Journal of Mathematical Imaging and Vision 17: 7–14, 2002

Digital Topology: Regular Sets and Root Images of the Cross-Median Filter

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Abstract. In the study of topological properties of digital images, which is the central topic of digital topology, one is often interested in special operations on object boundaries and their properties. Examples are contour filling or border following. In classical topology there exists the strong concept of regularity: regular sets in $\mathbb{R}^2$ show no "exotic behaviour" and are extensively used in the theory of boundary value problems. In this paper we transfer the concept of regularity to digital topology within the framework of semi-topology. It is shown that regular open sets in (a special) semi-topology can be characterized graphically. A relationship between digital topology and image processing is established by showing that regular open digital sets, interpreted as digital pictures, are left unchanged when the cross-median filter is applied.

Keywords: digital topology, median filter, regularity, semi-topology

1. Introduction

Over the last two decades, digital topology has proved to be a very strong concept in image analysis and image processing. It was Rosenfeld [13] who first presented the fundamentals of digital topology, which provides a sound mathematical basis for image processing operations such as image thinning, border following, contour filling and object counting (see [7] for a survey). Whenever spatial relations are modeled on a computer, a digital topology is needed.

Basically, digital topology aims to transfer concepts from classical topology (such as connectivity of objects, properties of their boundary and their neighbourhood, as well as continuity) to digital spaces (such as $\mathbb{Z}^2$), which are used to model computer images.

Among the different approaches to digital topology (see, e.g., [5, 8] or [13]), we use the concept of "semi-topology" introduced by Latecki [9]. In contrast to other concepts of digital topologies, the main advantage of this concept is that we can directly transfer many concepts from classical topology to digital topology, which is rather surprising since it is well known (see [10]) that in digital topology it is not possible to construct a classical topological theory. In this paper we use the 8-semi-topology, because it has the property that the 8-connected sets of Rosenfeld, which play a prominent role in connectivity issues in computer images, are exactly the 8-(semi-topological) connected sets in this semi-topology.

Within the framework of semi-topology we are able to introduce the concept of regularity in digital topology. Regularity is a strong concept in classical topology, and it is used in connection with boundary value problems and strict expansions of topological spaces.

We establish a connection between regular sets in the 8-semi-topology and root images of the two-dimensional cross-median filter in digital pictures. Root images in digital pictures are sets which are left invariant when a filter is applied.

For one-dimensional signals, the root images of median filters are well understood [4]. However, in the two-dimensional case, little is known. Astola, Heinonen and Neuvo [2] showed how to construct root images for two-dimensional median filters which operate on a rectangular window. However, Döhler [3] constructed special convex root images of more general two-dimensional median filters. We shall show in this paper that there exist also non-convex root images of the cross-median filter.

Median filters are frequently used in image processing, especially for de-noising computer images. Knowledge of these root images is of practical as well as of theoretical importance in digital image analysis: If we de-noise a computer image and if the objects of interest change their shape considerably, then it is not advisable to apply this special median filter. On the other hand, if the objects of interest in a computer image are the root images of the applied median filter then we have the guarantee that these objects remain unchanged in the de-noised image.

This paper is organized as follows: In Section 2 we introduce the semi-topology. In Section 3 we graphically characterize regular open sets in the 8-semi-topology, while the proof of the theorem (Theorem 1) is given in the appendix. Furthermore, we show a relationship between regular sets and root images of two-dimensional median filters in Section 4. We conclude with a short outlook in Section 5.

2. Semi-Topology

Definition 1. A **semi-topology** on a set X is a system $\mathcal{O}$ of subsets of X, which meets the following conditions:

1. $\emptyset, X \in \mathcal{O}$,
2. The union of every subsystem of $\mathcal{O}$ is a member of $\mathcal{O}$.

Clearly, this a more general definition than the definition of a topology, because the axiom about the intersection of two open sets is omitted. This concept was first introduced by Latecki [9]. He showed that it provides a suitable framework for topological issues in image processing. Here we refrain from reviewing the results of Latecki in detail which can be found in [9] and [10]. Instead, we deal with two special semi-topologies, the 8- and 4-semi-topology. What makes these semi-topologies important to image processing is that the 8- and 4-connected sets in the corresponding semi-topology are exactly the 8- and 4-connected sets in the common graph-theoretical interpretation. Before we define these two semi-topologies, we first mention what we understand by a digital picture. A two-dimensional digital picture is as usual [10] a tuple $(\mathbb{Z}^2, B)$, where $B \subseteq \mathbb{Z}^2$. The elements of $\mathbb{Z}^2$ are called points of the digital picture, and the elements of B are called the black points of the picture, and the points in $\mathbb{Z}^2 \setminus B$ are called the white points of the picture. The relationship between digital pictures and sets $B \subseteq \mathbb{Z}^2$ in a semi-topology is obvious: B can be regarded as the set of black (or white) points of a digital picture and vice versa. Now, we define the 8- and 4-semi-topologies by their "point bases" $U_8(p)$ and $U_4(p)$, respectively, for $p = (p_1, p_2) \in \mathbb{Z}^2$:

$$\begin{aligned} U_8(p_1,p_2) :&= \{(q_1,q_2) \in \mathbb{Z}^2 \mid \max(|q_1 - p_1|, \\ &\quad |q_2 - p_2|) \leq 1\} \text{and} \\ U_4(p_1,p_2) :&= \{(p_1,p_2-1),(p_1-1,p_2),(p_1,p_2), \\ &\quad (p_1+1,p_2),(p_1,p_2+1)\}. \end{aligned}$$

Then, $B_8 := \bigcup\{U_8(p) | p \in \mathbb{Z}^2\}$ is a base of $(\mathbb{Z}^2, \mathcal{O}_8)$ and $B_4 := \bigcup\{U_4(p) | p \in \mathbb{Z}^2\}$ is a base of $(\mathbb{Z}^2, \mathcal{O}_4)$. Here "$B_8$ is a base of $(\mathbb{Z}^2, \mathcal{O}_8)$" means that every member of $\mathcal{O}_8$ is the union of the sets belonging to B_8, like in the classical definition. The same applies to B_4. We notice that every point p has a smallest neighbourhood in the 8- and 4-semi-topology, namely $U_8(p)$ or $U_4(p)$, respectivly. We also notice that $(\mathbb{Z}^2, \mathcal{O}_8)$ and $(\mathbb{Z}^2, \mathcal{O}_4)$ are not classical topologies at all, because there are two open sets, e.g. $U_8(p_1,p_2)$ and $U_8(p_1+1,p_2)$, which have a non-open intersection. Among the topological properties and operators we consider the interior operator

$$\operatorname{int} B := \{x \in \mathbb{Z}^2 | \exists y \in \mathbb{Z}^2 \text{ with } x \in U(y) \subseteq B\}$$

and the closure operator

$$\begin{aligned} \operatorname{cl} B := \{x \in \mathbb{Z}^2 | \forall y \in \mathbb{Z}^2 \text{ with } x \in U(y) \\ \text{follows } U(y) \cap B \neq \emptyset\}. \end{aligned}$$

3. Regular Open Sets in the 8-Semi-Topology

In this section we introduce the concept of regularity in semi-topological spaces. We investigate here regular open sets in the 8-semi-topology and obtain a characterization which enables us to decide whether an open set is also regular open by comparing this set with special patterns. This gives us a graphical criterion for regularity (regular openness) in the

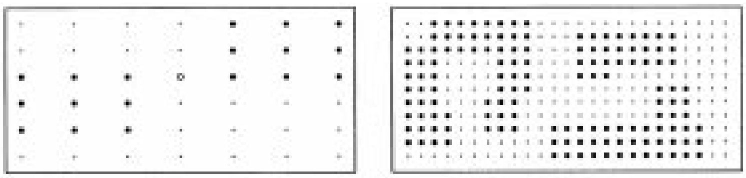

Figure 1. The examples of regular open sets (black points) in the 8-semi-topology.

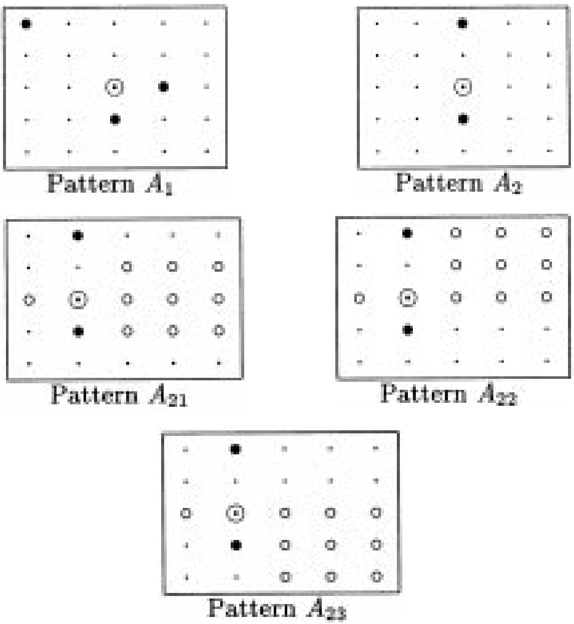

Figure 2. Patterns for characterizing regular open sets in the 8-semi-topology.

8-semi-topology at hand. Regularity in classical topology is a strong concept. Stone [14] defined an open set G of a topological space to be *regular open* if $\overline{G}^o = G$. He investigated spaces with a basis of regular open sets. This concept was useful in investigating strict expansions of topological spaces (see [12]). In the theory of boundary value problems, or more generally in the theory of Sobolev spaces, one usually assumes that the domain Ω of the differential operator is an open (simply connected) subset in $\mathbb{R}^d$. However, the boundary values are prescribed on the boundary of Ω. The closure $\overline{\Omega}$ of an open set is regular open in the sense that the closure of its interior is the set itself. So one also can require that the differential operator and its boundary values are defined on a regular open set $\overline{\Omega}$. For a more in-depth investigation there are more stringent conditions necessary. One example is the cone property which is needed in connection with interpolation, extension theorems [1] and embeddings.

The following definition is an analogue to the topological definition.

Definition 2. Let $(X, \mathcal{O})$ be a semi-topological space. $B \subseteq X$ is called a regular open set in $(X, \mathcal{O})$ if $B = \operatorname{int} \operatorname{cl} B$.

Of course, a regular open set is an open set.

Figure 1 shows two regular open sets in the 8-semi-topology. The set of black points is the set B, whereas the dots and the white point (○) represent points of $\mathbb{Z}^2 \setminus B$. Both examples show non-convex sets B.

The fact that the left example of Fig. 1 shows a regular open set can be seen as follows: The closure cl$B$ consists of the black points and the white point (○) in the center of the 7×6 grid (Every 3×3 block (8-semi-open set) containing this point hits a point of B, and thus is a member of cl$B$). This center point is again removed when cl$B$ is opened, because there is no black 3×3 block (8-semi-open set) containing it.

The five point patterns A_1, A_2, A_{21}, A_{22} and A_{23} are shown in Fig. 2. These patterns consist of four different types of points: black points (●), white points (○), dots (·) and a ⊙. Suppose a pattern A, as in the aforementioned figures, and a digital picture $C = (\mathbb{Z}^2, B)$ is given. For better understanding we only give an informal description of what we mean by the statement: "The pattern A occurs at point p". We lay the pattern A on top of C in such a way that p and the point ⊙ lie on top of each other. Then A occurs at p if we can apply reflections or rotations by a multiple of 90 degrees to A such that all black points of A are black points in C, and the white points of A are white points of C. The point ⊙ of A counts as a white point, and the dots in A indicate points in $\mathbb{Z}^2$ which can be coloured either white or black in C.

Now we may characterize regular open sets in the 8-semi-topology by means of patterns. The proof of the following theorem is given in the appendix.

Theorem 1. *Let* $(\mathbb{Z}^2, B)$ *be a digital picture and B be an open set in the 8-semi-topology.*

B is a regular open set $\Leftrightarrow$ $\forall p \notin B$ *it holds:*
(*i*) *Pattern* A_1 *does not occur at p*;
(*ii*) *If pattern* A_2 *occurs at p then pattern* A_{21}, A_{22} *or* A_{23} *occurs, too.*

With this characterization at hand we can easily verify whether a given digital picture $(\mathbb{Z}^2, B)$, which is an open set in the 8-semi-topology, is also a regular open set. This charaterization also leads to an algorithm for checking the regularity: We just have to check for every $p \notin B$ whether the given patterns A_1, A_2, A_{21}, A_{22} or A_{23} occur. If there is a point for which the pattern A_1 occurs (Property (i) of the above theorem is violated), then we can conclude that B is not a regular open set. If this is not the case and Property (ii) holds, then B is a regular open set.
Similar charaterizations can be found for regular sets in other semi-topologies, such as the 4-semi-topology, but we will not present this case here.

4. Root Images of the Cross-Median Filter

As a corollary of our charaterization theorem we show that digital pictures $(\mathbb{Z}^2, B)$, where B is a regular open set in the 8-semi-topology, are root images of the cross-median filter.

Median filters are quite popular tools in image processing. For example, they are often used to de-noise (or de-speckle) images (see [11]). Essentially, a median filter averages the pixel values of a computer image in a certain "window". Let us now define this filter mathematically. The cross-median filter Med_4 on digital pictures is a mapping which maps $(\mathbb{Z}^2, B)$ to $(\mathbb{Z}^2, B')$ with

$$B' = \{p \in \mathbb{Z}^2 : |U_4(p) \cap B| \geq 3\}.$$

The cross-median filter owes its name from the set $U_4(p)$ which is cross-shaped.

A root image of Med_4 is a digital picture $(\mathbb{Z}^2, B)$ with $Med_4((\mathbb{Z}^2, B)) = (\mathbb{Z}^2, B)$. This means, that a root image of Med_4 is a digital picture which is left unchanged by the cross-median filter. The term root image is a term that was originally introduced in the theory of signal processing. A digital picture can be regarded as a two-dimensional binary signal, and a filter is an operator which maps one signal to another. A root image of a filter is a signal which is left unchanged by applying this filter to the signal.

Root images play a prominent role in signal theory, because they help us to understand the properties of filters and to mathematically justify their use in different applications. This is demonstrated by the following example: Let us suppose that we want to detect an object of a given shape in a noisy digital picture. In order to de-noise this picture we apply a median filter. However, if this filter does not only remove the noise, but also the object of interest in the image (or if it changes the shape of the object considerably), then it is not advisable to apply this filter.

Clearly, the best choice of a median filter would be the median filter where our objects of interest are root images of the filter. The following corollary shows that if our objects are regular open sets in the 8-semi-topology then they are root images of the cross-median filter.

Corollary 1. *If* $B \subseteq \mathbb{Z}^2$ *is a regular open set in the 8-semi-topology, then* $(\mathbb{Z}^2, B)$ *is a root image of the cross-median filter, i.e.*

$$Med_4((\mathbb{Z}^2, B)) = (\mathbb{Z}^2, B).$$

Proof: Let us suppose that B is a regular open set in the 8-semi-topology and $Med_4((\mathbb{Z}^2, B)) = (\mathbb{Z}^2, B')$ with $B \neq B'$. We must consider two cases:

(a) Suppose, there is a point $p \in B \setminus B'$. This means that $p \in B$ has at least three 4-neighbours which are no elements of B. A contradiction to the openness of B.

(b) If $p \in B' \setminus B$, then p clearly has at least three 4-neighbours which are elements of B. Because of the openness of B, the pattern A_2 would occur at p, whereas patterns A_{21}, A_{22} and A_{23} would not occur at p. Thus Property (ii) of Theorem 1 does not hold, which contradicts that B is a regular open set.

Special root images of two-dimensional median filters were investigated in [2] and [3]. The previous corollary characterizes a rather large class of root images of the cross-median filter, which also includes some non-convex root images (see Fig. 1).

It is also possible to prove this corollary without referring to Theorem 1. Essentially one has to apply similar reasoning to that used in the proof of Theorem 1, which leads to a more technical proof of the corollary.

Note that there are also root images of the cross-median filter which are not regular open sets in the 8-semi-topology. Take, for example,

$$B = \{(p_1, p_2), (p_1+1, p_2), (p_1, p_2+1), (p_1+1, p_2+1)\}$$

which is not an open set, but in fact is a root image of the median filter.

5. Conclusions

In this paper we have shown how the topological concept of regularity can be transferred to the framework of semi-topology, and thus to digital topology. We have investigated regular open sets in the 8-semi-topology and obtained a graphical characterization of these sets (Theorem 1). Furthermore, we have shown that these sets, interpreted as a digital pictures, have the property that they remain unchanged when the cross-median filter is applied (Corollary 1). This is a new result on root images of a special two-dimensional median filter, which also shows, e.g., that there are non-convex root images for the cross-median filter.

By defining the point bases $U(p)$ of points $p \in \mathbb{Z}^2$ it is possible to introduce various semi-topologies. It seems likely, that regular sets in these semi-topologies could be characterized in the same graphical way as in the 8-semi-topology. This result can be obtained, e.g., in the 4-semi-topology. These characterizations give a graphical image of the abstract concept of regularity in digital topology and thus permit the mathematically modeling of objects with prescribed "regular" boundaries.

The fact that objects with "regular" boundaries can be of particular interest was illustrated by the result on root images of the cross-median filter in Corollary 1. This approach in digital topology provides a new way of investigating root images of two-dimensional median filters. Furthermore, it is a very promising approach for applications in digital image processing, since median filters are used for de-noising digital images. Knowing their root images, help us to specify acceptable and unacceptable changes caused by such filtering.

6. Acknowledgments

I am grateful to Prof. Dr. Ulrich Eckhardt from the Institut für Angewandte Mathematik, Universität Hamburg, for directing my attention to regular sets and providing me with useful information on the literature. The investigations reported in this paper were carried out in the framework of a diploma thesis at the University of Hamburg.

7. Appendix

Now we give the proof of Theorem 1 (Section 3).

Theorem 1. *Let* $(\mathbb{Z}^2, B)$ *be a digital picture and B be an open set in the 8-semi-topology.*

B is a regular open set $\Leftrightarrow$ $\forall p \notin B$ *it holds*:
(*i*) *Pattern* A_1 *does not occur at* p;
(*ii*) *If pattern* A_2 *occurs at p then pattern* A_{21}, A_{22} or A_{23} *occurs, too.*

Proof: Since B is an open set and $B \subseteq \mathrm{cl}\,B$, $B \subseteq \mathrm{int}\,\mathrm{cl}B$. Thus, B is a regular open set if and only if

$$\mathbb{Z}^2 \setminus B \subseteq \mathbb{Z}^2 \setminus \mathrm{int}\,\mathrm{cl}\,B. \tag{1}$$

"$\Rightarrow$": We want to show that if B is a regular open set then Properties (i) and (ii) hold. We show this indirectly: If a point $p = (p_1, p_2) \notin B$ does not meet Property (i) or Property (ii) then $p \in \mathrm{int}\,\mathrm{cl}B$ in contradiction to (1).

(a) Let pattern A_1 occur at p. Without loss of generality we can assume that pattern A_1 occurs at p and consequently $(p_1+1, p_2) \in B$. Since B is open there is a 3×3 block of black points containing (p_1+1, p_2) but not p. There are three different blocks of this type. And the reader may verify that in each case cl$B$ contains a black 3×3 block containing p. Thus $p \in$ int cl$B$.

(b) Let the pattern A_2 occur at p. We also assume that neither the patterns A_{21}, A_{22} nor A_{23} occur. Furthermore we can assume (p_1+1, p_2), $(p_1-1, p_2) \notin B$, for otherwise we could conclude in a similar way to (a) that $p \in \text{int}\,\text{cl}B$. It also holds that (p_1+1, p_2), $(p_1-1, p_2) \in \text{cl}B$, for otherwise one of the patterns A_{21}, A_{22} or A_{23} would occur. This implies, without loss of generality, (p_1+1, p_2+1), $(p_1+2, p_2+1) \in B$ (for otherwise $p \in \text{int}\,\text{cl}B$). It follows that $(p_1+2, p_2) \notin B$ is not contained in cl$B$, and the same holds for (p_1+2, p_2-1), (p_1+2, p_2-2), (p_1+3, p_2), (p_1+3, p_2-1) and (p_1+3, p_2-2) (for otherwise $p \in \text{int}\,\text{cl}B$). Because of $(p_1+1, p_2) \in \text{cl}B$ it follows that $(p_1+1, p_2-2) \in B$ and therefore $(p_1-1, p_2-2) \in B$. But this means that $(p_1-1+i, p_2-2+j) \in \text{cl}B$ for $0 \leq i, j \leq 2$. And thus leading to the contradiction $p \in$ int cl$B$. See Fig. 3.

"⇐": We assume that Property (i) and Property (ii) hold for every $p \notin B$. To show that B is a regular open set we have to show for $p \in \text{cl}B \setminus B$ that $p \notin \text{int}\,\text{cl}B$.

Let $p \in \text{cl}B \setminus B$. The point p could not have three or four black 4-neighbours because this would contradict Property (ii). Thus we have to distinguish five different cases. Figure 4 shows the two cases (a) and (b) which have to be investigated if p has two black 4-neighbours. It also shows a slightly more generalized version of two black neighbours in one line, which includes as a subcase the case of two black 4-neighbours in one line (because B is an open set).The two cases (c) and (d) of Fig. 4 have to be investigated if p has exactly one black 4-neighbour and when the Case (b) does not occur. Finally, the Case (e) of Fig. 4 has to be investigated which is the case that p has no black 4-neighbours.

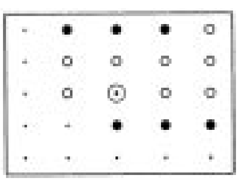

Figure 3. This figure illustrates our reasoning in Case (b).

(a) First we observe that (p_1, p_2-1), (p_1, p_2-2), (p_1-1, p_2) and (p_1-2, p_2) are white points, because otherwise pattern A_2 but not A_{21}, A_{22} and A_{23} would occur at p. The point (p_1-2, p_2-2) is white, because pattern A_1 does not occur. Because B is open it follows that (p_1-1, p_2-2), (p_1-1, p_2-1) and (p_1-2, p_1-1) are white points. But then $p \notin \text{cl}B$ which contradicts $p \in \text{cl}B$. Thus Case (a) could not occur under the assumption of Property (i) and Property (ii). See Fig. 5 for an illustration of our reasoning.

(b) Without loss of generality we can assume that $(p_1-1, p_2) \notin \text{int}\,\text{cl}B$. First we consider: (p_1-1, p_2+1), $(p_1+1, p_2+1) \in B$. Then (p_1+1, p_2-2), $(p_1+2, p_2-2) \in B$. But now occurs pattern A_2 at (p_1+1, p_2) but neither pattern A_{21}, A_{22} nor A_{23}. Thus, $(p_1+2, p_2) \notin B \cup \text{int}\,\text{cl}B \Rightarrow p \notin \text{int}\,\text{cl}B$. The second case to consider is: (p_1-2, p_2+1), $(p_1-1, p_2-1) \in B$. Then (p_1+1, p_2-2), $(p_1+2, p_2-2) \in B$. Furthermore $(p_1+1, p_2+1) \notin B$, because we just considered the other case. Now, $(p_1+3, p_2-1) \notin B$,

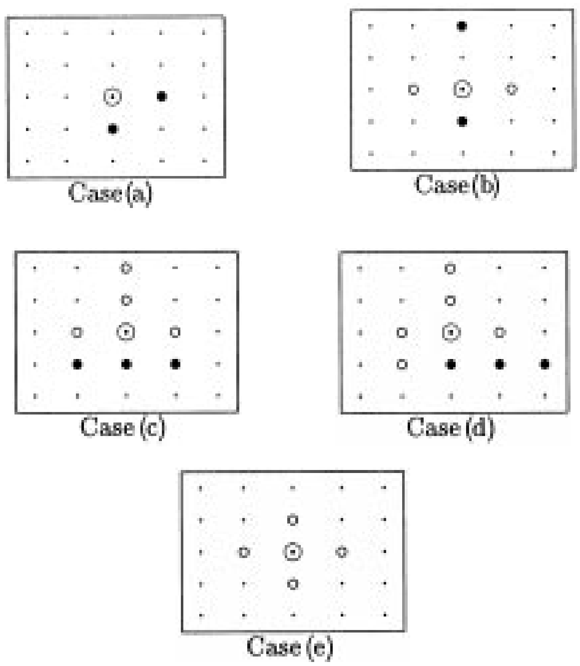

Figure 4. The considered cases in the "⇐" part of the proof of Theorem 1.

because otherwise we would obtain (p_1, p_2-1), (p_1+1, p_2-1), $(p_1+2, p_2-1) \in B$, and in the same way $(p_1+1+i, p_2+j) \notin B$, for $0 \le i,j \le 2$, which means $p \notin \mathrm{int\,cl}B$. Thus, $(p_1+1, p_2+1) \notin B$. If $(p_1+1, p_2-1) \notin B$, then $(p_1+1+i, p_2-1+j) \notin B$, for $0 \le i, j \le 2$, i.e. $p \notin \mathrm{int\,cl}B$. If $(p_1+1, p_2-1) \in B$, then (p_1+1, p_2+2), $(p_1+3, p_2+2) \notin B \Rightarrow (p_1+1+i, p_2+j) \notin B$, for $0 \le i,j \le 2$, i.e. $p \notin \mathrm{int\,cl}B$. The third case to consider is: (p_1+1, p_2+1), $(p_1+2, p_2+1) \in B$ and $(p_1-1, p_2+1) \notin B$. Consequently, $(p_1+1, p_2-2) \notin B$ (for otherwise $(p_1+2, p_2) \notin \mathrm{cl}B\ p \notin \mathrm{int\,cl}B$), which leads to (p_1-2, p_2-2), $(p_1-1, p_2-2) \in B$ and, as in the previous case, $(p_1+1+i, p_2-2+j) \notin B$, for $0 \le i,j \le 2$, i.e. $p \notin \mathrm{int\,cl}B$.

(c) Without loss of generality we have $(p_1+1, p_2-2) \in B$ because $p \in \mathrm{cl}B$. Because pattern A_1 does not occur at (p_1-1, p_2) it follows that $(p_1-2, p_2) \notin B$. Also $(p_1-2, p_2-2) \notin B$, because otherwise $(p_1-2, p_2-3) \in B$; and this contradicts that one of the patterns A_{21}, A_{22} or A_{23} occurs at (p_1, p_2-2). From the openness of B follows $(p_1-2+i, p_2-2+j) \notin B$, for $0 \le i, j \le 2$, i.e. $p \notin \mathrm{int\,cl}B$. See Fig. 6 for an illustration of our reasoning.

(d) In this case (p_1+1, p_2-1), $(p_1+1, p_2-2) \notin B$ (for otherwise the patterns A_{21}, A_{22} and A_{23} could not occur at (p_1+1, p_2)). Because of $p \in \mathrm{cl}B$ we can assume (p_1-1, p_2-2), $(p_1-1, p_2-3) \in B$ and $(p_1+2, p_2) \notin B$ (because of pattern A_1); and therefore (p_1+2, p_2-2), $(p_1+2, p_2-3) \in B$. But this contradicts that for (p_1, p_2-2) one of the patterns A_{21}, A_{22} or A_{23} has to occur (Property (ii). See Fig. 7 for an illustration of our reasoning.

(e) Without loss of generality we assume (p_1+1, p_2-1), $(p_1+2, p_2-1) \in B$, so that $(p_1+1, p_2+1) \notin B$ (otherwise pattern A_{21}, A_{22} and A_{23} do not occur at (p_1+1, p_2)). If $(p_1, p_2+2) \in B$, then $(p_1-1, p_2+2) \in B$ (otherwise pattern A_{21}, A_{22} and A_{23} could not occur at (p_1+1, p_2)) and (p_1-1, p_2+1), $(p_1-2, p_2+1) \notin B$ (otherwise pattern A_1 for (p_1, p_2+1) would occur), but then $(p_1-2, p_2-2) \notin B$ which leads to a contradiction to Property (ii) at (p_1, p_2-1). Thus, $(p_1, p_2+2) \notin B$ and also $(p_1+1, p_2+2) \notin B$ (otherwise pattern A_{21}, A_{22} and A_{23} could not occur at (p_1+1, p_2)). Because of $p \in \mathrm{cl}B$ we conclude that $(p_1-1, p_2+2) \in B$. Thus, $(p_1+2, p_2), (p_1+2, p_2+2) \notin B \Rightarrow (p_1+i, p_2+j) \notin B$, for $0 \le i,j \le 2$, *i.e.* $p \notin \mathrm{int\,cl}B$. See Fig. 8 for an illustration of our reasoning.

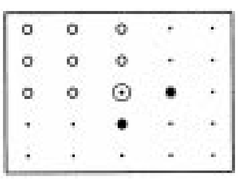

Figure 5. Illustration of our reasoning in Case (a).

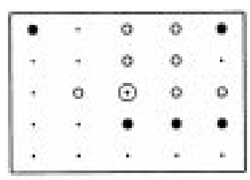

Figure 7. Illustration of our reasoning in Case (d).

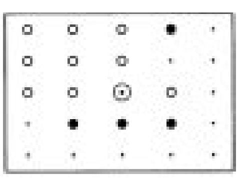

Figure 6. Illustration of our reasoning in Case (c).

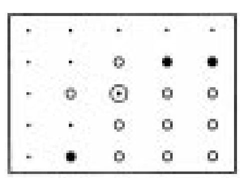

Figure 8. Illustration of our reasoning in Case (e).

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