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A note on the Kesten–Grincevičius–Goldie theorem

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Abstract

Consider the perpetuity equation $X \stackrel{\mathcal{D}}{=} AX + B$, where (A, B) and X on the right-hand side are independent. The Kesten–Grincevičius–Goldie theorem states that if $\mathbf{E}A^\kappa = 1$, $\mathbf{E}A^\kappa \log_+ A < \infty$, and $\mathbf{E}|B|^\kappa < \infty$, then $\mathbf{P}\{X > x\} \sim cx^{-\kappa}$. Assume that $\mathbf{E}|B|^\nu < \infty$ for some $\nu > \kappa$, and consider two cases (i) $\mathbf{E}A^\kappa = 1$, $\mathbf{E}A^\kappa \log_+ A = \infty$; (ii) $\mathbf{E}A^\kappa < 1$, $\mathbf{E}A^t = \infty$ for all $t > \kappa$. We show that under appropriate additional assumptions on A the asymptotic $\mathbf{P}\{X > x\} \sim cx^{-\kappa}\ell(x)$ holds, where ℓ is a nonconstant slowly varying function. We use Goldie's renewal theoretic approach.

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1 Introduction and results

Consider the perpetuity equation

$$X \stackrel{\mathcal{D}}{=} AX + B, \tag{1.1}$$

where (A, B) and X on the right-hand side are independent. To exclude degenerate cases as usual we assume that $\mathbf{P}\{Ax + B = x\} < 1$ for any $x \in \mathbb{R}$. We also assume that $A \geq 0$, $A \not\equiv 1$, and that $\log A$ conditioned on $A \neq 0$ is nonarithmetic.

The first results on existence and tail behavior of the solution are due to Kesten [23], who proved that if

$$\begin{gathered}\mathbf{E}A^\kappa = 1,\ \mathbf{E}A^\kappa \log_+ A < \infty,\ \log A \text{ conditioned on } A \neq 0 \text{ is nonarithmetic,} \\ \text{and } \mathbf{E}|B|^\kappa < \infty \text{ for some } \kappa > 0,\end{gathered} \tag{1.2}$$

where $\log_+ x = \max\{\log x, 0\}$, then the solution of (1.1) has Pareto-like tail, i.e.

$$\mathbf{P}\{X > x\} \sim c_+ x^{-\kappa} \text{ and } \mathbf{P}\{X < -x\} \sim c_- x^{-\kappa} \text{ as } x \to \infty \tag{1.3}$$

for some $c_+, c_- \geq 0, c_+ + c_- > 0$. (In the following any nonspecified limit relation is meant as $x \to \infty$.) Actually, Kesten proved a similar statement in d dimension. Later Goldie [16] simplified the proof of the same result in the one-dimensional case (for more general equations) using renewal theoretic methods. His method is based

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on ideas from Grincevičius [19], who partly rediscovered Kesten's results. We refer to the implication (1.2) $\Rightarrow$ (1.3) as the Kesten–Grincevičius–Goldie theorem. That is, under general conditions on A, if $\mathbf{P}\{A > 1\} > 0$ the tail decreases at least polynomially. Dyszewski [10] showed that the tail of the solution of (1.1) can even be slowly varying. On the other hand, Goldie and Grübel [17] showed that the solution has at least exponential tail under the assumption $A \leq 1$ a.s. For further results in the thin-tailed case see Hitczenko and Wesołowski [20]. Returning to the heavy-tailed case Grey [18] showed that if $\mathbf{E}A^\kappa < 1$, $\mathbf{E}A^{\kappa+\epsilon} < \infty$, then the tail of X is regularly varying with parameter $-\kappa$ if and only if the tail of B is. Grey's results are also based on previous results by Grincevičius [19].

That is, the regular variation of the solution X of (1.1) is either caused by A alone, or by B alone (under some weak condition on the other variable). Our intention in the present note is to explore more the role of A, i.e. to extend the Kesten–Grincevičius–Goldie theorem. More precisely, we assume that $\mathbf{E}|B|^\nu < \infty$ for some $\nu > \kappa$, and we obtain sufficient conditions on A that imply $\mathbf{P}\{X > x\} \sim \ell(x)x^{-\kappa}$, where $\ell(\cdot)$ is some nonconstant slowly varying function.

The perpetuity equation (1.1) has a wide range of applications; we only mention the ARCH and GARCH models in financial time series analysis, see Embrechts, Klüppelberg and Mikosch [11, Section 8.4 Perpetuities and ARCH Processes]. For a complete account on the equation (1.1) refer to Buraczewski, Damek and Mikosch [5].

The key idea in Goldie's proof is to introduce the new probability measure

$$\mathbf{P}_\kappa\{\log A \in C\} = \mathbf{E}[I(\log A \in C)A^\kappa], \tag{1.4}$$

where $I(\cdot)$ stands for the indicator function. Since $\mathbf{E}A^\kappa = 1$, this is indeed a probability measure. If F is the distribution function (df) of $\log A$ under $\mathbf{P}$, then under $\mathbf{P}_\kappa$

$$F_\kappa(x) = \mathbf{P}_\kappa\{\log A \leq x\} = \int_{-\infty}^x e^{\kappa y} F(\mathrm{d}y). \tag{1.5}$$

Under $\mathbf{P}_\kappa$ equation (1.1) can be rewritten as a renewal equation, where the renewal function corresponds to F_κ. If $\mathbf{E}_\kappa \log A = \mathbf{E}A^\kappa \log A \in (0, \infty)$, then a renewal theorem on the line implies the required tail asymptotics. Yet a smoothing transformation and a Tauberian argument is needed, since key renewal theorems apply only for direct Riemann integrable functions.

What we assume instead of the finiteness of the mean is that under $\mathbf{P}_\kappa$ the variable $\log A$ is in the domain of attraction of a stable law with index $\alpha \in (0, 1]$, i.e. $\log A \in D(\alpha)$. Since

$$F_\kappa(-x) = \mathbf{P}_\kappa\{\log A \leq -x\} = \mathbf{E}I(\log A \leq -x)A^\kappa \leq e^{-\kappa x}, \tag{1.6}$$

under $\mathbf{P}_\kappa$ the variable $\log A$ belongs to $D(\alpha)$ if and only if

$$1 - F_\kappa(x) = \overline{F}_\kappa(x) = \frac{\ell(x)}{x^\alpha}, \tag{1.7}$$

where ℓ is a slowly varying function. Let $U(x) = \sum_{n=0}^\infty F_\kappa^{*n}(x)$ be the renewal function of $\log A$ under $\mathbf{P}_\kappa$. Since the random walk $(\log A_1 + \ldots + \log A_n)_{n\geq 1}$ drifts to infinity under $\mathbf{P}_\kappa$ and $\mathbf{E}_\kappa[(\log A)_-]^2 < \infty$ by (1.6), we have $U(x) < \infty$ for all $x \in \mathbb{R}$; see Theorem 2.1 by Kesten and Maller [24]. Put

$$m(x) = \int_0^x [F_\kappa(-u) + \overline{F}_\kappa(u)]\mathrm{d}u \sim \int_0^x \overline{F}_\kappa(u)\mathrm{d}u \sim \frac{\ell(x)x^{1-\alpha}}{1-\alpha}$$

for the truncated expectation; the first asymptotic follows from (1.6), the second from (1.7), and holds only for $\alpha \neq 1$. To obtain the asymptotic behavior of the solution of the renewal equation we have to use a key renewal theorem for random variables with infinite mean. The infinite mean analogue of the strong renewal theorem (SRT) is the convergence

$$\lim_{x\to\infty} m(x)[U(x+h)-U(x)] = hC_\alpha, \quad \forall h>0, \quad \text{where } \; C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}. \tag{1.8}$$

The first infinite mean SRT was shown by Garsia and Lamperti [15] in 1963 for nonnegative integer valued random variables, which was extended to the nonarithmetic case by Erickson [12, 13]. In both cases it was shown that for $\alpha \in (1/2, 1]$ (in [15] $\alpha < 1$) assumption (1.7) implies the SRT, while for $\alpha \le 1/2$ further assumptions are needed. For $\alpha \le 1/2$ sufficient conditions for (1.8) were given by Chi [7], Doney [8], Vatutin and Topchii [28]. The necessary and sufficient condition for nonnegative random variables was given independently by Caravenna [6] and Doney [9]. They showed that if for a nonnegative random variable with df H (1.7) holds with $\alpha \le 1/2$, then (1.8) holds if and only if

$$\lim_{\delta\to 0}\limsup_{x\to\infty} x\overline{H}(x)\int_1^{\delta x}\frac{1}{y\overline{H}(y)^2}H(x-\mathrm{d}y)=0. \tag{1.9}$$

We need this result in our case, where the random variable is not necessarily positive, but the left tail is exponential. This is Theorem 3.1 in the Appendix. The proof follows along the same lines as the proof of the SRT in [6]. For further results and history about the infinite mean SRT we refer to [6, 9] and the references therein. In Lemma 2.2 below, which is a modification of Erickson's Theorem 3 [12], we prove the corresponding key renewal theorem. Since in the literature ([27, Lemma 3], [28, Theorem 4]) this lemma is stated incorrectly, we give a counterexample in the Appendix. We use the notation $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$, $x \in \mathbb{R}$. Summarizing, our assumptions on A are the following:

$$\begin{gathered}\mathbf{E}A^\kappa = 1, \text{ (1.7) and (1.9) holds for } F_\kappa \text{ for some } \kappa>0 \text{ and } \alpha\in(0,1],\\ \text{and } \log A \text{ conditioned on } A\neq 0 \text{ is nonarithmetic.}\end{gathered} \tag{1.10}$$

Theorem 1.1. *Assume (1.10) and $\mathbf{E}|B|^\nu < \infty$ for some $\nu > \kappa$. Then for the tail of the solution of the perpetuity equation (1.1) we have*

$$\begin{aligned}&\lim_{x\to\infty} m(\log x)x^\kappa \mathbf{P}\{X>x\} = C_\alpha \frac{1}{\kappa}\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa],\\ &\lim_{x\to\infty} m(\log x)x^\kappa \mathbf{P}\{X\le -x\} = C_\alpha \frac{1}{\kappa}\mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa].\end{aligned} \tag{1.11}$$

Moreover, $\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa] + \mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa] > 0$ *if* $\mathbf{P}\{Ax+B=x\}<1$ *for any* $x\in\mathbb{R}$.

Theorem 1.1 is stated as a conjecture/open problem in [21, Problem 1.4.2] by Iksanov.

The conditions of the theorem are stated in terms of the properties of A under the new measure $\mathbf{P}_\kappa$. Simple properties of regularly varying functions imply that if $e^{\kappa x}\overline{F}(x) = \alpha\,\ell(x)/(\kappa\, x^{\alpha+1})$ with a slowly varying function ℓ, then (1.7) holds. See the remark after Theorem 2 [26] by Korshunov.

Using the same methods, Goldie obtained tail asymptotics for solutions of more general random equations. The extension of these results to our setup is straightforward. We mention a particular example, because in the proof of the positivity of the constant in Theorems 1.1 and 1.3 we need a result on the maximum of a random walk.

Consider the equation

$$X \stackrel{\mathcal{D}}{=} AX \vee B, \tag{1.12}$$

where $a\vee b = \max\{a,b\}$, $A\ge 0$ and (A,B) and X on the right-hand side are independent. Theorem 5.2 in [16] states that if (1.2) holds, then there is a unique solution X to (1.12), and $\mathbf{P}\{X>x\}\sim cx^{-\kappa}$, with some $c\ge 0$, and $c>0$ if and only if $\mathbf{P}\{B>0\}>0$.

Theorem 1.2. *Assume (1.10),* $\mathbf{E}|B|^\nu < \infty$ *for some* $\nu > \kappa$*. Then for the tail of the solution of (1.12) we have*

$$\lim_{x\to\infty} m(\log x)x^\kappa \mathbf{P}\{X > x\} = C_\alpha \frac{1}{\kappa}\mathbf{E}[(AX_+ \vee B_+)^\kappa - (AX_+)^\kappa]. \tag{1.13}$$

Equation (1.12) has an important application in the analysis of the maximum of perturbed random walks; see Iksanov [22].

Finally, we note that the tail behavior (1.11) with nontrivial slowly varying function was noted before by Rivero for exponential functionals of Lévy processes; see [27, Counterexample 1].

Assume now that $\mathbf{E}A^\kappa = \theta < 1$ for some $\kappa > 0$, and $\mathbf{E}A^t = \infty$ for any $t > \kappa$. Consider the new probability measure

$$\mathbf{P}_\kappa\{\log A \in C\} = \theta^{-1}\mathbf{E}[I(\log A \in C)A^\kappa],$$

that is under the new measure $\log A$ has df

$$F_\kappa(x) = \theta^{-1}\int_{-\infty}^x e^{\kappa y}F(\mathrm{d}y).$$

The assumption $\mathbf{E}A^t = \infty$ for all $t > \kappa$ means that F_κ is heavy-tailed. Rewriting again (1.1) under the new measure $\mathbf{P}_\kappa$ leads now to a *defective* renewal equation for the tail of X. To analyze the asymptotic behavior of the resulting equation we use the techniques and results developed by Asmussen, Foss and Korshunov [4]. A slight modification of their setup is necessary, since our df F_κ is not concentrated on $[0,\infty)$.

For some $T \in (0,\infty]$ let $\Delta = (0,T]$. For a df H we put $H(x+\Delta) = H(x+T) - H(x)$. A df H on $\mathbb{R}$ is in the class $\mathcal{L}_\Delta$ if $H(x+t+\Delta)/H(x+\Delta) \to 1$ uniformly in $t \in [0,1]$, and it belongs to the class of Δ-subexponential distributions, $H \in \mathcal{S}_\Delta$, if $H(x+\Delta) > 0$ for x large enough, $H \in \mathcal{L}_\Delta$, and $(H*H)(x+\Delta) \sim 2H(x+\Delta)$. If $H \in \mathcal{S}_\Delta$ for every $T > 0$, then it is called *locally subexponential*, $H \in \mathcal{S}_{loc}$. The definition of the class $\mathcal{S}_\Delta$ is given by Asmussen, Foss and Korshunov [4] for distributions on $[0,\infty)$ and by Foss, Korshunov and Zachary [14, Section 4.7] for distributions on $\mathbb{R}$. In order to use a slight extension of Theorem 5 [4] we need the additional natural assumption $\sup_{y>x} F_\kappa(y+\Delta) = O(F_\kappa(x+\Delta))$ for x large enough. Our assumptions on A are the following:

$$\begin{gathered}\mathbf{E}A^\kappa = \theta < 1,\ \kappa > 0, \quad F_\kappa \in \mathcal{S}_{loc}, \quad \sup\{F_\kappa(y+\Delta) : y > x\} = O(F_\kappa(x+\Delta)) \\ \text{for } x \text{ large enough, and } \log A \text{ conditioned on } A \neq 0 \text{ is nonarithmetic.}\end{gathered} \tag{1.14}$$

Theorem 1.3. *Assume (1.14) and* $\mathbf{E}|B|^\nu < \infty$ *for some* $\nu > \kappa$*. Then for the tail of the solution of the perpetuity equation (1.1) we have*

$$\begin{aligned}\lim_{x\to\infty} g(\log x)^{-1}x^\kappa\mathbf{P}\{X > x\} &= \frac{\theta}{(1-\theta)^2\kappa}\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa], \\ \lim_{x\to\infty} g(\log x)^{-1}x^\kappa\mathbf{P}\{X \le -x\} &= \frac{\theta}{(1-\theta)^2\kappa}\mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa],\end{aligned} \tag{1.15}$$

where $g(x) = F_\kappa(x+1) - F_\kappa(x)$*. Moreover, if* $\mathbf{P}\{Ax+B = x\} < 1$ *for any* $x \in \mathbb{R}$*, then* $\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa] + \mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa] > 0$*.*

Note that the condition $F_\kappa \in \mathcal{L}_\Delta$ with $\Delta = (0,1]$ implies that $g(\log x)$ is slowly varying. Indeed, for any $\lambda > 0$

$$\frac{g(\log(\lambda x))}{g(\log x)} = \frac{F_\kappa(\log x + \log\lambda + \Delta)}{F_\kappa(\log x + \Delta)} \to 1.$$

The condition $F_\kappa \in \mathcal{S}_{loc}$ is much stronger than the corresponding regularly varying condition in Theorem 1.1. Typical examples satisfying this condition are the Pareto, lognormal and Weibull (with parameter less than 1) distributions, see [4, Section 4]. For example in the Pareto case, i.e. if for large enough x we have $\overline{F}_\kappa(x) = c\,x^{-\beta}$ for some $c > 0, \beta > 0$, then $g(x) \sim c\beta x^{-\beta-1}$, and so $\mathbf{P}\{X > x\} \sim c' x^{-\kappa}(\log x)^{-\beta-1}$. In the lognormal case, when $F_\kappa(x) = \Phi(\log x)$ for x large enough, with Φ being the standard normal df, (1.15) gives the asymptotic $\mathbf{P}\{X > x\} \sim c x^{-\kappa} e^{-(\log\log x)^2/2}/\log x$, $c > 0$. Finally, for Weibull tails $\overline{F}_\kappa(x) = e^{-x^\beta}$, $\beta \in (0,1)$, we obtain $\mathbf{P}\{X > x\} \sim c x^{-\kappa}(\log x)^{\beta-1} e^{-(\log x)^\beta}$, $c > 0$.

Theorem 1.4. *Assume (1.14), $\mathbf{E}|B|^\nu < \infty$ for some $\nu > \kappa$. Then for the tail of the solution of (1.12) we have*

$$\lim_{x\to\infty} g(\log x)^{-1} x^\kappa \mathbf{P}\{X > x\} = \frac{\theta}{(1-\theta)^2\kappa}\mathbf{E}[(AX_+ \vee B_+)^\kappa - (AX_+)^\kappa], \tag{1.16}$$

where $g(x) = F_\kappa(x+1) - F_\kappa(x)$.

In the special case $B \equiv 1$ we obtain a new result for the tail asymptotic of the maximum of a random walk.

In this direction we note that assuming (1.7) Korshunov [26] showed for $\alpha > 1/2$ (all he needs is the SRT, so the same holds under (1.9) for $\alpha \in (0,1)$) that for some constant $c > 0$

$$\lim_{x\to\infty} \mathbf{P}\{M > x\} e^{\kappa x} m(x) = c.$$

Thus Theorem 1.2 contains Korshunov's result [26]. However, note that Korshunov obtained the corresponding liminf result in (1.13), when the SRT does not hold. With our method the liminf result does not follow due to the smoothing transform (2.4). The problem is to 'unsmooth' the liminf version of (2.10). The same difficulty appears in the perpetuity case.

It turns out that in some special cases the regular variation of the tail of X and of e^M are equivalent. This can be deduced from Theorem 4 by Arista and Rivero [3].

Finally, we note that using Alsmeyer's sandwich method [1] it is possible to apply our results to iterated function systems.

2 Proofs

First, we prove the analogue of Goldie's implicit renewal theorem [16, Theorem 2.3] in both cases.

Theorem 2.1. *Assume (1.10), and for some random variable X*

$$\int_0^\infty |\mathbf{P}\{X > x\} - \mathbf{P}\{AX > x\}| x^{\kappa+\delta-1} \mathrm{d}x < \infty$$

for some $\delta > 0$, where X and A are independent. Then

$$\lim_{x\to\infty} m(\log x) x^\kappa \mathbf{P}\{X > x\} = C_\alpha \int_0^\infty [\mathbf{P}\{X > x\} - \mathbf{P}\{AX > x\}] x^{\kappa-1} \mathrm{d}x.$$

Proof. We follow closely Goldie's proof. Put

$$\psi(x) = e^{\kappa x}(\mathbf{P}\{X > e^x\} - \mathbf{P}\{AX > e^x\}),\ f(x) = e^{\kappa x}\mathbf{P}\{X > e^x\}. \tag{2.1}$$

Using that X and A are independent we obtain the equation

$$f(x) = \psi(x) + \mathbf{E} f(x - \log A) A^\kappa. \tag{2.2}$$

By (1.4) we have $\mathbf{E}_\kappa g(\log A) = \mathbf{E}(g(\log A)A^\kappa)$, thus under $\mathbf{P}_\kappa$ equation (2.2) reads as

$$f(x) = \psi(x) + \mathbf{E}_\kappa f(x - \log A). \tag{2.3}$$

Since ψ is not necessarily directly Riemann integrable (dRi), we introduce the smoothing transform of a function g as

$$\hat{g}(s) = \int_{-\infty}^{s} e^{-(s-x)} g(x)\mathrm{d}x. \tag{2.4}$$

Applying this transform to both sides of (2.3) we get the renewal equation

$$\hat{f}(s) = \hat{\psi}(s) + \mathbf{E}_\kappa \hat{f}(s - \log A). \tag{2.5}$$

Iterating (2.5) we obtain for any $n \geq 1$

$$\hat{f}(s) = \sum_{k=0}^{n-1} \int_{\mathbb{R}} \hat{\psi}(s-y) F_\kappa^{*k}(\mathrm{d}y) + \mathbf{E}_\kappa \hat{f}(s - S_n), \tag{2.6}$$

where $\log A_1, \log A_2, \ldots$ are independent copies of $\log A$ (under $\mathbf{P}$ and $\mathbf{P}_\kappa$), independent of X, and $S_n = \log A_1 + \ldots + \log A_n$. Since $S_n \to -\infty$ $\mathbf{P}$-a.s.

$$\mathbf{E}_\kappa \hat{f}(s - S_n) = e^{-s} \int_{-\infty}^{s} e^{(\kappa+1)y} \mathbf{P}\{X e^{S_n} > e^y\} \mathrm{d}y \to 0 \quad \text{as } n \to \infty,$$

where we also used that $\mathbf{E}_\kappa g(S_n) = \mathbf{E}(g(S_n)e^{\kappa S_n})$. Therefore as $n \to \infty$ from (2.6) we have

$$\hat{f}(s) = \int_{\mathbb{R}} \hat{\psi}(s-y) U(\mathrm{d}y), \tag{2.7}$$

where $U(x) = \sum_{n=0}^{\infty} F_\kappa^{*n}(x)$ is the renewal function of F_κ. The question is under what conditions of z the key renewal theorem

$$m(x) \int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) \to C_\alpha \int_{\mathbb{R}} z(y)\mathrm{d}y \tag{2.8}$$

holds. In the following lemma, which is a modification of Erickson's Theorem 3 [12], we give sufficient condition for z to (2.8) hold. We note that both in Lemma 3 [27] and in Theorem 4 of [28] the authors wrongly claim that (2.8) holds if z is dRi. A counterexample is given in the Appendix. The same statement under less restrictive condition is shown using stopping time argument in [21, Proposition 6.4.2]. For the sake of completeness we give a proof here.

Lemma 2.2. *Assume that z is dRi, $z(x) = O(x^{-1})$ as $x \to \infty$, and (1.10) holds. Then (1.8) implies (2.8).*

Proof. Using the decomposition $z = z_+ - z_-$ we may and do assume that z is nonnegative. Write

$$m(x) \int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) =: I_1(x) + I_2(x) + I_3(x),$$

where I_1, I_2, and I_3 are the integrals on (x, ∞), $(0, x]$, and on $(-\infty, 0]$, respectively.

We first show that $I_1(x) \to C_\alpha \int_{-\infty}^{0} z(y)\mathrm{d}y$ whenever z is dRi. Fix $h > 0$. Introduce the functions $z_k(x) = I(x \in [(k-1)h, kh))$, and put $a_k = \inf\{z(x) : x \in [(k-1)h, kh)\}$, and $b_k = \sup\{z(x) : x \in [(k-1)h, kh)\}$, $k \in \mathbb{Z}$. Simply

$$m(x) \sum_{k=-\infty}^{0} a_k (U * z_k)(x) \leq I_1(x) \leq m(x) \sum_{k=-\infty}^{0} b_k (U * z_k)(x).$$

As $x \to \infty$ by (1.8) for any fixed k

$$m(x)(U * z_k)(x) = \frac{m(x)}{m(x-kh)} m(x-kh)[U(x-kh+h) - U(x-kh)] \to C_\alpha h,$$

where the convergence $m(x)/m(x-kh) \to 1$ follows from the fact that m is regularly varying with index $1-\alpha$. Since m is nondecreasing and $k \le 0$ this also gives us an integrable majorant uniformly in $k \le 0$, i.e. $\sup_{k<0} m(x)(U * z_k)(x) \le 2C_\alpha h$ for x large enough. Thus by Lebesgue's dominated convergence theorem

$$\lim_{x\to\infty} m(x) \sum_{k=-\infty}^{0} a_k (U * z_k)(x) = C_\alpha \sum_{k=-\infty}^{0} a_k h,$$

and similarly for the upper bound. Since z is dRi the statement follows.

The convergence $I_2(x) \to C_\alpha \int_0^\infty z(x)\mathrm{d}x$ follows exactly as in the proof of [12, Theorem 3], since in that proof only formula (1.8) and its consequence $U(x) \sim C_\alpha x/(\alpha m(x))$ are used.

Finally, we show that $I_3(x) \to 0$. Indeed, with $K = \sup_{x>0} xz(x)$,

$$m(x) \int_{-\infty}^{0} z(x-y) U(\mathrm{d}y) \le K m(x) \int_{-\infty}^{0} (x-y)^{-1} U(\mathrm{d}y) \le K \frac{m(x)}{x} U(0) \to 0. \qquad \square$$

Recall (2.1). Next we show that $\hat{\psi}$ satisfies the condition of Lemma 2.2. Indeed,

$$\begin{aligned} \hat{\psi}(s) &= e^{-s} \int_{-\infty}^{s} e^{(\kappa+1)x} [\mathbf{P}\{X > e^x\} - \mathbf{P}\{AX > e^x\}]\mathrm{d}x \\ &\le e^{-s} \int_0^{e^s} y^\kappa |\mathbf{P}\{X > y\} - \mathbf{P}\{AX > y\}| \mathrm{d}y \\ &\le e^{-\delta s} \int_0^\infty y^{\kappa+\delta-1} |\mathbf{P}\{X > y\} - \mathbf{P}\{AX > y\}| \mathrm{d}y, \end{aligned} \tag{2.9}$$

and the last integral is finite due to our assumptions. The same calculation shows that

$$\int_{\mathbb{R}} \hat{\psi}(s)\mathrm{d}s = \int_{\mathbb{R}} \psi(x)\mathrm{d}x = \int_0^\infty y^{\kappa-1} [\mathbf{P}\{X > y\} - \mathbf{P}\{AX > y\}]\mathrm{d}y.$$

It follows from [16, Lemma 9.2] that $\hat{\psi}$ is dRi, thus from Lemma 2.2 and (2.9) we obtain that for the solution of (2.7)

$$\lim_{s\to\infty} m(s) \hat{f}(s) = C_\alpha \int_{\mathbb{R}} \psi(y)\mathrm{d}y. \tag{2.10}$$

From (2.10) the statement follows in the same way as in [16, Lemma 9.3]. $\square$

Theorem 2.3. *Assume (1.14), and for some random variable X*

$$\int_0^\infty |\mathbf{P}\{X > x\} - \mathbf{P}\{AX > x\}| x^{\kappa+\delta-1} \mathrm{d}x < \infty$$

for some $\delta > 0$, where X and A are independent. Then

$$\lim_{x\to\infty} g(\log x)^{-1} x^\kappa \mathbf{P}\{X > x\} = \frac{\theta}{(1-\theta)^2} \int_0^\infty [\mathbf{P}\{X > x\} - \mathbf{P}\{AX > x\}] x^{\kappa-1} \mathrm{d}x.$$

Proof. Following the same steps as in the proof of Theorem 2.1 we obtain

$$\hat{f}(s) = \int_{\mathbb{R}} \hat{\psi}(s-y) U(\mathrm{d}y),$$

where U is the defective renewal function $U(x) = \sum_{n=0}^{\infty} (\theta F_\kappa)^{*n}(x)$. Since $\theta < 1$ we have $U(\mathbb{R}) = (1-\theta)^{-1} < \infty$. A modification of Theorem 5 [4] gives the following. Recall g from Theorem 1.3.

Lemma 2.4. *Assume (1.14), z is dRi, and $z(x) = o(g(x))$. Then*

$$\int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) \sim \frac{\theta g(x)}{(1-\theta)^2} \int_{\mathbb{R}} z(y) \mathrm{d}y.$$

Proof. By the decomposition $z = z_+ - z_-$, we may and do assume that z is nonnegative. We again split the integral

$$\int_{\mathbb{R}} z(x-y) U(\mathrm{d}y) = I_1(x) + I_2(x) + I_3(x),$$

where I_1, I_2, and I_3 are the integrals on (x, ∞), $(0, x]$, and on $(-\infty, 0]$, respectively.

The asymptotics $I_1(x) \sim \theta g(x) \int_{-\infty}^{0} z(y)\mathrm{d}y/(1-\theta)^2$ follows along the same lines as in the proof of Lemma 2.2. Theorem 5(i) [4] gives $I_2(x) \sim \theta g(x) \int_0^\infty z(y)\mathrm{d}y/(1-\theta)^2$. (In the Appendix we explain why the results for Δ-subexponential distributions on $[0,\infty)$ remain true in our case.) Finally, for I_3 we have

$$I_3(x) \leq U(0) \sup\{z(y) : y \geq x\} = o(g(x)),$$

where we used that $\sup_{y \geq x} F_\kappa(y + \Delta) = O(F_\kappa(x+\Delta))$. □

As in (2.9) we have $\hat{\psi}(x) = O(e^{-\delta x})$ for some $\delta > 0$. Since F_κ is subexponential $\hat{\psi}(x) = o(g(x))$. That is, the condition of Lemma 2.4 holds, and we obtain the asymptotic

$$\hat{f}(s) \sim \frac{\theta g(s)}{(1-\theta)^2} \int_{\mathbb{R}} \psi(y)\mathrm{d}y \quad \text{as } s \to \infty.$$

Since $g(x)$ is subexponential, $g(\log x)$ is slowly varying, and the proof follows again along the same lines as in [16, Lemma 9.3]. □

The proofs of Theorems 1.1, 1.3, and 1.2, 1.4 are applications of the corresponding implicit renewal theorem.

Proofs of Theorems 1.2 and 1.4. The existence of the unique solution of (1.12) follows from [16, Proposition 5.1]. Choose $\delta \in (0, \nu - \kappa)$. Since $|\mathbf{P}\{AX \vee B > x\} - \mathbf{P}\{AX > x\}| = \mathbf{P}\{AX \vee B > x \geq AX\}$, Fubini's theorem gives

$$\begin{aligned} &\int_0^\infty |\mathbf{P}\{AX \vee B > x\} - \mathbf{P}\{AX > x\}| x^{\kappa+\delta-1}\mathrm{d}x = \int_0^\infty \mathbf{P}\{AX \vee B > x \geq AX\} x^{\kappa+\delta-1}\mathrm{d}x \\ &= (\kappa+\delta)^{-1}\mathbf{E}[(AX \vee B)_+^{\kappa+\delta} - (AX)_+^{\kappa+\delta}] \leq (\kappa+\delta)^{-1}\mathbf{E}B_+^{\kappa+\delta}. \end{aligned}$$

Therefore (1.13) and (1.16) follows from Theorem 2.1 and 2.3, respectively. The form of the limit constant follows similarly. Note that for $B \equiv 1$, i.e. when $\log X = M$ is the maximum of a random walk with negative drift, the constant is strictly positive. □

Proofs of Theorems 1.1 and 1.3. The existence of the unique solution of (1.1) is well-known. Let us choose $\delta > 0$ so small that

$$\kappa + \frac{3\kappa\delta}{1-\delta} < \nu \text{ for } \kappa \geq 1, \text{ and } \kappa + \delta \leq \min\{1, \nu\} \text{ for } \kappa < 1. \tag{2.11}$$

Note that

$$|\mathbf{P}\{AX+B>y\}-\mathbf{P}\{AX>y\}|\le \mathbf{P}\{AX+B>y\ge AX\}+\mathbf{P}\{AX>y\ge AX+B\}.$$

Now Fubini's theorem gives for the first term

$$\int_0^\infty \mathbf{P}\{AX+B>y\ge AX\}y^{\kappa-1+\delta}\mathrm{d}y\le(\kappa+\delta)^{-1}\mathbf{E}I(B\ge 0)((AX+B)_+^{\kappa+\delta}-(AX)_+^{\kappa+\delta}).$$

The same calculation for the second term implies

$$\int_0^\infty |\mathbf{P}\{AX+B>y\}-\mathbf{P}\{AX>y\}|y^{\kappa+\delta-1}\mathrm{d}y\le(\kappa+\delta)^{-1}\mathbf{E}|(AX+B)_+^{\kappa+\delta}-(AX)_+^{\kappa+\delta}|.$$

We show that the expectation on the right-hand side is finite. Indeed, for $a,b\in\mathbb{R}$ we have $|(a+b)_+^\gamma-a_+^\gamma|\le|b|^\gamma$ for $\gamma\le 1$ and $|(a+b)_+^\gamma-a_+^\gamma|\le 2\gamma|b|(|a|^{\gamma-1}+|b|^{\gamma-1})$ for $\gamma>1$. From Theorem 1.4 by Alsmeyer, Iksanov and Rösler [2] we know that $\mathbf{E}|X|^\gamma<\infty$ for any $\gamma<\kappa$. (We note that for $\kappa>1$ this also follows from Theorem 5.1 by Vervaat [29]. Actually, [2, Theorem 1.4] states equivalence.) Assume that $\kappa\ge 1$ and let $p=\kappa+2\kappa\delta/(1-\delta)$, $1/q=1-1/p$. By Hölder's inequality and by the choice of δ in (2.11)

$$\begin{aligned}\mathbf{E}|(AX+B)_+^{\kappa+\delta}-(AX)_+^{\kappa+\delta}|&\le 2(\kappa+\delta)\left[\mathbf{E}|B||AX|^{\kappa+\delta-1}+\mathbf{E}|B|^{\kappa+\delta}\right]\\&\le 2(\kappa+\delta)\left[\mathbf{E}|X|^{\kappa+\delta-1}(\mathbf{E}|B|^p)^{1/p}(\mathbf{E}A^{q(\kappa+\delta-1)})^{1/q}+\mathbf{E}|B|^{\kappa+\delta}\right]<\infty,\end{aligned}$$

which proves the statement for $\kappa\ge 1$. For $\kappa<1$ we choose δ such that $\kappa+\delta\le 1$, so

$$\mathbf{E}||AX+B|^{\kappa+\delta}-|AX|^{\kappa+\delta}|\le\mathbf{E}|B|^{\kappa+\delta}<\infty.$$

Finally, the positivity of the limit follows in exactly the same way as in [16]. Goldie shows [16, p.157] that for some positive constants $c,C>0$

$$\mathbf{P}\{|X|>x\}\ge c\,\mathbf{P}\left\{\max\{0,S_1,S_2,\ldots\}>C+\log x\right\}.$$

Now the positivity follows from Theorem 1.2 and 1.4, respectively, with $B\equiv 1$. □

3 Appendix

3.1 Strong renewal theorem

We state a slight extension of the strong renewal theorem by Caravenna [6] and Doney [9]. The proof follows along the same lines as the proof of Caravenna [6], and it is given in [25]. For convenience, we also use Caravenna's notation.

Theorem 3.1. *Assume that the distribution function H is nonarithmetic, and for some $c,\kappa>0$, $\alpha\in(0,1)$, and for a slowly varying function ℓ we have*

$$H(-x)\le ce^{-\kappa x},\quad 1-H(x)=\overline{H}(x)=\frac{\ell(x)}{x^\alpha},\quad x>0.$$

Then, for the renewal function $U(x)=\sum_{n=0}^\infty H^{*n}(x)$

$$\lim_{x\to\infty}m(x)[U(x+h)-U(x)]=hC_\alpha$$

holds for any $h>0$ with $m(x)=\int_0^x\overline{H}(u)\mathrm{d}u$, if and only if

$$\lim_{\delta\to 0}\limsup_{x\to\infty}x\overline{H}(x)\int_1^{\delta x}\frac{1}{y\overline{H}(y)^2}H(x-\mathrm{d}y)=0.$$

3.2 A counterexample

Here we give a counterexample to [27, Lemma 3] and [28, Theorem 4], which shows that alone from the direct Riemann integrability of z the key renewal theorem (2.8) does not follow.

Let $a_n = n^{-\beta}$, with some $\beta > 1$, and let $d_n \uparrow \infty$ a sequence of integers. Consider the function z that satisfies $z(d_n) = a_n$, $z(d_n \pm 1/2) = 0$, is linearly interpolated on the intervals $[d_n - 1/2, d_n + 1/2]$, and 0 otherwise. Since $\sum_{n=1}^{\infty} a_n < \infty$ the function z is directly Riemann integrable.

Consider a renewal measure U for which SRT (1.8) holds. Let $a > 0$ be such that $U(a + 1/4) - U(a - 1/4) > 0$. From the proof of [12, Theorem 3] it is clear that for any $\nu \in (0, 1)$

$$m(x) \int_{\nu x}^{x} z(x-y) U(\mathrm{d}y) \to C_\alpha \int_0^\infty z(y) \mathrm{d}y.$$

On the other hand for $x = a + d_n$

$$\int_{a-1/4}^{a+1/4} z(x-y) U(\mathrm{d}y) \geq \frac{a_n}{2} [U(a+1/4) - U(a-1/4)]$$

Choosing $d_n = n^2$ and β such that $2\alpha + \beta < 2$, and recalling that m is regularly varying with index $1 - \alpha$, we see that $m(a + d_n) a_n \to \infty$, so the asymptotic (2.8) cannot hold.

3.3 Local subexponentiality

We claim that Theorem 5 in [4] remains true in our setup. Additionally to the local subexponential property, we assume that $\sup_{y \geq x} H(y + \Delta) = O(H(x + \Delta))$. The main technical tool in [4] is the equivalence in Proposition 2. In our setup it has the following form.

Lemma 3.2. *Assume that* $H \in \mathcal{L}_\Delta$, *and* $\sup_{y \geq x} H(y + \Delta) = O(H(x + \Delta))$. *Let* X, Y *be iid* H. *The following are equivalent:*

(i) $H \in \mathcal{S}_\Delta$;

(ii) there is a function h *such that* $h(x) \to \infty$, $h(x) < x/2$, $H(x - y + \Delta) \sim H(x + \Delta)$ *uniformly in* $|y| \leq h(x)$, *and*

$$\mathbf{P}\{X + Y \in x + \Delta, X > h(x), Y > h(x)\} = o(H(x + \Delta)).$$

The proof is similar to the proof of Proposition 2 in [4], so it is omitted. Assuming the extra growth condition all the results in [4] hold true with the obvious modification of the proof.

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