TECHNISCHE UNIVERSITÄT MÜNCHEN

Fakultät für Mathematik

Lehr- und Forschungseinheit Algebra — Algebraische Geometrie

Automorphisms of Enriques Surfaces

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

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Die Dissertation wurde am 05.10.2017 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 18.02.2018 angenommen.

Table of Contents

Introduction 3

Chapter I. Enriques surfaces with finite automorphism group and smooth K3 cover 7
- Convention 7
- § 1. Summary 7
- § 2. Preliminaries 11
- § 3. Enriques surfaces of type I 22
- § 4. Enriques surfaces of type II 25
- § 5. Enriques surfaces of type III 27
- § 6. Enriques surfaces of type IV 32
- § 7. Enriques surfaces of type V 36
- § 8. Enriques surfaces of type VI 40
- § 9. Enriques surfaces of type VII 44
- § 10. The classification-theorem 46
- § 11. Arithmetic of Enriques surfaces with finite automorphism group 61
- § 12. Semi-symplectic automorphisms 63

Chapter II. Enriques surfaces with finite automorphism group in characteristic 2 65
- § 1. Summary 65
- § 2. Preliminaries 69
- § 3. Construction of vector fields 74
- § 4. Equations of Enriques surfaces and their automorphisms 76
- § 5. Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$ 82
- § 6. Enriques surfaces of type VII 93
- § 7. Enriques surfaces of type VIII 94
- § 8. Enriques surfaces of Type $\tilde{E}_8$ 98
- § 9. Enriques surfaces of type $\tilde{E}_7 + \tilde{A}_1$ 104
- § 10. Enriques surfaces of type $\tilde{D}_8$ 113
- § 11. Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$ 119
- § 12. Possible dual graphs 122

Chapter III. Numerically trivial automorphisms of Enriques surfaces 139
- § 1. Summary 139
- § 2. Generalities on numerically and cohomologically trivial automorphisms 140
- § 3. Half-fibers of genus one fibrations 141

§ 4. Automorphisms of genus one curves 144
§ 5. Bielliptic maps and bielliptic involutions 144
§ 6. Extra-special Enriques surfaces 147
§ 7. Cohomologically trivial automorphisms 149
§ 8. Numerically trivial automorphisms 153

Outlook 155

Bibliography 157

Introduction

Given a smooth and projective variety X over an algebraically closed field k, we can form its structure sheaf $\mathcal{O}_X$, its tangent sheaf T_X and its cotangent sheaf Ω_X. While all invariants associated to the three of them are invariants of X up to isomorphism, their behaviour under birational maps is more subtle. However, one can form the canonical sheaf $\omega_X = \det(\Omega_X)$ and define the geometric genus $p_g(X) = h^0(X, \omega_X)$. As it turns out, $p_g(X)$ is a birational invariant of X [Har77].

Now, a closer look at the case of curves reveals that there is a unique, smooth and projective curve with $p_g = 0$, namely $\mathbb{P}^1$, giving a simple way to check whether a given curve is birational and hence isomorphic to the projective line. However, already the case of dimension 2 is much harder: In the late 19th century, M. Noether introduced a new birational invariant of a surface, the arithmetic genus $p_a = \sum_{i=0}^{2}(-1)^i h^i(X, \mathcal{O}_X) - 1$, and A. Cayley gave an example of a surface showing that $h^1(X, \mathcal{O}_X) = q = p_g - p_a > 0$ is possible. This new invariant q is called the irregularity of the surface X. Observing that $q(\mathbb{P}^2) = p_g(\mathbb{P}^2) = 0$, Noether conjectured that any surface with $q = p_g = 0$ is rational. The first counterexample to this conjecture – and the main objects to be studied in this thesis – were constructed by F. Enriques as normalizations of a generic sextic in $\mathbb{P}^3$ passing doubly through the edges of a tetrahedron [Enr96] and surfaces of this type are now called Enriques surfaces in honor of Enriques. Using the plurigenera $p_n = H^0(X, \omega_X^{\otimes n})$, which are new birational invariants introduced by Enriques, G. Castelnuovo was able to formulate his famous rationality criterion for surfaces, saying that a surface is rational if and only if $q = p_2 = 0$. After Enriques' discovery, Castelnuovo found examples of surfaces with $q = p_g = 0$ and p_n growing linearly, followed by L. Campedelli [Cam32] and L. Godeaux [God35] giving such examples of surfaces where p_n grows quadratically.

In fact, as the plurigenera are birational invariants of a surface, so is the minimal number κ such that $p_n = O(n^\kappa)$ as $n \to \infty$. This κ is the Kodaira dimension of the surface and Enriques surfaces satisfy $\kappa = 0$. The other surfaces of Kodaira dimension zero are Abelian, bielliptic and K3 surfaces. With varieties of general type (with $\kappa = 2$) always having finite automorphism group and projective space (with $\kappa = -\infty$) always having infinite automorphism group, it is a natural question to ask for the behaviour of automorphism groups of surfaces of Kodaira dimension 0 and 1. The work related to this thesis completes the classification of Enriques surfaces with finite automorphism group in arbitrary characteristic.

Let us give some more background on Enriques surfaces: An Enriques surface is a smooth and projective surface X with second ℓ-adic betti number $b_2 = 10$ whose canonical divisor class is numerically trivial. The reason why Enriques surfaces have Kodaira dimension 0 is that $\omega_X^{\otimes 2} = \mathcal{O}_X$. In fact, one can show that the torsion component of the identity of the Picard scheme Pic^τ is of length 2 and if $\mathrm{Pic}^\tau(k)$ is non-trivial, it is generated by ω_X. By the classification of finite group schemes of prime order over algebraically closed fields [TO70], we know that $\mathrm{Pic}^\tau \in \{\mathbb{Z}/2\mathbb{Z}, \mu_2, \alpha_2\}$, where the first and second group schemes are isomorphic to each other if and only if $p = \mathrm{char}(k) \neq 2$ and the third one only exists in characteristic 2. If $\mathrm{Pic}^\tau \cong \mathbb{Z}/2\mathbb{Z}$, we call X classical. Otherwise, we call X singular (resp. supersingular) if $\mathrm{Pic}^\tau \cong \mu_2$ (resp. α_2). While the Pic^τ-torsor over X is étale and hence the total space is a K3 surface if $p \neq 2$ or X is singular, the torsor becomes inseparable with singular total space for the other cases. However, in any case, the cover $\tilde{X}$ is an integral and Gorenstein surface with trivial dualizing sheaf, hence it is sometimes called "K3-like". Even though it might seem to be the case that the singularities of the K3-like cover make the analysis of singular and supersingular Enriques surfaces more difficult, we will explain how to use these singularities, which are invariants of the surface X, to our advantage in Chapter II of this thesis.

Over the complex numbers, a [GH16] moduli space of unpolarized Enriques surfaces, which is 10-dimensional, quasi-affine [Bor96] and rational [Kon94] can be constructed using the period map for complex Enriques surfaces [Hor78a], [Hor78b]. While a general Enriques surface does not contain (-2)-curves, i.e. smooth rational curves with self-intersection (-2), there is a codimension-one subvariety parametrizing Enriques surfaces containing such curves. Note that a general Enriques surface with a (-2)-curve contains infinitely many such curves. On the boundary of the moduli space, there is a codimension-one subvariety corresponding to certain rational and smooth surfaces X with $h^0(X, \omega_X^{-1}) = 0$ and $h^0(X, (\omega_X^{-1})^{\otimes 2}) \neq 0$, which are called Coble surfaces. Both of these codimension-one subvarieties are rational [DK13]. In Chapter I of this thesis, we will give some explicit examples of 1-dimensional families of Enriques surfaces, which degenerate to Coble surfaces, as predicted by the complex period space.

Recently, C. Liedtke in [Lie15] and T. Ekedahl, J. Hyland and N. Shepherd-Barron in [EHS12] have studied the moduli space of Enriques surfaces in positive characteristic: The moduli space of Cossec-Verra polarized Enriques surfaces is a quasi-separated Artin stack of finite type over $\mathrm{Spec}\ \mathbb{Z}$, which is irreducible, unirational, 10-dimensional, smooth in odd characteristics and consists of two connected components with these properties in characteristic 2. These two connected components parametrize singular and classical Enriques surfaces, respectively. Their 9-dimensional intersection parametrizes supersingular Enriques surfaces. The stack of unpolarized Enriques surfaces is very badly behaved (see [Lie15, Remark 5.3]) because the automorphism group of a generic Enriques surface X is an infinite and, unless X is supersingular or an exceptional [ES04] and classical Enriques surface in characteristic 2, discrete group.

The automorphism group of a very general complex Enriques surface has been studied in the early 1980's. As we mentioned before, such an Enriques surface does not contain (-2)-curves, making it possible to study these automorphism groups using the Torelli Theorem for K3 surfaces. This has been carried out by W. Barth and C. Peters [BP83] and by V. V. Nikulin [Nik81]. The automorphism group is equal to the 2-congruence subgroup of the group of positive-cone-preserving automorphisms of the E_{10} lattice. In particular, it is infinite. However, as the Enriques surface acquires more independent classes of (-2)-curves, its automorphism group becomes smaller. Therefore, to find Enriques surfaces with finite automorphism group, we have to find surfaces with very special configurations of (-2)-curves. It is in fact a corollary of the classification, that the number of (-2)-curves on Enriques surfaces with finite automorphism group is finite.

The first example of an Enriques surface with finite automorphism group (Type VII) was found by G. Fano [Fan10] in 1910 and a second one (Type I) was found by I. Dolgachev [Dol84] in 1984. The systematic classification of these surfaces over the complex numbers was then carried out by V. Nikulin [Nik84] and S. Kondo [Kon86]. There are seven types I, . . . , VII of such Enriques surfaces, distinguished by their dual graphs of (-2)-curves, the first two of which form a 1-dimensional family and the others are unique [Kon86]. Now, we can explain the contents of this thesis:

Chapter I, which follows the exposition in [Mar17], gives the classification of Enriques surfaces with finite automorphism group in odd characteristic and of singular Enriques surfaces with finite automorphism group in characteristic 2. These are exactly the cases where the K3-like cover is smooth, so in fact K3, and our method also gives another proof of the classification of these surfaces over the complex numbers. The main tool – and the main difference to the other cases – is the existence of elliptic fibrations on these Enriques surfaces, which are separable twists of rational and elliptic fibrations with a section. Since the theory of separable twists is very well developed and the twisting can be controlled explicitly using the K3 cover, it is possible to obtain very precise information on these surfaces. We give an explicit description of the moduli spaces of these objects and give minimal fields of definition for the different types of Enriques surfaces with finite automorphism group. Here, the list of Enriques surfaces with finite automorphism group turned out to be more or less the same as over the complex numbers, except that some of the seven types are missing in small characteristics.

Chapter II – following the paper [KKM17], which is joint work with T. Katsura and S. Kondo – finishes the classification in characteristic 2 in the cases where the K3-like cover is singular. To obtain the classification of possible dual graphs of classical and supersingular Enriques surfaces with finite automorphism group in characteristic 2, we use the singularities of the canonical cover to our advantage. In particular, the technique of conductrices, which was developed by Ekedahl and Shepherd-Barron in [ES04] to study exceptional Enriques surfaces, will play an important role. On the other hand, the realization of these dual graphs is obtained by the following method, which can be thought of as an inseparable analogue of the method in odd characteristics: We start with a rational and elliptic or quasi-elliptic fibration with a section, take an inseparable double cover, which corresponds to the Frobenius on the base curve, and take the quotient by a suitable action of an infinitesimal group scheme of length 2, which can be done explicitly using vector fields as in [KK15b] for Type VII. As in the separable case, this corresponds to a quadratic twist of the generic fiber of the genus one fibration, but this time the twist is inseparable.

Chapter III – following the paper [DM17], which is joint work with I. Dolgachev – deals with the classification of groups G of automorphisms of Enriques surfaces X acting trivially on $\mathrm{Num}(X)$ resp. $\mathrm{Pic}(X)$. Over the complex numbers, all Enriques surfaces which admit such numerically resp. cohomologically trivial automorphisms are classified (see [MN84]). We obtain a list of possible finite groups G of numerically resp. cohomologically trivial automorphisms of Enriques surfaces in positive characteristic and show that most of them are realized on Enriques surfaces with finite automorphism group. Moreover, we describe a method for obtaining a full classification of surfaces, which admit such automorphisms, in any characteristic.

Acknowledgement. It is a pleasure for me to thank my Ph.D. advisor C. Liedtke for suggesting this research topic and for his support.

CHAPTER I

Enriques surfaces with finite automorphism group and smooth K3 cover

Up to minor modifications, this chapter is taken from the paper "Enriques surfaces with finite automorphism group in positive characteristic" of the author. Currently, the paper is submitted and a preprint can be found on the ArXiv (see [Mar17]).

Convention

Unless mentioned otherwise, we will work over an algebraically closed field k of arbitrary characteristic. By Enriques surface we will mean Enriques surface with a smooth K3 cover throughout this chapter. This means that we will not be dealing with classical and supersingular Enriques surfaces in characteristic 2 in this chapter.

1. Summary

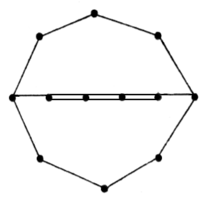
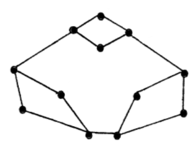
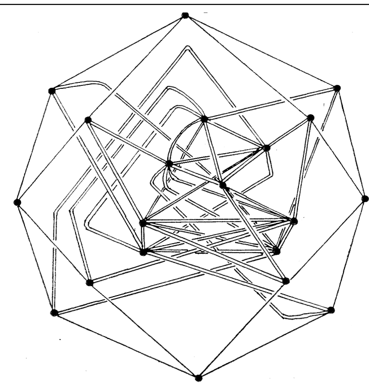
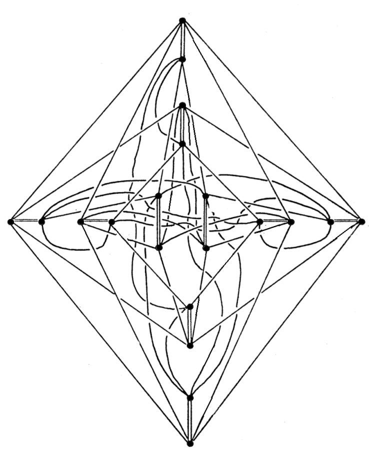
As we explained in the introduction to this thesis, the classification of Enriques surfaces with finite automorphism group over the complex numbers is due to Nikulin and Kondo. The key observation for Nikulin's approach to the classification is the fact that for a complex Enriques surface X the subgroup $W_X \subseteq \mathrm{O}(\mathrm{Num}(\mathrm{X}))$ generated by reflections along classes of (-2)-curves has finite index if and only if $\mathrm{Aut}(X)$ is finite. However, while in any characteristic W_X being of finite index in $\mathrm{O}(\mathrm{Num}(\mathrm{X}))$ implies that the automorphism group $\mathrm{Aut}(X)$ is finite [Dol84, Main Theorem], the converse uses the Global Torelli Theorem proven by E. Horikawa [Hor78a], [Hor78b], which is not available in positive characteristic. For this reason, we will not pursue Nikulin's approach. Nevertheless, it will follow from our explicit classification that $\mathrm{Aut}(X)$ being finite implies that $W_X \subseteq \mathrm{O}(\mathrm{Num}(\mathrm{X}))$ has finite index.

Kondō's approach is based on the observation – due to Dolgachev [Dol84, §4] – that the Mordell-Weil group of the Jacobian of every elliptic fibration of an Enriques surface X acts on X, hence it has to be finite if we want X to have finite automorphism group. Using this approach, we will obtain the classification of Enriques surfaces with finite automorphism group and smooth K3 cover in positive characteristic. Recall that the K3 cover of an Enriques surface X is smooth if and only if $\mathrm{char}(k) \neq 2$ or X is a singular Enriques surface.

MAIN THEOREM (Classification). *Let X be an Enriques surface with smooth K3 cover over an algebraically closed field k.*

(1) X has finite automorphism group if and only if the dual graph of all (-2)-curves on X is one of the seven dual graphs in Table 1.

(2) *The automorphism groups, the characteristics in which they exist, and the moduli of Enriques surfaces of each of the seven types are as in Table 1.*

Type	Dual Graph of (-2)-curves	Aut	Aut_{nt}	$\mathrm{char}(k)$	Moduli
I		D_4	$\mathbb{Z}/2\mathbb{Z}$	*any*	$\mathbb{A}^1 - \{0, -256\}$
II		$\mathfrak{S}_4$	$\{1\}$	*any*	$\mathbb{A}^1 - \{0, -64\}$
III		$(\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_4$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2$	*unique*
IV		$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$	$\{1\}$	$\neq 2$	*unique*

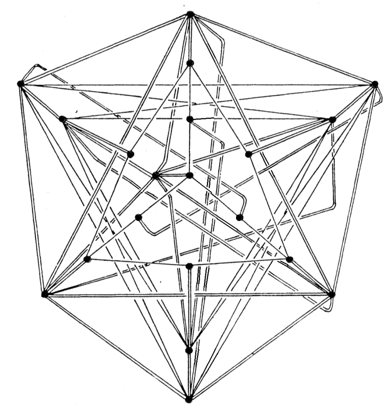
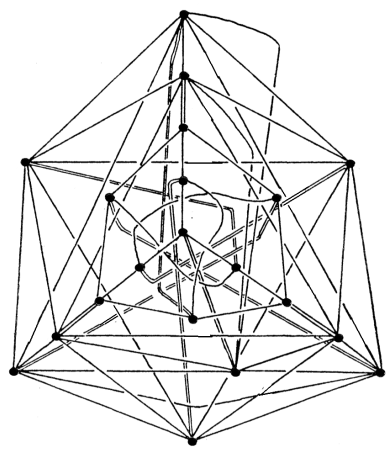
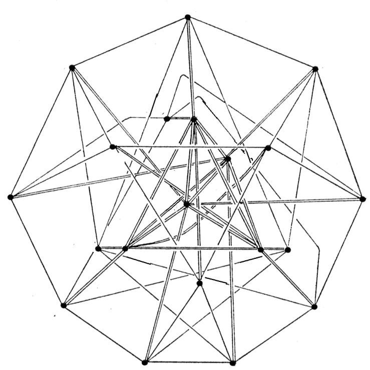
V		$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2, 3$	*unique*
VI		$\mathfrak{S}_5$	$\{1\}$	$\neq 3, 5$	*unique*
VII		$\mathfrak{S}_5$	$\{1\}$	$\neq 2, 5$	*unique*

TABLE 1. Classification

In Table 1, $\mathfrak{S}_n$ is the symmetric group on n letters, D_4 is the dihedral group of order 8, and for two groups N and H, $N \rtimes H$ denotes a semi-direct product of N and H.

In characteristic 2, the search for Enriques surfaces with finite automorphism group has been started recently by T. Katsura and S. Kondō [KK15b]. There, the question of existence of the seven types in characteristic 2 was settled. The classification in Chapter I shows that the examples of singular Enriques surfaces with finite automorphism group in [KK15b] are in fact all possible examples of such surfaces. The classification of classical and supersingular Enriques surfaces with finite automorphism group in characteristic 2 will be treated in Chapter II.

REMARK. As an application of our classification, we determine the semi-symplectic parts of the automorphism groups of Enriques surfaces with finite automorphism group. For the precise statement, we refer the reader to Theorem 12.2 and Table 6.

As explained in the introduction, we avoid the use of transcendental methods by exploiting a quadratic twist construction for special (i.e. with (-2)-curve as bisection) elliptic fibrations: We exhibit "critical" subgraphs, which are dual graphs of singular fibers of a special elliptic fibration π on X together with some special bisection N, for each of Kondō's seven types and we show that an Enriques surface whose dual graph of all (-2)-curves contains such a diagram is one of the seven types. Therefore, we can use the quadratic twist construction to construct π and N and hence the Enriques surface itself. Since the quadratic twist construction is universal, we can give an explicit description of the moduli of Enriques surfaces with finite automorphism group. Finally, the equations we give can actually be interpreted as integral models of these Enriques surfaces and some of them were found using the integral models of extremal and rational elliptic surfaces of T. Jarvis, W. Lang and J. Ricks [JLR12].

As we have just mentioned, a closer look at our equations reveals that they do in fact define integral models of these surfaces in the following sense.

THEOREM 11.3 (Integral models). *Let $K \in \{I, \ldots, VII\}$ and P_K be as in Table 2. There is a family $\varphi_K : \mathcal{X} \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{\mathrm{P_K}}])$ whose fibers are Enriques surfaces of type K with full Picard group over the prime field.*

Type	$\mathrm{P_K}$
I	$255, 257$
II	$63, 65$
III	2
IV	2
V	6
VI	15
VII	10

TABLE 2. Integral models

Note that for $K \neq \mathrm{I}, \mathrm{II}$, P_K is exactly the product over the characteristics where type K does not exist. If $K = \mathrm{I}, \mathrm{II}$, we give two integral models to obtain the following corollary, which solves the existence of the seven types over arbitrary fields.

COROLLARY 11.5. *Suppose that there exists an Enriques surface of type $K \in \{I, \dots, VII\}$ in characteristic p. Then, there exists an Enriques surface of type K with full Picard group over $\mathbb{F}_p$ (resp. over $\mathbb{Q}$ if $p = 0$).*

Moreover, we exhibit special generators of the automorphism groups of Enriques surfaces with finite automorphism group, leading to our third result.

THEOREM 11.6. *Let X be an Enriques surface of type $K \in \{I, \dots, VII\}$ over a field k such that $Pic(X) = Pic(X_{\bar{k}})$.*

- *If $K \neq III, IV$, then $Aut(X)$ is defined over k.*
- *If $K = III$, then $Aut(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 2$.*
- *If $K = IV$, then $Aut(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 16$.*

Let us now explain the structure of Chapter I: In §2, we extend Kondō's base change construction to positive characteristic after recalling several facts on Enriques surfaces and elliptic fibrations. In §3, ..., §9, we construct Enriques surfaces of types I, ..., VII and compute their automorphism groups as well as their moduli. After that, in §10, we classify the dual graphs of Enriques surfaces with finite automorphism group, finishing the proof of our Main Theorem. In §11, we explain how to obtain information on the arithmetic of these surfaces and in §12, we give the list of semi-symplectic automorphism groups of Enriques surfaces with finite automorphism group.

2. Preliminaries

2.1. Generalities on Enriques surfaces, dual graphs and elliptic fibrations. Here we recall some basic facts about Enriques surfaces, clarify our terminology, and refer the reader to [CD89] for proofs and to [Sil94] for anything related to elliptic curves. In the first ten sections, we will be working over an algebraically closed field k.

DEFINITION 2.1. A *K3 surface* is a smooth, projective surface $\tilde{X}$ over k with $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$ and $\mathrm{H}^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. An *Enriques surface* X with smooth K3 cover is the quotient of a K3 surface by a fixed point free involution σ. We call the K3 surface $\tilde{X}$ with $\tilde{X}/\sigma = X$ the *canonical cover* or *K3 cover* of X.

CONVENTION 2.2. From now on, we will drop the "with smooth K3 cover" and we will always assume that the Enriques surfaces we talk about have such a cover.

DEFINITION 2.3. An *elliptic fibration* (with base curve $\mathbb{P}^1$) of a smooth surface $\tilde{X}$ is a surjective morphism $\tilde{\pi} : \tilde{X} \to \mathbb{P}^1$ such that almost all fibers are smooth genus 1 curves, $\tilde{\pi}_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\mathbb{P}^1}$ and no fiber contains a (-1)-curve. We do not require that $\tilde{\pi}$ has a section.

PROPOSITION 2.4. (Bombieri and Mumford [BM76, Theorem 3]) *Every Enriques surface admits an elliptic fibration.*

The reason why we do not assume that elliptic fibrations have a section is that this is never the case for Enriques surfaces:

PROPOSITION 2.5. (Cossec and Dolgachev [CD89, Theorem 5.7.2, Theorem 5.7.5, Theorem 5.7.6]) *Let π be an elliptic fibration of an Enriques surfaces. Then,*

- *if char$(k) \neq 2$, π has exactly two tame double fibers, both of which are either of multiplicative type or smooth, and*
- *if char$(k) = 2$, π has exactly one wild double fiber, which is either of multiplicative type or a smooth ordinary elliptic curve.*

REMARK 2.6. Since being supersingular is an isogeny-invariant, one can check the type of the double fiber on the K3 cover.

Therefore, the intersection number of any curve with a fiber of an elliptic fibration of an Enriques surface is even. Thus, the best approximation to a section will be a bisection.

DEFINITION 2.7. Let N be an irreducible curve on an Enriques surface X and let π be an elliptic fibration of X.

- N is a (-2)-*curve* if $N^2 = -2$. Equivalently, $N \cong \mathbb{P}^1$.
- N is a *special bisection* of π if N is a (-2)-curve with $F.N = 2$, where F is a general fiber of π.
- If π admits a special bisection, we call π *special*.

In fact, special elliptic fibrations are much more common than one might think. More precisely, we have the following result of F. Cossec, which was shown by W. Lang also to hold in characteristic 2.

PROPOSITION 2.8. (Cossec [Cos85, Theorem 4], Lang [Lan88, Theorem A3]) *An Enriques surface contains a (-2)-curve if and only if it admits a special elliptic fibration.*

Now, we recall some facts on the Jacobian fibrations of elliptic fibrations of Enriques surfaces.

PROPOSITION 2.9. (Cossec and Dolgachev [CD89, Theorem 5.7.1]) *Let π be an elliptic fibration of an Enriques surface. Then, the Jacobian fibration $J(\pi)$ of π is an elliptic fibration of a rational surface.*

Since the group of sections of the Jacobian of an elliptic fibration of an Enriques surface acts on the surface, we will mostly be concerned with extremal and rational elliptic fibrations. The group of sections of an elliptic fibration π is also called the *Mordell-Weil group of* π [Sil94, III §9].

DEFINITION 2.10. Let π be an elliptic fibration of an Enriques surface and let $J(\pi)$ be its Jacobian. We call $J(\pi)$ and π *extremal* if the Mordell-Weil group $\mathrm{MW}(J(\pi))$ is finite.

We will use the Kodaira-symbols $\mathrm{I}_n (n \geq 1), \mathrm{I}_n^* (n \geq 0), \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{II}^*, \mathrm{III}^*$, and IV^* to denote the singular fibers of an elliptic fibration (see for example [Sil94, p.354]). The reducible fibers consist of (-2)-curves and their intersection behaviour will play an important role throughout this thesis.

DEFINITION 2.11. Let M be a set of (-2)-curves on a smooth surface X.

- The *dual graph of* M is the graph whose vertices are elements of M and two vertices $E_i, E_j \in M$ with $i \neq j$ are joined by an n-tuple line if $E_i.E_j = n$.

- If M is the set of all (-2)-curves on X, we will call the corresponding graph the *dual graph of all* (-2)*-curves on* X.
- If M is the set of all (-2)-curves contained in singular fibers of an elliptic fibration π of X, we call M the *dual graph of singular fibers of* π.

The dual graphs of the singular fibers of type $\mathrm{I}_n (n \geq 2), \mathrm{I}_n^* (n \geq 0), \mathrm{III}, \mathrm{IV}, \mathrm{II}^*, \mathrm{III}^*$, and IV^* are $\tilde{A}_{n-1}, \tilde{D}_{n+4}, \tilde{A}_1, \tilde{A}_2, \tilde{E}_8$, $\tilde{E}_7$, and $\tilde{E}_6$, respectively (see [Mir89, I.6]). Conversely, configurations of (-2)-curves whose dual graphs are extended Dynkin diagrams of these types give rise to elliptic fibrations.

PROPOSITION 2.12. (Kodaira [Kod63], Mumford [Mum69]) *A connected, reduced divisor D on an Enriques surface X is equal to the support of a fiber of an elliptic fibration if and only if D is an irreducible genus 1 curve or the irreducible components of D are (-2)-curves whose dual graph is an extended Dynkin diagram of type $\tilde{A}$-$\tilde{D}$-$\tilde{E}$.*

Note that one cannot always reconstruct the fiber type from the graph. Using this notation, we can give the list of extremal and rational elliptic fibrations in every characteristic due to R. Miranda, U. Persson and W. E. Lang.

PROPOSITION 2.13. (Miranda and Persson [MP86], Lang [Lan91], [Lan94]) *Let π be an extremal fibration of a rational surface. Then, the singular fibers of π are given in Table 3.*

The extremal and rational elliptic surfaces with singular fibers (I_0^*, I_0^*) in characteristic $\neq 2$ and the ones with singular fiber (I_4^*) in characteristic 2 form 1-dimensional families and all other fibrations are unique.

$\mathrm{char}(k) \neq 2,3,5$	$\mathrm{char}(k) = 5$	$\mathrm{char}(k) = 3$	$\mathrm{char}(k) = 2$
$(\mathrm{II}^*, \mathrm{II})$	$(\mathrm{II}^*, \mathrm{II})$	(II^*)	(II^*)
$(\mathrm{III}^*, \mathrm{III})$	$(\mathrm{III}^*, \mathrm{III})$	$(\mathrm{III}^*, \mathrm{III})$	–
$(\mathrm{IV}^*, \mathrm{IV})$	$(\mathrm{IV}^*, \mathrm{IV})$	–	$(\mathrm{IV}^*, \mathrm{IV})$
$(\mathrm{I}_0^*, \mathrm{I}_0^*)$	$(\mathrm{I}_0^*, \mathrm{I}_0^*)$	$(\mathrm{I}_0^*, \mathrm{I}_0^*)$	–
$(\mathrm{II}^*, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{II}^*, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{II}^*, \mathrm{I}_1)$	$(\mathrm{II}^*, \mathrm{I}_1)$
$(\mathrm{III}^*, \mathrm{I}_2, \mathrm{I}_1)$	$(\mathrm{III}^*, \mathrm{I}_2, \mathrm{I}_1)$	$(\mathrm{III}^*, \mathrm{I}_2, \mathrm{I}_1)$	$(\mathrm{III}^*, \mathrm{I}_2)$
$(\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$	$(\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$	$(\mathrm{IV}^*, \mathrm{I}_3)$	$(\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$
$(\mathrm{I}_4^*, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_4^*, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_4^*, \mathrm{I}_1, \mathrm{I}_1)$	(I_4^*)
$(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$	$(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$	$(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$	–
$(\mathrm{I}_1^*, \mathrm{I}_4, \mathrm{I}_1)$	$(\mathrm{I}_1^*, \mathrm{I}_4, \mathrm{I}_1)$	$(\mathrm{I}_1^*, \mathrm{I}_4, \mathrm{I}_1)$	$(\mathrm{I}_1^*, \mathrm{I}_4)$
$(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_9, \mathrm{II})$	$(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1)$
$(\mathrm{I}_8, \mathrm{I}_2, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_8, \mathrm{I}_2, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_8, \mathrm{I}_2, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_8, \mathrm{III})$
$(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_5, \mathrm{I}_5, \mathrm{II})$	$(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$	$(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$
$(\mathrm{I}_6, \mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_1)$	$(\mathrm{I}_6, \mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_1)$	$(\mathrm{I}_6, \mathrm{I}_3, \mathrm{III})$	$(\mathrm{I}_6, \mathrm{IV}, \mathrm{I}_2)$
$(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$	$(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$	$(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$	–
$(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$	$(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$	–	$(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$

TABLE 3. Extremal and rational elliptic fibrations

REMARK 2.14. From Table 3 we see that the fibrations in small characterstics differ from the characteristic 0 cases only if either a II^* fiber is involved or if the characteristic divides the number of simple components of some fiber of the fibration.

In fact, the Shioda-Tate formula implies that the dual graph of (-2)-curves contained in singular fibers of an elliptic fibration π determines whether π is extremal or not.

LEMMA 2.15. (Shioda, [Shi72, Corollary 1.5]) *Let π be an elliptic fibration of a rational surface or of an Enriques surface. Then, π is extremal if and only if the lattice spanned by the fiber components of π has rank 9.*

Extremal elliptic fibrations of Enriques surfaces over the complex numbers were studied by the author in [Mar16], where he classified those extremal fibrations with at least one reducible double fiber.

2.2. Base Change Construction.

NOTATION 2.16. Let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration with section of a rational surface or of a K3 surface. We denote the composition in $\mathrm{MW}(\pi)$ with respect to some fixed zero section by $\oplus$, the inverse of a section P is denoted by $\ominus P$ and the translation by a section P is denoted by t_P. By abuse of notation, we will also use t_P for the induced automorphism of X.

Over the complex numbers, the following is due to S. Kondō [Kon86, p.199]. There are generalizations of this result in [HS11] and [Sch16]. Since we need this construction for our classification, we will extend it to arbitrary characteristic.

LEMMA 2.17. *Let $f : \tilde{X} \to X$ be the canonical cover of an Enriques surface X and let σ be the covering involution. Let $\pi : X \to \mathbb{P}^1$ be a special elliptic fibration of X with a special bisection N, let F be a double fiber of π and let $J(\pi) : J(X) \to \mathbb{P}^1$ be the Jacobian fibration associated to π. Let $\tilde{\pi}$ be the fibration of $\tilde{X}$ induced by $|f^{-1}F|$ and denote by $\varphi : |f^{-1}F| = \mathbb{P}^1 \to \mathbb{P}^1 = |2F|$ the induced morphism on the base curve.*

Then,

(1) N splits into two sections N^+ and N^- of $\tilde{\pi}$. In particular, the minimal proper smooth models of the base changes of $J(\pi)$ and π along φ are isomorphic.
(2) Choose N^+ as the zero section of $\tilde{\pi}$. Then, $J(\sigma) = t_{\ominus N^-} \circ \sigma$ is an involution whose quotient, after minimalizing the obtained fibration, is $J(\pi)$.
(3) N^- satisfies $N^-.N^+ = 0$, $J(\sigma)(N^-) = \ominus N^-$ and it does not meet the preimage of a singular double fiber of π in the identity component.

The main tool to establish this result in arbitrary characteristic is the following lemma, which is a close study of how automorphisms of the generic fiber of an elliptic fibration with section extend to special fibers. For lack of a reference, we will give a proof.

LEMMA 2.18. *Let R be a discrete valuation ring and let $K = Quot(R)$. Let (E, O) be an elliptic curve over K and let $\mathcal{E}$ be the Néron model of E over R. Let E_0 be the identity component of the special fiber of $\mathcal{E}$. Let $\rho : Aut(E, O) \to Aut(E_0, \overline{O}|_{E_0})$ be the natural map obtained from the Néron mapping property and restriction. Then, ρ is injective if and only if one of the following holds:*

- *char*$(k) \notin \{2,3\}$
- *char*$(k) \in \{2,3\}$ *and* E_0 *is not additive.*

If E_0 *is additive, then* $\ker(\rho)$ *consists of all elements of order* p^n, *where* $p =$ *char*(k).

PROOF. We will compute the reduction of the automorphisms explicitly using Weierstrass equations and the description of automorphisms in [Sil09, p.411] (see also [Sil94, p.364] for an exposition of Tate's algorithm). Throughout, we denote by π a uniformizer of R.

If char$(k) \geq 5$, then we use a minimal and integral Weierstrass equation

$$y^2 = x^3 + a_4x + a_6.$$

Since all $g \in \mathrm{Aut}(E,O)$ are of the form $g : (x,y) \mapsto (\zeta^2 x, \zeta^3 y)$ for some 12-th root of unity ζ, they induce non-trivial automorphisms of E_0 independently of a_4 and a_6.

If char$(k) = 3$, then we use a minimal and integral Weierstrass equation

$$y^2 = x^3 + a_2x^2 + a_4x + a_6.$$

If $a_2 \neq 0$, then the same argument as before works, so we may assume $a_2 = 0$. Then, an automorphism $g \in \mathrm{Aut}(E,O)$ is given by $g : (x,y) \mapsto (\zeta^2 x + r, \zeta^3 y)$, where $\zeta^4 = 1$ and $r^3 + a_4 r + (1-\zeta^2)a_6 = 0$. If $\zeta \neq 1$, then $\rho(g) \neq \mathrm{id}$, since ζ does not depend on a_4 and a_6. But if $\zeta = 1$ and $r = \pm\sqrt{-a_4}$, then $\rho(g)$ is trivial if and only if $\pi \mid a_4$, i.e. if and only if E_0 is of additive type.

If char$(k) = 2$, then we use a minimal and integral Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The inversion involution $g \in \mathrm{Aut}(E,O)$ is given by $(x,y) \mapsto (x, y + a_1x + a_3)$. Thus, $\rho(g)$ is trivial if and only if $\pi \mid a_1, a_3$, i.e. if and only if E_0 is of additive type. Now if $j(E_0) = 0$, then we can assume $a_1 = a_2 = 0$. An automorphism $g \in \mathrm{Aut}(E,O)$ is given by $g : (x,y) \mapsto (\zeta^2 x + s^2, \zeta^3 y + \zeta^2 sx + t)$, where $\zeta^3 = 1$, $s^4 + a_3 s + (1-\zeta)a_4 = 0$ and $t^2 + a_3 t + s^6 + a_4 s^2 = 0$. If $\zeta \neq 1$, then we have $\rho(g) \neq \mathrm{id}$. Therefore, assume $\zeta = 1$ and $s^3 + a_3 = 0$. Now, $\rho(g) = \mathrm{id}$ and if and only if $\pi \mid a_3$, i.e. if and only if E_0 is additive. □

PROOF OF LEMMA 2.17. Since $\tilde{X} \to X$ is étale of degree 2, every (-2)-curve on X splits into two disjoint (-2)-curves on $\tilde{X}$. In particular, N splits into two (-2)-curves N^+ and N^-. We claim that a general fiber of π also splits into two components. Indeed, suppose that a general fiber does not split into two components. Then, char$(k) = 2$ and σ acts on every fiber of $\tilde{\pi}$. Since σ is fixed point free and additive and supersingular fibers do not admit fixed point free involutions, every fiber of $\tilde{\pi}$ would have to be multiplicative or ordinary, which is absurd. Both N^+ and N^- have to be sections of the fibration $\tilde{\pi}$, since a general fiber of π splits into two components F_1 and F_2, both of which are fibers of $\tilde{\pi}$, and therefore $2 = N.F = 2N^+.F_1 = 2N^-.F_1$.

Next, we show that $J(\sigma)$ is indeed an involution. Let F_0 be the identity component of a fiber of $\tilde{\pi}$ which is fixed (not necessarily pointwise) by σ. Note that F_0 is either multiplicative or smooth by Proposition 2.5. Since σ is fixed point free, it induces a translation on F_0 if F_0 is smooth. Moreover, because $J(\sigma)(N^+) = t_{\ominus N^-} \circ \sigma(N^+) = N^+$, $J(\sigma)|_{F_0}$ is the identity if F_0 is smooth, and it can have at most order 2 if F_0 is multiplicative. Together we obtain $J(\sigma)^2|_{F_0} = \mathrm{id}$ in any case. Now, $J(\sigma)^2$ fixes $\tilde{\pi}$ and hence it is an automorphism of the generic fiber of $\tilde{\pi}$ fixing the zero section N^+. By Lemma 2.18, $J(\sigma)^2 = \mathrm{id}$, because it restricts to the identity on F_0. Since $J(\sigma)(N^+) = N^+$, this section descends to the quotient and we obtain $J(\pi)$.

Finally, if π has a singular double fiber F of type I_n, the preimage of F in $\tilde{X}$ is a fiber F' of $\tilde{\pi}$ of type I_{2n}, since this happens with the corresponding fiber on the Jacobian. Now, σ has to act without fixed points, hence it acts as a rotation of order 2 on the corresponding $\tilde{A}_{2n-1}$ diagram, while $J(\sigma)$ fixes the diagram. In particular, the preimage of N meets two opposite curves of the diagram, i.e. N^- does not meet the identity component of F' if we choose N^+ to be the zero section of $\tilde{\pi}$. □

In particular, we obtain a distinguished non-zero section of $\tilde{\pi}$ if $\tilde{\pi}$ arises as the base change of a special elliptic fibration π of an Enriques surface with a given special bisection. Conversely, we will see that we can reconstruct π from $J(\pi)$ by exhibiting a suitable section on a degree 2 base change of $J(\pi)$. This has been studied by K. Hulek and M. Schütt in [HS11] using quadratic twists. Since in our case $J(\pi)$ is an extremal and rational elliptic fibration and extremal and rational elliptic surfaces are classified, we can approach the classification problem in a very explicit way. First, let us clarify what we mean by a "suitable section".

DEFINITION 2.19. Let $J(\pi) : J \to \mathbb{P}^1$ be an elliptic fibration of a rational surface J with zero section N^+. Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a separable degree 2 morphism such that no branch point of φ is a point of additive reduction of $J(\pi)$. If $\mathrm{char}(k) = 2$, assume further that the branch point is not a point of good supersingular reduction of $J(\pi)$. Then, a minimal proper smooth model of the base change $\tilde{\pi}$ of $J(\pi)$ along φ is an elliptic fibration of a K3 surface $\tilde{X}$. Denote the zero section of $\tilde{\pi}$ also by N^+ and let $J(\sigma)$ be a covering involution of $\tilde{X} \to J$ such that $J(\sigma)(N^+) = N^+$. A section N^- of $\tilde{\pi}$ is called a $J(\pi)$-*Enriques section* of $\tilde{\pi}$ if

(1) $N^-.N^+ = 0$,
(2) $J(\sigma)(N^-) = \ominus N^-$, and
(3) N^- does not meet the identity component of the fiber over $\varphi^{-1}(x)$ if φ is branched over a point x with $J(\pi)^{-1}(x)$ singular.

REMARK 2.20. Observe that these are exactly the properties satisfied by the section N^- in Lemma 2.17 (3).

REMARK 2.21. We will encounter several examples of such $J(\pi)$-Enriques sections throughout this chapter. The quickest way to achieve conditions (1) and (2) is to take for N^- an everywhere integral (i.e. $N^-.N^+ = 0$) 2-torsion section of $\tilde{\pi}$, since such a section will be a base change of a 2-torsion section of $J(\pi)$. However, this does not guarantee condition (3) to hold, as we will see later.

The following is the main ingredient in our approach to the classification. Over the complex numbers, this is implicitly contained in [Kon86] (for a variation of this, see [HS11]).

PROPOSITION 2.22. *With notation as in the above definition, let N^- be a section of $\tilde{\pi}$ such that $J(\sigma)(N^-) = \ominus N^-$ and $N^+.N^- = 0$. Then, the quotient of $\tilde{X}$ by the involution $\sigma := t_{N^-} \circ J(\sigma)$ is an Enriques surface X with a special elliptic fibration π induced by $\tilde{\pi}$ if and only if N^- is a $J(\pi)$-Enriques section. The Jacobian of π is $J(\pi)$ and the double fibers of π occur over the branch points of φ.*

PROOF. Let us first show that σ is an involution. Denote by F_0 a fiber which is fixed (not necessarily pointwise) by $J(\sigma)$. We have $\sigma^2|_{F_0} = t_{N^-}|_{F_0} \circ J(\sigma)|_{F_0} \circ t_{N^-}|_{F_0} \circ J(\sigma)|_{F_0} = t_{N^-}|_{F_0} \circ t_{\ominus N^-}|_{F_0} = \mathrm{id}|_{\mathrm{F}_0}$ and since F_0 is either multiplicative or smooth and σ^2 fixes $\tilde{\pi}$ and N^+, we obtain $\sigma^2 = \mathrm{id}$ by Lemma 2.18.

Since translation by a section fixes all fibers and $J(\sigma)$ fixes at most two fibers F_0 and F_1, we have $\mathrm{Fix}(\sigma) \subseteq F_0 \cup F_1$. If $F \in \{F_0, F_1\}$ is smooth, we claim that $J(\sigma)$ acts trivially on F. In characteristic different from 2, this follows because $J(\sigma)$ acts non-trivially on a global 2-form, and in characteristic 2, $J(\sigma)|_F$ is either the identity or a hyperelliptic involution, since it fixes N^+ and F is ordinary. The latter case is impossible by [DK01, Theorem 1]. Since $J(\sigma)$ acts trivially on a smooth fiber $F \in \{F_0, F_1\}$ and $N^-.N^+ = 0$, $\sigma|_F = t_{N^-}|_F$ will have no fixed points on F. As for a multiplicative fiber $F \in \{F_0, F_1\}$, $J(\sigma)$ fixes the components of F (not necessarily pointwise), hence σ has fixed points if and only if N^- meets the identity component of this fiber, i.e. if and only if N^- is not a $J(\pi)$-Enriques section.

Now, if N^- is a $J(\pi)$-Enriques section, this means that the quotient of $\tilde{X}$ by σ is an Enriques surface X. Moreover, the divisors F and $N^+ + N^-$ are fixed by σ and thus descend to X, giving a special elliptic fibration π on X. Additionally, F_0 and F_1 descend to the two double fibers of π and $J(\pi)$ is the Jacobian of π by construction. □

REMARK 2.23. If σ has fixed points , we claim that it actually has a fixed locus of dimension 1. To see this, note that σ fixing two points on a (-2)-curve in characteristic 2 means that the whole curve is fixed (see also [DK01]). For the other characteristics, we refer the reader to [Zha98]. After contracting the fixed locus, the quotient by σ is nothing but a rational log Enriques surface of index 2 [Zha91] and its minimal resolution is a Coble surface (see [DZ01]). We will not study these surfaces here, but the attentive reader will see them occur naturally as degenerations of the models we give for the surfaces in our Main Theorem.

REMARK 2.24. We see from the proof that one can also obtain an Enriques surface as quotient by σ if one weakens the assumption $N^+.N^- = 0$ to $N^+ \cap N^- \cap F_0 = N^+ \cap N^- \cap F_1 = \emptyset$. However, in general, this will not produce a *smooth* bisection. For more on this, see [HS11].

With this explicit and universal construction at our disposal, we can have a look at the relation between special bisections of an elliptic fibration of an Enriques surface and sections of its Jacobian.

COROLLARY 2.25. *Let π be a special elliptic fibration of an Enriques surface X with a special bisection N splitting into N^+ and N^- on the K3 cover $\tilde{X}$ of X. There is a map*

$$jac_2 : \mathit{MW}(J(\pi)) \to \{\textit{special bisections of } \pi\},$$

which is

- *injective if N^- is not 2-torsion after fixing N^+ as the zero section, and*
- *2-to-1 onto its image otherwise.*

Moreover, MW$(J(\pi))$ acts transitively on the image of jac_2 via its action on X.

PROOF. We use the notation of Lemma 2.17. There is a natural injection $\mathrm{MW}(J(\pi)) \to \mathrm{MW}(\tilde{\pi})$ and using this, we will consider sections of $J(\pi)$ as sections of $\tilde{\pi}$ by abuse of notation. Let

$P \in \mathrm{MW}(J(\pi))$. Since P comes from $J(\pi)$, it is fixed by $J(\sigma)$. Now, we compute

$$P.\sigma(P) = P.(t_{N^-} \circ J(\sigma))(P) = P.(P \oplus N^-) = N^+.N^- = 0.$$

Therefore, the divisor $P + \sigma(P)$ descends to a (-2) curve $jac_2(P)$ on X, which is necessarily a bisection of π, since $2 = (P + \sigma(P)).\tilde{F} = jac_2(P).F$, where $\tilde{F}$ (resp. F) is a general fiber of $\tilde{\pi}$ (resp. π). For the injectivity, observe that $\sigma(P) \in \mathrm{MW}(J(\pi))$ if and only if $J(\sigma)(\sigma(P)) = \sigma(P)$, i.e. if and only if

$$P \oplus N^- = (t_{N^-} \circ J(\sigma))(P) = \sigma(P) = J(\sigma)(\sigma(P)) = P \ominus N^-,$$

which happens if and only if N^- is 2-torsion. The statement about the action of $\mathrm{MW}(J(\pi))$ is clear by construction of jac_2. □

To compute the intersection behaviour of the special bisections obtained via jac_2, we will use the height pairing on $\mathrm{MW}(\tilde{\pi})$.

PROPOSITION 2.26. (Shioda [Shi90]) *Let $\tilde{\pi}$ be an elliptic fibration of a K3 surface with zero section N^+. The pairing*

$$\begin{array}{rcl} MW(\tilde{\pi}) \times MW(\tilde{\pi}) & \to & \mathbb{Q} \\ (P,Q) & \mapsto & \langle P, Q\rangle = 2 + P.N^+ + Q.N^+ - P.Q - \sum_{\nu \in \mathbb{P}^1} contr_\nu(P,Q), \end{array}$$

where the $contr_\nu(P,Q)$ are local correction terms depending on the intersection of P and Q with the fiber over ν, is a symmetric, bilinear pairing on MW($\tilde{\pi}$), which induces the structure of a positive definite lattice on MW($\tilde{\pi}$)/MW($\tilde{\pi}$)$_{tors}$. It is called the height pairing *on MW($\tilde{\pi}$). We write $h(P)$ for $\langle P, P\rangle$.*

REMARK 2.27. Note that this implies immediately that $h(P) = 0$ if and only if P is in $\mathrm{MW}(\tilde{\pi})_{tors}$. Moreover, $\langle P, Q\rangle = 0$ as soon as P or Q is in $\mathrm{MW}(\tilde{\pi})_{tors}$.

For the reader's convenience, we recall the correction terms of the height pairing following [SS10, p.52]. First, we have to fix a numbering of the simple components of a reducible fiber F_ν of an elliptic fibration π with zero section N^+ depending on the dual graph Γ of F_ν. In any case, denote the component of F_ν which meets N^+ by E_0.

- If $\Gamma = \tilde{A}_{n-1}$, denote the components of F_ν by $E_0, \ldots, E_{n-1}$ such that $E_i.E_j = 1$ if and only if $i - j = \pm 1$ mod n.
- If $\Gamma = \tilde{D}_{n+4}$, denote the simple components of F_ν by E_0, E_1, E_2, and E_3 such that E_1 is a simple component with minimal distance to E_0.

Now, let $P, Q \in \mathrm{MW}(\pi)$ such that P meets E_i and Q meets E_j and assume $i \leq j$. If $i = 0$, the correction term is 0. Otherwise, the value of $contr_\nu(P,Q)$ is given in the following Table 4.

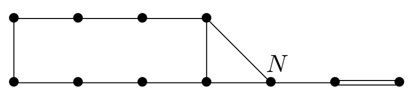
2.3. Example. We keep the notation introduced in the previous subsection. Since we know how sections coming from $J(\pi)$ intersect the fibers of $\tilde{\pi}$, we can compute the intersection behaviour of the corresponding bisections on X once we know how N^- intersects the fibers of $\tilde{\pi}$. But this is already determined by the intersection behaviour of the special bisection N on X with the fibers

Γ	$\tilde{E}_7$	$\tilde{E}_6$	$\tilde{D}_{n+4}$	$\tilde{A}_{n-1}$
Case $i = j$	$\frac{3}{2}$	$\frac{4}{3}$	$\begin{cases} 1 & \text{if } i = 1 \\ 1 + \frac{n}{4} & \text{else} \end{cases}$	$\frac{i(n-i)}{n}$
Case $i < j$	-	$\frac{2}{3}$	$\begin{cases} \frac{1}{2} & \text{if } i = 1 \\ \frac{1}{2} + \frac{n}{4} & \text{else} \end{cases}$	$\frac{i(n-j)}{n}$

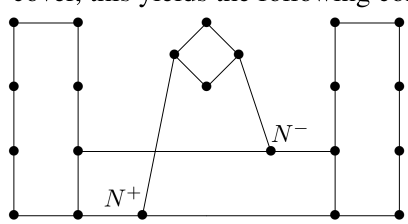
TABLE 4. Correction terms for the height pairing

of the morphism π. We will leave these computations to the reader but give a detailed description of the procedure in the following example.

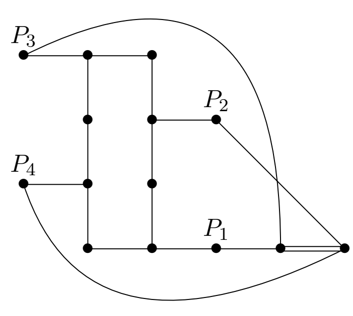
Suppose an Enriques surface contains the following dual graph of (-2)-curves with N as indicated:



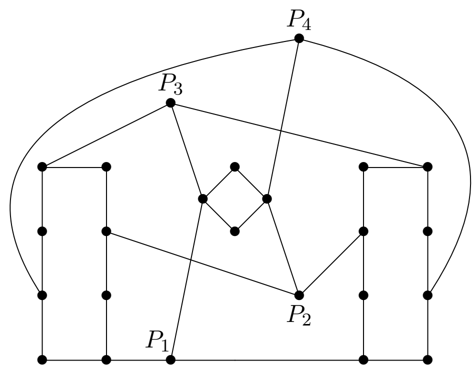
This is the dual graph of a special elliptic fibration with a singular fiber of type I_8 and a double fiber of type I_2. Note that the I_2 fiber has to be double, since N meets its components only once and N is a bisection. On the K3 cover, this yields the following configuration:



On the other hand, we know that the Jacobian of π together with its four sections P_1, P_2, P_3, and P_4 has the following dual graph:

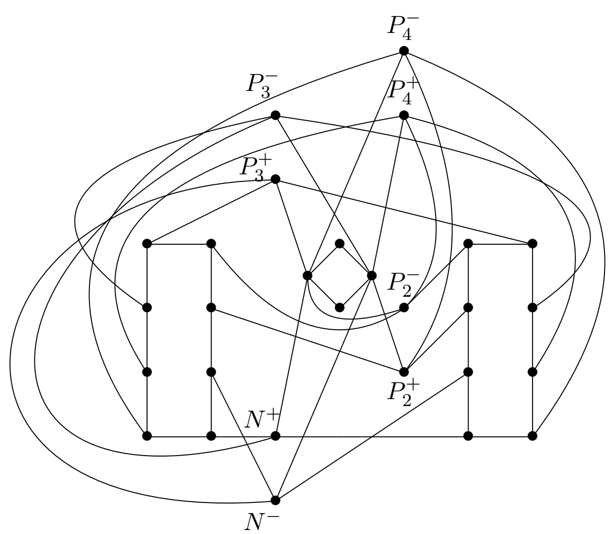


One can explicitly compute the dual graph of a degree 2 base change of $J(\pi)$ ramified over the I_2 fiber (and not ramified over I_8):

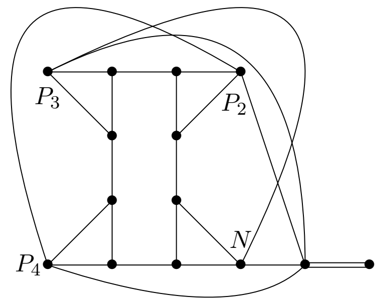


To put this picture together with the second one, we set $N^+ = P_1$ as the zero section, add the sections $N^- \oplus P_i$ for all i to the diagram and calculate the intersection of N^- with P_i using the height pairing and the equality $0 = \langle P_i, N^- \rangle = 2 - N^-.P_i - \sum_\nu contr_\nu(P_i, N^-)$ which follows from Remark 2.27. By using translations, we obtain the remaining intersection numbers and the following graph, where we denote P_i and $P_i \oplus N^-$ by P_i^+ and P_i^- respectively:

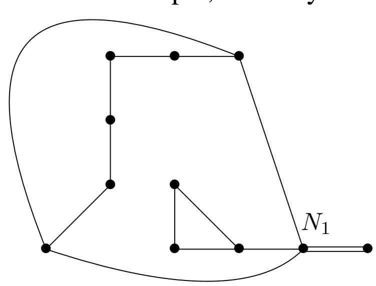
$$P_2.N^- = P_4.N^- = 2 - \left(\frac{6}{8} + \frac{2}{8} + 1\right) = 0; \quad P_3.N^- = 2 - \left(\frac{4}{8} + \frac{4}{8}\right) = 1$$



This yields the following configuration on the quotient Enriques surface, where we denote the special bisection corresponding to P_i again by P_i:



In fact, we can produce six more (-2)-curves using different fibrations with a double I_3 fiber to obtain the dual graph of type VII. For example, one may look at the following subgraph:



By Proposition 2.12, the (-2)-curve N_1 is a special bisection of a fibration with fibers I_6, I_3 (not IV, since it is double) and another reducible fiber. By Lemma 2.15, the corresponding fibration is extremal and by Table 3, the last reducible fiber is of type I_2 (resp. III in characteristic 3) and it is simple, since N_1 meets its reduced components twice. Hence, we can add the missing component of the I_2 (resp. III) fiber to the graph. Similarly, one finds five more (-2)-curves and finally obtains the dual graph of type VII. The configuration we started with is what we will later call the "critical subgraph of type VII", since we have shown that any Enriques surface containing this graph is of type VII.

REMARK 2.28. Note that the crucial point in all examples is the computation of the intersection numbers of the bisections using the height pairing. The intersection of the bisections obtained via jac_2 with the fibers is just a "translation" of the intersection of N with the fibers. In particular, the process is much easier if N^- is a 2-torsion section, since the bisections arising via jac_2 are disjoint.

2.4. Vinberg's criterion and numerically trivial automorphisms. In order to check that the (-2)-curves in the graphs for types $\mathrm{I}, \ldots, \mathrm{VII}$ are all (-2)-curves on the Enriques surface, one uses Vinberg's criterion.

PROPOSITION 2.29. (Vinberg [Vin75, Theorem 2.6]) *Let Γ be a dual graph of finitely many (-2)-curves on an Enriques surface X. Suppose that Γ contains no m-tuple lines with $m \geq 3$ and suppose that the cone $K = \{C \in Num(X)_{\mathbb{R}} | C.E \geq 0 \text{ for all } E \in \Gamma\}$ is strictly convex. Then, the group W_Γ generated by reflections along (-2)-curves in Γ has finite index in $O(Num(X))$ if and* *only if the fibration π induced by every subgraph F of Γ of type $\tilde{A}$-$\tilde{D}$-$\tilde{E}$ is extremal and Γ contains the dual graph of singular fibers of π. In this case, Γ is the dual graph of all (-2)-curves on X.*

REMARK 2.30. This is a reformulation of the version of Vinberg's criterion presented by Kondō [Kon86, Theorem 1.9]. The last statement is due to Namikawa [Nam85, (6.9)]. The strict convexity of K can be achieved, for example, if Γ contains the dual graph of singular fibers of an elliptic fibration π and also contains another (-2)-curve which is not contained in a fiber of π.

The following corollary is a straightforward application of Vinberg's criterion.

COROLLARY 2.31. *Let X be an Enriques surface whose dual graph of all (-2)-curves contains a graph Γ which is one of the seven dual graphs in the Main Theorem. Then, the (-2)-curves in Γ are all (-2)-curves on X.*

Therefore, we can check the action of Aut(X) on Num(X) directly on the dual graph of (-2)-curves on X.

DEFINITION 2.32. An automorphism of an Enriques surface X is called *numerically trivial* if it acts trivially on Num(X). It is called *cohomologically trivial* if it acts trivially on Pic(X). We denote the respective groups by $\mathrm{Aut}_{\mathrm{nt}}(X)$ and $\mathrm{Aut}_{\mathrm{ct}}(X)$.

Recall that Num(X) is a quotient of Pic(X), hence $\mathrm{Aut}_{\mathrm{ct}}(X)$ is a normal subgroup of $\mathrm{Aut}_{\mathrm{nt}}(X)$. Over the complex numbers a complete classification of such automorphisms is available (see [MN84] and [Muk10]). There are three types of Enriques surfaces X with numerically trivial automorphisms and they satisfy $\mathrm{Aut}_{nt}(X) \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}\}$. In positive characteristics, however, we only have bounds on the size of these groups.

PROPOSITION 2.33. (Dolgachev [Dol13]) *Let X be an Enriques surface. Then,*

$$|Aut_{\mathrm{ct}}(X)| \leq 2 \quad \text{and} \quad |Aut_{\mathrm{nt}}(X)/Aut_{\mathrm{ct}}(X)| \leq 2.$$

We will not use this result, since we are interested in the precise shape of the automorphism group. Therefore, we give explicit arguments in every case. In Chapter III, we explain how to deduce Dolgachev's results on numerically and cohomologically trivial automorphisms in arbitrary characteristic and we will see how a classification similar to the complex case can be obtained.

3. Enriques surfaces of type I

3.1. Main theorem for type I.

THEOREM 3.1. *Let X be an Enriques surface. The following are equivalent:*

(1) X is of type I.
(2) The dual graph of all (-2)-curves on X contains the graph in Figure 1.
(3) The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form

$$y^2 + \beta(s^2+s)xy = x^3 + \beta^3(s^2+s)^3 x$$

such that the covering morphism $\rho : \tilde{X} \to X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s-1$ and t_{N^-} is translation by $N^- = (0,0)$.

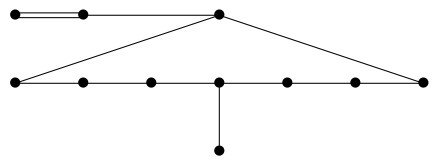
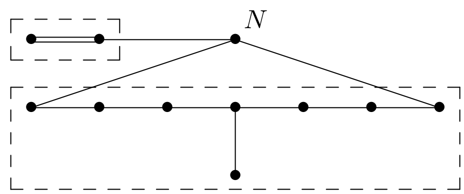


FIGURE 1. Critical subgraph for type I

PROOF. First, observe that the dual graph of type I (see Table 1) contains the graph in the above Figure 1.

This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers III* and I_2 (not III, since this fiber is a double fiber) and special bisection N as follows, where the dotted rectangles mark the fibers:



As explained in Lemma 2.17, N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we can compute $h(N^-) = 0$ and we see that N^- is a 2-torsion section of $\tilde{\pi}$. Starting from the subgraph in Figure 1, we get the last missing (-2)-curve from the elliptic fibration with a double fiber of type I_8, which is induced by the $\tilde{A}_7$ diagram, as follows: The fibration is extremal by Lemma 2.15, the second reducible fiber is of type I_2 (resp. III in characteristic 2) by Table 3 and the intersection behaviour can be determined from the dual graph. These are all (-2)-curves on X by Corollary 2.31.

Now, we pursue the converse process dictated by Proposition 2.22 and calculate all elliptic fibrations of K3 surfaces obtained as separable quadratic base changes of $J(\pi)$ together with a section having the same intersection behaviour as N^- with curves obtained from (-2)-curves on X.

By [JLR12] we have the following equation for the unique rational elliptic surface with singular fibers of type III* and I_2

$$y^2 + txy = x^3 + t^3 x,$$

where t is a coordinate on $\mathbb{P}^1$. The I_2 fiber is at $t = \infty$, while the III* fiber is at $t = 0$. Moreover, if $\mathrm{char}(k) \neq 2$, there is an I_1 fiber at $t = 64$ and all other fibers are smooth. The non-trivial 2-torsion section is $s = (0, 0)$.

In every characteristic, we can write a degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with $t = \infty$ as branch point and which is not branched over $t = 0$ in the form

$$t \mapsto \beta(s^2 + s),$$

where s is the new parameter on $\mathbb{P}^1$ and $\beta \in k - \{0\}$. We are allowed to assume that $t = 0$ is not a branch point, since the III* fiber is not multiple. The covering involution is given by $J(\sigma) : s \mapsto -s - 1$. The second branch point of this degree 2 cover in characteristic different from

2 is at $t = -\frac{\beta}{4}$, which corresponds to $s = -\frac{1}{2}$. Now, we get the equation

$$y^2 + \beta(s^2+s)xy = x^3 + \beta^3(s^2+s)^3 x$$

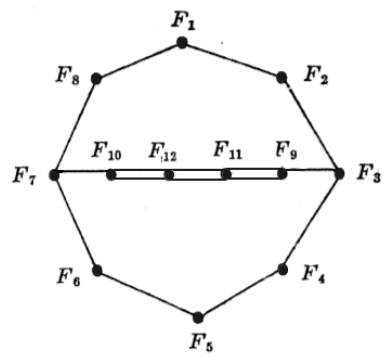
together with the 2-torsion section $s' = (0,0)$ obtained by pulling back s. This equation defines an elliptic fibration $\tilde{\pi}$ on a K3 surface. As explained in Section 2.2, if $\tilde{\pi}$ is obtained as base change of a fibration of an Enriques surface, then $s' = N^-$ and σ is the covering involution. □

REMARK 3.2. Note that we have not yet claimed existence of Enriques surfaces of type I. However, we have reduced this problem to the question whether N^- is a $J(\pi)$-Enriques section or not. We answer this question in the subsection on degenerations and moduli.

3.2. Automorphisms.

PROPOSITION 3.3. *Let X be an Enriques surface of type I. Then, $Aut(X) \cong D_4$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X. Moreover, $Aut_{nt}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and $Aut(X)/Aut_{nt}(X) = (\mathbb{Z}/2\mathbb{Z})^2$.*

PROOF. Recall that the dual graph of type I is as follows:



As has already been explained by Kondō [Kon86, p.205] and Dolgachev [Dol84, p.175], the symmetry group of the dual graph of (-2)-curves is $(\mathbb{Z}/2\mathbb{Z})^2$ and the 2-torsion section of the fibration π induced by the linear system $|2(F_9+F_{11})|$ acts as a reflection along the horizontal axis, while the 2-torsion section of the fibration induced by $|F_{11}+F_{12}|$ acts trivially on the graph. A non-trivial numerically trivial automorphism g fixes F_3 and F_7 pointwise, hence g fixes the fibration π and at least one geometric point on the generic fiber of π. Since π is non-isotrivial, g is the unique hyperelliptic involution of the generic fiber of π fixing the geometric points defined by F_3 and F_7. Since Aut(X) contains a translation by a 4-torsion section of the Jacobian of $|F_{11}+F_{12}|$, it suffices to observe that the 2-torsion section of a fibration with I_4^* fiber acts as a reflection along the vertical axis to show that Aut$(X) \cong D_4$. This follows from Corollary 2.25. □

3.3. Degenerations and Moduli.

PROPOSITION 3.4. *Let $\beta \neq 0$ and*

$$y^2 + \beta(s^2+s)xy = x^3 + \beta^3(s^2+s)^3 x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}_\beta$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s - 1$ and t_{N^-} is translation by the section $N^- = (0, 0)$. Then, the following statements are true:

(1) σ is fixed point free if and only if $\beta \neq -256$. If $\beta = -256$, then the fixed locus of σ is one (-2)-curve.
(2) Two fibrations $\tilde{\pi}_\beta$ and $\tilde{\pi}_{\beta'}$ are isomorphic up to automorphisms of $\mathbb{P}^1$ if and only if $\beta = \beta'$.

PROOF. For the first claim, by Lemma 2.22, we have to check whether N^- is a $J(\pi)$-Enriques section. First, observe that $N^-.N^+ = 0$, $J(\sigma)(N^-) = N^- = \ominus N^-$ and N^- does not meet the I_4 fiber in the identity component. Therefore, we are done if the second fiber fixed by $J(\sigma)$, namely $F_{-\frac{1}{2}}$, is smooth. This happens if and only if $\beta \neq -256$ and otherwise $F_{-\frac{1}{2}}$ is an I_2 fiber. In the latter case, N^- does not meet the singular point $(-2^9, 2^{14})$ of the Weierstrass equation at $s = -\frac{1}{2}$ and therefore it meets the identity component of $F_{-\frac{1}{2}}$. Hence, N^- is not a $J(\pi)$-Enriques section in this case and σ is not fixed point free by Proposition 2.22.

The second claim follows immediately from a comparison of j-invariants, since in any characteristic and independently of β, the locations of the III^* and I_4 fibers are at $s = -1, 0, \infty$. □

We have seen in the previous subsection that the two elliptic fibrations with singular fiber III^* on an Enriques surface of type I are isomorphic. Therefore, we can describe the moduli space of these Enriques surfaces using the previous proposition.

COROLLARY 3.5. *Enriques surfaces of type I are parametrized by $\mathbb{A}^1 - \{0, -256\}$ in every characteristic.*

While $\beta \in \{0, \infty\}$ leads to very degenerate surfaces, we still get an involution if $\beta = -256$, while the K3 surface acquires an additional rational double point. The minimal resolution of the quotient is a Coble surface (see also Remark 2.23).

4. Enriques surfaces of type II

4.1. Main theorem for type II.

THEOREM 4.1. *Let X be an Enriques surface. The following are equivalent:*

(1) X is of type II.
(2) The dual graph of all (-2)-curves on X contains the graph in Figure 2.

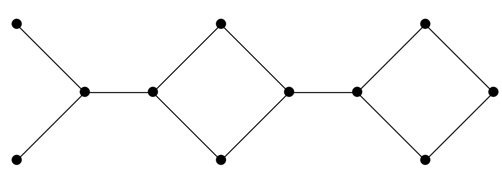


FIGURE 2. Critical subgraph for type II

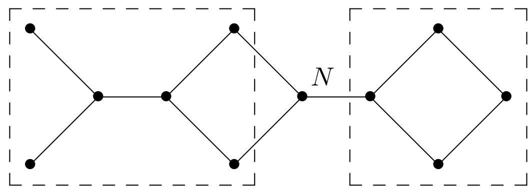
(3) The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form

$$y^2 + \beta(s^2+s)xy + \beta^2(s^2+s)^2 y = x^3 + \beta(s^2+s)x^2$$

such that the covering morphism $\rho : \tilde{X} \to X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s-1$ and t_{N^-} is translation by $N^- = (0,0)$.

PROOF. First, observe that the dual graph of type II (see Table 1) contains the graph in the above Figure 2.

This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers $\mathrm{I}_1^*, \mathrm{I}_4$ and special bisection N as follows, where the dotted rectangles mark the fibers:



Note that the I_4 fiber is a double fiber. Similarly to the case of type II, we compute $h(N^-) = 0$ and find the last missing (-2)-curves via jac_2.

We found the following equation for the unique rational elliptic surface with singular fibers of type I_1^* and I_4 in arbitrary characteristic

$$y^2 + txy + t^2 y = x^3 + tx^2,$$

where t is a coordinate on $\mathbb{P}^1$. The I_4 fiber is at $t = \infty$, while the I_1^* fiber is at $t = 0$. Moreover, if $\mathrm{char}(k) \neq 2$, then there is an I_1 fiber at $t = 16$ and all other fibers are smooth. The non-trivial 2-torsion section is $s = (0,0)$.

In every characteristic, we can write every degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with $t = \infty$ as branch point that is not branched over $t = 0$ in the following form

$$t \mapsto \beta(s^2+s),$$

where s is the new parameter on $\mathbb{P}^1$ and $\beta \in k - \{0\}$. The covering involution is given by $s \mapsto -s-1$. The second branch point of this degree 2 cover in characteristic different from 2 is at $t = -\frac{\beta}{4}$. Now, we get the equation

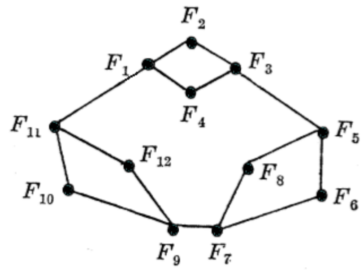
$$y^2 + \beta(s^2+s)xy + \beta^2(s^2+s)^2 y = x^3 + \beta(s^2+s)x^2$$

together with the 2-torsion section $s' = (0,0)$ obtained by pulling back s. This equation defines an elliptic fibration $\tilde{\pi}$ on a K3 surface. As explained in Section 2.2, if $\tilde{\pi}$ is obtained as base change of a fibration of an Enriques surface, then $s' = N^-$ and σ is the covering involution. □

4.2. Automorphisms.

PROPOSITION 4.2. *Let X be an Enriques surface of type II. Then, $Aut(X) \cong \mathfrak{S}_4$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X. Moreover, $Aut_{nt}(X) \cong \{1\}$.*

PROOF. Kondō's proof works in arbitrary characteristic [Kon86, p.208] once we show that the surface has no numerically trivial automorphisms. Recall that the dual graph of (-2)-curves for type II is as follows:



A numerically trivial automorphism g fixes the two bisections F_1 and F_7 of the non-isotrivial fibration π induced by the linear system $|2(F_9 + F_{10} + F_{11} + F_{12})|$ pointwise. Both F_1 and F_7 are separable (i.e. the projection to the base curve is separable) bisections of π, since they meet distinct points on the I_1^* fiber, hence g fixes at least four geometric points on the generic fiber of π. If $\mathrm{char}(k) = 2$, then g is trivial. If $\mathrm{char}(k) \neq 2$, then we may assume that g is non-trivial. Then, g is a hyperelliptic involution of π and the four geometric points on the generic fiber are 2-torsion points relative to each other. But in characteristic different from 2, π has an I_1 fiber which has only two 2-torsion points. Therefore, F_1 and F_7 would have to meet, but they do not. Hence, g is trivial. □

4.3. Degenerations and Moduli. As in the case of type I, one proves the following.

PROPOSITION 4.3. *Let $\beta \neq 0$ and*

$$y^2 + \beta(s^2 + s)xy + \beta^2(s^2 + s)^2 y = x^3 + \beta(s^2 + s)x^2$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}_\beta$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s - 1$ and t_{N^-} is translation by the section $N^- = (0, 0)$. Then, the following statements are true:

(1) σ is fixed point free if and only if $\beta \neq -64$. If $\beta = -64$, the fixed locus of σ is one (-2)-curve.
(2) Two fibrations $\tilde{\pi}_\beta$ and $\tilde{\pi}_{\beta'}$ are isomorphic up to automorphisms of $\mathbb{P}^1$ if and only if $\beta = \beta'$.

COROLLARY 4.4. *Enriques surfaces of type II are parametrized by $\mathbb{A}^1 - \{0, -64\}$ in every characteristic.*

As in the case of type I, the cases where $\beta \in \{0, \infty\}$ are very degenerate surfaces and $\beta = -64$ leads to a Coble surface.

5. Enriques surfaces of type III

5.1. Main theorem for type III.

THEOREM 5.1. *Let X be an Enriques surface. The following are equivalent:*

(1) X is of type III.
(2) The dual graph of all (-2)-curves on X contains the graph in Figure 3.

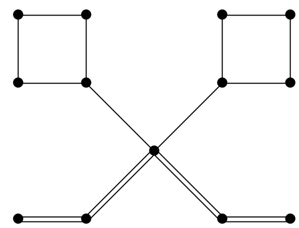


FIGURE 3. Critical subgraph for type III

(3) The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form

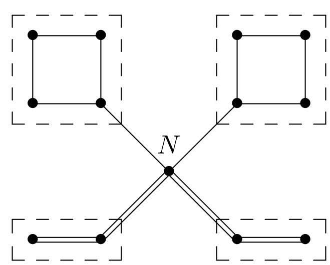
$$y^2 + xy = x^3 + 4s^4x^2 + s^4x$$

such that the covering morphism $\rho : \tilde{X} \to X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s$ and t_{N^-} is translation by $N^- = (0,0)$.

Moreover, Enriques surfaces of type III do not exist in characteristic 2.

PROOF. Note that the dual graph of type III (see Table 1) contains the graph in Figure 3.

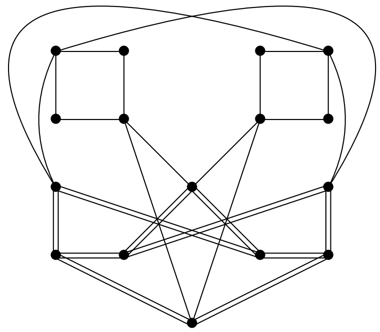
The subgraph in Figure 3 can be interpreted as the dual graph of a special elliptic fibration π with singular fibers $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$ and special bisection N as follows, where the dotted rectangles mark the fibers:



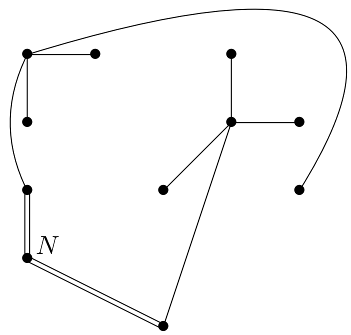
As before, the bisection N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we compute $h(N^-) = 0$ and we see that N^- is a 2-torsion section of $\tilde{\pi}$ meeting the I_8 fibers in a non-identity component.

Note that the existence of this fibration already implies non-existence of this type of Enriques surfaces in characteristic 2, since a fibration with singular fibers $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$ does not exist on rational surfaces in characteristic 2, as can be seen in Table 3.

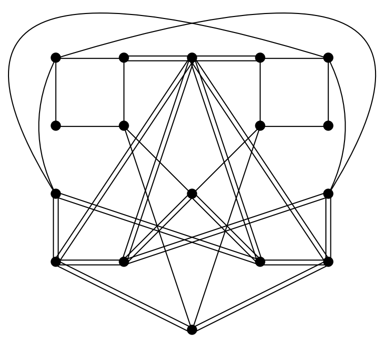
Now, Corollary 2.25 gives three more (-2)-curves resulting in the following graph:



We find a graph of an elliptic fibration with singular fibers $(\mathrm{I}_0^*, \mathrm{I}_0^*)$ and special bisection N:



With the usual notation, we compute $h(N^-) = 2$ and add two bisections coming from Corollary 2.25. In the following figure, we only added one of these bisections to maintain readability:



Note that one of the bisections arising via jac_2 has already been part of the graph to begin with. Hence, it remains to produce two more (-2)-curves using another fibration. We leave the details to the reader.

By [JLR12], we have the following equation for the unique rational elliptic surface with singular fibers of type $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$ in characteristic different from 2 (the equation can be simplified over $\mathbb{Z}$)

$$y^2 + xy = x^3 + 4t^2x^2 + t^2x,$$

where t is a coordinate on $\mathbb{P}^1$. The I_4 fibers are at $t = 0, \infty$, while the I_2 fibers are at $t = \pm\frac{1}{4}$. The non-trivial 2-torsion sections are $s_1 = (-4t^2, 2t^2)$, $s_2 = (0, 0)$ and $s_3 = (-\frac{1}{4}, \frac{1}{8})$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with $t = 0, \infty$ as branch points in the following form

$$t \mapsto s^2,$$

where s is the new parameter on $\mathbb{P}^1$. The covering involution $J(\sigma)$ is given by $s \mapsto -s$. Now, we get the equation

$$y^2 + xy = x^3 + 4s^4x^2 + s^4x \tag{5.1}$$

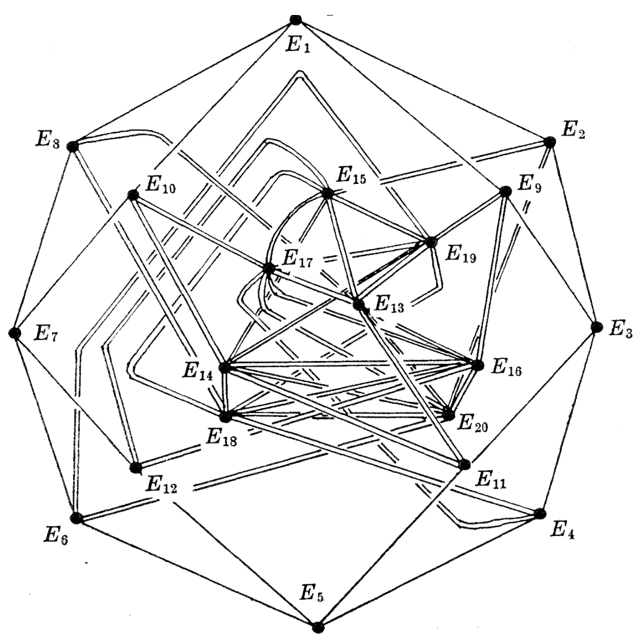
together with the 2-torsion sections $s'_1 = (-4s^4, 2s^4)$, $s'_2 = (0, 0)$ and $s'_3 = (-\frac{1}{4}, \frac{1}{8})$ obtained by pulling back s_1,s_2 and s_3. All of them are $J(\sigma)$-(anti-)invariant. However, s'_1 (resp. s'_3) meets the identity component of the fiber at $s = \infty$ (resp. $s = 0$). Therefore, s'_2 is the section we are looking for. $\square$

REMARK 5.2. Note that Equation (5.1) has an automorphism $\iota : s \mapsto \sqrt{-1}s$ which commutes with σ. Therefore, ι induces an automorphism of the Enriques surface, which we will also denote by ι. Moreover, ι fixes the 2-torsion sections of (5.1). Note also that this automorphism acts as $\sqrt{-1}$ on a non-zero global 2-form of the K3 surface.

5.2. Automorphisms.

PROPOSITION 5.3. *Let X be an Enriques surface of type III. Then, $Aut(X) \cong (\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_4$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of non-isotrivial elliptic fibrations of X and the automorphism exhibited in Remark 5.2. Moreover, $Aut_{nt} \cong \mathbb{Z}/2\mathbb{Z}$ and $Aut(X)/Aut_{nt}(X) = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes D_4$.*

PROOF. Recall that the dual graph of (-2)-curves for type III is as follows:



Let us first show that $|\mathrm{Aut}_{nt}(X)| \leq 2$. Consider the elliptic fibration π induced by the linear system $|2(E_3 + E_4 + E_5 + E_{11})|$ and let $g \in \mathrm{Aut}_{nt}(X)$ be a non-trivial automorphism. If g fixes one of the bisections E_2, E_9, E_6 and E_{12} pointwise, then it is the hyperelliptic involution of the generic fiber of π fixing the geometric points defined by the bisection. Moreover, g induces a unique involution on such a bisection if it acts non-trivially on it. In any case, $ord(g) = 2^n$ for some $n \in \mathbb{N}$ and, since $\mathrm{char}(k) \neq 2$, g is tame. The fixed locus of a tame automorphism is smooth by the Lefschetz fixed point formula [Ive72]. Since g fixes E_1, E_3, E_5 and E_7 pointwise, it has to act non-trivially on E_2, E_9, E_6 and E_{12}. In particular, g is unique.

As explained by Kondō [Kon86, p.214], the automorphism group of the graph is the same as the automorphism group of the subgraph Σ generated by $\{E_i\}_{i\in\{1,\ldots,12\}}$, which is $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes D_4$. Moreover, since the intersection behaviour of the curves is the same in any characteristic, it is still true that only an index 2 subgroup of $\mathrm{Aut}(\Sigma)$ may be realized.

As for the realization of the automorphisms, note the following:

- A reflection r_d along a diagonal axis is realized by a 2-torsion section of the Jacobian of $|E_2 + E_9 + E_6 + E_{12} + 2(E_1 + E_7 + E_8)|$.
- A reflection r_v along the vertical axis is realized by the 2-torsion section of the Jacobian of $|E_2 + E_9 + E_8 + E_{10} + 2(E_3 + E_4 + E_5 + E_6 + E_7)|$.
- There is a 2-torsion section of the Jacobian of the fibration $|2(E_3 + E_4 + E_5 + E_{11})|$ which interchanges E_2 and E_9 as well as E_6 and E_{12} while fixing E_4, E_{11}, E_8 and E_{10}. Another 2-torsion section of the same fibration induces the numerically trivial involution.
- After fixing E_6 as a special bisection N of $|2(E_3+E_4+E_5+E_{11})|$, the automorphism ι of Remark 5.2 fixes E_6 and E_{12} and interchanges E_2 and E_9. Moreover, it acts non-trivially on exactly one of the pairs (E_3, E_{10}) and (E_4, E_{11}).

These facts are checked by using Corollary 2.25 and following through the construction of jac_2. Now, note that we can compute the pointwise stabilizer G of the set $\{E_1, E_3, E_5, E_7\}$ using Equation (5.1). It is generated by t_{s_1}, t_{s_3} and t_{s_2} as well as ι and the inversion involution. All these automorphisms commute with each other and $\iota^2 = t_{s_2}$, hence $G \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, we have a short exact sequence

$$0 \longrightarrow G \longrightarrow \mathrm{Aut}(X) \longrightarrow D_4 \longrightarrow 0.$$

We claim that this sequence splits. Indeed, by [MO14, Corollary 4.7 and Section 7.1], a tame semi-symplectic automorphism (i.e. an automorphism acting trivially on $\mathrm{H}^0(\mathrm{X}, \omega_{\mathrm{X}}^{\otimes 2})$) has order at most 6. We have realized all reflections using translations by 2-torsion sections, which are semi-symplectic, since they fix the base of an elliptic fibration and act as translation on the fibers, and tame, since we are working in characteristic different from 2. Therefore, $r_d \circ r_v$ has order 4 and the group generated by r_d and r_v is a subgroup of $\mathrm{Aut}(X)$ isomorphic to D_4. Hence, the sequence splits and the proof is finished. □

REMARK 5.4. In particular, note that $\mathrm{Aut}(X)$ is not a semi-direct product $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes D_4$. This was already observed by H. Ohashi in [Oha15] and corrects a small mistake in [Kon86].

5.3. Degenerations and Moduli. This is similar to the first two cases. However, the involution is always fixed point free, since the branch points of the degree 2 map of $\mathbb{P}^1$s do not move.

PROPOSITION 5.5. *Assume char*$(k) \neq 2$. *Let*

$$y^2 + xy = x^3 + 4s^4x^2 + s^4x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ *with section on a K3 surface* $\tilde{X}$. *Define the involution* $\sigma = t_{N^-} \circ J(\sigma)$, *where* $J(\sigma) : s \mapsto -s$ *and* t_{N^-} *is translation by the section* $N^- = (0,0)$. *Then,* σ *is fixed point free.*

COROLLARY 5.6. *Enriques surfaces of type III exist if and only if char*$(k) \neq 2$. *Moreover, they are unique if they exist.*

REMARK 5.7. The equation we took from [JLR12] for $J(\pi)$ makes sense in characteristic 2, where it defines a rational elliptic surface with singular fibers I_4 at $t = 0$ and I_1^* at $t = \infty$. The degree 2 cover $t \mapsto s^2$ given in Proposition 5.5 is the Frobenius morphism and the base change along this morphism defines a rational elliptic surface with singular fibers $(\mathrm{I}_8, \mathrm{III})$. This surface is the minimal resolution of singularities of a surface covering a 1-dimensional family of classical Enriques surfaces with finite automorphism group of "type VIII", as is shown in Chapter II Section 7.

6. Enriques surfaces of type IV

6.1. Main theorem for type IV.

THEOREM 6.1. *Let* X *be an Enriques surface. The following are equivalent:*

(1) X *is of type IV.*
(2) The dual graph of all (-2)*-curves on* X *contains the graph in Figure 4.*

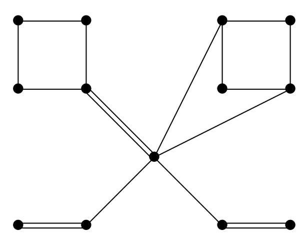


FIGURE 4. Critical subgraph for type IV

(3) The canonical cover $\tilde{X}$ *of* X *admits an elliptic fibration with a Weierstrass equation of the form*

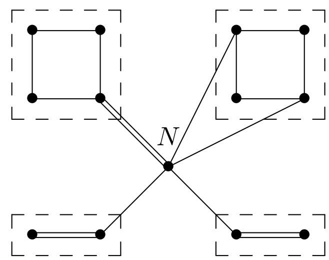
$$y^2 = x^3 + 2(s^4 + 1)x^2 + (s^4 - 1)^2 x$$

such that the covering morphism $\rho : \tilde{X} \to X$ *is given as quotient by the involution* $\sigma = t_{N^-} \circ J(\sigma)$, *where* $J(\sigma) : s \mapsto -s$ *and* t_{N^-} *is translation by* $N^- = (-(s^2 - 1)^2, 0)$.

Moreover, Enriques surfaces of type IV do not exist in characteristic 2.

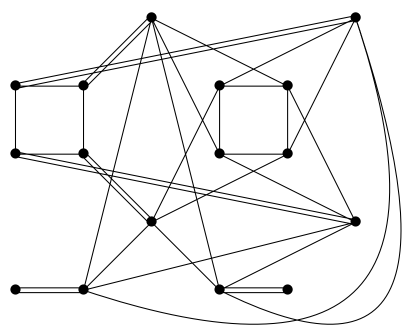
PROOF. First, we observe that the dual graph of type IV (see Table 1) contains the graph in Figure 4.

This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$ and special bisection N as follows:

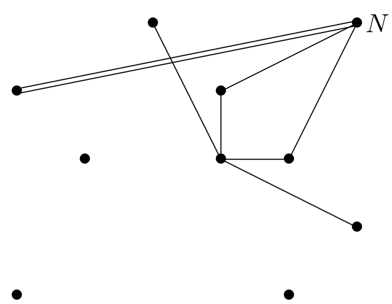


The bisection N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we compute $h(N^-) = 0$ and we see that N^- is a 2-torsion section of $\tilde{\pi}$ meeting the I_4 fibers coming from the I_2 fibers of π in a non-identity component. The same argument as for type III shows that this type cannot exist in characteristic 2.

Now, Corollary 2.25 gives three more (-2)-curves resulting in the following graph:



Again, to produce additional (-2)-curves, we find a different special fibration with special bisection N on this surface as follows:



This special fibration has one I_0^* fiber and four disjoint (-2)-curves contained in some other fibers. Such a fibration will be extremal in any case by Lemma 2.15, so by Table 3 the fibers are $(\mathrm{I}_0^*, \mathrm{I}_0^*)$. Hence, we obtain one more (-2)-curve. We leave it to the reader to find three more such diagrams and to check that the resulting graph is the one of type IV.

We use the same equation as for surfaces of type III

$$y^2 + xy = x^3 + 4t^2x^2 + t^2x,$$

where t is a coordinate on $\mathbb{P}^1$. Recall that the I_4 fibers are at $t = 0, \infty$, while the I_2 fibers are at $t = \pm\frac{1}{4}$. The non-trivial 2-torsion sections are $s_1 = (-4t^2, 2t^2)$, $s_2 = (0, 0)$ and $s_3 = (-\frac{1}{4}, \frac{1}{8})$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with $t = \pm\frac{1}{4}$ as branch points in the following form

$$t \mapsto \frac{1}{4}(\frac{s^2 - 1}{s^2 + 1}),$$

where s is the new parameter on $\mathbb{P}^1$. The covering involution $J(\sigma)$ is given by $s \mapsto -s$. After scaling x and y and simplifying we get the equation

$$y^2 = x^3 + 2(s^4 + 1)x^2 + (s^4 - 1)^2x \tag{6.1}$$

together with the 2-torsion sections $s_1' = (-(s^2 - 1)^2, 0)$, $s_2' = (0, 0)$ and $s_3' = (-(s^2 + 1)^2, 0)$ obtained by pulling back s_1,s_2 and s_3. All of them are $J(\sigma)$-anti-invariant. However, s_2' meets the identity component of the fiber at $s = 0$. Moreover, the surface defined by equation (6.1) has an automorphism ι interchanging s_1' and s_3' given by $\iota : s \mapsto \sqrt{-1}s$.

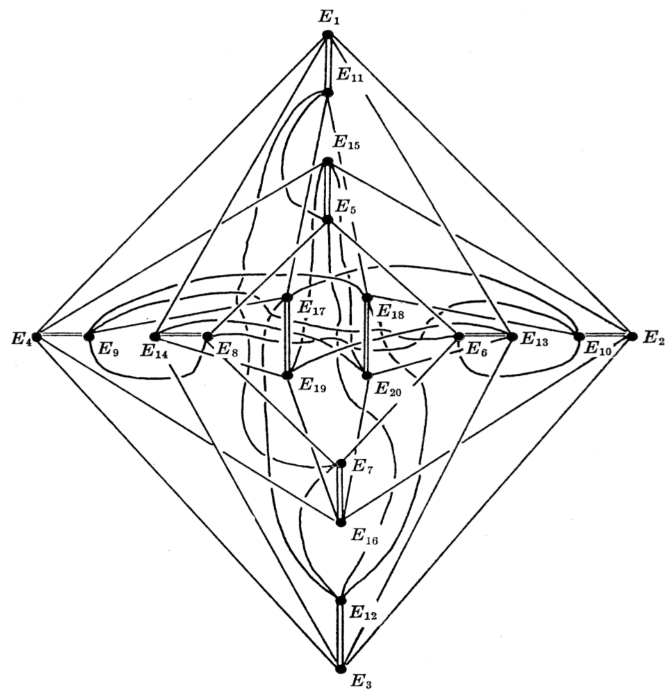
Therefore, we can choose s_1' as N^-. □

REMARK 6.2. It is important to observe that the fibration $\tilde{\pi}$ defined by Equation (6.1) has more torsion sections than the ones coming from the rational surface. For example, one can check that $P = (-(s - \sqrt{-1})^2(s^2 - 1), -2s(s - \sqrt{-1})^2(s^2 - 1))$ is a section satisfying $P \oplus P = N^-$. Since $t_P \circ \iota$ commutes with σ, it will induce an automorphism of the Enriques surface, which we will also denote by $t_P \circ \iota$. Moreover, $(t_P \circ \iota)^2 = t_Q \circ J(\sigma)$ for a 4-torsion section Q of $\tilde{\pi}$. Again, note that $t_P \circ \iota$ acts as $\sqrt{-1}$ on a non-zero global 2-form of the K3 surface.

6.2. Automorphisms.

PROPOSITION 6.3. *Let X be an Enriques surface of type IV. Then, $Aut(X) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$ and this group is generated by automorphisms induced by sections of the Jacobian fibrations of elliptic fibrations of X and an automorphism exhibited in Remark 6.2. More precisely, we can choose the sections in such a way that at most one of them is not 2-torsion and that none of them is a section of an isotrivial fibration. Moreover, $Aut_{nt} \cong \{1\}$.*

PROOF. Recall that the dual graph of (-2)-curves for type IV is as follows:



We claim that $\mathrm{Aut}_{nt}(X)$ is trivial. Indeed, a numerically trivial automorphism g acts trivially on the base of the fibration $|2(E_1 + E_{11})|$, since this fibration has four reducible fibers and g fixes the four bisections E_2, E_4, E_{13} and E_{14} pointwise, hence it is trivial.

Following [Kon86, p.217] we look at the action of $\mathrm{Aut}(X)$ on the set of five fibrations $\{\Delta_i | i = 1, \ldots, 5\}$ with $\Delta_1 = |2(E_1+E_{11})|$, $\Delta_2 = |2(E_2+E_{10})|$, $\Delta_3 = |2(E_5+E_{15})|$, $\Delta_4 = |2(E_6+E_{13})|$ and $\Delta_5 = |2(E_{17} + E_{19})|$. The kernel of the induced homomorphism $\psi : \mathrm{Aut}(X) \to \mathfrak{S}_5$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ and it is generated by translations by 2-torsion sections of the Jacobians of the Δ_i [Kon86, p.218]. From the dual graph, we see that an automorphism of X cannot act as a permutation of order 3 or as a transposition on $\{\Delta_1, \ldots, \Delta_5\}$. Now, we show that the image of ψ is the group G generated by

$$\begin{aligned} \varphi_1 : \quad & \Delta_1 \mapsto \Delta_3 \mapsto \Delta_4 \mapsto \Delta_2 \mapsto \Delta_5 \\ \varphi_2 : \quad & \Delta_1 \mapsto \Delta_3 \mapsto \Delta_2 \mapsto \Delta_4. \end{aligned}$$

Using Corollary 2.25, these permutations are realized as follows:

- The Jacobian of $|E_5 + E_6 + E_{10} + E_{18} + E_{11}|$ has a 5-torsion section which realizes φ_1.
- If we fix E_{11} as a special bisection of Δ_5, we obtain a section P by Remark 6.2 such that φ_2 is realized by the automorphism $t_P \circ \iota$. To see this, note that a 4-torsion section of the Jacobian of Δ_5 acts as $\Delta_1 \mapsto \Delta_2; \Delta_3 \mapsto \Delta_4$.

We have $G \cong \mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ and, since $[\mathfrak{S}_5 : G] \geq 6$, this yields the claim on the image of ψ. Now, note that we can compose $t_P \circ \iota$ with an involution interchanging the two I_2 fibers of the Δ_5

fibration to obtain an automorphism of order 4 realizing φ_2. Hence, we obtain a splitting of

$$0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^4 \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z} \longrightarrow 0.$$

This finishes the proof. □

6.3. Degenerations and Moduli. Similarly to the previous case, we obtain information about degenerations and moduli by direct calculation.

PROPOSITION 6.4. *Assume char$(k) \neq 2$. Let*

$$y^2 = x^3 + 2(s^4+1)x^2 + (s^4-1)^2 x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s$ and t_{N^-} is translation by the section $N^- = (-(s^2-1)^2, 0)$. Then, σ is fixed point free.

COROLLARY 6.5. *Enriques surfaces of type IV exist if and only if char$(k) \neq 2$. Moreover, they are unique if they exist.*

7. Enriques surfaces of type V

7.1. Main theorem for type V.

THEOREM 7.1. *Let X be an Enriques surface. The following are equivalent:*

(1) X is of type V.
(2) The dual graph of all (-2)-curves on X contains the graph in Figure 5.

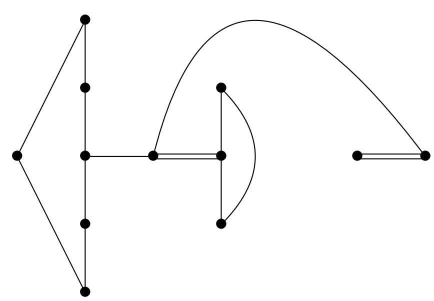


FIGURE 5. Critical subgraph for type V

(3) The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form

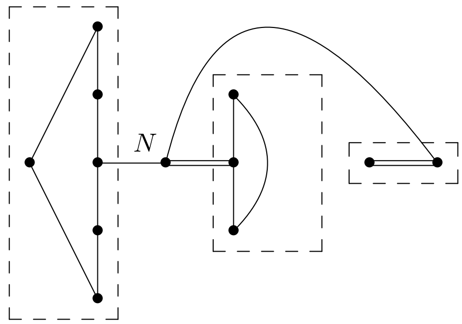
$$y^2 + (s^2+1)xy + (s^2+1)y = x^3 + (s^2+2)x^2 + (s^2+1)x$$

such that the covering morphism $\rho : \tilde{X} \to X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s$ and t_{N^-} is translation by $N^- = (-1, 0)$.

Moreover, Enriques surfaces of type V do not exist in characteristic 2 and 3.

PROOF. First, we observe that the dual graph of type V (see Table 1) contains the graph in Figure 5.

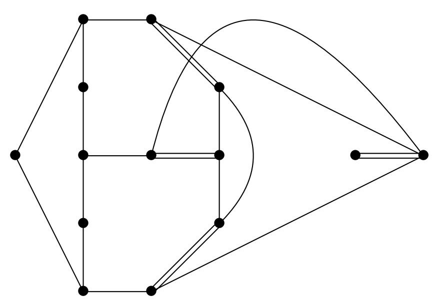
This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers $\mathrm{I}_6, \mathrm{I}_2$ (not III, since it is double) and I_3 (or IV) and special bisection N as follows, where the dotted rectangles mark the fibers:



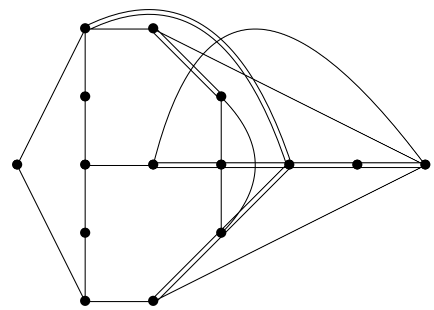
As before, the bisection N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we can compute $h(N^-) = 0$ and we see that N^- is a 2-torsion section of $\tilde{\pi}$ meeting the I_6 and I_2 fibers in a non-identity component.

Note that the existence of this fibration already gives non-existence of this type of Enriques surfaces in characteristic 2 and 3, since an extremal fibration with singular fibers I_6 and I_2 does not exist on rational surfaces in characteristic 3 (see Table 3) and because a fibration with two double fibers cannot exist in characteristic 2. Therefore, we will assume $\mathrm{char}(k) \neq 2, 3$ from now on.

Now, Corollary 2.25 gives two more (-2)-curves resulting in the following graph:



For this example, one can use a fibration with an I_2^* fiber to produce another (-2)-curve:



As usual, the remaining curves can be found similarly.

By [JLR12], we have, after simplifying, the following equation for the unique extremal and rational elliptic surface with singular fibers $(\mathrm{I}_6, \mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_1)$

$$y^2 + txy + ty = x^3 + (1+t)x^2 + tx,$$

where t is a coordinate on $\mathbb{P}^1$. The I_6 fiber is at $t = \infty$, the I_3 fiber is at $t = 0$, the I_2 fiber is at $t = 1$ and the I_1 fiber is at $t = -8$. The non-trivial 2-torsion section is $s = (-1, 0)$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with $t = 1, \infty$ as branch points in the following form

$$t \mapsto s^2 + 1,$$

where s is the new parameter on $\mathbb{P}^1$. The covering involution $J(\sigma)$ is given by $s \mapsto -s$. Now, we have the equation

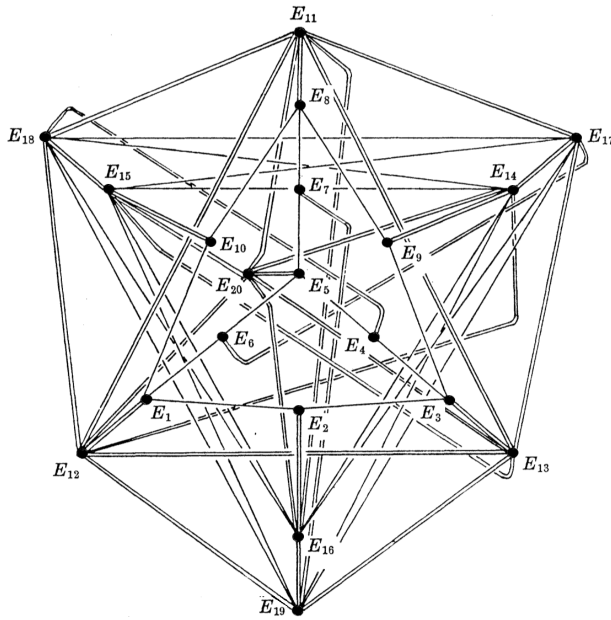
$$y^2 + (s^2+1)xy + (s^2+1)y = x^3 + (s^2+2)x^2 + (s^2+1)x \tag{7.1}$$

together with the 2-torsion sections $s' = (-1, 0)$ obtained by pulling back s. Since s' is $J(\sigma)$-(anti-)invariant and meets the fibers in the correct components, it is the section we are looking for. □

7.2. Automorphisms.

PROPOSITION 7.2. *Let X be an Enriques surface of type V. Then, Aut$(X) \cong \mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X. Moreover, Aut$_{nt}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and Aut$(X)/$Aut$_{nt}(X) \cong \mathfrak{S}_4$.*

PROOF. Recall that the dual graph of (-2)-curves for type V is as follows:



We claim that $|\mathrm{Aut}_{nt}(X)| = 2$. Indeed, a numerically trivial automorphism g acts trivially on the base of the fibration $|2(E_1 + E_2 + E_3 + E_4 + E_5 + E_6)|$, since this fibration has at least three singular fibers and g acts trivially or induces a unique involution on the three bisections E_{10},E_7 and E_9. By the same argument as for type III, there is at most one such g. Now, note that the 2-torsion section of the Jacobian of this fibration acts identically on the graph of (-2)-curves.

The automorphism group of the graph is $\mathfrak{S}_4$ [Kon86, p.223]. It suffices to look at the action of $\mathrm{Aut}(X)$ on the set $\{E_1, E_3, E_5, E_8\}$.

- Transpositions of E_5 with another curve of the set are induced by 2-torsion sections of fibrations with a singular fiber of type I_2^*. For example, there is a 2-torsion section of the Jacobian of $|E_2 + E_6 + E_7 + E_9 + 2(E_1 + E_8 + E_{10})|$ which interchanges E_3 and E_5 by Corollary 2.25.
- All transpositions of two curves different from E_5 are induced by 2-torsion sections of fibrations with a singular fiber of type III^*, e.g. the 2-torsion section of the Jacobian of $|E_{10} + E_9 + 2E_1 + 2E_3 + 2E_7 + 3E_6 + 3E_4 + 4E_5|$ interchanges E_{10} and E_9.

Finally, we claim that these transpositions generate a subgroup of $\mathrm{Aut}(X)$, which is isomorphic to $\mathfrak{S}_4$. Indeed, this can be checked by using Equation (7.1) to compute the stabilizer G of E_1 (which is D_6) and by using the fact that the maximal order of a tame semi-symplectic automorphism is 6 (see [MO14]). This finishes the proof. □

7.3. Degenerations and Moduli. As in the previous cases, we prove the existence of this type by explicit calculation.

PROPOSITION 7.3. *Assume char$(k) \neq 2, 3$. Let*

$$y^2 + (s^2 + 1)xy + (s^2 + 1)y = x^3 + (s^2 + 2)x^2 + (s^2 + 1)x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s$ and t_{N^-} is translation by the section $N^- = (-1, 0)$. Then, σ is fixed point free.

COROLLARY 7.4. *Enriques surfaces of type V exist if and only if char$(k) \neq 2, 3$. Moreover, they are unique if they exist.*

REMARK 7.5. Again, the equation makes sense in characteristic 2, where it defines a $K3$ surface covering a 1-dimensional family of classical and supersingular Enriques surfaces of type VII (see [KK15b] and Chapter II Section 6).

8. Enriques surfaces of type VI

8.1. Main theorem for type VI.

REMARK 8.1. In the first five cases, every base change with the correct ramification points produced an elliptic fibration of a K3 surface with $J(\pi)$-Enriques section N^-. This happened because the section N^- was a 2-torsion section. In the last two cases, however, we do not get this section for free.

LEMMA 8.2. *Let char$(k) \neq 3$, $J(\sigma) : s \mapsto -s - \beta$, and*

$$y^2 - 3(3(s^2 + \beta s) + 1)xy + (3(s^2 + \beta s) + 1)^2 y = x^3$$

with $\beta \in k - \{\pm\frac{2}{\sqrt{3}}\}$ be the Weierstrass equation of an elliptic fibration of a K3 surface. Then, an everywhere integral, $J(\sigma)$-anti-invariant section N^- meeting the fiber at $s = \infty$ in a non-identity component exists if and only if $\beta = \pm 1$. Moreover, it is unique up to sign if it exists. Both cases are isomorphic and if $\beta = 1$, the section is given by $N^- = (s + s^2, s^3)$.

PROOF. By [Shi10, Lemma 1.2], an everywhere integral section N^- is given by $(x(s), y(s))$, where $x(s)$ and $y(s)$ are polynomials in s with $\deg_s(x) \leq 4$ and $\deg_s(y) \leq 6$. Now, a lengthy, but straightforward calculation comparing coefficients gives the result. Finally, note that the automorphism $s \mapsto -s$ exchanges both cases. □

THEOREM 8.3. *Let X be an Enriques surface. The following are equivalent:*

1. *X is of type VI.*
2. *The dual graph of all (-2)-curves on X contains the graph in Figure 6.*
3. *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 - 3(3s^2 + 3s + 1)xy + (3s^2 + 3s + 1)^2 y = x^3$$

such that the covering morphism $\rho : \tilde{X} \to X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s - 1$ and t_{N^-} is translation by $N^- = (s + s^2, s^3)$.

Moreover, Enriques surfaces of type VI do not exist in characteristic 3.

PROOF. First, observe that the dual graph of type VI (see Table 1) contains the graph in the below Figure 6.

This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers IV^*, I_3 (not III, since it is double) and special 2-section N. With the same notation as in

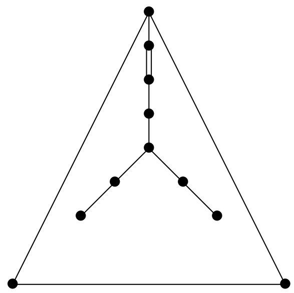
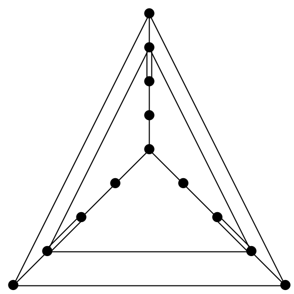
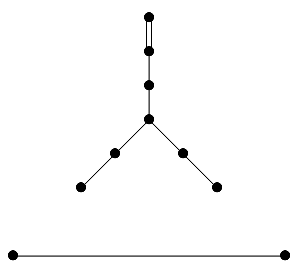


FIGURE 6. Critical subgraph for type VI

the previous cases, we can compute $h(N^-) \neq 0$ and from Corollary 2.25 we obtain two more (-2)-curves as follows:



There are three subgraphs of type $\tilde{A}_1$ such that the graph of (-2)-curves disjoint from this diagram together with a special bisection has the following form:



The only rational elliptic fibration with a singular fiber of type I_2 and some other singular fibers whose dual graphs contain an A_5 and an A_2 diagram is the one with singular fibers $(\mathrm{I}_6, \mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_1)$ (resp. $(\mathrm{I}_6, \mathrm{IV}, \mathrm{I}_2)$ in characteristic 2). Using the other (-2)-curves in the graph, one deduces that the I_6 and I_3 (resp. IV) fibers are simple. These fibrations give the seven remaining (-2)-curves for the dual graph of type VI. Observe that the existence of such a fibration excludes this case in characteristic 3, since the I_2 fiber is double.

We have found the following equation for the unique rational and extremal elliptic surface with singular fibers IV^* and I_3 in any characteristic

$$y^2 + txy + t^2y = x^3,$$

where t is a coordinate on $\mathbb{P}^1$. By a change of coordinates (valid away from characteristic 3) we obtain

$$y^2 - 3(3t+1)xy + (3t+1)^2 y = x^3.$$

The IV* fiber is at $t = -\frac{1}{3}$, the I_3 fiber is at $t = \infty$ and there is an I_1 fiber at $t = -\frac{2}{3}$.

In characteristic $\neq 3$, we can write a degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with $t = \infty$ as branch point and which is not branched over $-\frac{1}{3}$ as

$$t \mapsto s^2 + \beta s,$$

where s is the new parameter on $\mathbb{P}^1$ and $\beta \neq \pm\frac{2}{\sqrt{3}}$. The covering involution $J(\sigma)$ is given by $s \mapsto -s - \beta$. We obtain the equation

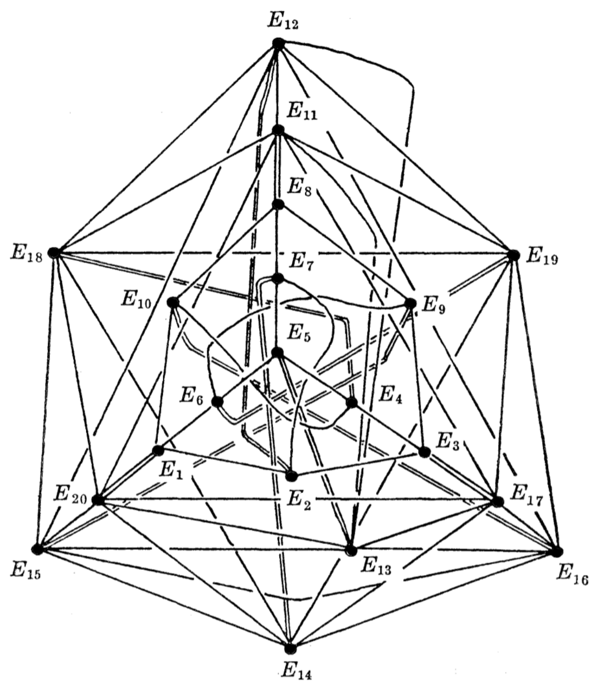
$$y^2 - 3(3(s^2+\beta s)+1)xy + (3(s^2+\beta s)+1)^2 y = x^3.$$

By Lemma 8.2, if a suitable section N^- exists, we can assume $\beta = 1$ and $N^- = (s + s^2, s^3)$. □

8.2. Automorphisms.

PROPOSITION 8.4. *Let X be an Enriques surface of type VI. Then, $Aut(X) \cong \mathfrak{S}_5$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X. Moreover, $Aut_{nt}(X) \cong \{1\}$.*

PROOF. Recall that the dual graph of (-2)-curves for type VI is as follows:



Let us first show that $\mathrm{Aut}_{nt}(X)$ is trivial. Indeed, the three separable bisections E_7, E_9 and E_{10} of $|E_1 + E_2 + E_3 + E_4 + E_5 + E_6|$ are fixed pointwise by any numerically trivial automorphism, which therefore has to be the identity.

The automorphism group of the graph is $\mathfrak{S}_5$ [Kon86, p.223]. We look at the induced action of $\mathrm{Aut}(X)$ on the set $\Sigma = \{E_1, \ldots, E_{10}\}$ and note the following points:

- The pointwise stabilizer of the set $\Gamma_1 = \{E_4, E_5, E_6, E_7\}$ is $\mathbb{Z}/2\mathbb{Z}$. It is realized by the 2-torsion section of the Jacobian of $|2(E_5 + E_{13})|$.
- The stabilizer of E_5 under the action of the automorphism group of the graph is $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$. It is realized by the stabilizer of Γ_1 and the 2-torsion sections of the Jacobian fibrations of fibrations with a fiber of type I_1^*. For example the Jacobian of $|E_6 + E_7 + E_3 + E_{10} + 2(E_4 + E_5)|$ has a 2-torsion section which interchanges E_6 and E_7.
- Since the stabilizer of E_5 has order 12, it suffices to show that the group generated by 2-torsion sections acts transitively on Σ. We show that we can map E_5 to E_{10}, E_3 and E_6. The rest can be done similarly.
- Indeed, the 2-torsion sections of the Jacobians of $|2(E_3 + E_{17})|$, $|2(E_{10} + E_{16})|$ and $|2(E_8 + E_{11})|$ interchange E_5 and E_{10}, E_5 and E_3 and E_3 and E_6, respectively.

□

8.3. Degenerations and Moduli.

PROPOSITION 8.5. *Assume char$(k) \neq 3$. Let*

$$y^2 - 3(3(s^2+s)+1)xy + (3(s^2+s)+1)^2 y = x^3$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s-1$ and t_{N^-} is translation by the section $N^- = (s+s^2, s^3)$. Then, σ is fixed point free if and only if char$(k) \neq 5$. If char$(k) = 5$, σ has exactly one (-2)-curve as fixed locus.

PROOF. The only possibility for σ to have fixed points is the case where $\varphi : t \mapsto s^2 + s$ is branched over the point lying under the nodal fiber. Hence, we may assume that $\mathrm{char}(k) \neq 2$. The branch points of φ are $t = \infty$ and $t = -\frac{1}{4}$, while the I_1 fiber of π lies over $t = -\frac{2}{3}$. Hence, φ is branched over the point lying under the nodal fiber if and only if $-\frac{2}{3} = -\frac{1}{4}$, i.e. if and only if $5 = 0$.

Now if $\mathrm{char}(k) = 5$, the location of the I_2 fiber of $\tilde{\pi}$ is $s = -\frac{1}{2} = 2$. The singular point of the Weierstrass equation at $s = 2$ is $(-1, 1)$, while N^- passes through $(1, 3)$. Hence, N^- meets the identity component of the I_2 fiber and therefore it is not a $J(\pi)$-Enriques section and σ fixes a (-2)-curve. □

COROLLARY 8.6. *Enriques surfaces of type VI exist if and only if char$(k) \neq 3, 5$. Moreover, they are unique if they exist.*

Similarly to the cases of type I and II, one obtains a Coble surface if σ has a fixed curve, i.e. if $\mathrm{char}(k) = 5$.

9. Enriques surfaces of type VII

9.1. Main theorem for type VII.

LEMMA 9.1. *Let $char(k) \neq 2$, $J(\sigma) : s \mapsto -s$, and*

$$y^2 = x^3 - (s_\beta^2 + s_\beta)x^2 + (2s_\beta^3 - 3s_\beta^2 + 4s_\beta - 2)x + (-s_\beta^3 + 2s_\beta^2 - 2s_\beta + 1),$$

where $s_\beta = s^2 + \beta$ with $\beta \in k - \{1\}$, be the Weierstrass equation of an elliptic fibration of a K3 surface. Then, an everywhere integral, $J(\sigma)$-anti-invariant section N^- meeting the fibers at $s = \infty$ and $s = \pm\sqrt{1-\beta}$ in a non-identity component exists if and only if $\beta \in \{0, 2\}$. Moreover, it is unique up to sign if it exists. Both cases are isomorphic and if $\beta = 0$, the section is $N^- = (1, s - s^3)$.

PROOF. Similarly to the previous case, one obtains conditions on β by direct calculation. Let us show the existence of the automorphism. The Weierstrass equation for the rational elliptic fibration

$$y^2 = x^3 - (t^2 + t)x^2 + (2t^3 - 3t^2 + 4t - 2)x + (-t^3 + 2t^2 - 2t + 1)$$

has an automorphism

$$t \mapsto 2 - t; \quad x \mapsto x - 2 + 2t.$$

This automorphism induces the desired isomorphism. □

THEOREM 9.2. *Let X be an Enriques surface. The following are equivalent:*

(1) X is of type VII.
(2) The dual graph of all (-2)-curves on X contains the graph in Figure 7.

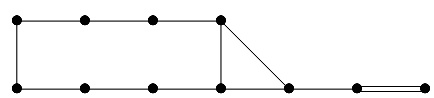


FIGURE 7. Critical subgraph for type VII

(3) The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form

$$y^2 = x^3 - (s^4 + s^2)x^2 + (2s^6 - 3s^4 + 4s^2 - 2)x + (-s^6 + 2s^4 - 2s^2 + 1)$$

such that the covering morphism $\rho : \tilde{X} \to X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s$ and t_{N^-} is translation by $N^- = (1, s - s^3)$.

Moreover, singular Enriques surfaces of type VII do not exist in characteristic 2.

PROOF. First, observe that the dual graph of type VII (see Table 1) contains the graph in the above Figure 7.

Conversely, we have shown in Example 2.3 that we recover type VII from the critical subgraph and, since an elliptic fibration with singular fibers I_8 and I_2 (not III, since it is a double fiber) does not exist in characteristic 2, this type cannot exist in characteristic 2.

We have found the following equation for the unique rational and extremal elliptic surface with singular fibers $(\mathrm{I}_8, \mathrm{I}_2, \mathrm{I}_1, \mathrm{I}_1)$ in characteristic different from 2

$$y^2 = x^3 - (t^2 + t)x^2 + (2t^3 - 3t^2 + 4t - 2)x + (-t^3 + 2t^2 - 2t + 1),$$

where t is a coordinate on $\mathbb{P}^1$. The I_8 fiber is at $t = 1$, the I_2 fiber is at $t = \infty$ and there are two I_1 fibers at $t = 1 \pm \frac{\sqrt{-1}}{2}$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with $t = \infty$ as branch point and which is not branched over $t = 0$ as

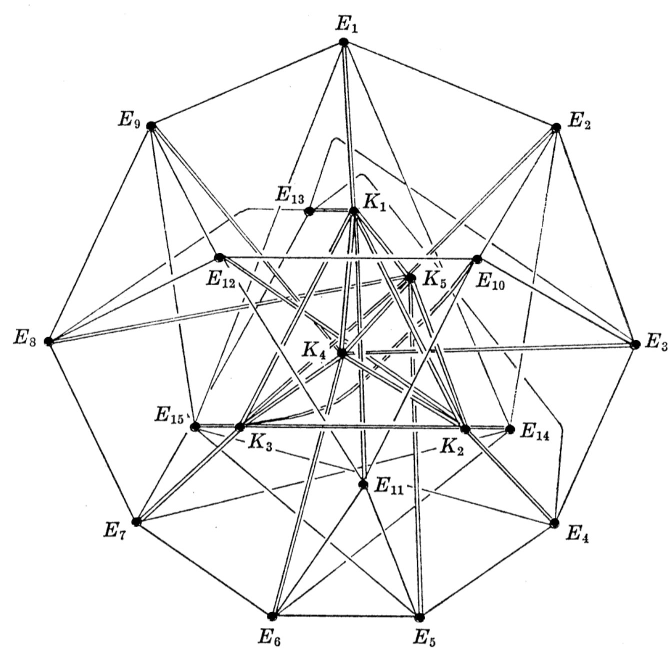
$$t \mapsto s^2 + \beta,$$

where s is the new parameter on $\mathbb{P}^1$ and $\beta \neq 0$. The covering involution $J(\sigma)$ is given by $s \mapsto -s$. Now, note that we are looking for a section N^- which meets the I_4 and I_8 fibers in non-identity components. By Lemma 9.1, if a suitable section N^- exists, we can assume $\beta = 0$ and $N^- = (1, s - s^3)$. Moreover, one can check that N^- has the correct intersection behaviour with the I_8 fibers. $\square$

9.2. Automorphisms.

PROPOSITION 9.3. *Let X be an Enriques surface of type VII. Then, $Aut(X) \cong \mathfrak{S}_5$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X. Moreover, $Aut_{nt}(X) \cong \{1\}$.*

PROOF. Recall that the dual graph of (-2)-curves for type VII is as follows:



We claim that $\mathrm{Aut}_{nt}(X)$ is trivial. Indeed, one can check that the bisections E_2, E_3, E_5, E_6, E_8 and E_9 of $|K_4 + K_5|$ are fixed pointwise by any numerically trivial automorphism, which therefore has to be trivial.

The automorphism group of the graph is $\mathfrak{S}_5$. Following [Kon86, p.232], we look at the induced action on the set $\Sigma = \{K_1, \ldots, K_5\}$ and observe that the pointwise stabilizer of Σ is trivial. Now, each K_i, $i \in \{1, \ldots, 5\}$, meets exactly three E_j, $j \in \{1, \ldots, 15\}$, twice. The 2-torsion sections of the Jacobians of the elliptic fibrations $|2(K_i + E_j)|$ act as permutations of cycle type $(2, 2)$ on $\Sigma - K_i$. Note that the 2-torsion section of the Jacobian of $|K_4 + K_5|$ interchanges K_4 and K_5 while fixing K_1, K_2 and K_3. Together, these involutions generate the full automorphism group. □

9.3. Degenerations and Moduli.

PROPOSITION 9.4. *Assume char$(k) \neq 2$. Let*

$$y^2 = x^3 - (s^4 + s^2)x^2 + (2s^6 - 3s^4 + 4s^2 - 2)x + (-s^6 + 2s^4 - 2s^2 + 1)$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma) : s \mapsto -s$ and t_{N^-} is translation by the section $N^- = (1, s - s^3)$. Then, σ is fixed point free if and only if char$(k) \neq 5$. If char$(k) = 5$, σ has exactly one (-2)-curve as fixed locus.

PROOF. The branch points of φ are $t = \infty$ and $t = 0$, while the I_1 fibers of π lie over $t = 1 \pm \frac{\sqrt{-1}}{2}$. Hence, φ is branched over a point lying under a nodal fiber if and only if $1 \pm \frac{\sqrt{-1}}{2} = 0$, i.e. if and only if $5 = 0$.

Now, if $\mathrm{char}(k) = 5$, the location of the I_2 fiber of $\tilde{\pi}$ is $s = 0$. The singular point of the Weierstrass equation at $s = 0$ is $(2, 0)$, while N^- passes through $(1, 0)$. Hence, N^- meets the identity component of the I_2 fiber and therefore it is not a $J(\pi)$-Enriques section and σ fixes a (-2)-curve. □

COROLLARY 9.5. *Enriques surfaces of type VII with smooth K3 cover exist if and only if char$(k) \neq 2, 5$. Moreover, they are unique if they exist.*

REMARK 9.6. Here, it is important to recall our convention on Enriques surfaces in characteristic 2. In fact, by [KK15b], there is a 1-dimensional family of classical and supersingular Enriques surfaces of type VII in characteristic 2 (see Chapter II Section 6). Note also that the involution σ produces a Coble surface in characteristic 5.

10. The classification-theorem

Now that we have completed the construction of the seven types of Enriques surfaces with finite automorphism group, it remains to show that these seven types are indeed all possible Enriques surfaces with finite automorphism group. Hence, the goal of this chapter is to prove the following classification-theorem, finishing the proof of the Main Theorem. Recall that all our Enriques surfaces are assumed to have a smooth canonical cover.

THEOREM 10.1. *Let X be an Enriques surface. The following are equivalent:*

(1) X has finite automorphism group.

(2) Every elliptic fibration of X is extremal.
(3) Every special elliptic fibration of X is extremal and X contains a (-2)-curve.
(4) The dual graph of all (-2)-curves on X contains a critical subgraph for one of the types I, ..., VII.
(5) The dual graph of all (-2)-curves on X is one of the seven types I, ..., VII.
(6) X contains only finitely many, but at least one, (-2)-curves.

Before giving the proof of Theorem 10.1, we need to introduce the tools for the classification of dual graphs.

10.1. Preparations for the proof of the classification-theorem. Corollary 2.25 and the height pairing of sections of elliptic fibrations of the K3 cover will play an important role. More precisely, we have the following lemma.

LEMMA 10.2. *Let $\pi : X \to \mathbb{P}^1$ be a special and extremal elliptic fibration of an Enriques surface X with special bisection N. Let $\tilde{\pi}$ be the corresponding elliptic fibration of the K3 cover $\tilde{X}$ of X. Denote the irreducible curves on $\tilde{X}$ mapping surjectively onto N by N^+ and N^-. Let $J(\pi)$ be the Jacobian of π. We choose N^+ as the zero section of $\tilde{\pi}$.*
Then,

- *either $h(N^-) = 0$ and N^- is a 2-torsion section in $MW(J(\pi)) \subseteq MW(\tilde{\pi})$*
- *or N^- satisfies*

$$\sum_{\nu} contr_{\nu}(N^-) < 4 \quad \text{and} \quad \sum_{\nu} contr_{\nu}(N^-, P) \in \{0, 1, 2\}$$

for all $P \in MW(J(\pi)) \subseteq MW(\tilde{\pi})$ with $P \neq N^-$.

PROOF. Since

$$0 \leq h(N^-) = 4 + 2N^-.N^+ - \sum_{\nu} contr_{\nu}(N^-) = 4 - \sum_{\nu} contr_{\nu}(N^-)$$

and N^- restricts to a 2-torsion section on a fiber F of $\tilde{\pi}$ lying over a double fiber of π, we either have $h(N^-) = 0$ and we claim that N^- is 2-torsion or $h(N^-) > 0$ and therefore we have $\sum_{\nu} contr_{\nu}(N^-) < 4$.

Indeed, suppose $h(N^-) = 0$ and N^- is not 2-torsion. Then, $N^- \oplus N^-$ meets the zero section in F, hence its order is divisible by $\text{char}(k) = 2$ by [IL13, Proposition 2.4]. But if $\text{char}(k) = 2$, the fiber F is either multiplicative or ordinary by Proposition 2.5, contradicting [IL13, Proposition 2.1].

Since every $P \in \text{MW}(J(\pi)) \subseteq \text{MW}(\tilde{\pi})$ is disjoint from N^+, we have

$$\begin{aligned} 0 = \langle P, N^- \rangle &= 2 + P.N^+ + N^-.N^+ - P.N^- - \sum_{\nu} contr_{\nu}(N^-, P) = \\ &= 2 - P.N^- - \sum_{\nu} contr_{\nu}(N^-, P), \end{aligned}$$

which yields the second claim. □

REMARK 10.3. By Table 4, the local contributions to the height pairing can be read off almost completely from the dual graph of singular fibers. However, a remark about the cases where π has a double fiber of type I_1 is in order. Since sections $P \in \mathrm{MW}(J(\pi))$ meet the corresponding I_2 fiber of $\tilde{\pi}$ in the identity component, N^- cannot be 2-torsion. Moreover, $\sum_\nu contr_\nu(N^-)$ will decrease by $\frac{1}{2}$, while $\sum_\nu contr_\nu(N^-, P)$ stays the same, hence N^- can only satisfy the conditions of the lemma if it does so, when we ignore the double I_1 fiber. We will do this from now on.

DEFINITION 10.4. Let Γ_1 be the dual graph of singular fibers of a rational and extremal elliptic fibration. A graph $\Gamma \supseteq \Gamma_1$ is called a fiber-bisection configuration for Γ_1 if the following two conditions hold:

(1) $\Gamma - \Gamma_1$ consists of one vertex N called the special bisection.
(2) N meets every connected component of Γ_1 of type $\tilde{D}$ and $\tilde{E}$ exactly twice and every component of type $\tilde{A}$ at least once and at most twice. Moreover, N meets at most two connected components of Γ_1 exactly once.

Given a fiber-bisection configuration Γ, we can check whether it could be the dual graph of a special elliptic fibration π on an Enriques surface as follows: Suppose it is the dual graph of π. Then, we can pass to the canonical cover, add the sections coming from the Jacobian $J(\pi)$ of π and check the conditions of Lemma 10.2. By Remark 10.3, it makes sense to say that a fiber-bisection configuration satisfies the conditions of Lemma 10.2.

DEFINITION 10.5. A fiber-bisection configuration is called *admissible* if it satisfies the conditions of Lemma 10.2

10.2. Outline of proof. In this section, we outline the proof of the following lemma, which is the main ingredient in the proof of Theorem 10.1.

LEMMA 10.6. *Let X be an Enriques surface such that every special elliptic fibration of X is extremal and X contains a (-2)-curve. Then, the dual graph of (-2)-curves on X contains a critical subgraph (see Figures 1, . . . , 7) for one of the types I, . . . , VII.*

PROOF OF THEOREM 10.1 (ASSUMING LEMMA 10.6). As observed by Dolgachev [Dol84, §4], if X has finite automorphism group, then every elliptic fibration π on X is extremal, since the Mordell-Weil group of $J(\pi)$ acts faithfully on X. In particular, since X admits an elliptic fibration by Proposition 2.4, X contains a (-2)-curve by Lemma 2.15 and every special elliptic fibration of X is extremal. From Lemma 10.6, we deduce that X contains a critical subgraph, which, by the earlier chapters, implies that the dual graph of (-2)-curves on X is one of the types I, . . . , VII.

The seven dual graphs in Table 1 consist of 12 (resp. 20) vertices, hence X contains finitely many and at least one (-2)-curve. Moreover, we have computed the automorphism groups of these surfaces. They are finite. Finally, by Corollary 2.25, the only special elliptic fibrations of Enriques surfaces with finitely many, but at least one, (-2)-curves are the extremal ones. □

Since we have constructed all seven types in the previous chapters, Theorem 10.1 will finish the classification. The strategy for the proof of Lemma 10.6 can be summarized as follows:

(1) Let X be an Enriques surface with a (-2)-curve such that every special elliptic fibration of X is extremal. By Proposition 2.8, X admits such a special elliptic fibration π.

(2) Pick a dual graph Γ_1 of singular fibers of a rational and extremal elliptic fibration and some admissible fiber-bisection configuration $\Gamma \supseteq \Gamma_1$. Suppose that Γ is the dual graph of fibers and special bisection of π.
(3) Apply Corollary 2.25 to find additional (-2)-curves and obtain a bigger graph Γ_2.
(4) If Γ_2 contains one of the critical subgraphs, we have shown in the previous chapters that X is of one of the seven types.
(5) If not, find a different subgraph Γ_3 of Γ_2 of type $\tilde{A}_n$ together with a vertex N meeting Γ_3 exactly once. By Proposition 2.12, Γ_3 is the dual graph of a singular fiber of a special elliptic fibration π_1 and N is a special bisection of π_1. By the assumption on X, π_1 is extremal, i.e. we can extend Γ_3 to a dual graph Γ_4 of singular fibers of an extremal elliptic fibration such that $\Gamma_4 \cup N$ is an admissible fiber-bisection configuration for Γ_4. Now, go back to step (3).

We will show that the above process will terminate at some point for every choice of Γ_1, either with a contradiction or with step (4).

10.3. Proof of the classification-theorem. The following lemma shows that the number of admissible fiber-bisection configurations we have to check is "not too big".

LEMMA 10.7. *Let X be an Enriques surface with a special and extremal elliptic fibration π. Then, X admits a special elliptic fibration with a double fiber of type $\mathrm{I_n}$ with $n \geq 2$. Moreover, if π has double fibers of type $\mathrm{I_{n_1}}$ and $\mathrm{I_{n_2}}$, then $n_1 + n_2 \leq 8$.*

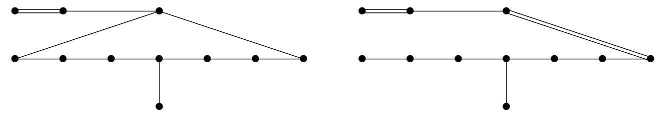
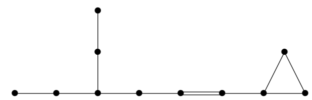
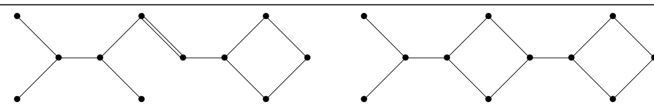
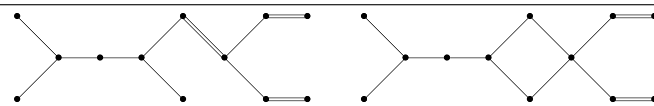
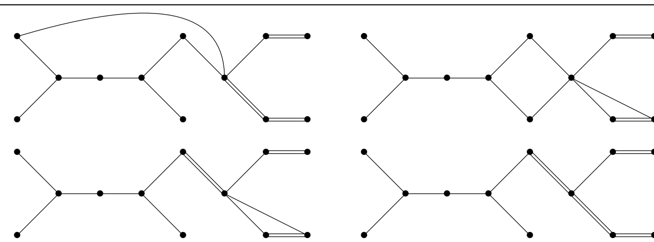
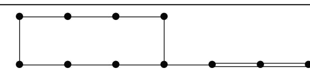
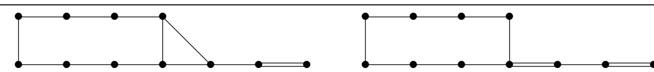
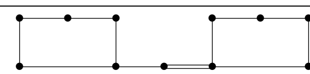
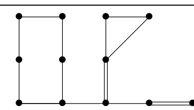
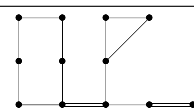
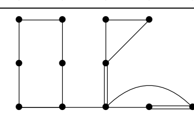
PROOF. For the first claim, let π be a special and extremal elliptic fibration of X and let N be a special bisection of π. If π has a fiber of type $\mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*, \mathrm{I}_n^*$, or I_n with $n \geq 5$, then N and fiber components form a fiber of type I_n and a component of the fiber takes the role of a special bisection. The remaining possibilities for π are the one with fibers $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$ and the one with fibers $(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$. These are checked similarly, using more than one fiber.

For the second claim, let π be a special elliptic fibration of X with double fibers of type I_{n_1} and I_{n_2}. Denote a special bisection by N and the corresponding curves on the K3 cover by N^+ and N^- as usual. Then, we compute $\sum_\nu contr_\nu(N^-) \geq (n_1 + n_2)/2$ using Table 4. Since $\sum_\nu contr_\nu(N^-) \leq 4$, this gives the second claim. □

It is straightforward to give a complete list of admissible fiber-bisection configurations for dual graphs of singular fibers of extremal elliptic fibrations. We leave the details to the reader. Note that it follows from the classification of extremal and rational elliptic surfaces (see Table 3) that we do not have to take special care of small characteristics.

LEMMA 10.8. *Let Adm_p be the set of admissible fiber-bisection configurations for dual graphs of extremal elliptic fibrations over an algebraically closed field of characteristic p. Then $Adm_p \subseteq Adm_0$.*

LEMMA 10.9. *Table 5 shows the list of all admissible fiber-bisection configurations for dual graphs of singular fibers of extremal elliptic fibrations, where the special bisection meets at least one $\tilde{A}$ subgraph (marked with a 2 in front) only once.*

Dual graph of fibers	Admissible fiber-bisection configurations
$\tilde{E}_7 \oplus 2\tilde{A}_1$	
$\tilde{E}_6 \oplus 2\tilde{A}_2$	
$\tilde{D}_5 \oplus 2\tilde{A}_3$	
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	
$2\tilde{A}_7 \oplus \tilde{A}_1$	
$\tilde{A}_7 \oplus 2\tilde{A}_1$	
$2\tilde{A}_4 \oplus \tilde{A}_4$	
$2\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	
$\tilde{A}_5 \oplus 2\tilde{A}_2 \oplus 2\tilde{A}_1$	
$2\tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1$	

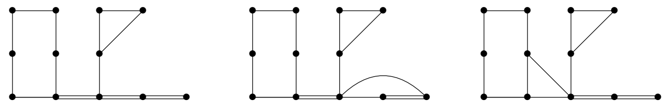
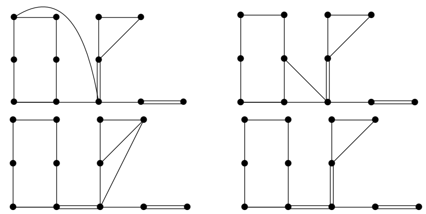
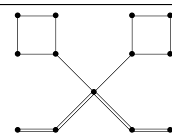
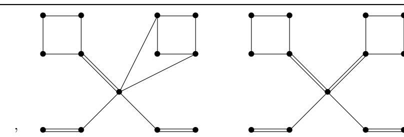
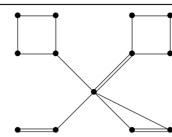
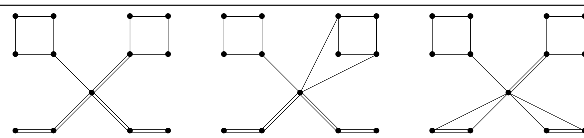
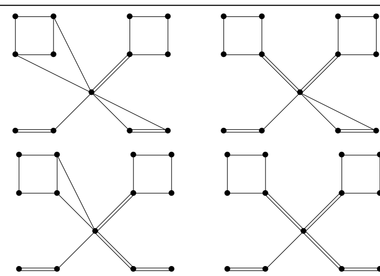
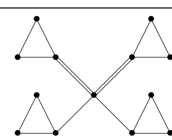
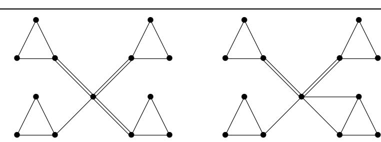
$\tilde{A}_5 \oplus 2\tilde{A}_2 \oplus \tilde{A}_1$	
$\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	
$2\tilde{A}_3 \oplus 2\tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$	
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	
$2\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	
$2\tilde{A}_3 \oplus \tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$	
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	
$2\tilde{A}_2 \oplus 2\tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2$	
$2\tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2$	

TABLE 5. Admissible fiber-bisection configurations for extremal fibrations

REMARK 10.10. In fact, many of these admissible fiber-bisection configurations are realizable over the complex numbers (see [Mar16]).

From these tables, we can deduce the following improvement of Lemma 10.7.

COROLLARY 10.11. *If an Enriques surface X admits a special and extremal elliptic fibration, then X is either of type II or it admits a special elliptic fibration with a double fiber of type I_2.*

PROOF. By Lemma 10.7, we know that X admits an elliptic fibration with a double fiber of type I_n for some n. Almost every graph in Lemma 10.9 admits an $\tilde{A}_1$ subgraph and a vertex meeting this subgraph exactly once; the only exception is the critical subgraph for type II. Hence, the claim follows. □

Before we start with the proof of Lemma 10.6, we need the following auxiliary result.

LEMMA 10.12. *There is no Enriques surface with a special elliptic fibration with singular fibers*

- (I_3, I_3, I_3, I_3) *such that two of the I_3 fibers are multiple or*
- (I_6, I_3, I_2, I_1) *such that the I_3 and I_2 fibers are multiple.*

PROOF. We will only show the first claim; the second one is similar. The claim is true if $\mathrm{char}(k) \in \{2, 3\}$, since there is no rational elliptic surface with singular fibers $(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$ in characteristic 3 and an elliptic fibration of an Enriques surface in characteristic 2 cannot have two multiplicative double fibers.

Let us assume $\mathrm{char}(k) \notin \{2, 3\}$. The rational elliptic surface $J(\pi)$ with singular fibers $(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$ has the Weierstrass equation

$$y^2 = x^3 + (-3t^4 + 24t)x + 2t^6 + 40t^3 - 16. \tag{10.1}$$

If an Enriques surface with this Jacobian and two double I_3 fibers exists, it is covered by the base change of (10.1) via $t \mapsto s^2 - 1$.

A $J(\pi)$-Enriques section $N^- = (x(s), y(s))$ meets the fibers of $J(\pi)$ at $s = 0$ and at $s = \infty$ in a non-identity component and is $J(\sigma)$-anti-invariant, where $J(\sigma) : s \mapsto -s$. Since the singular point of the fiber at $s = 0$ (resp. $s = \infty$) is $(-3, 0)$ (resp. $(1, 0)$), N^- has the form

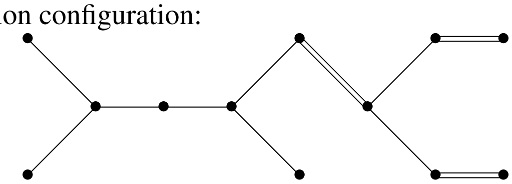
$$\begin{aligned} x &= -3 + x_2 s^2 + s^4 \\ y &= y_1 s + y_3 s^3 + y_5 s^5. \end{aligned}$$

Plugging this into the base change of equation (10.1), we additionally obtain $y_1 = y_5 = 0$, $y_3 = \pm 8$, $x_2 = -2$ and finally $144 = 0$, which is not allowed, since $\mathrm{char}(k) \neq 2, 3$. □

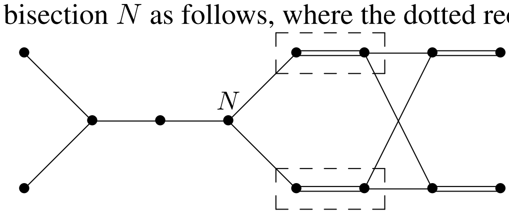
PROOF OF LEMMA 10.6. (For a detailed explanation of how to add (-2)-curves using jac_2, see Section 2.3.) By Corollary 10.3, it suffices to check the admissible fiber-bisection configurations with a $2\tilde{A}_1$ component. We will treat them in the following order:

Γ_1	# Admissible fiber-bisection configurations
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	2
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	4
$\tilde{E}_7 \oplus 2\tilde{A}_1$	2
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	2
$2\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	1
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	4
$2\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	1
$\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	4
$\tilde{A}_7 \oplus 2\tilde{A}_1$	2

- $\Gamma_1 = \tilde{D}_6 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$
  a) Fiber-bisection configuration:

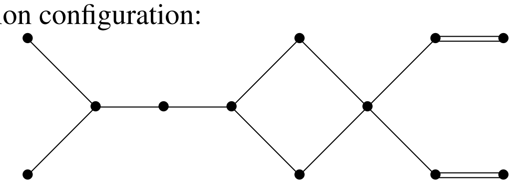


After adding a bisection with jac_2, we find another special fibration with two double I_2 fibers and bisection N as follows, where the dotted rectangles mark the fibers:

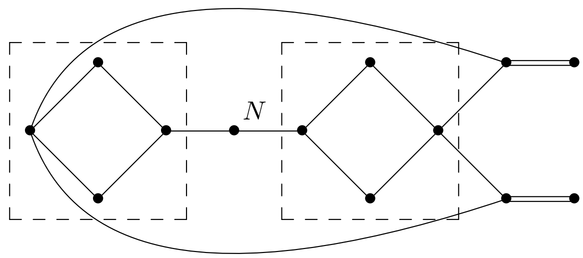


There is a D_4 diagram which is disjoint from the two $\tilde{A}_1$ subgraphs. By Table 3, the only extremal fibration with two singular fibers of type I_2 and one singular fiber whose dual graph contains a D_4 is the one with singular fibers $(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$. However, the bisection N cannot meet the I_2^* fiber in an admissible way, hence this fiber-bisection configuration does not occur.

  b) Fiber-bisection configuration:

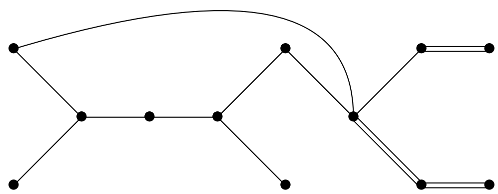


After adding a bisection with jac_2, we find another special fibration with two double I_4 fibers and bisection N as follows:

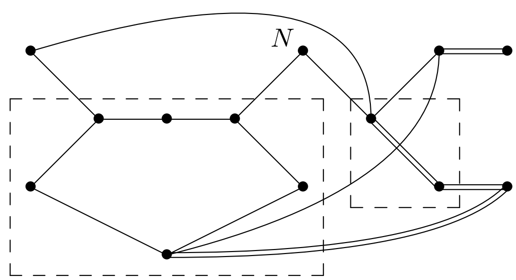


By Table 3, the only extremal fibration with two singular fibers of type I_4 is the one with singular fibers $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$ and the only admissible fiber-bisection configuration with $\Gamma_1 = 2\tilde{A}_3 \oplus 2\tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$ is the critical subgraph for type III.

- $\Gamma_1 = \tilde{D}_6 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$
  a) Fiber-bisection configuration:

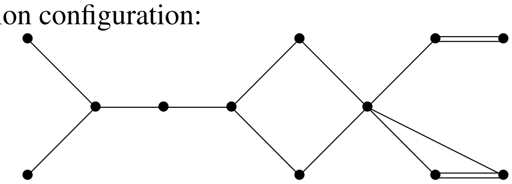


After adding a bisection corresponding to a 2-torsion section via jac_2, we obtain another special fibration with double singular fibers I_6 and I_2 and bisection N as follows:

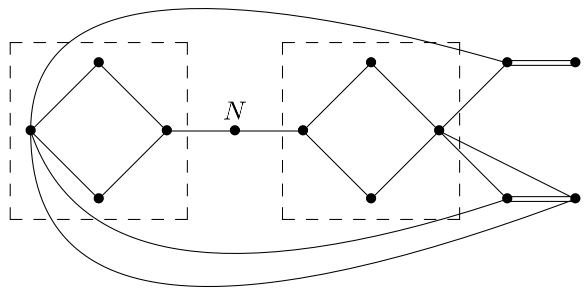


The only admissible fiber-bisection configuration for such a fibration is the critical subgraph for type V.

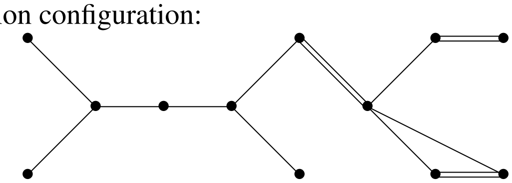
  b) Fiber-bisection configuration:



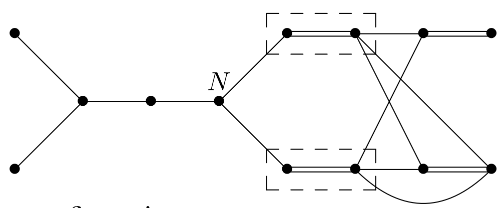
Adding another bisection corresponding to a 2-torsion section via jac_2, we obtain another special fibration with two singular double fibers of type I_4, giving the critical subgraph for type III:



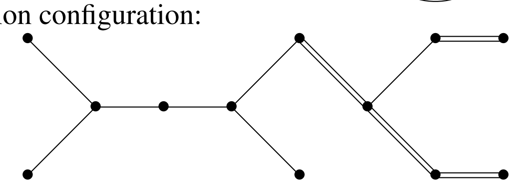
c) Fiber-bisection configuration:



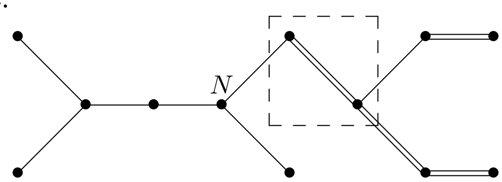
Adding another bisection corresponding to a 2-torsion section via jac_2, we obtain another special fibration with two singular double fibers of type I_2, bisection N, and some fiber whose dual graph contains a D_4. The only extremal fibration satisfying this is the one with fibers $(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$ and we have already treated the cases where both I_2 fibers are double.



d) Fiber-bisection configuration:

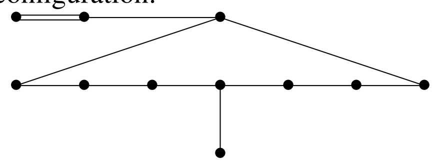


There is another special elliptic fibration with double fiber of type I_2 as in the following figure:



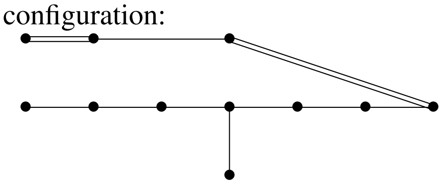
There is a D_4 diagram and three disjoint vertices, which are disjoint from the marked subgraph. The only extremal fibration whose dual graph of singular fibers contains these diagrams is the one with singular fibers $(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$. But the bisection N meets the fibers in such a way, that the fiber-bisection configuration will be one of the configurations we have already treated.

- $\Gamma_1 = \tilde{E}_7 \oplus 2\tilde{A}_1$
  a) Fiber-bisection configuration:

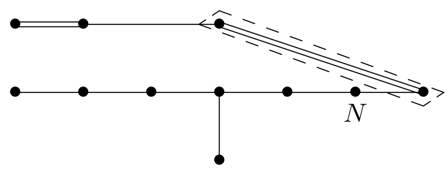


  This is the critical subgraph for type I.

  b) Fiber-bisection configuration:

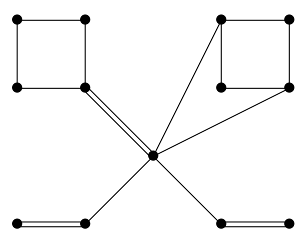


  There is another special elliptic fibration with a double fiber of type I_2 and a bisection N as follows:



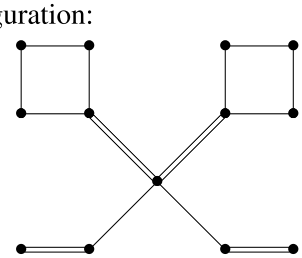
  There is a D_6 diagram and an isolated vertex which are disjoint from the marked subgraph. Moreover, from the intersection behaviour of N, we can exclude the case that the new fibration has a singular fiber of type III^*. The only extremal fibration satisfying these conditions is the one with singular fibers $(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$. We have already treated all fiber-bisection configurations for this fibration.

- $\Gamma_1 = \tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$
  a) Fiber-bisection configuration:

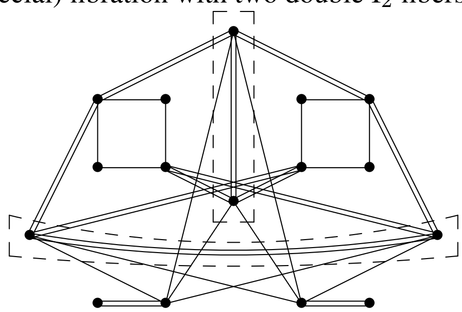


  This is the critical subgraph for type IV.

  b) Fiber-bisection configuration:



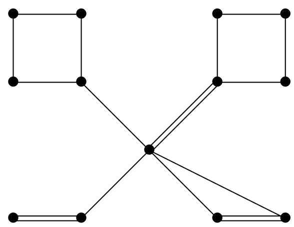
After adding bisections coming from 2-torsion sections via jac_2, we obtain another (maybe non-special) fibration with two double I_2 fibers as follows:



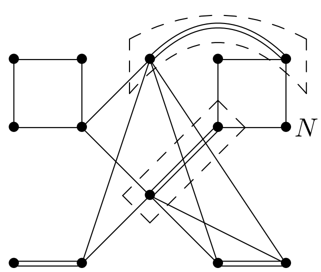
There are six disjoint vertices which are disjoint from the two I_2 fibers. There is no extremal elliptic fibration whose dual graph of singular fibers contains two $\tilde{A}_1$ diagrams and six disjoint vertices.

- $\Gamma_1 = 2\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$

 Fiber-bisection configuration:



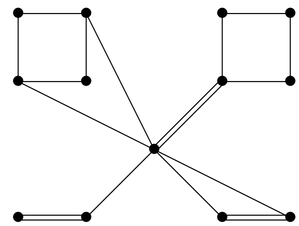
Adding a bisection corresponding to a 2-torsion section via jac_2, we find another special fibration with two double I_2 fibers and special bisection N.



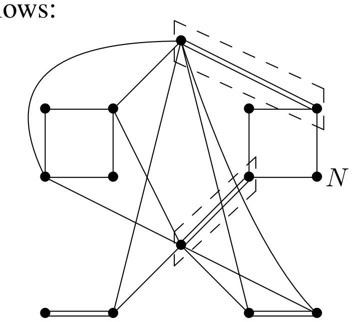
Since we have already treated all cases with two double I_2 fibers, we are done with this case.

- $\Gamma_1 = \tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$

a) Fiber-bisection configuration:



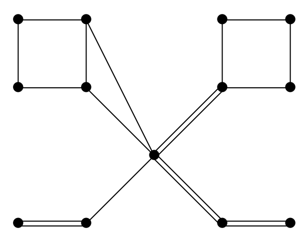
After adding a bisection corresponding to a different 2-torsion section via jac_2, we obtain another special elliptic fibration with two double fibers of type I_2 and a special bisection N as follows:



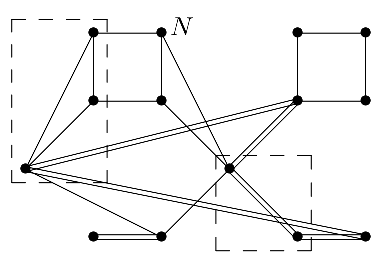
Since we have treated all fibrations with two double I_2 fibers, we are done.

b) The other fiber-bisection configuration where the bisection meets both components of the simple I_2 fiber is treated similarly to case a).

c) Fiber-bisection configuration:

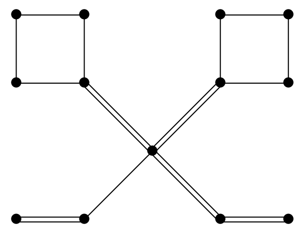


We add another bisection arising via jac_2 and find a special elliptic fibration with double fibers of type I_3 and I_2 and bisection N as follows:

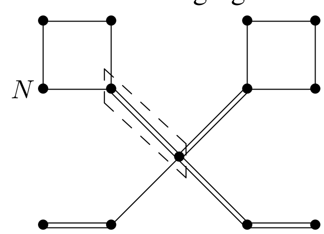


The only extremal fibration with these fibers is the one with singular fibers of type $(\mathrm{I}_6, \mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_1)$. But the I_3 and I_2 fibers cannot both be double by Lemma 10.12. Therefore, this fiber-bisection configuration does not occur.

d) Fiber-bisection configuration:



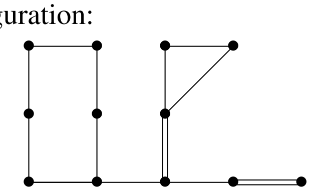
There is another special elliptic fibration with a double singular fiber of type I_2 and special bisection N as in the following figure:



There is an A_3 diagram and three disjoint vertices which are disjoint from the I_2 fiber. The extremal fibrations whose dual graphs of singular fibers satisfy these conditions are the ones with singular fibers $(\mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_2)$ and $(\mathrm{I}_4, \mathrm{I}_4, \mathrm{I}_2, \mathrm{I}_2)$. Since we have already treated the first fibration, we can assume that the second one occurs. But the bisection N and the fibers form a fiber-bisection configuration which we have already treated, hence this case is settled.

- $\Gamma_1 = 2\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$

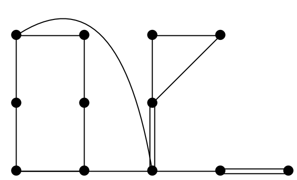
  Fiber-bisection configuration:



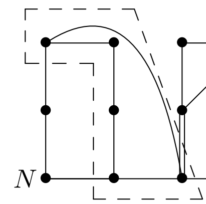
  This is the critical subgraph for type V.

- $\Gamma_1 = \tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$

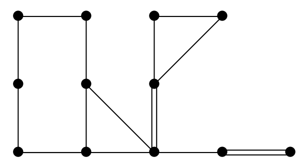
  a) Fiber-bisection configuration:



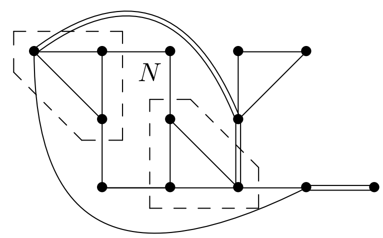
There is another special elliptic fibration with a double fiber of type I_5 and bisection N. We leave it to the reader to check that one obtains the critical subgraph for type VI from a fibration with singular fibers $\mathrm{I}_5, \mathrm{I}_5$ where one of the I_5 fibers is double.



b) Fiber-bisection configuration:

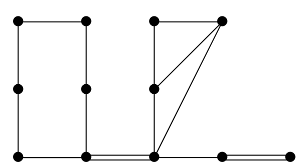


Adding another special bisection corresponding to the 2-torsion section via jac_2, we obtain another special elliptic fibration with two double singular fibers of type I_3 and bisection N as follows:

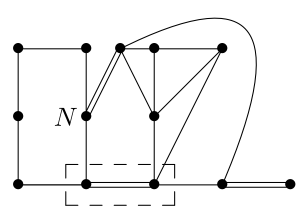


The only extremal and rational elliptic fibration with two fibers of type I_3 is the fibration with fibers $(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$. By Lemma 10.12, there is no such fibration with two double I_3 fibers.

c) Fiber-bisection configuration:

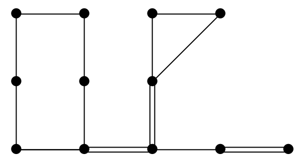


Adding another special bisection corresponding to a 6-torsion section via jac_2, we obtain another special fibration with a double fiber of type I_2 and a special bisection N as follows:



There are diagrams of type A_3, A_2, and A_1 which are disjoint from the double I_2 fiber. Therefore, the fibration cannot have an I_8 fiber. Since we have treated all the other cases with a double I_2 fiber, we can assume that the fibration has singular fibers of type $\mathrm{I}_6, \mathrm{I}_3$ (or IV) and I_2 such that the I_6 fiber is simple. But then, the fibers together with the bisection N form the admissible fiber-bisection configuration of case a) or b), since N meets distinct components of the I_6 fiber. Therefore, this case is settled.

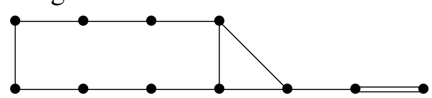
d) Fiber-bisection configuration:



Here, we can use the same (-2)-curves as in the previous case and the same argument right away without adding additional bisections.

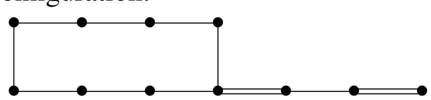
- $\Gamma_1 = \tilde{A}_7 \oplus 2\tilde{A}_1$

a) Fiber-bisection configuration:

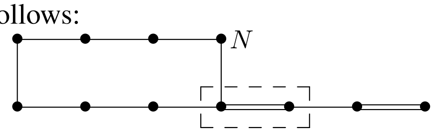


This is the critical subgraph for type VII.

b) Fiber-bisection configuration:



There is another special elliptic fibration with a double fiber of type I_2 and special bisection N as follows:



This fiber-bisection configuration is not the same as the one we started with and since this is the last case, we have already treated this. □

11. Arithmetic of Enriques surfaces with finite automorphism group

In this section, we explain how to derive the results on the arithmetic of Enriques surfaces with finite automorphism group, which we mentioned in the introduction, from the equations we gave in the earlier chapters (see §3, ..., §9). In particular, we establish explicit models of Enriques surfaces of every type over the prime fields $\mathbb{F}_p$ and $\mathbb{Q}$.

LEMMA 11.1. *The following integral Weierstrass models of elliptic K3 surfaces admit a resolution of singularities over the ring* R*, where* R *is as follows:*

PROOF. Let $f : \mathcal{X} \to \mathrm{Spec}(\mathrm{R})$ be one of the families defined by the above equations. Since the non-smooth locus of f is closed and f is proper, the non-smooth locus of f is proper. Hence,

Equation	R	Type
$y^2 + (s^2+s)xy = x^3 + (s^2+s)^3 x$	$\mathbb{Z}[\frac{1}{257}]$	I
$y^2 - (s^2+s)xy = x^3 - (s^2+s)^3 x$	$\mathbb{Z}[\frac{1}{255}]$	I
$y^2 + (s^2+s)xy + (s^2+s)^2 y = x^3 + (s^2+s)x^2$	$\mathbb{Z}[\frac{1}{65}]$	II
$y^2 - (s^2+s)xy + (s^2+s)^2 y = x^3 - (s^2+s)x^2$	$\mathbb{Z}[\frac{1}{63}]$	II
$y^2 + xy = x^3 + 4s^4x^2 + s^4 x$	$\mathbb{Z}[\frac{1}{2}]$	III
$y^2 = x^3 + 2(s^4+1)x^2 + (s^4-1)^2 x$	$\mathbb{Z}[\frac{1}{2}]$	IV
$y^2 + (s^2+1)xy + (s^2+1)y = x^3 + (s^2+2)x^2 + (s^2+1)x$	$\mathbb{Z}[\frac{1}{6}]$	V
$y^2 - 3(3s^2+3s+1)xy + (3s^2+3s+1)^2 y = x^3$	$\mathbb{Z}[\frac{1}{15}]$	VI
$y^2 = x^3 - (s^4+s^2)x^2 + (2s^6-3s^4+4s^2-2)x + (-s^6+2s^4-2s^2+1)$	$\mathbb{Z}[\frac{1}{10}]$	VII

every singular point of the generic fiber X_η of f is the generic point of a subscheme Z of $\mathcal{X}$ which is completely contained in the singular locus of f and flat over $\mathrm{Spec}(\mathrm{R})$. Since Z is flat over $\mathrm{Spec}(\mathrm{R})$, a local computation shows that blowing up along Z commutes with taking fibers of f. Moreover, we know that every fiber of f has the same types of rational double points, hence we can repeat the above argument and deduce that the minimal resolution of singularities of the generic fiber extends uniquely to a minimal resolution of the whole family. □

REMARK 11.2. The reason why we have to exclude some seemingly arbitrary characteristics is that the surface defined by the Weierstrass equation acquires additional singularities in these characteristics, because the degree 2 morphism to a rational elliptic surface we used to find the equations branches over a multiplicative fiber. This happens for the first four equations and for the last two, where the double cover branches over a nodal fiber, producing an additional A_1 singularity in some fibers. This singularity cannot be resolved in families without a base change to an algebraic space (see [Art74]).

THEOREM 11.3. *Let $K \in \{I, \ldots, VII\}$. There is a morphism $\varphi_K : \mathcal{X} \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{\mathrm{P_K}}])$ whose fibers are Enriques surfaces of type K with full Picard group, i.e. $Pic(\mathcal{X}_{\mathbb{F}_p}) = Pic(\mathcal{X}_{\overline{\mathbb{F}}_p})$. The numbers P_K are given in Table 2.*

PROOF. By Lemma 11.1, we have a family of K3 surfaces over $\mathbb{Z}[\frac{1}{P_K}]$. Now, observe that the Enriques involution is also defined over this ring. Hence, the only remaining claim is the one that the fibers of the family have full Picard group.

Let X_p be the fiber over p of one of the families of Enriques surfaces over R and let $\tilde{X}_p$ be its canonical cover. By Vinberg's criterion (Proposition 2.29), the geometric Picard group of X_p is generated by (-2)-curves, hence it suffices to check that all these curves are defined over $\mathbb{F}_p$ (resp. over $\mathbb{Q}$ if $p = 0$). Then, one uses our explicit equations to check that the Galois action preserves the preimages of these curves in $\tilde{X}_p$ and therefore all (-2)-curves on X_p are defined over $\mathbb{F}_p$ (resp. over $\mathbb{Q}$ if $p = 0$). Note that it suffices to check that the fiber components and special bisections of the fibration we used to construct the surfaces are fixed, since this will imply that the Galois action is trivial on the whole graph. □

REMARK 11.4. In particular, note that there are Enriques surfaces of type VI and VII with full Picard group over $\mathbb{Q}$, while this is not possible for their canonical cover due to a result of N. D. Elkies (see [Sch10]).

Moreover, Theorem 11.3 proves the existence of a model for every type of Enriques surfaces with finite automorphism group together with its dual graph of (-2)-curves over the prime fields.

COROLLARY 11.5. *Suppose that there exists an Enriques surface of type $K \in \{I, \ldots, VII\}$ in characteristic p. Then, there exists an Enriques surface X of type K with full Picard group over $\mathbb{F}_p$ (resp. over $\mathbb{Q}$ if $p = 0$).*

THEOREM 11.6. *Let X be an Enriques surface of type $K \in \{I, \ldots, VII\}$ over a field k such that $Pic(X) = Pic(X_{\bar{k}})$.*

- *If $K \neq III, IV$, then $Aut(X)$ is defined over k.*
- *If $K = III$, then $Aut(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 2$.*
- *If $K = IV$, then $Aut(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 16$.*

PROOF. Let X be an Enriques surface over k such that $|\mathrm{Aut}(X_{\bar{k}})| < \infty$ and $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\bar{k}})$. Since $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\bar{k}})$, every elliptic fibration of X is defined over k. Therefore, all Jacobian fibrations of elliptic fibrations of X are defined over k. Now, if X is of type I, II, V, VI or VII, the generic fiber of an elliptic fibration of X whose Jacobian has non-trivial sections has j-invariant $\neq 0, 1728$. Therefore, the Jacobian is unique up to quadratic twisting with elements in $\bar{k}$. We have shown in Propositions 3.3, 4.2, 7.2, 8.4, and 9.3 that $\mathrm{Aut}(X_{\bar{k}})$ is generated by the actions of 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X. Since quadratic twisting preserves 2-torsion sections and all extremal and rational elliptic fibrations have a model over k such that their 2-torsion is already defined over k, all such sections, and hence $\mathrm{Aut}(X)$, are defined over k.

If X is of type III, we need to realize the additional automorphism of Remark 5.2. For this, a quadratic extension is sufficient.

If X is of type IV, we need the automorphism of Remark 6.2 and one non-2-torsion section (see Proposition 6.3). As before, we need a field extension of degree at most 2 per non-2-torsion section. To define the automorphism of Remark 6.2, we need a field extension of degree at most eight, since we found a model of the corresponding fibration which acquires the required section after a quadratic extension and we need a quadratic extension to define ι (see Remark 6.2). □

REMARK 11.7. Over finite fields (and for our model), the proof shows that an extension of degree 4 suffices to realize all automorphisms for type IV.

12. Semi-symplectic automorphisms

As an application of our explicit classification of Enriques surfaces with finite automorphism group, we determine the semi-symplectic automorphism groups of these surfaces.

DEFINITION 12.1. Let X be an Enriques surface. An automorphism of X is called *semi-symplectic* if it acts trivially on $\mathrm{H}^0(\mathrm{X}, \omega_{\mathrm{X}}^{\otimes 2})$. We denote the group of all semi-symplectic automorphisms of X by $\mathrm{Aut}_{ss}(X)$.

These automorphisms are studied in [MO14]. There, the semi-symplectic automorphism groups of Enriques surfaces of type VI and VII have already been computed. See [Oha15] for a study of finite and non-semi-symplectic automorphisms.

THEOREM 12.2. *Let X be an Enriques surface of type $K \in \{I, \ldots, VII\}$. Then, the group $Aut_{ss}(X)$ is as given in the following table:*

Type	$Aut_{ss}(X)$
I	D_4
II	$\mathfrak{S}_4$
III	$(\mathbb{Z}/2\mathbb{Z})^3 \rtimes D_4$
IV	$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})$
V	$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$
VI	$\mathfrak{S}_5$
VII	$\mathfrak{S}_5$

TABLE 6. Semi-symplectic automorphism groups

PROOF. Note that an automorphism induced by a section of the Jacobian of an elliptic fibration of X is semi-symplectic, since it fixes the base of the fibration and acts as translation on the fibers. For all K, the group generated by such automorphisms is equal to the group given in Table 6. If $K \neq$ III, IV, these are all automorphisms, and if $K \in \{\text{III}, \text{IV}\}$, we have exhibited non-semi-symplectic automorphisms in Remarks 5.2 and 6.2. Since the groups in Table 6 have index 2 in Aut(X) for $K \in \{\text{III}, \text{IV}\}$, this finishes the proof. □

REMARK 12.3. The fact that surfaces of type III and IV admit non-semi-symplectic automorphisms is the reason why, in general, we need a field extension to realize all automorphisms of these surfaces. These non-semi-symplectic automorphisms act as $\sqrt{-1}$ on a non-zero global 2-form of the K3 cover, hence it is necessary to adjoin at least $\sqrt{-1}$ to k to realize all automorphisms of these surfaces. Since the K3 cover of Enriques surfaces of type III and IV is the Kummer surface associated to the self-product of an elliptic curve with j-invariant 1728 [Kon86, p.193], it is likely that this field extension always suffices.

CHAPTER II

Enriques surfaces with finite automorphism group in characteristic 2

Up to minor modifications, this chapter is taken from the paper ”Classification of Enriques surfaces with finite automorphism group in characteristic 2”, which is joint work of the author with T. Katsura and S. Kondo. Currently, the paper is submitted and a preprint can be found on the ArXiv (see [KKM17]).

1. Summary

In this chapter, we give the classification of supersingular and classical Enriques surfaces with finite automorphism group in characteristic 2.

These two cases differ drastically from the other types of Enriques surfaces: The K3-like cover is no longer smooth – sometimes not even normal – and some of these surfaces admit quasi-elliptic fibrations. Also, the total number of genus one fibrations on these surfaces might be very small and the Enriques surfaces with the smallest number of genus one fibrations are called ”extra-special” (see Chapter III §6) and are distinguished by the special configurations of (-2)-curves on them. Moreover, even though a generic classical Enriques surface in characteristic 2 does not admit global vector fields, there are some ”exceptional” surfaces that have global vector fields. These surfaces have been classified by Ekedahl, Shepherd-Barron and Salomonsson in [ES04] and [Sal] according to their dual graphs of (-2)-curves. Since Enriques surfaces with finite automorphism group have the most special configurations of (-2)-curves, it is natural that all these phenomena occur during the classification.

As a first step towards the classification of Enriques surfaces with finite automorphism group in characteristic 2, Katsura and Kondo [KK15b] checked whether the seven types of Enriques surfaces with finite automorphism group in characteristic 0 can also occur in characteristic 2. Their results are given in the following table.

Type	I	II	III	IV	V	VI	VII
singular	◯	◯	×	×	×	◯	×
classical	×	×	×	×	×	×	◯
supersingular	×	×	×	×	×	×	◯

TABLE 1. The seven types in characteristic 2

In Table 1, ◯ denotes the existence and × denotes the non-existence of an Enriques surface with the dual graph of type I, ..., VII. All examples in Table 1 are given explicitly.

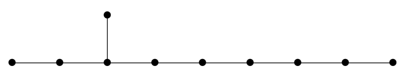
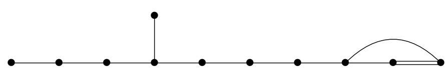
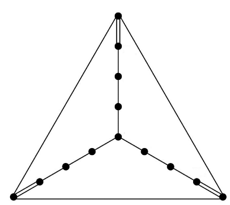
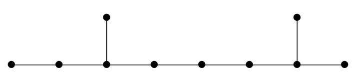
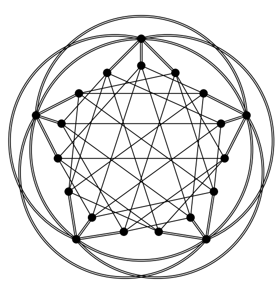
We have already seen in Chapter I that this list is complete in the singular case. The following theorems settle the remaining cases.

THEOREM 1.1. *Let X be a supersingular Enriques surface in characteristic* 2.

(A) *X has a finite group of automorphisms if and only if the dual graph of all (-2)-curves on X is one of the graphs in Table* 2 (A).

(B) *All cases exist. More precisely, we construct families of these surfaces whose automorphism groups and dimensions are given in Table* 2 (B).

(A) Classification

Type	Dual Graph of (-2)-curves
$\tilde{E}_8$	
$\tilde{E}_7 + \tilde{A}_1$	
$\tilde{E}_6 + \tilde{A}_2$	
$\tilde{D}_8$	
VII	

(B) Examples

$\mathrm{Aut}(X)$	$\mathrm{Aut}_{ct}(X)$	dim
$\mathbb{Z}/11\mathbb{Z}$	$\mathbb{Z}/11\mathbb{Z}$	0
$\mathbb{Z}/2\mathbb{Z}$ *or* $\mathbb{Z}/14\mathbb{Z}$	$\{1\}$ *or* $\mathbb{Z}/7\mathbb{Z}$	1
$\mathbb{Z}/5\mathbb{Z} \times \mathfrak{S}_3$	$\mathbb{Z}/5\mathbb{Z}$	0
Q_8	Q_8	1
$\mathfrak{S}_5$	$\{1\}$	0

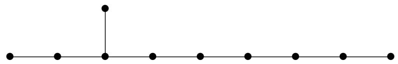
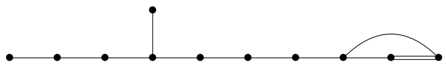
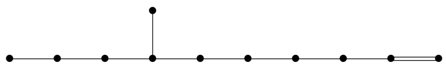
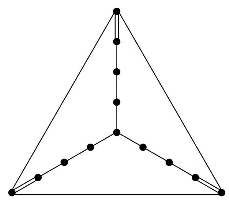
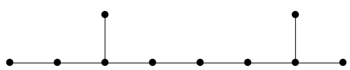
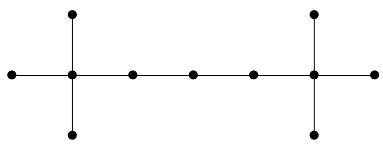
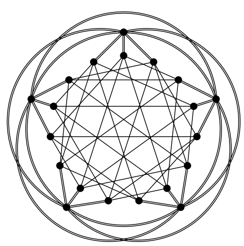
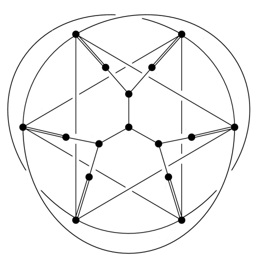
TABLE 2. (A) and (B)

THEOREM 1.2. *Let X be a classical Enriques surface in characteristic* 2.

(A) *X has a finite group of automorphisms if and only if the dual graph of all (-2)-curves on X is one of the graphs in Table* 3 (A).

(B) *All cases exist. More precisely, we construct families of these surfaces whose automorphism groups and dimensions are given in Table* 3 (B).

(A) Classification

Type	Dual Graph of (-2)-curves
$\tilde{E}_8$	
$\tilde{E}_7 + \tilde{A}_1$	
$\tilde{E}_7 + \tilde{A}_1$	
$\tilde{E}_6 + \tilde{A}_2$	
$\tilde{D}_8$	
$\tilde{D}_4 + \tilde{D}_4$	
VII	
VIII	

(B) Examples

$\mathrm{Aut}(X)$	$\mathrm{Aut}_{nt}(X)$	dim
$\{1\}$	$\{1\}$	1
$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	2
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1
$\mathfrak{S}_3$	$\{1\}$	1
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2
$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$	2
$\mathfrak{S}_5$	$\{1\}$	1
$\mathfrak{S}_4$	$\{1\}$	1

TABLE 3. (A) and (B)

In Theorem 1.1 and Theorem 1.2, $\mathrm{Aut}(X)$, $\mathrm{Aut}_{\mathrm{ct}}(X)$ or $\mathrm{Aut}_{\mathrm{nt}}(X)$ is the automorphism group of X, the cohomologically trivial automorphism group or the numerically trivial automorphism group (see Definition 2.3 and Chapter III), respectively, $\mathfrak{S}_n$ is the symmetric group of degree n and Q_8 is the quaternion group of order 8. The examples of supersingular Enriques surfaces of type $\tilde{E}_7+\tilde{A}_1$ form a 1-dimensional family, but in some cases the automorphism group jumps.

REMARK 1.3. We remark that the following families in Theorem 1.1 (B) and Theorem 1.2 (B) are non-isotrivial: $\tilde{E}_7+\tilde{A}_1$ supersingular, $\tilde{E}_6+\tilde{A}_2$ classical, VII classical, $\tilde{D}_4+\tilde{D}_4$ and VIII. The family of $\tilde{E}_7+\tilde{A}_1$ classical surfaces with simple type III fiber and the family of type $\tilde{D}_4+\tilde{D}_4$ are at least 1-dimensional. However, the problem of determining the moduli space of such Enriques surfaces is still open.

Recall that only the dual graph of type VII appears over the complex numbers. Moreover the Enriques surface with the dual graph of type VII is unique over the complex numbers, on the other hand, the example of Katsura and Kondo [KK15b] in characteristic 2 is a 1-dimensional non-isotrivial family of classical and supersingular Enriques surfaces with such dual graph (see Theorem 6.1). The canonical cover of any Enriques surface of type VII has 12 rational double points of type A_1 and its minimal resolution is the unique supersingular K3 surface with Artin invariant 1. The canonical covers of the other Enriques surfaces in Theorem 1.1 and Theorem 1.2 are non-normal rational surfaces.

Let us summarize the genus one fibrations on each of the above Enriques surfaces (for the notation, see Subsection 2.4, and Propositions 2.7, 2.8). We indicate that it is either elliptic (e) or quasi-elliptic (qe) after the type of singular fibers:

- Type $\tilde{E}_8$: $(2\mathrm{II}^*)$ (qe)
- Type $\tilde{E}_7+\tilde{A}_1$:
  supersingular : $(2\mathrm{III}^*, \mathrm{III})$ (qe), (II^*) (qe)
  classical-case 1: $(2\mathrm{III}^*, \mathrm{III})$ (qe), (II^*) (qe)
  classical-case 2: $(2\mathrm{III}^*, 2\mathrm{III})$ (qe), (II^*) (qe)
- Type $\tilde{E}_6+\tilde{A}_2$:
  supersingular: $(2\mathrm{IV}^*, \mathrm{IV})$ (e), $(\mathrm{III}^*, 2\mathrm{III})$ (qe)
  classical : $(2\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$ (e), $(\mathrm{III}^*, 2\mathrm{III})$ (qe)
- Type $\tilde{D}_8$:
  supersingular: $(2\mathrm{I}_4^*)$ (qe), $(2\mathrm{II}^*)$ (e), (II^*) (e)
  classical: $(2\mathrm{I}_4^*)$ (qe), $(2\mathrm{II}^*, \mathrm{I}_1)$ (e), $(\mathrm{II}^*, \mathrm{I}_1)$ (e)
- Type $\tilde{D}_4+\tilde{D}_4$: $(2\mathrm{I}_0^*, 2\mathrm{I}_0^*)$ (qe), (I_4^*) (e), $(2\mathrm{I}_4^*)$ (e)
- Type VII: $(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1)$ (e), $(\mathrm{I}_8, 2\mathrm{III})$ (e), $(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$ (e), $(\mathrm{I}_6, 2\mathrm{IV}, \mathrm{I}_2)$ (e)
- Type VIII: $(2\mathrm{I}_1^*, \mathrm{I}_4)$ (e), $(\mathrm{I}_2^*, 2\mathrm{III}, 2\mathrm{III})$ (qe), $(\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$ (e).

As we have explained above, the canonical cover of classical and supersingular Enriques surfaces in characteristic 2 is singular. Moreover, these Enriques surfaces admit non-zero global 1-forms. By definition of the K3 cover, the singular points of the cover are exactly the points mapping to zeroes of a global 1-form η on the Enriques surface. The divisorial part of the set of zeros of η is called the bi-conductrix and half of it is called the conductrix. Ekedahl and Shepherd-Barron

[ES04] classified possible conductrices of elliptic and quasi-elliptic fibrations on classical and supersingular Enriques surfaces.

The idea of the classification of dual graphs here is similar to the cases with smooth K3 cover in Chapter I: We use the classification of extremal elliptic fibrations. However, we also have to take quasi-elliptic fibrations into account. Moreover, it is harder to produce new (-2)-curves from a given fibration since we do not have a quadratic twist construction in the cases with singular cover. Instead, we can use the conductrix to cut down the number of cases to be checked considerably.

Due to the lack of the quadratic twist construction, we take another approach to the construction of examples, which is nevertheless inspired by the cases with smooth K3 cover. We start with some special genus one fibration on the Enriques surface we want to construct. Starting from its Jacobian, we base change along the Frobenius on $\mathbb{P}^1$ to obtain a singular surface. This will be birationally equivalent to the canonical cover of our Enriques surface. Now, we have to construct a rational vector field on this singular surface and take the quotient by this vector field to obtain another singular surface, which, after a minimal resolution of singularities, is the Enriques surface we are looking for.

In some cases, which will play an important role in Chapter III, it is not possible to determine the automorphism groups of the Enriques surfaces with finite automorphism group directly from their dual graphs of (-2)-curves. In these cases, we use our equations to calculate the automorphism group $\mathrm{Aut}(X)$ (see §4).

Let us now explain the structure of Chapter II: In §2, we recall basic facts on vector fields and conductrices. For the reader's convenience, we also repeat some of the facts, which are already used in Chapter I, but still hold for classical and supersingular Enriques surfaces. In §3 and §4 we explain how to construct the rational vector fields and give equations for our examples of Enriques surfaces with finite automorphism group. The construction of these examples is given in detail in $\S5, \ldots, \S11$. In these sections, we also compute the automorphism groups and give some non-isotriviality results. Finally, in §12, we give the classification of possible dual graphs of Enriques surfaces with finite automorphism group.

2. Preliminaries

2.1. Vector fields. Let k be an algebraically closed field of characteristic $p > 0$, and let S be a nonsingular complete algebraic surface defined over k. We denote by K_S a canonical divisor of S. A rational vector field D on S is said to be p-closed if there exists a rational function f on S such that $D^p = fD$. A vector field D for which $D^p = 0$ is called of additive type, while that for which $D^p = D$ is called of multiplicative type. Let $\{U_i = \mathrm{Spec}A_i\}$ be an affine open covering of S. We set $A_i^D = \{D(\alpha) = 0 \mid \alpha \in A_i\}$. The affine varieties $\{U_i^D = \mathrm{Spec}A_i^D\}$ glue together to define a normal quotient surface S^D.

Now, we assume that D is p-closed. Then, the natural morphism $\pi : S \longrightarrow S^D$ is a purely inseparable morphism of degree p. If the affine open covering $\{U_i\}$ of S is fine enough, then taking local coordinates x_i, y_i on U_i, we see that there exist $g_i, h_i \in A_i$ and a rational function f_i such that the divisors defined by $g_i = 0$ and by $h_i = 0$ have no common divisor, and such that

$$D = f_i \left(g_i \frac{\partial}{\partial x_i} + h_i \frac{\partial}{\partial y_i} \right) \quad \text{on } U_i.$$

By Rudakov and Shafarevich [RS76, Section 1], the divisors (f_i) on U_i give a global divisor (D) on S, and zero-cycles defined by the ideal (g_i, h_i) on U_i give a global zero cycle $\langle D\rangle$ on S. A point contained in the support of $\langle D\rangle$ is called an isolated singular point of D. If D has no isolated singular point, D is said to be divisorial. Rudakov and Shafarevich [RS76, Theorem 1, Corollary] showed that S^D is nonsingular if $\langle D\rangle = 0$, i.e., D is divisorial. When S^D is nonsingular, they also showed a canonical divisor formula

$$K_S \sim \pi^* K_{S^D} + (p-1)(D), \tag{2.1}$$

where $\sim$ means linear equivalence. As for the Euler number $c_2(S)$ of S, we have a formula

$$c_2(S) = \deg\langle D\rangle - K_S \cdot (D) - (D)^2 \tag{2.2}$$

(cf. Katsura and Takeda [KT89, Proposition 2.1]).

Now we consider an irreducible curve C on S and we set $C' = \pi(C)$. Take an affine open set U_i above such that $C \cap U_i$ is non-empty. The curve C is said to be integral with respect to the vector field D if $g_i\frac{\partial}{\partial x_i} + h_i\frac{\partial}{\partial y_i}$ is tangent to C at a general point of $C \cap U_i$. Then, Rudakov-Shafarevich [RS76, Proposition 1] showed the following proposition:

PROPOSITION 2.1. (i) *If C is integral, then $C = \pi^{-1}(C')$ and $C^2 = pC'^2$.*
(ii) *If C is not integral, then $pC = \pi^{-1}(C')$ and $pC^2 = C'^2$.*

2.2. Enriques surfaces in characteristic 2. Since we only dealt with Enriques surfaces with smooth $K3$ cover in Chapter I, we will recall the general definition here. A minimal algebaic surface with numerically trivial canonical divisor is called an Enriques surface if the second Betti number is equal to 10. Such surfaces X are divided into three classes in characteristic 2 (for details, see Bombieri and Mumford [BM76, Section 3]):

(i) K_X is not linearly equivalent to zero and $2K_X \sim 0$. Such an Enriques surface is called a classical Enriques surface.
(ii) $K_X \sim 0$, $\mathrm{H}^1(X, \mathcal{O}_X) \cong k$ and the Frobenius map acts on $\mathrm{H}^1(X, \mathcal{O}_X)$ bijectively. Such an Enriques surface is called a singular Enriques surface.
(iii) $K_X \sim 0$, $\mathrm{H}^1(X, \mathcal{O}_X) \cong k$ and the Frobenius map is the zero map on $\mathrm{H}^1(X, \mathcal{O}_X)$. Such an Enriques surface is called a supersingular Enriques surface.

It is known that Pic^τ_X is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if X is classical, μ_2 if X is singular or α_2 if X is supersingular (Bombieri-Mumford [BM76, Theorem 2]). As in the case of characteristic 0 or $p > 2$, an Enriques surface X in characteristic 2 has a canonical double cover $\pi : \tilde{X} \to X$, which is a separable $\mathbb{Z}/2\mathbb{Z}$-cover, a purely inseparable μ_2- or α_2-cover according to X being singular, classical or supersingular. The surface $\tilde{X}$ might have singularities and it might even be non-normal (see Proposition 2.13), but it is $K3$-like in the sense that its dualizing sheaf is trivial.

2.3. (-2)**-curves.** Let X be an Enriques surface and let $\mathrm{Num}(X)$ be the quotient of the Néron-Severi group $\mathrm{NS}(X)$ of X by torsion. Then $\mathrm{Num}(X)$ together with the intersection product is an even unimodular lattice of signature $(1, 9)$ (Illusie [Ill79, Corollary 7.3.7], Cossec and Dolgachev [CD89, Chap. II, Theorem 2.5.1]), and hence is isomorphic to $U \oplus E_8$. We denote by $\mathrm{O}(\mathrm{Num}(X))$ the orthogonal group of $\mathrm{Num}(X)$. The set

$$\{x \in \mathrm{Num}(X) \otimes \mathbb{R} \ : \ \langle x, x\rangle > 0\}$$

has two connected components. Denote by $P(X)$ the connected component containing an ample class of X. For $\delta \in \text{Num}(X)$ with $\delta^2 = -2$, we define an isometry s_δ of Num(X) by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta, \quad x \in \text{Num}(X),$$

which is nothing but the reflection with respect to the hyperplane perpendicular to δ. The isometry s_δ is called the reflection associated with δ. We call a nonsingular rational curve on an Enriques surface or a $K3$ surface a (-2)-curve. For a (-2)-curve E on an Enriques surface, we identify E with its class in Num(X). Let $W(X)$ be the subgroup of $\mathrm{O}(\text{Num}(X))$ generated by reflections associated with all (-2)-curves on S. Then $P(X)$ is divided into chambers each of which is a fundamental domain with respect to the action of $W(X)$ on $P(X)$. There exists a unique chamber containing an ample class which is nothing but the closure of the ample cone $D(X)$ of X. It is known that the natural map

$$\rho_n : \text{Aut}(X) \to \text{O}(\text{Num}(X)) \tag{2.3}$$

has a finite kernel. Since the image Im(ρ_n) preserves the ample cone, we see Im$(\rho_n) \cap W(X) = \{1\}$. Therefore Aut$(X)$ is finite if the index $[\text{O}(\text{Num}(X)) : W(X)]$ is finite. Thus we have the following Proposition (see Dolgachev [Dol84, Proposition 3.2]).

PROPOSITION 2.2. *If $W(X)$ is of finite index in $\mathrm{O}(\mathrm{Num}(X))$, then $\mathrm{Aut}(X)$ is finite.*

Over the field of complex numbers, the converse of Proposition 2.2 holds by using the Torelli type theorem for Enriques surfaces (Dolgachev [Dol84, Theorem 3.3]). As in Chapter I, we have the following definition.

DEFINITION 2.3. Denote by $\text{Aut}_{\text{nt}}(X)$ the kernel of the map ρ_n given by (2.3). Similarly denote by $\text{Aut}_{\text{ct}}(X)$ the kernel of the map

$$\rho_c : \text{Aut}(X) \to \text{O}(\text{NS}(X)). \tag{2.4}$$

A non-trivial automorphism is called cohomologically or numerically trivial if it is contained in $\text{Aut}_{\text{ct}}(X)$ or $\text{Aut}_{\text{nt}}(X)$, respectively. If S is not classical, then $\mathrm{NS}(X) = \mathrm{Num}(X)$ and hence $\text{Aut}_{\text{ct}}(X) = \text{Aut}_{\text{nt}}(X)$.

2.4. Genus one fibrations. We recall some facts on an elliptic or a quasi-elliptic fibrations on Enriques surfaces. For simplicity, we call an elliptic or a quasi-elliptic fibration a genus one fibration. For classical and supersingular Enriques surfaces, we have the following more general versions of Proposition 2.4 and Proposition 2.5 of Chapter I.

PROPOSITION 2.4. (Bombieri and Mumford [BM76, Theorem 3]) *Every Enriques surface has a genus one fibration.*

PROPOSITION 2.5. (Dolgachev and Liedtke [CDL, Theorem 4.8.3])

Let $f : X \to \mathbb{P}^1$ be a genus one fibration on an Enriques surface X in characteristic 2. Then the following hold.

(i) *If X is classical, then f has two tame double fibers, each is either an ordinary elliptic curve or a singular fiber of additive type.*

(ii) *If X is singular, then f has one wild double fiber which is a smooth ordinary elliptic curve or a singular fiber of multiplicative type.*

(iii) *If X is supersingular, then f has one wild double fiber which is a supersingular elliptic curve or a singular fiber of additive type.*

LEMMA 2.6. *Let $f : X \to \mathbb{P}^1$ be an isotrivial genus one fibration on an Enriques surface in characteristic 2. Let F be a double fiber of f such that the underlying reduced fiber F_{red} is an elliptic curve. Then F_{red} has j-invariant 0 if and only if the generic fiber of f also has j-invariant 0.*

PROOF. We can assume that the general fiber of f is an elliptic curve. Since f is isotrivial, it becomes trivial after passing to a finite cover of $\mathbb{P}^1$. Hence, F is isogeneous to the generic fiber of f. Since having j-invariant 0 is equivalent to being supersingular in characteristic 2 and being supersingular is an isogeny-invariant, we get the result. □

As in Chapter I, we use the symbols I_n $(n \geq 1)$, I_n^* $(n \geq 0)$, II, III, IV, II*, III*, IV* of singular fibers of an elliptic or a quasi-elliptic fibration in the sense of Kodaira. The dual graph of (-2)-curves on a singular fiber of type I_n $(n \geq 2)$, I_n^* $(n \geq 0)$, III, IV, II*, III* or IV* is an extended Dynkin diagram $\tilde{A}_{n-1}$, $\tilde{D}_{n+4}$, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{E}_8$, $\tilde{E}_7$ or $\tilde{E}_6$, respectively. For a double singular fiber of type F, we denote it by $2F$. Let $f : S \to \mathbb{P}^1$ be a genus one fibration on a surface S. If, for example, f has a double singular fiber of type III and a singular fiber of type IV*, then it is said that f has singular fibers $(2\mathrm{III}, \mathrm{IV}^*)$. If f has a section and its Mordell-Weil group is torsion, then f is called extremal. We use the following classifications of extremal rational elliptic and rational quasi-elliptic fibrations (compare Chapter I Table 3).

PROPOSITION 2.7. (Lang [Lan91], [Lan94]) *The following are the singular fibers of extremal elliptic fibrations on rational surfaces*:

$$(\mathrm{II}^*),\ (\mathrm{II}^*, \mathrm{I}_1),\ (\mathrm{III}^*, \mathrm{I}_2),\ (\mathrm{IV}^*, \mathrm{IV}),\ (\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1),\ (\mathrm{I}_4^*),\ (\mathrm{I}_1^*, \mathrm{I}_4),$$

$$(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1),\ (\mathrm{I}_8, \mathrm{III}),\ (\mathrm{I}_6, \mathrm{IV}, \mathrm{I}_2),\ (\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1),\ (\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3).$$

PROPOSITION 2.8. (Ito [Ito94]) *The following are the singular fibers of quasi-elliptic fibrations on rational surfaces*:

$$(\mathrm{II}^*),\ (\mathrm{III}^*, \mathrm{III}),\ (\mathrm{I}_4^*),\ (\mathrm{I}_2^*, \mathrm{III}, \mathrm{III}),\ (\mathrm{I}_0^*, \mathrm{I}_0^*),$$

$$(\mathrm{I}_0^*, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}),\ (\mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}).$$

Note that any quasi-elliptic fibration on a rational surface is extremal.

Consider a genus one fibration on an Enriques surface $\pi : X \to \mathbb{P}^1$. Then the Mordell-Weil group of the Jacobian of π acts on X effectively as automorphisms. This implies the following Proposition.

PROPOSITION 2.9. (Dolgachev [Dol84, §4]) *Assume that the automorphism group of an Enriques surface X is finite. Then any genus one fibration on X is extremal.*

Let X be an Enriques surface. A genus one fibration $f : X \to \mathbb{P}^1$ is called special if there exists a (-2)-curve R with $R \cdot f^{-1}(P) = 2$ $(P \in \mathbb{P}^1)$, that is, f has a (-2)-curve as a 2-section. In this case, R is called a special 2-section. The following result is due to Cossec [Cos85] in which he assumed the characteristic $p \neq 2$, but the assertion for $p = 2$ holds, too.

PROPOSITION 2.10. (Lang [Lan88, II, Theorem A3], Dolgachev and Liedtke [CDL, Theorem 5.3.4]) *Assume that an Enriques surface X contains a (-2)-curve. Then there exists a special genus one fibration on X.*

2.5. Vinberg's criterion. Let X be an Enriques surface. We recall Vinberg's criterion which guarantees that a group generated by a finite number of reflections is of finite index in $\mathrm{O}(\mathrm{Num}(X))$.

Let Δ be a finite set of (-2)-vectors in $\mathrm{Num}(X)$. Let Γ be the graph of Δ, that is, Δ is the set of vertices of Γ and two vertices δ and δ' are joined by m-tuple lines if $\langle \delta, \delta' \rangle = m$. We assume that the cone

$$K(\Gamma) = \{x \in \mathrm{Num}(X) \otimes \mathbb{R} \,:\, \langle x, \delta_i \rangle \geq 0,\ \delta_i \in \Delta\}$$

is a strictly convex cone. Such Γ is called non-degenerate. A connected parabolic subdiagram Γ' in Γ is a Dynkin diagram of type $\tilde{A}_m$, $\tilde{D}_n$ or $\tilde{E}_k$ (see Vinberg [Vin75, p. 345, Table 2]). If the number of vertices of Γ' is $r+1$, then r is called the rank of Γ'. A disjoint union of connected parabolic subdiagrams is called a parabolic subdiagram of Γ. We denote by $\tilde{K}_1 \oplus \cdots \oplus \tilde{K}_s$ a parabolic subdiagram which is a disjoint union of connected parabolic subdiagrams of type $\tilde{K}_1, \ldots, \tilde{K}_s$, where K_i is A_m, D_n or E_k. The rank of a parabolic subdiagram is the sum of the ranks of its connected components. Note that the dual graph of singular fibers of a genus one fibration on X gives a parabolic subdiagram. We denote by $W(\Gamma)$ the subgroup of $\mathrm{O}(\mathrm{Num}(X))$ generated by reflections associated with $\delta \in \Gamma$.

PROPOSITION 2.11. (Vinberg [Vin75, Theorem 2.3]) *Let Δ be a set of (-2)-vectors in $\mathrm{Num}(X)$ and let Γ be the graph of Δ. Assume that Δ is a finite set, Γ is non-degenerate and Γ contains no m-tuple lines with $m \geq 3$. Then $W(\Gamma)$ is of finite index in $\mathrm{O}(\mathrm{Num}(X))$ if and only if every connected parabolic subdiagram of Γ is a connected component of some parabolic subdiagram in Γ of rank 8 (= the maximal one).*

PROPOSITION 2.12. (Namikawa [Nam85, Proposition 6.9]) *Let Δ be a finite set of (-2)-curves on an Enriques surface X and let Γ be the graph of Δ. Assume that $W(\Gamma)$ is of finite index in $\mathrm{O}(\mathrm{Num}(X))$. Then Δ is the set of all (-2)-curves on X.*

2.6. Conductrix. Let X be a classical or supersingular Enriques surface. Then it is known that there exists a global regular 1-form η on X. The canonical cover $\pi : \tilde{X} \to X$ has a singularity at $P \in \tilde{X}$ if and only if η vanishes at $\pi(P)$. Since $c_2(X) = 12$, η always vanishes somewhere, and hence $\tilde{X}$ is singular. The divisorial part B of the zero set of η is called the bi-conductrix of X. The divisor B is of the form $2A$ where A is a divisor called the conductrix of X.

PROPOSITION 2.13. (Ekedahl and Shepherd-Barron [ES04, Proposition 0.5], Dolgachev and Liedtke [CDL, Proposition 1.3.8]) *Let X be a classical or supersingular Enriques surface and A its conductrix. Assume $A \neq 0$. Then A is 1-connected. Moreover $A^2 = -2$ and the normalization of the canonical cover has either four rational double points of type A_1 as singularities or one rational double point of type D_4.*

In the paper [ES04], Ekedahl and Shepherd-Barron gave possibilities of the conductrices for quasi-elliptic and elliptic fibrations in characteristic 2. In Section 12, we will use their classification of the conductrices ([ES04, Theorems 2.2, 3.1]). For simplicity, we say an A_1-singularity or a D_4-singularity for a rational double point of type A_1 or of type D_4 respectively. Also we will use the symbol nA_1 for n rational double points of type A_1.

3. Construction of vector fields

In this section, we explain two methods to construct a candidate of a vector field D on an algebraic surface Y such that the quotient surface Y^D becomes an Enriques surface.

3.1. Enriques surfaces with an elliptic pencil. Let $f : Y \longrightarrow \mathbb{P}^1$ be an elliptic surface with a section. Assume that Y is either a $K3$ surface or a rational surface. Then, the generic fiber is an elliptic curve E over the field $k(t)$ with one variable t. Therefore, there exists a non-zero regular vector field δ on E which we can regard as a non-zero rational vector field on Y. Taking a suitable vector field $g(t)\frac{\partial}{\partial t}$ and a suitable function $f(t)$ on $\mathbb{P}^1$, we look for a vector field

$$D = f(t)\{g(t)\frac{\partial}{\partial t} + \delta\}$$

such that Y^D is birationally isomorphic to an Enriques surface. In many cases, double fibers of the Enriques surface Y^D exist over the zero points of $g(t)$ by the theory of vector field (cf. Proposition 2.1). In this way, we construct Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$ in Section 5, of type VII in Section 6 and of type VIII in Section 7.

3.2. Enriques surfaces with a quasi-elliptic pencil. By Queen [Que71, Theorem 2], we have two normal forms of the generic fibers of a quasi-elliptic fiber space over the field $K = k(s)$ with a variable s:

(1) $u^2 = a + v + cv^2 + dv^4$ with $a, c, d \in K$ and $d \notin K$,

(2) $u^2 + u = a + dv^4$ with $a, d \in K$ and $d \notin K$.

Here, u, v are variables. Note that the case (3) in Queen [Que71, Theorem 2] doesn't occur in our case, because the trancendental degree of $K = k(s)$ over k is 1. As for the relative generalized Jacobians of these quasi-elliptic surfaces, Queen [Que72, Theorem 1] showed the following:

The generalized Jacobian for (1) : $u^2 = v + cv^2 + dv^4$,

The generalized Jacobian for (2) : $u^2 + u = dv^4$.

We use the case (1) to construct our Enriques surfaces. By the change of coordinates $x = 1/v + c,\ y = u/v^2$, the generalized Jacobian for (1) is birationally isomorphic to

$$y^2 = x^3 + c^2x + d,$$

which is a Weierstrass normal form. By Bombieri-Mumford [BM76], the relative Jacobian of the quasi-elliptic Enriques surface is a rational surface. Therefore, this surface is birationally isomorphic to the rational quasi-elliptic surface in the list of Ito [Ito02, Proposition 5.1].

Starting from Ito's list of rational quasi-elliptic surfaces, we pursue the converse procedure above to construct a candidate of an Enriques surface X, and using the candidate, we construct a vector field D on a rational surface Y such that Y^D is birationally isomorphic to the Enriques surface X. Using this technique, we will construct Enriques surfaces of type $\tilde{E}_8$ in Section 8, of type $\tilde{E}_7 + \tilde{A}_1$ in Section 9, of type $\tilde{D}_8$ in Section 10 and of type $\tilde{D}_4 + \tilde{D}_4$ in Section 11.

We will concretely show in the next subsection how to construct a vector field on a rational surface to make an Enriques surface of type $\tilde{D}_4 + \tilde{D}_4$.

3.3. Example: Vector fields for Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$. By Ito [Ito02, Proposition 5.1], we take the rational quasi-elliptic surface defined by

$$y^2 = x^3 + a^4 s^2 x + s^3 \quad \text{with } a \in k.$$

This quasi-elliptic surface has two singular fibers of type I_0^* (namely, of type $\tilde{D}_4$) over the points on $\mathbb{P}^1$ defined by $s = 0$ and $s = \infty$. Taking the change of coordinates

$$x = 1/v + a^2 s,\ y = s^2 u/v^2,\ s = 1/S$$

we get

$$u^2 = S^4 v + a^2 S^3 v^2 + S v^4$$

Now, we add a term $S^7 + S^3$ and a parameter b ($b \neq 0$) as follows:

$$u^2 = b^2 S^4 v + a^2 S^3 v^2 + S v^4 + S^7 + S^3. \tag{3.1}$$

We need to show that these surfaces are Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$. For this purpose, we take the base change by the Frobenius morphism:

$$S = t^2.$$

Then, the surface becomes

$$u^2 + b^2 t^8 v + a^2 t^6 v^2 + t^2 v^4 + t^{14} + t^6 = 0.$$

Therefore, by this equation we have

$$\{(u + at^3 v + tv^2 + t^7 + t^3)/bt^4\}^2 = v.$$

Now, by the change of coordinates

$$w = (u + at^3 v + tv^2 + t^7 + t^3)/bt^4,\ v = v,\ t = t,$$

we have

$$v = w^2.$$

This means we have $k(u, v, t) = k(w, t)$, which is a rational function field of two variables. Since

$$\begin{cases} u = bt^4 w + at^3 w^2 + tw^4 + t^7 + t^3 \\ S = t^2 \\ v = w^2, \end{cases}$$

we have

$$\begin{cases} \frac{\partial u}{\partial w} = bt^4 \\ \frac{\partial u}{\partial t} = at^2 w^2 + w^4 + t^6 + t^2. \end{cases}$$

We put

$$D' = (1/t^3)\left(bt^4 \frac{\partial}{\partial t} + (at^2 w^2 + w^4 + t^6 + t^2)\frac{\partial}{\partial w}\right).$$

Then, we see $D'(u) = 0$, $D'(v) = 0$, $D'(S) = 0$ and $k(t,w)^{D'} = k(u,v,S)$ with the equation (3.1). For the later use, taking new coordinates (x,y), we consider the change of coordinates

$$x = 1/t,\ y = t/w.$$

Then, we have

$$\frac{\partial}{\partial t} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\ \frac{\partial}{\partial w} = xy^2\frac{\partial}{\partial y}.$$

By this change of coordinates, D' becomes

$$D = \frac{1}{x^2y^2}\left(bx^3y^2\frac{\partial}{\partial x} + (ax^2y^2 + x^2 + x^4y^4 + y^4 + bx^2y^3)\frac{\partial}{\partial y}\right) \tag{3.2}$$

where $a, b \in k, b \neq 0$. We will show in Section 11 that the quotient surface with the function field $k(x,y)^D$ is an Enriques surface of type $\tilde{D}_4 + \tilde{D}_4$.

4. Equations of Enriques surfaces and their automorphisms

4.1. Generalities. Let X be an Enriques surface, and we assume that X has a structure of a quasi-elliptic fibration $\varphi : X \longrightarrow \mathbb{P}^1$. Let t be a parameter of an affine line $\mathbb{A}^1$ in the base curve $\mathbb{P}^1$. We denote by C the curve of cusps of the quasi-elliptic fibration, and assume that over the point defined by $t = \infty$ it has a double fiber $2F_\infty$. We assume that

$$y^2 = tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t) \quad (g_1(t), g_2(t), g_3(t) \in k[t]) \tag{4.1}$$

is the defining equation of an affine normal surface whose resolution of singularities is isomorphic to the open set $X \setminus (C \cup 2F_\infty)$ of X. Under these conditions, let σ be an automorphism of X which preserves the double fiber $2F_\infty$. Then, for large positive integers m, σ acts on the vector space $L(2mF_\infty)$ associated with the linear system $|2mF_\infty|$. Therefore, σ keeps the structure of the quasi-elliptic fibration $\varphi : X \longrightarrow \mathbb{P}^1$, and it acts on the base curve $\mathbb{P}^1$ with a fixed point at infinity:

$$\begin{array}{cccc} \sigma : & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ & \cup & & \cup \\ & \mathbb{A}^1 & & \mathbb{A}^1 \\ & t & \mapsto & c_1t + c_2 \end{array}$$

Here, c_1, c_2 are elements of k with $c_1 \neq 0$.

We set $A = k[t,x,y]/(y^2 + tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t))$. Then A is normal by our assumption. As $k[x,y]$-module, we have

$$A = k[t,x] \oplus k[t,x]y, \tag{4.2}$$

which is a free $k[t,x]$-module. Since σ preserve C and $2F_\infty$, σ acts on the open set $X \setminus (C \cup 2F_\infty)$ of X.

LEMMA 4.1. *σ induces an automorphism of* $\operatorname{Spec}(A)$.

PROOF. We consider the change of coordinates

$$u = \frac{1}{x},\ v = \frac{y}{x^2}.$$

Then, the equation becomes $v^2 = t + g_1(t)u^2 + g_2(t)u^3 + g_3(t)u^4$, and the curve C of cusps is given by $u = 0$. On the curve C, the affine surface is nonsingular. Therefore, the open set $X \setminus (C \cup 2F_\infty)$ is constructed by some blowing-ups of $\mathrm{Spec}(A)$:

$$\pi : X \setminus (C \cup 2F_\infty) \longrightarrow \mathrm{Spec}(A).$$

Note that π is surjective. Since σ is an automorphism of $X \setminus (C \cup 2F_\infty)$, we have a morphism

$$(\pi, \pi \circ \sigma) : X \setminus (C \cup 2F_\infty) \longrightarrow \mathrm{Spec}(A) \times \mathrm{Spec}(A).$$

We denote by Γ the image of the morphism $(\pi, \pi\circ\sigma)$. We denote bt p_1 (resp. p_2) the first projection (resp. the second projection) : $p_i : \mathrm{Spec}(A) \times \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A)$ $(i = 1, 2)$. Then, restricting the projection p_1 to Γ, we have a morphism

$$p_1|_\Gamma : \Gamma \longrightarrow \mathrm{Spec}(A).$$

Since $\mathrm{Spec}(A)$ is affine, the exceptional curves by blowing-ups collapse by the morphism $(\pi, \pi\circ\sigma)$. Therefore, the morphism $p_1|_\Gamma$ is a finite birational morphism. Since $\mathrm{Spec}(A)$ is normal by our assumption, we see that by the Zariski main theorem $p_1|_\Gamma$ is an isomorphism. Therefore, we have a morphism $p_2|_\Gamma \circ p_1|_\Gamma^{-1} : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A)$ which is the induced automorphism by σ. $\square$

By this lemma, σ acts on $\mathrm{Spec}(A)$ and induces an automorphism

(4.3) $$\sigma^* : A \longrightarrow A.$$

Now we consider the generic fiber of $\varphi : X \longrightarrow \mathbb{P}^1$. It is a curve of genus one over $k(t)$ whose affine part is given by the equation (4.1). The curve C of cusps gives a point P_∞ of degree 2 on the curve of genus one. We denote by $\tilde{L}(P_\infty)$ the vector space over $k(t)$ associated with the linear system $|P_\infty|$ on the curve of genus one. By the Riemann-Roch theorem, we have $\dim \tilde{L}(P_\infty) = 2$ and we see that 1 and x give the basis of $\tilde{L}(P_\infty)$. Since σ preserves the curve C of cusps, $\sigma^*(x)$ is contains in $\tilde{L}(P_\infty)$. Therefore, there exists $d_1(t), d_2(t) \in k(t)$ such that

$$\sigma^*(x) = d_1(t)x + d_2(t).$$

By (4.2) and (4.3), there exist $d_3(t,x), d_4(t,x) \in k[t,x]$ such that

$$\sigma^*(x) = d_3(t,x) + d_4(t,x)y.$$

Therefore, considering $\sigma^*(x)^2$, we have

$$d_1(t)^2x^2 + d_2(t)^2 = d_3(t,x)^2 + d_4(t,x)^2(tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t)).$$

Since the right-hand-side is in $k[t,x]$, we see that $d_1(t)$ and $d_2(t)$ are also polynomials of t. Therefore, we see that σ is of the following form:

(4.4) $$\sigma : \begin{cases} t & \mapsto \quad c_1t + c_2 \quad (c_1, c_2 \in k; c_1 \neq 0) \\ x & \mapsto \quad d_1(t)x + d_2(t) \quad (d_1(t), d_2(t) \in k[t]; d_1(t) \not\equiv 0) \\ y & \mapsto \quad e_1(t,x)y + e_2(t,x) \quad (e_1(t,x), e_2(t,x) \in k[t,x]; e_1(t,x) \not\equiv 0) \end{cases}$$

REMARK 4.2. Let X be an Enriques surface which has a structure of elliptic or quasi-elliptic fibration $\varphi : X \longrightarrow \mathbb{P}^1$ defined by

$$y^2 + g_0(t)y = tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t)$$

with $g_0(t), g_1(t), g_2(t), g_3(t) \in k[t]$. Here, t is a parameter of an affine line $\mathbb{A}^1$ in the base curve $\mathbb{P}^1$. We denote by C the 2-section defined by $x = \infty$, and by F_∞ the fiber over the point on $\mathbb{P}^1$ defined by $t = \infty$. We assume that the equation is the defining equation of an affine normal surface whose resolution of singularities is isomorphic to the open set $X \setminus (C \cup F_\infty)$ of X. Under these conditions, let σ be an automorphism of X which preserves the curve C and the fiber F_∞. Then, the automorphism σ is also expressed as the form (4.4), and a similar argument to the above works.

We use the following trivial lemma.

LEMMA 4.3. *$k[x, y]$ is a free $k[x^2, y^2]$-module of rank 4. A basis is given by $1, x, y, xy$.*

4.2. List of equations and automorphisms. In this subsection, we list up the equations of Enriques surfaces X with finite automorphism group and their automorphism groups except the case of type VII. We will use these equations to calculate the automorphism group in cases of type $\tilde{E}_6 + \tilde{A}_2$ (supersingular), type $\tilde{E}_8$ (supersingular and classical), type $\tilde{E}_7 + \tilde{A}_1$ (supersingular), type $\tilde{D}_8$ (supersingular and classical) and type $\tilde{D}_4 + \tilde{D}_4$. We will give the proofs of this list in Examples 4.3, 4.4, 4.5 and in Theorem 5.9, Theorem 8.4, Theorem 8.9, Theorem 9.10, Theorem 10.5, Theorem 10.11, Theorem 11.4. For the remaining cases, we do not use this list to determine the automorphism groups and hence omit the details.

(1) Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$.
(i) Supersingular case:

$$y^2 + ty = tx^4 + x^3 + t^3x + t^7, \quad \mathrm{Aut}(X) \cong \langle \sigma, \tau, \rho \rangle \cong \mathbb{Z}/5\mathbb{Z} \times \mathfrak{S}_3,$$

where $\sigma : \begin{cases} t \mapsto \zeta t \\ x \mapsto \zeta^4 x \\ y \mapsto \zeta y, \end{cases}$ $\tau : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + t, \end{cases}$ and ρ is an automorphism induced from the action of a section of order 3 of a relative Jacobian of the elliptic fibration on X with singular fibers $(\mathrm{IV}, \mathrm{IV}^*)$. Here, ζ is a primitive fifth root of unity and $\langle \tau, \rho \rangle \cong \mathfrak{S}_3$.

(ii) Classical case:

$$y^2 + c^2txy + \beta c^3t^2y = tx^4 + c^2t^3x^2 + (c^3t^4 + c^5\alpha t^3)x + t^7 + t^3 = 0,$$

where $c = \frac{1}{a + \sqrt[4]{a^3}}$ $(a \neq 0, 1)$, α is a root of $z^8 + z^6 + z^5 + a^2z^4 + a^4z^3 + a^8z^2 + a^{16} = 0$, and $\beta = \frac{\alpha^2 + a^4}{\alpha}$.

$\mathrm{Aut}(X) \cong \langle \sigma, \tau \rangle \cong \mathfrak{S}_3$, where $\sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + c^2tx + \beta c^3t^2 \end{cases}$ and τ is an automorphism induced from the action of a section of order 3 of the relative Jacobian of the elliptic fibration on X with singular fibers $(\mathrm{IV}, \mathrm{IV}^*)$.

(2) Enriques surfaces of type VII:

$$y^2 = t(t + 1)(t + a^2)(t + b^2)xy + \{(ab + 1)t + ab\}(t + 1)(t + a^2)(t + b^2)y + tx^4+$$

$$\{(ab+1)t+ab\}(t+1)(t+a^2)(t+b^2)x^3+\{t^2+(t+1)(t+a^2)(t+b^2)\}(t+1)(t+a^2)(t+b^2)x^2$$
$$+\{(ab+1)t+ab\}t(t+1)(t+a^2)(t+b^2)x+t^3+t^3(t+1)(t+a^2)(t+b^2)$$
$$+t(t+1)^2(t+a^2)^2(t+b^2)^2+t(t+1)^3(t+a^2)^3(t+b^2)^3,$$

where $a, b \in k,\ a+b=ab,\ a^3 \neq 1$. In this case, Katsura and Kondo calculated $\mathrm{Aut}(X)$ from the dual graph in [KK15b].

(3) Enriques surfaces of type VIII:
$y^2 = tx^4 + at^2x^3 + at^3(t+1)^2x + t^7 + t^3 \quad (a \neq 0).$
In this case we calculate $\mathrm{Aut}(X)$ from the dual graph in Section 7.

(4) Enriques surfaces of type $\tilde{E}_8$.
(i) Supersingular case:
$y^2 = tx^4 + x + t^7, \quad \mathrm{Aut}(X) \cong \langle\sigma\rangle \cong \mathbb{Z}/11\mathbb{Z}$
where $\sigma : \begin{cases} t \mapsto \zeta t \\ x \mapsto \zeta^7 x \\ y \mapsto \zeta^9 y \end{cases}$ and ζ is a primitive 11-th root root of unity.
(ii) Classical case:
$y^2 = tx^4 + at^3x + t^7 + t^3 \quad (a \neq 0), \quad \mathrm{Aut}(X) \simeq \{1\}.$

(5) Enriques surfaces of type $\tilde{E}_7 + \tilde{A}_1$
(i) Supersingular case:
$y^2 + y = tx^4 + ax + t^7 \quad (a \neq 0),$
$\mathrm{Aut}(X) \cong \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$ if $a^7 \neq 0, 1,$
$\mathrm{Aut}(X) \cong \langle\sigma\rangle \cong \mathbb{Z}/14\mathbb{Z}$ if $a^7 = 1.$
By the change of coordinates $t \mapsto t + a^4,\ y \mapsto y + a^2x^2 + ax,\ x \mapsto x$, the equation becomes

$$y^2 + y = tx^4 + (t+a^4)^7$$

and σ is given by

$$\sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y+1. \end{cases} \quad \text{if } a^7 \neq 0, 1,$$

$$\sigma : \begin{cases} t \mapsto \zeta t \\ x \mapsto \frac{1}{\sqrt[4]{\zeta}}x + \frac{(\sqrt[4]{\zeta}+1)}{\sqrt[4]{\zeta}}a^6 + \frac{(\sqrt[4]{\zeta^5}+1)}{\sqrt[4]{\zeta}}a^2t \\ y \mapsto y + 1 + (1+\zeta^2)a^6t^2 + (1+\zeta^3)a^2t^3 \end{cases} \quad \text{if } a^7 = 1,$$

where ζ is a primitive 7-th root of unity.
(ii) Classical one with singular fibers of type $(2\mathrm{III}^*, \mathrm{III})$:
$y^2 + at^2y = tx^4 + bt^3x + t^7 + t^3 \quad (a \neq 0,\ b \neq 0), \quad \mathrm{Aut}(X) \cong \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$
where $\sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + at^2. \end{cases}$
(iii) Classical one with singular fibers of type $(2\mathrm{III}^*, 2\mathrm{III})$:
$y^2 + at^2y = tx^4 + t^7 + t^3 \quad (a \neq 0), \quad \mathrm{Aut}(X) \cong \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$

where $\sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + at^2. \end{cases}$

(6) Enriques surfaces of type $\tilde{D}_8$
(i) Supersingular case:

$$y^2 = tx^4 + tx^2 + ax + t^7 \quad (a \neq 0), \quad \mathrm{Aut}(X) \cong \langle \{\sigma_{\omega,\alpha}\} \rangle \cong Q_4$$

$$\sigma_{\omega,\alpha} : \begin{cases} t \mapsto t + \omega \\ x \mapsto x + \alpha + \omega^2 t \\ y \mapsto y + \omega^2 x^2 + \omega^2 x + \omega^2 t^3 + \sqrt{a\alpha} + \sqrt{a} \end{cases}$$

Here, ω is a primitive cube root of unity and α is a root of the equation

$$z^2 + z + \omega\sqrt{a} + 1 = 0.$$

(ii) Classical case:

$$y^2 = tx^4 + at^3x^2 + bt^3x + t^7 + t^3 \quad (a \neq 0,\ b \neq 0), \quad \mathrm{Aut}(X) \cong \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

where $\sigma : \begin{cases} t \mapsto t \\ x \mapsto x + \sqrt{a}t \\ y \mapsto y + \sqrt[4]{a}\sqrt{b}t^2. \end{cases}$

(7) Classical Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$

$$y^2 = tx^4 + at^3x^2 + bt^4x + t^7 + t^3 \quad (b \neq 0), \quad \mathrm{Aut}(X) \cong \langle \{\sigma_\alpha\}, \tau \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3,$$

where $\sigma_\alpha : \begin{cases} t \mapsto t \\ x \mapsto x + \alpha t \\ y \mapsto y, \end{cases} \quad \tau : \begin{cases} t \mapsto 1/t \\ x \mapsto x/t^2 \\ y \mapsto y/t^5, \end{cases}$

and α is a root of the equation $z^3 + az + b = 0$.

4.3. Example 1. We calculate the defining equation of classical Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$. As in (5.3), (5.5) in Section 5, we take the elliptic surface defined by $y^2 + xy + t^2y = x^3$, and a vector field $D = (t + a)\frac{\partial}{\partial t} + (x + t^2)\frac{\partial}{\partial x}$ on it. Set $T = t^2$, $u = (t + a)x + t^3$, and $v = (t+a)^3(y+x^2)$. Then, we have $D(T) = 0$, $D(u) = 0$, $D(v) = 0$ and $k(t, x, y)^D = k(T, u, v)$. We have a relation

$$\begin{aligned} &v^2 + (T + a^2)uv + a(T + a^2)Tv \\ &\quad = (T + a^2)u^4 + (T^2 + a^4)(Tu^2 + T^4) + T^6(T + a^2), \end{aligned}$$

and this equation defines our classical Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$. We put $c = 1/(a + \sqrt[4]{a^3})$, and consider the change of new coordinates

$$\begin{cases} T = \frac{1}{c^2}t + a^2 \\ u = \frac{1}{c^3}x + \frac{\beta + a}{c^2}t + a^3 \\ v = \frac{1}{c^7}y + \frac{\delta}{c^5}tx + \frac{1}{c^6}t^3 + \frac{\alpha + \beta\delta}{c^4}t^2. \end{cases}$$

Here, δ is a root of the equation $z^2 + z + a^2 = 0$, α is a root of the equation $z^8 + z^6 + z^5 + a^2z^4 + a^4z^3 + a^8z^2 + a^{16} = 0$, and $\beta = \frac{\alpha^2 + a^4}{\alpha}$. Then, we get the normal form

$$y^2 + c^2txy + \beta c^3t^2y = tx^4 + c^2t^3x^2 + (c^3t^4 + c^5\alpha t^3)x + t^7 + t^3 = 0.$$

4.4. Example 2. We calculate the defining equation of classical Enriques surfaces of type VII. In [KK15b], to construct Enriques surfaces of type VII, we use an elliptic surface defined by $y^2+t^2xy+y=x^3+x^2+t^2$ and a vector field $D=(t+a)(t+b)\frac{\partial}{\partial t}+\frac{t^2x+1}{t+1}\frac{\partial}{\partial x}$, $a,b\in k,\ a+b=ab,\ a^3\neq 1$. Put

$$X=(t+1)(t+a)(t+b)x+t,\quad Y=(t+1)(t+a)(t+b)+tx^2,\quad T=t^2.$$

Then $k(x,y,t)^D=k(X,Y,T)$. Thus, replacing $X,\ Y(T+1)(T+a^2)(T+b^2),\ T$ by $x,\ y,\ t$, respectively, we have the equation of Enriques surfaces of type VII.

4.5. Example 3. We calculate the defining equation of classical Enriques surfaces of type VIII. We consider the elliptic surface Y defined by

$$y^2+txy+ty=x^3+x^2.$$

Then, we have

$$\frac{\partial y}{\partial t}=\frac{y}{t},\frac{\partial y}{\partial x}=\frac{ty+x^2}{t(x+1)}.$$

Therefore, considering x,y as local parameters instead of x,t, and using $t=\frac{x^3+x^2+y^2}{(x+1)y}$, we have

$$\begin{aligned} D &= t(at+1)\tfrac{\partial}{\partial t}+(x+1)\tfrac{\partial}{\partial x}\\ &= t(at+1)\{\tfrac{(x+1)y^2}{x^3+x^2+y^2}\tfrac{\partial}{\partial y}\}+(x+1)\{\tfrac{\partial}{\partial x}+\tfrac{ty+x^2}{t(x+1)}\tfrac{\partial}{\partial y}\}\\ &= \tfrac{x^3+x^2+y^2}{(x+1)y}\{a\tfrac{x^3+x^2+y^2}{(x+1)y}+1\}\{\tfrac{(x+1)y^2}{x^3+x^2+y^2}\tfrac{\partial}{\partial y}\}\\ &\quad +(x+1)\tfrac{\partial}{\partial x}+(x+1)\tfrac{(x+1)y}{x^3+x^2+y^2}\tfrac{1}{x+1}\{\tfrac{x^3+x^2+y^2}{(x+1)y}y+x^2\}\tfrac{\partial}{\partial y}\\ &= \tfrac{1}{(x+1)(x^3+x^2+y^2)}\{\{a(x^6+x^4+y^4)+x^4y+x^2y\}\tfrac{\partial}{\partial y}\\ &\quad +(x^5+x^4+x^2y^2+x^3+x^2+y^2)\tfrac{\partial}{\partial x}\} \end{aligned}$$

with $a\neq 0$. Putting

$$T=x^2,X=y^2,z=ax^7+ax^5+ay^4x+x^5y+x^3y+x^4y+x^2y^3+x^2y+y^3,$$

we have $D(T)=D(X)=D(z)=0$, and we have an equation

$$z^2=a^2TX^4+(T^2+1)X^3+(T^5+T^4+T^3+T^2)X+a^2T^5+a^2T^7,$$

which gives birationally the equation of Y^D. We consider the change of coordinates defined by

$$y=\frac{z}{a}+X^2+T^3+T^2,t=T+1,x=X.$$

with new variables x,y,t. Then the equation becomes

$$y^2=tx^4+\frac{1}{a^2}t^2x^3+\frac{1}{a^2}t^3(t+1)^2x+t^7+t^3$$

For the sake of simplicity, we replace $\frac{1}{a^2}$ by a. Then we have the normal form

$$y^2=tx^4+at^2x^3+at^3(t+1)^2x+t^7+t^3.$$

REMARK 4.4. This surface has an involution defined by

$$t \mapsto \frac{1}{t},\ x \mapsto \frac{x}{t^2},\ y \mapsto \frac{y}{t^5}.$$

Other results on the defining equations and their groups of automorphisms in Subsection 4.2 are obtained in a similar way.

5. Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$

From Section 5 to Section 11, we will construct examples of Enriques surfaces given in Theorem 1.1 and Theorem 1.2. First we consider the cases where the Enriques surfaces have a special elliptic fibration with a desired double fiber, that is, the case of type $\tilde{E}_6 + \tilde{A}_2$, of type VII and of type VIII. Next we consider the remaining cases that Enriques surfaces have a special quasi-elliptic fibration with a desired double fiber. In this section, we give Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$.

5.1. Supersingular case. We consider the relatively minimal nonsingular complete elliptic surface $\psi : R \longrightarrow \mathbb{P}^1$ associated with a Weierstrass equation

$$y^2 + sy = x^3$$

with a parameter s. This surface is a unique rational elliptic surface with a singular fiber of type IV over the point given by $s = 0$ and a singular fiber of type IV* over the point given by $s = \infty$ (Lang [Lan94, §2]). Note that all nonsingular fibers are supersingular elliptic curves. We consider the base change of $\psi : R \longrightarrow \mathbb{P}^1$ by $s = t^2$. Then, we have the elliptic surface defined by

$$y^2 + t^2 y = x^3. \tag{5.1}$$

We consider the relatively minimal nonsingular complete model of this elliptic surface :

$$f : \tilde{R} \longrightarrow \mathbb{P}^1. \tag{5.2}$$

By considering the change of coordinate defined by $x' = x/t^2, y' = y/t^3, t' = 1/t$, we have

$$y'^2 + t'y' = x'^3.$$

Thus the surface $\tilde{R}$ is isomorphic to R. The rational elliptic surface $f : \tilde{R} \to \mathbb{P}^1$ has a singular fiber of type IV* over the point given by $t = 0$ and a singular fiber of type IV over the point given by $t = \infty$.

The elliptic surface $f : \tilde{R} \longrightarrow \mathbb{P}^1$ has three sections s_i $(i = 0, 1, 2)$ given as follows:

s_0 : the zero section.
$s_1 : x = y = 0.$
$s_2 : x = 0, y = t^2.$

On the singular elliptic surface (5.1), we denote by F_0 the fiber over the point defined by $t = 0$, and by F_∞ the fiber over the point defined by $t = \infty$. Both F_0 and F_∞ are irreducible, and on each F_i $(i = 0, \infty)$ the surface (5.1) has only one singular point P_i. The surface $\tilde{R}$ is the surface obtained by the minimal resolution of singularities of the surface (5.1). We denote the proper transform of F_i on $\tilde{R}$ again by F_i, if confusion doesn't occur. We have six exceptional curves $E_{0,k}$ $(k = 1, 2, \ldots, 6)$ over the point P_0 such that F_0 and these six exceptional curves make a singular fiber of type IV* of the elliptic surface $f : \tilde{R} \longrightarrow \mathbb{P}^1$ as follows: The blowing-up at the singular point P_0 gives one exceptional curve $E_{0,1}$, and the surface is nonsingular along F_0 and has a unique singular point $P_{0,1}$ on $E_{0,1}$. The blowing-up at the singular point $P_{0,1}$ gives two exceptional curves $E_{0,2}$ and $E_{0,3}$. We denote the proper transform of $E_{0,1}$ by $E_{0,1}$. The three curves $E_{0,1}$, $E_{0,2}$ and $E_{0,3}$ meet at one point $P_{0,2}$ which is a singular point of the obtained surface. The blowing-up at the singular point $P_{0,2}$ again gives two exceptional curves $E_{0,4}$ and $E_{0,5}$. The three curves $E_{0,1}$, $E_{0,4}$ and $E_{0,5}$ meet at one point $P_{0,3}$ which is a singular point of the obtained surface. The curve $E_{0,2}$ (resp. $E_{0,3}$) intersects $E_{0,4}$ (resp. $E_{0,5}$) and does not meet other curves. Finally the blowing-up at the singular point $P_{0,3}$ gives an exceptional curve $E_{0,6}$ and the obtained surface is nonsingular over these curves. The curve $E_{0,6}$ meets $E_{0,1}$, $E_{0,4}$ and $E_{0,5}$ transversally. The dual graph of the curves $F_0, E_{0,1}, \ldots, E_{0,6}$ is of type $\tilde{E}_6$. The cycle

$$F_0 + E_{0,2} + E_{0,3} + 2(E_{0,1} + E_{0,4} + E_{0,5}) + 3E_{0,6}$$

forms a singular fiber of type IV*. On the other hand, the blowing-up at the singular point P_∞ gives two exceptional curves $E_{\infty,1}$ and $E_{\infty,2}$. The obtained surface is now nonsingular, that is, nothing but $\tilde{R}$. The three curves F_∞, $E_{\infty,1}$ and $E_{\infty,2}$ form a singular fiber of type IV. The configuration of these curves is as in the following Figure 1.

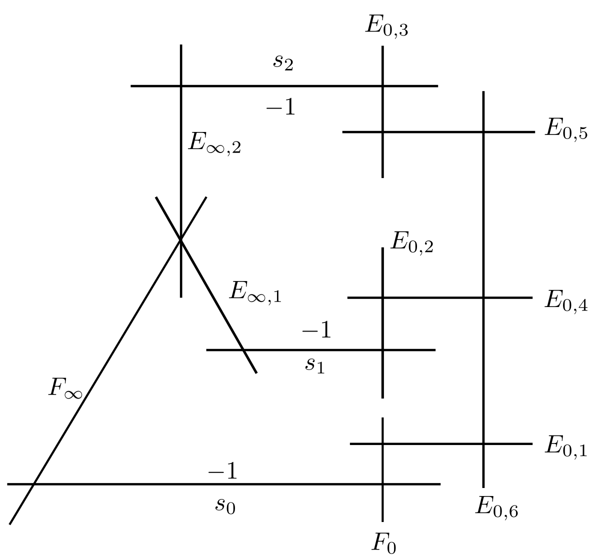


FIGURE 1

The sections s_i have the self-intersection number -1 and others have the self-intersection number -2.

Now, we consider a rational vector field on $\tilde{R}$ induced from

$$D = \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial x}.$$

Then, we have $D^2 = 0$, that is, D is 2-closed. However D has an isolated singularity at the point P which is the singular point of the fiber of type IV, that is, the intersection point of three curves F_∞, $E_{\infty,1}$ and $E_{\infty,2}$ (note that (t, x) is not a local parameter along the fiber defined by $t = 0$). To resolve this singularity, we first blow up at P. Denote by $E_{\infty,3}$ the exceptional curve. We denote the proper transforms of F_∞, $E_{\infty,1}$ and $E_{\infty,2}$ by the same symbols. Then blow up at three points $E_{\infty,3} \cap (F_\infty + E_{\infty,1} + E_{\infty,2})$. Let Y be the obtained surface and $\psi : Y \to \tilde{R}$ the successive blowing-ups. We denote by $E_{\infty,4}$, $E_{\infty,5}$ or $E_{\infty,6}$ the exceptional curve over the point $E_{\infty,3} \cap F_\infty$, $E_{\infty,3} \cap E_{\infty,1}$ or $E_{\infty,3} \cap E_{\infty,2}$ respectively. Then we have the following Figure 2. In this Figure 2 we give the self-intersection numbers of the curves except for the curves with the self-intersection number -2, and the thick lines are integral curves with respect to D.

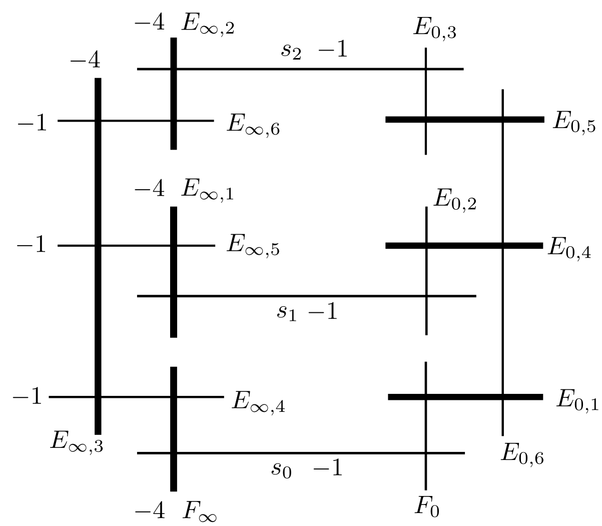


FIGURE 2

Now, according to the above blowing-ups, we see the following:

LEMMA 5.1. (i) *The divisorial part* (D) *on* Y *is given by*

$$-2(E_{0,1} + E_{0,4} + E_{0,5} + E_{0,6} + E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}) - (F_\infty + E_{\infty,1} + E_{\infty,2}).$$

(ii) *The integral curves in Figure* 2 *are*

$$E_{0,1}, E_{0,4}, E_{0,5}, F_\infty, E_{\infty,1}, E_{\infty,2}, E_{\infty,3}.$$

LEMMA 5.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor* K_Y *of* Y *is given by*

$$K_Y = -2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}) - (F_\infty + E_{\infty,1} + E_{\infty,2}).$$

(iii) $K_Y \cdot (D) = -4$.

LEMMA 5.3. *D is divisorial and the quotient surface Y^D is nonsingular.*

PROOF. Since $\tilde{R}$ is a rational elliptic surface and Y is the blowing-ups at 4 points, we have $c_2(Y) = 16$. Using $(D)^2 = -12$, $K_Y \cdot (D) = -4$ and the equation (2.2), we have

$$16 = c_2(Y) = \deg\langle D\rangle - K_Y \cdot (D) - (D)^2 = \deg\langle D\rangle + 4 + 12.$$

Therefore, we have $\deg\langle D\rangle = 0$. This means that D is divisorial, and that Y^D is nonsingular. □

Let $\pi : Y \to Y^D$ be the natural map. By the result on the canonical divisor formula (2.1), we have

$$K_Y = \pi^* K_{Y^D} + (D).$$

LEMMA 5.4. (i) *The images of the curves $E_{0,1}, E_{0,4}, E_{0,5}$ in Y^D are exceptional curves.*
(ii) *The self-intersection numbers of the images of $F_0, E_{0,2}, E_{0,3}, E_{0,6}$ in Y^D are -4.*
(iii) *The self-intersection numbers of the images of $F_\infty, E_{\infty,i}$ $(i = 1, \ldots, 6)$ and three sections s_i $(i = 0, 1, 2)$ in Y^D are -2.*

PROOF. The assertions follows from Proposition 2.1 and Lemma 5.1, (ii). □

Let $E'_{0,1}, E'_{0,4}, E'_{0,5}, E'_{0,6}$ be the image of $E_{0,1}, E_{0,4}, E_{0,5}, E_{0,6}$ in Y^D, respectively. Then we have the following Figure 3 in which we give the self-intersection numbers of the curves except the curves with the self-intersection number -2.

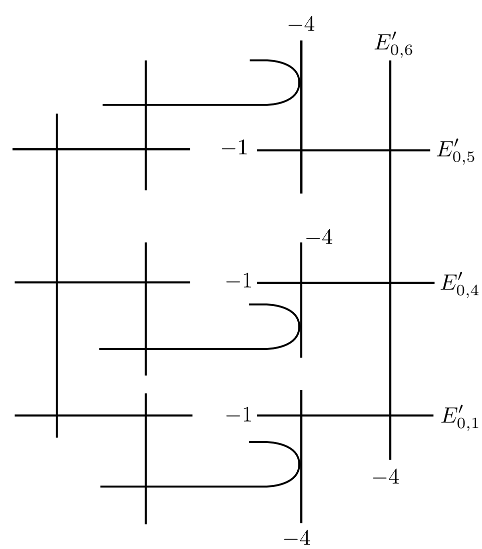


FIGURE 3

Let

$$\varphi_1 : Y^D \to X'$$

be the blowing-downs of $E'_{0,1}, E'_{0,4}, E'_{0,5}$. Then the image of $E'_{0,6}$ in X' is an exceptional curve. Let

$$\varphi_2 : X' \to X$$

be the blowing-down of this exceptional curve. Now we have the following diagram

$$\begin{array}{ccc} Y^D & \xleftarrow{\pi} & Y \\ \varphi_1 \downarrow & & \downarrow \psi \\ X' & & \tilde{R} \\ \varphi_2 \downarrow & & \\ X & & \end{array}$$

We have thirteen (-2)-curves $E_1, \ldots, E_{13}$ with the self-intersection number -2 which form the following Figure 4.

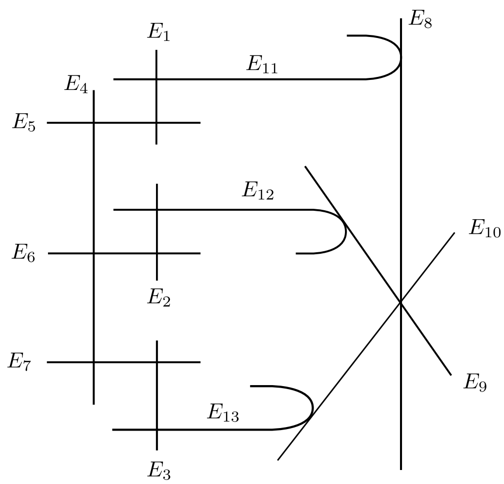


FIGURE 4

Then, we have

$$K_{Y^D} = \varphi_1^*(K_{X'}) + E_{0,1} + E_{0,4} + E_{0,5}$$
$$= \varphi_1^* \circ \varphi_2^*(K_X) + E_{0,6} + 2(E_{0,1} + E_{0,4} + E_{0,5}).$$

LEMMA 5.5. *The canonical divisor K_X of X is numerically equivalent to* 0*.*

PROOF. By Lemma 5.2, (ii),

$$K_Y = -2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}) - (F_\infty + E_{\infty,1} + E_{\infty,2}).$$

On the other hand,

$$K_Y = \pi^* K_{Y^D} + (D) = \pi^*(\varphi_1^* \circ \varphi_2^*(K_X) + E_{0,6} + 2(E_{0,1} + E_{0,4} + E_{0,5})) + (D) =$$
$$\pi^*(\varphi_1^* \circ \varphi_2^*(K_X)) + 2(E_{0,6} + E_{0,1} + E_{0,4} + E_{0,5}) + (D) = \pi^*(\varphi_1^* \circ \varphi_2^*(K_X)) + K_Y.$$

Here we use the fact that $E_{0,1}, E_{0,4}, E_{0,5}$ are integral and $E_{0,6}$ is non-integral (Lemma 5.1, (ii) and Lemma 2.1). Therefore, K_X is numerically equivalent to zero. □

LEMMA 5.6. *The surface X has $b_2(X) = 10$.*

PROOF. Since $\pi : Y \longrightarrow Y^D$ is finite and purely inseparable, the étale cohomology of $\tilde{Y}$ is isomorphic to the étale cohomology of Y^D. Therefore, we have $b_1(Y^D) = b_1(Y) = 0$, $b_3(Y^D) = b_3(Y) = 0$ and $b_2(Y^D) = b_2(Y) = 14$. Since $\varphi_2 \circ \varphi_1$ is the blowing-downs of four exceptional curves, we see $b_0(X) = b_4(X) = 1$, $b_1(X) = b_3(X) = 0$ and $b_2(X) = 10$. □

THEOREM 5.7. *With the notation above, X is a supersingular Enriques surface.*

PROOF. Since K_X is numerically trivial, X_a is minimal and the Kodaira dimension $\kappa(X)$ is equal to 0. Since $b_2(X) = 10$, X is an Enriques surface. Since $\tilde{Y}$ is a rational surface, X_a is either supersingular or classical. Consider the elliptic fibration $g : X \longrightarrow \mathbb{P}^1$ induced by $f : \tilde{R} \longrightarrow \mathbb{P}^1$. Note that the fiber over the point given by $t = \infty$ is a double fiber of type IV* and the fiber over the point given by $t = 0$ is simple. Since the other fibers are smooth and supersingular elliptic curves by Lemma 2.6, they are simple by Proposition 2.5. Therefore X is a supersingular Enriques surface by Proposition 2.5. □

The dual graph of the thirteen (-2)-curves $E_1, \ldots, E_{13}$ is as in the following Figure 5.

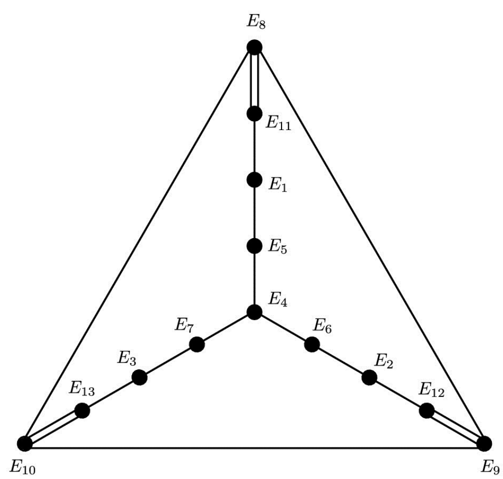


FIGURE 5

On X_a, there exist exactly one elliptic fibration with singular fibers of type $(2\text{IV}^*, \text{IV})$ defined by the linear system $|E_8 + E_9 + E_{10}|$ and three quasi-elliptic fibrations with singular fibers of type $(\text{III}^*, 2\text{III})$ defined by $|2(E_9 + E_{12})|, |2(E_8 + E_{11})|, |2(E_{10} + E_{13})|$ respectively.

By Proposition 2.2, we now have the following theorem.

THEOREM 5.8. *The Enriques surface X contains exactly thirteen (-2)-curves.*

PROOF. Consider the dual graph Γ of 13 (-2)-curves in Figure 5. We can easily prove that any maximal parabolic subdiagrams in Γ is of type $\tilde{E}_6 + \tilde{A}_2$ or of type $\tilde{E}_7 + \tilde{A}_1$. It follows from Proposition 2.2 that $\mathrm{Aut}(X)$ is finite and X contains exactly 13 (-2)-curves. □

THEOREM 5.9. *The automorphism group* $\mathrm{Aut}(X)$ *is isomorphic to* $\mathbb{Z}/5\mathbb{Z} \times \mathfrak{S}_3$ *and the numerically trivial automorphism group* $\mathrm{Aut}_{nt}(X)$ *is isomorphic to* $\mathbb{Z}/5\mathbb{Z}$.

PROOF. To calculate $\mathrm{Aut}(X)$ we first give an equation of X mentioned in Section 4 and then determine its automorphism group. As in Subsection 4.1, we consider the elliptic surface defined by $y^2+t^2y = x^3$ and the vector field $D = \frac{\partial}{\partial t}+t^2\frac{\partial}{\partial x}$. Put $T = t^2$, $u = x+t^3$, $v = y+tx^2$. Then, we have $D(T) = 0$, $D(u) = 0$, $D(v) = 0$ and we have the relation $v^2 + Tv = Tu^4 + u^3 + T^3u + T^7$. Since we have $k(x, y, t)^D = k(u, v, T)$, the quotient surface by D is birationally isomorphic to the surface defined by $v^2+Tv = Tu^4+u^3+T^3u+T^7$. We replace variables u, v, T by new variables x, y, t, respectively for convenience. Then, the equation $y^2 + ty = tx^4 + x^3 + t^3x + t^7$ gives a normal affine surface. Note that the minimal normal completion of this surface is a normal elliptic surface $f : X \longrightarrow \mathbb{P}^1$ which is birationally isomorphic to our Enriques surface. Set

$$A = k[t, x, y]/(y^2 + ty + tx^4 + x^3 + t^3x + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2)-curves, σ preserves $2F_\infty$. Therefore, σ preserves the structure of this elliptic surface. Since there are three 2-sections for this elliptic surface by the configuration of nordal curves, σ acts on these three 2-sections as a permutation. We denote by $\tilde{C}$ be the 2-section at infinity and assume that σ preserves $\tilde{C}$. Then, as in the case of a quasi-elliptic surface, σ has the form in (4.4) in Subsection 4.1. Moreover, this elliptic surface has a singular fiber over the point defined by $t = 0$, σ preserves also the singular fiber. Therefore, we know $c_2 = 0$ and we have $\sigma^*(t) = c_1t$.

Therefore, together with the equation $y^2 = ty + tx^4 + x^3 + t^3x + t^7$, we have an identity

$$\begin{aligned}
&e_1(t,x)^2(ty + tx^4 + x^3 + t^3x + t^7) + e_2(t,x)^2 + c_1t(e_1(t,x)y + e_2(t,x)) \\
&= c_1t(d_1(t)x + d_2(t))^4 + (d_1(t)x + d_2(t))^3 \\
&\quad +(c_1t)^3(d_1(t)x + d_2(t)) + (c_1t)^7.
\end{aligned}$$

A is a free $k[x, y]$-module, and 1 and y are linearly independent over $k[x, y]$. Taking the coefficient of y, we have $e_1(t,x)^2t + c_1te_1(t,x) = 0$. Since $e_1(t,x) \neq 0$, we have $e_1(t,x) = c_1$, which is a constant. Therefore, we have

$$\begin{aligned}
&c_1^2(tx^4 + x^3 + t^3x + t^7) + e_2(t,x)^2 + c_1te_2(t,x) \\
&= c_1t(d_1(t)x + d_2(t))^4 + (d_1(t)x + d_2(t))^3 \\
&\quad +(c_1t)^3(d_1(t)x + d_2(t)) + (c_1t)^7.
\end{aligned}$$

As a polynomial of x, if $e_2(t,x)$ has a term of degree greater than or equal to 3, then $e_2(t,x)^2$ has a term greater than or equal to 6. We cannot kill this term in the equation. Therefore, we can put $e_2(t,x) = a_0(t) + a_1(t)x + a_2(t)x^2$ with $a_0(t), a_1(t), a_2(t) \in k[t]$. We take terms which contain only the variable t. Then, we have an equality

$$c_1^2t^7 + a_0(t)^2 + c_1ta_0(t) = c_1td_2(t)^4 + d_2(t)^3 + c_1^3t^3d_2(t) + c_1^7t^7.$$

Put deg $d_2(t) = \ell$. Suppose $\ell \geq 2$. Then, the right-hand-side has an odd term whose degree is equal to $4\ell + 1 \geq 9$. Therefore, the left-hand-side must have an odd term which is of degree $4\ell + 1$. This means deg $a_0(t) = 4\ell + 1$. However, in the equation we cannot kill the term of degree $8\ell + 2$ which comes from $a_0(t)^2$. Therefore, we can put $d_2(t) = b_0 + b_1 t$ with $b_0, b_1 \in k$. Then, the equation becomes

$$\begin{aligned} &a_0(t)^2 + c_1 t a_0(t) + c_1^2 t^7 \\ &= c_1 b_0^4 t + c_1 b_1^4 t^5 + b_0^3 + b_0^2 b_1 t + b_0 b_1^2 t^2 + b_1^3 t^3 + c_1^3 b_0 t^3 + c_1^3 b_1 t^4 + c_1^7 t^7. \end{aligned}$$

If deg $a_0(t) \geq 4$, we cannot kill the term of degree greater than or equal to 8 which comes from $a_0(t)^2$. Therefore, we can put $a_0(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$. Then, we have equations:

$$\begin{aligned} &c_1^2 = c_1^7, \alpha_3^2 = 0, 0 = c_1 b_1^4, \alpha_2^2 + c_1\alpha_3 = c_1^3 b_1, c_1\alpha_2 = b_1^3 + c_1^3 b_0, \\ &\alpha_1^2 + c_1\alpha_1 = b_0 b_1^2, c_1\alpha_0 = c_1 b_0^4 + b_0^2 b_1, \alpha_0^2 = b_0^3. \end{aligned}$$

Solving these equations, we have

$$b_0 = 0, b_1 = 0, \alpha_0 = 0, \alpha_2 = 0, \alpha_3 = 0, c_1^5 = 1, \alpha_1 = 0 \text{ or } c_1.$$

Therefore, we have $c_1 = \zeta, e_1(t,x) = \zeta, a_0(t) = 0$ or $\zeta t, d_2(t) = 0$. with $\zeta^5 = 1, \zeta \in k$. Putting these date into the original equation, we have

$$\begin{aligned} &\zeta^2(tx^4 + x^3 + t^3 x) + a_1(t)^2 x^2 + a_2(t)^2 x^4 + \zeta t a_1(t) x + \zeta t a_2(t) x^2 \\ &\quad = \zeta t d_1(t)^4 x^4 + d_1(t)^3 x^3 + \zeta^3 t^3 d_1(t) x. \end{aligned}$$

Considering the coefficients of x^4, we have $\zeta^2 t + a_2(t)^2 + \zeta t d_1(t)^4 = 0$. Therefore, we have $a_2(t) = 0$ and $d_1(t) = \zeta^4$. Considering the coefficients of x^2, we have $a_1(t) = 0$. Therefore we have

$$c_1 = \zeta, \quad d_1(t) = \zeta^4, \quad d_2(t) = 0, \quad e_1(t,x) = \zeta, \quad e_2(t,x) = 0 \text{ or } \zeta t.$$

Fixing a fifth primitive root ζ of unity, we set

$$\begin{aligned} &\sigma : t \mapsto \zeta t, \quad x \mapsto \zeta^4 x, \quad y \mapsto \zeta y \\ &\tau : t \mapsto t, \quad x \mapsto x, \quad y \mapsto y + t. \end{aligned}$$

Then, we have

$$\sigma \circ \tau : t \mapsto \zeta t, \quad x \mapsto \zeta^4 x, \quad y \mapsto \zeta y + \zeta t$$

and $\langle \sigma \circ \tau \rangle \cong \mathbb{Z}/10\mathbb{Z}$. We now take the relative Jacobian variety of $f : X \longrightarrow \mathbb{P}^1$. It has singular fibers of types IV, IV*, and the Mordell Weil group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ (cf. Ito [Ito02]). We denote by ρ a generator of the group. It acts on X and permutes three 2-sections. On the other hand, τ is induced from the action of the Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$ of a quasi-elliptic fibration p with singular fibers of type (III*, III) (cf. Ito [Ito02]) and it interchanges two 2-sections not equal to the curve of cusps of p. Therefore, considering the action of the subgroup $\langle \tau, \rho \rangle$ generated by τ and ρ on the dual graph of (-2)-curves, we see $\langle \tau, \rho \rangle$ is isomorphic to the symmetric group $\mathfrak{S}_3$ of degree 3. Considering the commutation relations of σ, τ, ρ, we conclude $\mathrm{Aut}(X) \cong \mathbb{Z}/5\mathbb{Z} \times \mathfrak{S}_3$ (see Subsection 4.2). The automorphism σ is numerically trivial by construction. □

REMARK 5.10. Note that $\mathrm{Aut}_{ct}(X) = \mathrm{Aut}_{nt}(X)$ because X is supersingular. The numerically trivial automorphism σ of order 5 is a new example of such automorphisms.

5.2. Classical case. We consider the relatively minimal nonsingular complete elliptic surface $\psi : R \longrightarrow \mathbb{P}^1$ associated with the Weierstrass equation

$$y^2 + xy + sy = x^3$$

with a parameter s. This surface is a rational elliptic surface with a singular fiber of type I_3 over the point given by $s = 0$, a singular fiber of type I_1 over the point given by $s = 1$ and a singular fiber of type IV^* over the point given by $s = \infty$ (cf. Lang [Lan94, §2]). We consider the base change of $\psi : R \longrightarrow \mathbb{P}^1$ by $s = t^2$. Then, we have the elliptic surface associated with the Weierstrass equation

$$y^2 + xy + t^2y = x^3. \tag{5.3}$$

We consider the relatively minimal nonsingular complete model of this elliptic surface :

$$f : \tilde{R} \longrightarrow \mathbb{P}^1. \tag{5.4}$$

The rational elliptic surface $f : \tilde{R} \to \mathbb{P}^1$ has a singular fiber of type I_6 over the point given by $t = 0$, a singular fiber of type I_2 over the point given by $t = 1$ and a singular fiber of type IV over the point given by $t = \infty$ (see Figure 6). The fibration f has six sections. In Figure 6, (-1)-curves denote the 0-section and two sections defined by the equations

$$x = y = 0, \quad x = y + t^2 = 0$$

respectively.

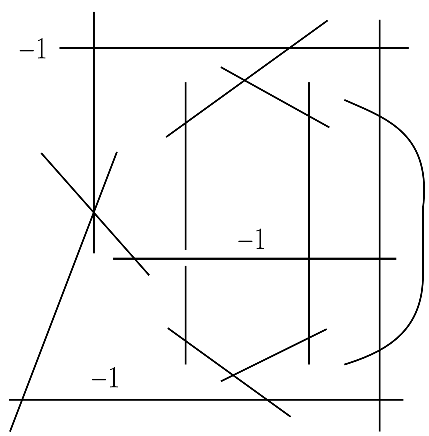


FIGURE 6

Now, we consider a rational vector field on $\tilde{R}$ defined by

$$D = D_a = (t + a)\frac{\partial}{\partial t} + (x + t^2)\frac{\partial}{\partial x} \tag{5.5}$$

where $a \in k,\ a \neq 0, 1$. We see that $D^2 = D$, that is, D is 2-closed. Note that the nonsingular fiber F_a over the point defined by $t = a$ is integral with respect to D. The vector field D has an isolated singularity at the point P which is the singular point of the fiber of type IV. Denote by F_∞, $E_{\infty,1}$ and $E_{\infty,2}$ the three components of the singular fiber of type IV. Then P is the intersection point of these three curves. To resolve this singularity, we first blow up at P. Denote by $E_{\infty,3}$ the exceptional curve. We denote the proper transforms of F_∞, $E_{\infty,1}$ and $E_{\infty,2}$ by the same symbols. Then blow up at three points $E_{\infty,3} \cap (F_\infty + E_{\infty,1} + E_{\infty,2})$. Let Y be the obtained surface and $\psi : Y \to \tilde{R}$ the successive blowing-ups. We denote by the same symbol D the induced vector field on Y. We denote by $E_{\infty,4}$, $E_{\infty,5}$ or $E_{\infty,6}$ the exceptional curve over the point $E_{\infty,3} \cap F_\infty$, $E_{\infty,3} \cap E_{\infty,1}$ or $E_{\infty,3} \cap E_{\infty,2}$ respectively. Then we have the following Figure 7 in which we give the self-intersection numbers of the curves, and the thick curves are integral with respect to D.

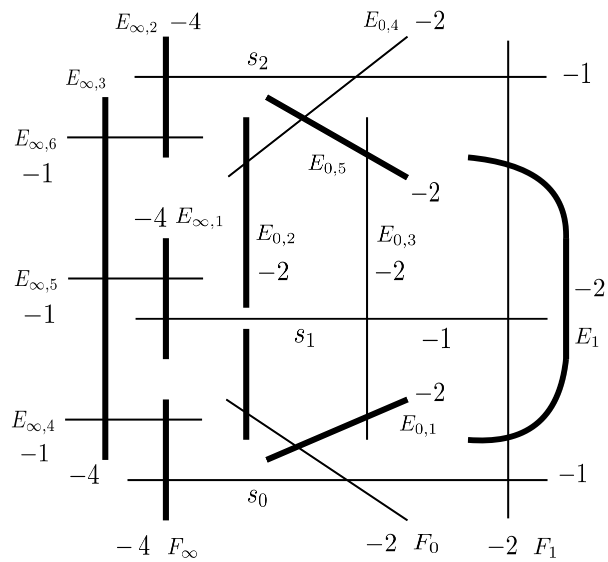


FIGURE 7

A direct calculation shows the following Lemmas.

LEMMA 5.11. (i) *The divisorial part* (D) *of* D *on* Y *is given by*

$$-(E_1 + E_{0,1} + E_{0,2} + E_{0,5} + F_\infty + E_{\infty,1} + E_{\infty,2}) - 2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}).$$

(ii) *The integral curves in Figure* 7 *are*

$$E_{0,1}, E_{0,2}, E_{0,5}, F_\infty, E_{\infty,1}, E_{\infty,2}, E_{\infty,3}, E_1.$$

LEMMA 5.12. (i) $(D)^2 = -12$.
(ii) *The canonical divisor* K_Y *of* Y *is given by*

$$K_Y = -(F_\infty + E_{\infty,1} + E_{\infty,2}) - 2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}).$$

(iii) $K_Y \cdot (D) = -4$.

Now, by taking the quotient by D, we have the following Figure 8. Here the numbers $-1, -4$ denote the self-intersection numbers of curves. The other curves have the self-intersection number -2.

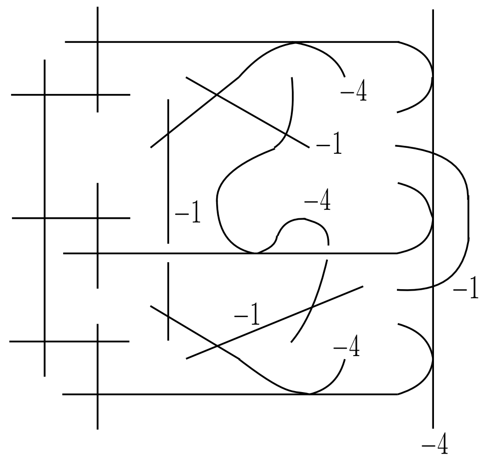


FIGURE 8

We now contract four (-1)-curves in Figure 8, and denote by X_a the obtained surface which has the dual graph of (-2)-curves given in Figure 5 (recall that the vector field (5.5) contains a parameter a). We use the notation of Figure 8. On X_a, there exist exactly one elliptic fibration with singular fibers of type $(2\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$ defined by the linear system $|E_8 + E_9 + E_{10}|$ and three quasi-elliptic fibrations with singular fibers of type $(\mathrm{III}^*, 2\mathrm{III})$ defined by $|2(E_9 + E_{12})|, |2(E_8 + E_{11})|, |2(E_{10} + E_{13})|$ respectively.

THEOREM 5.13. *The surfaces* $\{X_a\}$ *form a* 1*-dimensional non-isotrivial family of classical Enriques surfaces with the dual graph given in Figure* 5*.*

PROOF. By using Lemmas 5.11, 5.12 and the same argument as in the case of the supersingular surface in the previous subsection, X_a is an Enriques surface. Since the image of F_a and the singular fiber of type VI* are double fibers, X_a is classical by Proposition 2.5. Moreover the double fiber F_a varies and hence this family is non-isotrivial. By the same proof as that of Theorem 5.8, we prove that X_a contains exactly 13 (-2)-curves whose dual graph is given in Figure 5. □

LEMMA 5.14. *The map* $\rho_n : \mathrm{Aut}(X_a) \to \mathrm{O}(\mathrm{Num}(X_a))$ *is injective.*

PROOF. Let $g \in \mathrm{Ker}(\rho_n)$. Then g preserves each of the thirteen curves E_1,..., E_{13} (see Figure 5). First note that g fixes three points on each of E_8, E_9, E_{10} (in contrast to the supersingular case, where only two distinct points are fixed). Hence, g fixes E_8, E_9 and E_{10} pointwisely. Let p be the quasi-elliptic fibration with singular fibers of type $(\mathrm{III}^*, 2\mathrm{III})$ defined by the linear system $|2(E_8 + E_{11})|$ and let F be a general fiber of p. The two curves E_9, E_{10} are 2-sections of the fibration p. Then, g fixes at least three points on F which are the intersection with E_9 and E_{10} and the cusp of F. Hence, g fixes F pointwisely. Thus ρ_n is injective. □

By Proposition 2.2, we now have the following theorem.

THEOREM 5.15. *The automorphism group* $\mathrm{Aut}(X_a)$ *is isomorphic to the symmetric group* $\mathfrak{S}_3$ *of degree three and* X_a *contains exactly thirteen* (-2)*-curves.*

PROOF. By Lemma 5.14, $\mathrm{Aut}(X_a)$ is a subgroup of the symmetry group of the dual graph of (-2)-curves which is isomorphic to $\mathfrak{S}_3$. By considering the actions of the Mordell-Weil groups of the Jacobian fibrations of genus one fibrations on X_a, any symmetry of the dual graph can be realized by an automorphism of X_a. □

6. Enriques surfaces of type VII

Katsura and Kondo proved the following theorem based on a method given in [KK15a].

THEOREM 6.1. *([KK15b]) There exists a* 1*-dimensional non-isotrivial family of Enriques surfaces with the dual graph of* (-2)*-curves given in Figure* 9*. A general member of this family is classical and a special member is supersingular. The automorphism group of any member in this family is isomorphic to the symmetric group* $\mathfrak{S}_5$ *of degree* 5*. The canonical cover of any member in this family has* 12 *ordinary nodes and its minimal resolution is the supersingular* $K3$ *surface with Artin invariant* 1.

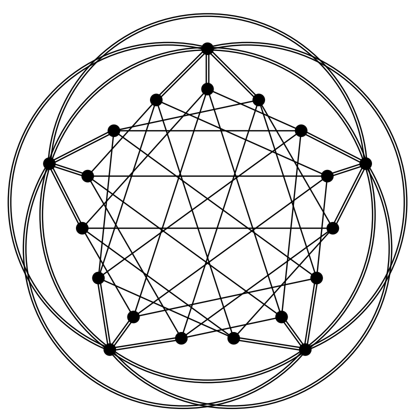


FIGURE 9

There exist elliptic fibrations with singular fibers of type $(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1)$, $(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$, $(\mathrm{I}_8, 2\mathrm{III})$ or $(\mathrm{I}_6, 2\mathrm{IV}, \mathrm{I}_2)$ on Enriques surfaces of type VII. For more details, we refer the reader to [KK15b].

7. Enriques surfaces of type VIII

In this section we give a construction of a one-dimensional family of classical Enriques surfaces with the dual graph of type VIII.

We consider the relatively minimal nonsingular complete elliptic surface $\psi : R \longrightarrow \mathbb{P}^1$ associated with the Weierstrass equation

$$y^2 + sxy = x^3 + s^2x$$

with a parameter s. This surface is a rational elliptic surface with a singular fiber of type I_1^* over the point given by $s = 0$ and a singular fiber of type I_4 over the point given by $s = \infty$ (Lang [Lan94, §2]). We consider the base change of $\psi : R \longrightarrow \mathbb{P}^1$ by $s = t^2$. Then, we have the Weierstrass model defined by

$$y^2 + txy + ty = x^3 + x^2 \tag{7.1}$$

(see Lang [Lan94, §2]). We consider the relatively minimal nonsingular complete model of this elliptic surface :

$$f : \tilde{R} \longrightarrow \mathbb{P}^1. \tag{7.2}$$

The rational elliptic surface $f : \tilde{R} \to \mathbb{P}^1$ has a singular fiber of type III over the point given by $t = 0$ and a singular fiber of type I_8 over the point given by $t = \infty$.

On the singular elliptic surface (7.1), we denote by F_0 the fiber over the point defined by $t = 0$, and by E_0 the fiber over the point defined by $t = \infty$. Both F_0 and E_0 are irreducible, and on each F_0 and E_0, the surface (7.1) has only one singular point P_0 and P_∞ respectively. The surface $\tilde{R}$ is a surface obtained by the minimal resolution of singularities of (7.1). We use the same symbol for the proper transforms of curves on $\tilde{R}$. The blowing-up at the singular point P_0 gives one exceptional curve F_1, and the surface is nonsingular along F_0 and F_1. The two curves F_1 and F_0 make a singular fiber of type III of the elliptic surface $f : \tilde{R} \longrightarrow \mathbb{P}^1$. On the other hand, the blowing-up at the singular point P_∞ gives two exceptional curves E_1, E_2, and the surface is nonsingular along E_0 and has a unique singular point P_1 which is the intersection of E_1 and E_2. The blowing-up at the singular point P_1 gives two exceptional curves E_3 and E_4. The curves E_3 and E_4 meet at one point P_2 which is a singular point of the obtained surface. The blowing-up at the singular point P_2 again gives two exceptional curves E_5 and E_6. The curves E_5 and E_6 meet at one point P_3 which is a singular point of the obtained surface. Finally the blowing-up at the singular point P_3 gives an exceptional curve E_7 and the obtained surface is nonsingular over these curves. The cycle

$$E_0 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$$

forms a singular fiber of type I_8 given in Figure 10.

The elliptic surface $f : \tilde{R} \longrightarrow \mathbb{P}^1$ has four sections s_i $(i = 0, 1, 2, 3)$ given as follows:

s_0 : the zero section.
$s_1 : x = y = 0.$
$s_2 : x = t,\ y = 0.$
$s_3 : x = 0,\ y = t.$

Also we consider the following two 2-sections b_1, b_2 defined by:

$$b_1 : x + y = 0,\ x^2 + tx + t = 0.$$
$$b_2 : x + y + tx + t = 0,\ x^2 + tx + t = 0.$$

The configuration of singular fibers, three sections and two 2-sections is given in the following Figure 10:

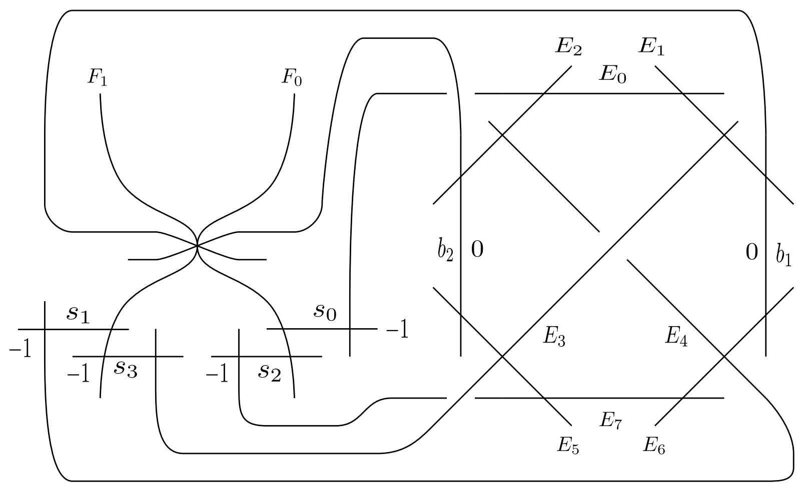


FIGURE 10

Now, we consider a rational vector field on $\tilde{R}$ defined by

$$D = D_a = t(at+1)\frac{\partial}{\partial t} + (x+1)\frac{\partial}{\partial x},\ a \neq 0 \in k.$$

Then, we have $D^2 = D$, that is, D is 2-closed. However D has an isolated singularity at the point P which is the singular point of the fiber of type III, that is, the intersection point of two curves F_0 and F_1 (note that (x, t) is not a local parameter along F_0). To resolve this singularity, we first blow up at P. Denote by F_2 the exceptional curve. We denote the proper transforms of F_0 and F_1 by the same symbols. Then the induced vector field has three isolated singularities one of which is the intersection of three curves and other two points lie on the curve F_2. Blow up at these three points. Let Y be the obtained surface and $\psi : Y \to \tilde{R}$ the successive blowing-ups. We denote the induced vector field by the same symbol D, and the four exceptional curves by F_2, F_3, F_4, F_5. Then we have the following Figure 11.

In the Figure 11 we give the self-intersection numbers of the curves except the curves with the self-intersection number -2. Also the thick lines are integral curves with respect to D. Denote by F_a the fiber over the point defined by $at = 1$. Then F_a is integral with respect to D. Now, according to the above blowing-ups, we see the following lemmas.

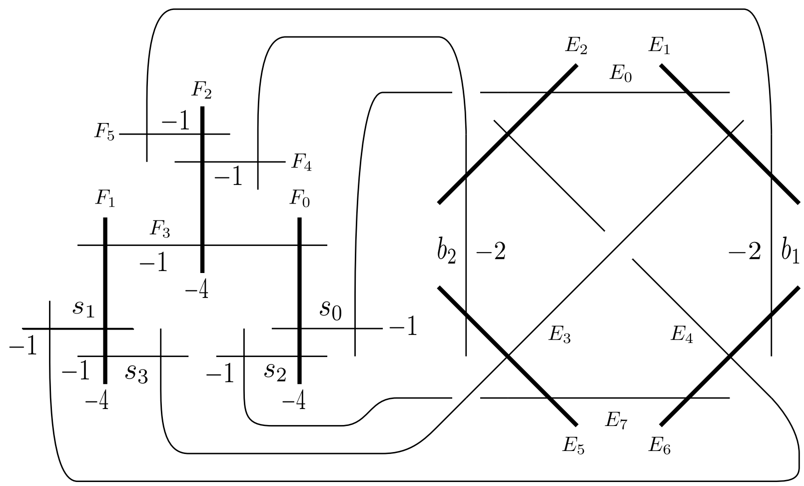


FIGURE 11

LEMMA 7.1. (i) *The divisorial part* (D) *of the vector field* D *on* Y *is given by*

$$-(F_0 + F_1 + F_2 + 2F_3 + E_1 + E_2 + E_5 + E_6).$$

(ii) *The integral curves in Figure* 11 *are*

$$F_0, F_1, F_2, E_1, E_2, E_5, E_6.$$

LEMMA 7.2. (i) $(D)^2 = -12$.
(ii) *The canonical divisor* K_Y *of* Y *is given by*

$$K_Y = -(F_0 + F_1 + F_2 + 2F_3).$$

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, D is divisorial and Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 12. In the Figure 12 we give the self-intersection numbers of the curves except the curves with the self-intersection number -2.

Let X_a be the surface obtained by contracting four exceptional curves in Figure 12 (Recall that the vector field D contains a parameter a). Then we have the following configuration of (-2)-curves in Figure 13.

The dual graph of the sixteen (-2)-curves in Figure 13 is nothing but the one given in Figure 14. Note that any maximal parabolic subdiagram of this diagram is of type $\tilde{D}_5 \oplus \tilde{A}_3$, $\tilde{D}_6 \oplus \tilde{A}_1 \oplus \tilde{A}_1$ or $\tilde{E}_6 \oplus \tilde{A}_2$.

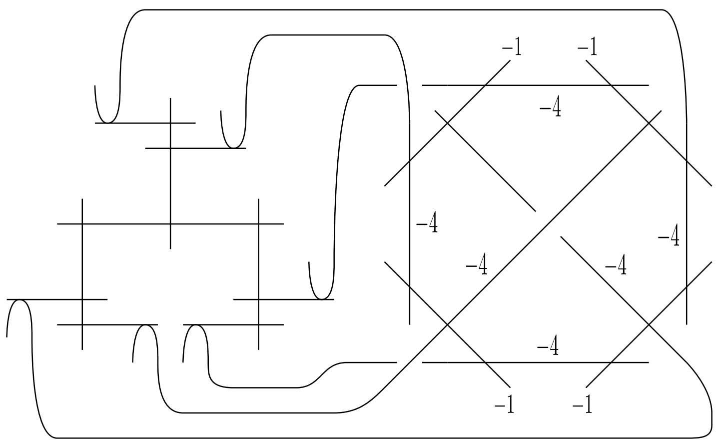


FIGURE 12

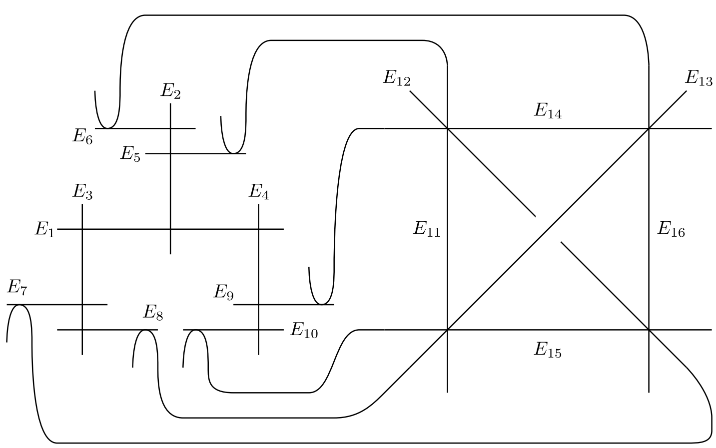


FIGURE 13

THEOREM 7.3. *The surfaces* $\{X_a\}$ *form a non-isotrivial* 1*-dimensional family of classical Enriques surfaces with the dual graph given in Figure* 14*. The automorphism group* $\mathrm{Aut}(X_a)$ *is isomorphic to* $\mathfrak{S}_4$*.*

PROOF. By using Lemmas 7.1 and 7.2 and the same argument as in the proof of Theorem 1, X_a is an Enriques surface. Since X_a has a quasi-elliptic fibration defined by $|2(E_5+E_{11})| = |2(E_6+E_{16})|$ with two double fibers, X_a is classical (Proposition 2.5). Note that the image of F_a is a

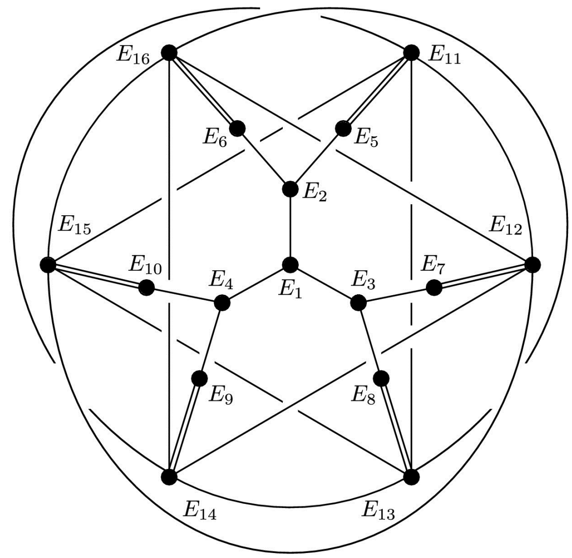


FIGURE 14

double fiber of an elliptic fibration with singular fibers of type $(2\mathrm{I}_1^*, \mathrm{I}_4)$. Since F_a varies, this family is non-isotrivial. By the same proof as that of Theorem 5.8, X_a contains exactly 16 (-2)-curves whose dual graph given in Figure 14. The quasi-elliptic fibration defined by $|2(E_5+E_{11})|$ has five 2-sections $E_2, E_{12}, E_{13}, E_{14}, E_{15}$. Each of these 2-sections meets another (-2)-curves at three different points, and hence they are fixed by any numerically trivial automorphisms. Therefore, by the same proof as that of Lemma 5.14, the natural map $\rho_n : \mathrm{Aut}(X_a) \to \mathrm{O}(\mathrm{Num}(X_a))$ is injective. Note that the automorphism group of the dual graph is isomorphic to the symmetric group $\mathfrak{S}_4$. By considering the actions of the Mordell-Weil groups of the Jacobian fibrations of genus one fibrations on X_a, we have proved that $\mathrm{Aut}(X_a) \cong \mathfrak{S}_4$. $\square$

On X_a, there are three types of genus one fibrations: three elliptic fibrations with singular fibers of type $(2\mathrm{I}_1^*, \mathrm{I}_4)$, three quasi-elliptic fibrations with singular fibers of type $(\mathrm{I}_2^*, 2\mathrm{III}, 2\mathrm{III})$ and eight elliptic fibrations with singular fibers of type $(\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$.

8. Enriques surfaces of Type $\tilde{E}_8$

In this section we give constructions of supersingular and classical Enriques surfaces with the following dual graph given of all (-2)-curves in Figure 15.

8.1. Supersingular case. Let (x, y) be an affine coordinate of $\mathbb{A}^2 \subset \mathbb{P}^2$. Consider a rational vector field D defined by

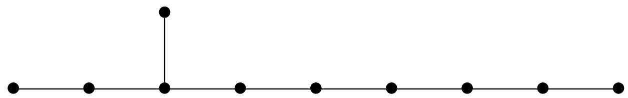


FIGURE 15

$$D = \frac{1}{x^5}\left((xy^6 + x^3)\frac{\partial}{\partial x} + (x^6 + y^7 + x^2 y)\frac{\partial}{\partial y}\right). \tag{8.1}$$

Then $D^2 = 0$, that is, D is 2-closed. Note that D has a pole of order 5 along the line ℓ defined by $x = 0$ and this line is integral with respect to D. We see that D has a unique isolated singularity $(x, y) = (0, 0)$. First blow up at the point $(0, 0)$. Then we see that the exceptional curve is not integral and the induced vector field has a pole of order 2 along the exceptional curve. Moreover the induced vector field has a unique isolated singularity at the intersection of the proper transform of ℓ and the exceptional curve. Then continue this process until the induced vector field has no isolated singularities. The final configuration of curves is given in Figure 16. Here F_0 is the proper transform of ℓ and the suffix i of the exceptional curve E_i corresponds to the order of successive blowing-ups.

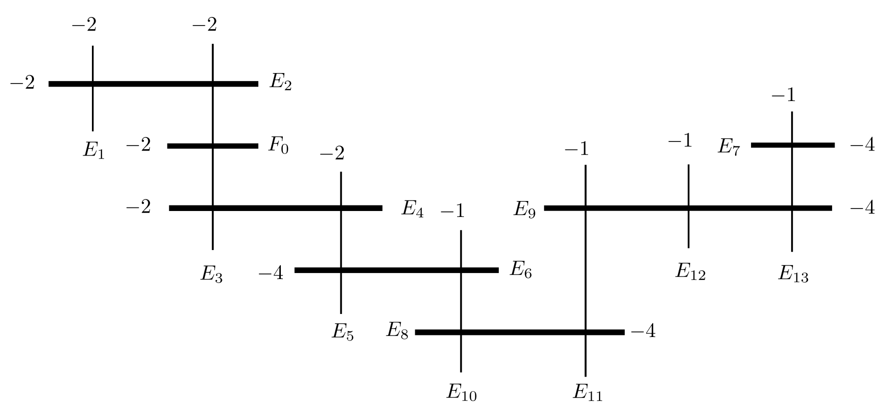


FIGURE 16

We denote by Y the surface obtained by this process. Also we denote by the same symbol D the induced vector field on Y. By direct calculations, we have the following lemmas.

LEMMA 8.1. (i) *The integral curves with respect to D in Figure* 16 *are all horizontal curves* (*thick lines*).

(ii) $(D) = -(5F_0 + 2E_1 + 6E_2 + 8E_3 + 7E_4 + 4E_5 + 3E_6 + 2E_7 + 4E_8 + 5E_9 + 6E_{10} + 8E_{11} + 4E_{12} + 6E_{13})$.

LEMMA 8.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by* $K_Y = -(3F_0 + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 4E_8 + 5E_9 + 6E_{10} + 8E_{11} + 4E_{12} + 6E_{13})$.

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, D is divisorial and hence Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 17:

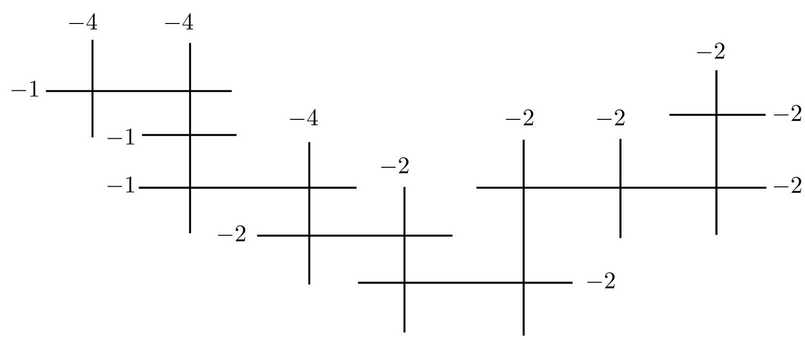


FIGURE 17

By contracting three exceptional curves, we get a new exceptional curve which is the image of the (-4)-curve meeting three exceptional curves. Let X be the surface obtained by contracting the exceptional curve. The surface X contains 10 (-2)-curves whose dual graph is given by Figure 15. Note that this diagram contains a unique maximal parabolic subdiagram which is of type $\tilde{E}_8$. The pencil of lines in $\mathbb{P}^2$ through $(x, y) = (0, 0)$ induces a quasi-elliptic fibration on X with a double fiber of type II^*.

THEOREM 8.3. *The surface X is a supersingular Enriques surfaces with the dual graph given in Figure* 15.

PROOF. By using Lemmas 8.1, 8.2 and the same arguments as in the proofs of Theorems 5.7, 5.8, X is an Enriques surface with the dual graph given in Figure 15. Note that the normalization of the canonical cover of X is obtained from Y by contracting the divisor $F_0 + E_2 + E_3 + E_4$, and hence it has a rational double point of type D_4. It follows from Lemma 12.6 (Section 12) that X is supersingular. □

THEOREM 8.4. $\mathrm{Aut}(X) = \mathrm{Aut}_{nt}(X) = \mathrm{Aut}_{ct}(X) \cong \mathbb{Z}/11\mathbb{Z}$.

PROOF. First note that the dual graph has no symmetries and hence $\mathrm{Aut}(X) = \mathrm{Aut}_{nt}(X)$. Since X is supersingular, $\mathrm{Aut}_{ct}(X) = \mathrm{Aut}_{nt}(X)$.

Now we consider the vector field (8.1), and we set $u = x^2$, $v = y^2$, $z = x^7 + xy^7 + x^3y$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = u^7 + uv^7 + u^3v$. Therefore, the quotient surface $\mathbb{P}^2$ by D is birationally isomorphic to the surface defined by $z^2 = u^7 + uv^7 + u^3v$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = 1/u, \quad y = z/u^4, \quad t = v/u.$$

Then, the equation becomes $y^2 + tx^4 + x + t^7 = 0$. This equation gives a nonsingular affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + x + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type II* exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2)-curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1.

Therefore, together with the equation $y^2 = tx^4 + x + t^7$, we have an identity

$$\begin{aligned}&e_1(t,x)^2(tx^4 + x + t^7) + e_2(t,x)^2\\ &= (c_1t + c_2)(d_1(t)x + d_2(t))^4 + (d_1(t)x + d_2(t)) + (c_1t + c_2)^7.\end{aligned}$$

Using Lemma 4.3 and taking the coefficients of x, we have $e_1(t,x)^2 + d_1(t) = 0$. Therefore, $e_1(t,x)$ is a polynomial of t, i.e., we can put $e_1(t,x) = e_1(t)$, and $d_1(t) = e_1(t)^2$. Taking the coefficients of t, we have $e_1(t)^2x^4 + e_1(t)^2t^6 + c_1(d_1(t)x + d_2(t))^4 + d_2(t)_{odd}/t + c_1(c_1t + c_2)^6 = 0$. Here, $d_2(t)_{odd}$ is the odd terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1d_1(t)^4 = c_1e_1(t)^8$. Since we have $e_1(t) \not\equiv 0$, we have $e_1(t)^6 = 1/c_1$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = 1/c_1$. Therefore, we have an identity $e_1^2t^6 + c_1d_2(t)^4 + d_2(t)_{odd}/t + c_1(c_1t + c_2)^6 = 0$ with $e_1^6 = 1/c_1$. Let $d_2(t)$ be of degree m. If $m \geq 2$, then we have $\deg d_2(t)^4 \geq 8$ and we cannot kill the highest term of $d_2(t)^4$ in the equation. Therefore, we can put $d_2(t) = b_0 + b_1t$ $(b_0, b_1 \in k)$ and we have an identity

$$(e_1^2 + c_1^7)t^6 + (c_1b_1^4 + c_1^5c_2^2)t^4 + c_1^3c_2^4t^2 + (c_1b_0^4 + b_1 + c_1c_2^6) = 0.$$

Therefore, we have $e_1^2 + c_1^7 = 0$, $c_1b_1^4 + c_1^5c_2^2 = 0$, $c_1^3c_2^4 = 0$, $c_1b_0^4 + b_1 + c_1c_2^6 = 0$ with $e_1^6 = 1/c_1$. Since $c_1 \neq 0$, we have $c_2 = b_1 = b_0 = 0$ and $c_1 = \zeta$, $e_1 = \zeta^9$, $d_1 = \zeta^7$ with $\zeta^{11} = 1$. Putting these data into the original equation, we have $e_2(t,x) = 0$. These σ's are really automorphisms of X and we conclude $\mathrm{Aut}(X) \cong \mathbb{Z}/11\mathbb{Z}$ (see Subsection 4.2). □

REMARK 8.5. The automorphism σ is a new example of a cohomologically trivial automorphism.

8.2. Classical case. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a nonsingular quadric and let $((u_0, u_1), (v_0, v_1))$ be a homogeneous coordinate of Q. Let $x = u_0/u_1,\ x' = u_1/u_0,\ y = v_0/v_1, y' = v_1/v_0$. Consider a rational vector field D defined by

$$D = \frac{1}{x^3y^2}\left(x^4y^2\frac{\partial}{\partial x} + (x^2 + ax^4y^4 + y^4)\frac{\partial}{\partial y}\right), \quad a \neq 0 \in k. \tag{8.2}$$

Then $D^2 = D$, that is, D is 2-closed. Note that D has a pole of order 3 along the divisor defined by $x = 0$, a pole of order 1 along the divisor defined by $x = \infty$ and a pole of order 2 along the divisor defined by $y = 0$. Moreover D has two isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. As in the case of supersingular Enriques surfaces of type E_8, we blow up the points of isolated singularities of D and those of associated vector field, and finally get a vector field D, denoted by the same symbol, without isolated singularities. The configuration of curves is given in Figure 18. Here F_0, E_1, or E_2 is the proper transform of the curve defined by $y = 0$, $x = 0$, or $x = \infty$, respectively, and the suffix i of the other exceptional curve E_i corresponds to the order of successive blowing-ups. We denote by Y the surface obtained by these successive blowing-ups.

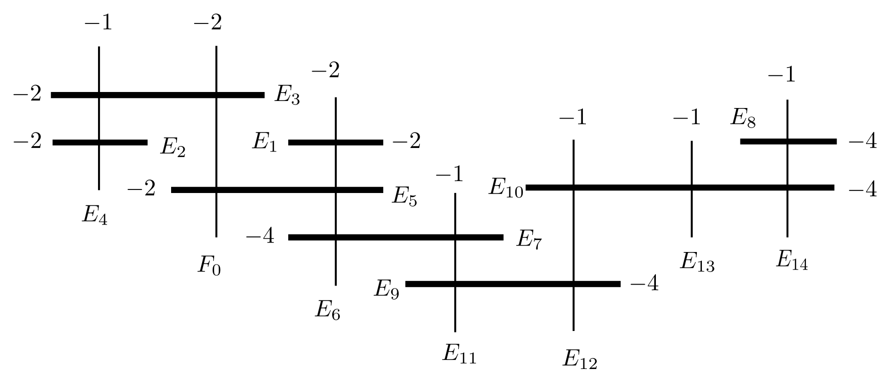


FIGURE 18

A direct calculation shows the following two Lemmas 8.6 and 8.7.

LEMMA 8.6. (i) *The integral curves with respect to D in Figure* 18 *are all horizontal curves* (*thick lines*).

(ii) $(D) = -(2F_0 + 3E_1 + E_2 + 2E_3 + 4E_5 + 4E_6 + 3E_7 + 2E_8 + 4E_9 + 5E_{10} + 6E_{11} + 8E_{12} + 4E_{13} + 6E_{14})$.

LEMMA 8.7. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by* $K_Y = -(2F_0 + 2E_1 + E_3 + 3E_5 + 4E_6 + 3E_7 + 2E_8 + 4E_9 + 5E_{10} + 6E_{11} + 8E_{12} + 4E_{13} + 6E_{14})$.

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in the below Figure 19:

Let X_a be the surface obtained by contracting four exceptional curves in Figure 19 (Recall that the vector field D contains one parameter a (see (8.2))). Then X_a contains 10 (-2)-curves whose

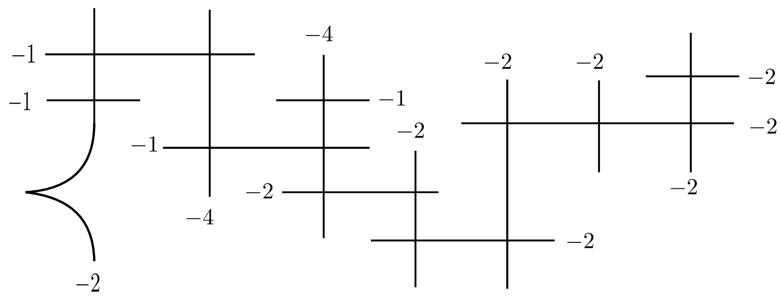


FIGURE 19

dual graph is given by Figure 15. Recall that this diagram contains a unique maximal parabolic subdiagram which is of type $\tilde{E}_8$. The first projection from Q to $\mathbb{P}^1$ induces a quasi-elliptic fibration on X_a with two double fibers of type II^* and of type II.

THEOREM 8.8. *The surfaces* $\{X_a\}$ *form a* 1*-dimensional family of classical Enriques surfaces with the dual graph given in Figure* 15*.*

PROOF. By using Lemmas 8.6, 8.7 and the same arguments as in the proofs of Theorems 5.7, 5.8, X_a is an Enriques surface with the dual graph given in Figure 15. Since $X_{a,b}$ has a genus one fibration with two double fibers of type $\mathrm{II}^*, \mathrm{II}$, X_a is classical (Proposition 2.5). □

THEOREM 8.9. *The automorphism group* $\mathrm{Aut}(X_a)$ *is trivial.*

PROOF. We consider the vector field (8.2), and we set $u = x^2$, $v = y^2$, $z = x^3 + ax^5y^4 + xy^4 + x^4y^3$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = u^3 + a^2u^5v^4 + uv^4 + u^4v^3$ with $a \neq 0$. Therefore, the quotient surface $\mathbb{P}^1 \times \mathbb{P}^1$ by D is birationally isomorphic to the surface defined by $z^2 = u^3 + a^2u^5v^4 + uv^4 + u^4v^3$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = 1/a^{\frac{3}{4}}uv, \quad y = z/a^{\frac{7}{4}}u^4v^2, \quad t = 1/\sqrt{a}u$$

and we replace $1/a^{\frac{5}{4}}$ by a for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + at^3x + t^3 + t^7 = 0$. This equation gives a normal affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + at^3x + t^3 + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type II^* exists over the point defined by $t = \infty$. Since σ preserves the dual graph of (-2)-curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1. Moreover, this quasi-elliptic surface has a singular fiber over the point defined by $t = 0$, σ preserves also the singular fiber. Therefore, we know $c_2 = 0$ and we have $\sigma^*(t) = c_1t$.

Therefore, together with the equation $y^2 + tx^4 + at^3x + t^3 + t^7 = 0$, we have an identity

$$\begin{aligned}&e_1(t,x)^2(tx^4 + at^3x + t^3 + t^7) + e_2(t,x)^2\\ &= c_1t(d_1(t)x + d_2(t))^4 + a(c_1t)^3(d_1(t)x + d_2(t))\\ &\quad +(c_1t)^3 + (c_1t)^7.\end{aligned}$$

Differentiating both sides by x, we have $ae_1(t,x)^2t^3 + ac_1^3d_1(t)t^3 = 0$, that is, $e_1(t,x)^2 = c_1^3d_1(t)$. Therefore, $e_1(t,x)$ is a polynomial of t, i.e., we can put $e_1(t,x) = e_1(t)$, and $d_1(t) = c_1^{-3}e_1(t)^2$. Using Lemma 4.3 and taking the coefficients of t, we have $e_1(t)^2x^4 + e_1(t)^2t^2 + e_1(t)^2t^6 + c_1(c_1^{-3}e_1(t)^2x + d_2(t))^4 + ac_1^3d_2(t)_{even}t^2 + c_1^3t^2 + c_1^7t^6 = 0$. Here, $d_2(t)_{even}$ is the even terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1^{-11}e_1(t)^8$. Since we have $e_1(t) \not\equiv 0$, we have $e_1(t)^6 = c_1^{11}$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = c_1^{11}$ and the equation becomes $e_1^2t^2 + e_1^2t^6 + c_1d_2(t)^4 + ac_1^3d_2(t)_{even}t^2 + c_1^3t^2 + c_1^7t^6 = 0$. If the degree of $d_2(t)$ is greater than or equal to 2, then the highest term of $d_2(t)^4$ cannot be killed in the equation. Therefore, we can put $d_2(t) = b_0 + b_1t$ $(b_0, b_1 \in k)$ and we have an identity

$$e_1^2t^2 + e_1^2t^6 + c_1(b_0 + b_1t)^4 + ac_1^3b_0t^2 + c_1^3t^2 + c_1^7t^6 = 0.$$

Therefore, we have $e_1^2 = c_1^7$, $c_1b_1^4 = 0$, $e_1^2 + ac_1^3b_0 + c_1^3 = 0$ and $c_1b_0^4 = 0$. Therefore, considering $e_1^6 = c_1^{11}$, we have $b_0 = b_1 = 0$, or $c_1 = e_1 = 1$. Therefore, we have $d_1(t) = 1$, $d_2(t) = 0$, $e_1(t,x) = 1$ and $e_2(t,x) = 0$. Hence, $\mathrm{Aut}(X_a)$ is trivial. □

9. Enriques surfaces of type $\tilde{E}_7 + \tilde{A}_1$

9.1. Classical case with a double fiber of type III*. In this subsection we give a construction of an Enriques surface with the following dual graph of all (-2)-curves given in Figure 20.

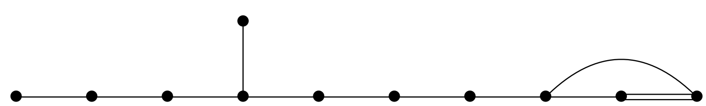


FIGURE 20

Let $(X_0, X_1, X_2) \in \mathbb{P}^2$ and $(S,T) \in \mathbb{P}^1$ be homogeneous coordinates. Consider the surface R defined by

$$S(aX_0^2 + bX_2^2) + T(X_1^2 + aX_1X_2 + bX_0X_2) = 0 \ (a, b \in k,\ a \neq 0, b \neq 0). \tag{9.1}$$

Note that the projection to $\mathbb{P}^1$ defines a fiber space $\pi : R \to \mathbb{P}^1$ whose general fiber is a nonsingular conic. Let E_1 be the fiber over the point $(S,T) = (0,1)$ which is nonsingular. The fiber over the point $(S,T) = (1,0)$ is a double line denoted by $2E_2$ and the fiber over the point (b^2, a^3) is a union of two lines denoted by E_3, E_4. The line defined by $X_2 = 0$ is a 2-section of the fiber space which is denoted by F_0. The surface R has two rational double points $Q_i = ((\alpha, \beta_i, 1), (1,0))$ $(i = 1,2)$ of type A_1, where $\alpha = \sqrt{b/a}$ and β_i is a root of the equation $y^2 + ay + \sqrt{b^3/a} = 0$.

Let $(x = X_0/X_2,\ y = X_1/X_2, s = S/T)$ be an affine coordinate. Define

$$D = \frac{1}{s}\left(a(s^2+c)\frac{\partial}{\partial x} + (as^2x^2+bc)\frac{\partial}{\partial y}\right) \quad (b \neq a^2c) \tag{9.2}$$

where c is a root of the equation of $t^2 + (b/a)t + 1 = 0$. Then $D^2 = aD$, that is, D is 2-closed. A direct calculation shows that D has two isolated singularities at the intersection points of F_0 and E_1, E_2. As in the case of supersingular Enriques surfaces of type E_8, we blow up the two rational double points and the points of isolated singularities of D successively, and finally get a vector field, denoted by the same symbol D, without isolated singularities. The configuration of curves is given in Figure 21. Here the suffix i of the exceptional curve E_i corresponds to the order of successive blowing-ups.

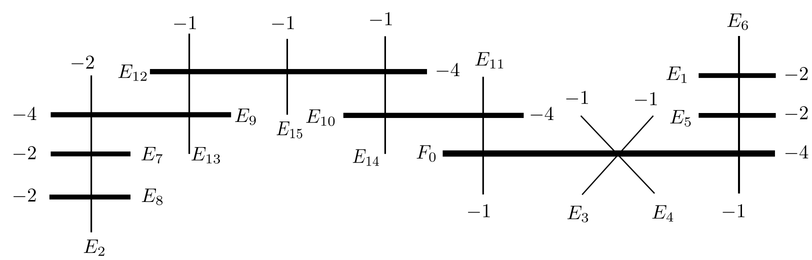


FIGURE 21

Now we denote by Y the surface obtained by successive blowing-ups. By direct calculations, we have the following lemmas.

LEMMA 9.1. (i) *The integral curves with respect to D in Figure* 21 *are all horizontal curves* (*thick lines*).

(ii) $(D) = -(F_0 + E_1 + 2E_2 + E_5 + 2E_7 + 2E_8 + 2E_9 + 2E_{10} + 2E_{11} + 3E_{12} + 4E_{13} + 4E_{14} + 2E_{15})$.

LEMMA 9.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by* $K_Y = -(F_0 + 2E_2 + E_7 + E_8 + 2E_9 + 2E_{10} + 2E_{11} + 3E_{12} + 4E_{13} + 4E_{14} + 2E_{15})$.

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, D is divisorial and hence Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 22:

Let $X_{a,b}$ be the surface obtained by contracting four exceptional curves. The surface $X_{a,b}$ contains 11 (-2)-curves whose dual graph is given by Figure 20. Note that any maximal parabolic

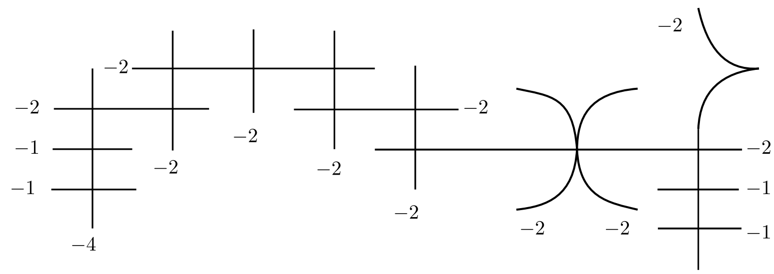


FIGURE 22

subdiagram of this diagram is of type $\tilde{E}_7 \oplus \tilde{A}_1$ or $\tilde{E}_8$. On the surface $X_{a,b}$, there exist a quasi-elliptic fibration with singular fibers of type $(2\mathrm{III}^*, \mathrm{III})$ induced from the fiber space $\pi : R \to \mathbb{P}^1$ and two quasi-elliptic fibrations with a singular fiber of type II^*.

THEOREM 9.3. *The surfaces* $\{X_{a,b}\}$ *form a* 2*-dimensional family of classical Enriques surfaces with the dual graph given in Figure* 20*. It contains an at least* 1*-dimensional, non-isotrivial family. The automorphism group* $\mathrm{Aut}(X_{a,b})$ *is* $\mathbb{Z}/2\mathbb{Z}$ *which is not numerically trivial.*

PROOF. By using Lemmas 9.1, 9.2 and the same arguments as in the proofs of Theorems 5.7, 5.8, $X_{a,b}$ is an Enriques surface with the dual graph given in Figure 20. Let p_1 be the genus one fibration with singular fibers $(2\mathrm{III}^*, \mathrm{III})$. By construction, p_1 has two double fibers (see Figure 22). Hence $X_{a,b}$ is classical (Proposition 2.5). In the next subsection 9.2, we give classical Enriques surfaces with double fibers of type III^* and III which are specializations of $\{X_{a,b}\}$. It follows from Matsusaka and Mumford [MM64, Theorem 1] that the family $\{X_{a,b}\}$ contains an at least 1-dimensional non-isotrivial family.

Next we determine the automorphism group. First we show that there are no numerically trivial automorphisms. Consider a genus one fibration p_2 with a singular fiber of type II^*. By using the classification of conductrices (Ekedahl and Shepherd-Barron [ES04], see also Table 5 in the later Section 12), p_2 is quasi-elliptic and the fiber of type II^* is simple. The simple component of the fiber of type III^* not meeting the special 2-section is the curve of cusps of the fibration p_2. Let C_1, C_2 be the double fibers of p_2 both of which are rational curves with a cusp. Let g be any numerically trivial automorphism. First assume that g is of order 2. Note that g preserves the double fiber C of type II of the fibration p_1 and g fixes two points on C which are the cusp of C and the intersection of C and the curve of cusps of p_1. Hence g fixes C pointwisely. Since C is a 2-section of p_2, C_i is preserved by g. Thus g fixes three points on C_i, which are the cusp of C_i and the intersection points of C_i with the two double fibers of p_1, and hence g fixes C_1 and C_2 pointwisely. Therefore g fixes at least three points on a general fiber F of p_1 which are its cusp and the intersection points with C_1 and C_2, and hence g fixes F pointwisely. Hence g is identity, that is, there are no numerically trivial automorphisms of even order. In the case where the order of g is odd, obviously, g preserves each C_i, and hence the above argument works well. Therefore, there are no numerically trivial automorphisms of $X_{a,b}$. Obviously the symmetry group of the dual graph of (-2)-curves is $\mathbb{Z}/2\mathbb{Z}$ (see Figure 20). By considering the action of the Mordell-Weil group of the Jacobian fibration of p_1, we have $\mathrm{Aut}(X_{a,b}) \cong \mathbb{Z}/2\mathbb{Z}$. □

9.2. Classical case with double fibers of type III* and of type III. In this subsection we give a construction of classical Enriques surfaces with the following dual graph of all (-2)-curves given in Figure 23.

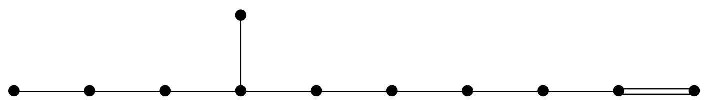


FIGURE 23

In the previous equations (9.1), (9.2), we set $b = 0$. Then $c = 1$ and the surface is defined by

(9.3) $$aSX_0^2 + T(X_1^2 + aX_1X_2) = 0 \ (a \in k,\ a \neq 0)$$

The fiber over the point $(S, T) = (0, 1)$ is a union of two lines, denoted by E_1, E_2, defined by $X_1(X_1 + aX_2) = 0$. The fiber over the point $(S, T) = (1, 0)$ is a double line denoted by $2E_3$. The line defined by $X_2 = 0$ is a 2-section of the fiber space which is denoted by F_0. The surface R has two rational double points $Q_1 = ((0, 0, 1), (1, 0)), Q_2 = ((0, a, 1), (1, 0))$ of type A_1.

Let $(x = X_0/X_2,\ y = X_1/X_2, s = S/T)$ be an affine coordinate. Define

(9.4) $$D = \frac{1}{s}\left((s^2 + 1)\frac{\partial}{\partial x} + s^2x^2\frac{\partial}{\partial y}\right).$$

Then $D^2 = D$, that is, D is 2-closed. A direct calculation shows that D has two isolated singularities at the intersection points of the 2-section F_0 and two fibers over the points $(S, T) = (1, 0), (0, 1)$. As in the previous case, we blow up the two rational double points and the points of isolated singularities of D successively, and finally get a vector field D, denoted by the same symbol, without isolated singularities. The configuration of curves is given in Figure 24.
Here we use the same symbols F_0, E_1, E_2, E_3 for their proper transforms, and the suffix i of the other exceptional curve E_i corresponds to the order of successive blowing-ups. The thick lines are integral curves. We denote by Y the surface obtained by successive blowing-ups. By direct calculations, we have the following lemmas.

LEMMA 9.4. (i) *The integral curves with respect to D in Figure* 24 *are* $F_0, E_1, E_2, E_6, E_7,$ E_8, E_9, E_{11} (*thick lines*).

(ii) $(D) = -(F_0+E_1+E_2+2E_3+2E_6+2E_7+2E_8+2E_9+2E_{10}+3E_{11}+4E_{12}+4E_{13}+2E_{14})$.

LEMMA 9.5. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by* $K_Y = -(F_0 + 2E_3 + E_6 + E_7 + 2E_8 + 2E_9 + 2E_{10} + 3E_{11} + 4E_{12} + 4E_{13} + 2E_{14})$.

(iii) $K_Y \cdot (D) = -4$.

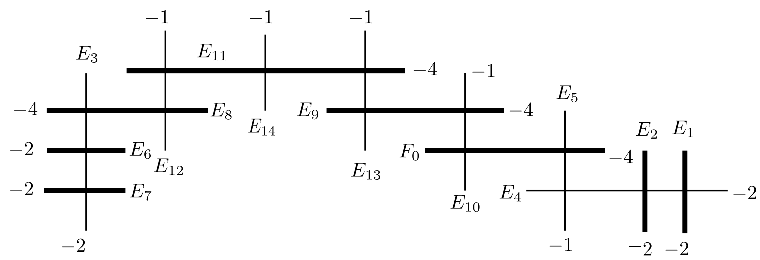


FIGURE 24

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in the below Figure 25:

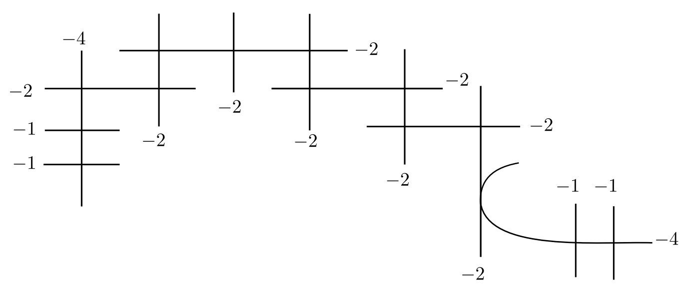


FIGURE 25

Let X_a be the surface obtained by contracting four exceptional curves. The surface X_a contains 11 (-2)-curves whose dual graph is given by Figure 23. Note that any maximal parabolic subdiagram of this diagram is of type $\tilde{E}_7 \oplus \tilde{A}_1$ or $\tilde{E}_8$. The surface X_a has a quasi-elliptic fibration with singular fibers of type $(2\mathrm{III}^*, 2\mathrm{III})$ induced from the fiber space $\pi : R \to \mathbb{P}^1$ and a quasi-elliptic fibration with a singular fiber of type (II^*).

THEOREM 9.6. *The surfaces $\{X_a\}$ form a 1-dimensional family of classical Enriques surfaces with the dual graph given in Figure 23. The automorphism group $\mathrm{Aut}(X_a)$ is $\mathbb{Z}/2\mathbb{Z}$ which is numerically trivial.*

PROOF. By using Lemmas 9.4 and 9.5 and the same arguments as in the proofs of Theorems 1 and 5.8, X_a is an Enriques surface with the dual graph given in Figure 23. Since X_a has a quasi-elliptic fibration with two double fibers, X_a is classical (Proposition 2.5). By the same argument as in the case of Theorem 9.3, we see that $|\mathrm{Aut}_{nt}(X_a)| \leq 2$. Since the dual graph of (-2)-curves on X_a has no symmetries (see Figure 23), we have $\mathrm{Aut}(X_a) = \mathrm{Aut}_{nt}(X_a)$. Let p be the quasi-elliptic fibration with singular fibers of type $(2\mathrm{III}^*, 2\mathrm{III})$. By considering the action of the Mordell-Weil group of the Jacobian fibration of p, we have $\mathrm{Aut}(X_a) \cong \mathbb{Z}/2\mathbb{Z}$. □

9.3. Supersingular case with a double fiber of type III*. In this subsection we give a construction of supersingular Enriques surfaces with the dual graph of all (-2)-curves given in Figure 20.

Let $(X_0, X_1, X_2) \in \mathbb{P}^2$ and $(S, T) \in \mathbb{P}^1$ be homogeneous coordinates. Consider the surface R defined by

$$S(X_0^2 + a^3X_2^2) + T(X_1^2 + X_1X_2 + a^2X_0X_2) = 0 \ (a \in k, \ a \neq 0). \tag{9.5}$$

Note that the projection to $\mathbb{P}^1$ defines a fiber space $\pi : R \to \mathbb{P}^1$ whose general fiber is a nonsingular conic. The fiber over the point $(S, T) = (a^4, 1)$ is a union of two lines denoted by E_1, E_2 and the fiber over the point $(S, T) = (1, 0)$ is a double line denoted by $2E_3$. The line defined by $X_2 = 0$ is a 2-section, denoted by F_0, of the fiber space.

The surface R has two rational double points $Q_i = ((\alpha, \beta_i, 1), (1, 0))$ $(i = 1, 2)$ where $\alpha = \sqrt{a^3}$ and β_i's are roots of the equation $y^2 + y + a^3\sqrt{a} = 0$.

Let $(x = X_0/X_2, \ y = X_1/X_2, s = S/T)$ be an affine coordinate. Define

$$D = (s^2 + a)\frac{\partial}{\partial x} + (x^2 + a^2s^2)\frac{\partial}{\partial y}. \tag{9.6}$$

Then $D^2 = 0$, that is, D is 2-closed. A direct calculation shows that D has an isolated singularity at the intersection point of the 2-section F_0 and the fiber over the point $(S, T) = (1, 0)$. As in the case of the previous section, we blow up the two rational double points and the point of isolated singularity of D successively, and finally get a vector field without isolated singularities. The configuration of curves is given in Figure 26.

Here we use the same symbols F_0, E_1, E_2, E_3 for their proper transforms, and the suffix i of the other exceptional curve E_i corresponds to the order of successive blowing-ups.

We denote by Y the surface obtained by successive blowing-ups. By direct calculations, we have the following lemmas.

LEMMA 9.7. (i) *The integral curves with respect to D in Figure* 26 *are all horizontal curves* (*thick lines*).

(ii) $(D) = -(F_0 + 4E_3 + 3E_4 + 3E_5 + 4E_6 + 2E_7 + 2E_8 + 2E_9 + 2E_{10} + 3E_{11} + 4E_{12} + 4E_{13} + 2E_{14})$.

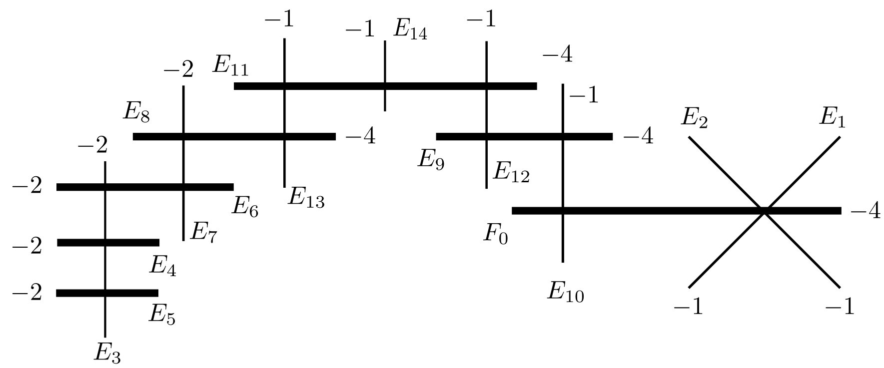


FIGURE 26

LEMMA 9.8. (i) $(D)^2 = -12$.

(ii) *The canonical divisor* K_Y *of* Y *is given by* $K_Y = -(F_0 + 2E_3 + E_4 + E_5 + 2E_6 + 2E_7 + 2E_8 + 2E_9 + 2E_{10} + 3E_{11} + 4E_{12} + 4E_{13} + 2E_{14})$.

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, Y is divisorial and hence Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 27.

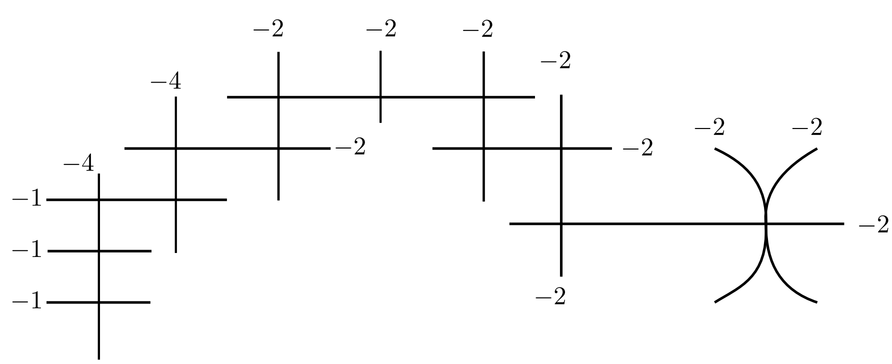


FIGURE 27

Let X_a be the surface obtained by contracting the three exceptional curves and the curve meeting the three exceptional curves. The surface X_a contains 11 (-2)-curves whose dual graph is given by Figure 20. Recall that any maximal parabolic subdiagram of this diagram is of type $\tilde{E}_7 \oplus \tilde{A}_1$ or $\tilde{E}_8$.

THEOREM 9.9. *The surfaces $\{X_a\}$ form a 1-dimensional non-isotrivial family of supersingular Enriques surfaces with the dual graph given in Figure 20.*

PROOF. By using Lemmas 9.7, 9.8 and the same arguments as in the proofs of Theorems 5.7, 5.8, X_a is an Enriques surface with the dual graph given in Figure 20. By construction, the normalization of the canonical cover has a rational double point of type D_4. It now follows from Lemma 12.6 (Section 12) that X_a is supersingular. It follows from the following Theorem 9.10 and Matsusaka and Mumford [MM64, Theorem 1] that the family $\{X_a\}$ is non-isotrivial. □

The surface X_a contains a unique quasi-elliptic fibration with singular fibers of type $(2\mathrm{III}^*, \mathrm{III})$ induced from the fiber space $\pi : R \to \mathbb{P}^1$ and two quasi-elliptic fibrations with a singular fiber of type (II^*).

THEOREM 9.10. *If $a^7 \neq 1$, then the automorphism group $\mathrm{Aut}(X_a)$ is $\mathbb{Z}/2\mathbb{Z}$ which is not numerically trivial. If $a^7 = 1$, then the automorphism group $\mathrm{Aut}(X_a)$ is $\mathbb{Z}/14\mathbb{Z}$ and $\mathrm{Aut}_{nt}(X_a)$ is $\mathbb{Z}/7\mathbb{Z}$.*

PROOF. We consider the vector field (9.6), and we set $T = s^2$, $u = x + as + s^3$ and $v = y + sx^2 + a^2s^3$. Here, $s = (y^2 + y + a^2x)/(x^2 + a^3)$ by (9.5). Then, we have $D(T) = 0$, $D(u) = 0$, $D(v) = 0$ with the equation $v^2 + v = Tu^4 + a^2u + T^7$ with $a \neq 0$ and the quotient surface $\mathbb{P}^2$ by D is birationally isomorphic to the surface defined by $v^2 + v = Tu^4 + a^2u + T^7$, which is birationally isomorphic to our Enriques surface. For the sake of simplicity, we replace a^2 by a. Then , the normal form becomes $v^2 + v = Tu^4 + au + T^7$. To calculate the automorphism group, we consider the change of coordinates with new coordinates x, y, t:

$$T = t + a^4, \quad v = y + a^2x^2 + ax, \quad u = x.$$

Then, the equation becomes $y^2 + y = tx^4 + (t + a^4)^7$ with $a \neq 0$. This equation gives a nonsingular affine surface. Set

$$A = k[t, x, y]/(y^2 + y + tx^4 + (t + a^4)^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2)-curves, σ preserves $2F_\infty$. Therefore, σ preserves the structure of this quasi-elliptic surface. σ has the form in (4.4) in Subsection 4.1. Moreover, this quasi-elliptic surface has a singular fiber over the point defined by $t = 0$ and σ preserves also the singular fiber. Therefore, we have $\sigma^*(t) = c_1t$.

Therefore, together with the equation $y^2 + y + tx^4 + (t + a^4)^7 = 0$, we have an identity

$$\begin{aligned}&e_1(t,x)^2(y + tx^4 + (t + a^4)^7) + e_2(t,x)^2 + (e_1(t,x)y + e_2(t,x))\\ &= c_1t(d_1(t)x + d_2(t))^4 + (c_1t + a^4)^7.\end{aligned}$$

A is a free $k[x, y]$-module, and 1 and y are linearly independent over $k[x, y]$. Taking the coefficient of y, we have $e_1(t, x)^2 + e_1(t, x) = 0$. Since $e_1(t, x) \neq 0$, we have $e_1(t, x) = 1$. Therefore, we have

$$\begin{aligned} &tx^4 + (t + a^4)^7 + e_2(t, x)^2 + e_2(t, x) \\ &= c_1 t(d_1(t)x + d_2(t))^4 + (c_1 t + a^4)^7. \end{aligned}$$

As a polynomial of x, if $e_2(t, x)$ has a term of degree greater than or equal to 3, then $e_2(t, x)^2$ has a term greater than or equal to 6. We cannot kill this term in the equation. By the equation, we know that $e_2(t, x)$ doesn't have terms of x of odd degree. Therefore, we can put $e_2(t, x) = a_0(t) + a_2(t)x^2$ with $a_0(t), a_2(t) \in k[t]$. We take the coefficients of x^4. Then, we have $t + a_2(t)^2 + c_1 t d_1(t)^4 = 0$. Therefore, we have two equations $1 + c_1 d_1(t)^4 = 0$ and $a_2(t)^2 = 0$. Therefore, we have $a_2(t) = 0$ and $d_1(t) = \frac{1}{\sqrt[4]{c_1}}$. The equation becomes $(t + a^4)^7 + a_0(t)^2 + a_0(t) = c_1 t d_2(t)^4 + (c_1 t + a^4)^7$. Put deg $d_2(t) = \ell$. Suppose $\ell \geq 2$. Then, the right-hand-side has an odd term whose degree is equal to $4\ell + 1 \geq 9$. Therefore, the left-hand-side must have an odd term which is of degree $4\ell + 1$. This means deg $a_0(t) = 4\ell + 1$. However, in the equation we cannot kill the term of degree $8\ell + 2$ which comes from $a_0(t)^2$. Therefore, we can put $d_2(t) = b_0 + b_1 t$ with $b_0, b_1 \in k$. Then, the equation becomes

$$(t + a^4)^7 + a_0(t)^2 + a_0(t) = c_1 b_0^4 t + c_1 b_1^4 t^5 + (c_1 t + a^4)^7$$

If deg $a_0(t) \geq 4$, we cannot kill the term of degree greater than or equal to 8 in the equation which comes from $a_0(t)^2$. Therefore, we can put $a_0(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$. Then, we have equations:

$$\begin{aligned} &1 = c_1^7, \quad a^4 + \alpha_3^2 = c_1^6 a^4, \quad a^8 = c_1 b_1^4 + c_1^5 a^8, a^{12} + \alpha_2^2 = c_1^4 a^{12}, \\ &a^{16} + \alpha_3 = c_1^3 a^{16}, \quad a^{20} + \alpha_1^2 + \alpha_2 = c_1^2 a^{20}, \\ &a^{24} + \alpha_1 = c_1 b_0^4 + c_1 a^{24}, \quad a^{28} + \alpha_0^2 + \alpha_0 = a^{28}. \end{aligned}$$

Assume $a^7 \neq 1$. Since $\alpha_3 = (c_1^3 + 1)a^2 = (c_1^3 + 1)a^{16}$, we have $(c_1^3 + 1)a^2(a^7 + 1)^2 = 0$. By $a^7 \neq 1$ and $a \neq 0$, we have $c_1^3 = 1$. Since $1 = c_1^7$, we have $c_1 = 1$. Therefore, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $b_0 = b_1 = 0$, and $\alpha_0 = 1$ or 0. Therefore, we see that σ is given by either $t \mapsto t,\ x \mapsto x,\ y \mapsto y + 1$ or the identity. Hence, we have $\mathrm{Aut}(X_a) \cong \mathbb{Z}/2\mathbb{Z}$ if $a^7 \neq 1$. Now, assume $a^7 = 1$. By $c_1^7 = 1$, c_1 is a seventh root of unity. We denote by ζ a primitive seventh root of unity. Then we have a solution

$$\begin{aligned} &c_1 = \zeta, \alpha_1 = 0, \quad \alpha_2 = (1 + \zeta^2)a^6, \quad \alpha_3 = (1 + \zeta^3)a^2, \\ &b_0 = \frac{(\sqrt[4]{\zeta} + 1)a^6}{\sqrt[4]{\zeta}}, \quad b_1 = \frac{(\sqrt[4]{\zeta^5} + 1)a^2}{\sqrt[4]{\zeta}}. \end{aligned}$$

We have also $\alpha_0 = 1$ or 0. Using these data, we have an automorphism σ which is defined by

$$\begin{aligned} t &\mapsto \zeta t \\ x &\mapsto \frac{1}{\sqrt[4]{\zeta}} x + \frac{(\sqrt[4]{\zeta} + 1)}{\sqrt[4]{\zeta}} a^6 + \frac{(\sqrt[4]{\zeta^5} + 1)}{\sqrt[4]{\zeta}} a^2 t \\ y &\mapsto y + 1 + (1 + \zeta^2)a^6 t^2 + (1 + \zeta^3)a^2 t^3. \end{aligned}$$

This σ is of order 14, and by our argument the automorphism group is generated by σ. This means $\mathrm{Aut}(X_a) \cong \mathbb{Z}/14\mathbb{Z}$ if $a^7 = 1$. By our construction, we have $\mathrm{Aut}_{nt}(X_a) = \mathrm{Aut}_{ct}(X_a) \cong \mathbb{Z}/7\mathbb{Z}$ if $a^7 = 1$.

Finally we show that $\mathbb{Z}/2\mathbb{Z}$ is not numerically trivial. Assume that $g = \sigma^7$ is numerically trivial. Let p_1 be the quasi-elliptic fibration with singular fibers of type $(2\mathrm{III}^*, \mathrm{III})$ and let p_2 be a genus one fibration with singular fiber of type (II^*). By using the classification of conductrices (Ekedahl and Shepherd-Barron [ES04], see also Table 5 in the later Section 12), we see that p_2 is quasi-elliptic and the fiber of type II^* is simple. Note that the simple component E of the singular fiber of type III^* not meeting the special 2-section is the curve of cusps of p_2. Since g preserves the double fiber C of p_2, g fixes two points on C which are the cusp of C and the intersection point of C and E. Thus g fixes C pointwisely. Obviously, g preserves a general fiber F of p_1 and fixes two points on F which are the cusp of F and the intersection with C. Hence g fixes F pointwisely. Thus we obtain $g = 1$ which is a contradiction. □

REMARK 9.11. The automorphism of order 7 is a new example of cohomologically trivial automorphisms.

10. Enriques surfaces of type $\tilde{D}_8$

In this section we give a construction of Enriques surfaces with the following dual graph of all (-2)curves given in Figure 28.

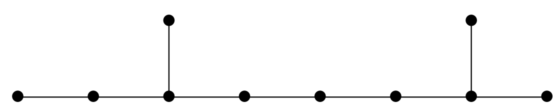


FIGURE 28

10.1. Supersingular case. Let (x, y) be an affine coordinate of $\mathbb{A}^2 \subset \mathbb{P}^2$. Consider a rational vector field D defined by

$$D = D_a = \frac{1}{x^5}\left(x(x^4 + x^2 + y^6)\frac{\partial}{\partial x} + (ax^6 + y(x^4 + x^2 + y^6))\frac{\partial}{\partial y}\right) \tag{10.1}$$

where $a \in k,\ a \neq 0$. Then $D^2 = 0$, that is, D is 2-closed. Note that D has poles of order 5 along the line ℓ defined by $x = 0$, and this line is integral. We see that D has a unique isolated singularity $(x, y) = (0, 0)$. First blow up at the point $(0, 0)$. Then we see that the exceptional curve is not integral and the induced vector field has poles of order 2 along the exceptional curve. Moreover the induced vector field has a unique isolated singularity at the intersection of the proper transform of ℓ and the exceptional curve. Continue this process until the induced vector field has no isolated singularities. The final configuration of curves is given in Figure 29. Here F_0 is the proper transform of ℓ and the suffix i of the exceptional curve E_i corresponds to the order of successive blowing-ups.

We denote by Y the surface obtained by this process. Also we denote by the same symbol D the induced vector field on Y. By direct calculations, we have the following lemmas.

LEMMA 10.1. (i) *The integral curves with respect to D in Figure 29 are all horizontal curves (thick lines).*

(ii) $(D) = -(5F_0 + 2E_1 + 6E_2 + 8E_3 + 7E_4 + 4E_5 + 3E_6 + 2E_7 + 2E_8 + 4E_9 + E_{10} + 2E_{11})$.

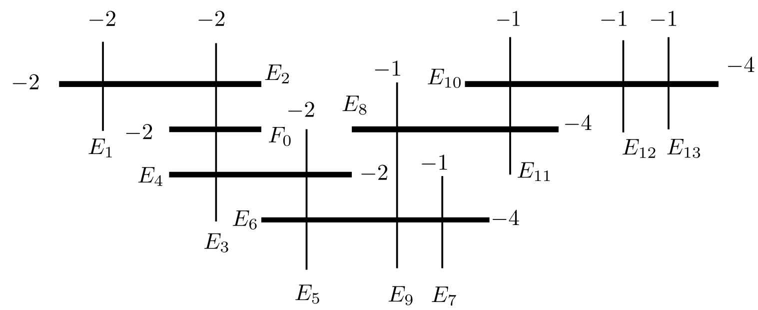


FIGURE 29

LEMMA 10.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor* K_Y *of* Y *is given by* $K_Y = -(3F_0 + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 2E_8 + 4E_9 + E_{10} + 2E_{11})$.

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, D is divisorial and Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 30.

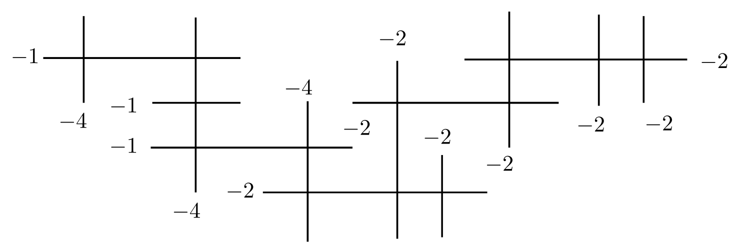


FIGURE 30

By contracting three exceptional curves, we get a new exceptional curve which is the image of the (-4)-curve meeting three exceptional curves. Let X_a be the surface obtained by contracting the new exceptional curve (Recall that the vector field (8.1) contains a parameter a). The surface X_a

contains 10 (-2)-curves whose dual graph is given by Figure 28. Note that any maximal parabolic subdiagram of this diagram is of type $\tilde{D}_8$ or $\tilde{E}_8$. On X_a there exist a quasi-elliptic fibration with singular fibers of type (I_4^*) induced from the pencil of lines in $\mathbb{P}^2$ through $(x, y) = (0, 0)$.

THEOREM 10.3. *The surfaces $\{X_a\}$ form a 1-dimensional family of supersingular Enriques surfaces with the dual graph given in Figure* 28.

PROOF. By using Lemmas 10.1, 10.2 and the same arguments as in the proofs of Theorems 5.7, 5.8, X is an Enriques surface with the dual graph given in Figure 28. By construction, the normalization of the canonical cover has a rational double point of type D_4. Hence X_a is supersingular (Lemma 12.6, Section 12). □

REMARK 10.4. Note that X_a contains exactly three genus one fibrations. Let p_1 be the genus one fibration with a double singular fiber $2F_1$ of type I_4^*, and let p_i $(i = 2, 3)$ be two genus one fibrations with a singular fiber F_i of type II^*. Note that p_1 is quasi-elliptic because its pull back on the canonical cover is a $\mathbb{P}^1$-bundle, and p_2 and p_3 are elliptic because the conductrix is contained in the singular fiber of type II^* (see Lemma 12.2). Note that $F_1 \cdot F_2 = F_1 \cdot F_3 = F_2 \cdot F_3 = 2$. If both F_2 and F_3 are double fibers, then there are no canonical U-pairs on this Enriques surface which is a contradiction (Cossec and Dolgachev [CD89, Theorem 3.4.1]). Hence one of them, for example, F_2 is double and other F_3 is simple. Since there are no automorphisms which change a double fiber and a simple fiber, any automorphism of X_a is cohomologically trivial.

THEOREM 10.5. *The automorphism group* $\mathrm{Aut}(X_a)$ *is a quaternion group* Q_8 *of order* 8 *which is cohomologically trivial.*

PROOF. We consider the vector field (10.1), and we set $u = x^2$, $v = y^2$, $z = ax^7 + x^5y + x^3y + xy^7$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = a^2u^7 + u^5v + u^3v + uv^7$. Therefore, the quotient surface $\mathbb{P}^2$ by D is birationally isomorphic to the surface defined by $z^2 = a^2u^7 + u^5v + u^3v + uv^7$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = 1/u, \quad y = z/u^4, \quad t = v/u$$

and we replace a^2 by a for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + tx^2 + ax + t^7 = 0$. This equation gives a nonsingular affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + tx^2 + ax + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type I_4^* exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2)-curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1.

Therefore, together with the equation $y^2 = tx^4 + tx^2 + ax + t^7$, we have an identity

$$\begin{aligned}
&e_1(t,x)^2(tx^4 + tx^2 + ax + t^7) + e_2(t,x)^2 \\
&= (c_1t + c_2)(d_1(t)x + d_2(t))^4 + (c_1t + c_2)(d_1(t)x + d_2(t))^2 \\
&\quad + a(d_1(t)x + d_2(t)) + (c_1t + c_2)^7.
\end{aligned}$$

Using Lemma 4.3 and taking the coefficients of x, we have $ae_1(t,x)^2 + ad_1(t) = 0$. Therefore, $e_1(t,x)$ is a polynomial of t, i.e., we can put $e_1(t,x) = e_1(t)$, and $d_1(t) = e_1(t)^2$. Taking the coefficients of t, we have $e_1(t)^2x^4 + e_1(t)^2x^2 + e_1(t)^2t^6 + c_1(d_1(t)x + d_2(t))^4 + c_1(d_1(t)x + d_2(t))^2 + ad_2(t)_{odd}/t + c_1(c_1t + c_2)^6 = 0$. Here, $d_2(t)_{odd}$ is the odd terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1d_1(t)^4 = c_1e_1(t)^8$. Since we have $e_1(t) \not\equiv 0$, we have $e_1(t)^6 = 1/c_1$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = 1/c_1$. Considering the coefficients of x^2, we have $e_1^2 = e_1(t)^2 = c_1d_1(t)^2 = c_1e_1^4$. Therefore, $e_1^2 = 1/c_1$. Therefore, we have $c_1 = 1$ and so $e_1 = d_1 = 1$. The equation becomes $t^6 + d_2(t)^4 + d_2(t)^2 + ad_2(t)_{odd}/t + (t + c_2)^6 = 0$. If the degree of $d_2(t)$ is greater than or equal to 2, then the highest term of $d_2(t)^4$ cannot be killed in the equation. Therefore, we can put $d_2(t) = b_0 + b_1t$ $(b_0, b_1 \in k)$ and we have an identity

$$t^6 + (b_0 + b_1t)^4 + (b_0 + b_1t)^2 + ab_1 + (t + c_2)^6 = 0.$$

Therefore, we have $c_2 = b_1^2$, $c_2^2 = b_1$ and $b_0^4 + b_0^2 + ab_1 + c_2^6 = 0$. Therefore, we have either $c_2 = 0$, $b_1 = 0$, $b_0 = 0, 1$, or $c_2 = \omega$, $b_1 = \omega^2$ and $b_0 = \alpha$ is any root of $z^2 + z + \omega\sqrt{a} + 1 = 0$. Here, ω is any cube root of unity. There exist 8 solutions. Putting these data into the original equation, we have $e_2(t, x) = \sqrt{a}$ or $\omega^2x^2 + \omega^2x + \omega^2t^3 + \sqrt{a\alpha} + \sqrt{a}$. These σ's are really automorphisms of X and we conclude $\mathrm{Aut}(X) \cong Q_4$ (see Subsection 4.2). The cohomologically trivialness follows from Remark 10.4. □

REMARK 10.6. The group Q_8 is a new example of cohomologically trivial automorphsims.

10.2. Classical case. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a nonsingular quadric and let $((u_0, u_1), (v_0, v_1))$ be a homogeneous coordinate of Q. Let $x = u_0/u_1,\ x' = u_1/u_0,\ y = v_0/v_1, y' = v_1/v_0$. Consider a rational vector field D defined by

$$D = \frac{1}{xy^2}\left(ax^2y^2\frac{\partial}{\partial x} + (x^4y^4 + by^4 + x^2y^2 + x^2)\frac{\partial}{\partial y}\right) \tag{10.2}$$

where $a, b \in k, a, b \neq 0$. Then $D^2 = aD$, that is, D is 2-closed. Note that D has a pole of order 1 along the divisor defined by $x = 0$, a pole of order 3 along the divisor defined by $x = \infty$ and a pole of order 2 along the divisor defined by $y = 0$. Moreover D has isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. As in the case of supersingular Enriques surfaces of type $\tilde{E}_8$, we blow up the points of isolated singularities of D and those of associated vector fields, and finally get a vector field without isolated singularities. The configuration of curves is given in Figure 31.
Here F_0, E_1, or E_2 is the proper transform of the curve defined by $y = 0$, $x = 0$, or $x = \infty$, respectively.

We denote by Y the surface obtained by the successive blowing-ups. A direct calculation shows the following two lemmas.

LEMMA 10.7. (i) *The integral curves with respect to D in Figure* 31 *are all horizontal curves* (*thick lines*).

(ii) $(D) = -(2F_0 + E_1 + 3E_2 + 2E_3 + 4E_5 + 4E_6 + 3E_7 + 2E_8 + 2E_9 + 4E_{10} + E_{11} + 2E_{12})$.

LEMMA 10.8. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by* $K_Y = -(2F_0 + 2E_2 + E_3 + 3E_5 + 4E_6 + 3E_7 + 2E_8 + 2E_9 + 4E_{10} + E_{11} + 2E_{12})$.

(iii) $K_Y \cdot (D) = -4$.

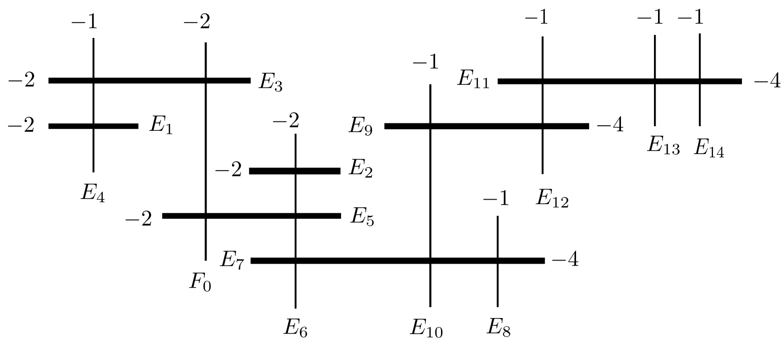


FIGURE 31

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, D is divisorial and hence Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 32.

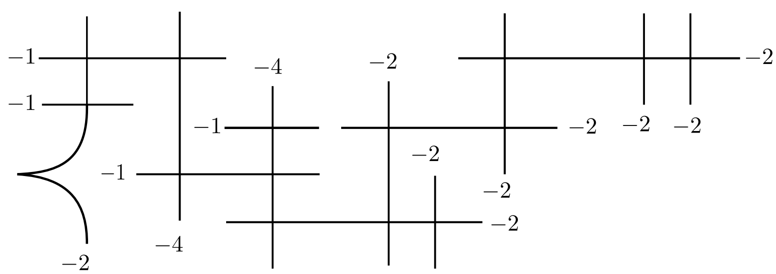


FIGURE 32

Let $X_{a,b}$ be the surface obtained by contracting four exceptional curves in Figure 32 (Recall that the vector field D contains two parameters a, b (see (10.2))). On $X_{a,b}$, there exist 10 (-2)-curves whose dual graph is given by Figure 28. Recall that any maximal parabolic subdiagram of this diagram is of type $\tilde{D}_8$ or $\tilde{E}_8$. On $X_{a,b}$ there exists a quasi-elliptic fibration with singular fibers of type (I_4^*) induced from the first projection from Q to $\mathbb{P}^1$.

THEOREM 10.9. *The surfaces* $\{X_{a,b}\}$ *form a* 2*-dimensional family of classical Enriques surfaces with the dual graph given in Figure* 28*.*

PROOF. By using Lemmas 10.7, 10.8 and the same arguments as in the proofs of Theorems 5.7 and 5.8, $X_{a,b}$ is an Enriques surface with the dual graph given in Figure 28. Since $X_{a,b}$ has a genus one fibration with two double fibers (see Figure 32), $X_{a,b}$ is classical (Proposition 2.5). □

REMARK 10.10. There are two genus one fibrations with a singular fiber of type II^*. As we explained in Remark 10.4, one of them is double and the other is simple. If its only singular fiber is (II^*), then its j-invariant is zero (Lang [Lan94]) and hence all nonsingular fibers are supersingular elliptic curves by Lemma 2.6. This contradicts to the fact that a double fiber of a genus one fibration on a classical Enriques surface is an ordinary elliptic curve or an additive type (Proposition 2.4). Thus this fibration has singular fibers of type $(\mathrm{II}^*, \mathrm{I}_1)$ by Lang [Lan94].

THEOREM 10.11. *The automorphism group* $\mathrm{Aut}(X_{a,b})$ *is* $\mathbb{Z}/2\mathbb{Z}$ *which is numerically trivial.*

PROOF. It follows from Remark 10.10 that $\mathrm{Aut}(X_{a,b}) = \mathrm{Aut}_{nt}(X_{a,b})$. We consider the vector field (10.2), and we set $u = x^2$, $v = y^2$, $z = x^5y^4 + bxy^4 + x^3y^2 + x^3 + ax^2y^3$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = u^5v^4 + b^2uv^4 + u^3v^2 + u^3 + a^2u^2v^3$ with $a, b \neq 0$. Therefore, the quotient surface of $\mathbb{P}^1 \times \mathbb{P}^1$ by D is birationally isomorphic to the surface defined by $z^2 = u^5v^4 + b^2uv^4 + u^3v^2 + u^3 + a^2u^2v^3$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = \sqrt[4]{b}/uv, \quad y = \sqrt[4]{b^3}z/u^4v^2, \quad t = \sqrt{b}/u.$$

and we replace $\frac{1}{\sqrt{b}}$ and $\frac{a^2}{\sqrt[4]{b^5}}$ by a and b, respectively, for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + at^3x^2 + bt^3x + t^3 + t^7 = 0$. This equation gives a normal affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + at^3x^2 + bt^3x + t^3 + t^7 = 0)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type I_4^* exists over the point defined by $t = \infty$. Since σ preserves the dual graph of (-2)-curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1. Moreover, this quasi-elliptic surface has a singular fiber over the point defined by $t = 0$, σ preserves also the singular fiber. Therefore, we know $c_2 = 0$ and we have $\sigma^*(t) = c_1t$.

Therefore, together with the equation $y^2 + tx^4 + at^3x^2 + bt^3x + t^3 + t^7 = 0$, we have an identity

$$\begin{aligned} &e_1(t,x)^2(tx^4 + at^3x^2 + bt^3x + t^3 + t^7) + e_2(t,x)^2 \\ &= c_1t(d_1(t)x + d_2(t))^4 + a(c_1t)^3(d_1(t)x + d_2(t))^2 \\ &\quad + b(c_1t)^3(d_1(t)x + d_2(t)) + (c_1t)^3 + (c_1t)^7. \end{aligned}$$

Differentiate both sides by x, and we have $be_1(t,x)^2t^3 + bc_1^3d_1(t)t^3 = 0$, that is, $e_1(t,x)^2 = c_1^3d_1(t)$. Therefore, $e_1(t,x)$ is a polynomial of t, i.e., we can put $e_1(t,x) = e_1(t)$, and $d_1(t) = c_1^{-3}e_1(t)^2$. Using Lemma 4.3 and taking the coefficients of t, we have $e_1(t,x)^2(x^4 + at^2x^2 + t^2 + t^6) + c_1(c_1^{-3}e_1(t)^2x + d_2(t))^4 + ac_1^3t^2(c_1^{-3}e_1(t)^2x + d_2(t))^2 + bc_1^3d_2(t)_{even}t^2 + c_1^3t^2 + c_1^7t^6 = 0$. Here, $d_2(t)_{even}$ is the even terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1^{-11}e_1(t)^8$. Since we have $e_1(t) \not\equiv 0$, we have $e_1(t)^6 = c_1^{11}$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, we have $e_1^6 = c_1^{11}$. Considering the coefficients of x^2 of this equation, we have $ae_1^2t^2 = ac_1^{-3}e_1^4t^2$, i.e., $e_1^2 = c_1^3$. Therefore, we have $c_1^9 = c_1^{11}$. Since $c_1 \neq 0$, we have $c_1 = 1$. Therefore, we have $e_1 = 1$ and $d_1(t) = 1$. Then, the equation becomes $d_2(t)^4 + at^2d_2(t)^2 + bd_2(t)_{even}t^2 = 0$. If the degree of $d_2(t)$ is greater than or equal to 2, then the highest term of $d_2(t)^4$ cannot be killed in the equation. Therefore, we can put $d_2(t) = b_0 + b_1t$ $(b_0, b_1 \in k)$ and we have an identity $(b_0 + b_1t)^4 + a(b_0 + b_1t)^2t^2 + bb_0t^2 = 0$. Therefore, we have $b_1^4 = ab_1^2$, $ab_0^2 = bb_0$ and $b_0^4 = 0$. Therefore, we have $b_0 = 0$, and $b_1 = \sqrt{a}$ or 0. Going to the original equality, we have $e_2(t,x)^2 = bt^3\sqrt{a}t$, i.e., $e_2(t,x) = \sqrt[4]{a}\sqrt{b}t^2$. Therefore, we conclude that σ is given by either $t \mapsto t,\ x \mapsto x + \sqrt{a}t,\ y \mapsto y + \sqrt[4]{a}\sqrt{b}t^2$ or the identity. Hence, we have $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z}$. □

11. Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$

In this section we give a construction of Enriques surfaces with the following dual graph of all (-2)-curves given in Figure 33 .

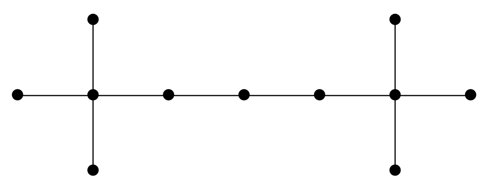


FIGURE 33

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a nonsingular quadric and let $((u_0, u_1), (v_0, v_1))$ be a homogeneous coordinate of Q. Let $x = u_0/u_1,\ x' = u_1/u_0,\ y = v_0/v_1, y' = v_1/v_0$. Consider a rational vector field D defined by the equation (3.2):

$$D = \frac{1}{x^2y^2}\left(bx^3y^2\frac{\partial}{\partial x} + (ax^2y^2 + x^2 + x^4y^4 + y^4 + bx^2y^3)\frac{\partial}{\partial y}\right)$$

where $a, b \in k, b \neq 0$. Note that $D^2 = bD$, that is, D is 2-closed. Denote by E_1, E_2 and F_0 the curves defined by $x = 0$, $x' = 0$ and $y = 0$, respectively. The vector field D has poles of order 2 along E_1, E_2, E_3, and has isolated singularities $(x, y) = (0, 0)$ and $(x', y) = (0, 0)$. The curves E_1, E_2 are integral. Now blow up at two points $(x, y) = (0, 0)$ and $(x', y) = (0, 0)$. The both exceptional curves are integral with respect to the induced vector field. The induced vector field has poles of order 3 along two exceptional curves and has isolated singularities at the intersections of the exceptional curves and the proper transforms of E_1 and E_2. Then blow up at the isolated singularities of the induced vector field and continue this process until the induced vector field has no isolated singularities. We denote by Y the surface obtained by this process and by the same symbols E_1, E_2, F_0 the their proper transforms. Also we denote by the same symbol D the induced vector field on Y. The final configuration of curves is given in Figure 34.

LEMMA 11.1. (i) *The integral curves with respect to D in Figure* 34 *are all horizontal curves* (*thick lines*).

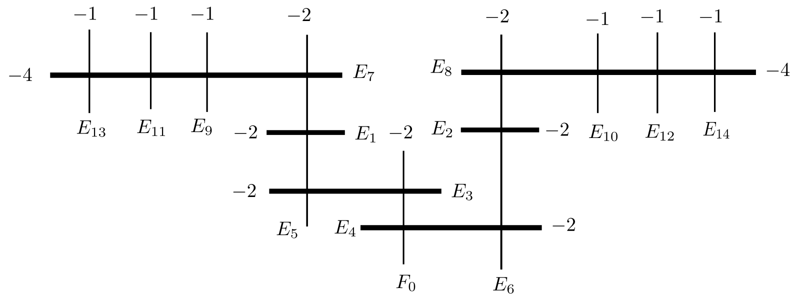


FIGURE 34

(ii) $(D) = -(2F_0 + 2E_1 + 2E_2 + 3E_3 + 3E_4 + 2E_5 + 2E_6 + E_7 + E_8)$.

LEMMA 11.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor* K_Y *of* Y *is given by* $K_Y = -(2F_0 + E_1 + E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6 + E_7 + E_8)$.

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D. By using the same argument as in the proof of Lemma 5.3, D is divisorial and Y^D is nonsingular. By Proposition 2.1, we have the following Figure 35.

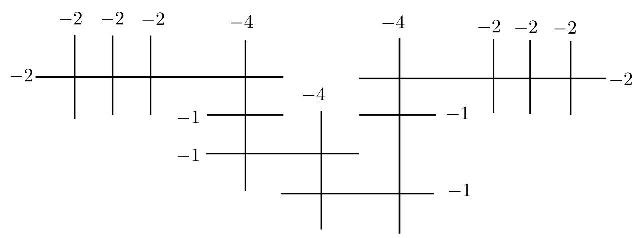


FIGURE 35

Let $X_{a,b}$ be the surface obtained by contracting four exceptional curves which contains 11 (-2)-curves whose dual graph is given by Figure 33. Note that any maximal parabolic subdaigram of this diagram is of type $\tilde{D}_8$ or $\tilde{D}_4 \oplus \tilde{D}_4$. The surface $X_{a,b}$ contains a quasi-elliptic fibration p_1 with singular fibers of type $(2\mathrm{I}_0^*, 2\mathrm{I}_0^*)$ induced from the first projection from Q to $\mathbb{P}^1$ and nine genus one fibrations with a singular fiber of type (I_4^*). These nine genus one fibrations are elliptic by comparing to the conductrix given in Ekedahl and Shepherd-Barron [ES04, Theorem 2.2, Theorem 3.1] (see Tables 4 and 5 in the Section 12).

THEOREM 11.3. *The surfaces $\{X_{a,b}\}$ form a 2-dimensional family of classical Enriques surfaces with the dual graph given in Figure 33. It contains an at least 1-dimensional, non-isotrivial family.*

PROOF. By using Lemmas 11.1, 11.2 and the same arguments as in the proofs of Theorems 5.7, 5.8, $X_{a,b}$ is an Enriques surface with the dual graph given in Figure 33.

By (3.1) in Subsection 3.3, the surface $X_{a,b}$ is the quasi-elliptic surface given by the equation

$$u^2 + Sv^4 + a^2S^3v^2 + b^2S^4v + S^3 + S^7 = 0$$

By Queen [Que71, Theorem 2], its Jacobian is the quasi-elliptic surface given by

$$u^2 + Sv^4 + a^2S^3v^2 + b^2S^4v = 0$$

Now we change coordinates

$$Y = u/bS^2v^2, X = 1/v + a^2/b^2S, T = 1/S$$

which yields

$$Y^2 = X^3 + (a^4/b^4)T^2X + (1/b^2)T^3$$

Since these Jacobian quasi-elliptic surfaces form a 1-dimensional, non-isotrivial family by Ito [Ito02], the family $\{X_{a,b}\}$ contains an at least 1-dimensional, non-isotrivial family. □

THEOREM 11.4. *The automorphism group $\mathrm{Aut}(X_{a,b})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Moreover $\mathrm{Aut}_{nt}(X_{a,b}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

PROOF. Equations of our classical Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$ are given by (3.1) in Subsection 3.3. For our use, we set $x = v,\ y = u,\ t = S$ and we replace a^2 (resp. b^2) by a (resp. b) for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + at^3x^2 + bt^4x + t^3 + t^7 = 0$. This equation gives a normal affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + at^3x^2 + bt^4x + t^3 + t^7).$$

Our quasi-elliptic surface $\varphi : X \longrightarrow \mathbb{P}^1$ has two double fibers of type I_0^* over the points defined by $t = 0$ (resp. $t = \infty$). First, we consider an automorphism τ defined by

$$\tau :\ t \mapsto 1/t, \quad x \mapsto x/t^2, \quad y \mapsto y/t^5.$$

This automorphism is of order 2 and exchanges two double fibers. Let σ be an automorphism of our Enriques surface. σ either keeps the double fibers or exchanges them. If σ exchanges the double fibers, then we consider $\tau \circ \sigma$. This keeps the double fibers. Therefore, we assume that σ keeps the double fibers. Since σ preserves the diagram of (-2)-curves, σ preserves the curve C of cusps and the double fiber $2F_\infty$ over $t = \infty$. Therefore, σ has the form in (4.4) in Subsection 4.1. Moreover, by our assumption, σ preserves the double fiber over the point defined by $t = 0$. Therefore, we may assume $\sigma^*(t) = c_1t$. Using these data, together with the equation $y^2 = tx^4 + at^3x^2 + bt^4x + t^3 + t^7$, we have an identity

$$\begin{aligned} &e_1(t,x)^2(tx^4 + at^3x^2 + bt^4x + t^3 + t^7) + e_2(t,x)^2 \\ &= c_1t(d_1(t)x + d_2(t))^4 + a(c_1t)^3(d_1(t)x + d_2(t))^2 \\ &\quad + b(c_1t)^4(d_1(t)x + d_2(t)) + (c_1t)^3 + (c_1t)^7. \end{aligned}$$

Using Lemma 4.3 and taking the coefficients of x, we have $be_1(t,x)^2t^4 + bc_1^4t^4d_1(t) = 0$. Therefore, we have $e_1(t,x)^2 + c_1^4d_1(t) = 0$ and $e_1(t,x)$ is a polynomial of t, i.e., we can put $e_1(t,x) = e_1(t)$, and $d_1(t) = e_1(t)^2/c_1^4$. Taking the coefficients of t, we have

$$\begin{aligned}&e_1(t)^2x^4 + ae_1(t)^2t^2x^2 + e_1(t)^2t^2 + e_1(t)^2t^6 + c_1(d_1(t)x + d_2(t))^4\\&+ac_1^3t^2(d_1(t)x + d_2(t))^2 + bc_1^4t^4d_2(t)_{odd}/t + c_1^3t^2 + c_1^7t^6 = 0.\end{aligned}$$

Here, $d_2(t)_{odd}$ is the odd terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1d_1(t)^4 = e_1(t)^8/c_1^{15}$. Since we have $e_1(t) \not\equiv 0$, we have $e_1(t)^6 = c_1^{15}$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = c_1^{15}$. Considering the coefficients of x^2, we have $at^2e_1^2 = ac_1^3t^2d_1(t)^2 = ac_1^3t^2(e_1^2/c_1^4)^2 = at^2e_1^4/c_1^5$. Therefore, $e_1^2 = c_1^5$ and $d_1(t) = c_1$. The equation becomes

$$e_1^2t^2 + e_1^2t^6 + c_1d_2(t)^4 + ac_1^3t^2d_2(t)^2 + bc_1^4t^4d_2(t)_{odd}/t + c_1^3t^2 + c_1^7t^6 = 0.$$

If deg $d_2(t) \geq 2$, then we cannot kill the hightest term of $c_1d_2(t)^4$ in the equation.

Therefore, we can put $d_2(t) = b_0 + b_1t$, and we have equations

$$e_1^2 = c_1^7,\ c_1b_1^4 + ac_1^3b_1^2 + bc_1^4b_1 = 0, e_1^2 + ac_1^3b_0^2 + c_1^3 = 0, c_1b_0^4 = 0.$$

Solving these equations with $e_1^2 = c_1^5$, we have $b_0 = 0$, $c_1 = e_1 = d_1 = 1$, and b_1 is either 0 or root of the equation $z^3 + az + b = 0$. Putting this data into the original equation, we have $e_2(t,x) = 0$. Hence, we have 4 automorphisms, which are the identity and three automorphisms of order 2. The involution τ and these automorphisms are commutative with each other. We now conclude $\mathrm{Aut}(X) \cong (\mathbb{Z}/2\mathbb{Z})^3$ (see Subsection 4.2).

Obviously τ is not numerically trivial. We show that any involution σ preserving each double fiber of type I_0^* is numerically trivial. Let F be a double fiber of type I_0^* and let E be the component with multiplicity 2 of F. Then σ preserves E and a simple component C of F meeting with the special 2-section of the fibration, and hence it preserves one more simple component C' of F. This implies that σ fixes two points on E which are intersection points of E with C and C'. Therefore σ fixes E pointwisely and hence σ preserves all components of F. Thus σ is numerically trivial. □

12. Possible dual graphs

In this section, unless mentioned otherwise, all our Enriques surfaces are classical or supersingular.

12.1. Singularities of the canonical cover. In [ES04], Ekedahl and Shepherd-Barron studied "exceptional" Enriques surfaces using the conductrix associated to their canonical cover. In this section, we show that the non-normal locus as well as the isolated singularities of the canonical cover can be used to determine the dual graphs of (-2)-curves on Enriques surfaces with finite automorphism group. For this, we first need some preliminaries.

LEMMA 12.1. (*Ekedahl and Shepherd-Barron [ES04, Lemma 0.9]*) *Let X be an Enriques surface, $\rho : \tilde{X} \to X$ its canonical cover and $\pi : X \to \mathbb{P}^1$ a genus one fibration. Then the morphism ρ factors through the pullback X_F of π by the Frobenius map on $\mathbb{P}^1$. The map $\tilde{X} \to X_F$ is an isomorphism outside of the double fibers of π.*

LEMMA 12.2. *Let X be an Enriques surface with conductrix A. Let π be a genus one fibration on X.*

(1) *If π is a quasi-elliptic fibration, then the curve of cusps of π is a component of A with multiplicity 1.*
(2) *If π is an elliptic fibration, then A is contained in one fiber of π.*

In particular, π is elliptic if and only if A is contained in a fiber of π.

PROOF. By Katsura [Kat82], a non-zero regular 1-form ω on X is given by the pullback of a regular 1-form on $\mathbb{P}^1$. Assume π is quasi-elliptic. Let F be a general cuspidal fiber and t a local parameter at $\pi(F)$. Then locally around the cusp of F is given by the equation $\pi^*t = y^2 + x^3$ (Bombieri-Mumford [BM77, Proposition 4]), hence $\omega = x^2dx$ which vanishes twice at the cusp. Therefore, the curve of cusps is a component of A with multiplicity 1. Similarly one shows that ω does not vanish on any smooth point of an elliptic fiber of π if π is an elliptic fibration. Since A is connected, this yields the second claim. □

Recall that the minimal dissolution of a double cover $Y \to X$ of surfaces with X smooth and Y normal is the successive blowing-ups of points on X lying under singular points of Y. For an Enriques surface X we call the minimal dissolution of the double cover $\tilde{X}_{norm} \to X$, where $\tilde{X}_{norm}$ is the normalization of the canonical cover $\tilde{X}$, the minimal dissolution of X and denote it by X_{diss}. The normalization $\tilde{X}_{sm}$ of X_{diss} in $K(\tilde{X})$ is the minimal resolution of singularities of $\tilde{X}_{norm}$ if $\tilde{X}_{norm}$ has only rational singularities.

Now, we recall the results of Ekedahl and Shepherd-Barron [ES04] on what happens to (-2)-curves on X when taking their inverse image in $\tilde{X}_{sm}$ and additionally study curves of arithmetic genus 1.

LEMMA 12.3. *With the notation introduced above, let C be an irreducible curve of arithmetic genus at most 1 on an Enriques surface X with conductrix A. Denote the irreducible curve on $\tilde{X}_{sm}$ mapping surjectively to C by $\tilde{C}$ and let $\rho : \tilde{X}_{sm} \to \tilde{X}$ and $\pi : \tilde{X} \to X$ be the morphisms from the normalization of the minimal dissolution of X to $\tilde{X}$ and from $\tilde{X}$ to X respectively. We fix the following invariants*:

(i) *The degree s of $(\pi \circ \rho)|_{\tilde{C}} : \tilde{C} \to C$.*
(ii) *The number r of points (including infinitely near ones) on C which are blown up during the minimal dissolution of X and their multiplicity m.*
(iii) *The intersection number $A \cdot C$.*
(iv) *The self-intersection numbers $\tilde{C}^2$ and C^2.*
(v) *The arithmetic genera $p_a(C)$ and $p_a(\tilde{C})$.*
(vi) *If $p_a(C) = 1$, the type* Sing *of singularity of C. This is either nodal n, cuspidal c or smooth sm.*

Then $\tilde{C}$ satisfies the following:

(1) $\tilde{C}^2 = (C^2 - m^2r)s^2/2$ *and* $2p_a(\tilde{C}) - 2 = \tilde{C}^2 - sA \cdot C$
(2) *If two curves meet transversally on X and both have s-invariant 1, then they do not meet on X_{diss}.*
(3) *For $A \cdot C \geq -2$ and $p_a(C) = 0$, we have the following possibilities*

r	s	$A \cdot C$	$\tilde{C}^2$	$p_a(\tilde{C})$
0	1	1	-1	0
0	2	-1	-4	0
2	1	0	-2	0
4	1	-1	-3	0
6	1	-2	-4	0
1	2	-2	-6	0

(4) *For $p_a(C) = 1$, we have the following possibilities*

Sing	r	m	s	$A \cdot C$	$\tilde{C}^2$	$p_a(\tilde{C})$
sm	0		1	0	0	1
sm	0		2	0	0	1
n	1	2	1	0	-2	0
c	0		1	0	0	1
c	0		2	0	0	1
c	1	2	1	0	-2	0
c	4	1	1	0	-2	0
c	2	1	1	1	-1	0
c	0		1	2	0	0

(5) *If C is a cuspidal curve such that*
- *$|C|$ defines a quasi-elliptic fibration, then $r = 0$ and $s = 1$*
- *$|C|$ defines an elliptic fibration, then $r = 1$, $m = 2$ and $s = 1$*
- *$|C|$ does not define a quasi-elliptic fibration and $|2C|$ defines a quasi-elliptic fibration, then $r = 2$, $m = 1$ and $s = 1$.*

PROOF. Similar to Ekedahl and Shepherd-Barron [ES04], the formulas for the self-intersection number and the genus of $\tilde{C}$ are obtained by observing that the self-intersection number of C drops by m^2 for every point of multiplicity m on C which is blown up during the minimal dissolution and from $\omega_{\tilde{X}/X} = \pi^*(\mathcal{O}_X(-A))$. Also the claim (2) is in [ES04].

The first table is contained in [ES04] and we will only establish the second one. Therefore, assume that $p_a(C) = 1$. If C is smooth, then $A \cdot C = 0$ by Lemma 12.2 which only leaves the two possibilities listed. If C is nodal, then $|C|$ defines an elliptic fibration φ with C as a simple fiber. Therefore, formally locally around C, X is isomorphic to the Jacobian of φ and by Lemma 12.1 we can find $\tilde{C}$ by doing Frobenius pullback along the base. But on an I_1 fiber, an elliptic surface acquires an A_1-singularity at the singular point of the nodal curve after Frobenius pullback. Therefore, the node of C is blown up during the minimal dissolution. A similar argument works if C is cuspidal and $|C|$ defines an elliptic fibration.

If C is cuspidal, we have enumerated all numerical possibilities except for the ones where $p_a(\tilde{C}) = 0$ and $s = 2$. These cases do not occur. In fact, assume that $s = 2$ and $p_a(\tilde{C}) = 0$. Denote the image of $\tilde{C}$ on $\tilde{X}_{norm}$ by $\tilde{C}'$. Since the singular point of C is not blown up during the dissolution (by the self-intersection formula), we have $\tilde{C}' \cong \tilde{C} \cong \mathbb{P}^1$. Then, the flat morphism $\varphi : \tilde{X}_{norm} \to X$ restricts to a morphism $\varphi|_{\tilde{C}'} : \tilde{C}' \to C$. Since $s = 2$, we have $\varphi^* C = \tilde{C}'$ so

$\varphi|_{\tilde{C}'}$ is nothing but the base change of φ along the closed immersion $C \to X$ and as such it is a flat morphism. But a morphism from $\mathbb{P}^1$ to the cuspidal cubic is never flat.

For the last statement (5), observe that $|C|$ defines a quasi-elliptic fibration if and only if $A \cdot C = 2$, and $|2C|$ defines a quasi-elliptic fibration if and only if $A \cdot C = 1$. This follows immediately from Lemma 12.2, which implies that $A \cdot C = D \cdot C$ where D is the curve of cusps of $|C|$ (resp. $|2C|$). □

REMARK 12.4. Several of the numerical possibilities in Lemma 12.3 might be excluded by using Lang's list of possible configurations of singular fibers on rational elliptic surfaces in characteristic 2 [Lan00] together with Lemma 12.1. However, we will not pursue this here.

COROLLARY 12.5. *Let X be an Enriques surface with a quasi-elliptic fibration φ. Let F be a fiber of φ. If F is a double fiber, then two points on F (including infinitely near ones) are blown up during the minimal dissolution. If F is simple, then no point on F is blown up.*

PROOF. If F is reducible, this can be read off from the table in [ES04, p.13], since every (-2)-curve on a simple fiber has r-invariant 0 and exactly one (-2)-curve on a double fiber has r-invariant 2 while the others have r-invariant 0. If F is irreducible, this is the last statement of Lemma 12.3. □

COROLLARY 12.6. *Let X be an Enriques surface with a quasi-elliptic fibration. Then the normalization $\tilde{X}_{norm}$ of the canonical cover has an isolated D_4-singularity if and only if X is supersingular.*

PROOF. Let φ be a quasi-elliptic fibration on X. Since the conductrix is non-empty by Lemma 12.2, $\tilde{X}$ is not normal. Therefore, $\tilde{X}_{norm}$ has either four A_1- or one D_4-singularity by Proposition 2.13. If φ has two double fibers, at least two distinct points on X are blown up during the minimal dissolution by Lemma 12.5. In this case, X is classical (Proposition 2.5) and $\tilde{X}$ has four A_1-singularities. If φ has only one double fiber, at most two distinct points on X are blown up. In this case, X is supersingular and $\tilde{X}$ has one D_4-singularity. □

12.2. Special extremal genus one fibrations. In this section, we present a detailed study of Enriques surfaces with special genus one fibrations, their conductrices and isolated singularities on their canonical cover. Throughout, we will use the observations summed up in the following Lemma.

LEMMA 12.7. *Let X be an Enriques surface with a conductrix A and $\tilde{X}$ its canonical cover. The following hold.*

(1) *If two (-2)-curves which meet transversally have s-invariant 1, then their intersection is blown up.*
(2) *Every (-2)-curve meets the conductrix at most once.*
(3) *Every (-2)-curve which is not a component of the conductrix has s-invariant 1.*

Now let $\pi : X \to \mathbb{P}^1$ be a genus one pencil. Then the following hold.

(a) *A singular fiber of type I_n of π gives n A_1-singularities on $\tilde{X}$.*
(b) *If $A \neq \emptyset$ and π has a singular fiber of type I_n, then $\tilde{X}$ has four A_1-singularities.*

(c) *If* $A \neq \emptyset$ *and two disjoint* (-2)*-curves have positive* r*-invariant, then* $\tilde{X}$ *has four* A_1*-singularities.*

(d) *If* $A \neq \emptyset$ *and the sum of all* r*-invariants of fiber components is less than* 4*, then* $\tilde{X}$ *has one* D_4*-singularity.*

PROOF. The first claim is obtained by checking intersection numbers, as was done by Ekedahl and Shepherd-Barron in [ES04] and the second is a consequence of Lemma 12.3. Since a curve C which is not contained in A has $A \cdot C \geq 0$, the third claim follows from Lemma 12.3.

For the statements about π: The first can be checked using the Jacobian of π, since an I_n fiber is simple. The second claim follows immediately from the first, since $\tilde{X}$ has either four A_1-singularities or one D_4-singularity if $A \neq \emptyset$ (see Proposition 2.13). Two disjoint curves having positive r-invariant means that distinct points are blown up during the dissolution, excluding the possibility of a D_4-singularity on the cover. For the last claim, the sum of r-invariants of fiber components being less than 4 means that less than 4 distinct points are blown up, so the singularity can only be a D_4-singularity. □

REMARK 12.8. Observe that we have used that the singularities lying over a simple fiber of π can be read off from the Frobenius base change of the Jacobian fibration.

LEMMA 12.9. *There are no special elliptic fibrations on Enriques surfaces with a double fiber of type* $2\mathrm{III}^*, 2\mathrm{II}^*$ *or* $2\mathrm{I}_4^*$*. Moreover, if the conductrix is nonempty, a special elliptic fibration with a double fiber of type* IV *can not exist.*

PROOF. The statement about $\mathrm{II}^*, \mathrm{III}^*$ and I_4^* is contained in Ekedahl and Shepherd-Barron [ES04, Corollary 3.2]. We will give another argument here. Let N be a special 2-section and C the simple component of the double fiber we want to exclude. By checking all possible conductrices of [ES04, Theorem 3.1], we obtain that C and N have s-invariant 1. Moreover, $A \cdot C = 0$ if C is a component of A with multiplicity 1, whereas $A \cdot C = 1$ if C does not occur in the conductrix. Therefore, $N \cdot A = 1$ if and only if $C \cdot A = 0$. Now by Lemma 12.7 (1), the intersection of N and C is blown up. But one of them has r-invariant 0 by Lemma 12.3. This is a contradiction.

Now we prove the second claim. Since N has s-invariant 1 by Lemma 12.7 (3) and every component of the fiber of type IV also has s-invariant 1 by the same Lemma, the intersection of N and the fiber of type IV is blown up. Additionally, the intersection of the three components of the fiber of type IV is blown up. Therefore, the canonical cover has four A_1-singularities by Proposition 2.13. But every component of the fiber of type IV and N have r-invariant 2. This can not be achieved by blowing-ups at only 4 distinct points. □

LEMMA 12.10. *The isolated singularities on the normalization of the canonical cover of an Enriques surface with a special extremal elliptic fibration and the conductrix are summed up in Table* 4*. The self-intersection number of the reduced inverse image of the curve on the minimal resolution of singularities of the canonical cover is given as an index to the multiplicity.*

PROOF. For the list of rational extremal elliptic fibrations see Proposition 2.7. We will use the tables in [ES04, p.16-18] for the possibilites of the conductrix A. In every case, we denote the special 2-section by N. Recall that $A^2 = -2$ by Proposition 2.13.

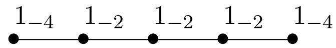
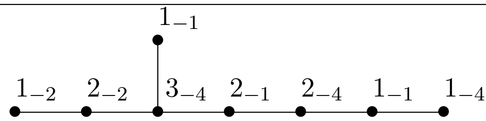
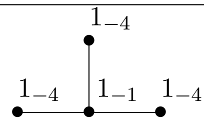
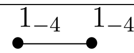
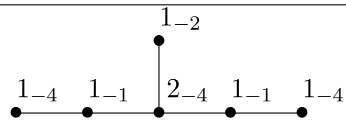
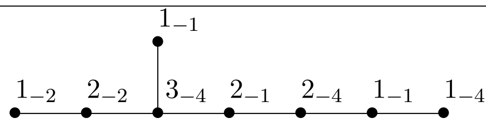
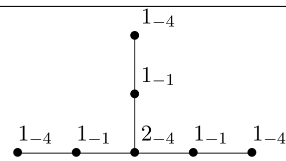
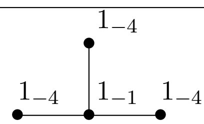
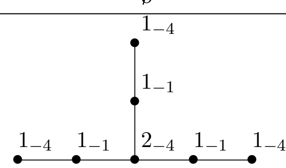
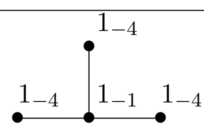
Singular fibers	Conductrix	Isolated singularities
(I_4^*)		$4A_1$
(II^*)		D_4
$(2\mathrm{III}, \mathrm{I}_8)$	$\emptyset$	$12A_1$
$(\mathrm{III}, \mathrm{I}_8)$	$\emptyset$	$D_4, 8A_1$
$(2\mathrm{I}_1^*, \mathrm{I}_4)$		$4A_1$
$(\mathrm{I}_1^*, \mathrm{I}_4)$		$4A_1$
$(\mathrm{III}^*, \mathrm{I}_2)$		$4A_1$
$(\mathrm{II}^*, \mathrm{I}_1)$		$4A_1$
$(\mathrm{IV}, 2\mathrm{IV}^*)$		D_4
$(\mathrm{IV}, \mathrm{IV}^*)$		D_4
$(2\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$	$\emptyset$	$12A_1$
$(\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$	$\emptyset$	$D_4, 8A_1$
$(2\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$		$4A_1$
$(\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$		$4A_1$
$(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1)$	$\emptyset$	$12A_1$
$(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$	$\emptyset$	$12A_1$
$(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$	$\emptyset$	$12A_1$

TABLE 4. Singularities on the canonical cover of an Enriques surface with an extremal, special, elliptic fibration

- (I_4^*) : There is only one possibility for A with $A^2 = -2$. The canonical cover has four A_1-singularities by Lemma 12.7 (c).
- (II^*) : There are two possible conductrices with $A^2 = -2$. However, since $N \cdot A \leq 1$ by Lemma 12.7 (2), we get the one in the table. Since all fibers different from the fiber of type II* are smooth and no point on a smooth fiber is blown up during the dissolution by Lemma 12.3, the sum of all r-invariants of fibers is less than 4. Hence the cover has one D_4-singularity by Lemma 12.7 (d).
- $(2\mathrm{III}, \mathrm{I}_8)$: In this case $A = \emptyset$. Since the intersection of N with a component of the fiber of type III is blown up, there are at least 9 distinct points which are blown up during the dissolution by Lemma 12.7 (a). Therefore, the cover has 12 A_1-singularities.
- $(\mathrm{III}, \mathrm{I}_8)$: Again, we have $A = \emptyset$. By [Lan00], the fiber of type III acquires a D_4-singularity after Frobenius pullback. The 8 A_1-singularities come from the fiber of type I_8 by Lemma 12.7 (a).
- $(2\mathrm{I}_1^*, \mathrm{I}_4)$: By Lemma 12.7 (b), we have 4 A_1-singularities. Since every point which is blown up lies on the fiber of type I_4 , the r-invariant of N is at most 1 and therefore $N \cdot A = 1$. This is only possible for the conductrix in our table.
- $(\mathrm{I}_1^*, \mathrm{I}_4)$: By the same argument as in the previous case, we have $N \cdot A = 1$. Moreover, N can not meet distinct components of the fiber of type I_1^* since we would obtain a different fibration with a double fiber of type I_4 or I_5 in these cases. Therefore, N meets a multiplicity 2 component of the fiber of type I_1^* . Now N and some components of the fiber of type I_1^* form a fiber of type I_0^* of a different fibration and the only possible conductrix for this behaviour is the one in our table.
- $(\mathrm{III}^*, \mathrm{I}_2)$: There are two possible conductrices with $A^2 = -2$. If the conductrix has the full fiber as support, N meets the central multiplicity 2 component since $N \cdot A \leq 1$ by Lemma 12.7 (2). But then, there is a fiber of type IV* of a different fibration such that two components of the conductrix meet the fiber without being contained in it. This is not possible by Lemma 12.2. Hence, we have the conductrix in our table and the isolated singularities because of Lemma 12.7 (b).
- $(\mathrm{II}^*, \mathrm{I}_1)$: The conductrix is the one in the table by the same argument as in the (II^*) case. By Lemma 12.7 (b), we get the types of isolated singularities.
- $(\mathrm{IV}, 2\mathrm{IV}^*)$: Since N meets a simple component of the fiber of type IV*, we can exclude the case where the conductrix does not have the full fiber as support, since in this case every simple component of the fiber of type IV* has s-invariant 1 and r-invariant 0 while N has s-invariant 1, contradicting Lemma 12.7 (1). The isolated singularities are as in the table, since by [Lan00] the fibers of type IV acquires a D_4-singularity after Frobenius pullback.
- $(\mathrm{IV}, \mathrm{IV}^*)$: Suppose that A has the full fiber of type IV* as support. Then N meets a multiplicity 2 component of this fiber by $A \cdot N \leq 1$. But then N and components of the fiber of type IV* form a fiber of type I_1^* of a different elliptic fibration such that two components of the conductrix meet the fiber without being contained in it. This is not possible by Lemma 12.2. As in the previous case, we get a D_4-singularity.
- $(2\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$ and $(\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$: The argument is essentially the same as in the $(2\mathrm{III}, \mathrm{I}_8)$ and $(\mathrm{III}, \mathrm{I}_8)$ cases.

- $(2\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$ amd $(\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$: The argument is similar to the cases with singular fibers $(\mathrm{IV}, 2\mathrm{IV}^*)$ and $(\mathrm{IV}, \mathrm{IV}^*)$, except that the fibers of type I_n give 4 A_1-singularities by Lemma 12.7 (a).
- All singular fibers multiplicative: In these cases, we get 12 A_1-singularities by Lemma 12.7 (a). □

For the convenience of the reader we give the corresponding table for quasi-elliptic fibrations. This does not require proof, since the conductrices are uniquely determined (see [ES04]) and the isolated singularities depend on the number of double fibers (see Corollary 12.5).

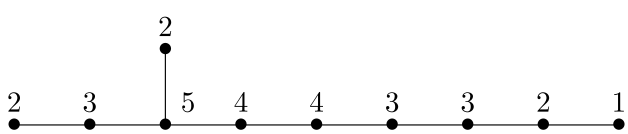
LEMMA 12.11. *The isolated singularities on the normalization of the canonical cover of an Enriques surface with a quasi-elliptic fibration and the conductrix are summed up in Table* 5. *The self-intersection number of the reduced inverse image of the curve on the minimal resolution of singularities of the canonical cover is given as an index to the multiplicity. We do not give multiplicities of the fibers of type* III. *The curve of cusps is encircled.*

REMARK 12.12. Recall that any Enriques surface has a genus one fibration (Proposition 2.4) and if an Enriques surface X has a finite group of automorphisms, then any genus one fibration on X is extremal (Proposition 2.9). Therefore, X has an extremal, special genus one fibration by Proposition 2.10. Lemmas 12.10 and 12.11 imply that the canonical cover of any Enriques surface with finite automorphism group has only A_1- or D_4-singularities as isolated singularities.

12.3. Determination of possible dual graphs.

THEOREM 12.13. *Let X be a classical or supersingular Enriques surface with finite automorphism group. Then, the dual graph of (-2)-curves on X is one of the dual graphs given in Theorem* 1.1 (A) *and Theorem* 1.2 (A).

PROOF. We start with a tuple (A, I) where A is one of the possible conductrices and I is either D_4 or $4A_1$. We consider all possible special extremal genus one fibrations and check, if an Enriques surface with finite automorphism group with conductrix A and canonical double cover whose normalization has isolated singularities of type I can exist and determine its dual graph of (-2)-curves. We will make use of Lemma 12.2 very often without mentioning it from now on. Also we denote by N a special (-2)-section for a given special genus one fibration. If the fibration is quasi-elliptic, then N denotes the curve of cusps.

- Conductrix: 

  Singularities: D_4 or $4A_1$
  Possible special, extremal fibrations: $(2\mathrm{II}^*)$ quasi-elliptic

  This is nothing but the dual graph of type $\tilde{E}_8$. The Enriques surfaces are supersingular or classical according to the type of singularities (Corollary 12.6). These are the $\tilde{E}_8$ exceptional surfaces studied in [ES04].

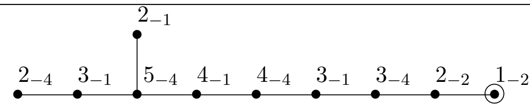
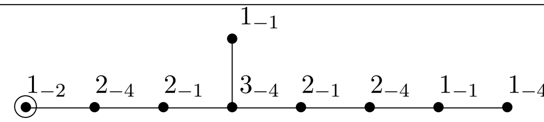
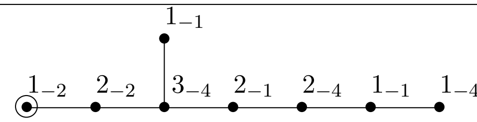
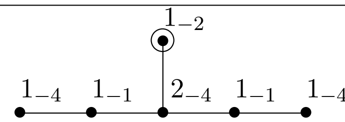
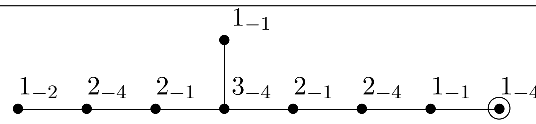
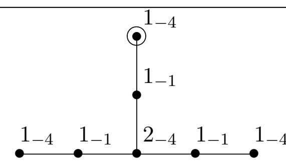
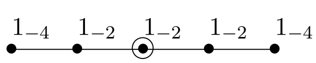
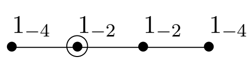
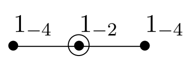
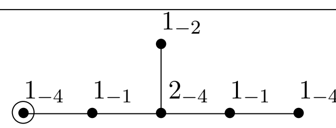
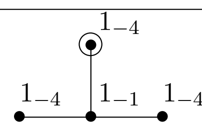
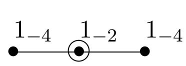
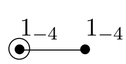
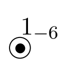
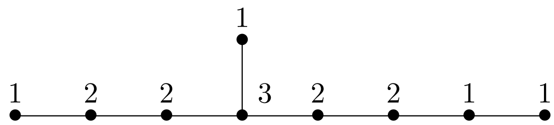
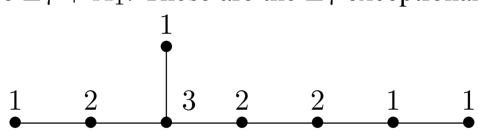
Singular fibers	Conductrix	Isolated singularities
$(2\mathrm{II}^*)$		$4A_1$ *or* D_4
(II^*)		$4A_1$ *or* D_4
$(2\mathrm{I}_4^*)$		$4A_1$ *or* D_4
(I_4^*)		$4A_1$ *or* D_4
$(2\mathrm{III}^*, \mathrm{III})$		$4A_1$ *or* D_4
$(\mathrm{III}^*, \mathrm{III})$		$4A_1$ *or* D_4
$(2\mathrm{I}_0^*, 2\mathrm{I}_0^*)$		$4A_1$
$(2\mathrm{I}_0^*, \mathrm{I}_0^*)$		$4A_1$ *or* D_4
$(\mathrm{I}_0^*, \mathrm{I}_0^*)$		$4A_1$ *or* D_4
$(2\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$		$4A_1$ *or* D_4
$(\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$		$4A_1$ *or* D_4
$(2\mathrm{I}_0^*, 4 \times \mathrm{III})$		$4A_1$ *or* D_4
$(\mathrm{I}_0^*, 4 \times \mathrm{III})$		$4A_1$ *or* D_4
$(8 \times \mathrm{III})$		$4A_1$ *or* D_4

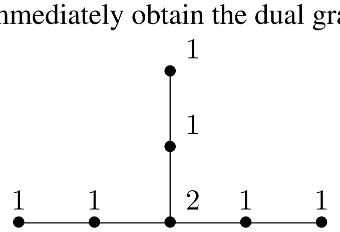
TABLE 5. Singularities on the canonical cover of an Enriques surface with a quasi-elliptic fibration

- Conductrix: 
  Singularities: D_4 or $4A_1$
  Possible special, extremal fibrations: (II^*) quasi-elliptic, $(2\mathrm{III}^*, \mathrm{III})$ quasi-elliptic and $(2\mathrm{III}^*, 2\mathrm{III})$ quasi-elliptic.

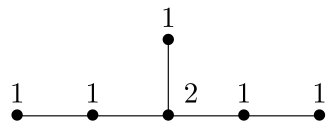
First note that in case of $(2\mathrm{III}^*, \mathrm{III})$ the 2-section N meets each component of the singular fiber of type III because otherwise there is a (-2)-curve meeting the conductrix more than once. Now for each special genus one fibration we immediately obtain the dual graph of type $\tilde{E}_7 + \tilde{A}_1$. These are the $\tilde{E}_7$ exceptional surfaces of [ES04].

- Conductrix: 
  Singularities: D_4 or $4A_1$
  Possible special, extremal fibrations: $(2\mathrm{I}_4^*)$ quasi-elliptic, (II^*) elliptic and $(\mathrm{II}^*, \mathrm{I}_1)$ elliptic

If we start with a special elliptic fibration with a singular fiber of type II^*, the 2-section N has to meet this fiber in a component with multiplicity 2, for otherwise there is a quasi-elliptic fibration with a double fiber of type 2III. This is not allowed. Thus we either get a quasi-elliptic fibration with a double fiber of type III^* or a quasi-elliptic fibration with a double fiber of type I_4^*. Again, the first case is not allowed. Therefore, this is an Enriques surface of type $\tilde{D}_8$. Starting from the quasi-elliptic fibration of type $(2\mathrm{I}_4^*)$, we immediately obtain the dual graph of type $\tilde{D}_8$.

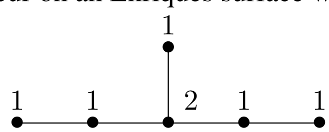
- Conductrix: 
  Singularities: D_4 or $4A_1$
  Possible special extremal fibrations: $(\mathrm{III}^*, 2\mathrm{III})$ quasi-elliptic, $(\mathrm{III}^*, \mathrm{III})$ quasi-elliptic, $(\mathrm{IV}, 2\mathrm{IV}^*)$ elliptic and $(2\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$ elliptic

If we start with a special genus one fibration $(\mathrm{III}^*, \mathrm{III})$ together with the 2-section N, then we find a genus one fibration with a double fiber of type IV^*, and if we start with $(\mathrm{IV}, 2\mathrm{IV}^*)$ or $(2\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1)$, then we find a fibration $(\mathrm{III}^*, 2\mathrm{III})$. In the case of $(\mathrm{III}^*, 2\mathrm{III})$, we immediately obtain the dual graph of type $\tilde{E}_6 + \tilde{A}_2$ (we can prove the non-existence of $(\mathrm{III}^*, \mathrm{III})$ quasi-elliptic case, but we omit the details). This is an $\tilde{E}_6$ exceptional Enriques surface of [ES04].

- Conductrix: 

  Singularities: D_4
  Possible special extremal fibrations: (I_4^*) quasi-elliptic, $(2\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$ quasi-elliptic, and $(2\mathrm{I}_2^*, 2\mathrm{III}, \mathrm{III})$ quasi-elliptic

If we start with (I_4^*), then we find a special fibration with a double fiber of type I_2^*. In cases $(2\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$ and $(2\mathrm{I}_2^*, 2\mathrm{III}, \mathrm{III})$, there exists a genus one fibration with a fiber of type III^* which is elliptic since the conductrix is contained in a fiber. Hence it is of type $(\mathrm{III}^*, \mathrm{I}_2)$ which contradicts the type of singularities (Lemma 12.7, (b)). Thus this case does not occur on an Enriques surface with finite automorphism group.

- Conductrix: 

  Singularities: $4A_1$
  Possible special extremal fibrations: $(2\mathrm{I}_2^*, 2\mathrm{III}, \mathrm{III})$ quasi-elliptic, $(2\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$ quasi-elliptic, (I_4^*) quasi-elliptic and $(\mathrm{III}^*, \mathrm{I}_2)$ elliptic

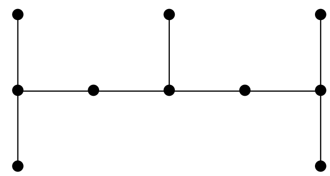
In every case, there is a quasi-elliptic fibration with a singular fiber of type (I_4^*) and with the curve of cusps meeting the central component.

To see this in the case of the special elliptic fibration with singular fibers of type $(\mathrm{III}^*, \mathrm{I}_2)$, note that if the 2-section meets a simple component of the fiber of type III^*, we get a quasi-elliptic fibration with a singular fiber of type 2III, if it meets a component of multiplicity 2 on one of the long arms, we get a quasi-elliptic fibration with a singular fiber of type $2\mathrm{I}_2^*$ and if it meets the component of multiplicity 2 in the center, there would be a special elliptic fibration with a double fiber of type IV^*, which we have excluded.

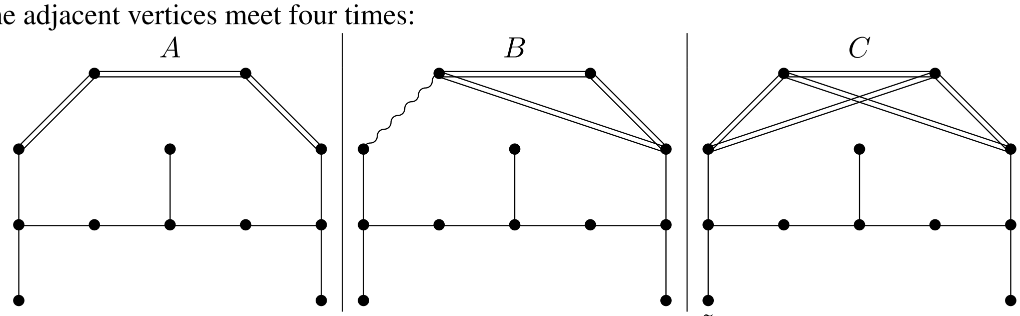
In the cases with a double fiber of type I_2^*, observe that the curve of cusps can not meet a component of a simple fiber of type III twice, because of Lemma 12.7 (2). Hence we obtain a quasi-elliptic fibration with a singular fiber of type I_4^*.

We will now start from a quasi-elliptic fibration with a singular fiber of type I_4^* and exclude this case. Two of the blown up points lie on the conductrix and two do not. Any (-2)-curve not meeting the conductrix has r-invariant 2 and therefore it passes through the 2 blown up points not lying on the conductrix. In particular, any two (-2)-curves not meeting the conductrix meet each other at least twice.

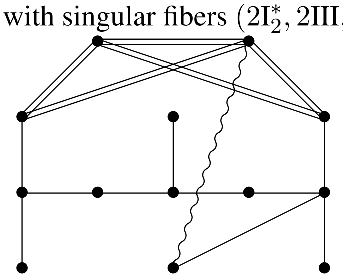
The configuration we start with is the following:



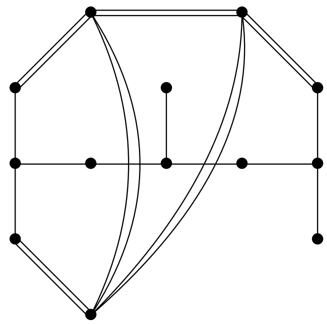
There are four subdiagrams of type $\tilde{E}_7$. If the automorphism group of an Enriques surface with this conductrix is finite, the elliptic fibrations induced by those subdiagrams have singular fibers of type $(\mathrm{III}^*, \mathrm{I}_2)$. For any of these diagrams of type $\tilde{E}_7$, the two remaining curves are either 2- or 4-sections of the fibration, depending on whether the fiber of type III^* is double or not. If such a multisection meets a component of the fiber of type I_2 only once, we obtain a quasi-elliptic fibration with singular fiber of type II^*, which is not allowed. If one of the multisections meets only one component of the fiber of type I_2, the other multisection and the other component of the fiber of type I_2 are disjoint from a diagram of type $\tilde{D}_6$, hence they meet each other twice. This leaves us with the following three possible dual graphs, where a wiggly line means that the two curves corresponding to the adjacent vertices meet four times:



- We first exclude Case C. Using one of the diagrams of type $\tilde{A}_1$, which yields a quasi-elliptic fibration with singular fibers $(2\mathrm{I}_2^*, 2\mathrm{III}, \mathrm{III})$, we get the following graph:

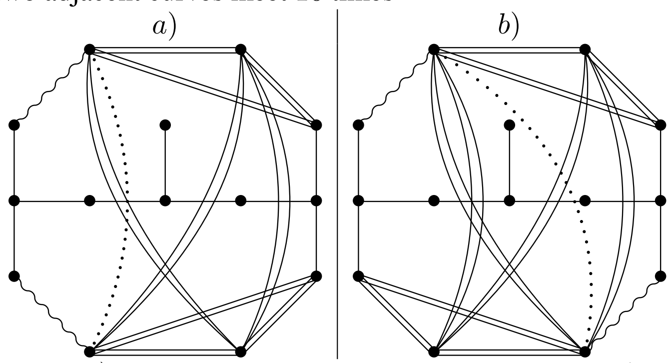
  

  Therefore, there is a subdiagram of type $\tilde{D}_4$. This is not allowed for an Enriques surface with finite automorphism group having this conductrix.
- Now we exclude Case A. We get another (-2)-curve as in the following diagram from one of the other fibrations with singular fibers of type $(\mathrm{III}^*, \mathrm{I}_2)$

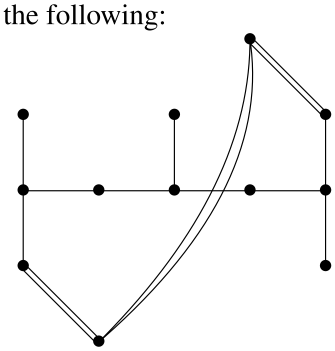


But then the orthogonal complement of a diagram of type $\tilde{D}_6$ contains a 2-connected path of four (-2)-curves, which is not possible.

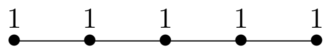
– Lastly, let us exclude Case B. Again, looking at another fibration with singular fibers of type $(\mathrm{III}^*, \mathrm{I}_2)$, we get the following two cases, where a dotted line denotes that the two adjacent curves meet 10 times



In case $a)$ we get the same contradiction as for Case A. In case $b)$ there is a special elliptic fibration with singular fibers of type $(\mathrm{III}^*, \mathrm{I}_2)$ having intersection graph of Case A, namely the following:



Therefore, an Enriques surface with finite automorphism group and this conductrix can not exist.

- Conductrix: 

Singularities: $4A_1$
Possible special extremal fibrations: $(2\mathrm{I}_0^*, 2\mathrm{I}_0^*)$ quasi-elliptic and (I_4^*) elliptic

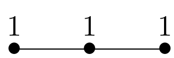
If we start with a special elliptic fibration with a singular fiber of type (I_4^*), we have to observe that a special 2-section N has to meet the conductrix, for otherwise we obtain a quasi-elliptic fibration with a singular fiber of type 2III. Now if the 2-section N meets the conductrix, we obtain a special genus one fibration with a singular fiber of type $2\mathrm{I}_2^*$, $2\mathrm{I}_1^*$ or $2\mathrm{I}_0^*$. The first two are not allowed. Thus, we get a quasi-elliptic fibration with a double fiber of type I_0^* and an Enriques surface of type $\tilde{D}_4 + \tilde{D}_4$.

The same graph is immediately obtained when starting with the quasi-elliptic fibration with singular fibers of type $(2\mathrm{I}_0^*, 2\mathrm{I}_0^*)$.

- Conductrix: 

Singularities: D_4 or $4A_1$
Possible special extremal fibrations: $(2\mathrm{I}_0^*, \mathrm{I}_0^*)$ quasi-elliptic

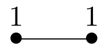
Starting with a fibration with singular fibers of type $(2\mathrm{I}_0^*, \mathrm{I}_0^*)$, the special 2-section N meets the component with multiplicity 2 of the singular fiber of type I_0^* (otherwise there exists a fibration with a fiber of type III containing a component N of the conductrix), and hence there is a subdiagram of type $\tilde{D}_7$ which defines a non-extremal fibration (Propositions 2.7 and 2.8). Therefore, an Enriques surface with this conductrix can not have a finite automorphism group.

- Conductrix: 

Singularities: D_4 or $4A_1$
Possible special extremal fibrations: $(2\mathrm{I}_0^*, 2\mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III})$ quasi-elliptic and $(2\mathrm{I}_0^*, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III})$ quasi-elliptic

Starting with a quasi-elliptic fibration with singular fibers of type $(\mathrm{I}_0^*, \mathrm{I}_0^*)$, we obtain an elliptic fibration with a singular fiber of type I_2^*, which is not allowed.

As for the fibrations with a double fiber of type $2\mathrm{I}_0^*$, by the same reason as in the previous case, a special 2-section N meets two components of each simple fiber of type III. Therefore there is a diagram of type $\tilde{D}_6$ containing the conductrix. But an elliptic fibration with a fiber of type I_2^* can not be extremal by Propositions 2.7 and 2.8.

- Conductrix: 

Singularities: D_4 or $4A_1$
Possible special extremal fibrations: $(\mathrm{I}_0^*, 2\mathrm{III}, 2\mathrm{III}, \mathrm{III}, \mathrm{III})$ quasi-elliptic, $(\mathrm{I}_0^*, 2\mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III})$ quasi-elliptic, $(\mathrm{I}_0^*, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III})$ quasi-elliptic and $(\mathrm{I}_1^*, \mathrm{I}_4)$ elliptic.

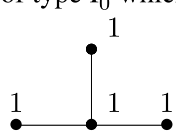
If there is a quasi-elliptic fibration on this surface, then there is a configuration of type I_0^* containing the conductrix. The induced elliptic fibration is not extremal.

Starting with a special elliptic fibration with singular fibers of type $(\mathrm{I}_1^*, \mathrm{I}_4)$, we look at the intersection of N with the fiber of type I_1^*. If the special 2-section N meets distinct components, we obtain a configuration giving a double fiber of type I_4 or I_5, which is a contradiction. If N meets a simple component twice, we get a double fiber of type III of a quasi-elliptic fibration and we have excluded this case before. If N meets a double component once, then there is a configuration of type I_0^* containing the conductrix giving the same contradiction as in the first paragraph.

- Conductrix: 

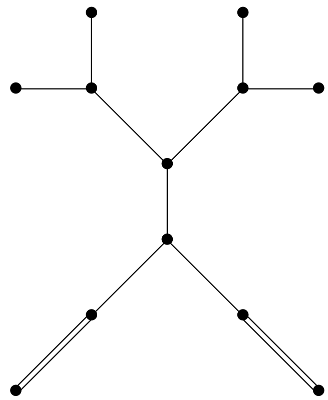
Singularities: D_4 or $4A_1$
Possible special extremal fibrations: $(\mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III}, \mathrm{III})$ quasi-elliptic, any multiplicities

The 2-section N is nothing but the conductrix and hence N meets two components of each simple fiber of type III as in the previous cases. Thus we have an elliptic fibration with a fiber of type I_0^* which is not extremal by Proposition 2.7.

- Conductrix: 

Singularities: $4A_1$
Possible special extremal fibrations: $(\mathrm{I}_2^*, 2\mathrm{III}, 2\mathrm{III})$ quasi-elliptic, $(\mathrm{I}_2^*, \mathrm{III}, 2\mathrm{III})$ quasi-elliptic, $(\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$ quasi-elliptic, $(2\mathrm{I}_1^*, \mathrm{I}_4)$ elliptic and $(\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$ elliptic

If there is a quasi-elliptic fibration with singular fibers of type $(\mathrm{I}_2^*, 2\mathrm{III}, 2\mathrm{III})$, we have the following configuration of (-2)-curves:

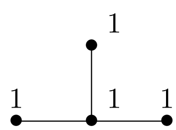


The special elliptic fibration induced by the diagram of type $\tilde{D}_5$ meeting the two curves at the bottom gives four more (-2)-curves. We leave it to the reader to check that the resulting intersection graph is of type VIII.

If there is a special elliptic fibration with singular fibers of type $(2\mathrm{I}_1^*, \mathrm{I}_4)$, the 2-section N has to meet a component of the fiber of type I_4 twice, since a special elliptic fibration with a double fiber of type IV is not allowed. Therefore, there is a quasi-elliptic fibration with a double singular fiber of type III, which has to be a fibration with singular fibers of type $(\mathrm{I}_2^*, 2\mathrm{III}, 2\mathrm{III})$, since the curve of cusps does not meet one of the components of the second fiber of type III which is a component of the fiber of type I_4 and the curve of cusps may not meet the other component twice.

Starting with a quasi-elliptic fibration with singular fibers of type $(\mathrm{I}_2^*, \mathrm{III}, 2\mathrm{III})$ or $(\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$, we immediately get the existence of a special elliptic fibration with a singular double fiber of type I_1^*, returning us to the case above.

If there is a special elliptic fibration with singular fibers of type $(\mathrm{IV}^*, \mathrm{I}_1, \mathrm{I}_3)$, the 2-section meets either a simple component of the fiber of type IV^* twice or a double component once. In the first case, we get a quasi-elliptic fibration with a singular fiber of type 2III and in the second case, we get a special elliptic fibration with a double fiber of type I_1^*. Both cases have already been dealt with.

- Conductrix: 

Singularities: D_4
Possible special extremal fibrations: $(\mathrm{I}_2^*, \mathrm{III}, 2\mathrm{III})$ quasi-elliptic, $(\mathrm{I}_2^*, \mathrm{III}, \mathrm{III})$ quasi-elliptic and $(\mathrm{IV}, \mathrm{IV}^*)$ elliptic

If we start with a quasi-elliptic fibration, we get a special elliptic fibration with a double fiber of type I_1^*, which is not allowed.

In the case of the fibration with singular fibers of type $(\mathrm{IV}, \mathrm{IV}^*)$, the 2-section either meets a simple component of the fiber of type IV^* twice, or a double component once.

The first case leads to a special genus one fibration with a double fiber of type III and the second one to a special elliptic fibration with a double fiber of type I_1^*. Both cases have already been treated.

- Conductrix: $\emptyset$
  Singularities: $D_4, 8A_1$
  Possible special extremal fibrations: $(\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$ elliptic and $(\mathrm{III}, \mathrm{I}_8)$ elliptic

  We start from any of the two special fibrations and a special 2-section N. By considering the intersection of N with the fibers of type I_6, I_8, we can find a special genus one fibration with an additive double fiber of type III or IV no matter how the 2-section intersects the fibers. However, these fibrations are not allowed by our list. Hence a surface with these singularities can not have finite automorphism group.

- Conductrix: $\emptyset$
  Singularities: $12A_1$
  Possible special extremal fibrations: $(\mathrm{I}_9, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1)$ elliptic, $(\mathrm{I}_5, \mathrm{I}_5, \mathrm{I}_1, \mathrm{I}_1)$ elliptic, $(2\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$ elliptic, $(2\mathrm{III}, \mathrm{I}_8)$ elliptic and $(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3)$ elliptic

  If we start with a special fibration with singular fibers of type $(2\mathrm{III}, \mathrm{I}_8)$, the 2-section has to meet two adjacent components of the fiber of type I_8. Indeed, the twelve blowing-ups for the dissolution all happen on the singular fibers and the eight of them occuring on the fiber of type I_8 are the blowing-ups of the intersections of any two adjacent components. Since we have to blow up two points on the special 2-section, it has to meet such a point of intersection. From this configuration we leave it to the reader to verify, using the above list, that the dual graph we obtain is the one of type VII.

  Starting with a special extremal fibration with singular fibers of type $(2\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$, we can check that there is a special fibration with double fiber of type 2III, which returns us to the case above. Indeed, if the 2-section meets distinct components of every fiber, we obtain a fibration with a singular fiber of type II^* which is not allowed by the assumption $A = \emptyset$.

  For the other configurations, we also obtain a special elliptic fibration with a degenerate double fiber from the 2-section and components of the fiber of type I_n with $n \geq 3$. Hence, the argumentation of the previous two cases applies. □

CHAPTER III

Numerically trivial automorphisms of Enriques surfaces

Up to minor modifications, this chapter is taken from the paper "Numerically trivial automorphisms of Enriques surfaces in characteristic 2", which is joint work of the author with I. Dolgachev. Currently, the paper is submitted and a preprint can be found on the ArXiv (see [DM17]).

1. Summary

We have seen in the previous chapters that the automorphism group of an Enriques surface with finite automorphism group can be read off almost completely from its dual graph of (-2)-curves. However, in some cases we had to calculate the automorphism group explicitly using the equations since we could not exclude automorphisms acting trivially on the graph of (-2)-curves. Such cohomologically and numerically trivial automorphisms are the topic of this Chapter III.

Quite generally, if X is a smooth and projective algebraic surface over an algebraically closed field k of characteristic $p \geq 0$, we call an automorphism g of X cohomologically trivial (resp. numerically trivial) if it acts trivially on the ℓ-adic étale cohomology $H^2_{\text{ét}}(X, \mathbb{Z}_l)$ (resp. $H^2_{\text{ét}}(X, \mathbb{Z}_l)$ modulo torsion).

Apart from automorphisms which belong to the connected group of automorphisms that preserves an ample divisor class, cohomologically and numerically trivial automorphisms are very rare. For example, over the field of complex numbers X must be either an elliptic surface with $q = p_g = 0$ or with $c_2 = 0$, or a surface of general type whose canonical linear system has a base point or its Chern classes satisfy $c_1^2 = 2c_2$ or $c_1^2 = 3c_2$ (see [Pet79]). In particular, a complex K3 surface does not admit non-trivial numerically trivial automorphisms, while a complex Enriques surface could. In the case of algebraic surfaces over a field of positive characteristic we know less. However, we know, for example, that K3 surfaces do not admit any numerically trivial automorphisms by work of A. Ogus [Ogu78], J. Keum [Keu12] and J. Rizov [Riz06].

The first example of a numerically trivial automorphism of an Enriques surface was constructed by D. Lieberman in 1976 [Lie78]. After that, the classification of complex Enriques surfaces with cohomologically or numerically trivial automorphisms was carried out by S. Mukai and Y. Namikawa: An Enriques surface has a numerically trivial automorphism if and only if it contains one of three critical configurations of (-2)-curves (see Remark 7.3) and Enriques surfaces with finite automorphism group, namely type I, III and V, give examples of Enriques surfaces realizing these three configurations.

In this chapter, we give the classification of numerically and cohomologically trivial automorphism groups of Enriques surfaces in arbitrary characteristic. As in the case of Enriques surfaces with finite automorphism group, the classification over the complex numbers uses transcendental methods. In positive characteristics, as in the other chapters, we use genus one fibrations instead of transcendental methods. A first attempt to give a classification in arbitrary characteristic was the content of the paper [Dol13] of I. Dolgachev. Although the main result of the paper is correct when $p \neq 2$, the analysis of possible groups in characteristic 2 is erroneous. In fact, we have seen many groups of cohomologically trivial automorphisms in Chapter II of this thesis, which contradict Dolgachev's claims. The goal of this chapter is to use a different approach to the study of cohomologically and numerically trivial automorphisms to obtain the complete classification of these groups in arbitrary characteristic:

THEOREM. *Let X be an Enriques surface over an algebraically closed field of characteristic $p \geq 0$.*

(1) If $p \neq 2$, then $|\mathrm{Aut}_{\mathrm{ct}}(X)| \leq 2$ and $\mathrm{Aut}_{\mathrm{nt}}(X) \cong \mathbb{Z}/2^a\mathbb{Z}$ with $a \leq 2$.
(2) If $p = 2$ and X is singular, then $|\mathrm{Aut}_{\mathrm{ct}}(X)| = |\mathrm{Aut}_{\mathrm{nt}}(X)| \leq 2$.
(3) If $p = 2$ and X is classical and not E_8-extra-special, then $|\mathrm{Aut}_{\mathrm{ct}}(X)| \leq 2$ and $\mathrm{Aut}_{\mathrm{nt}}(X) \cong (\mathbb{Z}/2\mathbb{Z})^a$ with $a \leq 2$.
(4) If $p = 2$ and X is supersingular, then $|\mathrm{Aut}_{\mathrm{ct}}(X)| = |\mathrm{Aut}_{\mathrm{nt}}(X)| \leq 2$, unless X is one of five types of exceptions distinguished by their dual graphs of (-2)-curves.

Moreover, if X is unnodal, then $\mathrm{Aut}_{\mathrm{ct}}(X) = \{1\}$.

Let us now explain the structure of Chapter III: In §2, ..., §4, we give the necessary background material on numerically trivial automorphisms, on genus one curves and on genus one fibrations of Enriques surfaces. In §5 we recall bielliptic models of Enriques surfaces, which are the main tool in our classification. After explaining the classification of extra-special Enriques surfaces in §6, we prove our main results in §7 and §8.

2. Generalities on numerically and cohomologically trivial automorphisms

Let X be an Enriques surface. It is known that

$$H^2_{\text{ét}}(X, \mathbb{Z}_l) \cong \mathrm{NS}(X) \otimes \mathbb{Z}_l, \quad H^2_{\text{ét}}(X, \mathbb{Z}_l)/\text{torsion} \cong \mathrm{Num}(X) \otimes \mathbb{Z}_l,$$

where $\mathrm{Num}(X) = \mathrm{NS}(X)/(K_X)$ is the group of divisor classes modulo numerical equivalence and $\mathrm{NS}(X)$ is the Néron-Severi group that coincides with the Picard group of X (see [CD89], Chapter 1, §2). The automorphism group $\mathrm{Aut}(X)$ is discrete in the sense that the connected component of the identity of the scheme of automorphisms $\mathbf{Aut}_{X/k}$ of X consists of one point, and admits natural representations

$$\rho : \mathrm{Aut}(X) \to \mathrm{Or}(\mathrm{NS}(X)), \quad \rho_n : \mathrm{Aut}(X) \to \mathrm{Or}(\mathrm{Num}(X)),$$

in the group of automorphisms of the corresponding abelian groups preserving the intersection form. We set

$$\mathrm{Aut}_{\mathrm{ct}}(X) = \mathrm{Ker}(\rho), \quad \mathrm{Aut}_{\mathrm{nt}}(X) = \mathrm{Ker}(\rho_n).$$

An automorphism in $\mathrm{Ker}(\rho)$ (resp. $\mathrm{Ker}(\rho_n)$) is called *cohomologically trivial* (resp. *numerically trivial*).

We start with the following general result that applies to any surface with discrete scheme of automorphisms and discrete Picard scheme.

PROPOSITION 2.1. *The groups $\mathrm{Aut}_{\mathrm{ct}}(X)$ and $\mathrm{Aut}_{\mathrm{nt}}(X)$ are finite groups.*

PROOF. We know that $\mathrm{NS}(X) = \mathrm{Pic}(X)$ and $\mathrm{Num}(X)$ is the quotient of $\mathrm{NS}(X)$ by its finite torsion subgroup $\mathrm{Tors}(\mathrm{NS}(X))$. Thus, the elementary theory of abelian groups gives us

$$\mathrm{Or}(\mathrm{NS}(X)) \cong \mathrm{Hom}(\mathrm{Num}(X), \mathrm{Tors}(\mathrm{NS}(X))) \rtimes \mathrm{Or}(\mathrm{Num}(X)).$$

This implies that

$$\mathrm{Aut}_{\mathrm{nt}}(X)/\mathrm{Aut}_{\mathrm{ct}}(X) \subseteq \mathrm{Tors}(\mathrm{NS}(X))^{\oplus \rho(X)}. \tag{2.1}$$

So, it is enough to prove that $G = \mathrm{Aut}_{\mathrm{ct}}(X)$ is a finite group. The group acts trivially on $\mathrm{Pic}(X)$, hence leaves invariant any very ample invertible sheaf $\mathcal{L}$. For any $g \in G$ let $\alpha_g : g^*(\mathcal{L}) \to \mathcal{L}$ be an isomorphism. Define a structure of a group on the set $\tilde{G}$ of pairs (g, α_g) by

$$(g, \alpha_g) \circ (g', \alpha_{g'}) = (g \circ g', \alpha_{g'} \circ g'^*(\alpha_g)).$$

The homomorphism $(g, \alpha_g) \to g$ defines an isomorphism $\tilde{G} \cong k^* \rtimes G$. The sheaf $\mathcal{L}$ admits a natural $\tilde{G}$-linearization, and hence the group $\tilde{G}$ acts linearly on the space $H^0(X, \mathcal{L})$ and the action defines an injective homomorphism $G \to \mathrm{Aut}(\mathbb{P}(H^0(X, \mathcal{L}))$. The group of projective transformations of X embedded by $|\mathcal{L}|$ is a linear algebraic group that has finitely many connected components. We know that G is discrete. Thus, the group G is finite. □

In our case, when X is an Enriques surface, we know that the torsion subgroup of $\mathrm{NS}(X)$ is generated by the canonical class K_X and $2K_X = 0$. Moreover, $K_X \neq 0$ if $p \neq 2$. Recall from Chapter II that, in characteristic 2, Enriques surfaces come in three types:

- classical surfaces,
- singular Enriques surfaces or $\boldsymbol{\mu}_2$-surfaces,
- supersingular surfaces or $\boldsymbol{\alpha}_2$-surfaces

Surfaces of the first type are characterized by the condition $K_X \neq 0$ if $p = 2$. Surfaces of the second and the third type satisfy $K_X = 0$. They are distinguished by the action of the Frobenius endomorphism on the cohomology space $H^2(X, \mathcal{O}_X) \cong k$. It is trivial in the third case and it is not trivial in the second case.

Applying (2.1), we obtain the following.

COROLLARY 2.2. *The quotient group $\mathrm{Aut}_{\mathrm{nt}}(X)/\mathrm{Aut}_{\mathrm{ct}}(X)$ is a 2-elementary abelian group.*

3. Half-fibers of genus one fibrations

In this section, we recall basic facts on genus one fibrations of Enriques surfaces, some of which we have already seen in the previous chapters. We will emphasize the relation to sequences of primitive isotropic vectors in $\mathrm{Num}(X)$, which will play an important role throughout this chapter.

Recall that an Enriques surface always admits a fibration $f : X \to \mathbb{P}^1$ with general fiber X_η an elliptic curve or a quasi-elliptic curve over the field K of rational functions on $\mathbb{P}^1$ (i.e. a regular non-smooth irreducible curve of arithmetical genus one) (see [CD89], Corollary 3.2.1). To treat both cases, we call such a fibration a *genus one fibration*, specifying when needed whether it is an *elliptic fibration* or a *quasi-elliptic fibration*.

A genus one fibration is defined by a base-point-free pencil $|D|$ of divisors of arithmetic genus one satisfying $D^2 = 0$. The numerical class $[D]$ in $\mathrm{Num}(X)$ is always divisible by two, so $D = 2F$, where $[F]$ is a primitive isotropic vector in the lattice $\mathrm{Num}(X)$. There are two representatives

F, F' of $[F]$ if $p \neq 2$ or X is classical Enriques surface in characteristic 2. Otherwise, there is only one representative. We call these representatives *half-fibers* of $|2F|$, of the pencil or of the corresponding fibration.

Conversely, let W_X^{nod} be the group of isometries of $\mathrm{Num}(X)$ generated by reflections into the classes of smooth rational curves ((-2)-*curves*, for short). Any primitive isotropic vector in $\mathrm{Num}(X)$ can be transformed by an element of W_X^{nod} to the numerical class of a half-fiber. Hence, any nef divisor F such that $[F]$ is a primitive isotropic vector in $\mathrm{Num}(X)$ defines a genus one pencil $|2F|$ and a corresponding genus one fibration $f : X \to \mathbb{P}^1$. An Enriques surface is called *unnodal* if it does not contain (-2)-curves. In this case $W_X^{\mathrm{nod}} = \{1\}$ and there is a bijective correspondence between primitive isotropic vectors in $\mathrm{Num}(X)$ and genus one fibrations on X.

A general fiber of an elliptic (resp. quasi-elliptic) fibration is a smooth elliptic curve (resp. irreducible curve of arithmetic genus one with one ordinary cusp). We will use Kodaira's notations for singular (resp. reducible) fibers of elliptic (resp. quasi-elliptic) fibrations, namely $\mathrm{I}_1, \mathrm{I}_n, \mathrm{I}_n^*, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$. Fibers of type I_n are called of *multiplicative type*, all others of *additive type*.

We have the following (see [CD89], Chapter 5. §7).

PROPOSITION 3.1. *Let F be a half fiber of a genus one fibration on an Enriques surface.*

- *If $p \neq 2$ or X is a singular Enriques surface in characteristic 2, then F is of multiplicative type or a smooth elliptic curve, which is ordinary if $p = 2$.*
- *If $p = 2$ and $K_X \neq 0$, then F is of additive type or a smooth ordinary elliptic curve.*
- *If $p = 2$ and X is a supersingular Enriques surface, then F is of additive type or a supersingular elliptic curve.*

A (-2)-curve is called a *special bisection* of a half-fiber F or of the corresponding pencil $|2F|$, or of the corresponding genus one fibration, if it intersects F with multiplicity 1.

A relatively minimal model of the Jacobian variety J_η of the generic fiber X_η of an elliptic fibration is a rational elliptic surface $j : J \to \mathbb{P}^1$. The group $J_\eta(\eta)$ is called the *Mordell-Weil group* of the elliptic fibration. It is a finitely generated abelian group. It acts on X_η by translation, and by the properties of a relative minimal model, the action extends to a biregular action on X.

The type of a singular fiber J_t of $j : J \to \mathbb{P}^1$ coincides with the type of the fiber X_t (see [CD89], Theorem 5.3.1 and [LLR04], Theorem 6.6). Similarly, if the fibration is quasi-elliptic, the Jacobian variety J_η of its general fiber is a unipotent group scheme, a non-trivial inseparable form of the additive group scheme. Its Mordell-Weil group is a finite p-elementary abelian group. The theory of minimal models of surfaces allows us to construct a rational surface with a quasi-elliptic fibration whose generic fiber with the singular point deleted is isomorphic to J_η.

An ordered sequence $(f_1, \ldots, f_n)$ of isotropic vectors in $\mathrm{Num}(X)$ with $f_i \cdot f_j = 1 - \delta_{ij}$ and $f_i \cdot h > 0$ for the class of an ample divisor h can always be transformed by an element $w \in W_X^{\mathrm{nod}}$ to a sequence where $f_1 + \cdots + f_n$ is the class of a nef divisor. A lift $(F_1, \ldots, F_n)$ of such a sequence to $\mathrm{NS}(X)$ is called a $U_{[n]}$-*sequence*. After reordering, we may assume that F_1 is a half-fiber of a genus one fibration and either $F_{i+1} = F_i + R$, where R is a (-2)-curve with $R \cdot F_i = 1$ or F_{i+1} is a half-fiber of a genus one fibration. A $U_{[n]}$-sequence is called *c-degenerate*, if it contains exactly c half-fibers. If $c = n$, it is called *non-degenerate*. We say that a $U_{[m]}$-sequence A *extends* a $U_{[n]}$-sequence B if, after reordering, A contains B. For a given Enriques surface X, the maximal length of a non-degenerate $U_{[n]}$-sequence is denoted by $\mathrm{nd}(X)$ and is called the *non-degeneracy invariant* of X.

REMARK 3.2. Note that, by definition, the R_i that occur in a $U_{[n+1]}$-sequence of the form $(F_1, F_1 + R_1, \ldots, F_1 + \sum_{i=1}^{n} R_i)$ form a Dynkin diagram of type A_n and the R_i with $i \geq 2$ are contained in fibers of $|2F_1|$.

For the following Proposition, see [DL] Proposition 5.1.5.

PROPOSITION 3.3. *Let $n \leq 8$. Then, any c-degenerate $U_{[n]}$-sequence can be extended to a c'-degenerate $U_{[10]}$-sequence with $c' \geq c$.*

It is a much more difficult question whether a non-degenerate $U_{[n]}$-sequence can be extended to a non-degenerate $U_{[m]}$-sequence (see e.g. Section 5). However, the following is known (see [Cos85], Theorem 3.5 or [DL], Theorem 5.1.17).

THEOREM 3.4. *Suppose $p \neq 2$ or X is a singular Enriques surface. Then, any half-fiber can be extended to a non-degenerate $U_{[3]}$-sequence.*

LEMMA 3.5. *Let F_1, F_2 form a non-degenerate $U_{[2]}$-pair. Then, F_1 and F_2 do not have common irreducible components.*

PROOF. We use that a fiber F_1 is numerically 2-connected, i.e. if we write F_1 as a sum of two proper effective divisors $F_1 = D_1 + D_2$, then $D_1 \cdot D_2 \geq 2$. To see this, we use that $D_1^2 < 0, D_2^2 < 0$ and $F_1^2 = F_1 \cdot D_1 = F_1 \cdot D_2 = 0$. Now, if D_1 is the maximal effective divisor with $D_1 \leq F_1$ and $D_1 \leq F_2$ and if we let $F_1 = D_1 + D_2$ and $F_2 = D_1 + D_2'$ be decompositions into effective divisors, we have $D_2.D_2' \geq 0$. Therefore $1 = F_1 \cdot F_2 = (D_1 + D_2) \cdot F_2 = (D_2 \cdot D_1 + D_2 \cdot D_2') \geq D_2 \cdot D_1$, hence $D_1 = 0$. □

Let (F_1, F_2) be a non-degenerate $U_{[2]}$-sequence. Since $F_1 \cdot F_2 = 1$, by the previous lemma, $F_1 \cap F_2$ consists of one point.

LEMMA 3.6. *Let (F_1, F_2, F_3) be a non-degenerate $U_{[3]}$-sequence. Suppose that $|F_2 + F_3 - F_1 + K_X| = \emptyset$. Then, $F_1 \cap F_2 \cap F_3 = \emptyset$.*

PROOF. Consider the natural exact sequence coming from restriction of the sheaf $\mathcal{O}_X(F_1 - F_2)$ to F_3:

$$0 \to \mathcal{O}_X(F_1 - F_2 - F_3) \to \mathcal{O}_X(F_1 - F_2) \to \mathcal{O}_{F_3}(F_1 - F_2) \to 0.$$

We have $(F_1 - F_2 - F_3) \cdot F_1 = -2$. Since F_1 is nef, the divisor class $F_1 - F_2 - F_3$ is not effective. Thus, by Riemann-Roch and Serre's Duality, $h^1(\mathcal{O}_X(F_1 - F_2 - F_3)) = 0$ since $h^0(\mathcal{O}_X(K_X + F_3 + F_2 - F_1)) = 0$ by assumption. Now, $h^0(\mathcal{O}_X(F_1 - F_2)) = 0$, because $(F_1 - F_2).F_1 = -1$ and F_1 is nef. Suppose $F_1 \cap F_2 \cap F_3 \neq \emptyset$, then $\mathcal{O}_{F_3}(F_1 - F_2) \cong \mathcal{O}_{F_3}$ and $h^0(\mathcal{O}_{F_3}(F_1 - F_2)) = 1$. It remains to consider the exact sequence of cohomology and get a contradiction. □

REMARK 3.7. Note that for any $D \in |F_2 + F_3 - F_1 + K_X|$, we have $D^2 = -2$ and $D.F_2 = D.F_3 = 0$, so D consists of (-2)-curves contained in fibers of $|2F_2|$ and $|2F_3|$.

4. Automorphisms of genus one curves

Let us recall some known results about automorphism groups of elliptic curves over algebraically closed fields which we will use frequently. The proof of the following result can be found in [Sil09], III, §10 and Appendix A.

PROPOSITION 4.1. *Let E be an elliptic curve over an algebraically closed field with automorphism group G and absolute invariant j. For $g \in G$, let E^g be the set of fixed points of g.*

(1) If $p \neq 2, 3$

j	G	*ord*(g)	$\lvert E^g \rvert$
$\neq 0, 1$	$\mathbb{Z}/2\mathbb{Z}$	2	4
1	$\mathbb{Z}/4\mathbb{Z}$	$\begin{cases} 2 \\ 4 \end{cases}$	$\begin{cases} 4 \\ 2 \end{cases}$
0	$\mathbb{Z}/6\mathbb{Z}$	$\begin{cases} 2 \\ 3 \\ 4 \\ 6 \end{cases}$	$\begin{cases} 4 \\ 3 \\ 2 \\ 1 \end{cases}$

(2) If $p = 3$

j	G	*ord*(g)	$\lvert E^g \rvert$
$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	2	4
0	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$	$\begin{cases} 2 \\ 3 \\ 4 \end{cases}$	$\begin{cases} 4 \\ 1 \\ 2 \end{cases}$

(3) If $p = 2$

j	G	*ord*(g)	$\lvert E^g \rvert$
$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	2	2
0	$Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$	$\begin{cases} 2, 4 \\ 3 \end{cases}$	$\begin{cases} 1 \\ 3 \end{cases}$

5. Bielliptic maps and bielliptic involutions

Let (F_1, F_2) be a non-degenerate $U_{[2]}$-pair of half-fibers. The linear system $|2F_1 + 2F_2|$ defines a morphism of degree 2 from X to a surface D of degree 4 in $\mathbb{P}^4$ (it is called a superelliptic map in [CD89], renamed as a bielliptic map in [CDL]). The surface D is an anti-canonical model of a unique (up to isomorphism) weak del Pezzo surface of degree 4 obtained by blowing up 5 points $p_1, \ldots, p_5$ in the projective plane $\mathbb{P}^2$.

If $K_X \neq 0$, the point p_3 is infinitely near to p_2 and p_5 is infinitely near to p_4. The points p_1, p_2, p_3 and p_1, p_4, p_5 lie on lines ℓ_1 and ℓ_1'. The proper inverse transform of the pencil of lines through p_1 and the pencil of conics through p_2, p_3, p_4, p_5 on $\mathbb{P}^2$ are pencils of conics on D. The proper inverse transforms of the lines ℓ_1, ℓ_1' (resp. the lines ℓ_2, ℓ_2' passing through p_2, p_4 and the exceptional curve over p_1) on $\mathbb{P}^2$ are the four lines L_1, L_1' (resp. L_2, L_2') on D. The proper inverse transforms of the two pencils of conics on D are the genus one pencils $|2F_1|$ and $|2F_2|$ of X. The half-fibers F_1, F_1' (resp. F_2, F_2') are the proper inverse transforms of the lines L_1, L_1' (resp. L_2, L_2'). One can choose projective coordinates in $\mathbb{P}^4$ so that D is given by equations

$$x_0^2 + x_1x_2 = x_0^2 + x_3x_4 = 0. \tag{5.1}$$

The pencils of conics that give rise to the pencils $|2F_1|$ and $|2F_2|$ are cut out by the linear pencils of planes

$$ax_2 + bx_3 = ax_4 + bx_1 = 0, \quad ax_2 + bx_4 = ax_3 + bx_1 = 0. \tag{5.2}$$

The lines are given by equations $x_0 = x_i = x_j = 0, i \in \{1,2\}, j \in \{3,4\}$. They correspond to the parameters $(a:b) = (1:0)$ and $(0:1)$.

If $K_X = 0$ and X is singular (resp. supersingular), the surface D has a unique singular point, which is a rational double of type $D_4^{(1)}$ (resp. $D_4^{(0)}$). The surface is again an anti-canonical model of a unique (up to isomorphism) weak del Pezzo surface of degree 4, which is the blow-up of 5 points $p_1, \ldots, p_5$ in $\mathbb{P}^2$, where p_5 is infinitely near to p_4, p_4 is infinitely near to p_3 and p_3 is infinitely near to p_2. The surface has only two lines. Their proper inverse transforms on X are the half-fibers of the genus one fibrations $|2F_1|$ and $|2F_2|$. The fibrations themselves are defined by the pencils of conics on D obtained from the pencil of lines through p_1 and the pencil of conics through the points p_2, p_3, p_4, p_5. The surface D is isomorphic to a surface given by equations

$$x_0^2 + x_1x_2 = x_1x_3 + x_4(ex_0 + x_2 + x_4) = 0, \tag{5.3}$$

where $e = 1$ if X is singular, and $e = 0$ if X is supersingular. The pencils of concis that give rise to our pencils are given by the equations

$$ax_3 + b(ex_0 + x_2 + x_4) = ax_4 + bx_1 = 0, \quad a(ex_0 + x_2 + x_4) + bx_1 = ax_3 + bx_4 = 0. \tag{5.4}$$

If the map ϕ is separable, the birational automorphism of X defined by the degree two separable extension of the fields of rational functions $k(X)/\phi^*k(\mathsf{D})$ extends to a biregular automorphism of X which we call a *bielliptic involution* of X.

The group of automorphisms of the surface D is a subgroup of projective transformations of $\mathbb{P}^4$ that leaves the surface D invariant. The following proposition describes the group of automorphisms of the quartic surface D.[1]

PROPOSITION 5.1. *Let* $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ *be the image of a bielliptic map defined by the linear system* $|2F_1 + 2F_2|$, *where* $K_X \neq 0$, X *is singular, or* X *is supersingular, respectively. Then*

- *Aut*$(\mathsf{D}_1) \cong \mathbb{G}_m^2 \rtimes D_8$;
- *Aut*$(\mathsf{D}_2) \cong \mathbb{G}_a^2 \rtimes \mathbb{Z}/2\mathbb{Z}$;
- *Aut*$(\mathsf{D}_3) \cong (\mathbb{G}_a^2 \rtimes \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Here, $\mathbb{G}_m$ *(resp.* $\mathbb{G}_a$*) denote the multiplicative (resp. additive) one-dimensional algebraic group over* k *and* D_8 *denotes the dihedral group of order* 8.

[1] The computation of these groups in the cases of surfaces $\mathsf{D}_2, \mathsf{D}_3$ in [CD89] is erroneous. The correct computation can be found in [CDL], Proposition 0.6.26.

REMARK 5.2. Note that the connected component $\mathrm{Aut}(\mathsf{D})^0$ of $\mathrm{Aut}(\mathsf{D})$ is the group of automorphisms preserving each line on D. Using equations (5.1) and (5.3), we can write the action of $\mathrm{Aut}(\mathsf{D})^0$ explicitly as follows, with $\lambda, \mu \in \mathbb{G}_m$ and $\alpha, \beta \in \mathbb{G}_a$:

- Action of $\mathrm{Aut}(\mathsf{D}_1)^0$:

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 : \lambda x_1 : \lambda^{-1} x_2 : \mu x_3 : \mu^{-1} x_4)$$

- Action of $\mathrm{Aut}(\mathsf{D}_2)^0$:

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0+\alpha x_1 : x_1 : \alpha^2 x_1+x_2 : \beta x_0+(\alpha\beta+\alpha^2\beta+\beta^2)x_1+\beta x_2+x_3+(\alpha+\alpha^2)x_4 : \beta x_1+x_4)$$

- Action of $\mathrm{Aut}(\mathsf{D}_3)^0$:

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 + \alpha x_1 : x_1 : \alpha^2 x_1 + x_2 : (\alpha^2\beta + \beta^2)x_1 + \beta x_2 + x_3 + \alpha^2 x_4, \beta x_1 + x_4)$$

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 : \lambda^{-1} x_1 : \lambda x_2 : \lambda^3 x_3, \lambda x_4)$$

Moreover, we can compute the group of automorphism fixing the pencils given by equations (5.2) (resp. (5.4)) on D. They are obtained by setting $\lambda = \mu$ (resp. $\alpha \in \{0, 1\}, \beta = 0$, resp. $\alpha = \beta = 0, \lambda = 1$).

The known information about the automorphism group of the surfaces D allows us to give a criterion for an automorphism to be a bielliptic involution.

COROLLARY 5.3. *Let (F_1, F_2) be a non-degenerate $U_{[2]}$-sequence and let g be a non-trivial automorphism of X. Assume that g preserves F_1, F_2 and a (-2)-curve E with $E.F_1 = E.F_2 = 0$, which is not a component of one of the half-fibers F_1, F_2, F_1', F_2'. If X is supersingular, assume additionally that g has order 2^n. Then, g is the bielliptic involution associated to the linear system $|2F_1 + 2F_2|$.*

PROOF. Let $\phi : X \to \mathsf{D}$ be a bielliptic map defined by the linear system $|2F_1 + 2F_2|$. Since g leaves $|2F_1 + 2F_2|$ invariant, it descends to an automorphism of $\mathbb{P}^4 = |2F_1 + 2F_2|^*$ that leaves D invariant. Moreover, the induced automorphism preserves the lines on D by assumption. Recall that $E.F_1 = E.F_2 = 0$, hence $\phi(E)$ is a point P. Since E is not a component of one of the half-fibers, P does not lie on any of the lines of D. If $\mathsf{D} = \mathsf{D}_1$, this means that P is not on the hypersurface $x_0 = 0$ and if $\mathsf{D} \in \{\mathsf{D}_2, \mathsf{D}_3\}$, it means that P is not on the hypersurface $x_1 = 0$.

If $\mathsf{D} = \mathsf{D}_1$, the x_0 coordinate $x_0(P)$ of P is non-zero, hence so are all $x_i(P)$ by Equation (5.1). By Remark 5.2, there is no automorphism of D_1 fixing P and preserving the lines except the identity. Therefore, g coincides with the covering involution of ϕ.

If $\mathsf{D} \in \{\mathsf{D}_2, \mathsf{D}_3\}$, we have $x_1(P) \neq 0$. Again, by Remark 5.2, there is no automorphism of D_2 fixing P and preserving the lines except the identity. For D_3, we use the additional assumption to exclude the case that g acts on D_3 via $\mathbb{G}_m$. □

REMARK 5.4. In fact, the failure of this criterion without the additional assumption in the supersingular case leads to the existence of cohomologically trivial automorphisms of odd order (see Section 7).

LEMMA 5.5. *Let τ be the bielliptic involution associated to a linear system $|2F_1 + 2F_2|$. Suppose τ is numerically trivial. Then, Num$(X)_{\mathbb{Q}}$ is spanned by the numerical classes $[F_1], [F_2]$ and eight smooth rational curves that are contained in fibers of both $|2F_1|$ and $|2F_2|$.*

PROOF. We have a finite degree 2 cover $X' = X - E \to D' = D - P$, where E is spanned by (-2)-curves blown down to a finite set of points P on D. We have Pic$(\mathsf{D}')_{\mathbb{Q}}$ = Pic$(\mathsf{D})_{\mathbb{Q}}$ and Pic$(X')^g_{\mathbb{Q}}$ (the invariant part) = f^*(Pic$(\mathsf{D}')_{\mathbb{Q}}$ is spanned by the restriction of F_1, F_2 to X'. Since Pic(X) is spanned by Pic(X') and the classes of components of E, we can write any any invariant divisor class as a linear combination of $[F_1], [F_2]$ and invariant components of E. In our case all divisors classes are invariant. Since $\dim(\text{Pic}(X)_{\mathbb{Q}}) - \dim(\langle F_1, F_2\rangle_{\mathbb{Q}}) = 8$, E consists of eight (-2)-curves. □

Denote the number of irreducible components of a fiber D of $|2F|$ by m_D. Since $\mathrm{rk}(\text{Pic}(X)) = 10$, we have $\sum_{D\in|2F|}(m_D - 1) \le 8$, and, by the Shioda-Tate formula, the Jacobian of $|2F|$ has finite Mordell-Weil group if and only if equality holds. In the latter case, $|2F|$ is called *extremal*.

COROLLARY 5.6. *Let (F_1, F_2) be a $U_{[2]}$-pair of half-fibers such that the bielliptic involution τ associated to $|2F_1 + 2F_2|$ is numerically trivial. Then, $|2F_1|$ and $|2F_2|$ are extremal.*
Moreover, the following hold:

(1) For every fiber D of $|2F_1|$, all but one component C of D is contained in fibers of $|2F_2|$.
(2) C has multiplicity at most 2.
(3) Neither $|2F_1|$ nor $|2F_2|$ have a multiplicative fiber with more than two components.

PROOF. By the previous lemma, there are eight (-2)-curves contained in fibers of both $|2F_1|$ and $|2F_2|$. Since a fiber of $|2F_1|$ cannot contain a full fiber of $|2F_2|$, this implies $8 \le \sum_{D\in|2F|}(m_D - 1) \le 8$. Hence, $|2F_1|$ is extremal and so is $|2F_2|$. Moreover, if, for some fiber D of $|2F_1|$, two components of D are not contained in fibers of $|2F_2|$, then, by the same formula, $|2F_1|$ and $|2F_2|$ share less than eight (-2)-curves. This contradicts Lemma 5.5.

For (2), note that the remaining component C of multiplicity m in D satisfies $2 = D.F_2 = mC.F_2$. Since $C.F_2 > 0$, this yields (2).

As for (3), assume that D is multiplicative with more than 2 components. Note that C meets distinct points on distinct components of D. The connected divisor $D' = D - C$ satisfies $D'.(2F_1 + 2F_2) = 0$, hence it is contained in the exceptional locus of the bielliptic map ϕ. Since τ preserves the components of D', $\phi(C)$ is an irreducible curve with a node. But C is contained in the pencil of conics induced by $|2F_1|$. This is a contradiction. □

6. Extra-special Enriques surfaces

Throughout this section, we assume that $p = 2$ and X is either classical or supersingular. An Enriques surface X is called *extra-special* if $\mathrm{nd}(X) \le 2$.

It is claimed in [CD89], Theorem 3.5.1 that Theorem 3.4 is true in any characteristic unless the surface is extra-special with finitely many (-2)-curves with the dual graph defined by one of the diagrams from the following Table 1. The surfaces of type $\tilde{E}_8$, $\tilde{E}_7^1$ and $\tilde{D}_8$ are called E_8, E_7 and D_8-extra-special, respectively. However, the surface of type $\tilde{E}_7^2$ was erroneously asserted to have $\mathrm{nd}(X) = 2$, although, in fact, it is not extra-special and has $\mathrm{nd}(X)$ equal to 3 (see [DL, Proposition

5.2.4]).[2]. We refer the reader to [DL] for a new proof due to the author of the classification of extra-special surfaces and collect the results we need in the context of numerically trivial automorphisms in this section.

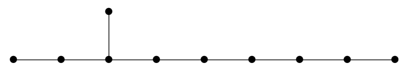
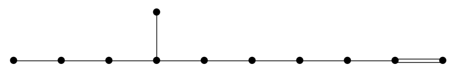
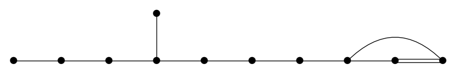
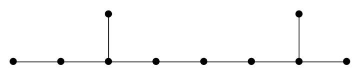
Type	Configuration
$\tilde{E}_8$	
$\tilde{E}_7^1$	
$\tilde{E}_7^2$	
$\tilde{D}_8$	

TABLE 1. E_8,E_7 and D_8-extra-special surfaces and the $\tilde{E}_7^2$ surface

THEOREM 6.1. *Assume that X is not E_8-extra-special. Then, any half-fiber can be extended to a non-degenerate $U_{[2]}$-sequence.*

THEOREM 6.2. *Assume that X is not E_8,E_7 or D_8-extra-special. Then,* $\mathrm{nd}(X) \geq 3$.

REMARK 6.3. In Chapter II, the cohomologically trivial and numerically trivial automorphism groups of extra-special surfaces have been calculated. For our examples, the groups are given in Table 2.

Type	$\mathrm{Aut}_{\mathrm{ct}}(X)$	$\mathrm{Aut}_{\mathrm{nt}}(X)$
classical $\tilde{E}_8$	$\{1\}$	$\{1\}$
supersingular $\tilde{E}_8$	$\mathbb{Z}/11\mathbb{Z}$	$\mathbb{Z}/11\mathbb{Z}$
classical $\tilde{D}_8$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
supersingular $\tilde{D}_8$	Q_8	Q_8
classical $\tilde{E}_7^1$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$

TABLE 2. Numerically trivial automorphisms of extra-special surfaces

However, as we have seen in the previous chapter, it is not known whether there are more surfaces of these types than the ones given in Chapter II. Note that the calculation of these groups in the case where X is classical of type $\tilde{D}_8$ or $\tilde{E}_7^1$ only depends on the dual graph of (-2)-curves (see Chapter II, Theorem 10.11 and Theorem 9.6)

[2]So far, this is the only known example of an Enriques surface with $\mathrm{nd}(X) = 3$.

7. Cohomologically trivial automorphisms

Now that we have treated the necessary background material, we can proceed to the heart of this chapter. In this section, we prove our main results on cohomologically trivial automorphisms.

7.1. Cohomologically trivial automorphisms of even order.

THEOREM 7.1. *Let X be an Enriques surface which is not extra-special.*

(1) If X is classical or singular, then $|Aut_{\mathrm{ct}}(X)| \leq 2$. If X is also unnodal, then $Aut_{\mathrm{ct}} = \{1\}$.
(2) If X is supersingular, then the statements of (1) hold for the 2-Sylow subgroup G of $Aut_{\mathrm{ct}}(X)$.

Moreover, if a non-trivial $g \in Aut_{\mathrm{ct}}$ (resp. G) exists, then g is a bielliptic involution.

PROOF. Let $g \in \mathrm{Aut}_{\mathrm{ct}}(X)$ and assume that g has order 2^n if X is supersingular. Note that, by definition, g preserves all half-fibers on X. We will show that there is a $U_{[2]}$-pair such that g satisfies the conditions of Corollary 5.3. Note that g preserves all half-fibers and (-2)-curves, since it is cohomologically trivial, so it suffices to find a (-2)-curve, which is contained in two simple fibers of genus one fibrations forming a $U_{[2]}$-pair.

Take a c-degenerate $U_{[10]}$-sequence on X with c maximal. If $3 \leq c \leq 9$, then there is a (-2)-curve R in this sequence such that $F.R = 0$ for at least 3 half-fibers F in the sequence. Now, Lemma 3.5 shows that R is contained in a simple fiber of two pencils $|2F_1|$ and $|2F_2|$. By Corollary 5.3, g is the bielliptic involution associated to $|2F_1 + 2F_2|$. In particular, g is unique.

If $c = 10$, assume that one of the half-fibers, say F_1, is reducible. Then, by Lemma 3.5, for every F_i in the sequence, all but one component of F_1 is contained in simple fibers of $|2F_i|$. Hence, we find some component R with $R.F_i = 0$ for at least 3 half-fibers and the same argument as before applies.

If $|F_i + F_j - F_k| \neq \emptyset$ or $|F_i + F_j - F_k + K_X| \neq \emptyset$ for some half-fibers F_i, F_j, F_k occuring in the sequence, by Remark 3.7, there is an effective divisor D with $D.F_i = D.F_j = 0$ and $D^2 = -2$. Since F_i and F_j can be assumed to be irreducible, D contains a (-2)-curve which is contained in a simple fiber of both $|2F_i|$ and $|2F_j|$. Again, Corollary 5.3 applies.

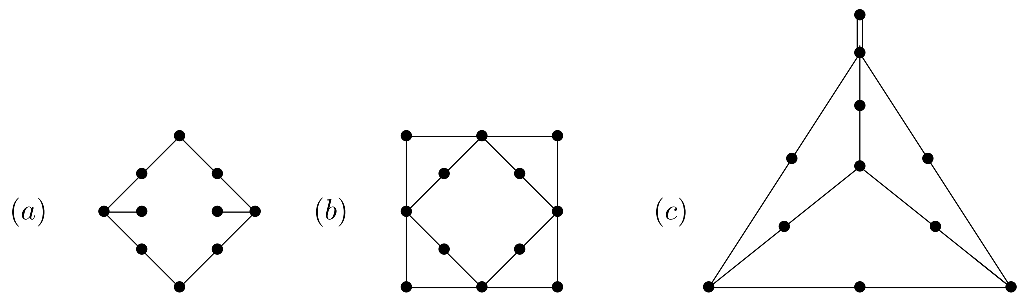
Therefore, we can assume that all half-fibers are irreducible and $F_i \cap F_j \cap F_k = \emptyset$ by Lemma 3.6. This is immediate if X is unnodal. Then, g fixes all F_i pointwise by Proposition 4.1, hence it is trivial, as can be seen by applying the same Proposition to a general fiber of, say, $|2F_1|$. □

In the case of classical Enriques surface in characteristic 2, we can say more, using the classification of Enriques surfaces with finite automorphism group.

COROLLARY 7.2. *Let X be a classical Enriques surface in characteristic 2 which is not E_8-extra-special. Then, $Aut_{\mathrm{ct}}(X) \cong \mathbb{Z}/2\mathbb{Z}$ if and only if X is D_8-extra-special.*

PROOF. Let F_1 be a half-fiber on X. By Theorem 3.4, we can extend F_1 to a non-degenerate $U_{[2]}$-sequence. Assume that there exists a non-trivial $g \in \mathrm{Aut}_{\mathrm{ct}}(X)$. Then, g acts on D_1 via its action on $|2F_1 + 2F_2|^*$. By Proposition 5.1, g acts via $\mathbb{G}_m^2$ on D_1. But g has order 2 by Theorem 7.1, hence it acts trivially on D_1. Therefore, g is the covering involution of the bielliptic map and by Corollary 5.6, $|2F_1|$ is extremal. Therefore, every genus one fibration on X is extremal. In particular, by Chapter II Section 12, X has finite automorphism group. We have calculated the groups $\mathrm{Aut}_{\mathrm{ct}}(X)$ of these surfaces and the only surfaces for which the calculation of the groups depends on the specific example are the ones of type $\tilde{E}_8$ and $\tilde{D}_4+\tilde{D}_4$ (see Remark 6.3 and Remark 8.4). In the latter case, there is a $U_{[2]}$-pair of fibrations with simple I_4^* fibers, which share only 7 components. By Corollary 5.6, the corresponding bielliptic involution is not cohomologically trivial. Therefore, our calculation of the groups shows that the D_8-extra-special surface is the only classical Enriques surface which is not E_8-extra-special and has a non-trivial cohomologically trivial automorphism. □

REMARK 7.3. Using Theorem 7.1 and Corollary 5.6 may lead to an explicit classification of Enriques surfaces X with non-trivial $\mathrm{Aut}_{\mathrm{nt}}(X)$. For example, in characteristic $p \neq 2$, one can show that the surface must contain (-2)-curves with one of the following dual graphs:



In the case $k = \mathbb{C}$ this is an assertion from [Kon86, Theorem (1.7)].

7.2. Cohomologically trivial automorphisms of odd order. Before we start with the treatment of cohomologically trivial automorphisms of odd order of supersingular Enriques surfaces, let us collect the known examples. These surfaces have finite automorphism groups and a detailed study can be found in Chapter II. In Table 3, we recall the group of cohomologically trivial automorphisms of these examples. Again, it is not known whether there are more examples of these surfaces than the ones given in Chapter II.

Type	$\mathrm{Aut}_{\mathrm{ct}}(X)$
$\tilde{E}_8$	$\mathbb{Z}/11\mathbb{Z}$
$\tilde{E}_7^2$	$\mathbb{Z}/7\mathbb{Z}$ or $\{1\}$
$\tilde{E}_6$	$\mathbb{Z}/5\mathbb{Z}$

TABLE 3. Examples of cohomologically trivial automorphisms of odd order

LEMMA 7.4. *Let X be a supersingular Enriques surface which is not E_8-extra-special and let $G \subseteq \mathrm{Aut}_{\mathrm{ct}}(X)$ be a non-trivial subgroup of odd order. Then, G is cyclic and acts non-trivially on the base of every genus one fibration of X.*

PROOF. Take any half-fiber F_1 and extend it to a non-degenerate $U_{[2]}$-sequence (F_1, F_2) on X. Since G has odd order, it acts on D_3 via a finite subgroup of $\mathbb{G}_m$, hence G is cyclic. By Remark 5.2, a generator g of G acts on the image D_3 of the bielliptic map as

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 : \lambda^{-1}x_1 : \lambda x_2 : \lambda^3 x_3, \lambda x_4).$$

Such an automorphism acts non-trivially on the pencils of conics given by Equation (5.4), hence g acts non-trivially on $|2F_1|$. □

LEMMA 7.5. *Let F be a fiber of a genus one fibration and let g be a tame automorphism of finite order that fixes the irreducible components of F. Then, the Lefschetz fixed-point formula*

$$e(F^g) = \sum_{i=0}^{2}(-1)^i Tr(g^*|H^i_{\text{ét}}(F, \mathbb{Q}_l)).$$

holds for F. If F is reducible and not of type I_2, then $e(F^g) = e(F)$. If F is of type I_2, then $e(F^g) = e(F) = 2$ or $e(F^g) = 4$. The latter case can only occur if g has even order.

PROOF. In case the order is equal to 2, this is proven in [Dol13] by a case-by-case direct verification. The proof uses only the fact that a tame non-trivial automorphism of finite order of $\mathbb{P}^1$ has two fixed points. Also note that the verification in case F is of type I_2 and g interchanges the two singular points of F was missed, but it still agrees with the Lefschetz formula. □

PROPOSITION 7.6. *Let $g \in Aut_{\mathrm{ct}}(X)$ be an automorphism of odd order. Then, every genus one pencil $|D|$ of X has one of the following combinations of singular fibers*

(7.1) $\quad I_0^* + I_0^*\ I_4^* + II,\ IV^* + IV,\ III^* + III,\ II^* + II,\ I_9 + I_1 + I_1 + I_1,\ I_3^*,\ III^*$

The last three configurations can only occur if g has order 3.

PROOF. The claim is clear if X is E_8-extra-special, hence we can apply Lemma 7.4 and find that g acts non-trivially on all genus one pencils. Since the order of g is prime to p, it fixes two members F_1, F_2 of the pencil, one of which is a double fiber. Since all other fibers are moved, the set of fixed points X^g is contained in $F_1 \cup F_2$. Applying the Lefschetz fixed-point formula, we obtain

$$e(X) = 12 = e(X^g) = e(F_1^g) + e(F_2^g), \tag{7.2}$$

where $e()$ denotes the l-adic topological Euler-Poincaré characteristic.

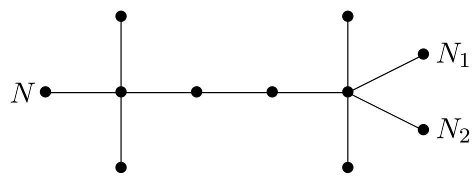
If one of the fibers, say F_1 is smooth, then, since g has odd order and $e(F_2^g) \leq 10$, σ acts as an automorphism of order 3 on F_1. Hence, by Proposition 4.1, g has three fixed points on F_1. Therefore, F_2 is of type I_9, I_3^* or III^* and g has order 3. By [Lan00], we get the last three configurations of the Proposition.

If both fibers or the corresponding half-fibers are singular curves, then $e(F_i) = e(F_i^g)$. Indeed, for irreducible and singular curves, this follows from $e(F_2^g) \leq 10$ and for reducible fibers, this is Lemma 7.5 for automorphisms of odd order. The formula for the Euler-Poincaré characteristic of an elliptic surface from [CD89], Proposition 5.1.6 implies that F_1 and F_2 are the only singular fibers of $|D|$. In this case, denoting the number of irreducible components of F_i by m_i, we have $m_1 + m_2 \geq 8$, hence $|2F|$ is extremal and both fibers are of additive type. The classification of singular fibers of extremal rational genus one fibrations is known [Lan91], [Lan94], [Ito02]. Since the types of singular fibers of a genus one fibration and of its Jacobian fibration are the same, it is straightforward to check that the list given in the proposition is complete. □

COROLLARY 7.7. *If X admits an automorphism $g \in Aut_{\rm ct}(X)$ of odd order at least 5, then X is one of the surfaces in Table 3.*

PROOF. By Proposition 7.6, every genus one fibration on X is extremal. It is shown in Chapter II Section 12, that such an Enriques surface has finite automorphism group. Using the list of Proposition 7.6, the claim follows from the classification of supersingular Enriques surfaces with finite automorphism group. □

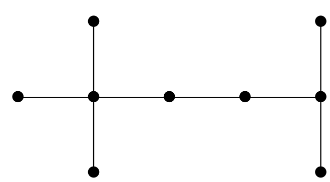
PROPOSITION 7.8. *Assume that X is not one of the surfaces in Table 3. If X admits an automorphism $g \in Aut_{\rm ct}(X)$ of order 3, then X contains the following diagram of (-2)-curves*



In this case, $Aut_{\rm ct}(X) = \mathbb{Z}/3\mathbb{Z}$.

PROOF. If every special genus one fibration on X is extremal, then X has finite automorphism group by Chapter II Section 12. Therefore, we observe that, by Proposition 7.6, X has to admit a special genus one fibration with special bisection N such that g fixes two fibers F_1 and F_2, where F_1 is a smooth supersingular elliptic curve and F_2 is of type III^* or I_3^*. If F_1 is a simple fiber, then N meets two distinct points of F_1, since g does not fix the tangent line at any point of F_1. But then, g fixes three points on N, hence it fixes N pointwise, which contradicts Corollary 7.4.

Therefore, F_1 is a double fiber and an argument similar to the above also shows that N meets a component of multiplicity 2 of F_2. Now, depending on the intersection behaviour of N with F_2, we see that N and components of F_2 form a half-fiber of type I_n^* or IV^* of some other genus one fibration. Using the list of Proposition 7.6, we conclude that F_2 is of type I_3^* and N intersects F_2 as follows:



The five leftmost vertices form a fiber of type I_0^*. By Proposition 7.6, this diagram is a half-fiber of a fibration with singular fibers I_0^* and I_0^*. Adding the second fiber to the diagram, we arrive at the diagram of the Proposition.

Now, observe that the fibration we started with has three (-2)-curves as bisections. They are the curves N, N_1, N_2 in the diagram from the assertion of the proposition. All of them are fixed pointwise by any cohomologically trivial automorphism of order 2, since such an automorphism fixes their intersection with F_1 and F_2. Hence, no such automorphism can exist by Proposition 4.1 applied to a general fiber of the fibration. Since no cohomologically trivial automorphisms of higher order can occur on X by Corollary 7.7, we obtain $\mathrm{Aut}_{\mathrm{ct}}(X) = \mathbb{Z}/3\mathbb{Z}$. $\square$

REMARK 7.9. In fact, using conductrices as in Chapter II, one can show that the only genus one fibrations on the supersingular Enriques surface of Proposition 7.8 are quasi-elliptic fibrations with singular fibers of types I_0^* and I_0^* or elliptic fibrations with a unique singular fiber of type I_3^*.

THEOREM 7.10. *Assume that the automorphism groups of surfaces of type $\tilde{E}_8, \tilde{D}_8, \tilde{E}_7^2$ and $\tilde{E}_6$, are as in Table 2 and Table 3. Then, for any supersingular Enriques surface X in characteristic* 2, *we have Aut*$_{\mathrm{ct}}(X) \in \{1, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/11\mathbb{Z}, Q_8\}$,

8. Numerically trivial automorphisms

If $K_X = 0$, $\mathrm{Aut}_{\mathrm{nt}}(X) = \mathrm{Aut}_{\mathrm{ct}}(X)$, so we only have to treat the case that $K_X \neq 0$, i.e. X is classical.

By definition, any $g \in \mathrm{Aut}_{\mathrm{nt}}(X)$ leaves invariant any genus one fibration, however, it may act non-trivially on its base, or equivalently, it may act non-trivially on the corresponding pencil $|D|$. Also, by definition, any $g \in \mathrm{Aut}_{\mathrm{ct}}(X)$ fixes the half-fibers of a genus one fibration (their difference in $\mathrm{NS}(X)$ is equal to K_X). The following lemma proves the converse.

LEMMA 8.1. *A numerically trivial automorphism g that fixes all half-fibers on X is cohomologically trivial.*

PROOF. Since g is numerically trivial, it fixes any smooth rational curve, because they are the unique representatives in $\mathrm{NS}(X)$ of their classes in $\mathrm{Num}(X)$. By assumption, it fixes the linear equivalence class of all irreducible genus one curves. Applying Enriques Reducibility Lemma from [CD89], Corollary 3.2.3 we obtain that g fixes the linear equivalence classes of all curves on X. $\square$

LEMMA 8.2. *Let G be a finite, tame group of automorphisms of an irreducible curve C fixing a nonsingular point x. Then, G is cyclic.*

PROOF. Since G is finite and tame, one can linearize the action in the formal neighborhood of the point x. It follows that the action of G on the tangent space of C at x is faithful. Since x is nonsingular, the tangent space is one-dimensional and therefore the group is cyclic. $\square$

THEOREM 8.3. *Let X be an Enriques surface and assume that $p \neq 2$. Then, Aut*$_{\mathrm{nt}}(X) \cong \mathbb{Z}/2^a\mathbb{Z}$ *with $a \leq 2$. Moreover, if X is unnodal, then Aut*$_{\mathrm{nt}}(X) = \{1\}$.

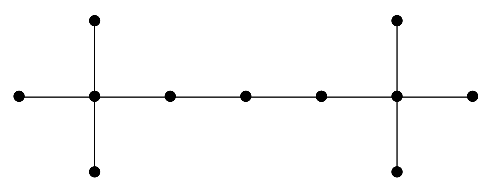
PROOF. By Theorem 7.1 and Lemma 8.1, any $g \in \mathrm{Aut}_{\mathrm{nt}}(X)$ has order 2 or 4, so it suffices to show that $\mathrm{Aut}_{\mathrm{nt}}(X)$ is cyclic. Since $\mathrm{Aut}_{\mathrm{nt}}(X)$ is tame, every numerically trivial automorphism has smooth fixed locus.

Assume that there is some $g \in \mathrm{Aut}_{\mathrm{nt}}(X) \setminus \mathrm{Aut}_{\mathrm{ct}}(X)$. Then, g switches the half-fibers of some elliptic fibration $|2F_1|$ on X by Lemma 8.1. The argument with the Euler-Poincaré characteristics from the proof of Proposition 7.6 applies and shows that one of the two fibers F', F'' of $|2F_1|$ fixed by g, say F', has at least 5 components. Hence, if X is unnodal, then $\mathrm{Aut}_{\mathrm{nt}}(X) = \{1\}$ follows from Theorem 7.1.

If F' is additive, then it has some component R, which is fixed pointwise by $\mathrm{Aut}_{\mathrm{nt}}(X)$, because it is adjacent to at least three other components. Since the fixed loci are smooth, any automorphism fixing a (-2)-curve adjacent to R is trivial. Hence, the claim follows from Lemma 8.2.

If F' is multiplicative, the fixed point formula shows that F' is of type I and g has four fixed points on F''. Extend F_1 to a non-degenerate $U_{[2]}$-sequence (F_1, F_2). Since $F'.F_2 = 2$, F' contains a cycle of 3 (-2)-curves contained in a fiber D of $|2F_2|$. Now, as in the additive case, we find a (-2)-curve, which is fixed pointwise by $\mathrm{Aut}_{\mathrm{nt}}(X)$. Indeed, if D is additive, we use the same argument as before and if D is multiplicative, then some component of D meets a component of F' exactly once in a nonsingular point of F'. This component is fixed pointwise by $\mathrm{Aut}_{\mathrm{nt}}(X)$. □

REMARK 8.4. The previous Theorem is not true if $p = 2$. Indeed, in Chapter II we have seen an Enriques surface X of type $\tilde{D}_4 + \tilde{D}_4$ with the dual graph of (-2) curves



that satisfies $\mathrm{Aut}_{\mathrm{nt}}(X) = (\mathbb{Z}/2\mathbb{Z})^2$ (see Chapter II Section 11). Moreover, we have seen in the proof of Corollary 7.2 that $\mathrm{Aut}_{\mathrm{ct}}(X) = \{1\}$.

If $p = 2$, even though we still have the same bound on the size of $\mathrm{Aut}_{\mathrm{nt}}(X)$, the cyclic group of order 4 can not occur.

THEOREM 8.5. *Let X be a classical Enriques surface in characteristic 2 which is not E_8-extra-special. Then, $\mathrm{Aut}_{\mathrm{nt}}(X) \cong (\mathbb{Z}/2\mathbb{Z})^b$ with $b \leq 2$.*

PROOF. By Corollary 7.2, $\mathrm{Aut}_{\mathrm{ct}}(X) \neq \{1\}$ if and only if X is D_8-extra-special and for such a surface we have $\mathrm{Aut}_{\mathrm{nt}}(X) = \mathrm{Aut}_{\mathrm{ct}}(X) = \mathbb{Z}/2\mathbb{Z}$. Hence, we can assume $\mathrm{Aut}_{\mathrm{ct}}(X) = \{1\}$. By Lemma 8.1, we have $\mathrm{Aut}_{\mathrm{nt}}(X) = (\mathbb{Z}/2\mathbb{Z})^b$ and we have to show $b \leq 2$. Suppose that $b \geq 3$ and take some half-fiber F_1. By Theorem 3.4, we can extend F_1 to a non-degenerate $U_{[2]}$-sequence (F_1, F_2). Since $|\mathrm{Aut}_{\mathrm{nt}}(X)| > 4$, there is some numerically trivial involution g that preserves F_1 and F_2. By Remark 5.2, such an automorphism acts trivially on D_1, hence it has to coincide with the bielliptic involution associated to $|2F_1 + 2F_2|$. Both fibrations have a unique reducible fiber F (resp. F') which has to be simple, since there is some numerically trivial involution which does not preserve F_i. By Corollary 5.6, F and F' are additive and share 8 components. This is only possible if they are of type I_4^* or II^*. Note that $F.F' = 4$ is impossible if both of them are of type I_4^*. In the remaining cases, it is easy to check that the surface is D_8-extra-special. We have already treated this surface. □

Outlook

In this last chapter, we want to summarize our results on automorphisms of Enriques surfaces and point the reader to some questions we have left open as well as give some ideas on how to solve them.

In Chapter I, despite the absence of transcendental methods, it was possible to solve the classification problem of Enriques surfaces with finite automorphism group and smooth K3 cover completely, using a separable quadratic twist construction. On the way, we also gave a complete description of the corresponding moduli spaces as well as minimal fields of definition for the surfaces and their automorphisms. Even in characteristic 0, this arithmetic information is new. However, it is not quite clear how the seven families of Enriques surfaces with finite automorphism group in characteristic 0 are connected to the classical and supersingular Enriques surfaces with finite automorphism group in characteristic 2. While the techniques of Chapter I only work in the case of Enriques surfaces with smooth canonical cover whereas the techniques of Chapter II only work in the other cases, the approach of Chapter III using bielliptic maps might make it possible to give explicit models connecting these two worlds.

In Chapter II, we completed the classification of Enriques surfaces with finite automorphism group in characteristic 2. Nevertheless, there are still some open problems related to the classification. For example, the problem of describing the corresponding moduli space is still open and we did not give minimal fields of definition for the Enriques surfaces with finite automorphism group. To solve these two problems, one could either try to extend the quadratic twist construction to the inseparable case and argue as in Chapter I or one could use bielliptic maps as in Chapter III and, by a very careful study of the branch loci of these bielliptic maps, give a description of the moduli space of Enriques surfaces with finite automorphism group.

In Chapter III, we gave the classification of possible groups of cohomologically and numerically trivial automorphisms of Enriques surfaces in arbitrary characteristic. Of course, the next step would be a classification of Enriques surfaces, which admit such numerically trivial automorphisms. We have already taken some steps into this direction in Chapter III and a classification of these surfaces in odd characteristics will be given in [DL]. As in the case of Enriques surfaces with finite automorphism group, the classification of classical and supersingular Enriques surfaces with numerically trivial automorphisms in characteristic 2 may be obtainable by keeping track of the singularities of their canonical cover. We hope to address this problem in a future paper.

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