Value of Information in Minimum-Rate LQG Control

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Abstract: This study concerns fundamental limitations in control of mobile cyber-physical systems. In these systems, communication between a node and its base station due to limited power of the node is asymmetrical in terms of bandwidth and signal-to-noise ratio. The framework we develop in this paper is for partially observed linear quadratic Gaussian (LQG) control over communication networks in which the forward channel transporting measurements is modeled by a zero-delay packet-deletion channel and the feedback channel transporting control inputs is assumed ideal. Making use of dynamic programming, we characterize the optimal control and the optimal sampling policies that achieve the minimum data rate required for a guaranteed level of control performance. In particular, we prove that in the presence of event-driven sampling the adopted filter is optimal and the separation principle between control and estimation holds. We show that the optimal control policy is a certainty equivalent policy and the optimal sampling policy is a threshold policy expressed in terms of the value of information. Furthermore, we prove that the value of information is a quadratic function of the innovation.

Keywords: Cyber-Physical Systems, Entropy Coding, Event-Driven Sampling, High Resolution Quantization, Rate-Distortion Trade-Off, Value of Information.

1. INTRODUCTION

This study concerns fundamental limitations in control of mobile cyber-physical systems. In these systems, communication between a node and its base station due to limited power of the node is “asymmetrical” in terms of bandwidth and signal-to-noise ratio. Therefore, while the downlink (base-station to node) channel can be assumed ideal, the uplink (node to base-station) channel is constrained and often unreliable. These constraints make feedback control of mobile cyber-physical systems over networks a challenge that demands new foundations both in control and information theories.

We believe a key to improving our understanding about these systems is the “value of information”. Two main information-theoretic measures of information for a pair of random sequences are “mutual information” due to Shannon (1958) and “directed information” due to Massey (1990). In fact, directed information, which first appeared in the context of feedback communication, is a generalization of mutual information in which the causality condition is considered. Nonetheless, the above measures fail to show the difference between important and unimportant events with similar distributions. In other words, they quantify only the amount of reduction in the uncertainty of the recipient about a stochastic process in the environment given the side information regardless of its application. This motivates us to investigate a new measure of information that takes into account operational and economic aspects of information. This value of information should quantify the improvement in the future expenditure of the system given the causal side information.

It is conceived that not every sampled measurement (i.e., causal side information) of a stochastic process has the same effect on the performance of the control system (Åström and Bernhardsson (2002)), and that a sampler can be designed in a way that is adaptive to the changes in the process (Rabi et al. (2012)). In particular, in their seminal work Åström and Bernhardsson (2002) show that for a stochastic scalar linear system under a sampling rate constraint “event-driven sampling” (Lebesgue sampling) outperforms “uniform sampling” (Riemann sampling). This result has received much attention leading to the development of different event-driven sampling policies for estimation and control problems with communication costs or constraints (Miskowicz (2015)). Interestingly, Soleymani et al. (2016) (see also Lipsa and Martins (2011), Molin and Hirche (2012)) show that for the estimation problem of a linear system the optimal sampling policy, without presuming any structure, samples a measurement whenever the value of information exceeds a threshold. This work connects the value of information to event-driven sampling, and is in accordance with the idea of using the value of information for optimal information acquisition (Poole and Mackworth (2010)).

Recently, Schenato et al. (2007) studied LQG control over lossy uplink and downlink channels, and showed that the separation principle does not hold without control packet acknowledgment. Here, we consider the effect of data packet loss due to the unreliability of the uplink channel, and assume that the downlink channel is ideal. In general, there is no mechanism in the network to explicitly signal lost packets to the receiver (Diggavi and Grossglauser (2001)). Network protocols like transmission control protocol (TCP) use sequence and acknowledgment numbers in the packet header to detect and retransmit lost packets. Unfortunately, these numbers introduce communication overhead. Moreover, retransmission of lost packets in our real-time control application because of significant delays is not favorable. In this study, we adopt protocols like user datagram protocol (UDP) with minimum communication overhead. This subsequently changes the channel model from a "packet-erasure channel" to a basic "packet-deletion channel" (Mitzenmacher (2009)). As we will see later, considering a packet-deletion channel not only reduces the communication overhead but yields a simpler optimal filter design.

The framework we develop in this paper is for partially observed linear quadratic Gaussian (LQG) control over communication networks in which the forward channel transporting measurements is modeled by a zero-delay packet-deletion channel and the feedback channel transporting control inputs is assumed ideal. Following the rate-distortion trade-off for control under communication constraints (Tatikonda et al. (2004)), here for the first time we study minimum-rate LQG control with optimal event-driven sampling. Making use of dynamic programming, we characterize the optimal control and the optimal sampling policies that achieve the minimum average data rate required for a guaranteed level of control performance. In particular, we prove that in the presence of event-driven sampling the adopted filter is optimal and the separation principle between control and estimation holds. We show that the optimal control policy is a "certainty equivalent" policy and the optimal sampling policy is a "threshold" policy expressed in terms of the value of information. Furthermore, we prove that the value of information is a quadratic function of the innovation.

The outline of the paper is as follows. After an introduction on notations, the problem is formulated in Section 2. In Section 3, we obtain the optimal filter, prove the separation principle, and characterize the optimal control and the optimal sampling policies. In addition, we determine the architecture of the value of information. Finally, concluding remarks are made in Section 4.

2. PROBLEM FORMULATION

2.1 Notations

In the sequel, we represent an n dimensional vector at time k with x_k. We write x^T to denote the transpose of the vector x. The identity matrix with dimension n is denoted by I_n. We use A^+ to denote the Moore-Penrose inverse of the matrix A. We write $\delta_{kk'}$ to denote the Kronecker delta function. We denote the probability distribution of the stochastic vector x by $\mathbb{P}(x)$, and the expected value and the covariance matrix of the vector x by $\mathbb{E}[x]$ and $\mathbb{C}\mathrm{ov}[x]$, respectively. For matrices A and B, we write $A \succ 0$ and $B \succeq 0$ to mean that A and B are positive definite and positive semi-definite, respectively.

2.2 System Model

Consider a mobile agent with discrete-time stochastic dynamics and measurements generated by the following linear state system:

$$x_{k+1} = Fx_k + Bu_k + w_k, \tag{1a}$$

$$z_k = Hx_k + v'_k, \tag{1b}$$

for the integer $k \geq 0$ where $x_k \in \mathbb{R}^n$ is the state of the system at time k, F is the state matrix, B is the input matrix, $u_k \in \mathbb{R}^m$ is the control input to be decided by a controller at a base station, $w_k \in \mathbb{R}^n$ is a white noise process with zero mean and covariance $R_1\delta_{kk'}$ where $R_1 \succ 0$, $z_k \in \mathbb{R}^p$ is the output of the system observed by a sensor at the agent, H is the output matrix assumed to have full rank, and $v'_k \in \mathbb{R}^p$ is a white noise process with zero mean and covariance $R'_2\delta_{kk'}$ where $R'_2 \succ 0$. It is assumed that the initial state x_0 is normal with mean m_0 and covariance R_0, and that x_0, w_k, and v_k are mutually independent. In addition, it is assumed that (F, B) is controllable, and (F, H) is observable.

The agent and the base station are connected via a communication network in which the feedback channel transporting control inputs is assumed ideal, while the forward channel transporting measurements is modeled by a zero-delay packet-deletion channel (Mitzenmacher (2009)). In the packet-deletion channel, packets are either received by the receiver or lost without any notifications at the receiver according to a Bernoulli arrival process γ_k with mean $\bar{\gamma}$ such that

$$\gamma_k = \begin{cases} 1, & \text{if packet is received,} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Consider a source encoder at the agent that samples the measurements z_k at times k_s for the integer $s \geq 1$. The sampling action δ_k to be decided by the encoder at time k is defined as

$$\delta_k = \begin{cases} 1, & \text{if } \exists s : k = k_s, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

At time $k = k_s$, the sampled measurement z_k is quantized by a high resolution lattice vector quantizer which is equivalent to a scalar uniform quantizer applied p times. Under high resolution quantization, we can adopt the Bennett quantization model (Bennett (1948)) to acquire the quantizer output:

$$y_k = z_k + n_k, \tag{4}$$

where n_k is a white noise process with zero mean and covariance $\Delta^2 I_p \delta_{kk'}/12$ where Δ is the quantization resolution assumed to be small and fixed. The asymptotic validity of this model is rigorously demonstrated by Marco and Neuhoff (2005).

Remark 1. In our setting, sampling and quantization are mathematically commutable operations. Therefore, it can be supposed that when $\delta_k = 0$ nothing is transmitted, and when $\delta_k = 1$ then

$$y_k = Hx_k + v_k, \tag{5}$$

is transmitted, where $v_k = v'_k + n_k$.

The quantizer is followed by a prefix-free entropy coding. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_M\}$ be the alphabet of a binary noiseless code where M is the size of the alphabet. The Kraft inequality (Gersho and Gray (1992)) determines the necessary condition for unique decodability of this code:

$$\sum_{i=1}^{M} 2^{-l_k^i} \leq 1, \tag{6}$$

where l_k^i are codeword lengths for $i = 1, 2, \dots, M$ at time k. Later, we shall exploit the conditional distribution of the measurement y_k to obtain the entropy codeword length that minimizes at each time k the expected codeword length subject to (6).

The measurement y_k transmitted in the form of a packet is instantaneously received by a source decoder at the base station if $\gamma_k = 1$, or is lost otherwise. To provide an acknowledgement of receipt, a control input generated by the controller is then sent along with the corresponding measurement.

Definition 1. The decoder information set at time k is a set of prior transmitted measurements, i.e.,

$$\mathcal{I}_k = \{y_\ell \mid \ell \leq k-1, \delta_\ell = 1, \gamma_\ell = 1\}. \tag{7}$$

Definition 2. The encoder information set at time k is a set of the current measurement and prior transmitted measurements, i.e.,

$$\mathcal{J}_k = y_k \cup \mathcal{I}_k \tag{8}$$

The encoder decides upon transmission of the measurement y_k after its observation, and neglects any prior non-transmitted measurements.

2.3 Performance Index

Consider the sampling policy $\pi = \{\delta_0, \dots, \delta_{N-1}\}$ where $\delta_k = \delta_k(\mathcal{J}_k)$ and the control policy $\mu = \{u_0, \dots, u_{N-1}\}$ where $u_k = u_k(\mathcal{I}_k)$. The control performance over the time horizon N associated with the policies π and μ is measured by the following quadratic functional:

$$I_{\pi,\mu} = \mathbb{E}\Big[x_N^T Q_0 x_N + \sum_{k=0}^{N-1} x_k^T Q_1 x_k + u_k^T Q_2 u_k\Big], \tag{9}$$

where $Q_0 \succeq 0$, $Q_1 \succeq 0$, and $Q_2 \succ 0$ are weighting matrices. Moreover, the transferred information associated with the policy π over the time horizon N is measured by

$$C_\pi = \mathbb{E}\Big[\sum_{k=0}^{N-1} \delta_k L_k\Big], \tag{10}$$

where L_k denotes the entropy codeword length of the measurement y_k.

Remark 2. From the communication viewpoint, transmission of rare events is more costly. This is because they are less probable and have longer codewords.

Our main objective is to achieve the minimum average data rate required for a guaranteed level of control performance I_0. This can succinctly be formulated as

$$\begin{aligned} &\text{minimize} \quad C_\pi \\ &\text{subject to} \quad I_{\pi,\mu} \leq I_0, \end{aligned} \tag{11}$$

over policies π and μ. Following the method of Lagrange multipliers (Bertsekas (1999)), we can recast the problem in (11) as

$$\text{minimize} \quad J_{\pi,\mu} = C_\pi + \lambda I_{\pi,\mu}, \tag{12}$$

where $\lambda \geq 0$. The optimal solution of the problem in (12) for a given value of the Lagrange multiplier λ corresponds to the optimal solution of the problem in (11) for a certain control performance I_0.

3. MAIN RESULTS

3.1 Optimal Filter

Filtration at the decoder is based on the decoder information set $\mathcal{I}_k$. Define $\hat{x}_k = \mathbb{E}[x_k|\mathcal{I}_k]$ and $P_k = \mathbb{C}\text{ov}[x_k|\mathcal{I}_k]$. It is known that Kalman-like filters are not optimal in the presence of event-driven sampling (Sijs and Lazar (2012)). The following lemma proves the optimality of the adopted Kalman-like filter with event-driven sampling in packet deletion channels. The result here is similar to that of Sinopoli et al. (2004).

Lemma 1. Let the forward channel transporting measurements be packet-deletion. The following Kalman-like filter adopted by the decoder minimizes the mean-square error in system (1), (5) under event-driven sampling in (3):

$$\hat{x}_{k+1} = F\hat{x}_k + Bu_k + \delta_k\gamma_k F K_k(y_k - H\hat{x}_k), \tag{13a}$$

$$P_{k+1} = FP_kF^T + R_1 - \delta_k\gamma_k F K_k H P_k F^T, \tag{13b}$$

with initial conditions $\hat{x}_0 = m_0$ and $P_0 = R_0$ where

$$K_k = P_k H^T (HP_kH^T + R_2)^{-1}, \tag{13c}$$

$$R_2 = R_2' + \Delta^2 I_p/12. \tag{13d}$$

Proof. The proof follows directly from the Kolmogorov forward equation and Bayes' rule (Åström (2006)). Notice that in a packet-deletion channel, the decoder does not infer when it does not receive any measurement, because in this case $\delta_k = 0$ and $\gamma_k = 0$ are not distinguishable. Since the conditional distribution remains Gaussian, the Kalman-like filter in (13) is optimal. ■

Remark 3. We assume that the duplicate of the Kalman-like filter in (13) is run locally at the encoder.

3.2 Separation Principle and Optimal Policies

The optimal sampling policy $\pi^\star$ and the optimal control policy $\mu^\star$ are the solution of the following finite horizon stochastic optimization problem:

$$\begin{aligned} \text{minimize} \quad & J_{\pi,\mu} = C_\pi + \lambda I_{\pi,\mu} \\ \text{subject to} \quad & x_{k+1} = Fx_k + Bu_k + w_k, \\ & y_k = Hx_k + v_k, \\ & \hat{x}_{k+1} = F\hat{x}_k + Bu_k + \delta_k\gamma_k F K_k(y_k - H\hat{x}_k), \\ & P_{k+1} = FP_kF^T + R_1 - \delta_k\gamma_k F K_k H P_k F^T, \end{aligned} \tag{14}$$

over polices π and μ with initial conditions x_0, $\hat{x}_0$, and P_0.

The next theorem gives the optimal control policy, and shows a separation between control and estimation. A point that should be noticed through the subsequent derivations is that due to event-driven sampling $\mathbb{E}[e_k|\mathcal{J}_k] \neq 0$ and $\mathbb{E}[e_k e_k^T|\mathcal{J}_k] \neq P_k$ where $e_k = x_k - \hat{x}_k$ is the estimation error.

Definition 3. The value of information α_k at time k is a $\mathcal{J}_k$-measurable function that expresses the maximum value the encoder would be willing to pay for the transmission of the measurement y_k with the codeword length L_k.

Proposition 1. The entropy codeword length L_k of the measurement y_k at time k is obtained by

$$L_k = -\log_2 \big(\mathbb{P}(y_k|\mathcal{I}_k)\Delta\big),$$

where $\mathbb{P}(y_k|\mathcal{I}_k)$ is the conditional distribution of y_k given $\mathcal{I}_k$ and evaluated at y_k.

Proof. See Gersho and Gray (1992). ■

Remark 4. The conditional distribution $\mathbb{P}(y_k|\mathcal{I}_k)$ at time k is calculated as:

$$\mathbb{P}(y_k|\mathcal{I}_k)=\det(2\pi\Sigma_k)^{-\frac{1}{2}}\exp\Big(-\frac{1}{2}(y_k-\hat{y}_k)^T\Sigma_k^{-1}(y_k-\hat{y}_k)\Big),$$

where $\hat{y}_k$ and Σ_k are

$$\hat{y}_k = \mathbb{E}[y_k|\mathcal{I}_k] = H\hat{x}_k,$$
$$\Sigma_k = \mathbb{C}\mathrm{ov}[y_k|\mathcal{I}_k] = HP_kH^T + R_2.$$

Theorem 1. For the minimum-rate LQG control problem characterized in (14), (i) the optimal control policy is a certainty equivalent policy given by $u_k^\star = -G_k\hat{x}_k$ where

$$G_k = (Q_2 + B^TS_{k+1}B)^{-1}B^TS_{k+1}F, \tag{15a}$$
$$S_k = Q_1 + F^TS_{k+1}F - G_k^T(Q_2 + B^TS_{k+1}B)G_k, \tag{15b}$$

where $S_k \succeq 0$ with $S_N = Q_0$, and (ii) the optimal sampling policy is a threshold policy given by

$$\delta_k^\star = \begin{cases} 1, & \text{if } \alpha_k \geq -\log_2\big(\mathbb{P}(y_k|\mathcal{I}_k)\Delta\big), \\ 0, & \text{otherwise,}\end{cases} \tag{16}$$

where α_k is the value of information at time k.

Proof. Part (i). We can write the optimal cost as

$$J^\star = \min_{\pi,\mu} J_{\pi,\mu} = \min_{\pi} J'_\pi,$$

where

$$J'_\pi = \min_{\mu}\{C_\pi + \lambda I_{\pi,\mu}\}.$$

Notice that C_π does not depend on μ. At time k, the control input u_k must be determined as a function of $\mathcal{I}_k$. Define the control cost-to-go V_k as

$$V_k=\min_{\mu}\mathbb{E}\Big[x_N^TQ_0x_N+\sum_{\ell=k}^{N-1}x_\ell^TQ_1x_\ell+u_\ell^TQ_2u_\ell\Big|\mathcal{I}_k\Big].$$

We obtain

$$\begin{aligned} V_k &= \min_{u_k(\mathcal{I}_k)}\mathbb{E}\Big[x_k^TQ_1x_k + u_k^TQ_2u_k \\ &+ \min_{u_{k+1}(\mathcal{I}_{k+1})}\mathbb{E}[x_{k+1}^TQ_1x_{k+1}+u_{k+1}^TQ_2u_{k+1}+\dots|\mathcal{I}_{k+1}]\Big|\mathcal{I}_k\Big] \\ &= \min_{u_k(\mathcal{I}_k)}\mathbb{E}[x_k^TQ_1x_k + u_k^TQ_2u_k + V_{k+1}|\mathcal{I}_k], \end{aligned} \tag{17}$$

with initial condition $V_N = \mathbb{E}[x_N^TQ_0x_N|\mathcal{I}_N]$. We shall prove that the control cost-to-go is of the form $V_k = \hat{x}_k^TS_k\hat{x}_k + \mathbb{E}[s_k|\mathcal{I}_k]$ where $S_k \succeq 0$. For time N, we see

$$V_N = \hat{x}_N^TQ_0\hat{x}_N + \mathbb{E}[e_N^TQ_0e_N|\mathcal{I}_N]. \tag{18}$$

We assume that the hypothesis holds for time $k+1$, and show that it also holds for time k. From the hypothesis, we have

$$\begin{aligned} V_k = \min_{u_k(\mathcal{I}_k)}\mathbb{E}\Big[&x_k^TQ_1x_k + u_k^TQ_2u_k \\ &+ \hat{x}_{k+1}^TS_{k+1}\hat{x}_{k+1} + \mathbb{E}[s_{k+1}|\mathcal{I}_{k+1}]\Big|\mathcal{I}_k\Big]. \end{aligned} \tag{19}$$

We see that

$$\mathbb{E}[x_k^TQ_1x_k|\mathcal{I}_k] = \hat{x}_k^TQ_1\hat{x}_k + \mathbb{E}[e_k^TQ_1e_k|\mathcal{I}_k].$$

In addition, from tower property

$$\mathbb{E}\Big[\mathbb{E}[s_{k+1}|\mathcal{I}_{k+1}]\Big|\mathcal{I}_k\Big] = \mathbb{E}[s_{k+1}|\mathcal{I}_k].$$

Next, we shall obtain the third term in (19). Let us write the estimate dynamics as

$$\hat{x}_{k+1} = F\hat{x}_k + Bu_k + \delta_k\xi_k,$$

where $\xi_k = \gamma_kFK_k(He_k + v_k)$. We can write

$$\begin{aligned}\mathbb{E}[\hat{x}_{k+1}^TS_{k+1}\hat{x}_{k+1}|\mathcal{I}_k]&=(F\hat{x}_k+Bu_k)^TS_{k+1}(F\hat{x}_k+Bu_k) \\ &+\mathbb{E}[\delta_k\xi_k^TS_{k+1}\xi_k|\mathcal{I}_k]+2(F\hat{x}_k+Bu_k)^TS_{k+1}\,\mathbb{E}[\delta_k\xi_k|\mathcal{I}_k],\end{aligned}$$

where the last term is zero. Hence,

$$\begin{aligned} V_k = \min_{u_k(\mathcal{I}_k)}\Big\{&\hat{x}_k^TQ_1\hat{x}_k+u_k^TQ_2u_k+\mathbb{E}[e_k^TQ_1e_k|\mathcal{I}_k] \\ &+ (F\hat{x}_k+Bu_k)^TS_{k+1}(F\hat{x}_k+Bu_k) \\ &+ \mathbb{E}[\delta_k\xi_k^TS_{k+1}\xi_k|\mathcal{I}_k] + \mathbb{E}[s_{k+1}|\mathcal{I}_k]\Big\}. \end{aligned}$$

Note that e_k and ξ_k are not functions of u_k. The minimum exists for all k, because the relevant functions are quadratic and $Q_2 \succ 0$ and $S_{k+1} \succeq 0$. Taking the derivative of the expression in the minimum with respect to u_k and setting it equal to zero, we obtain $u_k^\star = -G_k\hat{x}_k$ where

$$G_k = (Q_2 + B^TS_{k+1}B)^{-1}B^TS_{k+1}F, \tag{20}$$

and

$$V_k = \hat{x}_kS_k\hat{x}_k + \mathbb{E}[s_k|\mathcal{I}_k], \tag{21a}$$

where

$$S_k = Q_1 + F^TS_{k+1}F - G_k^T(Q_2 + B^TS_{k+1}B)G_k, \tag{21b}$$
$$s_k = e_k^TQ_1e_k + \delta_k\xi_k^TS_{k+1}\xi_k + s_{k+1}, \tag{21c}$$

with $S_N = Q_0$ and $s_N = e_N^TQ_0e_N$. This completes the proof of Part (i).

Part (ii). From the definition of the control cost-to-go V_k, we can construct the cost J'_π as follows:

$$\begin{aligned} J'_\pi &= \lambda\,\mathbb{E}[V_0] + \mathbb{E}\Big[\sum_{k=0}^{N-1}\delta_kL_k\Big] \\ &= \lambda\,\mathbb{E}\Big[\hat{x}_0^TS_0\hat{x}_0 + \mathbb{E}[s_0|\mathcal{I}_0]\Big] + \mathbb{E}\Big[\sum_{k=0}^{N-1}\delta_kL_k\Big] \\ &= \lambda m_0^TS_0m_0 + \lambda\,\mathbb{E}[s_0] + \mathbb{E}\Big[\sum_{k=0}^{N-1}\delta_kL_k\Big] \\ &= \lambda m_0^TS_0m_0 + \mathbb{E}\Big[\lambda e_N^TQ_0e_N \\ &\qquad + \sum_{k=0}^{N-1}\lambda e_k^TQ_1e_k + \delta_k\lambda\xi_k^TS_{k+1}\xi_k + \delta_kL_k\Big], \end{aligned}$$

where the first term is known and does not depend on π. At time k, we desire to determine the sampling action δ_k as a function of $\mathcal{J}_k$. Let us introduce the estimation cost-to-go W_k as

$$W_k = \min_{\pi} \mathbb{E}\Big[\lambda e_N^T Q_0 e_N + \sum_{\ell=0}^{N-1} \lambda e_\ell^T Q_1 e_\ell + \delta_\ell \lambda \xi_\ell^T S_{\ell+1} \xi_\ell + \delta_\ell L_\ell \Big| \mathcal{J}_k\Big].$$

We can write

$$\begin{aligned} W_k &= \min_{\delta_k(\mathcal{J}_k)} \mathbb{E}\Big[\lambda e_k^T Q_1 e_k + \delta_k \lambda \xi_k^T S_{k+1} \xi_k + \delta_k L_k \\ &+ \min_{\delta_{k+1}(\mathcal{J}_{k+1})} \mathbb{E}[\lambda e_{k+1}^T Q_1 e_{k+1} + \delta_{k+1} \lambda \xi_{k+1}^T S_{k+2} \xi_{k+1} \\ &\qquad + \delta_{k+1} L_{k+1} + \ldots | \mathcal{J}_{k+1}] \Big| \mathcal{J}_k\Big] \\ &= \min_{\delta_k(\mathcal{J}_k)} \mathbb{E}[\lambda e_k^T Q_1 e_k + \delta_k \lambda \xi_k^T S_{k+1} \xi_k + \delta_k L_k + W_{k+1} | \mathcal{J}_k], \end{aligned} \tag{22}$$

with initial condition $W_N = \lambda\, \mathbb{E}[e_N^T Q_0 e_N | \mathcal{J}_N]$. Notice that the cost-to-go W_{k+1} at time $k+1$ is a function of δ_k, i.e., $W_{k+1} = W_{k+1}(\delta_k)$. However, the error e_k does not depend on δ_k. Therefore, the minimizer $\delta_k^\star$ in (22) is given by

$$\delta_k^\star = \begin{cases} 1, & \text{if } \alpha_k \geq L_k, \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

where

$$\alpha_k = \mathbb{E}\Big[W_{k+1}(0) - W_{k+1}(1) - \lambda \xi_k^T S_{k+1} \xi_k \Big| \mathcal{J}_k\Big]. \tag{24}$$

Moreover, the entropy codeword length of the measurement y_k is $L_k = -\log_2\big(\mathbb{P}(y_k|\mathcal{I}_k)\Delta\big)$. This completes the proof of Part (ii). ■

3.3 Value of Information

Let $\nu_k = y_k - H\hat{x}_k$ be the innovation at time k, associated with the system in (1), (5) and the filter in (13). The next theorem proves that the value of information α_k is a quadratic function of the innovation.

Theorem 2. The value of information α_k defined in (24) associated with the minimum-rate LQG control problem in (14) is a quadratic function of the innovation ν_k, i.e., there exists a weighting matrix $\Lambda_k \succeq 0$ such that

$$\alpha_k = \lambda \nu_k^T \Lambda_k \nu_k. \tag{25}$$

Proof. First, we shall prove that the estimation cost-to-go is of the form $\mathbb{E}[W_k|\mathcal{J}_k] = \lambda \nu_k^T T_k \nu_k + t_k$. Let us assume the hypothesis holds for time $k+1$. We show that it also holds for time k. First, we see that

$$\begin{aligned} \mathbb{E}[W_{k+1}|\mathcal{J}_k] &= \mathbb{E}\left[\mathbb{E}[W_{k+1}|\mathcal{J}_{k+1}]|\mathcal{J}_k\right] \\ &= \mathbb{E}[\lambda \nu_{k+1}^T T_{k+1} \nu_{k+1} + t_{k+1} | \mathcal{J}_k]. \end{aligned} \tag{26}$$

Following the equation (22) and from the hypothesis (26), we have

$$\begin{aligned} W_k = \min_{\delta_k(\mathcal{J}_k)} \mathbb{E}\Big[&\lambda e_k^T Q_1 e_k + \delta_k \lambda \xi_k^T S_{k+1} \xi_k + \delta_k L_k \\ &+ \lambda \nu_{k+1}^T T_{k+1} \nu_{k+1} + t_{k+1} \Big| \mathcal{J}_k\Big]. \end{aligned} \tag{27}$$

In the following, we shall obtain the conditional expectation of the terms in (27). But before that we derive the conditional mean and covariance of the measurement noise v_k and of the error e_k. For the measurement noise v_k and the measurement y_k at time k, it is easy to show that

$$\mathbb{E}\left[v_k, y_k | \mathcal{I}_k\right] = \begin{bmatrix} 0 \\ H\hat{x}_k \end{bmatrix},$$

$$\mathbb{C}\text{ov}\left[v_k, y_k | \mathcal{I}_k\right] = \begin{bmatrix} R_2 & R_2 \\ R_2 & H^T P_k H + R_2 \end{bmatrix}.$$

We can find the mean and the covariance of the measurement noise v_k conditioned on $\mathcal{J}_k$:

$$\mathbb{E}[v_k|\mathcal{J}_k] = R_2(HP_kH^T + R_2)^{-1}\nu_k, \tag{28}$$

$$\mathbb{C}\text{ov}[v_k|\mathcal{J}_k] = R_2 - R_2(HP_kH^T + R_2)^{-1}R_2. \tag{29}$$

Let us write

$$He_k = Hx_k - H\hat{x}_k = y_k - H\hat{x}_k - v_k = \nu_k - v_k.$$

Notice that ν_k is $\mathcal{J}_k$-measurable. Thus,

$$\mathbb{E}[e_k|\mathcal{J}_k] = \Gamma_k \nu_k, \tag{30}$$

$$\mathbb{C}\text{ov}[e_k|\mathcal{J}_k] = \Gamma_k R_2 H^{+T}. \tag{31}$$

where $\Gamma_k = H^+(I_p - R_2(HP_kH^T + R_2)^{-1})$.

For the first term in (27), we have

$$\begin{aligned} \mathbb{E}[e_k^T Q_1 e_k | \mathcal{J}_k] &= \mathbb{E}[e_k|\mathcal{J}_k]^T Q_1\, \mathbb{E}[e_k|\mathcal{J}_k] \\ &\quad + \operatorname{tr}\left(Q_1\, \mathbb{C}\text{ov}[e_k|\mathcal{J}_k]\right) \\ &= \nu_k^T \Gamma_k^T Q_1 \Gamma_k \nu_k + \operatorname{tr}(Q_1 \Gamma_k R_2 H^{+T}). \end{aligned} \tag{32}$$

In fact, $\xi_k = \gamma_k F K_k \nu_k$. Hence, for the second term in (27) we find

$$\mathbb{E}[\delta_k \xi_k^T S_{k+1} \xi_k | \mathcal{J}_k] = \delta_k \bar{\gamma}^2 \nu_k^T K_k^T F^T S_{k+1} F K_k \nu_k. \tag{33}$$

Moreover, we observe that $\nu_k = He_k + v_k$. Thus, for the fourth term in (27) we can write

$$\begin{aligned} \mathbb{E}[\nu_{k+1}^T T_{k+1} \nu_{k+1} | \mathcal{J}_k] &= \mathbb{E}[e_{k+1}^T H^T T_{k+1} H e_{k+1} | \mathcal{J}_k] \\ &\quad + \mathbb{E}[v_{k+1}^T T_{k+1} v_{k+1} | \mathcal{J}_k] \\ &\quad + 2\, \mathbb{E}[e_{k+1}^T H^T T_{k+1} v_{k+1} | \mathcal{J}_k], \end{aligned} \tag{34}$$

where the last term is zero, and the second term is written as

$$\mathbb{E}[v_{k+1}^T T_{k+1} v_{k+1} | \mathcal{J}_k] = \operatorname{tr}(T_{k+1} R_2).$$

Next, we obtain the first term in (34). Observe that the dynamics of the error e_k is given by

$$e_{k+1} = Fe_k - \delta_k \gamma_k F K_k \nu_k + w_k.$$

Since ν_k is $\mathcal{J}_k$-measurable and w_k is independent of y_k, we can obtain

$$\begin{aligned} \mathbb{E}[e_{k+1}|\mathcal{J}_k] &= F\, \mathbb{E}[e_k|\mathcal{J}_k] - \mathbb{E}[\delta_k \gamma_k F K_k \nu_k | \mathcal{J}_k] \\ &= F\Gamma_k \nu_k - \delta_k \bar{\gamma} F K_k \nu_k, \end{aligned}$$

and

$$\begin{aligned} \mathbb{C}\text{ov}[e_{k+1}|\mathcal{J}_k] &= F\, \mathbb{C}\text{ov}[e_k|\mathcal{J}_k] F^T + \mathbb{C}\text{ov}[w_k|\mathcal{J}_k] \\ &= F\Gamma_k R_2 H^{+T} F^T + R_1. \end{aligned}$$

Therefore, we find

$$\begin{aligned} &\mathbb{E}[e_{k+1}^T H^T T_{k+1} H e_{k+1} | \mathcal{J}_k] \\ &\quad = \mathbb{E}[e_{k+1}|\mathcal{J}_k]^T H^T T_{k+1} H\, \mathbb{E}[e_{k+1}|\mathcal{J}_k] \\ &\qquad + \operatorname{tr}\left(H^T T_{k+1} H\, \mathbb{C}\text{ov}[e_{k+1}|\mathcal{J}_k]\right) \\ &\quad = \nu_k^T (\Gamma_k - \delta_k \bar{\gamma} K_k)^T F^T H^T T_{k+1} H F (\Gamma_k - \delta_k \bar{\gamma} K_k) \nu_k \\ &\qquad + \operatorname{tr}\left(H^T T_{k+1} H (F\Gamma_k R_2 H^{+T} F^T + R_1)\right). \end{aligned}$$

Bringing everything together, we have

$$
\begin{aligned}
W_k = \min_{\delta_k(\mathcal{J}_k)} \Big\{ & \lambda \nu_k^T \Gamma_k^T Q_1 \Gamma_k \nu_k + \lambda \operatorname{tr}(Q_1 \Gamma_k R_2 H^{+T}) \\
& + \delta_k L_k + \delta_k \lambda \bar{\gamma}^2 \nu_k^T K_k^T F^T S_{k+1} F K_k \nu_k \\
& + \lambda \nu_k^T (\Gamma_k - \delta_k \bar{\gamma} K_k)^T F^T H^T T_{k+1} H F \\
& \quad \times (\Gamma_k - \delta_k \bar{\gamma} K_k) \nu_k \\
& + \lambda \operatorname{tr}\big(H^T T_{k+1} H (F \Gamma_k R_2 H^{+T} F^T + R_1)\big) \\
& + \lambda \operatorname{tr}(T_{k+1} R_2) + t_{k+1} \Big\},
\end{aligned} \tag{35}
$$

that for the minimizer $\delta_k^\star$ can be written as

$$
W_k = \begin{cases} \lambda \nu_k^T T_k^1 \nu_k + t_k^1, & \text{if } \delta^\star = 1, \\ \lambda \nu_k^T T_k^0 \nu_k + t_k^0, & \text{otherwise}, \end{cases} \tag{36}
$$

where $T_k^1, T_k^0 \succeq 0$. One can show that

$$
\mathbb{E}[W_k | \mathcal{J}_k] = \lambda \nu_k^T T_k \nu_k + t_k, \tag{37}
$$

where $T_k \succeq 0$. Moreover, for time N we have

$$
\mathbb{E}[e_N^T Q_0 e_N | \mathcal{J}_N] = \lambda \nu_N^T \Gamma_N^T Q_0 \Gamma_k \nu_N + \operatorname{tr}(Q_0 \Gamma_N R_2 H^{+T}).
$$

Hence, the estimation cost-to-go is of the form $\mathbb{E}[W_k|\mathcal{J}_k] = \lambda \nu_k^T T_k \nu_k + t_k$. Following the definition of the value of information in (24) and from the equation (35), we conclude that the value of information is a quadratic function of the innovation, i.e., $\alpha_k = \lambda \nu_k^T \Lambda_k \nu_k$ where

$$
\begin{aligned}
\Lambda_k = {} & 2\bar{\gamma} \Gamma_k^T F^T H^T T_{k+1} H F K_k \\
& - \bar{\gamma}^2 K_k^T F^T H^T T_{k+1} H F K_k - \bar{\gamma}^2 K_k^T F^T S_{k+1} F K_k.
\end{aligned} \tag{38}
$$

This completes the proof. ■

Remark 5. The parametric architecture of the estimation cost-to-go provided by Theorem 2 can be tuned offline using techniques from approximate dynamic programming (Bertsekas and Tsitsiklis (1996)).

4. CONCLUSION

In this work, we studied the minimum-rate partially observed LQG control with optimal event-driven sampling over communication networks in which the forward channel is modeled by a zero-delay packet-deletion channel and the feedback channel is assumed ideal. We proved the separation principle between control and estimation, and showed that the optimal control policy is a certainty equivalent policy and the optimal sampling policy is a threshold policy expressed in terms of the value of information. Finally, we proved that the value of information is a quadratic function of the innovation.

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