz regular vector fields $b$ if the initial value $u_0$ and $Y_k$, $k \in \{2, \dots, K\}$ satisfy the following condition:

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) = 0,$$

where $X$ denotes the unique flow of $b$ and $\mathcal{J}_g$ denotes the jump set of some function $g \in BV(\Omega)$.
Finally, for the case that this condition is not fulfilled, we will prove that $G$ is still one-sided directional differentiable.

## 6.1. Fréchet differentiability of the control-to-state operator $L$

We start this section with several auxiliary lemmas. These lemmas will be helpful in the proof of the main statement and in the identification of the well-posedness of the Fréchet derivative.

**Lemma 6.1.1** *Let $r > 1$ and let $(f_n)$ be a bounded sequence in $L^r((0,T), \mathcal{M}(\Omega))$. Then, there exists a subsequence $(f_{n_k})$ and some $f \in L^r((0,T), \mathcal{M}(\Omega))$ such that*

$$f_{n_k} \stackrel{*}{\rightharpoonup} f \quad \text{in } \mathcal{M}((0,T) \times \Omega).$$

**Proof:** The proof can be found in the appendix.

□

**Lemma 6.1.2** *Let $g \in L^p((0,T), C(\Omega))$ and $h \in L^q((0,T), \mathcal{M}(\Omega))$ with $1 \leq p, q \leq \infty$. Then the product $gh : (0,T) \to \mathcal{M}(\Omega)$ is weak*-measurable. In addition, if $\frac{1}{p} + \frac{1}{q} = 1$, then $gh$ lies in $L^1((0,T), \mathcal{M}(\Omega))$ and if $q = \infty$ and $p$ are arbitrary, then $gh \in L^p((0,T), \mathcal{M}(\Omega))$.*

**Proof:** The proof can be found in the appendix.

□

**Lemma 6.1.3** *Let $1 \leq p < \infty$, $q > N$, $g \in L^p((0,T), W^{1,q}(\Omega)) \cap L^\infty((0,T) \times \Omega)$ and $h \in C([0,T], BV(\Omega) - w^*) \cap L^\infty((0,T) \times \Omega)$. Then the product $gh$ is an element of $L^p((0,T), BV(\Omega))$.*

**Proof:** The proof can be found in the appendix.

□

**Lemma 6.1.4** *Let $p \geq 1$ and $q > N$. Furthermore, let $b \in \mathrm{V}^p$ and denote $u$ the unique solution of* (5.13) *for some initial data $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$. Then, if*

$$(b_n) \subset \left( \mathrm{V}^{p,q} \cap L^\infty((0,T) \times \Omega)^N \right) \setminus \{b\}$$

*is a sequence being convergent to* $b$ *in* $L^p((0,T),C(\Omega))^N$*, we obtain that*

$$\left(\frac{\operatorname{Div}((b_n-b)u)-u\operatorname{div}(b_n-b)\mathcal{L}^N}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}}\right)_n$$

*is bounded in* $L^p((0,T),\mathcal{M}(\Omega))$*.*

**Proof:** As mentioned in equation (5.17), we know that

$$(b_n-b)\cdot\nabla u=\operatorname{Div}((b_n-b)u)-u\operatorname{div}(b_n-b)\mathcal{L}^N$$

and since $u\in C([0,T],BV(\Omega)-w^*)$ we have that $\nabla u\in C([0,T],\mathcal{M}(\Omega)-w^*)^N$. Lemma 6.1.2 then yields that

$$\frac{1}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}}(b_n-b)\cdot\nabla u\in L^p((0,T),\mathcal{M}(\Omega)).$$

Therefore, we estimate:

$$\begin{aligned}
&\frac{1}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}}\left(\int_0^T\|(b_n(t,\cdot)-b(t,\cdot))\cdot\nabla u(t,\cdot)\|^p_{\mathcal{M}(\Omega)}\ dt\right)^{\frac{1}{p}}\\
&\le\frac{1}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}}\left(\int_0^T\|b_n(t,\cdot)-b(t,\cdot))\|^p_{C(\Omega)^N}\|\nabla u(t,\cdot))\|^p_{\mathcal{M}(\Omega)^N}\ dt\right)^{\frac{1}{p}}\\
&\le\frac{1}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}}\|b_n-b\|_{L^p((0,T),C(\Omega))^N}\|\nabla u\|_{L^\infty((0,T),\mathcal{M}(\Omega))^N}\\
&\le\|u\|_{L^\infty((0,T),BV(\Omega))}\,.
\end{aligned}$$

Thus,

$$\frac{\operatorname{Div}((b_n-b)u)-u\operatorname{div}(b_n-b)\mathcal{L}^N}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}}\in L^p((0,T),\mathcal{M}(\Omega))$$

for all $n\in\mathbb{N}$ and represents therein a bounded sequence.

□

**Lemma 6.1.5** *Let* $p>1$*,* $q>N$ *and let* $(b_n)\subset \mathrm{V}^p$ *be a sequence such that*

$$b_n\to b\in \mathrm{V}^p\quad in\ L^p((0,T),C(\Omega))^N$$

*and* $(b_n)$ *is bounded with respect to* $\|\cdot\|_{L^p((0,T),Lip(\Omega))^N}$*. Then, for vector fields* $\tilde{b}\in \mathrm{V}^{p,q}\cap L^\infty((0,T)\times\Omega)^N$*, the sequence*

$$\operatorname{Div}(\tilde{b}u_n)\overset{*}{\rightharpoonup}\operatorname{Div}(\tilde{b}u)\quad in\ \mathcal{M}((0,T)\times\Omega),$$

*where* $u_n$ *and* $u$ *denote the unique solutions in* $C([0,T],BV(\Omega)-w^*)$ *of* (5.13) *with vector fields* $b_n$ *and* $b$*, respectively, and initial value* $u_0\in L^\infty(\Omega)\cap BV(\Omega)$*.*

**Proof:** Since $b_n \to b$ in $L^p((0,T),C(\Omega))^N$ and $(\operatorname{div} b_n)$ is bounded in $L^1((0,T),L^\infty(\Omega))$ due to the boundedness of $(b_n)$ in $L^p((0,T),Lip(\Omega))^N$, Theorem 3.2.9 yields that $u_n \to u$ in $C([0,T],L^2(\Omega))$ and $\|u_n(t,\cdot)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$ for all $n \in \mathbb{N}$ and $t \in [0,T]$. Furthermore, Remark 5.1.16 shows that for all $t \in [0,T]$

$$\|u_n(t,\cdot)\|_{BV(\Omega)} \leq c\,\|u_0\|_{L^\infty(\Omega)} + C_n^{N-1}\,\|u_0\|_{\mathcal{M}(\Omega)} \leq C < \infty$$

for some $C > 0$ since $C_n := e^{\int_0^T \mathbb{L}(b_n(r,\cdot))dr}$ represents a bounded set due to the boundedness of $(b_n)$ in $L^p((0,T),Lip(\Omega))^N$. Then, Proposition 3.13 in [AFP00] yields that

$$u_n(t,\cdot) \overset{*}{\rightharpoonup} u(t,\cdot) \quad \text{in } BV(\Omega)$$

for all $t \in [0,T]$. Thus, if we choose $\tilde{b} \in \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N$ we get that

$$\tilde{b}(t,\cdot)u_n(t,\cdot) \overset{*}{\rightharpoonup} \tilde{b}(t,\cdot)u(t,\cdot)$$

in $BV(\Omega)^N$ for almost all $t \in (0,T)$ since $BV(\Omega) \cap L^\infty(\Omega)$ is an algebra. As a consequence, we have $\operatorname{Div}(\tilde{b}(t,\cdot)u_n(t,\cdot)) \overset{*}{\rightharpoonup} \operatorname{Div}(\tilde{b}(t,\cdot)u(t,\cdot))$ in $\mathcal{M}(\Omega)$ for almost all $t \in (0,T)$. Now, using Lemma 6.1.3, we deduce that the mappings

$$t \mapsto \int_\Omega \varphi(t,x)\ d\operatorname{Div}(\tilde{b}(t,\cdot)u_n(t,\cdot))(x)$$

are measurable for $\varphi \in C_0((0,T)\times\Omega)$ and pointwise limited by

$$\begin{aligned}\left|\int_\Omega \varphi(t,x)\ d\operatorname{Div}(\tilde{b}(t,\cdot)u_n(t,\cdot))(x)\right| &\leq \|\varphi(t,\cdot)\|_{C(\Omega)}\,\|u_n(t,\cdot)\|_{BV(\Omega)\cap L^\infty(\Omega)}\,\left\|\tilde{b}(t,\cdot)\right\|_{BV(\Omega)^N\cap L^\infty(\Omega)^N}\\ &\leq C\,\|\varphi\|_{C((0,T)\times\Omega)}\,\left\|\tilde{b}(t,\cdot)\right\|_{BV(\Omega)^N}\end{aligned}$$

for some $C > 0$. Thus, we use Lebesgue's dominated convergence theorem and obtain that

$$\int_0^T\int_\Omega \varphi(t,x)\ d\operatorname{Div}(\tilde{b}(t,\cdot)u_n(t,\cdot))(x)dt \to \int_0^T\int_\Omega \varphi(t,x)\ d\operatorname{Div}(\tilde{b}(t,\cdot)u(t,\cdot))(x)dt.$$

□

With this lemma our preliminary studies end and we turn towards the main statement of this section. Before we present the main theorem, we introduce a special case of the general inhomogeneous continuity equation. The solution operator of this equation represents the Fréchet derivative of the control-to-state operator $L$ at some vector field $b \in \mathrm{V}^p$.

Let $p > 1$, $q > N$ and $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$. We choose $b \in \mathrm{V}^p$ and denote $u \in C([0,T],BV(\Omega)-w^*)$ the unique weak solution of (5.13) with vector field $b$ and initial value $u_0$. Then, we consider the equation system:

$$\begin{aligned}\partial_t\mu_{\tilde{b}} + \operatorname{div}(b\mu_{\tilde{b}}) - \mu_{\tilde{b}}\operatorname{div} b + \operatorname{div}(\tilde{b}u) - u\operatorname{div}\tilde{b} &= 0 \qquad \text{in } (0,T)\times\Omega,\\ \mu_{\tilde{b}}(0,\cdot) &= 0 \qquad \text{in } \Omega.\end{aligned} \tag{6.6}$$

Remark 5.2.6 shows that there exists a well-defined solution operator

$$\begin{aligned} S_{p,b} : \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N &\to C([0,T],\mathcal{M}(\Omega)-w^*), \\ \tilde{b} &\mapsto \mu_{\tilde{b}}, \end{aligned}$$

where $\mu_{\tilde{b}}$ denotes the unique weak solution of (6.6). Linearity of this operator is given by the equation. Continuity in $L^p((0,T),C(\Omega))^N$ follows from the following argument:
let $(\tilde{b}_n) \subset \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N$ be such that

$$\tilde{b}_n \to \tilde{b} \quad \text{in } L^p((0,T),C(\Omega))^N.$$

Then, the proof of Lemma 6.1.4 yields for $\varphi \in C_0((0,T)\times\Omega)$

$$\begin{aligned} &\int_0^T \langle \mathrm{Div}((\tilde{b}_n(t,\cdot)-\tilde{b}(t,\cdot))u(t,\cdot)) - u(t,\cdot)\,\mathrm{div}(\tilde{b}_n(t,\cdot)-\tilde{b}(t,\cdot))\mathcal{L}^N, \varphi(t,\cdot)\rangle \; dt \\ &= \int_0^T \langle (\tilde{b}_n(t,\cdot)-\tilde{b}(t,\cdot))\cdot\nabla u(t,\cdot), \varphi(t,\cdot)\rangle \; dt \\ &\le \|\varphi\|_{C((0,T)\times\Omega)} \left\|\tilde{b}_n-\tilde{b}\right\|_{L^p((0,T),C(\Omega))^N} \|u\|_{L^\infty(0,T),BV(\Omega))}\,. \end{aligned}$$

Hence,

$$\mathrm{Div}(\tilde{b}_n u) - u\,\mathrm{div}(\tilde{b}_n)\mathcal{L}^N \overset{*}{\rightharpoonup} \mathrm{Div}(\tilde{b}u) - u\,\mathrm{div}(\tilde{b})\mathcal{L}^N \quad \text{in } \mathcal{M}((0,T)\times\Omega).$$

Consequently, Theorem 5.2.10 yields that $\mu_{\tilde{b}_n} \to \mu_{\tilde{b}}$ in $C([0,T],\mathcal{M}(\Omega)-w^*)$. Thus,

$$S_{p,b} \in \mathcal{L}(\mathrm{V}^{p,q}\cap L^\infty((0,T)\times\Omega)^N, C([0,T],\mathcal{M}(\Omega)-w^*))$$

for any $b \in \mathrm{V}^p$. As we will show in the next theorem, $S_{p,b}$ represents the Fréchet derivative of $L$ at the vector field $b$.

**Theorem 6.1.6 (Continuous Fréchet differentiability of control-to-state operator $L$)** *Let $p > 1$, $q > N$ and $u_0 \in L^\infty(\Omega)\cap BV(\Omega)$. Then the control-to-state operator*

$$L : \mathrm{V}^{p,q}\cap L^\infty((0,T)\times\Omega)^N \to C([0,T],\mathcal{M}(\Omega)-w^*), \quad b \mapsto L(b) = S(u_0,b)$$

*is Fréchet-differentiable at vector fields $b \in \mathrm{V}^p$ with respect to convergence of $b$ in $L^p((0,T),C(\Omega))^N$ and* $\mathrm{div}\, b$ *in $L^1((0,T),C(\Omega))$. The derivative is given by*

$$D_b L(b)\tilde{b} = S_{p,b}(\tilde{b}),$$

*where $S_{p,b}(\tilde{b}) \in C([0,T],\mathcal{M}(\Omega)-w^*)$ is the unique solution of* (6.6). *Furthermore, the mapping*

$$\begin{aligned} S_{p,\cdot} : \mathrm{V}^p &\to \mathcal{L}\left(\mathrm{V}^{p,q}\cap L^\infty((0,T)\times\Omega)^N, C([0,T],\mathcal{M}(\Omega)-w^*)\right) \\ b &\mapsto S_{p,b} = D_b L(b) \end{aligned}$$

*is continuous: if $(b_n) \subset \mathrm{V}^p$ such that $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))}\right)$ is bounded,*

$$b_n \to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{and} \quad \mathrm{div}\, b_n \to \mathrm{div}\, b \text{ in } L^1((0,T),C(\Omega)),$$

*then $S_{p,b_n} \overset{*}{\rightharpoonup} S_{p,b}$, i.e.*

$$S_{p,b_n}(\tilde{b}) \to S_{p,b}(\tilde{b}) \quad \text{in } C([0,T],\mathcal{M}-w^*)$$

*for any $\tilde{b} \in \mathrm{V}^{p,q}\cap L^\infty((0,T)\times\Omega)^N$.*

The idea of the proof is based on the idea of the proof of Theorem 5.2.1 in [Ulb01].

**Proof:** Let $b \in \mathrm{V}^p$, $(b_n) \subset \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N \backslash \{b\}$ and let $u \in C([0,T], BV(\Omega)-w^*)$ and $(u_n) \subset C([0,T], L^2(\Omega))$, respectively, be the corresponding weak solutions of the transport equation with initial data $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ such that

$$b_n \to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T),C(\Omega)).$$

We set $\delta b_n := b_n - b$ and $\delta u_n := u_n - u$ as well as

$$\delta w_n := \frac{\delta u_n}{\|\delta b_n\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(\delta b_n)\|_{L^1((0,T),C(\Omega))}}$$

and

$$\delta f_n := \frac{\delta b_n}{\|\delta b_n\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(\delta b_n)\|_{L^1((0,T),C(\Omega))}}.$$

Then, $\delta w_n$ satisfies

$$\begin{aligned} \partial_t \delta w_n + \operatorname{div}(b_n \delta w_n) - \delta w_n \operatorname{div} b_n + \operatorname{div}(\delta f_n u) - u \operatorname{div} \delta f_n &= 0 \qquad \text{in } (0,T)\times\Omega, \\ \delta w_n(0,\cdot) &= 0 \qquad \text{in } \Omega. \end{aligned}$$

In addition, we consider the equations

$$\begin{aligned} \partial_t v_n + \operatorname{div}(b v_n) - v_n \operatorname{div} b + \operatorname{div}(\delta b_n u) - u \operatorname{div} \delta b_n &= 0 \qquad \text{in } (0,T)\times\Omega, \\ v_n(0,\cdot) &= 0 \qquad \text{in } \Omega \end{aligned}$$

with unique solutions $v_n = S_{p,b}(\delta b_n) \in C([0,T], \mathcal{M}(\Omega)-w^*)$ and set

$$z_n := \frac{v_n}{\|\delta b_n\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(\delta b_n)\|_{L^1((0,T),C(\Omega))}}.$$

Obviously, $z_n \in C([0,T], \mathcal{M}(\Omega)-w^*)$ are the unique solutions of

$$\begin{aligned} \partial_t z_n + \operatorname{div}(b z_n) - z_n \operatorname{div} b + \operatorname{div}(\delta f_n u) - u \operatorname{div} \delta f_n &= 0 \qquad \text{in } (0,T)\times\Omega, \\ z_n(0,\cdot) &= 0 \qquad \text{in } \Omega. \end{aligned}$$

Then, we have:

$$\frac{u_n - u - S_{p,b}(\delta b_n)}{\|\delta b_n\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(\delta b_n)\|_{L^1((0,T),C(\Omega))}} = \delta w_n - z_n.$$

We assume that $\delta w_n - z_n \nrightarrow 0$ in $C([0,T], \mathcal{M}(\Omega)-w^*)$ as $n \to \infty$. Then, there exists a subsequence $(\delta w_{n_k} - z_{n_k})$, some $\varphi_0 \in C_0(\Omega)$ and $\varepsilon > 0$ such that

$$\sup_{t\in[0,T]} |\langle \delta w_{n_k}(t,\cdot) - z_{n_k}(t,\cdot), \varphi_0\rangle| \geq \varepsilon$$

for all $k \in \mathbb{N}$. Due to Lemma 6.1.4 and the following estimate

$$\frac{\left\|\operatorname{Div}(\delta b_{n_k} u) - u \operatorname{div}(\delta b_{n_k})\mathcal{L}^N\right\|_{L^p((0,T),\mathcal{M}(\Omega))}}{\|\delta b_{n_k}\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(\delta b_{n_k})\|_{L^1((0,T),C(\Omega))}} \leq \frac{\left\|\operatorname{Div}(\delta b_{n_k} u) - u \operatorname{div}(\delta b_{n_k})\mathcal{L}^N\right\|_{L^p((0,T),\mathcal{M}(\Omega))}}{\|\delta b_{n_k}\|_{L^p((0,T),C(\Omega))^N}}$$

for all $k \in \mathbb{N}$, we have that $\mathrm{Div}(\delta f_{n_k} u) - u \operatorname{div}(\delta f_{n_k})\mathcal{L}^N$ is bounded in $L^p((0,T), \mathcal{M}(\Omega))$ and thus, Lemma 6.1.1 yields that the sequence $(\mathrm{Div}(f_{n_k} u) - u \operatorname{div}(f_{n_k})\mathcal{L}^N)$ contains a subsequence (which is labeled by $k$ again) that converges weakly* to some $f \in L^p((0,T), \mathcal{M}(\Omega))$ in $\mathcal{M}((0,T) \times \Omega)$. Let us denote $z$ the unique solution of (5.18) with vector field $b$, $g = \operatorname{div} b$, inhomogeneous term $f$ and zero initial data. Then, Theorem 5.2.10 yields that $z_{n_k}$ converges to $z$ in $C([0,T], \mathcal{M}(\Omega) - w^*)$. Furthermore, since $u_{n_k} \in C([0,T], L^2(\Omega))$ are unique, they can be approximated in $C([0,T], L^2(\Omega))$ by smooth solutions $u_{n_k,l}$ of the transport equation with smooth vector fields $b_{n_k,l} \subset C^\infty((0,T), C_0^m(\Omega))^N$ with $m \geq 2$ such that

$$b_{n_k,l} \to b_{n_k} \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad \operatorname{div}(b_{n_k,l}) \to \operatorname{div}(b_{n_k}) \quad \text{in } L^1((0,T), C(\Omega)).$$

Hence, the difference $\delta y_l := u_{n_k,l} - u$ solves

$$\begin{aligned} \partial_t \delta y_l + \operatorname{div}(b_{n_k,l}\delta y_l) - \delta y_l \operatorname{div}(b_{n_k,l}) + \operatorname{div}((b_{n_k,l} - b)u) - u \operatorname{div}(b_{n_k,l} - b) &= 0 \qquad \text{in } (0,T) \times \Omega, \\ \delta y_l(0,\cdot) &= 0 \qquad \text{in } \Omega, \end{aligned}$$

i.e. $\delta y_l$ is the unique measure solution of this equation and converges to $u_{n_k} - u$ in $C([0,T], \mathcal{M}(\Omega) - w^*)$ as $l \to \infty$ due to Theorem 5.2.10 since obviously

$$\operatorname{div}((b_{n_k,l} - b)u) - u \operatorname{div}(b_{n_k,l} - b) = (b_{n_k,l} - b) \cdot \nabla u$$

is bounded in $L^p((0,T), \mathcal{M}(\Omega))$ and converges weakly* to $\operatorname{div}((b_{n_k} - b)u) - u \operatorname{div}(b_{n_k} - b)$ in $\mathcal{M}((0,T) \times \Omega)$. Thus, the solution $u_{n_k} - u$ is approximable and therefore

$$\delta w_{n_k} = \frac{\delta u_{n_k}}{\|\delta b_n\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(\delta b_n)\|_{L^1((0,T),C(\Omega))}}$$

is also approximable. Consequently, due to Theorem 5.2.10, $\delta w_{n_k}$ also converges to $z$ in $C([0,T], \mathcal{M}(\Omega) - w^*)$. But this is a contradiction to our assumption since for $\varphi_0$ we have

$$\varepsilon \leq \sup_{t \in [0,T]} |\langle \delta w_{n_k}(t,\cdot) - z_{n_k}(t,\cdot), \varphi_0 \rangle| \to 0 \qquad \text{as } k \to \infty.$$

Thus, our control-to-state operator is Fréchet differentiable with derivative

$$D_b L(b)\tilde{b} = S_{p,b}(\tilde{b}) \quad \text{for } \tilde{b} \in \mathrm{V}^{p,q} \cap L^\infty((0,T) \times \Omega)^N.$$

It remains to show continuity. Let $(b_n) \subset \mathrm{V}^p$ be a sequence such that $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))^N}\right)$ is bounded,

$$b_n \to b \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \text{ in } L^1((0,T), C(\Omega)).$$

Furthermore, let $\tilde{b} \in \mathrm{V}^{p,q} \cap L^\infty((0,T) \times \Omega)^N$. Then, Lemma 6.1.5 yields that

$$\mathrm{Div}(\tilde{b} u_n) \overset{*}{\rightharpoonup} \mathrm{Div}(\tilde{b} u) \quad \text{in } \mathcal{M}((0,T) \times \Omega)$$

and using Theorem 5.2.10, we obtain that

$$S_{p,b_n}(\tilde{b}) \to S_{p,b}(\tilde{b})$$

in $C([0,T], \mathcal{M}(\Omega) - w^*)$. Thus, $(S_{p,b_n})$ is weakly* convergent to $S_{p,b}$ in

$$\mathcal{L}\left(\mathrm{V}^{p,q} \cap L^\infty((0,T) \times \Omega)^N, C([0,T], \mathcal{M}(\Omega) - w^*)\right).$$

□

In the subsequent sections, we are faced with the situation that

$$\sup_{t\in[0,T]} \frac{|\langle L(b_n)(t,\cdot)-L(b)(t,\cdot)-S_{p,b}(b_n-b)(t,\cdot),\varphi\rangle|}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}$$

shall converge to zero as $n\to\infty$ not only for $\varphi\in C_0(\Omega)$ but also for $\varphi\in C(\Omega)$. In general, this is not true but for the assumption that

$$b\in \mathrm{V}_0^p := \left\{\tilde b\in \mathrm{V}^p \middle|\ \operatorname{div}\tilde b\in L^1((0,T),Lip_0(\Omega))\right\}$$

and

$$b_n\in \mathrm{V}_0^{p,q} := \left\{\tilde b\in \mathrm{V}^{p,q} \middle|\ \operatorname{div}\tilde b\in L^1((0,T),C_0(\Omega))\right\}$$

for all $n\in\mathbb{N}$, we deduce the required result from Theorem 6.1.6.

**Corollary 6.1.7** *Let $p>1$, $q>N$ and $u_0\in L^\infty(\Omega)\cap BV(\Omega)$. Then for any $\varphi\in C(\Omega)$*

$$\sup_{t\in[0,T]} \frac{|\langle L(b_n)(t,\cdot)-L(b)(t,\cdot)-S_{p,b}(b_n-b)(t,\cdot),\varphi\rangle|}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}\to 0$$

*as $n\to\infty$, where $(b_n)\subset \mathrm{V}_0^{p,q}\cap L^\infty((0,T)\times\Omega)^N\backslash\{b\}$, $b\in\mathrm{V}_0^p$ and*

$$b_n\to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{as well as} \quad \operatorname{div} b_n\to\operatorname{div} b \quad \text{in } L^1((0,T),C(\Omega)).$$

*Furthermore, if $(b_n)\subset\mathrm{V}_0^p$ such that $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))}\right)$ is bounded,*

$$b_n\to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n\to\operatorname{div} b \text{ in } L^1((0,T),C(\Omega)),$$

*then for any $\varphi\in C(\Omega)$ and $\tilde b\in\mathrm{V}_0^{p,q}\cap L^\infty((0,T)\times\Omega)^N$*

$$\sup_{t\in[0,T]}\left|\langle S_{p,b_n}(\tilde b)(t,\cdot)-S_{p,b}(\tilde b)(t,\cdot),\varphi\rangle\right|\to 0$$

*as $n\to\infty$.*

**Proof:** We take an open, convex and bounded set $\hat\Omega\subset\mathbb{R}^N$ such that $\Omega\subset\hat\Omega$ is a proper subset. Then, we extend $u_0$ and $b$ by zero to the spatial domain $\hat\Omega$ and denote these extensions $\hat u_0$ and $\hat b$. Obviously, $\hat u_0\in BV(\hat\Omega)\cap L^\infty(\hat\Omega)$,

$$\hat b\in L^p((0,T),Lip_0(\hat\Omega))^N\cap L^\infty((0,T)\times\hat\Omega)^N$$

and $\operatorname{div}\hat b$ is given by

$$\operatorname{div}\hat b(t,x)=\begin{cases}\operatorname{div} b & \text{if } (t,x)\in(0,T)\times\Omega,\\ 0 & \text{if } (t,x)\in(0,T)\times\hat\Omega\backslash\Omega\end{cases},$$

which is an element of $L^1((0,T),Lip(\hat\Omega))$. In the same way, we do this for the sequence $(b_n)$. We deduce from Remark 3.1.5 that the solutions $\hat u_n:=\hat L(\hat b_n)$ and $\hat u:=\hat L(\hat b)$ are constant with value zero in $\hat{\Omega}\backslash\overline{\Omega}$, where $\hat{L} = L_{\hat{u}_0}$. Due to Lemma 5.1.5, we know that the unique flow $\hat{X}$ of $\hat{b}$ is constant for any $x \in \hat{\Omega}\backslash\Omega$ and thus we deduce using Corollary 3.89 in [AFP00]

$$\begin{aligned}\nabla\hat{L}(\hat{b})(t,\cdot) &= \nabla(\hat{u}(t,\cdot)) \\ &= \nabla(u(t,\cdot)) + Tu(t,\cdot) \otimes \nu_\Omega \mathcal{H}^{N-1} \llcorner \partial\Omega \\ &= \nabla(u(t,\cdot)) + Tu_0 \otimes \nu_\Omega \mathcal{H}^{N-1} \llcorner \partial\Omega,\end{aligned}$$

where $u := L_{u_0}(b)$, $T$ is the trace operator and $\nu_\Omega$ denotes the outer normal on $\partial\Omega$. Hence, we have that the solution operator $\hat{S}_{p,\hat{b}}$ of (6.6) on the spatio-temporal domain $(0,T)\times\hat{\Omega}$ with vector field $\hat{b}$ and unique solution $\hat{u}$ is given by

$$\begin{aligned}\hat{S}_{p,\hat{b}}(\hat{b}_n - \hat{b})(t,\cdot) &= -\int_0^t e^{\int_s^t \operatorname{div}\hat{b}(\tau,\hat{X}(\tau,t,\cdot))d\tau}(\hat{b}_n - \hat{b})(s,\hat{X}(s,t,\cdot)) \cdot (\nabla\hat{u}(s,\cdot))(\hat{X}(s,t,\cdot))\, ds \\ &= -\int_0^t e^{\int_s^t \operatorname{div} b(\tau,X(\tau,t,\cdot))d\tau}(b_n - b)(s,X(s,t,\cdot)) \cdot (\nabla u(s,\cdot))(X(s,t,\cdot))\, ds\end{aligned} \tag{6.7}$$

for all $t \in [0,T]$, where $X$ denotes the unique flow of $b$, i.e. $\hat{S}_{p,\hat{b}}(\hat{b}_n - \hat{b})(t,\cdot)$ is concentrated on the set $\Omega$. We set

$$\hat{h}_n := \left\|\hat{b}_n - \hat{b}\right\|_{L^p((0,T),C(\hat{\Omega}))^N} + \left\|\operatorname{div}(\hat{b}_n - \hat{b})\right\|_{L^1((0,T),C(\hat{\Omega}))}$$

and

$$h_n := \|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(b_n - b)\|_{L^1((0,T),C(\Omega))}.$$

Then, we apply Theorem 6.1.6 and obtain that for any $\hat{\varphi} \in C_0(\hat{\Omega})$

$$\begin{aligned}0 \leftarrow \sup_{t\in[0,T]} \frac{1}{\hat{h}_n}&\left|\int_{\hat{\Omega}} \hat{\varphi}(x)\left(\hat{L}(\hat{b}_n)(t,x) - \hat{L}(\hat{b})(t,x)\right)\, dx - \int_{\hat{\Omega}} \hat{\varphi}(x)\, d\hat{S}_{p,\hat{b}}(\hat{b}_n - \hat{b})(t,\cdot)(x)\right| \\ = \sup_{t\in[0,T]} \frac{1}{h_n}&\left|\int_\Omega \hat{\varphi}(x)\left(L(b_n)(t,x) - L(b)(t,x)\right)\, dx - \int_\Omega \hat{\varphi}(x)\, dS_{p,b}(b_n - b)(t,\cdot)(x)\right|\end{aligned}$$

as $n \to \infty$. Now, let $\psi \in C_0(\hat{\Omega})$ such that $\psi \equiv 1$ on $\overline{\Omega}$ and $\varphi \in C(\Omega)$. Extending $\varphi$ in a continuous way to $\hat{\Omega}$ yields that $\psi\varphi \in C_0(\hat{\Omega})$. Then, choosing $\hat{\varphi} = \psi\varphi$ proves the first result of the statement. The proof of the second statement works in the same way by extending the involved functions to $\hat{\Omega}$ and using Theorem 6.1.6 as well as the measure representation (6.7). □

## 6.2. Fréchet differentiability of the smoothed tracking term

So far we showed continuous Fréchet differentiability of $L$ at Lipschitz regular vector fields in the spatial domain. In vector fields with less regularity in $\Omega$, this result is not valid in general anymore due to two reasons:

(i) $BV$-regularity of initial data is not propagated in general and thus the inhomogeneity in the measure equation must not be a measure anymore.

(ii) Uniqueness of measure solutions is known in general for vector fields with Lipschitz regularity in space. Thus, only in this case the solution operator $S_{p,b}$ is well-defined.

In this section, our focus rests on differentiability properties of the composition of $L$ with the tracking term of $J$. As mentioned at the beginning of this chapter the regularity of $L$ seen as a mapping with codomain $C([0,T],\mathcal{M}(\Omega)-w^*)$ is too weak for proving Fréchet differentiability of the composition. Thus, we smooth the control-to-state operator $L$: we take the standard mollifier $\rho \in C_c^\infty(B_1(0))$ and define for $p > 1$, $q > N$ and for some $\varepsilon > 0$:

$$L_\varepsilon : \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N \to C([0,T],L^2(\Omega)), \quad b \mapsto L_\varepsilon(b),$$

where

$$(L_\varepsilon(b))(t,\cdot) := L(b)(t,\cdot) * \rho_\varepsilon|_\Omega \quad \text{for all } t \in [0,T].$$

Then, the smoothed tracking term $G_\varepsilon$ introduced in (6.4) is given by

$$G_\varepsilon(b) = \sum_{k=2}^{K} G_{\varepsilon,k}(b) = \frac{1}{2}\sum_{k=2}^{K} \|L_\varepsilon(b)(t_k,\cdot) - Y_k\|_{L^2(\Omega)}^2 . \tag{6.8}$$

In the subsequent parts, we first show continuous Fréchet differentiability of $L_\varepsilon$ at vector fields $b \in \mathrm{V}_0^p$ which then immediately leads to continuous Fréchet differentiability of $G_\varepsilon$. We start with an auxiliary lemma which sums up results about functions in $C([0,T],\mathcal{M}(\Omega)-w^*)$.

**Lemma 6.2.1** *Let $\rho$ be the standard mollifier and $\varepsilon > 0$. Then we have:*

*(i) For $\sigma \in C([0,T],\mathcal{M}(\Omega)-w^*)$ such that*

$$t \mapsto \langle \sigma(t,\cdot),\varphi\rangle \in C([0,T]) \quad \textit{for any } \varphi \in C(\Omega),$$

*the function*

$$(t,x) \mapsto (\sigma(t,\cdot) * \rho_\varepsilon)(x) = \int_\Omega \rho_\varepsilon(x-z)\, d\sigma(t,\cdot)(z), \quad \textit{for } (t,x) \in [0,T]\times\Omega$$

*lies in $C([0,T],L^q(\Omega))$ for any $1 \le q < \infty$.*

*(ii) If $(\sigma_n) \subset C([0,T],\mathcal{M}(\Omega)-w^*)$ such that*

$$\sup_{t\in[0,T]} |\langle \sigma_n(t,\cdot) - \sigma(t,\cdot),\varphi\rangle| \to 0 \quad \textit{as } n \to \infty$$

*for any $\varphi \in C(\Omega)$, where $\sigma \in C([0,T],\mathcal{M}(\Omega)-w^*)$, then*

$$\sigma_n * \rho_\varepsilon|_{[0,T]\times\Omega} \to \sigma * \rho_\varepsilon|_{[0,T]\times\Omega} \quad \textit{in } C([0,T],L^q(\Omega))$$

*for any $1 \le q < \infty$.*

**Proof:** The proof can be found in the appendix.

□

**Theorem 6.2.2 (Fréchet differentiability of $L_\varepsilon$ and $G_\varepsilon$)** *Let $p > 1$, $q > N$ and let $\rho$ be the standard mollifier. Then the following holds for $\varepsilon > 0$:*

*(i) The mapping*

$$L_\varepsilon : \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N \to C([0,T], L^r(\Omega)),$$
$$b \mapsto L(b) * \rho_\varepsilon|_{[0,T]\times\Omega}$$

*is Fréchet differentiable at vector fields $b \in \mathrm{V}_0^p$ with respect to convergence of $b$ in $L^p((0,T), C(\Omega))^N$ and $\operatorname{div} b$ in $L^1((0,T), C(\Omega))$ for any $1 \le r < \infty$. The derivative is given by*

$$D_b L_\varepsilon(b)\tilde{b} = S_{p,b}(\tilde{b}) * \rho_\varepsilon|_{[0,T]\times\Omega}.$$

*for any $\tilde{b} \in \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N$. Furthermore, if $(b_n) \subset \mathrm{V}_0^p$ is a sequence such that $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))^N}\right)$ is bounded,*

$$b_n \to b \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T), C(\Omega)),$$

*then*

$$S_{p,b_n}(\tilde{b}) * \rho_\varepsilon|_{[0,T]\times\Omega} \to S_{p,b}(\tilde{b}) * \rho_\varepsilon|_{[0,T]\times\Omega}$$

*in $C([0,T), L^r(\Omega))$ for any $1 \le r < \infty$ and for all $\tilde{b} \in \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N$.*

*(ii) The mapping $G_\varepsilon$, defined in (6.8) is Fréchet differentiable at vector fields $b \in \mathrm{V}_0^p$ with respect to convergence of $b$ in $L^p((0,T), C(\Omega))^N$ and $\operatorname{div} b$ in $L^1((0,T), C(\Omega))$. The derivative is given by*

$$D_b G_\varepsilon(b)\tilde{b} = \sum_{k=2}^{K} \int_\Omega (L_\varepsilon(b)(t_k,x) - Y_k(x))(S_{p,b}(\tilde{b}) * \rho_\varepsilon)(t_k,x)\, dx$$
$$= \sum_{k=2}^{K} \int_\Omega F_{b,k,\varepsilon}(x)\, dS_{p,b}(\tilde{b})(t_k,\cdot)(x),$$

*where for $x \in \Omega$*

$$F_{b,k,\varepsilon}(x) := (L_\varepsilon(b)(t_k,\cdot) - Y_k) * \rho_\varepsilon(x)$$
$$= \int_\Omega (L_\varepsilon(b)(t_k,z) - Y_k(z))\rho_\varepsilon(x-z)\, dz.$$

*In addition, if $(b_n) \subset \mathrm{V}_0^p$ is a sequence such that $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))^N}\right)$ is bounded,*

$$b_n \to b \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T), C(\Omega)),$$

*then*

$$D_b G_\varepsilon(b_n)\tilde{b} \to D_b G_\varepsilon(b)\tilde{b}$$

*for all $\tilde{b} \in \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N$.*

**Proof:**

(i) Let $b \in \mathrm{V}_0^p$ and $(b_n) \subset \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N \setminus \{b\}$ with

$$b_n \to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T),C(\Omega))$$

as $n \to \infty$. Then Corollary 6.1.7 yields that the control-to-state operator $L$ is Fréchet differentiable in $b$ such that

$$\sup_{t\in[0,T]} \frac{|\langle L(b_n)(t,\cdot) - L(b)(t,\cdot) - S_{p,b}(b_n - b)(t,\cdot), \varphi\rangle|}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(b_n - b)\|_{L^1((0,T),C(\Omega))}} \to 0$$

for any $\varphi \in C(\Omega)$ as $n \to \infty$. Then, we conclude for $r \in [1,\infty)$ using point (ii) in Lemma 6.2.1 that

$$\begin{aligned}
&\frac{\|L_\varepsilon(b_n) - L_\varepsilon(b) - S_{p,b}(b_n - b) * \rho_\varepsilon\|_{C([0,T],L^r(\Omega))}}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(b_n - b)\|_{L^1((0,T),C(\Omega))}} \\
&\qquad = \frac{\|(L(b_n) - L(b) - S_{p,b}(b_n - b)) * \rho_\varepsilon\|_{C([0,T],L^r(\Omega))}}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(b_n - b)\|_{L^1((0,T),C(\Omega))}} \to 0
\end{aligned}$$

as $n \to \infty$. The statement about pointwise continuity of the derivative is obvious since it is the composition of pointwise continuous functions.

(ii) The mapping

$$b \mapsto \frac{1}{2}\sum_{k=2}^{K} \|L_\varepsilon(b)(t_k,\cdot) - Y_k\|_{L^2(\Omega)}^2$$

is obviously Fréchet differentiable as a composition of Fréchet differentiable functions. Hence the chain rule and Fubini yields

$$\begin{aligned}
D_b G_\varepsilon(b)\tilde{b} &= \sum_{k=2}^{K} \int_\Omega (L_\varepsilon(b)(t_k,x) - Y_k(x))(S_{p,b}(\tilde{b}) * \rho_\varepsilon)(t_k,x)\ dx \\
&= \sum_{k=2}^{K} \int_\Omega \int_\Omega (L_\varepsilon(b)(t_k,x) - Y_k(x))\rho_\varepsilon(x-y)\ d(S_{p,b}(\tilde{b})(t_k,\cdot))(y)dx \\
&= \sum_{k=2}^{K} \int_\Omega \int_\Omega (L_\varepsilon(b)(t_k,x) - Y_k(x))\rho_\varepsilon(y-x)\ dx d(S_{p,b}(\tilde{b})(t_k,\cdot))(y) \\
&= \sum_{k=2}^{K} \int_\Omega F_{b,k,\rho_\varepsilon}(y)\ d(S_{p,b}(\tilde{b})(t_k,\cdot))(y).
\end{aligned}$$

Due to the boundedness of $(b_n)$ in $L^p((0,T),Lip_0(\Omega))^N$,

$$b_n \to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T),C(\Omega)),$$

we know that

$$S_{p,b_n}(\tilde{b}) * \rho_\varepsilon|_{[0,T]\times\Omega} \to S_{p,b}(\tilde{b}) * \rho_\varepsilon|_{[0,T]\times\Omega} \quad \text{in } C([0,T],L^2(\Omega))$$

and thus strong convergence of $S_{p,b_n}(\tilde{b})(t_k,\cdot) * \rho_\varepsilon|_\Omega$ to $S_{p,b}(\tilde{b})(t_k,\cdot) * \rho_\varepsilon|_\Omega$ in $L^2(\Omega)$ for any $k = 2,\dots,K$. Therefore, $D_b G_\varepsilon(b_n)\tilde{b} \to D_b G_\varepsilon(b)\tilde{b}$ for any $\tilde{b} \in \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N$.

□

## 6.3. Differentiability properties of the tracking term $G$

In the previous sections, we showed that the composition of a smoothed control-to-state operator with the tracking term of the objective function is continuously Fréchet differentiable. Our next aim is to show differentiability properties of the tracking term in the non-smoothed case. In this case, Fréchet differentiability of $G$ does not hold in general anymore. One main reason for this is that $BV$-functions in general have jumps sets which can lead to possible different limits of the function values when a point on the jump set is approached by different directions. Under some specific condition, this case appears in the attempt to prove Fréchet differentiability since different sequences of vector fields lead to different sequences of flows and thus to different directions in approaching points on jump sets. In this case, we will only be able to show one-sided directional differentiability of $G$ with a special defined derivative for each side and direction. It will be shown that this situation appears when the jump set of the initial value $u_0$ and the union of the jump sets of $Y_k(X(t_k, 0, \cdot))$ are not disjoint on a set with positive $\mathcal{H}^{N-1}$-measure value, where $k \in \{2, \dots, K\}$ and $X$ is the unique flow of some vector field $b \in \mathrm{V}_0^p$. If these sets are disjoint except for a $\mathcal{H}^{N-1}$ null-set, we will be able to show Fréchet differentiability of $G$. For the proof of both statements we need some further refinements on the set of initial values $u_0$ for our control-to-state operator $L$: our first additional assumption is the demand that

$$u_0 \in SBV(\Omega) \cap L^\infty(\Omega).$$

Then, the derivative of $u_0$ with respect to the spatial variable only consists of the absolute continuous part (with respect to the Lebesgue measure $\mathcal{L}^N$) and the jump part. Furthermore, we require continuity of $u_0$ in the parts of $\Omega$ where it is approximately continuous and for $\mathcal{H}^{N-1}$-almost all $x$ in the jump set $\mathcal{J}_{u_0}$ of $u_0$ we demand the existence of a ball $B_{r_x}(x)$ such that

$$B_{r_x}(x) \backslash J_{u_0} = \tilde{B}^+_{r_x}(x) \cup \tilde{B}^-_{r_x}(x),$$

where each of the subsets $\tilde{B}^+_{r_x}(x)$ and $\tilde{B}^-_{r_x}(x)$ are connected sets and $u_0$ constrained to these sets represents continuous functions. We require the second and third assumptions also for the functions $Y_k$, $k \in \{2, \dots, K\}$. Before we introduce precise definitions for the set of initial values and for the set of functions $Y_k$, $k \in \{2, \dots, K\}$, we first have some further considerations and introduce the set of discontinuity points $\mathcal{D}_f$ as well as the set of jump points $\mathcal{J}_f$ for some function $f \in BV(\Omega)$.

### 6.3.1. Products of $BV$-regular functions

Considering the derivative of the smoothed composition $G_\varepsilon(b) = \sum_{k=2}^K G_{\varepsilon,k}(b)$ at some vector field $b \in \mathrm{V}_0^p$ with $p > 1$, we observe that for each $k = 2, \dots, K$, it is the product of some smoothed $BV(\Omega)$-function $(L_\varepsilon(b)(t_k, \cdot) - Y_k) * \rho_\varepsilon$ with the derivative of some $BV(\Omega)$-function:

$$S_{p,b}(\tilde{b})(t_k, \cdot) = -\int_0^{t_k} \tilde{b}(s, X(s, t_k, \cdot) e^{\int_s^{t_k} (\operatorname{div} b)(\tau, X(\tau, t_k, \cdot)) d\tau} (\nabla L(b)(s, \cdot))(X(s, t_k, \cdot))\; ds.$$

In the limiting case the product would consist of some $BV(\Omega)$-function and a measure descending from $BV(\Omega)$-functions. We now show that this kind of product is well-defined. Therefore, we present some theory about $BV$-functions as it can be found in [AFP00].

A function $f \in L^1(\Omega)$ has an approximate limit at $x \in \Omega$ if there exists some $z_x \in \mathbb{R}$ such that

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - z_x| \; dy = 0.$$

The set $\mathcal{D}_f \subset \Omega$ where this property is not satisfied is called approximate discontinuity set and for each $x \in \Omega \backslash \mathcal{D}_f$ the uniquely determined vector $z_x \in \mathbb{R}$ is called approximate limit. We set $f^*(x) := z_x$. Then, Proposition 3.64 in [AFP00] yields:

(i) $\mathcal{D}_f$ is a $\mathcal{L}^N$-negligible Borel set and $f^* : \Omega \backslash \mathcal{D}_f \to \mathbb{R}$ is a Borel function, coinciding $\mathcal{L}^N$-almost everywhere in $\Omega \backslash D_f$ with $f$.

(ii) If $x \in \Omega \backslash \mathcal{D}_f$, then $f * \rho_\varepsilon(x)$ converges to $f^*(x)$ as $\varepsilon \to 0$.

Furthermore, $x \in \Omega$ is an approximate jump point of $f$ if there exist $w_x, v_x \in \mathbb{R}$ and $\nu_x \in \mathbb{S}^{N-1}$ such that $w_x \neq v_x$ and

$$\begin{aligned} \lim_{r \to 0} \frac{1}{\left|B_r^+(x, \nu_x)\right|} \int_{B_r^+(x,\nu)} |f(y) - w_x| \; dy = 0, \\ \lim_{r \to 0} \frac{1}{\left|B_r^-(x, \nu_x)\right|} \int_{B_r^-(x,\nu)} |f(y) - v_x| \; dy = 0 \end{aligned} \tag{6.9}$$

hold. The sets $B_r^+(x, \nu_x)$ and $B_r^-(x, \nu_x)$ are defined as

$$\begin{aligned} B_r^+(x, \nu_x) &:= \{ y \in B_r(x) | \; (y - x, \nu_x) > 0 \}, \\ B_r^-(x, \nu_x) &:= \{ y \in B_r(x) | \; (y - x, \nu_x) < 0 \}. \end{aligned}$$

The triplet $(w_x, v_x, \nu_x)$ is unique up to a switch of $(w_x, v_x)$ and a change of sign of $\nu_x$. We set $f^+(x) := w_x$ and $f^-(x) := v_x$ and denote the set of approximate jump points by $\mathcal{J}_f$. For $\mathcal{J}_f$, Proposition 3.69 in [AFP00] yields:

(i) The set $\mathcal{J}_f$ is a Borel subset of $\mathcal{D}_f$ and there exist Borel functions

$$(f^+, f^-, \nu_x) : \mathcal{J}_f \to \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{N-1}$$

such that (6.9) is satisfied at any $x \in \mathcal{J}_f$.

(ii) If $x \in \mathcal{J}_f$, then $f * \rho_\varepsilon(x)$ converges to $(f^+(x) + f^-(x))/2$ as $\varepsilon \to 0$.

Now, Theorem 3.78 in [AFP00] yields that for $f \in BV(\Omega)$ the set $\mathcal{D}_f \backslash \mathcal{J}_f$ is negligible with respect to $\mathcal{H}^{N-1}$, i.e. $\mathcal{H}^{N-1}(\mathcal{D}_f \backslash \mathcal{J}_f) = 0$. Summing up, we obtain for a function $f \in BV(\Omega)$ that

$$f * \rho_\varepsilon(x) \to f^*(x) \quad \text{for all } x \in \Omega \backslash (\mathcal{D}_f \backslash \mathcal{J}_f) \quad \text{as } \varepsilon \to 0,$$

where we define

$$f^* : \Omega \backslash (\mathcal{D}_f \backslash \mathcal{J}_f) \to \mathbb{R}, \quad x \mapsto f^*(x) := \begin{cases} f^*(x) & \text{if } x \in \Omega \backslash D_f \\ (f^+(x) + f^-(x))/2 & \text{if } x \in J_f \end{cases}$$

and $\mathcal{H}^{N-1}(\mathcal{D}_f \backslash \mathcal{J}_f) = 0$. Furthermore, Lemma 3.76 in [AFP00] tells us that for $f \in BV(\Omega)$ and any Borel set $A \subset \Omega$ the following implication holds:

$$\mathcal{H}^{N-1}(A) = 0 \Rightarrow |Df|(A) = 0.$$

Consequently, for any two functions $g, h \in BV(\Omega)$ and $i \in \{1, \dots, N\}$ we obtain that $g^*$ is defined for $D_{x_i}h$-almost all $x \in \Omega$ since $\mathcal{H}^{N-1}(\mathcal{D}_g \backslash \mathcal{J}_g) = 0$ and hence the set is negligible with respect to $|D_{x_i}h|$ (and thus also with respect to $D_{x_i}h$). Now, let $g$ be additionally in $L^\infty(\Omega)$. Then, for $x \in \Omega$

$$|g * \rho_\varepsilon(x)| \leq \int\limits_\Omega |g(z)| \, |\rho_\varepsilon(x - z)| \; dz \leq \|g\|_{L^\infty(\Omega)}$$

and thus $|g^*(x)| \leq \|g\|_{L^\infty(\Omega)}$ for all $x \in \Omega \backslash (\mathcal{D}_g \backslash \mathcal{J}_g)$. Therefore, we obtain for $i = 1, \dots, N$

$$\int\limits_\Omega |g^*(x)| \; d\,|D_{x_i}h|\,(x) \leq \|g\|_{L^\infty(\Omega)} \, |D_{x_i}h|\,(\Omega) < \infty$$

and using Lebesgue's dominated convergence theorem yields that

$$\int\limits_\Omega |g * \rho_\varepsilon(x) - g^*(x)| \; d\,|D_{x_i}h|\,(x) \to 0 \quad \text{as } \varepsilon \to 0.$$

We summarize the above results in the following lemma.

**Lemma 6.3.1** *Let $g \in BV(\Omega) \cap L^\infty(\Omega)$ and let $h \in BV(\Omega)$. Then, $g^*$ is integrable with respect to $|D_{x_i}h|$ for all $i \in \{1, \dots, N\}$ and*

$$\int\limits_\Omega |g * \rho_\varepsilon(x) - g^*(x)| \; d\,|D_{x_i}h|\,(x) \to 0 \quad \textit{as } \varepsilon \to 0.$$

### 6.3.2. Sets for the initial value $u_0$ and functions $Y_k$

In the introductive text of this section we announced that we need some more refinements on the assumptions of the initial values $u_0$ for the control-to-state operator as well as on the functions $Y_k,\ k \in \{2, \dots, K\}$ to show the various differentiability results for $G$. The previous subsection yields the necessary tools to give precise definitions of the assumptions and to motivate them. We start with the introduction of the assumptions.

**Definition 6.3.2** *The set $\mathcal{Y} \subset BV(\Omega) \cap L^\infty(\Omega)$ is the set of functions $Y \in \mathcal{Y}$ satisfying*

*(i) $Y$ is continuous in an open subset $\tilde{\Omega} \subset \Omega \backslash \mathcal{D}_Y$ such that*

$$\mathcal{H}^{N-1}((\Omega \backslash \mathcal{D}_Y) \backslash \tilde{\Omega}) = 0,$$

*(ii) for $\mathcal{H}^{N-1}$-almost all $x \in \mathcal{J}_Y$, there exists a ball $B_{r_x}(x)$ such that*

$$B_{r_x}(x) \backslash \mathcal{J}_Y = \tilde{B}^+_{r_x}(x) \cup \tilde{B}^-_{r_x}(x),$$

*where each of the sets $\tilde{B}^+_{r_x}(x)$ and $\tilde{B}^-_{r_x}(x)$ represents a connected set and $Y$ is continuous in each of the sets.*

*We choose the notation for the sets $\tilde{B}^+_{r_x}(x)$ and $\tilde{B}^-_{r_x}(x)$ such that $\nu_x$ points into the set $\tilde{B}^+_{r_x}(x)$ (and correspondingly $-\nu_x$ points into $\tilde{B}^-_{r_x}(x)$). Furthermore, we define the set of initial values as $\mathcal{U}_0 := SBV(\Omega) \cap \mathcal{Y}$.*

At first glance, it may be not clear why the set of functions $\mathcal{Y}$ is chosen according to Definition 6.3.2. We illustrate reasons for the choice of the properties (i) and (ii) in the following examples: our examples are of the general form $g \in BV(\Omega)$ with $\Omega := (-5,5) \times (5,5) \subset \mathbb{R}^2$ given by

$$g(x) = g_1 \chi_{B_1((0,0))}(x) + g_2 \chi_{B_1((0,2))}(x),$$

where $g_1, g_2 \in \mathbb{R}$ are fixed values.

In the first example, we choose $g_1 = g_2 \neq 0$ and obtain that

$$\mathcal{D}_g = \mathcal{J}_g = (\partial B_1((0,0)) \cup \partial B_1((0,2))) \setminus \{(0,1)\}$$

as well as the continuity of $g$ in $\Omega \setminus (\mathcal{J}_g \cup \{(0,1)\})$. The function $g$ is not continuous in the point $(0,1)$: for $(x_n) \subset \{(t,1) |\ t \in \mathbb{R}^+\} \cap \Omega$ converging to $(0,1)$ we have that

$$\lim_{n\to\infty} g(x_n) = 0$$

but for $(y_n) \subset \{(0,t) |\ t \in \mathbb{R}^+\} \cap \Omega$ being convergent to $(0,1)$, we obtain that

$$\lim_{n\to\infty} g(y_n) = g_1 = g_2 \neq 0 = \lim_{n\to\infty} g(x_n).$$

Thus, the point $(0,1)$ plays a special role in this example: it belongs to the set of approximately continuous points, but $g$ is not continuous in this point although it is continuous in every other approximately continuous point. In our proof for Fréchet differentiability of $G$ we need two distinguish between the jump points and the approximately continuous points of $Y_k$, $k \in \{2, \dots, K\}$: at approximately continuous points $x$ of $Y_k$ we need that $Y_k$ is continuous since arbitrary sequences of vector fields lead to the situation that the corresponding sequences of flows approaches $x$ by arbitrary directions. This requirement is obviously not satisfied in $(0,1)$ in our first example. As mentioned at the beginning of the previous subsection, this situation will appear in an integral term with the derivative of some $BV$ function as the integral measure. Thus, continuity is needed in all approximately continuous points of $Y_k$ except for a $\mathcal{H}^{N-1}$ null-set. Therefore, the first example would satisfies this requirement. This examples illustrates the need of the first demand in Definition 6.3.2.
Now, in the second example, we choose $g_1 = 1$ and $g_2 = 1/2$. Then,

$$\mathcal{D}_g = \mathcal{J}_g = (\partial B_1((0,0)) \cup \partial B_1((0,2)))$$

and $g$ is continuous in $\Omega \setminus \mathcal{J}_g$. As in the first example, we are interested in the point $(0,1)$ which again plays a special role in this example: in contrast to all other points of the jump set, $g^*|_{\mathcal{J}_g}$ is not continuous in $(0,1)$: let $(x_n) \subset \partial B_1((0,0)) \setminus (0,1)$ and $(y_n) \subset \partial B_1((0,2)) \setminus (0,1)$ be two sequences being convergent to $(0,1)$. Then, we have

$$\lim_{n\to\infty} g^*|_{\mathcal{J}_g}(x_n) = \frac{1}{2} \neq \frac{3}{4} = g^*|_{\mathcal{J}_g}((0,1)) \neq \frac{1}{4} = \lim_{n\to\infty} g^*|_{\mathcal{J}_g}(y_n).$$

In the proof of one-sided directional differentiability of $G$, the derivative contains specifically defined functions. These functions are the pointwise limits of function values $Y_k^*(x_n)$, $k \in$

$\{2, \dots, K\}$ of point sequences $(x_n) \subset \Omega$ approaching points $x$ in $\Omega$ on continuous curves given by specific flows. Therefore, if a jump point $x \in \mathcal{J}_{Y_k}$ is approached by a sequence $(x_n)$ whose elements lie in the jump set $\mathcal{J}_{Y_k}$ except for finitely many elements, then $Y_k^*|_{\mathcal{J}_{Y_k}}$ must be continuous in a neighborhood $U \subset \mathcal{J}_{Y_k}$ of $x$ to be able to define a meaningful limit for the sequence $(Y_k^*(x_n))$. This requirement is not satisfied in the point $(0, 1)$ in our second example. The reason lies in the fact that two different parts of the jump set touch at the point $(0, 1)$. Therefore, since $g^*$ is defined in $x \in \mathcal{J}_g$ as the arithmetic mean $g^*(x) = (g^+(x) + g^-(x))/2$ of the „left“ and „right“ approximate jump points $g^-(x)$ and $g^+(x)$, respectively, $g^*$ cannot be continuous in a neighborhood $U \subset \mathcal{J}_g$ of some $x \in \mathcal{J}_g$ if the „left“ or „right“ approximate jump function $g^-$ or $g^+$ itself is not continuous in a neighborhood $U \subset \mathcal{J}_g$ of $x \in \mathcal{J}_g$. In our example, we have

$$g^+ \equiv 0 \quad \text{in } \partial B_1((0,0))\backslash\{(0,1)\}, \quad g^- \equiv 1 \quad \text{in } \partial B_1((0,0))\backslash\{(0,1)\},$$

but

$$g^+((0,1)) = \frac{1}{2} \quad \text{and} \quad g^-((0,1)) = 1.$$

Thus, $g^+$ is not continuous in a neighborhood $U \subset \mathcal{J}_g$ of $(0, 1)$. A requirement preventing this situation is given in (ii) in Definition 6.3.2: the function $Y$ is continuous in $\mathcal{H}^{N-1}$-almost all points of the sets $\tilde{B}^+_{r_x}(x)$ and $\tilde{B}^-_{r_x}(x)$ due to (i) and thus, we obtain continuous „left“ and „right“ approximate jump functions $Y^-$ and $Y^+$ in a neighborhood $U \subset \mathcal{J}_Y$ of $x \in \mathcal{J}_Y$. Again, this requirement must only be demanded for $\mathcal{H}^{N-1}$-almost all $x \in \Omega$ since the limit functions appear in an integral term with the derivative of some $BV$ function as the integral measure. Thus, this second example would also satisfy the requirements of Definition 6.3.2.

We are now prepared for the first main result of the section shown in the successive subsection: the Fréchet differentiability of $G$. This result, however, is only valid if the jump sets of $u_0$ and $Y_k(X(t_k, 0, \cdot))$, $k \in \{2, \dots, K\}$ for a certain flow $X$ satisfy some condition.

### 6.3.3. Frechét differentiability of $G$

In the previous section 6.2, we have shown that $G_\varepsilon$ is continuously Fréchet differentiable. In this subsection, we want to prove that the result is also valid for the non-smoothed mapping

$$G : \mathrm{V}_0^p \to \mathbb{R}, \quad b \mapsto G(b)$$

if the initial value $u_0 \in \mathcal{U}_0$ and $Y_k \in \mathcal{Y}$, $k \in \{2, \dots, K\}$ fulfill the following condition:

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) = 0, \tag{6.10}$$

where $X$ denotes the unique flow of the vector field $b \in \mathrm{V}_0^p$ with $p > 1$ at which we want to prove Fréchet differentiability of $G$. If this condition is not satisfied, i.e.

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) > 0, \tag{6.11}$$

we will only be able to show one-sided directional differentiability of $G$. However, this result needs some more preparations and thus will be shown in the successive subsections. Therefore, we only consider in this subsection the case that condition (6.10) is fulfilled. For this case, we define for $b, \tilde{b} \in \mathrm{V}_0^p$ with $p > 1$ and $k \in \{2, \ldots, K\}$

$$G'_{b,k}(\tilde{b}) := \frac{1}{2} \left\langle \mathbb{1}, \hat{S}_{p,b}(\tilde{b})(t_k, \cdot) \right\rangle - \left\langle Y_k^*, S_{p,b}(\tilde{b})(t_k, \cdot) \right\rangle , \tag{6.12}$$

where $\hat{S}_{p,b}$ denotes the Fréchet derivative of the control-to-state operator $L_{u_0^2}$ and $\mathbb{1}$ the constant function in $\Omega$ with value 1. Obviously, the term (6.12) represents a linear bounded mapping from $\mathrm{V}_0^p$ into $\mathbb{R}$. Before we present the main statement of this subsection we first regard an auxiliary lemma.

**Lemma 6.3.3** *Let $p > 1$, $g \in BV(\Omega) \cap L^\infty(\Omega)$ and $(b_n) \subset \mathrm{V}_0^p$ be a sequence such that*

$$b_n \to b \quad \text{in } L^p((0,T), C(\Omega))^N, \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T), C(\Omega))$$

*and $\sup_{n\in\mathbb{N}} \int_0^T \mathbb{L}(b_n(t,\cdot))\, dt < \infty$. Then, for any $t \in [0,T]$*

$$\begin{aligned}\langle g, L(b_n)(t,\cdot) - L(b)(t,\cdot)\rangle \\ = -\Big\langle g^*, \int_0^1 \int_0^t & e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau, X_{r,n}(\tau,t,\cdot))d\tau} (b_n - b)(s, X_{r,n}(s,t,\cdot)) \\ & \cdot\nabla(L(b + r(b_n - b))(s,\cdot))(X_{r,n}(s,t,\cdot))\, dsdr\Big\rangle ,\end{aligned}$$

*where $X_{r,n}$ denotes the unique flow of the vector field $b + r(b_n - b)$.*

**Proof:** We mollify $g$ and obtain that

$$r \mapsto \langle g * \rho_\varepsilon, L(b + r(b_n - b))(t,\cdot)\rangle$$

is continuously Fréchet differentiable due to Corollary 6.1.7. Thus, we obtain

$$\begin{aligned}&\langle g * \rho_\varepsilon, L(b_n)(t,\cdot) - L(b)(t,\cdot)\rangle \\ &= -\Big\langle g * \rho_\varepsilon, \int_0^1 \int_0^t e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau, X_{r,n}(\tau,t,\cdot))d\tau} (b_n - b)(s, X_{r,n}(s,t,\cdot)) \\ &\qquad \cdot\nabla(L(b + r(b_n - b))(s,\cdot))(X_{r,n}(s,t,\cdot))\, dsdr\Big\rangle , \\ &= -\int_0^1 \int_0^t \Big\langle g * \rho_\varepsilon, e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau, X_{r,n}(\tau,t,\cdot))d\tau} (b_n - b)(s, X_{r,n}(s,t,\cdot)) \\ &\qquad \cdot\nabla(L(b + r(b_n - b))(s,\cdot))(X_{r,n}(s,t,\cdot))\Big\rangle\, dsdr.\end{aligned}$$

Since $g * \rho_\varepsilon(x) \to g^*(x)$ and $|g * \rho_\varepsilon(x)| \leq \|g\|_{L^\infty(\Omega)}$ for $\mathcal{H}^{N-1}$-almost all $x \in \Omega$, Lebesgue's dominated convergence theorem yields that

$$\begin{aligned}&\Big\langle g * \rho_\varepsilon, e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau, X_{r,n}(\tau,t,\cdot))d\tau} (b_n - b)(s, X_{r,n}(s,t,\cdot)) \\ &\quad \cdot\nabla(L(b + r(b_n - b))(s,\cdot))(X_{r,n}(s,t,\cdot))\Big\rangle \\ &\quad \to \Big\langle g^*, e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau, X_{r,n}(\tau,t,\cdot))d\tau} (b_n - b)(s, X_{r,n}(s,t,\cdot))\end{aligned}$$

$$\cdot\nabla(L(b+r(b_n-b))(s,\cdot))(X_{r,n}(s,t,\cdot))\rangle$$

for almost all $s \in (0,t)$ as $\varepsilon \to 0$. As $\nabla L(b+r(b_n-b)) \in C([0,T],\mathcal{M}(\Omega)-w^*)^N$ and $e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau,\cdot)d\tau} \in C(\Omega)$ for any $s,t \in (0,T)$, Lebesgue's dominated convergence theorem again yields that

$$\int_0^1\int_0^t \left\langle g*\rho_\varepsilon, e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau,X_{r,n}(\tau,t,\cdot))d\tau}(b_n-b)(s,X_{r,n}(s,t,\cdot))\right.$$
$$\left.\cdot\nabla(L(b+r(b_n-b))(s,\cdot))(X_{r,n}(s,t,\cdot))\right\rangle\ dsdr$$
$$\to \int_0^1\int_0^t \left\langle g^*, e^{\int_s^t \operatorname{div}((b+r(b_n-b))(\tau,X_{r,n}(\tau,t,\cdot))d\tau}(b_n-b)(s,X_{r,n}(s,t,\cdot))\right.$$
$$\left.\cdot\nabla(L(b+r(b_n-b))(s,\cdot))(X_{r,n}(s,t,\cdot))\right\rangle\ dsdr.$$

Since

$$\langle g*\rho_\varepsilon, L(b_n)(t,\cdot)-L(b)(t,\cdot)\rangle \to \langle g^*, L(b_n)(t,\cdot)-L(b)(t,\cdot)\rangle = \langle g, L(b_n)(t,\cdot)-L(b)(t,\cdot)\rangle$$

holds, the statement is proven.

□

We are now prepared to prove continuous Fréchet differentiability of $G$ in the case that condition (6.10) is satisfied.

**Theorem 6.3.4 (Fréchet differentiability of $G$)** *Let $p>1$, $\tilde b \in \mathrm{V}_0^p$ and $u_0 \in \mathcal{U}_0$ as well as $Y_k \in \mathcal{Y}$ for $k \in \{2,\dots,K\}$. Then, the mapping*

$$G : \mathrm{V}_0^p \to \mathbb{R},$$
$$b \mapsto G(b) = \sum_{k=2}^K \|L(b)(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}$$

*is Fréchet differentiable at the vector field $b \in \mathrm{V}_0^p$ with respect to convergence of $b$ in $L^p((0,T),C(\Omega))^N$, of $\operatorname{div} b$ in $L^1((0,T),C(\Omega))$ and boundedness in $L^p((0,T),Lip_0(\Omega))^N$ if*

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^K \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \textit{with } \mathcal{H}^{N-1}(A)=0$$

*holds, where $X$ denotes the unique flow of $b$. The derivative is given by*

$$DG(b)\tilde b = \sum_{k=2}^K G'_{b,k}(\tilde b) = \sum_{k=2}^K \left[\frac{1}{2}\left\langle \mathbb{1}, \hat S_{p,b}(\tilde b)(t_k,\cdot)\right\rangle - \left\langle Y_k^*, S_{p,b}(\tilde b)(t_k,\cdot)\right\rangle\right],$$

*where $\hat S_{p,b}$ denotes the Fréchet derivative of the control-to-state operator $L_{u_0^2}$. Moreover, if $(b_n) \subset \mathrm{V}_0^p$ is a sequence such that $G$ is Fréchet differentiable in $b_n$ for all $n \in \mathbb{N}$,*

$$b_n \to b \quad \textit{in } L^p((0,T),C(\Omega))^N, \quad \operatorname{div} b_n \to \operatorname{div} b \quad \textit{in } L^1((0,T),C(\Omega))$$

*and $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))^N}\right)$ is bounded, then*

$$DG(b_n)\tilde b \to DG(b)\tilde b$$

*for all $\tilde b \in \mathrm{V}_0^p$.*

**Proof:** We take a sequence $(b_n) \subset \mathrm{V}_0^p$ such that $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))^N}\right)$ is bounded and

$$b_n \to b \quad \text{in } L^p((0,T),C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T),C(\Omega)).$$

We prove the statement for each $k \in \{2,\dots,K\}$. So, let $k \in \{2,\dots,K\}$. Then, we estimate for the difference quotient

$$\begin{aligned}
&2\frac{G_k(b_n)-G(b)}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}\\
&\quad=\frac{\langle L(b_n)(t_k,\cdot)-Y_k, L(b_n)(t_k,\cdot)-Y_k\rangle-\langle L(b)(t_k,\cdot)-Y_k, L(b)(t_k,\cdot)-Y_k\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}\\
&\quad=\frac{\langle L(b_n)(t_k,\cdot)+L(b)(t_k,\cdot)-2Y_k, L(b_n)(t_k,\cdot)-L(b)(t_k,\cdot)\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}. \qquad (6.13)
\end{aligned}$$

We split term (6.13) into two terms:

$$\begin{aligned}
&\frac{\langle L(b_n)(t_k,\cdot)+L(b)(t_k,\cdot)-2Y_k, L(b_n)(t_k,\cdot)-L(b)(t_k,\cdot)\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}\\
&\quad=\frac{\langle L(b_n)(t_k,\cdot)+L(b)(t_k,\cdot), L(b_n)(t_k,\cdot)-L(b)(t_k,\cdot)\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}} \qquad (6.14)\\
&\quad-\frac{\langle 2Y_k, L(b_n)(t_k,\cdot)-L(b)(t_k,\cdot)\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}. \qquad (6.15)
\end{aligned}$$

We first consider term (6.14) and obtain

$$\begin{aligned}
&\frac{\langle L(b_n)(t_k,\cdot)+L(b)(t_k,\cdot), L(b_n)(t_k,\cdot)-L(b)(t_k,\cdot)\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}\\
&\quad=\frac{\left\langle (L(b_n)(t_k,\cdot))^2-(L(b)(t_k,\cdot))^2, \mathbb{1}\right\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}\\
&\quad=\frac{\left\langle \mathbb{1}, L_{u_0^2}(b_n)(t_k,\cdot)-L_{u_0^2}(b)(t_k,\cdot)\right\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}, \qquad (6.16)
\end{aligned}$$

where $\mathbb{1}$ denotes the constant one function in $\Omega$. Then, Corollary 6.1.7 yields that

$$\frac{\left|\left\langle \mathbb{1}, L_{u_0^2}(b_n)(t_k,\cdot)-L_{u_0^2}(b)(t_k,\cdot)-\hat{S}_{p,b}(b_n-b)(t_k,\cdot)\right\rangle\right|}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}} \to 0 \qquad (6.17)$$

as $n \to \infty$. Here, $\left\langle \mathbb{1}, \hat{S}_{p,b}(b_n-b)(t_k,\cdot)\right\rangle$ is given by

$$\begin{aligned}
\left\langle \mathbb{1}, \hat{S}_{p,b}(b_n-b)(t_k,\cdot)\right\rangle = -\int_0^{t_k} &\Big\langle (b_n-b)(s,X(s,t_k,\cdot))e^{\int_s^{t_k}\operatorname{div}(b)(\tau,X(\tau,t_k,\cdot))d\tau}\\
&\left(\nabla(u_0^2(X(0,s,\cdot)))\right)(X(s,t_k,\cdot))\Big\rangle\, ds.
\end{aligned}$$

Now for term (6.15) we consider the splitting

$$-\frac{\langle 2Y_k, L(b_n)(t_k,\cdot)-L(b)(t_k,\cdot)\rangle}{\|b_n-b\|_{L^p((0,T),C(\Omega))^N}+\|\operatorname{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}$$

$$= -\frac{2\,\langle Y_k - Y_k * \rho_\varepsilon, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\mathrm{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}} \tag{6.18}$$

$$- \frac{2\,\langle Y_k * \rho_\varepsilon, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\mathrm{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}}. \tag{6.19}$$

For term (6.19) we know that for fixed $\varepsilon > 0$

$$\frac{2\,|\langle Y_k * \rho_\varepsilon, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot) - S_{p,b}(b_n - b)(t_k,\cdot)\rangle|}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\mathrm{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}} \to 0 \tag{6.20}$$

as $n \to \infty$. Setting

$$h_n = \frac{2}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\mathrm{div}(b_n-b)\|_{L^1((0,T),C(\Omega))}},$$

we estimate for term (6.18) using Lemma 6.3.3

$$\begin{aligned}
&h_n\,|\langle Y_k - Y_k * \rho_\varepsilon, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot)\rangle| \\
&\quad\le h_n \int_0^1 \int_0^{t_k} \Big\langle |Y_k^* - Y_k * \rho_\varepsilon| \left|(b_n - b)(s, X_{r,n}(s,t_k,\cdot)) e^{\int_s^{t_k} \mathrm{div}(b + r(b_n - b))(\tau, X_{r,n}(\tau,t_k,\cdot))d\tau}\right|, \\
&\qquad\qquad |\nabla(u_0(X_{r,n}(0,s,\cdot)))|\,(X_{r,n}(s,t_k,\cdot))\Big\rangle\; dsdr \\
&\quad\le C h_n \,\|b_n - b\|_{L^p((0,T),C(\Omega))^N} \\
&\qquad \int_0^1 \left( \int_0^{t_k} \langle |Y_k^* - Y_k * \rho_\varepsilon|\,, |\nabla(u_0(X_{r,n}(0,s,\cdot)))|\,(X_{r,n}(s,t_k,\cdot))\rangle^{p'}\; ds \right)^{1/p'} dr \\
&\quad\le \tilde{C} \int_0^1 \left( \int_0^{t_k} \langle |Y_k^*(X_{r,n}(t_k,0,\cdot)) - Y_k^*(X(t_k,0,\cdot))|\,, |\nabla u_0|\rangle^{p'}\; ds \right)^{1/p'} dr \qquad (6.21) \\
&\quad + \tilde{C} \int_0^1 \left( \int_0^{t_k} \langle |Y_k^*(X(t_k,0,\cdot)) - Y_k * \rho_\varepsilon(X(t_k,0,\cdot))|\,, |\nabla u_0|\rangle^{p'}\; ds \right)^{1/p'} dr \qquad (6.22) \\
&\quad + \tilde{C} \int_0^1 \left( \int_0^{t_k} \langle |Y_k * \rho_\varepsilon(X(t_k,0,\cdot)) - Y_k * \rho_\varepsilon(X_{r,n}(t_k,0,\cdot)|\,, |\nabla u_0|\rangle^{p'} ds \right)^{1/p'} dr. \qquad (6.23)
\end{aligned}$$

Term (6.23) vanishes as $n \to \infty$ and Term (6.22) vanishes uniformly in $n$ as $\varepsilon \to 0$. Now, for term (6.21) we obtain that

$$\int_0^1 \left( \int_0^{t_k} \langle |Y_k^*(X_{r,n}(t_k,0,\cdot)) - Y_k^*(X(t_k,0,\cdot))|\,, |\nabla u_0|\rangle^{p'}\; ds \right)^{1/p'} dr \to 0 \tag{6.24}$$

as $n \to \infty$: as $\mathcal{J}_{u_0} \cap \mathcal{J}_{Y_k(X(t_k,0,\cdot))} \subset A$ with $\mathcal{H}^{N-1}(A) = 0$, we conclude that

$$\langle |Y_k^*(X_{r,n}(t_k,0,\cdot)) - Y_k^*(X(t_k,0,\cdot))|\,, |\nabla u_0|\rangle$$

$$= \int_{\Omega \setminus \mathcal{D}_{u_0}} |Y_k(X_{r,n}(t_k,0,x)) - Y_k(X(t_k,0,x))| \; d|D^a u_0|(x) \tag{6.25}$$

$$+ \int_{\mathcal{J}_{u_0} \setminus A} |Y_k^*(X_{r,n}(t_k,0,x)) - Y_k^*(X(t_k,0,x))| \; d|D^s u_0|(x). \tag{6.26}$$

Term (6.25) vanishes since $Y_k(X_{r,n}(t_k,0,\cdot)) \to Y_k(X(t_k,0,\cdot))$ in $L^2(\Omega)$. In the set $\mathcal{J}_{u_0} \setminus A$ the function $Y_k^*(X(t_k,0,\cdot))$ is continuous in an open neighborhood for $\mathcal{H}^{N-1}$-almost all $x$ due to condition (6.10) and thus

$$Y_k^*(X_{r,n}(t_k,0,x)) - Y_k^*(X(t_k,0,x)) \to 0 \quad \text{for } \mathcal{H}^{N-1}\text{-almost all } x \text{ in } \mathcal{J}_{u_0} \setminus A.$$

Then, Lebesgue's dominated convergence theorem yields the convergence predicted in (6.24). Thus,

$$\begin{aligned} & h_n \left| \langle Y_k, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot) \rangle - \langle Y_k^*, S_{p,b}(b_n - b)(t_k,\cdot) \rangle \right| \\ & \quad \le h_n \left| \langle Y_k - Y_k * \rho_\varepsilon, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot) \rangle \right| && (6.27) \\ & \quad + h_n \left| \langle Y_k * \rho_\varepsilon, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot) - S_{p,b}(b_n - b)(t_k,\cdot) \rangle \right| && (6.28) \\ & \quad + h_n \left| \langle Y_k * \rho_\varepsilon - Y_k^*, S_{p,b}(b_n - b)(t_k,\cdot) \rangle \right|. && (6.29) \end{aligned}$$

Term (6.27) can be estimated by two terms (6.21) and (6.23) vanishing as $n \to \infty$ and a term (6.22) vanishing uniformly in $n \in \mathbb{N}$ as $\varepsilon \to 0$. Term (6.28) is equal to term (6.20) and vanishes as $n \to \infty$. Finally, we obtain for term (6.29):

$$\begin{aligned} & h_n \left| \langle Y_k * \rho_\varepsilon - Y_k^*, S_{p,b}(b_n - b)(t_k,\cdot) \rangle \right| \\ & \le h_n \int_0^{t_k} \Bigg[ \int_\Omega |Y_k * \rho_\varepsilon(X(t_k,s,x)) - Y_k^*(X(t_k,s,x))| \, |(b_n(s,x) - b(s,x)| \\ & \qquad e^{\int_s^{t_k} |\operatorname{div} b(\tau, X(\tau,s,x))| \; d\tau} \; d \left| \nabla(u_0(X(0,s,\cdot))) \right| (x) \Bigg] \, ds \\ & \le 2 \left( \int_0^{t_k} \left( \int_\Omega |Y_k * \rho_\varepsilon(X(t_k,s,x)) - Y_k^*(X(t_k,s,x))| \right.\right. \\ & \qquad \left.\left. e^{\int_s^{t_k} |\operatorname{div} b(\tau, X(\tau,s,x))| \; d\tau} \; d \left| \nabla(u_0(X(0,s,\cdot))) \right| (x) \right)^{p'} \; ds \right)^{1/p'} \to 0 \end{aligned}$$

uniformly in $n \in \mathbb{N}$ as $\varepsilon \to 0$. Hence,

$$h_n \left| \langle Y_k, L(b_n)(t_k,\cdot) - L(b)(t_k,\cdot) \rangle - \langle Y_k^*, S_{p,b}(b_n - b)(t_k,\cdot) \rangle \right| \to 0 \tag{6.30}$$

as $n \to \infty$. Summing up, we obtain with (6.17) and (6.30) that

$$\frac{\left| G_k(b_n) - G(b) - G'_{b,k}(b_n - b) \right|}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(b_n - b)\|_{L^1((0,T),C(\Omega))}}$$

$$\frac{\left| G_k(b_n) - G(b) - \left( \frac{1}{2} \langle \mathbb{1}, \hat{S}_{p,b}(b_n - b)(t_k,\cdot) \rangle - \langle Y_k^*, S_{p,b}(b_n - b)(t_k,\cdot) \rangle \right) \right|}{\|b_n - b\|_{L^p((0,T),C(\Omega))^N} + \|\operatorname{div}(b_n - b)\|_{L^1((0,T),C(\Omega))}} \to 0$$

as $n \to \infty$. This proves Fréchet differentiability of $G$. It remains to show the continuity property. Let $(b_n) \subset \mathrm{V}_0^p$ be a sequence such that $G$ is Fréchet differentiable in $b_n$ for all $n \in \mathbb{N}$,

$$b_n \to b \quad \text{in } L^p((0,T), C(\Omega))^N, \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T), C(\Omega))$$

and $\left(\|b_n\|_{L^p((0,T),Lip(\Omega))^N}\right)$ is bounded. Then, for each $k \in \{2, \dots, K\}$ we know from Corollary 6.1.7 that for all $\tilde{b} \in \mathrm{V}_0^p$

$$\frac{1}{2}\left\langle \mathbb{1}, \hat{S}_{p,b_n}(\tilde{b})(t_k, \cdot)\right\rangle \to \frac{1}{2}\left\langle \mathbb{1}, \hat{S}_{p,b}(\tilde{b})(t_k, \cdot)\right\rangle$$

as $n \to \infty$. Furthermore, we have

$$\begin{aligned}\left\langle Y_k^*, S_{p,b_n}(\tilde{b})(t_k,\cdot) - S_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle &= \left\langle Y_k * \rho_\varepsilon, S_{p,b_n}(\tilde{b})(t_k,\cdot) - S_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle && (6.31)\\ &\quad + \left\langle Y_k^* - Y_k * \rho_\varepsilon, S_{p,b_n}(\tilde{b})(t_k,\cdot)\right\rangle && (6.32)\\ &\quad - \left\langle Y_k^* - Y_k * \rho_\varepsilon, S_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle. && (6.33)\end{aligned}$$

Obviously, term (6.31) vanishes as $n \to \infty$. Furthermore, term (6.33) tends to zero uniformly in $n$ as $\varepsilon \to 0$. For term (6.32) we estimate

$$\begin{aligned}\left|\left\langle Y_k^* - Y_k * \rho_\varepsilon, S_{p,b_n}(\tilde{b})(t_k,\cdot)\right\rangle\right| &\le C t_k \left\langle \left|Y_k^* - Y_k * \rho_\varepsilon\right|, |\nabla u_0|(X_n(0,t_k,\cdot))\right\rangle \\ &\le C t_k \left\langle \left|Y_k^*(X_n(t_k,0,\cdot)) - Y_k^*(X(t_k,0,\cdot))\right|, |\nabla u_0|\right\rangle && (6.34)\\ &\quad + C t_k \left\langle \left|Y_k^* - Y_k * \rho_\varepsilon\right|, |\nabla u_0|(X(0,t_k,\cdot))\right\rangle && (6.35)\\ &\quad + C t_k \left\langle \left|Y_k * \rho_\varepsilon(X(t_k,0,\cdot)) - Y_k * \rho_\varepsilon(X_n(t_k,0,\cdot))\right|, |\nabla u_0|\right\rangle. && (6.36)\end{aligned}$$

Term (6.35) vanishes uniformly in $n$ as $\varepsilon \to 0$. In addition, term (6.36) vanishes as $n \to \infty$. As in (6.24), we obtain that term (6.34) vanishes as $n \to \infty$. Summing up, we have a group of terms vanishing as $n \to \infty$ and a group of terms vanishing uniformly in $n$ as $\varepsilon \to 0$. Thus, we conclude for any $\tilde{b} \in \mathrm{V}_0^p$

$$\left\langle Y_k^*, S_{p,b_n}(\tilde{b})(t_k,\cdot) - S_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle \to 0$$

as $n \to \infty$.

□

**Remark 6.3.5** *For initial values $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, the condition*

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) = 0$$

*is always satisfied since $\mathcal{J}_{u_0} = \emptyset$. Thus, $G$ with initial value $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ is always Fréchet differentiable.*

### 6.3.4. $\mathcal{H}^{N-1}$-almost everywhere limits for specific $BV$-regular functions

In the previous subsection, we proved that $G$ is Fréchet differentiable at some vector field $b \in \mathrm{V}_0^p$ with $p > 1$ and unique flow $X$ if the jump set of $u_0$ and the union of the jump sets of $Y_k(X(t_k, 0, \cdot))$, $k \in \{2, \dots, K\}$ are disjoint except for a $\mathcal{H}^{N-1}$ null-set, i.e. if

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) = 0$$

holds. If the intersection is a set with positive $\mathcal{H}^{N-1}$-measure value, we will only be able to show that $G$ is one-sided directional differentiable. In the proof for this result, we will be confronted with the following situation: given some function $g \in BV(\Omega)$, some vector fields $b, \tilde{b} \in \mathrm{V}_0^p$ with $p > 1$ and a sequence of vector fields

$$b_n = b + r_n \tilde{b} \in \mathrm{V}_0^p$$

with $(r_n) \subset \mathbb{R}^+$ converging to zero. Then, the questions arising are: for $s, t \in [0, T]$, does the sequence of functions $(g(X_{r_n}(t, s, \cdot)))$ converge pointwise $\mathcal{H}^{N-1}$-almost everywhere to some function and when this is true, how does this function look like? Here, $X_{r_n}$ denotes the unique flow of the vector field $b_n$. In the following, we will investigate these questions.

We start our considerations with a look at the mapping

$$R_{t,s,x} : [0, \infty) \to \Omega, \quad r \mapsto X_r(t, s, x) \tag{6.37}$$

for each $x \in \Omega$ and for some fixed $s, t \in [0, T]$, where $X_r$ denotes the unique flow of the vector field $b + r\tilde{b}$ with $b, \tilde{b} \in \mathrm{V}_0^p$. For this mapping, we have the subsequent result.

**Lemma 6.3.6** *Let $b, \tilde{b} \in \mathrm{V}_0^p$, $p > 1$, $s, t \in [0, T]$ and $x \in \Omega$. Then, the mapping $R_{t,s,x}$ defined in (6.37), is Lipschitz continuous on bounded sets.*

**Proof:** We take $r_1, r_2 \in [0, \infty)$. For fixed $x \in \Omega$ and fixed $s, t \in [0, T]$ with $s \le t$ we obtain with Grönwall's lemma

$$\begin{aligned}
|R_{t,s,x}(r_1) - R_{t,s,x}(r_2)| &\le \int_s^t \mathbb{L}(b(z, \cdot)) \, |X_{r_1}(z, s, x) - X_{r_2}(z, s, x)| \, dz \\
&\quad + |r_1 - r_2| \left\|\tilde{b}\right\|_{L^\infty((0,T)\times\Omega)^N} \\
&\quad + \max(r_1, r_2) \int_s^t \mathbb{L}(\tilde{b}(z, \cdot)) \, |X_{r_1}(z, s, x) - X_{r_2}(z, s, x)| \, dz \\
&\le |r_1 - r_2| \left\|\tilde{b}\right\|_{L^\infty((0,T)\times\Omega)^N} e^{\int_s^t \mathbb{L}(b(z,\cdot)) + \max(r_1,r_2)\mathbb{L}(\tilde{b}(z,\cdot)) dz} \\
&\le C \, |r_1 - r_2|
\end{aligned}$$

for some $C > 0$. In the same way we obtain for $s \ge t$ the above estimate.

□

Lemma 6.3.6 states that the sequence $(X_{r_n}(t, s, x))$ lies on some continuous curve for some sequence $(r_n) \subset \mathbb{R}^+$ with $r_n \to 0$. Our next aim is to show that for point sequences $(x_n) \subset \Omega$ being convergent to some $x \in \Omega$ and for functions $g \in \mathcal{Y}$ a precise limit of $(g(x_n))$ can be defined if $(x_n)$ lies in the range of some continuous function.

Let $f : [0, \infty) \to \Omega$ be a continuous function and let $(x_n) \subset \Omega$ be a convergent sequence with limit $x \in \Omega$ such that

$$x_n = f(r_n) \quad \text{with } r_n \in \mathbb{R}^+ \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad r_n \to 0. \tag{6.38}$$

Obviously, we have that $f(0) = x$. Now, let $g \in \mathcal{Y}$. Denote $\mathcal{N}_{g,c} \subset \Omega \backslash \mathcal{D}_g$ the set of points $x \in \Omega \backslash \mathcal{D}_g$, where point (i) in Definition 6.3.2 is not satisfied and $\mathcal{N}_{g,j} \subset \mathcal{J}_g$ the set of points $x \in \mathcal{J}_g$, where point (ii) in Definition 6.3.2 is not fulfilled. Then, we distinguish two cases:

(i) $x \in \Omega \backslash (\mathcal{D}_g \cup \mathcal{N}_{g,c})$ and

(ii) $x \in \mathcal{J}_g \backslash \mathcal{N}_{g,j}$.

In the first case, we obtain for the sequence $(x_n)$ in (6.38) that

$$g(x_n) \to g(x) = g^*(x) \quad \text{as } n \to \infty.$$

In the second case, the limit depends on the direction from where we approach to the point $x$. From our choice $g \in \mathcal{Y}$ we know that for any $x \in \mathcal{J}_g \backslash \mathcal{N}_{g,j}$, there exists a ball $B_{r_x}(x)$ satisfying the condition (ii) in Definition 6.3.2. Then, we distinguish two cases:

(a) there exists $\bar{r} \in \mathbb{R}_+$ such that $f(r) \in \tilde{B}^+_{r_x}(x)$ or $f(r) \in \tilde{B}^-_{r_x}(x)$ for all $r \in (0, \bar{r})$.

(b) there exists $\bar{r} \in \mathbb{R}_+$ such that $f(r) \in \mathcal{J}_g$ for all $r \in (0, \bar{r})$.

In case (a) we know that $g$ is continuous on $\{f(r) |\ r \in (0, \bar{r})\}$ and bounded and thus

$$g(x_n) \to \hat{g}_f \quad \text{as } n \to \infty$$

for some $\hat{g} \in \mathbb{R}$. Obviously,

$$\hat{g}_f = \begin{cases} g^+(x) & \text{if } f(r) \in \tilde{B}^+_{r_x}(x) \text{ for } r \in (0, \bar{r}), \\ g^-(x) & \text{if } f(r) \in \tilde{B}^-_{r_x}(x) \text{ for } r \in (0, \bar{r}). \end{cases}$$

In case (b) we know that

$$g^*(x) = \frac{1}{2}(g^+(x) + g^-(x)) \quad \text{for all } x \in \mathcal{J}_g \cap B_{r_x}(x).$$

Since $g|_{\tilde{B}^+_{r_x}(x)}$ and $g|_{\tilde{B}^-_{r_x}(x)}$ are continuous, we obtain that $g^+$ and $g^-$ are continuous in $\mathcal{J}_g \cap B_{r_x}(x)$ and thus $g^*$ is continuous in $\mathcal{J}_g \cap B_{r_x}(x)$. Hence, we get in case (b) that

$$g^*(x_n) \to g^*(x) \quad \text{as } n \to \infty.$$

Summing up, we define the limit

$$\bar{g}_f(x) = \begin{cases} g^*(x) & \text{if case (i) or case (ii) (b) holds,} \\ g^+(x) & \text{if case (ii) (a) holds with } x_n \in \tilde{B}^+_{r_x}(x) \text{ for all } n \geq N, \\ g^-(x) & \text{if case (ii) (a) holds with } x_n \in \tilde{B}^-_{r_x}(x) \text{ for all } n \geq N \end{cases} \tag{6.39}$$

for the sequence $(x_n)$ approaching to $x$ on the curve given by $f$ and some $N \in \mathbb{N}$.

Now, for the sequence $(X_{r_n}(t,s,x))$ we know that it lies on some continuous curve given by the continuous function $R_{t,s,x}$ with left limit $R_{t,s,x}(0) = X(t,s,x)$. Thus, we obtain

$$g^*(X_{r_n}(t,s,x)) \to \bar{g}_{R_{t,s,x}}(X(t,s,x)) \quad \text{as } n \to \infty$$

for $\mathcal{H}^{N-1}$-almost all $x \in \Omega$, which we abbreviate to $\bar{g}(X(t,s,x))$ in the following if it is clear on which continuous curves $R_{t,s,x}$ the points $X(t,s,x)$ are approached. The so-defined function $\bar{g}(X(t,s,\cdot))$ is measurable as the pointwise limit of measurable functions $g^*(X_{r_n}(t,s,\cdot))$. This limit will appear in the one-sided directional derivative of $G$ in the case that (6.11) holds which will be shown in the following subsection.

### 6.3.5. One-sided directional differentiability of $G$

As pointed up in the previous subsections, our aim for this subsection is to show one-sided directional differentiability of $G$ at some vector field $b \in \mathrm{V}_0^p$ with $p > 1$ in the case that the initial value $u_0 \in \mathcal{U}_0$ and the functions $Y_k \in \mathcal{Y}$, $k \in \{2,\dots,K\}$ satisfy condition (6.11). For the proof of this statement, we need to restrict us to vector fields $b \in \mathrm{V}_0^p$ which are continuously differentiable instead of Lipschitz continuous in the spatial variable, i.e. we consider vector fields

$$b \in \mathrm{V}_0^p \cap L^1((0,T), C^1(\Omega))^N \quad \text{with } p > 1$$

in this subsection. We start with some auxiliary lemma.

**Lemma 6.3.7** *Let $g \in BV(\Omega)$ and let $b \in L^1((0,T), Lip_0(\Omega))^N \cap L^1((0,T), C^1(\Omega))^N$. Then, for $s,t \in [0,T]$ we have that*

$$\nabla(g(X(t,s,\cdot))) = (\nabla X(t,s,\cdot))^\top (\nabla g)(X(t,s,\cdot)),$$

*where $X$ denotes the unique flow of $b$.*

**Proof:** We know from Theorem 5.1.12 that

$$(g * \rho_\varepsilon)(X(t,s,\cdot)) \overset{*}{\rightharpoonup} (X(t,s,\cdot)) \quad \text{in } BV(\Omega)$$

as $\varepsilon \to 0$. Thus, we obtain using Lemma 5.1.10

$$\begin{aligned}\nabla(g(X(t,s,\cdot))) &\overset{*}{\leftharpoonup} \nabla(g * \rho_\varepsilon(X(t,s,\cdot))) \\ &= (\nabla X(t,s,\cdot))^\top \nabla(g * \rho_\varepsilon)(X(t,s,\cdot)) \\ &\overset{*}{\rightharpoonup} (\nabla X(t,s,\cdot))^\top (\nabla g)(X(t,s,\cdot)) \quad \text{in } \mathcal{M}(\Omega)^N.\end{aligned}$$

□

For the main statement, we consider vector fields $b, \tilde{b} \in \mathrm{V}_0^p \cap L^1((0,T), C^1(\Omega))^N$ with $p > 1$ and set the vector fields

$$b_z := b + z\tilde{b} \quad \text{with } z \in (0,\infty)$$

and unique flows $X_z$. Then, we define the following terms for $u_0 \in \mathcal{U}_0$ and $Y_k \in \mathcal{Y}$ with $k \in \{2,\dots,K\}$:

$$G_{1,k} := \frac{1}{2}\left\langle \mathbb{1}, \hat{S}_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle \text{ and} \tag{6.40}$$

$$G_{2,k} := \left\langle \overline{Y_k}(X(t_k,0,\cdot)), S_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle . \tag{6.41}$$

With these definitions we turn to the main result of this subsection.

**Theorem 6.3.8 (One-sided directional differentiability of $G$)** *Let $p > 1$ and $u_0 \in \mathcal{U}_0$ as well as $Y_k \in \mathcal{Y}$ for $k \in \{2,\dots,K\}$. Furthermore, let $b, \tilde{b} \in \mathrm{V}_0^p \cap L^1((0,T), C^1(\Omega))^N$. Assume that*

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) > 0 \tag{6.42}$$

*holds. Then, G is one-sided right directional differentiable with derivative*

$$\begin{aligned}\delta_+ G(b,\tilde{b}) &= \sum_{k=2}^{K} (G_{1,k} - G_{2,k}) \\ &= \sum_{k=2}^{K} \frac{1}{2}\left\langle \mathbb{1}, \hat{S}_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle - \left\langle \overline{Y_k}, S_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle .\end{aligned}$$

**Proof:** For $b, \tilde{b} \in \mathrm{V}_0^p \cap L^1((0,T), C^1(\Omega))^N$ we take a sequence $(z_n) \subset (0,\infty)$ with $z_n \to 0$ and consider $b_{z_n} = b + z_n \tilde{b}$. Obviously, $\left(\|b_{z_n}\|_{L^p((0,T),Lip(\Omega))^N}\right)$ is bounded and

$$b_{z_n} \to b \quad \text{in } L^p((0,T),C^1(\Omega))^N \quad \text{and} \quad \operatorname{div} b_{z_n} \to \operatorname{div} b \quad \text{in } L^1((0,T),C(\Omega)).$$

As in the proof of Theorem 6.3.4, we obtain

$$\begin{aligned}&2\frac{G_k(b_{z_n}) - G(b)}{z_n} \\ &\quad = \frac{\langle L(b_{z_n})(t_k,\cdot) - Y_k, L(b_{z_n})(t_k,\cdot) - Y_k\rangle - \langle L(b)(t_k,\cdot) - Y_k, L(b)(t_k,\cdot) - Y_k\rangle}{z_n} \\ &\quad = \frac{\langle L(b_{z_n})(t_k,\cdot) + L(b)(t_k,\cdot) - 2Y_k, L(b_{z_n})(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{z_n} \end{aligned}\tag{6.43}$$

and split term (6.43) into two terms:

$$\begin{aligned}&\frac{\langle L(b_{z_n})(t_k,\cdot) + L(b)(t_k,\cdot) - 2Y_k, L(b_{z_n})(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{z_n} \\ &\quad = \frac{\langle L(b_{z_n})(t_k,\cdot) + L(b)(t_k,\cdot), L(b_{z_n})(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{z_n} &(6.44)\\ &\quad - \frac{\langle 2Y_k, L(b_{z_n})(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{z_n}. &(6.45)\end{aligned}$$

For term (6.44) we obtain that

$$\frac{\langle L(b_{z_n})(t_k,\cdot) + L(b)(t_k,\cdot), L(b_{z_n})(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{z_n} \to \left\langle \mathbb{1}, \hat{S}_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle \tag{6.46}$$

as $n \to \infty$. Now, for term (6.45) we deduce using Lemma 6.3.3, Lemma 6.3.7 and Lemma 5.1.10

$$-\frac{\langle 2Y_k, L(b_{z_n})(t_k,\cdot) - L(b)(t_k,\cdot)\rangle}{z_n}$$

$$
\begin{aligned}
&= 2 \int_0^1 \int_0^{t_k} \left\langle Y_k^*(X_{r,n}(t_k,0,\cdot))\tilde{b}(s,X_{r,n}(s,0,\cdot))e^{\int_s^{t_k} \operatorname{div}(b+rz_n\tilde{b})(\tau,X_{r,n}(\tau,0,\cdot))d\tau} \right. \\
&\qquad\qquad \left. \nabla X_{r,n}(0,s,X_{r,n}(s,0,\cdot))^\top, \nabla u_0 \right\rangle \, dsdr \\
&\to 2 \int_0^{t_k} \left\langle \tilde{b}(s,X(s,0,\cdot))\overline{Y_k}(X(t_k,0,\cdot))e^{\int_s^{t_k} \operatorname{div}(b)(\tau,X(\tau,0,\cdot))d\tau} \nabla X(0,s,X(s,0,\cdot))^\top, \nabla u_0 \right\rangle \, ds \\
&= -2 \left\langle \overline{Y_k}, S_{p,b}(\tilde{b})(t_k,\cdot) \right\rangle
\end{aligned}
\tag{6.47}
$$

as $n \to \infty$ since

$$
\begin{aligned}
&\tilde{b}(s,X_{r,n}(s,0,x))e^{\int_s^{t_k} \operatorname{div}(b+rz_n\tilde{b})(\tau,X_{r,n}(\tau,0,x))d\tau} \nabla X_{r,n}(0,s,X_{r,n}(s,0,x))^\top \\
&\to \tilde{b}(s,X(s,0,x))e^{\int_s^{t_k} \operatorname{div}(b)(\tau,X(\tau,0,x))d\tau} \nabla X(0,s,X(s,0,x))^\top
\end{aligned}
$$

as a composition of continuous functions and

$$
Y_k^*(X_{r,n}(t_k,0,\cdot)) \to \overline{Y_k}(X(t_k,0,\cdot)) \quad \text{for } \mathcal{H}^{N-1}\text{-almost all } x \in \Omega
$$

as $n \to \infty$. Summing up, we have

$$
\begin{aligned}
\frac{G_k(b_{z_n}) - G(b)}{z_n} \to& \frac{1}{2} \left\langle \mathbb{1}, \hat{S}_{p,b}(\tilde{b})(t_k,\cdot) \right\rangle - \left\langle \overline{Y_k}, S_{p,b}(\tilde{b})(t_k,\cdot) \right\rangle \\
&= G_{1,k} - G_{2,k}
\end{aligned}
$$

as $n \to \infty$.

□

# 7. Gradient representation and optimality conditions

In this chapter, our aim is to find a gradient representation of the tracking term $G$ of our objective function. Such a gradient representation is normally obtained by using a duality relation between solutions of the investigated partial differential equation and its adjoint equation. In our case of the forward transport equation, the adjoint equation is given by the following backward continuity equation

$$\begin{aligned} \partial_t v + \operatorname{div}(bv) &= 0 \qquad \text{in } (0,T)\times\Omega, \\ v(T,\cdot) &= v_T \qquad \text{in } \Omega. \end{aligned}$$

The duality relation then yields a gradient representation involving solutions of the adjoint equation. Beside this adjoint based representation we give a second representation of the gradient based on solutions of the following backward transport equation

$$\begin{aligned} \partial_t w + \operatorname{div}(bw) - w\operatorname{div}(b) &= 0 \qquad \text{in } (0,T)\times\Omega, \\ w(T,\cdot) &= w_T \qquad \text{in } \Omega. \end{aligned}$$

We obtain this representation in the same way via a relation between solutions of the forward and backward equation. The second gradient representation, however, is more complex since it contains an additional term compared to the first one. But this second representation can be interesting for numerical applications: for its computation only the solution of a backward transport equation is needed which simplifies the numerical code since only a solver for the forward transport equation needs to be implemented which then can also be used for computing the solution of the backward transport equation.
In the second part of this chapter, we apply the results of the previous chapters to give necessary optimality conditions of first order for the optimal control problems presented in section 4.2 under some stricter assumptions on the involved functions. We use the gradient representations of the first section to formulate these conditions.

## 7.1. Relations between forward and backward equations and gradient representations of $G$

The first section contains three subsections: in the first one, we present some results about existence, uniqueness and stability for the backward adjoint equation as well as for the backward transport equation for final values $v_T$ and $w_T$ with some specific regularity which have not been presented so far in this thesis. In the second subsection, we first give general relation results involving general measure solutions and smooth solutions of both backward equations. In a second step, we use these results to prove relations for special measure solutions and solutions of the backward equations with less regularity. In the last subsection, we will apply these relations to obtain the two above mentioned representations of the gradient $G$.

### 7.1.1. Adjoint equation and backward transport equation

We start this subsection with a closer look on the following forward continuity equation

$$\begin{aligned} \partial_t u + \operatorname{div}(bu) &= 0 && \text{in } (0,T)\times\Omega, \\ u(0,\cdot) &= u_0 && \text{in } \Omega \end{aligned} \tag{7.1}$$

with some initial value $u_0 \in L^\infty(\Omega)$. For this equation we have the following result about existence and uniqueness of solutions.

**Theorem 7.1.1** *Let $u_0 \in L^\infty(\Omega)$ and $b \in \mathrm{V}^1$. Then, the unique solution $u \in C([0,T], L^\infty(\Omega)-w^*)$ of (7.1) is given by*

$$u(t,x) = u_0(X(0,t,x))e^{-\int_0^t \operatorname{div} b(s,X(s,t,x))ds} \quad \text{for } (t,x) \in [0,T]\times\Omega. \tag{7.2}$$

*In particular, if $u_0 \in C(\Omega)$, then $u \in C([0,T]\times\Omega)$.*

**Proof:** We prove this statement by using Theorem 5.2.3 and by showing that the unique measure solution is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$. We consider the initial measure $\mu_0 = u_0\mathcal{L}^N$. Then, the partial differential equation (7.1) is equal to the partial differential equation (5.18) if we choose $g = f = 0$. Therefore, Theorem 5.2.3 yields that the unique solution of (7.1) is given by

$$\mu(t,\cdot) = \mu_0(X(0,t,\cdot)) = u_0(X(0,t,\cdot))\mathcal{L}^N(X(0,t,\cdot))$$

for all $t \in [0,T]$. Incorporating the relation (5.21) in Remark 5.2.4 gives us

$$\mu(t,\cdot) = u_0(X(0,t,\cdot))e^{-\int_0^t \operatorname{div} b(s,X(s,t,\cdot))ds}\mathcal{L}^N =: u(t,\cdot)\mathcal{L}^N \quad \text{for } t \in [0,T].$$

We obtain for the Radon-Nikodym derivative $u$: for all $t \in [0,T]$

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}\, e^{\int_0^T \|\operatorname{div} b(s,\cdot)\|_{C(\Omega)}ds}$$

and since $\mu \in C([0,T], \mathcal{M}(\Omega)-w^*)$, i.e.

$$\langle \mu(t,\cdot) - \mu(s,\cdot), \varphi\rangle \to 0 \quad \text{as } t \to s$$

for any $\varphi \in C_0(\Omega)$, we obtain that

$$\langle u(t,\cdot)\mathcal{L}^N - u(s,\cdot)\mathcal{L}^N, \varphi\rangle \to 0 \quad \text{as } t \to s$$

for any $\varphi \in C_0(\Omega)$. Since $C_c(\Omega)$ is a dense subset of $L^1(\Omega)$, we conclude that $u \in C([0,T], L^\infty(\Omega)-w^*)$. Finally, if $u_0 \in C(\Omega)$, we immediately obtain that $u \in C([0,T]\times\Omega)$ as a composition of continuous functions.

□

For the general duality relation, we need the adjoint equation of the forward transport equation which is the backward continuity equation, i.e. we consider

$$\begin{aligned} \partial_t v + \operatorname{div}(bv) &= 0 && \text{in } (0,T)\times\Omega, \\ v(T,\cdot) &= v_T && \text{in } \Omega \end{aligned} \tag{7.3}$$

with final value $v_T \in C(\Omega)$. Using Theorem 7.1.1 and a time inversion $t' \mapsto T - t$ yields that the unique weak solution $v \in C([0,T] \times \Omega)$ of (7.3) is given by

$$v(t,x) = v_T(X(T,t,x))e^{\int_t^T \operatorname{div} b(s,X(s,t,x))ds} \quad \text{for } (t,x) \in [0,T] \times \Omega.$$

For the duality statements we also need a stability result for solutions of the backward adjoint equation.

**Lemma 7.1.2** *Let $1 \leq r < \infty$ and let $(b_n) \subset \mathrm{V}^1$ be a sequence such that $\left(\int_0^T \mathbb{L}(b_n(t,\cdot))dt\right)$ is bounded,*

$$b_n \to b \quad \text{in } L^1((0,T),C(\Omega))^N \quad \text{and} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T),C(\Omega))$$

*for some $b \in \mathrm{V}^1$ as $n \to \infty$. Furthermore, let $(v_{T,n}) \subset C(\Omega)$ be such that $v_{T,n} \to v_T$ in $C(\Omega)$ for some $v_T \in C(\Omega)$. Then, we have for the unique solutions $v_n$ of (7.3) with vector fields $b_n$ and final values $v_{T,n}$ that for any $t \in [0,T]$*

$$v_n(t,\cdot) \to v(t,\cdot) \quad \text{in } C(\Omega) \quad \text{and} \quad v_n \to v \quad \text{in } L^r((0,T),C(\Omega))$$

*as $n \to \infty$, where $v$ denotes the unique solution of (7.3) with vector field $b$ and final value $v_T$.*

**Proof:** We know that for $t \in [0,T]$

$$v_n(t,\cdot) = v_{T,n}(X_n(T,t,\cdot))e^{\int_t^T \operatorname{div} b_n(s,X_n(s,t,\cdot))ds}$$

and

$$v(t,\cdot) = v_T(X(T,t,\cdot))e^{\int_t^T \operatorname{div} b(s,X(s,t,\cdot))ds}.$$

Then, using Lemma 5.1.9, we obtain that

$$v_n(t,\cdot) \to v(t,\cdot) \quad \text{in } C(\Omega)$$

as $n \to \infty$. Since

$$\|v_n(t,\cdot)\|_{C(\Omega)} \leq \sup_{n\in\mathbb{N}} \|v_{T,n}\|_{C(\Omega)}\, e^{\int_0^T \|\operatorname{div} b_n(s,\cdot)\|_{C(\Omega)}ds} < \infty,$$

Lebesgue's dominated convergence theorem finishes the proof of the statement.

□

As mentioned in the introductive text of this chapter we give a further representation of the gradient $G$ based on solutions of the backward transport equation

$$\begin{aligned} \partial_t w + \operatorname{div}(bw) - w \operatorname{div}(b) &= 0 \qquad \text{in } (0,T) \times \Omega, \\ w(T,\cdot) &= w_T \quad\;\; \text{in } \Omega \end{aligned} \tag{7.4}$$

with final values $w_T \in C(\Omega)$ and vector fields $b \in \mathrm{V}^1$. Using Theorem 5.1.11 and a time inversion $t' \mapsto T - t$ yields that there exists a unique weak solution $w \in C([0,T] \times \Omega)$ of (7.4) given by

$$w(t,x) = w_T(X(T,t,x)) \quad \text{for } (t,x) \in [0,T] \times \Omega.$$

**Lemma 7.1.3** *Let $1 \le r < \infty$ and let $(b_n) \subset L^1((0,T), Lip_0(\Omega))^N$ be a sequence such that $\left(\int_0^T \mathbb{L}(b_n(t,\cdot))dt\right)$ is bounded and*

$$b_n \to b \quad \text{in } L^1((0,T), C(\Omega))^N$$

*for some $b \in L^1((0,T), Lip_0(\Omega))^N$ as $n \to \infty$. Furthermore, let $(w_{T,n}) \subset C(\Omega)$ be such that $w_{T,n} \to w_T$ in $C(\Omega)$ for some $w_T \in C(\Omega)$. Then, we have for the unique solutions $w_n$ of (7.4) with vector fields $b_n$ and final values $w_{T,n}$, that for any $t \in [0,T]$*

$$w_n(t,\cdot) \to w(t,\cdot) \quad \text{in } C(\Omega) \quad \text{and} \quad w_n \to w \quad \text{in } L^r((0,T), C(\Omega))$$

*as $n \to \infty$, where $w$ denotes the unique solution of (7.4) with vector field $b$ and final value $w_T$.*

**Proof:** The statement can be proven in the same way as Lemma 7.1.2.

□

### 7.1.2. Duality relations between measure solutions and solutions of backward equations

In this subsection, we present two main relations: the first one is in a general form, where general measure solutions of

$$\begin{aligned} \partial_t \mu + \operatorname{div}(b\mu) + g\mu + f = 0 \qquad & \text{in } (0,T) \times \Omega, \\ \mu(0,\cdot) = \mu_0 \qquad & \text{on } \Omega \end{aligned} \tag{7.5}$$

and continuous solutions of the adjoint equation as well as of the backward transport equation are involved. In the second one, we give a generalized result of the first one for special measure solutions descending from the Fréchet derivative of the control-to-state operator and for solutions of the two backward equations with $BV$-regularity in the spatial domain. We start with the first duality relation.

**Theorem 7.1.4 (General relations)** *Let $p \ge 1$ and let $b \in \mathrm{V}^p$ as well as $g \in L^1((0,T), Lip(\Omega))$ and $f \in L^r((0,T), \mathcal{M}(\Omega))$ for some $r > 1$ such that*

$$t \mapsto \langle f(t,\cdot), \varphi \rangle$$

*is measurable for any $\varphi \in C(\Omega)$. Furthermore, let $\mu \in C([0,T], \mathcal{M}(\Omega) - w^*)$ be a weak measure solution of (7.5) with initial data $\mu_0 \in \mathcal{M}(\Omega)$. If $v \in C([0,T] \times \Omega)$ is a weak solution of (7.3) with final data $v_T \in C(\Omega)$, the following duality relation holds for $0 \le s < t \le T$:*

$$\begin{aligned} \int_\Omega v(t,x)\ d\mu(t,\cdot)(x) = \int_\Omega v(s,x)\ d\mu(s,\cdot)(x) - \int_s^t \int_\Omega v(z,x)\ df(z,\cdot)(x)dz \\ - \int_s^t \int_\Omega v(z,x) \left(\operatorname{div} b(z,x) + g(z,x)\right)\ d\mu(z,\cdot)(x)dz. \end{aligned} \tag{7.6}$$

*Furthermore, if* $w \in C([0,T] \times \Omega)$ *is a weak solution of* (7.4) *with final data* $w_T \in C(\Omega)$, *we have the following relation for* $0 \leq s < t \leq T$:

$$\begin{aligned}\int_{\Omega} w(t,x)\ d\mu(t,\cdot)(x) = \int_{\Omega} w(s,x)\ d\mu(s,\cdot)(x) - \int_s^t \int_{\Omega} w(z,x)\ df(z,\cdot)(x) dz \\ - \int_s^t \int_{\Omega} w(z,x) g(z,x)\ d\mu(z,\cdot)(x) dz.\end{aligned} \tag{7.7}$$

For the proof, we need two auxiliary lemmas which are presented in the following.

**Lemma 7.1.5** *Let* $\sigma \in \mathcal{M}(\Omega)$ *and* $\rho$ *be the standard mollifier. Then, the sequence* $(\sigma_n) \subset C^\infty(\Omega)$, *given by*

$$\sigma_n = \sigma * \rho_{1/n}|_\Omega$$

*satisfy*

$$\langle \sigma_n \mathcal{L}^N - \sigma, \varphi \rangle \to 0 \quad \textit{for all } \varphi \in C(\Omega) \quad \textit{and} \quad \left\| \sigma_n \mathcal{L}^N \right\|_{\mathcal{M}(\Omega)} \leq \|\sigma\|_{\mathcal{M}(\Omega)}$$

*for all* $n \in \mathbb{N}$.

**Proof:** The proof can be found in the appendix.

□

**Lemma 7.1.6** *Let* $1 < p \leq \infty$ *and let* $f \in L^p((0,T), \mathcal{M}(\Omega))$ *such that*

$$t \mapsto \langle f(t,\cdot), \psi \rangle$$

*is measurable for any* $\psi \in C(\Omega)$. *Then, there exists a sequence* $(f_n) \subset C^\infty((0,T), L^1(\Omega))$ *such that for any* $m \in \mathbb{N}$ $f_n(t,\cdot) \in C^m(\Omega)$ *for almost all* $t \in (0,T)$ *and for any* $\varphi \in L^\infty((0,T), C(\Omega))$

$$\int_0^T |\langle f_n(t,\cdot) - f(t,\cdot), \varphi(t,\cdot) \rangle|\ dt \to 0 \qquad \textit{as } n \to \infty.$$

*In addition,* $\|f_n\|_{L^p((0,T),L^1(\Omega))} \leq \|f\|_{L^p((0,T),\mathcal{M}(\Omega))}$ *for all* $n \in \mathbb{N}$.

**Proof:** The proof can be found in the appendix.

□

With the aid of these lemmas we are able to prove Theorem 7.1.4:

**Proof:** Due to Lemma 5.2.2, we take a sequence of vector fields $(b_n) \subset C^\infty((0,T), C_0^m(\Omega))^N$ with $m \geq 2$ such that

$$\|b_n\|_{L^p((0,T),Lip_0(\Omega))^N} \leq C$$

for some $C > 0$ and

$$b_n \to b \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{as well as} \quad \operatorname{div} b_n \to \operatorname{div} b \quad \text{in } L^1((0,T), C(\Omega))$$

as $n \to \infty$. In the same way, we find a sequence $(g_n) \subset C^\infty((0,T), C^m(\Omega))$ such that

$$\|g_n\|_{L^1((0,T),Lip(\Omega))} \leq C \text{ for some } C > 0 \text{ and } g_n \to g \text{ in } L^1((0,T), C(\Omega)).$$

Beside these sequences, we choose sequences $(\mu_{0,n}) \subset C^\infty(\Omega)$ and $(v_{T,n}), (w_{T,n}) \subset C^\infty(\Omega)$ such that those are convergent: the first one such that

$$\langle \mu_{0,n}, \varphi \rangle \to \langle \mu_0, \varphi \rangle$$

as $n \to \infty$ for any $\varphi \in C(\Omega)$ and the other ones with respect to norm-convergence in $C(\Omega)$ and limits $v_T \in C(\Omega)$ and $w_T \in C(\Omega)$, respectively. Finally, due to Lemma 7.1.6 and the fact that $f \in L^r((0,T), \mathcal{M}(\Omega))$ for some $r > 1$, we find a sequence $(f_n) \subset C^\infty((0,T), L^1(\Omega))$ such that

$$\int_0^T |\langle f_n(t,\cdot) - f(t,\cdot), \varphi(t,\cdot)\rangle| \; dt \to 0 \quad \text{as } n \to \infty$$

for any $\varphi \in L^\infty((0,T), C(\Omega))$. Hence, we conclude for the sequence of unique solutions $(\mu_n)$ of the inhomogeneous continuity equation with vector fields $b_n$, $g_n$ and inhomogeneous terms $f_n$:

$$\begin{aligned}
&|\langle \mu_n(t,\cdot) - \mu(t,\cdot), \varphi\rangle| \\
&\quad \leq \left|\left\langle \mu_{0,n} - \mu_0, \varphi(X(t,0,\cdot))e^{-\int_0^t g(s,X(s,0,\cdot))ds}\right\rangle\right| \\
&\quad + C\left\|\varphi(X_n(t,0,\cdot))e^{-\int_0^t g_n(s,X_n(s,0,\cdot))ds} - \varphi(X(t,0,\cdot))e^{-\int_0^t g(s,X(s,0,\cdot))ds}\right\|_{C(\Omega)} \\
&\quad + \left|\int_0^t \left\langle f_n(s,\cdot) - f(s,\cdot), \varphi(X(t,s,\cdot))e^{-\int_s^t g(\tau,X(\tau,s,\cdot))d\tau}\right\rangle \; ds\right| \\
&\quad + \int_0^t \|f(s,\cdot)\|_{\mathcal{M}(\Omega)} \left\|\varphi(X_n(t,s,\cdot))e^{-\int_s^t g_n(\tau,X_n(\tau,s,\cdot))d\tau} - \varphi(X(t,s,\cdot))e^{-\int_s^t g(\tau,X(\tau,s,\cdot))d\tau}\right\|_{C(\Omega)} \; ds \\
&\quad \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

for any $\varphi \in C(\Omega)$ and for all $t \in [0,T]$, where $\mu$ denotes the unique weak solution of the same equation with vector field $b$, $g$ and inhomogeneous term $f$. In addition, due to Lemmas 7.1.2 and 7.1.3, the sequences of unique solutions $(v_n)$ and $(w_n)$ of the backward continuity and transport equation with vector fields $b_n$ and initial values $v_{T,n}$ and $w_{T,n}$ converge in $L^r((0,T), C(\Omega))$ with $1 \leq r < \infty$ to the unique solutions of the same equations with vector field $b$ and initial values $v_T$ and $w_T$, respectively. These solutions, given by

$$\mu_n(t,\cdot) = \mu_{0,n}(X_n(0,t,\cdot))e^{-\int_0^t g_n(s,X_n(s,t,\cdot))ds} - \int_0^t f_n(s, X_n(s,t,\cdot))e^{-\int_s^t g_n(\tau,X_n(\tau,t,\cdot))d\tau} \; ds,$$

$$v_n(t,\cdot) = v_{T,n}(X_n(T,t,\cdot))e^{\int_t^T \operatorname{div} b_n(s,X_n(s,t,\cdot))ds} \quad \text{and} \quad w_n(t,\cdot) = w_{T,n}(X_n(T,t,\cdot))$$

are obviously smooth as a composition of smooth functions. Thus, we conclude for $\mu_n$ and $v_n$:

$$0 = v_n\left(\partial_t \mu_n + \operatorname{div}(\mu_n b_n) + g_n \mu_n + f_n\right) + \mu_n\left(\partial_t v_n + \operatorname{div}(v_n b_n)\right)$$

$$
\begin{aligned}
&= v_n \left(\partial_t \mu_n + \nabla \mu_n \cdot b_n + \mu_n \operatorname{div}(b_n) + g_n \mu_n + f_n\right) + \mu_n \left(\partial_t v_n + \nabla v_n \cdot b_n + v_n \operatorname{div}(b_n)\right) \\
&= \partial_t(v_n \mu_n) + \operatorname{div}(v_n \mu_n b_n) + (\operatorname{div}(b_n) + g_n) v_n \mu_n + v_n f_n.
\end{aligned}
$$

Integration by parts over $(s,t) \times \Omega$ yields

$$
\begin{aligned}
0 = &\int_\Omega v_n(t,x)\ d\mu_n(t)(x) - \int_\Omega v_n(s,x)\ d\mu_n(s)(x) \\
&+ \int_s^t \int_\Omega \left(\operatorname{div} b_n(z,x) + g_n(z,x)\right) v_n(z,x)\ d\mu_n(z,\cdot)(x) dz + \int_s^t \int_\Omega v_n(z,x)\ df_n(z,\cdot)(x) dz.
\end{aligned}
$$

Now, we obtain for the first two terms with $z \in \{s,t\}$ using Lemma 7.1.2

$$
\begin{aligned}
\left|\langle \mu_n(z,\cdot), v_n(z,\cdot)\rangle - \langle \mu(z,\cdot), v(z,\cdot)\rangle\right| &\leq \|\mu_n(z,\cdot)\|_{\mathcal{M}(\Omega)} \|v_n(z,\cdot) - v(z,\cdot)\|_{C(\Omega)} \\
&\quad + \left|\langle \mu_n(z,\cdot) - \mu(z,\cdot), v(z,\cdot)\rangle\right| \\
&\leq C \|v_n(z,\cdot) - v(z,\cdot)\|_{C(\Omega)} \\
&\quad + \left|\langle \mu_n(z) - \mu(z), v(z,\cdot)\rangle\right|.
\end{aligned}
$$

The right side above converges to zero as $n \to \infty$. For the third term we conclude

$$
\begin{aligned}
&\int_s^t \left|\langle \mu_n(z,\cdot), \left(\operatorname{div} b_n(z,\cdot) + g_n(z,\cdot)\right) v_n(z,\cdot)\rangle - \langle \mu(z,\cdot), \left(\operatorname{div} b(z,\cdot) + g(z,\cdot)\right) v(z,\cdot)\rangle\right|\ dz \\
&\quad \leq \sup_{n\in\mathbb{N}} \sup_{z\in[0,T]} \|\mu_n(z,\cdot)\|_{\mathcal{M}(\Omega)} \left\|\left(\operatorname{div} b_n + g_n\right) v_n - \left(\operatorname{div} b + g\right) v\right\|_{L^1((0,T),C(\Omega))} \\
&\quad + \int_s^t \left|\langle \mu_n(z,\cdot) - \mu(z,\cdot), \left(\operatorname{div} b(z,\cdot) + g(z,\cdot)\right) v(z,\cdot)\rangle\right|\ dz \to 0.
\end{aligned}
$$

Finally, for the last term we get:

$$
\begin{aligned}
\int_s^t \left|\langle f_n(z,\cdot), v_n(z,\cdot)\rangle - \langle f(z,\cdot), v(z,\cdot)\rangle\right|\ dz &\leq \|f\|_{L^r((0,T),\mathcal{M}(\Omega))} \|v_n - v\|_{L^{r'}((0,T),C(\Omega))} \\
&\quad + \int_s^t \left|\langle f_n(z,\cdot) - f(z,\cdot), v(z,\cdot)\rangle\right|\ dz \to 0
\end{aligned}
$$

as $n \to \infty$. In a similar way, we deduce relation (7.7) for the functions $\mu$ and $w$.

□

In the successive generalized relation statement involving the solution of the adjoint equation, the third term on the right side in (7.6) will vanish since $g$ will be equal to $-\operatorname{div} b$. But this is not valid for the third term on the right side of the relation (7.7). However, we deduce a simplified relation in this case if the measure solution has the initial value $\mu_0 = 0$ in $\mathcal{M}(\Omega)$.

**Corollary 7.1.7** *Let the assumptions of Theorem 7.1.4 hold. If* $\mu_0 = 0$*, then the relation* (7.7) *is of the form*

$$\begin{aligned}\int_\Omega w(t,x)\ d\mu(t,\cdot)(x) &= \int_\Omega w(s,x)\ d\mu(s,\cdot)(x) - \int_s^t \int_\Omega w(z,x)\ df(z,\cdot)(x)dz \\ &+ \int_s^t \int_0^z \int_\Omega w(z,x)g(z,x)e^{-\int_r^z g(\tau,X(\tau,z,x))d\tau}\ df(r,X(r,z,\cdot))(x)drdz.\end{aligned} \tag{7.8}$$

**Proof:** Due to Remark 5.2.6, we know that in this case

$$\mu(z,\cdot) = -\int_0^z f(r,X(r,z,\cdot))e^{-\int_r^z g(X(\tau,z,\cdot))d\tau}\ dr.$$

Inserting this equation in (7.7) yields the statement.

□

Our aim for the successive subsection is to use such relations to get two representations of the gradient of $G$. In the case where we want to apply the theorem, the measure solution $\mu$ is given by the Fréchet derivative of the control-to-state operator $L$ at a given vector field $b \in \mathrm{V}^p$ and thus, the measure descends from the derivative of a $BV$-function. Consequently, the product with some other $BV$-function is well-defined as we have shown in subsection 6.3.1 and we can improve the previous statement for this case.

**Theorem 7.1.8 (Relations for Fréchet derivatives of $L$)** *Let* $p > 1, q > N$*,* $b \in \mathrm{V}^p$*,* $u_0, v_T, w_T \in L^\infty(\Omega) \cap BV(\Omega)$ *and let* $\tilde{b} \in \mathrm{V}^{p,q} \cap L^\infty((0,T)\times\Omega)^N$*. Furthermore, let* $v \in C([0,T], L^\infty(\Omega) - w^*)$ *and* $w \in C([0,T], BV(\Omega) - w^*)$ *be the unique weak solutions of the adjoint equation* (7.3) *and of the backward transport equation* (7.4)*, respectively, both with vector field* $b$*. Then, the duality relation* (7.6) *of Theorem 7.1.4 is also valid for* $\mu_{\tilde{b}} = S_{p,b}(\tilde{b}) \in C([0,T], \mathcal{M}(\Omega) - w^*)$ *and* $v$*, i.e. for* $0 \le s < t \le T$ *it holds*

$$\begin{aligned}\int_\Omega v^*(t,x)\ d\mu_{\tilde{b}}(t,\cdot)(x) &= \int_\Omega v^*(s,x)\ d\mu_{\tilde{b}}(s,\cdot)(x) \\ &- \int_s^t \int_\Omega v^*(z,x)\tilde{b}(z,x)\ d\nabla L(b)(z,\cdot)(x)dz.\end{aligned} \tag{7.9}$$

*Furthermore, for* $\mu_{\tilde{b}}$ *and* $w$*, the following relation holds for* $0 \le s < t \le T$*:*

$$\begin{aligned}&\int_\Omega w^*(t,x)\ d\mu_{\tilde{b}}(t,\cdot)(x) \\ &= \int_\Omega w^*(s,x)\ d\mu_{\tilde{b}}(s,\cdot)(x) - \int_s^t \int_\Omega w^*(z,x)\tilde{b}(z,x)\ d\nabla L(b)(z,\cdot)(x)dz \\ &- \int_0^s \int_\Omega w^*(z,x)\tilde{b}(z,x) \int_s^t \operatorname{div} b(r,X(r,z,x))e^{\int_z^r \operatorname{div} b(\tau,X(\tau,z,x))d\tau}\ dr\ d\nabla L(b)(z,\cdot)(x)dz\end{aligned}$$

$$-\int_s^t \int_\Omega w^*(z,x)\tilde{b}(z,x) \int_z^t \operatorname{div} b(r, X(r,z,x)) e^{\int_z^r \operatorname{div} b(\tau, X(\tau,z,x))d\tau}\, dr\ d\nabla L(b)(z,\cdot)(x)dz.$$

**Proof:** Our aim is to use Theorem 7.1.4 to prove the statement. Therefore we smooth the final values $v_{T,n} = v_T * \rho_{1/n}|_\Omega$, $w_{T,n} = w_T * \rho_{1/n}|_\Omega$ and obtain that the smoothed solutions $v_n, w_n \in C([0,T] \times \Omega)$ are given by

$$v_n(t,x) = v_{T,n}(X(T,t,x))e^{\int_t^T \operatorname{div} b(s,X(s,t,x))ds} \quad \text{and} \quad w_n(t,x) = w_{T,n}(X(T,t,x))$$

for $(t,x) \in [0,T] \times \Omega$. Furthermore, we know that $g = -\operatorname{div} b \in L^1((0,T), Lip(\Omega))$ and that $f$ is given by

$$f = \tilde{b} \cdot \nabla L(b) \in L^p((0,T), \mathcal{M}(\Omega)).$$

We need to show that $t \mapsto \langle f(t,\cdot), \varphi\rangle$ is measurable for all $\varphi \in C(\Omega)$: we know that $\nabla L(b) \in C([0,T], \mathcal{M}(\Omega) - w^*)^N$ and since $\tilde{b} \in L^p((0,T), C_0(\Omega))^N$, we find a sequence of simple functions $(b_k) \subset L^p((0,T), C_0(\Omega))^N$ such that $b_k(t,\cdot) \to \tilde{b}(t,\cdot)$ in $C_0(\Omega)^N$ as $k \to \infty$ for almost all $t \in (0,T)$. Thus,

$$t \mapsto \langle b_k(t,\cdot) \cdot \nabla L(b)(t,\cdot), \varphi\rangle = \langle \nabla L(b)(t,\cdot), b_k(t,\cdot)\varphi\rangle$$

is measurable since $b_k(t,\cdot) \in C_0(\Omega)^N$ for almost all $t \in (0,T)$ and $b_k$ is simple. Then,

$$t \mapsto \langle \tilde{b}(t,\cdot) \cdot \nabla L(b)(t,\cdot), \varphi\rangle$$

is measurable as the pointwise limit of measurable functions. Now, applying relation (7.6) yields

$$\int_\Omega v_n(t,x)\ d\mu_{\tilde{b}}(t,\cdot)(x) = \int_\Omega v_n(s,x)\ d\mu_{\tilde{b}}(s,\cdot)(x) - \int_s^t \int_\Omega v_n(z,x)\tilde{b}(z,x)\ d\nabla L(b)(z,\cdot)(x)dz.$$

Since $\mu_{\tilde{b}} = S_{p,b}(\tilde{b})$ has initial value $\mu_0 = 0$, we apply Corollary 7.1.7 and obtain from relation (7.7)

$$\begin{aligned}\int_\Omega & w_n(t,x)\ d\mu_{\tilde{b}}(t,\cdot)(x) \\ &= \int_\Omega w_n(s,x)\ d\mu_{\tilde{b}}(s,\cdot)(x) - \int_s^t \int_\Omega w_n(z,x)\tilde{b}(z,x)\ d\nabla L(b)(z,\cdot)(x)dz \\ &- \int_s^t \int_0^z \int_\Omega w_n(z,x) \operatorname{div} b(z,x) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,z,x))d\tau}\tilde{b}(r, X(r,z,x))\ d\nabla L(b)(r, X(r,z,\cdot))(x)drdz.\end{aligned} \tag{7.10}$$

For term (7.10) we conclude by switching the order of integration

$$\int_s^t \int_0^z \int_\Omega w_n(z,x) \operatorname{div} b(z,x) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,z,x))d\tau}\tilde{b}(r, X(r,z,x))\ d\nabla L(b)(r, X(r,z,\cdot))(x)drdz$$

$$= \int_s^t \int_0^z \int_\Omega w_{T,n}(X(T,r,x)) \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x)) d\tau} \tilde{b}(r,x) \; d\nabla L(b)(r,\cdot)(x) dr dz$$
$$= \int_0^s \int_s^t \int_\Omega w_n(r,x) \tilde{b}(r,x) \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x)) d\tau} \; d\nabla L(b)(r,\cdot)(x) dz dr$$
$$+ \int_s^t \int_r^t \int_\Omega w_n(r,x) \tilde{b}(r,x) \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x)) d\tau} \; d\nabla L(b)(r,\cdot)(x) dz dr$$
$$= \int_0^s \int_\Omega w_n(r,x) \tilde{b}(r,x) \int_s^t \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x)) d\tau} \; dz \; d\nabla L(b)(r,\cdot)(x) dr$$
$$+ \int_s^t \int_\Omega w_n(r,x) \tilde{b}(r,x) \int_r^t \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x)) d\tau} \; dz \; d\nabla L(b)(r,\cdot)(x) dr.$$

As

$$\tilde{v}_n(t,x) := v_{T,n}(X(T,t,x)) \to v_T^*(X(T,t,x)) =: \tilde{v}^*(t,x) \quad \text{and} \quad w_n(t,x) \to w^*(t,x)$$

for all $t \in [0,T]$ and for $\mathcal{H}^{N-1}$-almost all $x \in \Omega$,

$$|\tilde{v}_n(t,x)| \le \|v_T\|_{L^\infty(\Omega)} \quad \text{and} \quad |w_n(t,x)| \le \|w_T\|_{L^\infty(\Omega)}$$

for all $(t,x) \in [0,T] \times \Omega$ and for $z \in [0,T]$

$$\mu_{\tilde{b}}(z,\cdot) = -\int_0^z e^{\int_r^z \operatorname{div} b(X(\tau,z,\cdot)) d\tau} \tilde{b}(r, X(r,z,\cdot)) \; d(\nabla L(b))(z, X(r,z,\cdot)),$$

Lebesgue's dominated convergence theorem yields the statement.

□

### 7.1.3. Two representations of the gradient of $G$

The second relation statements about unique measure solutions of the forward equation (7.1) and unique solutions of the backward continuity and transport equation enable us to give two gradient representations of the derivatives of $G_\varepsilon$ and $G$ at some vector field $b \in \mathrm{V}_0^p$: one adjoint based and a second one based on the backward transport equation. These representations are presented in the following theorems.

**Theorem 7.1.9 (Adjoint based gradient representation)** *Let $p > 1$, $u_0 \in \mathcal{U}_0$, $Y_k \in \mathcal{Y}$ for $k \in \{k, \dots, K\}$, $\rho$ be the standard mollifier and $\varepsilon > 0$. For vector fields $b \in \mathrm{V}_0^p$, the Fréchet derivative $D_b G_\varepsilon(b)$ has the following adjoint based representation*

$$D_b G_\varepsilon(b) = -\sum_{k=2}^K \hat{v}_{k,\varepsilon} \nabla L(b). \tag{7.11}$$

*Here, the functions* $\hat{v}_{k,\varepsilon} : [0,T] \times \Omega \to \mathbb{R}$ *for* $k = 2, \ldots, K$ *are given by*

$$\hat{v}_{k,\varepsilon}(t,x) = \begin{cases} v_{k,\varepsilon}(t,x) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T, \end{cases}$$

*where* $v_{k,\varepsilon}$ *are the weak solutions of*

$$\begin{aligned} (v_{k,\varepsilon})_t + \operatorname{div}(v_{k,\varepsilon} b) &= 0 && \text{in } (0,t_k) \times \Omega, \\ v_{k,\varepsilon}(t_k, \cdot) &= F_{b,k,\varepsilon} && \text{in } \Omega \end{aligned} \tag{7.12}$$

*with*

$$F_{b,k,\varepsilon}(x) = \int_\Omega (L_\varepsilon(b)(t_k,z) - Y_k(z))\rho_\varepsilon(x-z)\, dz \quad \text{for } x \in \Omega.$$

*If*

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) = 0$$

*holds, then the Fréchet derivative* $D_bG(b)$ *has the following adjoint based representation*

$$D_bG(b) = -\sum_{k=2}^{K} \left[ \frac{1}{2} \hat{v}_{k,1}^* \nabla L_{u_0^2}(b) - \hat{v}_{k,2}^* \nabla L(b) \right], \tag{7.13}$$

*where for* $k = 2, \ldots, K$ *the functions* $\hat{v}_{k,1}, \hat{v}_{k,2} : [0,T] \times \Omega \to \mathbb{R}$ *are given by*

$$\hat{v}_{k,1}(t,x) = \begin{cases} v_{k,1}(t,x) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T, \end{cases} \quad \hat{v}_{k,2}(t,x) = \begin{cases} v_{k,2}(t,x) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T. \end{cases}$$

*Here,* $v_{k,1}$ *are the weak solutions of*

$$\begin{aligned} (v_{k,1})_t + \operatorname{div}(v_{k,1} b) &= 0 && \text{in } (0,t_k) \times \Omega, \\ v_{k,1}(t_k, \cdot) &= \mathbb{1} && \text{in } \Omega, \end{aligned} \tag{7.14}$$

*and* $v_{k,2}$ *are the weak solutions of*

$$\begin{aligned} (v_{k,2})_t + \operatorname{div}(v_{k,2} b) &= 0 && \text{in } (0,t_k) \times \Omega, \\ v_{k,2}(t_k, \cdot) &= Y_k && \text{in } \Omega. \end{aligned} \tag{7.15}$$

**Proof:** For $k \in \{2, \ldots, K\}$, we use Theorem 7.1.8 and we set $t = t_k,\ s = 0$ and

$$v_{T,k,\varepsilon} = F_{b,k,\varepsilon}(X(t_k,T,\cdot)) e^{-\int_{t_k}^T \operatorname{div} b(s,X(s,T,\cdot))ds}.$$

Then,

$$\begin{aligned} v_{k,\varepsilon}(t_k,\cdot) &= v_{T,k,\varepsilon}(X(T,t_k,\cdot)) e^{\int_{t_k}^T \operatorname{div} b(s,X(s,t_k,\cdot))ds} \\ &= F_{b,k,\varepsilon}(X(t_k,T,X(T,t_k,\cdot))) e^{-\int_{t_k}^T \operatorname{div} b(s,X(s,T,X(T,t_k,\cdot)))ds} e^{\int_{t_k}^T \operatorname{div} b(s,X(s,t_k,\cdot))ds} \\ &= F_{b,k,\varepsilon} \end{aligned}$$

and we obtain for $\tilde{b} \in \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N$:

$$\begin{aligned} D_b G_\varepsilon(b)\tilde{b} &= \sum_{k=2}^{K} \langle F_{b,k,\varepsilon}, S_{p,b}(\tilde{b})(t_k,\cdot)\rangle \\ &= -\sum_{k=2}^{K} \int_0^{t_k} \langle v_{k,\varepsilon}(r,\cdot)\tilde{b}(r,\cdot), \nabla L(b)(r,\cdot)\rangle \; dr \\ &= -\sum_{k=2}^{K} \int_0^{T} \langle \hat{v}_{k,\varepsilon}(r,\cdot)\tilde{b}(r,\cdot), \nabla L(b)(r,\cdot)\rangle \; dr. \end{aligned}$$

In the same way, we obtain for $\tilde{b} \in \mathrm{V}_0^p$

$$\begin{aligned} D_b G(b)\tilde{b} &= \sum_{k=2}^{K} \left[ \frac{1}{2}\langle \mathbb{1}, \tilde{S}_{p,b}(\tilde{b})(t_k,\cdot)\rangle - \left\langle Y_k^*, S_{p,b}(\tilde{b})(t_k,\cdot)\right\rangle \right] \\ &= -\sum_{k=2}^{K} \left[ \frac{1}{2}\int_0^{t_k} \langle v_{k,1}^*(r,\cdot)\tilde{b}(r,\cdot), \nabla L_{u_0^2}(b)(r,\cdot)\rangle \; dr - \int_0^{t_k} \langle v_{k,2}^*(r,\cdot)\tilde{b}(r,\cdot), \nabla L(b)(r,\cdot)\rangle \; dr \right] \\ &= -\sum_{k=2}^{K} \left[ \frac{1}{2}\int_0^{T} \langle \hat{v}_{k,1}^*(r,\cdot)\tilde{b}(r,\cdot), \nabla L_{u_0^2}(b)(r,\cdot)\rangle \; dr - \int_0^{T} \langle \hat{v}_{k,2}^*(r,\cdot)\tilde{b}(r,\cdot), \nabla L(b)(r,\cdot)\rangle \; dr \right]. \end{aligned}$$

□

**Theorem 7.1.10 (Alternative gradient representation)** *Let $p > 1$, $u_0 \in \mathcal{U}_0$, $Y_k \in \mathcal{Y}$ for $k \in \{k, \dots, K\}$, $\rho$ be the standard mollifier and $\varepsilon > 0$. For vector fields $b \in \mathrm{V}_0^p$, the Fréchet derivative $D_b G_\varepsilon(b)$ has the following representation*

$$D_b G_\varepsilon(b) = -\sum_{k=2}^{K} \hat{w}_{k,\varepsilon} \nabla L(b). \tag{7.16}$$

*Here, the functions $\hat{w}_{k,\varepsilon} : [0,T]\times\Omega \to \mathbb{R}$ for $k = 2, \dots, K$ are given by*

$$\hat{w}_{k,\varepsilon}(t,x) = \begin{cases} w_{k,\varepsilon}(t,x)\left(1 + \int_t^{t_k} \operatorname{div} b(z, X(z,t,x)) e^{\int_t^z \operatorname{div} b(\tau, X(\tau,t,x))d\tau} \; dz\right) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T, \end{cases}$$

*where $w_{k,\varepsilon}$ are the weak solutions of*

$$\begin{aligned} (w_{k,\varepsilon})_t + \operatorname{div}(w_{k,\varepsilon} b) - w_{k,\varepsilon} \operatorname{div} b &= 0 && \text{in } (0,t_k)\times\Omega, \\ w_{k,\varepsilon}(t_k,\cdot) &= F_{b,k,\varepsilon} && \text{in } \Omega \end{aligned} \tag{7.17}$$

*with*

$$F_{b,k,\varepsilon}(x) = \int_\Omega (L_\varepsilon(b)(t_k,z) - Y_k(z))\rho_\varepsilon(x-z) \; dz \quad \text{for } x \in \Omega.$$

*If*

$$\mathcal{J}_{u_0} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) = 0$$

*holds, then the Fréchet derivative $D_bG(b)$ has the following representation*

$$D_bG(b) = -\sum_{k=2}^{K} \left[\frac{1}{2}\hat{w}_{k,1}^{*}\nabla L_{u_0^2}(b) - \hat{w}_{k,2}^{*}\nabla L(b)\right], \tag{7.18}$$

*where for $k = 2,\dots,K$ the functions $\hat{w}_{k,1}, \hat{w}_{k,2} : [0,T]\times\Omega \to \mathbb{R}$ are given by*

$$\hat{w}_{k,1}(t,x) = \begin{cases} w_{k,1}(t,x)\left(1 + \int\limits_t^{t_k} \operatorname{div} b(z, X(z,t,x)) e^{\int_t^z \operatorname{div} b(\tau, X(\tau,t,x))d\tau}\, dz\right) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T \end{cases}$$

*and*

$$\hat{w}_{k,2}(t,x) = \begin{cases} w_{k,2}(t,x)\left(1 + \int\limits_t^{t_k} \operatorname{div} b(z, X(z,t,x)) e^{\int_t^z \operatorname{div} b(\tau, X(\tau,t,x))d\tau}\, dz\right) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T. \end{cases}$$

*Here, $w_{k,1}$ are the weak solutions of*

$$\begin{aligned} (w_{k,1})_t + \operatorname{div}(w_{k,1}b) - w_{k,1}\operatorname{div} b &= 0 && \text{in } (0,t_k)\times\Omega, \\ w_{k,1}(t_k,\cdot) &= \mathbb{1} && \text{in } \Omega, \end{aligned} \tag{7.19}$$

*and $w_{k,2}$ are the weak solutions of*

$$\begin{aligned} (w_{k,2})_t + \operatorname{div}(w_{k,2}b) - w_{k,2}\operatorname{div} b &= 0 && \text{in } (0,t_k)\times\Omega, \\ w_{k,2}(t_k,\cdot) &= Y_k && \text{in } \Omega. \end{aligned} \tag{7.20}$$

**Proof:** For $k \in \{2,\dots,K\}$, we use Theorem 7.1.8 and we set $t = t_k$, $s = 0$ and $w_{T,k,\varepsilon} = F_{b,k,\varepsilon}(X(t_k,T,\cdot))$. Then,

$$w_{k,\varepsilon}(t_k,\cdot) = w_{T,k,\varepsilon}(X(T,t_k,\cdot)) = F_{b,k,\varepsilon}(X(t_k,T,X(T,t_k,\cdot))) = F_{b,k,\varepsilon}$$

and we obtain for $\tilde{b} \in \mathrm{V}_0^{p,q} \cap L^\infty((0,T)\times\Omega)^N$:

$$\begin{aligned} D_bG_\varepsilon(b)\tilde{b} &= \sum_{k=2}^{K} \langle F_{b,k,\varepsilon}, S_{p,b}(\tilde{b})(t_k,\cdot)\rangle \\ &= -\sum_{k=2}^{K} \int\limits_0^{t_k} \langle w_{k,\varepsilon}(r,\cdot)\left(1 + \int\limits_r^{t_k} \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x))d\tau}\, dz\right)\tilde{b}(r,\cdot), \nabla L(b)(r,\cdot)\rangle\, dr \\ &= -\sum_{k=2}^{K} \int\limits_0^{T} \langle \hat{w}_{k,\varepsilon}(r,\cdot)\tilde{b}(r,\cdot), \nabla L(b)(r,\cdot)\rangle\, dr. \end{aligned}$$

In the same way, we obtain for $\tilde{b} \in \mathrm{V}_0^p$

$$
\begin{aligned}
&D_b G(b)\tilde{b} \\
&\quad = \sum_{k=2}^{K} \left[ \frac{1}{2} \langle \mathbb{1}, \tilde{S}_{p,b}(\tilde{b})(t_k, \cdot) \rangle - \left\langle Y_k^*, S_{p,b}(\tilde{b})(t_k, \cdot) \right\rangle \right] \\
&\quad = - \sum_{k=2}^{K} \left[ \frac{1}{2} \int_0^{t_k} \langle w_{k,1}^*(r, \cdot) \left( 1 + \int_r^{t_k} \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x)) d\tau} \; dz \right) \right. \\
&\qquad\qquad \cdot \tilde{b}(r, \cdot), \nabla L_{u_0^2}(b)(r, \cdot) \rangle \; dr \\
&\qquad\qquad - \int_0^{t_k} \langle w_{k,2}^*(r, \cdot) \left( 1 + \int_r^{t_k} \operatorname{div} b(z, X(z,r,x)) e^{\int_r^z \operatorname{div} b(\tau, X(\tau,r,x)) d\tau} \; dz \right) \\
&\qquad\qquad \left. \cdot \tilde{b}(r, \cdot), \nabla L(b)(r, \cdot) \rangle \; dr \; \right] \\
&\quad = - \sum_{k=2}^{K} \left[ \frac{1}{2} \int_0^{T} \langle \hat{w}_{k,1}^*(r, \cdot) \tilde{b}(r, \cdot), \nabla L_{u_0^2}(b)(r, \cdot) \rangle \; dr - \int_0^{T} \langle \hat{w}_{k,2}^*(r, \cdot) \tilde{b}(r, \cdot), \nabla L(b)(r, \cdot) \rangle \; dr \right].
\end{aligned}
$$

□

## 7.2. Optimality conditions

In this final section, we will apply the results of the previous chapters to give optimality conditions of first order for the optimal control problems presented in chapter 4. In section 4.2 of this chapter, the existence of minimizing points is proven under spatial $BV$-regularity assumptions on the vector fields $b$. For optimality conditions of first order, we need Fréchet differentiability of the involved functions, i.e. of the control-to-state operator as well as its composition with the objective function. Therefore, we need to require stricter assumptions on the spatial regularity of the vector fields $b$. We start with the reduced optimal control problems and the requirements regarded in this section.

### 7.2.1. Optimal control problems and existence of optimal controls

As in the last two chapters, we require $\Omega \subset \mathbb{R}^N$ as a bounded, open and convex subset with Lipschitz boundary $\partial\Omega$. In this case, we have that $W^{1,\infty}(\Omega) \simeq Lip(\Omega)$. We consider the optimal control problems of section 4.2 in a reduced, slightly changed form, i.e. we consider

$$
\begin{aligned}
\min_{b \in \mathrm{V}_0^2} F(b) = &\frac{1}{2} \sum_{k=2}^{K} \Upsilon_k \left( \| L_{Y_1}(b)(t_k, \cdot) - Y_k \|_{L^2(\Omega)}^2 \right) + \frac{\alpha}{2} \int_0^T \Gamma_1 \left( \| Db(t, \cdot) \|_{L^2(\Omega)^{N \times N}}^2 \right) \; dt \\
&+ \frac{\beta}{2} \int_0^T \Gamma_2 \left( \| \partial_t b(t, \cdot) \|_{L^2(\Omega)^N}^2 \right) \; dt + \frac{\gamma}{2} \int_0^T \Gamma_3 \left( \| \operatorname{div} b(t, \cdot) \|_{L^2(\Omega)}^2 \right) \; dt, \\
&\text{s.t. } b \in \mathrm{S}_{ad,\partial_t}^M
\end{aligned}
\tag{7.21}
$$

with regularization parameter $\alpha > 0$ and $\beta, \gamma \geq 0$. The given functions $Y_k$, $k \in \{1, \ldots, K\}$ shall satisfy $Y_1 \in \mathcal{U}_0$ and $Y_k \in \mathcal{Y}$ for $k \in \{2, \ldots, K\}$, where the sets $\mathcal{U}_0$ and $\mathcal{Y}$ are defined in Definition 6.3.2. For the functions $\Upsilon_k : \mathbb{R} \to \mathbb{R}$, $k = 2, \ldots, K$ and $\Gamma_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, we demand stricter assumptions than in section 4.2:

(a) the functions $\Upsilon_k : \mathbb{R} \to \mathbb{R}_0^+$ are continuously differentiable,

(b) the functions $\Gamma_i : \mathbb{R} \to \mathbb{R}_0^+$ are continuously differentiable, convex, monotonically increasing, in $\mathcal{O}(x)$ and

$$\lim_{x \to \infty} \Gamma_i(x) = \infty.$$

Finally, we define for $M > 0$ the admissible set $\mathrm{S}_{ad}^M$ as

$$\mathrm{S}_{ad}^M = \left\{ b \in \mathrm{V}_0^2 |\ \|b\|_{L^\infty((0,T)\times\Omega)^N} + \|b\|_{L^2((0,T),Lip(\Omega))^N} + \|\operatorname{div} b\|_{L^2((0,T),Lip(\Omega))} \leq M \right\} \tag{7.22}$$

and the admissible subsets

$$\mathrm{S}_{ad,\partial_t}^M = \left\{ b \in \mathrm{S}_{ad}^M |\ \partial_t b \in L^2((0,T) \times \Omega)^N \right\} \tag{7.23}$$

and

$$\mathrm{S}_{ad,\partial_t,0}^M = \left\{ b \in \mathrm{S}_{ad,\partial_t}^M |\ \operatorname{div} b = 0 \right\}. \tag{7.24}$$

In section 4.2, we have required one additional technical assumption beside these assumptions. This assumption aims at the behavior of the vector fields close to the spatial boundary to enforce zero spatial boundary for any weak*-limit of vector fields of the admissible set. With our assumptions in this section, this technical assumption is not necessary anymore. This is a direct consequence of the following theorem.

**Theorem 7.2.1** *Let $(b_n) \subset L^2((0,T), Lip_0(\Omega))^N$ with $(\operatorname{div} b_n) \subset L^2((0,T), Lip_0(\Omega))^N$ be bounded, i.e.*

$$\sup_{n \in \mathbb{N}} \left( \|b_n\|_{L^2((0,T),Lip(\Omega))^N} + \|\operatorname{div} b_n\|_{L^2((0,T),Lip(\Omega))} \right) < \infty.$$

*Then, there exists a subsequence $(b_{n_k})$ and a function $b \in L^2((0,T), Lip(\Omega))^N$ with $\operatorname{div} b \in L^2((0,T), Lip(\Omega))$ such that the following properties hold:*

*(i)*

$$b(t) \in \overline{\operatorname{conv}(\overline{\{b_n(t) | n \in \mathbb{N}\}}^{w^*})}^{w^*} \quad \textit{and} \quad \operatorname{div} b(t) \in \overline{\operatorname{conv}(\overline{\{\operatorname{div} b_n(t) | n \in \mathbb{N}\}}^{w^*})}^{w^*}$$

*for almost all $t \in (0,T)$ with respect to the weak*-topology in $W^{1,\infty}(\Omega)^N$ and $W^{1,\infty}(\Omega)$, respectively.*

*(ii) For any measurable set $B \in \mathcal{B}((0,T))$*

$$\int_B b_n(t,\cdot)\ dt \overset{*}{\rightharpoonup} \int_B b(t,\cdot)\ dt \quad \textit{in } W^{1,\infty}(\Omega)^N$$

*and*

$$\int_B \operatorname{div} b_n(t,\cdot)\ dt \overset{*}{\rightharpoonup} \int_B \operatorname{div} b(t,\cdot)\ dt \quad \textit{in } W^{1,\infty}(\Omega).$$

*(iii) For any measurable set $B \in \mathcal{B}((0,T))$ and any monotonically increasing and convex function $g : \mathbb{R} \to \mathbb{R}_0^+$ with $g \in \mathcal{O}(x)$*

$$\int\limits_B g\left(\|Db(t,\cdot)\|^2_{L^2(\Omega)^{N\times N}}\right)\,dt \le \liminf_{n\to\infty} \int\limits_B g\left(\|Db_n(t,\cdot)\|^2_{L^2(\Omega)^{N\times N}}\right)\,dt$$

*and*

$$\int\limits_B g\left(\|\operatorname{div} b(t,\cdot)\|^2_{L^2(\Omega)}\right)\,dt \le \liminf_{n\to\infty} \int\limits_B g\left(\|\operatorname{div} b_n(t,\cdot)\|^2_{L^2(\Omega)}\right)\,dt.$$

**Proof:** The proof works in the same way as the proof of Theorem 4.1.2.

□

**Corollary 7.2.2** *Let the assumptions of Theorem 7.2.1 hold. Then, the limit vector field $b$ is an element of $L^2((0,T), Lip_0(\Omega))^N$ with $\operatorname{div} b \in L^2((0,T), Lip_0(\Omega))$.*

**Proof:** We know that $W^{1,\infty}(\Omega) \simeq Lip(\Omega)$ compactly imbeds into $C(\Omega)$. Hence, for almost all $t \in (0,T)$, any function

$$f \in \overline{\{b_n(t)|n \in \mathbb{N}\}}^{w^*}$$

has zero boundary, since there exists a sequence $(f_k) \subset \{b_n(t)|n \in \mathbb{N}\} \subset C_0(\Omega)^N$ such that $f_k \to f$ in $C_0(\Omega)^N$. Again, in the same way, for almost all $t \in (0,T)$ any function

$$f \in \overline{\operatorname{conv}(\overline{\{b_n(t)|n \in \mathbb{N}\}}^{w^*})}^{w^*}$$

has zero boundary. Thus, $b \in L^2((0,T), Lip_0(\Omega))^N$. Applying the argument again for the divergence proves the statement.

□

Our aim is to show optimality conditions of first order for the optimal control problems given in (7.21). Therefore, we need the existence of minimizing points of these problems. We state this result in the following theorem

**Theorem 7.2.3** *The optimal control problems given by (7.21) for $\alpha > 0$ and $\beta, \gamma \ge 0$ attain its minima in $\mathrm{S}^M_{ad,\partial_t}$ and in $\mathrm{S}^M_{ad,\partial_t,0}$ for any $M > 0$.*

**Proof:** The statement can be proven in the same way as Theorem 4.2.4 by using Theorem 7.2.1, Corollary 7.2.2 and the direct method.

□

### 7.2.2. Fréchet differentiability of the reduced objective functions

Under our requirements on the vector fields and the functions $Y_k$, $k \in \{1,\ldots,K\}$, we know from chapter 6, that the function

$$b \mapsto \frac{1}{2}\sum_{k=2}^{K} \Upsilon_k\left(\|L_{Y_1}(b)(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)}\right)$$

is Fréchet differentiable as a composition of Fréchet differentiable functions at some vector field $b \in \mathrm{S}_{ad}^M$ with respect to convergence of $b$ in $L^2((0,T),C(\Omega))^N$, of $\operatorname{div} b$ in $L^1((0,T),C(\Omega))$ and boundedness in $L^2((0,T),Lip_0(\Omega))^N$ if

$$\mathcal{J}_{Y_1} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(X(t_k,0,\cdot))} = A \quad \text{with } \mathcal{H}^{N-1}(A) = 0$$

holds, where $X$ denotes the unique flow of $b$. Furthermore, we have the following statement.

**Lemma 7.2.4** *Let $Z$ be some Hilbert space, $g \in L^2((0,T),Z)$ and let $f : \mathbb{R} \to \mathbb{R}_0^+$ be continuously differentiable. Then,*

$$H : L^2((0,T),Z) \to \mathbb{R}, \quad g \mapsto \frac{1}{2}\int_0^T f\left(\|g(t,\cdot)\|_Z^2\right)\, dt$$

*is Fréchet differentiable with Fréchet derivative*

$$D_g H(g)\tilde{g} = \int_0^T f'\left(\|g(t,\cdot)\|_Z^2\right) \langle g(t,\cdot), \tilde{g}(t,\cdot)\rangle_Z \, dt.$$

**Proof:** The proof can be found in the appendix. □

The above lemma shows that

$$b \mapsto \frac{\alpha}{2}\int_0^T \Gamma_1\left(\|Db(t,\cdot)\|_{L^2(\Omega)^{N\times N}}^2\right)\, dt + \frac{\beta}{2}\int_0^T \Gamma_2\left(\|\partial_t b(t,\cdot)\|_{L^2(\Omega)^N}^2\right)\, dt$$
$$+ \frac{\gamma}{2}\int_0^T \Gamma_3\left(\|\operatorname{div} b(t,\cdot)\|_{L^2(\Omega)}^2\right)\, dt$$

is Fréchet differentiable at vector fields

$$b \in \left\{b \in \mathrm{V}_0^2 |\ \partial_t b \in L^2((0,T)\times\Omega)^N\right\}$$

with respect to convergence of $Db$ in $L^2((0,T)\times\Omega)^{N\times N}$, of $\partial_t b$ in $L^2((0,T)\times\Omega)^N$ and of $\operatorname{div} b$ in $L^2((0,T)\times\Omega)$.

### 7.2.3. Optimality conditions of first order for the optimal control problems

We are now prepared for the following result.

**Theorem 7.2.5** *Let $\bar{b}$ be an optimal control of the optimal control problems* (7.21) *lying in $\mathrm{S}_{ad,\partial_t}^M$ or in $\mathrm{S}_{ad,\partial_t,0}^M$. If*

$$\mathcal{J}_{Y_1} \cap \bigcup_{k=2}^{K} \mathcal{J}_{Y_k(\bar{X}(t_k,0,\cdot))} = A \quad \textit{with } \mathcal{H}^{N-1}(A) = 0$$

*holds, where $\bar{X}$ denotes the unique flow of $\bar{b}$, then the following optimality conditions holds for both cases*

*(a)*

$$\bar{b} \in \mathrm{S}^M_{ad,\partial_t}, \quad D_b F\left(\bar{b}\right)\left(\tilde{b} - \bar{b}\right) \geq 0 \quad \textit{for all } \tilde{b} \in \mathrm{S}^M_{ad,\partial_t}.$$

*(b)*

$$\bar{b} \in \mathrm{S}^M_{ad,\partial_t,0}, \quad D_b F\left(\bar{b}\right)\left(\tilde{b} - \bar{b}\right) \geq 0 \quad \textit{for all } \tilde{b} \in \mathrm{S}^M_{ad,\partial_t,0}.$$

**Proof:** We show the statement for the first case (a). The proof for the second case works in the same way.
Obviously, the set $\mathrm{S}^M_{ad,\partial_t}$ is convex and thus, $\bar{b} + s(\tilde{b} - \bar{b}) \in \mathrm{S}^M_{ad,\partial_t}$ for all $s \in [0,1]$ and for any $\tilde{b} \in \mathrm{S}^M_{ad,\partial_t}$. Furthermore, $F$ is Fréchet differentiable in $\bar{b}$ and thus also Gâteaux differentiable. The optimality of $\bar{b}$ then yields

$$F(\bar{b} + s(\tilde{b} - \bar{b})) - F(\bar{b}) \geq 0 \quad \text{for all } s \in [0,1]$$

and thus,

$$D_b F\left(\bar{b}\right)\left(\tilde{b} - \bar{b}\right) = \delta\left(F(\bar{b}, \left(\tilde{b} - \bar{b}\right)\right) = \lim_{s \to 0} \frac{F(\bar{b} + s(\tilde{b} - \bar{b})) - F(\bar{b})}{s} \geq 0.$$

□

#### Optimality conditions based on the adjoint equation

Using the gradient representation based on the adjoint equation gives us the following optimality conditions of first order:

(i)

$$(u_1)_t + \operatorname{div}(u_1\bar{b}) - u_1 \operatorname{div}\bar{b} = 0 \quad \text{in } (0,T)\times\Omega, \qquad (u_2)_t + \operatorname{div}(u_2\bar{b}) - u_2 \operatorname{div}\bar{b} = 0 \quad \text{in } (0,T)\times\Omega,$$
$$u_1(0,\cdot) = Y_1^2 \text{ in } \Omega, \qquad u_2(0,\cdot) = Y_1 \text{ in } \Omega.$$

(ii) For $k \in \{2,\ldots,K\}$

$$(v_{k,1})_t + \operatorname{div}(v_{k,1}\bar{b}) = 0 \quad \text{in } (0,t_k)\times\Omega, \qquad (v_{k,2})_t + \operatorname{div}(v_{k,2}\bar{b}) = 0 \quad \text{in } (0,t_k)\times\Omega,$$
$$v_{k,1}(t_k,\cdot) = \mathbb{1} \quad \text{in } \Omega, \qquad v_{k,2}(t_k,\cdot) = Y_k \quad \text{in } \Omega,$$

and

$$\hat{v}_{k,1}(t,x) = \begin{cases} v_{k,1}(t,x) & \text{if } 0 \leq t \leq t_k, \\ 0 & \text{if } t_k < t \leq T, \end{cases} \qquad \hat{v}_{k,2}(t,x) = \begin{cases} v_{k,2}(t,x) & \text{if } 0 \leq t \leq t_k, \\ 0 & \text{if } t_k < t \leq T. \end{cases}$$

(iii) For case
   (a) $\bar{b} \in \mathrm{S}^M_{ad,\partial_t}$ and for all $b \in \mathrm{S}^M_{ad,\partial_t}$

$$0 \leq -\frac{1}{2}\sum_{k=2}^{K}\int_0^T\int_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot) - Y_k\|^2_{L^2(\Omega)}\right)(b-\bar{b})(t,x)\hat{v}^*_{k,1}(t,x)\; d\left(\nabla u_1(t,\cdot)\right)(x)dt$$

$$+\sum_{k=2}^{K}\int_0^T\int_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right)(b-\bar b)(t,x)\hat v^*_{k,2}(t,x)\ d\left(\nabla u_2(t,\cdot)\right)(x)dt$$
$$+\alpha\int_0^T\int_\Omega \Gamma_1'\left(\left\|D\bar b(t,\cdot)\right\|^2_{L^2(\Omega)^{N\times N}}\right)D\bar b(t,x)\otimes D(b-\bar b)(t,x)\ dxdt$$
$$+\beta\int_0^T\int_\Omega \Gamma_2'\left(\left\|\partial_t\bar b(t,\cdot)\right\|^2_{L^2(\Omega)^{N}}\right)\partial_t\bar b(t,x)\cdot\partial_t(b-\bar b)(t,x)\ dxdt$$
$$+\gamma\int_0^T\int_\Omega \Gamma_3'\left(\left\|\operatorname{div}\bar b(t,\cdot)\right\|^2_{L^2(\Omega)}\right)\operatorname{div}\bar b(t,x)\operatorname{div}(b-\bar b)(t,x)\ dxdt.$$

(b) $\bar b\in \mathrm{S}^M_{ad,\partial_t,0}$ and for all $b\in \mathrm{S}^M_{ad,\partial_t,0}$

$$0\le -\frac{1}{2}\sum_{k=2}^{K}\int_0^T\int_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right)(b-\bar b)(t,x)\hat v^*_{k,1}(t,x)\ d\left(\nabla u_1(t,\cdot)\right)(x)dt$$
$$+\sum_{k=2}^{K}\int_0^T\int_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right)(b-\bar b)(t,x)\hat v^*_{k,2}(t,x)\ d\left(\nabla u_2(t,\cdot)\right)(x)dt$$
$$+\alpha\int_0^T\int_\Omega \Gamma_1'\left(\left\|D\bar b(t,\cdot)\right\|^2_{L^2(\Omega)^{N\times N}}\right)D\bar b(t,x)\otimes D(b-\bar b)(t,x)\ dxdt$$
$$+\beta\int_0^T\int_\Omega \Gamma_2'\left(\left\|\partial_t\bar b(t,\cdot)\right\|^2_{L^2(\Omega)^{N}}\right)\partial_t\bar b(t,x)\cdot\partial_t(b-\bar b)(t,x)\ dxdt.$$

In the above representation, we use for two matrices $M_1, M_2\in\mathbb{R}^{N\times N}$ the notation $M_1\otimes M_2$ for the sum of the coordinate-wise product of their entries, i.e.

$$M_1\otimes M_2 := \sum_{i=1}^{N}\sum_{j=1}^{N} M_{1,ij}M_{2,ij}.$$

**Optimality conditions based on the backward transport equation**

Similarly, using the gradient representation based on the backward transport equation gives us the following optimality conditions of first order:

(i)

$$(u_1)_t+\operatorname{div}(u_1\bar b)-u_1\operatorname{div}\bar b=0 \quad \text{in } (0,T)\times\Omega,\qquad (u_2)_t+\operatorname{div}(u_2\bar b)-u_2\operatorname{div}\bar b=0 \quad \text{in } (0,T)\times\Omega,$$
$$u_1(0,\cdot)=Y_1^2 \text{ in } \Omega,\qquad\qquad u_2(0,\cdot)=Y_1 \ \text{ in } \Omega.$$

(ii) For $k\in\{2,\dots,K\}$

$$(w_{k,1})_t+\operatorname{div}(w_{k,1}\bar b)-w_{k,1}\operatorname{div}\bar b=0 \qquad \text{in } (0,t_k)\times\Omega,$$

$$\begin{aligned} w_{k,1}(t_k,\cdot) &= \mathbb{1} && \text{in } \Omega, \\ (w_{k,2})_t + \operatorname{div}(w_{k,2}\bar{b}) - w_{k,2}\operatorname{div}\bar{b} &= 0 && \text{in } (0,t_k)\times\Omega, \\ w_{k,2}(t_k,\cdot) &= Y_k && \text{in } \Omega, \end{aligned}$$

and

$$\hat{w}_{k,1}(t,x) = \begin{cases} w_{k,1}(t,x)\left(1 + \int\limits_t^{t_k} \operatorname{div}\bar{b}(z,X(z,t,x))e^{\int_t^z \operatorname{div}\bar{b}(\tau,X(\tau,t,x))d\tau}\,dz\right) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T, \end{cases}$$

$$\hat{w}_{k,2}(t,x) = \begin{cases} w_{k,2}(t,x)\left(1 + \int\limits_t^{t_k} \operatorname{div}\bar{b}(z,X(z,t,x))e^{\int_t^z \operatorname{div}\bar{b}(\tau,X(\tau,t,x))d\tau}\,dz\right) & \text{if } 0 \le t \le t_k, \\ 0 & \text{if } t_k < t \le T. \end{cases}$$

(iii) For case

(a) $\bar{b} \in \mathrm{S}^M_{ad,\partial_t}$ and for all $b \in \mathrm{S}^M_{ad,\partial_t}$

$$\begin{aligned} 0 \le &-\frac{1}{2}\sum_{k=2}^{K}\int\limits_0^T\int\limits_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right)(b-\bar{b})(t,x)\hat{w}^*_{k,1}(t,x)\;d\left(\nabla u_1(t,\cdot)\right)(x)dt \\ &+\sum_{k=2}^{K}\int\limits_0^T\int\limits_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right)(b-\bar{b})(t,x)\hat{w}^*_{k,2}(t,x)\;d\left(\nabla u_2(t,\cdot)\right)(x)dt \\ &+\alpha\int\limits_0^T\int\limits_\Omega \Gamma_1'\left(\left\|D\bar{b}(t,\cdot)\right\|^2_{L^2(\Omega)^{N\times N}}\right)D\bar{b}(t,x)\otimes D(b-\bar{b})(t,x)\;dxdt \\ &+\beta\int\limits_0^T\int\limits_\Omega \Gamma_2'\left(\left\|\partial_t\bar{b}(t,\cdot)\right\|^2_{L^2(\Omega)^N}\right)\partial_t\bar{b}(t,x)\cdot\partial_t(b-\bar{b})(t,x)\;dxdt \\ &+\gamma\int\limits_0^T\int\limits_\Omega \Gamma_3'\left(\left\|\operatorname{div}\bar{b}(t,\cdot)\right\|^2_{L^2(\Omega)}\right)\operatorname{div}\bar{b}(t,x)\operatorname{div}(b-\bar{b})(t,x)\;dxdt. \end{aligned}$$

(b) $\bar{b} \in \mathrm{S}^M_{ad,\partial_t,0}$ and for all $b \in \mathrm{S}^M_{ad,\partial_t,0}$

$$\begin{aligned} 0 \le &-\frac{1}{2}\sum_{k=2}^{K}\int\limits_0^T\int\limits_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right)(b-\bar{b})(t,x)\hat{w}^*_{k,1}(t,x)\;d\left(\nabla u_1(t,\cdot)\right)(x)dt \\ &+\sum_{k=2}^{K}\int\limits_0^T\int\limits_\Omega \Upsilon_k'\left(\|u_2(t_k,\cdot)-Y_k\|^2_{L^2(\Omega)}\right)(b-\bar{b})(t,x)\hat{w}^*_{k,2}(t,x)\;d\left(\nabla u_2(t,\cdot)\right)(x)dt \\ &+\alpha\int\limits_0^T\int\limits_\Omega \Gamma_1'\left(\left\|D\bar{b}(t,\cdot)\right\|^2_{L^2(\Omega)^{N\times N}}\right)D\bar{b}(t,x)\otimes D(b-\bar{b})(t,x)\;dxdt \end{aligned}$$

$$+ \beta \int_0^T \int_\Omega \Gamma_2' \left( \left\| \partial_t \bar{b}(t, \cdot) \right\|^2_{L^2(\Omega)^N} \right) \partial_t \bar{b}(t,x) \cdot \partial_t (b - \bar{b})(t,x) \; dxdt.$$

# A. Appendix

We present here some technical proofs of statements appearing in the previous chapters.

### A.0.1. Proofs for auxiliary statements of chapter 5

For the proof of Lemma 5.2.2, we need the following statement.

**Lemma:** *Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded, open and convex and let*

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N | dist(x, \overline{\Omega}) < \varepsilon\}.$$

*for $\varepsilon > 0$. Then there exists a function $f_\varepsilon \in C^\infty(\mathbb{R}^N)^N$ such that*

$$\overline{\Omega_\varepsilon} \subset f_\varepsilon(\overline{\Omega})$$

*and $f_\varepsilon \to id$ uniformly on compact subsets of $\mathbb{R}^N$ as $\varepsilon \to 0$.*

**Proof:** We take a fixed $x_0 \in \Omega$ such that $B_R(x_0) \subset \Omega$ for some fixed $R$. Obviously, $\Omega_\varepsilon$ is open and convex. Furthermore, each point in $\overline{\Omega_\varepsilon}$ lies on a line segment

$$LS(z) := \{(1-\lambda)x_0 + \lambda z | \ \lambda \in [0,1]\}.$$

for some $z \in \partial\Omega_\varepsilon$. Due to the convexity of $\Omega$, there is a unique $x_{r,\varepsilon} \in \partial\Omega$ such that $LS(z) \cap \partial\Omega = \{x_{r,\varepsilon}\}$. We define

$$\mu_\varepsilon(z) := \frac{|z - x_0|}{|x_{r,\varepsilon} - x_0|} \leq \frac{\operatorname{diam}(\overline{\Omega_\varepsilon})}{R} =: C < \infty \qquad \text{for } z \in \partial\Omega_\varepsilon$$

and

$$\lambda_\varepsilon := \sup_{z \in \partial\Omega_\varepsilon} \mu_\varepsilon(z) \leq C.$$

Then, we set

$$f_\varepsilon(x) = x_0 + \lambda_\varepsilon(x - x_0) \qquad \text{for } x \in \mathbb{R}^N.$$

Obviously, $f_\varepsilon \in C^\infty(\mathbb{R}^N)^N$ and $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0$. Thus, we have for a compact subset $K \subset \mathbb{R}^N$:

$$|f_\varepsilon(x) - x| = |1 - \lambda_\varepsilon||x - x_0| \leq |1 - \lambda_\varepsilon| \max_{x \in K}(|x - x_0|) \to 0 \quad \text{as } \varepsilon \to 0.$$

In addition, $\overline{\Omega_\varepsilon} \subset f_\varepsilon(\overline{\Omega})$: we have that

$$x_{r,\varepsilon} = \frac{1}{\mu_\varepsilon(z)} z + \left(1 - \frac{1}{\mu_\varepsilon(z)}\right) x_0$$

and hence

$$x_\varepsilon := \frac{1}{\lambda_\varepsilon} z + \left(1 - \frac{1}{\lambda_\varepsilon}\right) x_0 \in \overline{\Omega}.$$

Now, we choose for $y \in LS(z)$ $\lambda_y \in [0,1]$ such that $y = \lambda_y z + (1-\lambda_y)x_0$. Then for

$$x_y := (\lambda_y x_\varepsilon + (1-\lambda_y)x_0) \in \overline{\Omega},$$

we have

$$\begin{aligned} f_\varepsilon(x_y) &= x_0 + \lambda_\varepsilon(x_y - x_0) = x_0 + \lambda_\varepsilon(\lambda_y x_\varepsilon + (1-\lambda_y)x_0 - x_0) \\ &= x_0 + \lambda_\varepsilon(\lambda_y(x_\varepsilon - x_0)) = x_0 + \lambda_\varepsilon \lambda_y \left(\frac{1}{\lambda_\varepsilon}(z - x_0) + x_0 - x_0\right) \\ &= x_0 + \lambda_y(z - x_0) = y. \end{aligned}$$

□

**Proof of Lemma 5.2.2:**

**Lemma:** *Let $m \in \mathbb{N}$, $1 \le p < \infty$ and $g \in L^p((0,T), Lip_0(\Omega))$.*

*(i) There exists a sequence $(g_n) \subset C^\infty((0,T), C_0^m(\Omega))$ such that*

$$g_n \to g \quad \text{in } L^p((0,T), C(\Omega))$$

*and $\left(\int_0^T \mathbb{L}(g_n(t,\cdot))^p \, dt\right)$ is bounded.*

*(ii) If $g \in L^p((0,T), Lip_0(\Omega))^N$ with $\operatorname{div} g \in L^1((0,T), Lip(\Omega))$, then there exists a sequence $(g_n) \subset C^\infty((0,T), C_0^m(\Omega))^N$ such that $\left(\int_0^T \mathbb{L}(g_n(t,\cdot))^p \, dt\right)$ is bounded,*

$$g_n \to g \quad \text{in } L^p((0,T), C(\Omega))^N \quad \text{and} \quad \operatorname{div} g_n \to \operatorname{div} g \quad \text{in } L^1((0,T), C(\Omega)).$$

**Proof:** We just prove point (i). Point (ii) can be proven in the same way.
Let $m \in \mathbb{N}$, $g \in L^p((0,T), Lip_0(\Omega))$ and $\rho$ be the standard mollifier. We set for almost all $t \in (0,T)$

$$h_n(t,\cdot) = (g(t,\cdot) * \rho_{1/n}) \circ f_{2/n} \in C_0^m(\Omega),$$

where $f_{2/n}$ denotes the function of the previous lemma. In addition, let $(\tilde{g}_k) \subset L^p((0,T), C_0(\Omega))$ be a sequence of simple functions such that

$$\tilde{g}_k(t,\cdot) \to g(t,\cdot) \quad \text{in } C_0(\Omega)$$

for almost all $t \in (0,T)$ and $\tilde{g}_k \to g$ in $L^p((0,T), C_0(\Omega))$. Such a sequence exists due to Theorem 10.4 in [Sch13]. Then, the sequence of simple functions $((\tilde{g}_k * \rho_{1/n}) \circ f_{2/n}) \subset L^p((0,T), C_0^m(\Omega))$ and

$$(\tilde{g}_k(t,\cdot) * \rho_{1/n}) \circ f_{2/n} \to (g(t,\cdot) * \rho_{1/n}) \circ f_{2/n} = h_n(t,\cdot) \quad \text{in } C_0^m(\Omega)$$

for almost all $t \in (0,T)$, i.e. $h_n$ is Bochner measurable for any $n \in \mathbb{N}$. We estimate for a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$ and for almost all $t \in (0,T)$

$$\|D^\alpha(h_n(t,\cdot))\|_{C_0(\Omega)} \le C \left\|D^\alpha \rho_{1/n}\right\|_{C(\mathbb{R}^N)} \|g(t,\cdot)\|_{C(\Omega)}$$

for some $C > 0$, independent of $t$ and $n$ and thus $h_n \in L^p((0,T), C_0^m(\Omega))$. In particular for $|\alpha| = 1$, we estimate

$$\|D^\alpha h_n(t,\cdot)\|_{C_0(\Omega)} \leq C \|D^\alpha g(t,\cdot)\|_{L^\infty(\Omega)} \leq C \|g(t,\cdot)\|_{Lip(\Omega)}$$

and thus

$$\mathbb{L}(h_n(t,\cdot)) \leq \|h_n(t,\cdot)\|_{C_0^1(\Omega)} \leq C \|g(t,\cdot)\|_{Lip(\Omega)},$$

where $C > 0$ is a constant independent of $n \in \mathbb{N}$ and $t \in (0,T)$. Hence, $\left(\int_0^T \mathbb{L}(h_n(t,\cdot))\, dt\right)$ is bounded in $L^p((0,T))$. Furthermore, due to Theorem 10.5 in [Sch13], we find a sequence $(h_{n,k}) \subset C^\infty((0,T), C_0^m(\Omega))$ for each $n \in \mathbb{N}$ such that

$$h_{n,k} \to h_n \quad \text{in } L^p((0,T), C_0^m(\Omega)).$$

These sequences are obtained by convolution with mollifiers, i.e. $h_{n,k} := h_n * \nu_{1/k}$ where $\nu$ denotes the standard mollifier in $\mathbb{R}$ and $h_n(t,\cdot) = 0$ for $t \in \mathbb{R}\backslash[0,T]$. We estimate for $d(t) = \|g(t,\cdot)\|_{Lip(\Omega)}$

$$\|h_{n,k}(t,\cdot)\|_{Lip(\Omega)} \leq C d * \nu_{1/k}(t)$$

and since $(d * \nu_{1/k})$ converges to $d$ in $L^p((0,T))$, we know that $(d * \nu_{1/k})$ is a bounded sequence in $L^p((0,T))$ and thus, the functions $h_{n,k}$ are uniformly bounded in $L^p((0,T), Lip(\Omega))$ for $n, k \in \mathbb{N}$. We choose $k(n) \in \mathbb{N}$ such that

$$\left\|h_{n,k(n)} - h_n\right\|_{L^p((0,T),C_0^m(\Omega))} \leq \frac{1}{n}$$

and set $g_n := h_{n,k(n)}$. Then, we estimate

$$\begin{aligned}\|g_n - g\|_{L^p((0,T),C(\Omega))} &\leq \|g_n - h_n\|_{L^p((0,T),C(\Omega))} + \|h_n - g\|_{L^p((0,T),C(\Omega))} \\ &\leq \frac{1}{n} + \|h_n - g\|_{L^p((0,T),C(\Omega))}.\end{aligned}$$

The right side converges to zero as $n \to \infty$ since for almost all $t \in (0,T)$

$$h_n(t,\cdot) \to g(t,\cdot) \quad \text{in } C(\Omega) \quad \text{and} \quad \|h_n(t,\cdot)\|_{C(\Omega)} \leq C \|g(t,\cdot)\|_{C(\Omega)}$$

and thus, Lebesgue's dominated convergence theorem shows the convergence.

□

## A.0.2. Proofs for auxiliary statements of chapter 6

### Proof of Lemma 6.1.1:

**Lemma:** *Let $r > 1$ and let $(f_n)$ be a bounded sequence in $L^r((0,T), \mathcal{M}(\Omega))$. Then, there exists a subsequence $(f_{n_k})$ and some $f \in L^r((0,T), \mathcal{M}(\Omega))$ such that*

$$f_{n_k} \overset{*}{\rightharpoonup} f \quad \textit{in } \mathcal{M}((0,T) \times \Omega).$$

**Proof:** We first show that for any $\varphi \in C_0(\Omega)$ the set of mappings

$$t \mapsto |\langle f_n(t,\cdot), \varphi\rangle|$$

is uniformly integrable. For any measurable set $B \in \mathcal{B}((0,T))$ we estimate

$$\int_B |\langle f_n(t,\cdot), \varphi \rangle| \; dt \leq \|\varphi\|_{C(\Omega)} \|f_n(t,\cdot)\|_{L^r((0,T),\mathcal{M}(\Omega))} |B|^{1/r'}$$

and thus, the above set of mappings is uniformly integrable. Then, Theorem 3.1 in [CdR04] yields that there exists a subsequence $(f_{n_k})$ and some $f \in L^1((0,T),\mathcal{M}(\Omega))$ such that

$$\int_B f_{n_k}(t,\cdot) \; dt \overset{*}{\rightharpoonup} \int_B f(t,\cdot) \; dt \quad \text{in } \mathcal{M}(\Omega)$$

for any $B \in \mathcal{B}((0,T))$. In the same way as in the proof of Theorem 4.1.2 we can show that $f \in L^r((0,T),\mathcal{M}(\Omega))$. It remains to show that $f_{n_k} \overset{*}{\rightharpoonup} f$ in $\mathcal{M}((0,T)\times\Omega)$. We take some $\varphi \in C_0((0,T)\times\Omega)$ and since $C_0((0,T)\times\Omega) \subset L^{r'}((0,T),C_0(\Omega))$ we find a sequence of simple functions

$$\varphi_l(t) = \sum_{i=1}^{N(l)} \chi_{A_{l,i}}(t)\varphi_{l,i} \quad \text{for almost all } t \in (0,T),$$

where for each $l \in \mathbb{N}$ and $i \in \{1,\dots,N(l)\}$, $A_{l,i} \in \mathcal{B}((0,T))$ are pairwise disjoint sets with $\bigcup_{i=1}^{N(l)} A_{l,i} = (0,T)$ and $\varphi_{l,i} \in C_0(\Omega)$ such that

$$\int_0^T \|\varphi_l(t,\cdot) - \varphi(t,\cdot)\|_{C(\Omega)}^{r'} \; dt \to 0 \quad \text{as } l \to 0.$$

Then, we estimate

$$\begin{aligned}
|\langle f_{n_k} - f, \varphi \rangle| &\leq |\langle f_{n_k} - f, \varphi - \varphi_l \rangle| + |\langle f_{n_k} - f, \varphi_l \rangle| \\
&\leq \|f_{n_k} - f\|_{L^r((0,T),\mathcal{M}(\Omega))} \|\varphi - \varphi_l\|_{L^{r'}((0,T),C_0(\Omega))} + |\langle f_{n_k} - f, \varphi_l \rangle| \\
&\leq C \|\varphi - \varphi_l\|_{L^{r'}((0,T),C_0(\Omega))} + |\langle f_{n_k} - f, \varphi_l \rangle| \qquad \text{(A.1)}
\end{aligned}$$

for some $C > 0$ since $(f_{n_k})$ is bounded in $L^r((0,T),\mathcal{M}(\Omega))$. We obtain for the second term on the right side

$$|\langle f_{n_k} - f, \varphi_l \rangle| \leq \sum_{i=1}^{N(l)} \left| \left\langle \int_{A_{l,i}} f_{n_k}(t,\cdot) - f(t,\cdot) \; dt, \varphi_{l,i} \right\rangle \right| \to 0 \quad \text{as } n \to \infty.$$

The uniform convergence of the first term in (A.1) and the convergence of the second term for fixed $l \in \mathbb{N}$ yields the convergence of $(|\langle f_{n_k} - f, \varphi \rangle|)$ to zero as $n \to \infty$. Thus, $f_{n_k} \overset{*}{\rightharpoonup} f$ in $\mathcal{M}((0,T)\times\Omega)$.

□

**Proof of Lemma 6.1.2:**

**Lemma:** *Let $g \in L^p((0,T),C(\Omega))$ and $h \in L^q((0,T),\mathcal{M}(\Omega))$ with $1 \leq p,q \leq \infty$. Then the product $gh : (0,T) \to \mathcal{M}(\Omega)$ is weak*-measurable. In addition, if $\frac{1}{p} + \frac{1}{q} = 1$, then $gh$ lies in*

$L^1((0,T),\mathcal{M}(\Omega))$ *and if* $q=\infty$ *and* $p$ *are arbitrary, then* $gh \in L^p((0,T),\mathcal{M}(\Omega))$.

**Proof:** For almost all $t \in (0,T)$ the product $g(t,\cdot)h(t,\cdot)$ lies in $\mathcal{M}(\Omega)$ since the product of a continuous function and a Radon measure is a Radon measure. Furthermore, $g$ is Bochner measurable with respect to $\|\cdot\|_{C(\Omega)}$, i.e. there is a sequence $(g_n)$ of simple functions

$$g_n = \sum_{i=1}^{k(n)} \chi_{A_{n,i}}(t) g_{n,i},$$

where $A_{n,i} \subset (0,T)$ are Lebesgue measurable sets, pairwise disjoint and $g_{n,i} \in C(\Omega)$ for all $i = 1, \dots, k(n)$ and $n \in \mathbb{N}$ such that

$$g_n(t,\cdot) \to g(t,\cdot) \quad \text{in } C(\Omega)$$

for almost all $t \in (0,T)$. Then the product $(g_n h)$ is a sum of weak*-measurable functions, since for $\varphi \in C_0(\Omega)$

$$t \mapsto \left\langle \chi_{A_{n,i}}(t) g_{n,i} h(t,\cdot), \varphi \right\rangle = \chi_{A_{n,i}}(t) \left\langle h(t,\cdot), g_{n,i}\varphi \right\rangle$$

is measurable due to the assumptions on $h$. Consequently $(g_n h)$ is weak*-measurable and thus $gh$ is weak*-measurable since for any $\varphi \in C_0(\Omega)$ the function

$$t \mapsto \langle g(t,\cdot)h(t,\cdot), \varphi \rangle$$

is the pointwise limit of $t \mapsto \langle g_n(t,\cdot)h(t,\cdot), \varphi\rangle$. As $C_0(\Omega)$ is separable and the pointwise supremum of measurable functions is measurable, we conclude that $t \mapsto \|g(t,\cdot)h(t,\cdot)\|_{\mathcal{M}(\Omega)}$ is measurable. Now, in the case $\frac{1}{p} + \frac{1}{q} = 1$, we estimate:

$$\begin{aligned}\int_0^T \|g(t,\cdot)h(t,\cdot)\|_{\mathcal{M}(\Omega)} \, dt &\le \int_0^T \|g(t,\cdot)\|_{C(\Omega)} \|h(t,\cdot)\|_{\mathcal{M}(\Omega)} \, dt \\ &\le \|g\|_{L^p((0,T),C(\Omega))} \|h\|_{L^q((0,T),\mathcal{M}(\Omega))} .\end{aligned}$$

Thus $gh \in L^1((0,T),\mathcal{M}(\Omega))$. If $p$ is arbitrary and $q = \infty$, we obtain that

$$\begin{aligned}\left(\int_0^T \|g(t,\cdot)h(t,\cdot)\|^p_{\mathcal{M}(\Omega)} \, dt\right)^{\frac{1}{p}} &\le \left(\int_0^T \|g(t,\cdot)\|^p_{C(\Omega)} \|h(t,\cdot)\|^p_{\mathcal{M}(\Omega)} \, dt\right)^{\frac{1}{p}} \\ &\le \|g\|_{L^p((0,T),C(\Omega))} \|h\|_{L^\infty((0,T),\mathcal{M}(\Omega))} .\end{aligned}$$

□

**Proof of Lemma 6.1.3:**

**Lemma:** *Let* $1 \le p < \infty$, $q > N$, $g \in L^p((0,T),W^{1,q}(\Omega)) \cap L^\infty((0,T)\times\Omega)$ *and* $h \in C([0,T],BV(\Omega)\text{–}w^*) \cap L^\infty((0,T)\times\Omega)$. *Then the product* $gh$ *is an element of* $L^p((0,T),BV(\Omega))$.

**Proof:** Using the fact that $BV(\Omega)\cap L^\infty(\Omega)$ is an algebra (see the remark before Definition 3.11 in [AFP00], the representation of the derivative of the product of two $BV(\Omega)$-functions, given in Example 3.97 [AFP00] and the representation (4.4) of $BV(\Omega)$-functions applied on functions of its predual, yields the statement in the same way as in the proof of Lemma 6.1.2. □

**Proof of Lemma 6.2.1:**

**Lemma:** *Let $\rho$ be the standard mollifier and $\varepsilon > 0$. Then we have:*

*(i) For $\sigma \in C([0,T], \mathcal{M}(\Omega) - w^*)$ such that*

$$t \mapsto \langle \sigma(t,\cdot), \varphi\rangle \in C([0,T]) \quad \textit{for any } \varphi \in C(\Omega),$$

*the function*

$$(t,x) \mapsto (\sigma(t,\cdot) * \rho_\varepsilon)(x) = \int_\Omega \rho_\varepsilon(x-z)\ d\sigma(t,\cdot)(z), \quad \textit{for } (t,x) \in [0,T]\times\Omega$$

*lies in $C([0,T], L^q(\Omega))$ for any $1 \le q < \infty$.*

*(ii) If $(\sigma_n) \subset C([0,T], \mathcal{M}(\Omega) - w^*)$ such that*

$$\sup_{t\in[0,T]} |\langle \sigma_n(t,\cdot) - \sigma(t,\cdot), \varphi\rangle| \to 0 \quad \textit{as } n \to \infty$$

*for any $\varphi \in C(\Omega)$, where $\sigma \in C([0,T], \mathcal{M}(\Omega) - w^*)$, then*

$$\sigma_n * \rho_\varepsilon|_{[0,T]\times\Omega} \to \sigma * \rho_\varepsilon|_{[0,T]\times\Omega} \quad \textit{in } C([0,T], L^q(\Omega))$$

*for any $1 \le q < \infty$.*

**Proof:**

(i) The function $\rho_\varepsilon$, considered as a function in $\mathbb{R}^N$ is uniformly continuous and we conclude that for any $\delta > 0$ there is some $\gamma(\delta) > 0$ such that

$$|\rho_\varepsilon(x-z) - \rho_\varepsilon(y-z)| \le \delta \qquad \forall\ x,y,z \in \Omega \text{ with } |x-y| \le \gamma(\delta).$$

Taking the supremum in $z$ yields that

$$\sup_{z\in\Omega} \rho_\varepsilon(\cdot - z) : \Omega \to \mathbb{R}$$

is continuous. Then, we deduce for a fixed $t \in [0,T]$ and $x, y \in \Omega$

$$\begin{aligned} |(\sigma(t,\cdot) * \rho_\varepsilon)(x) - (\sigma(t,\cdot) * \rho_\varepsilon)(y)| &\le \int_\Omega |\rho_\varepsilon(x-z) - \rho_\varepsilon(y-z)|\ d\,|\sigma(t,\cdot)|\,(z) \\ &\le \sup_{z\in\Omega} |\rho_\varepsilon(x-z) - \rho_\varepsilon(y-z)|\, \|\sigma(t,\cdot)\|_{\mathcal{M}(\Omega)}\,. \end{aligned}$$

Therefore, $\sigma(t,\cdot) * \rho_\varepsilon|_\Omega \in C(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q < \infty$. In addition, we obtain that for all $x \in \Omega$

$$|(\sigma(t,\cdot) * \rho_\varepsilon)(x) - (\sigma(s,\cdot) * \rho_\varepsilon)(x)| = |\langle \sigma(t,\cdot) - \sigma(s,\cdot), \rho_\varepsilon(x-\cdot)\rangle| \to 0$$

as $t \to s$ and

$$\begin{aligned} |(\sigma(t,\cdot) * \rho_\varepsilon)(x)| &\le \sup_{z\in\Omega} |\rho_\varepsilon(x-z)| \, \|\sigma(t,\cdot)\|_{\mathcal{M}(\Omega)} \\ &\le \|\rho_\varepsilon\|_{C(\mathbb{R}^N)} \sup_{t\in[0,T]} \|\sigma(t,\cdot)\|_{\mathcal{M}(\Omega)} . \end{aligned} \tag{A.2}$$

Lebesgue's dominated convergence theorem then yields that

$$\|\sigma(t,\cdot) * \rho_\varepsilon - \sigma(s,\cdot) * \rho_\varepsilon\|_{L^q(\Omega)} \to 0 \qquad \text{as } t \to s$$

and thus $\sigma * \rho_\varepsilon|_{[0,T]\times\Omega} \in C([0,T], L^q(\Omega))$ for all $q \in [1,\infty)$.

(ii) Using estimate (A.2), we obtain that

$$\begin{aligned} \sup_{t\in[0,T]} |\langle \sigma_n(t,\cdot) - \sigma(t,\cdot), \rho_\varepsilon(x-\cdot)\rangle| &\le \|\rho_\varepsilon\|_{C(\mathbb{R}^N)} \sup_{t\in[0,T]} \|\sigma_n(t,\cdot) - \sigma(t,\cdot)\|_{\mathcal{M}(\Omega)} \\ &\le C \end{aligned}$$

for some $C > 0$ since

$$\sup_{n\in\mathbb{N}, t\in[0,T]} \|\sigma_n(t,\cdot)\|_{\mathcal{M}(\Omega)} < \infty$$

due to the uniform boundedness principle. Then, we conclude with Lebesgue's dominated convergence theorem for any $1 \le q < \infty$

$$\begin{aligned} &\sup_{t\in[0,T]} \int_\Omega |(\sigma_n(t,\cdot) * \rho_\varepsilon)(x) - (\sigma(t,\cdot) * \rho_\varepsilon)(x)|^q \, dx \\ &\le \int_\Omega \left( \sup_{t\in[0,T]} |\langle \sigma_n(t,\cdot) - \sigma(t,\cdot), \rho_\varepsilon(x-\cdot)\rangle| \right)^q dx \\ &\to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

□

### A.0.3. Proofs for auxiliary statements of chapter 7

#### Proof of Lemma 7.1.5:

**Lemma:** *Let $\sigma \in \mathcal{M}(\Omega)$ and $\rho$ be the standard mollifier. Then, the sequence $(\sigma_n) \subset C^\infty(\Omega)$, given by*

$$\sigma_n = \sigma * \rho_{1/n}|_\Omega$$

*satisfy*

$$\langle \sigma_n \mathcal{L}^N - \sigma, \varphi \rangle \to 0 \quad \textit{for all } \varphi \in C(\Omega) \quad \textit{and} \quad \|\sigma_n \mathcal{L}^N\|_{\mathcal{M}(\Omega)} \le \|\sigma\|_{\mathcal{M}(\Omega)}$$

*for all $n \in \mathbb{N}$.*

**Proof:** Let $\rho$ be the standard mollifier and set $\tilde{\Omega} = \{x \in \mathbb{R}^N | \text{dist}(x, \overline{\Omega}) < 2\}$. Then $\tilde{\Omega}$ is a bounded open set containing $\Omega$ and we define the measure $\nu \in \mathcal{M}(\tilde{\Omega})$ by

$$\nu(A) := \sigma(A \cap \Omega) \qquad \text{for all Borel sets } A \subset \tilde{\Omega}.$$

Obviously, we have that

$$\|\nu\|_{\mathcal{M}(\tilde{\Omega})} = |\nu|(\tilde{\Omega}) = |\sigma|(\Omega) = \|\sigma\|_{\mathcal{M}(\Omega)}.$$

We set $\nu_n := \nu * \rho_{1/n}$ and Theorem 2.2 in [AFP00] yields that $\sigma_n := \nu_n|_\Omega \in C^\infty(\Omega)$ and that $\nu_n \mathcal{L}^N$ converges locally weakly* to $\nu$ in $\tilde{\Omega}$ as $n \to \infty$. Hence, we obtain for all $\varphi \in C(\Omega)$, extended in a continuous way to $\tilde{\Omega}$ and $\psi \in C_0(\tilde{\Omega})$ with $\psi|_\Omega \equiv 1$:

$$\int_\Omega \sigma_n(x)\varphi(x)\ dx = \int_{\tilde{\Omega}} \nu_n(x)\psi(x)\varphi(x)\ dx \to \int_{\tilde{\Omega}} \psi(x)\varphi(x)\ d\nu(x) = \int_\Omega \varphi(x)\ d\sigma(x)$$

as $n \to \infty$. Thus, $\langle \sigma_n \mathcal{L}^N - \sigma, \varphi \rangle \to 0$ for any $\varphi \in C(\Omega)$ as $n \to \infty$. In addition, we obtain for $\varphi \in C_0(\Omega)$ with $\|\varphi\|_{C(\Omega)} \leq 1$

$$\begin{aligned}\left|\int_\Omega \sigma_n(x)\varphi(x)\ dx\right| &= \left|\int_{\tilde{\Omega}} \nu_n(x)\varphi(x)\ dx\right| \leq \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} |\varphi(x)|\,\rho_{1/n}(x-y)\ d|\nu|(y)dx \\ &\leq \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} |\varphi(x)|\,\rho_{1/n}(x-y)\ dxd|\nu|(y) \leq \int_{\tilde{\Omega}} d|\nu|(y) \\ &= |\nu|(\tilde{\Omega}) = \|\sigma\|_{\mathcal{M}(\Omega)}.\end{aligned}$$

Taking the supremum over all $\varphi \in C_0(\Omega)$ with $\|\varphi\|_{C(\Omega)} \leq 1$ yields that $\left|\sigma_n \mathcal{L}^N\right|(\Omega) = \left\|\sigma_n \mathcal{L}^N\right\|_{\mathcal{M}(\Omega)} \leq \|\sigma\|_{\mathcal{M}(\Omega)}$.

□

**Proof of Lemma 7.1.6:**

**Lemma:** *Let $1 < p \leq \infty$ and let $f \in L^p((0,T), \mathcal{M}(\Omega))$ such that*

$$t \mapsto \langle f(t,\cdot), \psi \rangle$$

*is measurable for any $\psi \in C(\Omega)$. Then, there exists a sequence $(f_n) \subset C^\infty((0,T), L^1(\Omega))$ such that for any $m \in \mathbb{N}$ $f_n(t,\cdot) \in C^m(\Omega)$ for almost all $t \in (0,T)$ and for any $\varphi \in L^\infty((0,T), C(\Omega))$*

$$\int_0^T |\langle f_n(t,\cdot) - f(t,\cdot), \varphi(t,\cdot)\rangle|\ dt \to 0 \qquad \text{as } n \to \infty.$$

*In addition, $\|f_n\|_{L^p((0,T),L^1(\Omega))} \leq \|f\|_{L^p((0,T),\mathcal{M}(\Omega))}$ for all $n \in \mathbb{N}$.*

**Proof:** Let $m \in \mathbb{N}$ and $\rho$ be the standard mollifier. We define the functions

$$h_n : (0,T) \to L^1(\Omega), \quad t \mapsto h_n(t,\cdot) = f(t,\cdot) * \rho_{1/n}|_\Omega.$$

Then, due to Lemma 7.1.5 $h_n(t,\cdot) \in C^m(\Omega)$ for almost all $t \in (0,T)$ and

$$\langle h_n(t,\cdot) - f(t,\cdot), \varphi\rangle \to 0 \quad \text{as } n \to \infty$$

for any $\varphi \in C(\Omega)$. In addition, for $\psi \in L^\infty(\Omega)$ we have

$$t \mapsto \langle h_n(t,\cdot), \psi\rangle = \left\langle f(t,\cdot), \psi * \rho_{1/n}\right\rangle,$$

which is measurable for each $n \in \mathbb{N}$. Since

$$\int_0^T \|h_n(t,\cdot)\|^p_{L^1(\Omega)} \, dt \leq \int_0^T \|f(t,\cdot)\|^p_{\mathcal{M}(\Omega)} \, dt,$$

$h_n \in L^p((0,T), L^1(\Omega))$ and represents a bounded sequence in this space. Furthermore, Proposition 10.5 in [Sch13] yields that there are sequences $(h_{n,k}) \subset C^\infty((0,T), L^1(\Omega))$ such that $h_{n,k} \to h_n$ in $L^p((0,T), L^1(\Omega))$ as $k \to \infty$. These sequences are obtained by convolution with mollifiers, i.e. $h_{n,k} := h_n * \nu_{1/k}$ where $\nu$ denotes the standard mollifier in $\mathbb{R}$ and $h_n(t,\cdot) = 0$ for $t \in \mathbb{R}\backslash[0,T]$. Obviously, $h_{n,k}(t,\cdot) \in C^m(\Omega)$ for almost all $t \in (0,T)$ and for all $k \in \mathbb{N}$. We obtain for these sequences:

$$\begin{aligned}\int_0^T \left\|h_n * \nu_{1/k}(t,\cdot)\right\|^p_{L^1(\Omega)} \, dt &\leq \int_0^T \int_0^T \nu_{1/k}(t-s) \, \|h_n(s,\cdot)\|^p_{L^1(\Omega)} \, ds dt \\ &= \int_0^T \int_0^T \nu_{1/k}(t-s) \, dt \, \|h_n(s,\cdot)\|^p_{L^1(\Omega)} \, ds \\ &\leq \int_0^T \|h_n(s,\cdot)\|^p_{L^1(\Omega)} \, ds\end{aligned}$$

and thus

$$\|h_{n,k}\|_{L^p((0,T),L^1(\Omega))} \leq \|h_n\|_{L^p((0,T),L^1(\Omega))} \leq \|f\|_{L^p((0,T),L^1(\Omega))}.$$

We choose $k(n) \in \mathbb{N}$ such that

$$\int_0^T \left\|h_{n,k(n)}(t,\cdot) - h_n(t,\cdot)\right\|^p_{L^1(\Omega)} \, dt \leq \frac{1}{n}$$

and set $f_n := h_{n,k(n)}$. For $\varphi \in L^\infty((0,T, C(\Omega))$, there exists a sequence of simple functions $(\varphi_l) \subset L^{p'}((0,T), C(\Omega))$ such that $\varphi_l \to \varphi$ in $L^{p'}((0,T), C(\Omega))$ due to Theorem 10.4 in [Sch13]. Denote $A_{l,i} \subset (0,T)$, $i \in \{1, \ldots, K(l)\}$, the measurable sets on which $\varphi_l$ is constant with value $\varphi_{l,i} \in C(\Omega)$. Then, we estimate

$$\int_0^T |\langle h_n(t,\cdot) - f(t,\cdot), \varphi(t,\cdot)\rangle| \, dt \leq \sum_{i=1}^{K(l)} \int_{A_{l,i}} |\langle h_n(t,\cdot) - f(t,\cdot), \varphi_{l,i}\rangle| \, dt$$

$$+ 2\,\|f\|_{L^p((0,T),\mathcal{M}(\Omega))}\,\|\varphi_l - \varphi\|_{L^{p'}((0,T),C(\Omega))}$$
$$\to 0$$

as $n \to \infty$ since the second term converges uniformly in $n \in \mathbb{N}$ to zero as $l \to \infty$. Thus, we conclude

$$\begin{aligned}
\int_0^T |\langle f_n(t,\cdot) - f(t,\cdot), \varphi(t,\cdot)\rangle| \; dt &\le \int_0^T |\langle f_n(t,\cdot) - h_n(t,\cdot), \varphi(t,\cdot)\rangle| \; dt \\
&+ \int_0^T |\langle h_n(t,\cdot) - f(t,\cdot), \varphi(t,\cdot)\rangle| \; dt \\
&\le \|\varphi\|_{L^\infty((0,T),C(\Omega))} \int_0^T \|f_n(t,\cdot) - h_n(t,\cdot)\|_{L^1(\Omega)} \; dt \\
&+ \int_0^T |\langle h_n(t,\cdot) - g(t,\cdot), \varphi(t,\cdot)\rangle| \; dt \\
&\le \frac{\|\varphi\|_{L^\infty((0,T),C(\Omega))}}{n} + \int_0^T |\langle h_n(t,\cdot) - f(t,\cdot), \varphi(t,\cdot)\rangle| \; dt \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

□

**Proof of Lemma 7.2.4:**

**Lemma:** *Let $Z$ be some Hilbert space, $g \in L^2((0,T),Z)$ and let $f : \mathbb{R} \to \mathbb{R}_0^+$ be continuously differentiable. Then,*

$$H : L^2((0,T),Z) \to \mathbb{R}, \quad g \mapsto \frac{1}{2} \int_0^T f\left(\|g(t,\cdot)\|_Z^2\right) \; dt$$

*is Fréchet differentiable with Fréchet derivative*

$$D_g H(g)\tilde{g} = \int_0^T f'\left(\|g(t,\cdot)\|_Z^2\right) \langle g(t,\cdot), \tilde{g}(t,\cdot)\rangle_Z \; dt.$$

**Proof:** Let $\tilde{g} \in L^2((0,T),Z)$. Then, we define the Nemytskii operator

$$k : L^1((0,T)) \to L^2((0,T)), \quad h \mapsto f'(h),$$

which is well-defined since $f' \in C(\mathbb{R})$. Then, due to Theorem 7.19 in [DN11], the operator $k$ is continuous between these spaces. Now, we estimate for $s \in [0,T]$

$$\int_0^T \left| \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 - \|g(t,\cdot)\|_Z^2 \right| \; dt$$

$$= \int_0^T |\|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z - \|g(t,\cdot)\|_Z| \, |\|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z + \|\tilde{g}(t,\cdot)\|_Z| \ dt$$

$$\leq \left( \int_0^T |\|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z - \|g(t,\cdot)\|_Z|^2 \ dt \right)^{1/2} \left( \int_0^T |\|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z + \|g(t,\cdot)\|_Z|^2 \ dt \right)^{1/2}$$

$$\leq s \left( \int_0^T \|\tilde{g}(t,\cdot)\|_Z^2 \ dt \right)^{1/2} \left( \int_0^T |\|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z + \|g(t,\cdot)\|_Z|^2 \ dt \right)^{1/2}$$

$$\to 0$$

as $\tilde{g} \to 0$ in $L^2((0,T),Z)$. Thus,

$$\int_0^T \left| f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \right|^2 \ dt \to 0$$

for any $s \in [0,1]$ as $\tilde{g} \to 0$ in $L^2((0,T),Z)$. Since

$$\left| f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \right|^2 \leq 4 \|f'\|_{C(\mathbb{R})}$$

for almost all $t \in (0,T)$, we deduce that

$$t \mapsto \left| f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \right|^2$$

converges weakly* to 0 in $L^\infty((0,T))$ as $\tilde{g} \to 0$ in $L^2((0,T),Z)$, i.e.

$$\int_0^T \left| f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \right|^2 h(t) \ dt \to 0$$

for any $h \in L^1((0,T))$. Finally, we conclude

$$|H(g+\tilde{g}) - H(g) - D_g H(g)\tilde{g}|$$

$$= \left| \int_0^T \frac{1}{2} f\left( \|g(t,\cdot) + \tilde{g}(t,\cdot)\|_Z^2 \right) - \frac{1}{2} f\left( \|g(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \langle g(t,\cdot), \tilde{g}(t,\cdot)\rangle_Z \ dt \right|$$

$$= \left| \int_0^T \int_0^1 f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) \langle g(t,\cdot) + s\tilde{g}(t,\cdot), \tilde{g}(t,\cdot)\rangle_Z \ ds \right.$$
$$\left. - f'\left( \|g(t,\cdot)\|_Z^2 \right) \langle g(t,\cdot), \tilde{g}(t,\cdot)\rangle_Z \ dt \right|$$

$$\leq \left| \int_0^T \int_0^1 \left[ f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \right] \ ds \langle g(t,\cdot), \tilde{g}(t,\cdot)\rangle_Z \ dt \right|$$

$$+ \left| \int_0^1 \int_0^T s f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) \langle \tilde{g}(t,\cdot), \tilde{g}(t,\cdot)\rangle_Z \ dt ds \right|$$

$$\leq \left( \int_0^1 \int_0^T \left| f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \right|^2 \|g(t,\cdot)\|_Z^2 \; dtds \right)^{1/2} \|\tilde{g}\|_{L^2((0,T),Z)}$$
$$+ \frac{1}{2} \left\| f' \right\|_{C(\mathbb{R})} \|\tilde{g}\|^2_{L^2((0,T),Z)} .$$

Dividing by $\|\tilde{g}\|_{L^2((0,T),Z)}$ yields

$$\frac{|H(g + r\tilde{g}) - H(g) - D_g H(g,\tilde{g})|}{\|\tilde{g}\|_{L^2((0,T),Z)}}$$
$$\leq \left( \int_0^1 \int_0^T \left| f'\left( \|g(t,\cdot) + s\tilde{g}(t,\cdot)\|_Z^2 \right) - f'\left( \|g(t,\cdot)\|_Z^2 \right) \right|^2 \|g(t,\cdot)\|_Z^2 \; dtds \right)^{1/2}$$
$$+ \frac{1}{2} \left\| f' \right\|_{C(\mathbb{R})} \|\tilde{g}\|_{L^2((0,T),Z)} ,$$

which converges to zero as $\tilde{g} \to 0$ in $L^2((0,T),Z)$.

□

# Acknowledgements

First of all I would like to express my sincere gratitude to my supervisor Prof. Dr. Michael Ulbrich for providing me with an enthralling topic for my PhD thesis. I am deeply thankful for his help, advice, patience and guidance throughout the last few years. In particular, I gratefully acknowledge my continued employment at his chair after my funding by IGDK 1754 ended, which he made possible.

My second gratitude addresses to the International Research Training Group IGDK 1754 for including me in the research program and the resulting funding I received which enabled me to make this PhD thesis. Beside the funding, IGDK 1754 provided me with an environment where I could work with great colleagues at the Technical University in Munich as well as at the Karl-Franzens University in Graz. In the graduate seminar, the student workshops and various compact courses I could encounter new topics which helped me a lot for my own research. In particular, I would like to mention Prof. Dr. Kristian Bredies for inviting me to Graz and to work with him on interesting topics there.

Furthermore, I would like to thank all my colleagues at my chair and in the IGDK 1754 program for helping me in numerous conversations to tackle smaller but also bigger problems in the course of the last years. Especially, I want to mention Sebastian E., Sebastian A., Sebastian G., Dominik H., Johannes, Andre, Florian, Lukas H., Martin, Daniel, Behzad and Philip. I highly enjoyed the time I could spend with you.

Finally, I am grateful to my family to encourage and to enable me to study mathematics and to all my friends for their permanent support during my PhD time. In particular, I would like to mention Felix Küng for the fascinating discussions about analysis and algebra we had in countless night hours.

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