SIAM J. DISCRETE MATH.
Vol. 20, No. 1, pp. 227–239

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ON STABILITY, ERROR CORRECTION, AND NOISE COMPENSATION IN DISCRETE TOMOGRAPHY*

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Abstract. The task of reconstructing binary images from the knowledge of their line sums (discrete X-rays) in a given finite number m of directions is ill-posed. Even some small noise in the physical measurements can lead to dramatically different yet still unique solutions.

The present paper addresses in particular the following problems. Does discrete tomography have the power of error correction? Can noise be compensated by taking more X-ray images, and, if so, what is the quantitative effect of taking one more X-ray? Our main theorem gives the first nontrivial unconditioned (and best possible) stability result. In particular, we show that the Hamming distance between any two different sets of m X-ray images of the same cardinality is at least $2(m-1)$, and this is best possible. As a consequence, this result implies a Rényi-type theorem for denoising and shows that the noise compensating effect of X-rays is linear in their number.

Our theoretical results are complemented by determining the computational complexity of some underlying algorithmic tasks. In particular, we show that while there always is a certain inherent stability, the possibility of making (worst-case) efficient use of it is rather limited.

Key words. discrete tomography, stability, discrete inverse problems, computational complexity

AMS subject classifications. 90C31, 68R05, 11P05

DOI. 10.1137/040617443

1. Introduction.

Discrete tomography deals with the reconstruction of finite sets from knowledge about their interaction with certain query sets. The most prominent example is that of the reconstruction of a finite subset F of $\mathbb{Z}^d$ from its X-rays (i.e., line sums) in a small positive integer number m of directions. Applications of discrete tomography include quality control in semiconductor industry, image processing, graph theory, scheduling, statistical data security, game theory, etc. (see, e.g., [6], [8], [9], [13], [14], [17], [19]). The reconstruction task is an ill-posed discrete inverse problem, depicting (suitable variants of) all three Hadamard criteria [12] for ill-posedness. In fact, for general data there need not exist a solution, if the data is consistent, the solutions need not be uniquely determined, and even in the case of uniqueness, the solution may change dramatically with small changes of the data.

The papers [1] and [2] show just how unstable the reconstruction task really is: For arbitrarily large lattice sets even of the same cardinality, a total error of only $2(m-1)$ in the measurements can lead to unique but disjoint solutions. Clearly, this is an important issue for all practical applications where noise in the data cannot be avoided, particularly if the data stems from physical measurements.

The main theorem of the present paper shows that this number $2(m-1)$ is best possible in an ultimate sense. In Theorem 2.1 we prove that two finite sets of the same cardinality whose X-rays in a given set of m directions differ by a total of less than $2(m-1)$ are "tomographically equivalent." This means that either the X-rays differ by at least $2(m-1)$, or they do not differ at all. Note that the situation becomes trivial if the assumption on the equal cardinality of the lattice sets is omitted. Indeed, if the cardinalities of the two sets differ by k, then the total difference of the X-rays is

*Received by the editors October 21, 2004; accepted for publication (in revised form) August 8, 2005; published electronically March 15, 2006.
http://www.siam.org/journals/sidma/20-1/61744.html

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at least km, and this is best possible (just delete k points of an arbitrary finite lattice set of cardinality at least k to obtain the second set).

Theorem 2.1 enables us to derive stability versions of all known uniqueness theorems, providing uniqueness even for somewhat noisy data. Complementing the theoretical results, we deal with the computational complexity of trying to take advantage of the inherent stability. The precise statements of our results will be given in the next section. Here we only summarize them qualitatively.

While it is clear that the total sum over all X-rays is a multiple of m and hence a small enough error in this number can be corrected, the problem of determining how the individual measurements should be corrected in order to provide consistency of the data is $\mathbb{NP}$-complete whenever $m \geq 3$ but easy for $m \leq 2$. Also, finding a set which best fits the data is $\mathbb{NP}$-hard for $m \geq 3$ but can be solved in polynomial time for $m \leq 2$.

The paper is organized as follows: After introducing some notation we state our main stability theorem, some of its corollaries, and the related algorithmic results in section 2. In sections 3 and 4 we give the proofs of our stability result and of the algorithmic results, respectively.

2. Main results: A stability theorem and some of its relatives. Let $d, m \in \mathbb{N}$, $d \geq 2$, and let $\mathbb{F}$ be a field with $\mathbb{Z} \subseteq \mathbb{F}$. Our underlying vector space will always be $\mathbb{F}^d$ but certain restrictions to the subring $\mathbb{Z}^d$ of all lattice points will also be relevant. Hence we will formulate some definitions and results in terms of $\mathbb{K} \in \{\mathbb{F}, \mathbb{Z}\}$. In particular, set

$$\mathcal{F}^d(\mathbb{K}) = \{F : F \subset \mathbb{K}^d \wedge F \text{ is finite}\}$$

and $\mathcal{F}^d = \mathcal{F}^d(\mathbb{Z})$. The elements of $\mathcal{F}^d$ are called *lattice sets*. Let $\mathcal{S}^d$ denote the set of all 1-dimensional linear subspaces of $\mathbb{F}^d$, and let $\mathcal{L}^d$ be the subset of $\mathcal{S}^d$ of all such subspaces that are spanned by vectors from $\mathbb{Z}^d$. The elements of $\mathcal{L}^d$ will be referred to as *lattice lines*. Further, for $S \in \mathcal{S}^d$ let $\mathcal{A}_{\mathbb{K}}(S) = \{v + S : v \in \mathbb{K}^d\}$.

Then, for $F \in \mathcal{F}^d(\mathbb{K})$ and $S \in \mathcal{S}^d$, the *(discrete 1-dimensional) X-ray of F parallel to S* is the function

$$X_S F : \mathcal{A}_{\mathbb{K}}(S) \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

defined by

$$X_S F(T) = |F \cap T| = \sum_{x \in T} \mathbf{1}_F(x)$$

for each $T \in \mathcal{A}_{\mathbb{K}}(S)$.

Two sets $F_1, F_2 \in \mathcal{F}^n(\mathbb{F})$ are called *tomographically equivalent* with respect to $S_1, \ldots, S_m \in \mathcal{S}^d$ if $X_{S_i} F_1 = X_{S_i} F_2$ for $i = 1, \ldots, m$.

Given m different lines $S_1, \ldots, S_m \in \mathcal{S}^d$, the basic questions in discrete tomography are as follows. What kind of information about a finite (lattice) set $F \in \mathbb{K}^d$ can be retrieved from its X-ray images $X_{S_1} F, \ldots, X_{S_m} F$? How difficult is the reconstruction algorithmically? How sensitive is the task to data errors? Here the data is given in terms of functions

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$$f_i : \mathcal{A}_{\mathbb{K}}(S_i) \to \mathbb{N}_0, \qquad i = 1, \ldots, m,$$

with finite support $\mathcal{T}_i \subseteq \mathcal{A}_{\mathbb{K}}(S_i)$ represented by appropriately chosen data structures; see [8]. Hence the difference of two data functions with respect to the same line $S \in \mathcal{S}^d$ is a function $h : \mathcal{A}_{\mathbb{K}}(S) \to \mathbb{Z}$; its size will be measured in terms of its ℓ_1-norm

$$\|h\|_1 = \sum_{T \in \mathcal{A}_{\mathbb{K}}(S)} |h(T)|.$$

For surveys on various aspects of discrete tomography see [10], [11], [13].

Our main stability result can now be formulated as follows.

THEOREM 2.1. *Let $S_1, \ldots, S_m \in \mathcal{S}^d$ be different and $F_1, F_2 \in \mathcal{F}^d(\mathbb{K})$ with $|F_1| = |F_2|$. If*

$$\sum_{i=1}^{m} ||X_{S_i} F_1 - X_{S_i} F_2||_1 < 2(m-1),$$

then F_1 and F_2 are tomographically equivalent.

The proof will be given in section 3. Clearly, Theorem 2.1 is equivalent to the following theorem.

THEOREM 2.2. *Let $S_1, \ldots, S_m \in \mathcal{S}^d$ be different. Then there do not exist $F_1, F_2 \in \mathcal{F}^d(\mathbb{K})$ with $|F_1| = |F_2|$ and $0 < \sum_{i=1}^{m} ||X_{S_i} F_1 - X_{S_i} F_2||_1 < 2(m-1)$.*

As corollaries to this stability result we may derive "noisy versions" of all known uniqueness theorems. In the following we give two such examples.

Rényi's well-known theorem [16] states that if we know the cardinality $|F|$ of a finite set F we can guarantee uniqueness from X-rays taken in any $m \geq |F| + 1$ different directions. Our first corollary shows that we can guarantee uniqueness, *even if the X-rays are not given precisely.*

COROLLARY 2.3. *Let $F_1, F_2 \in \mathcal{F}^d(\mathbb{K})$ with $|F_1| = |F_2|$, $m \in \mathbb{N}$ with $m \geq |F_1| + 1$, and let $S_1, \ldots, S_m \in \mathcal{S}^d$ be different. If $\sum_{i=1}^{m} ||X_{S_i} F_1 - X_{S_i} F_2||_1 < 2|F_1|$, then $F_1 = F_2$.*

Proof. By Theorem 2.1, F_1 and F_2 are tomographically equivalent; hence the assertion follows from Rényi's theorem [16]. ∎

Corollary 2.3 shows the potential power of error correction in the setting of Rényi's theorem: A total error smaller than $2n$ can be compensated without increasing the number of X-rays taken if the cardinality n of the original set F is known. But even without knowing n precisely we can correct errors—at the expense, however, of taking more X-rays.

COROLLARY 2.4. *Let $F_1, F_2 \in \mathcal{F}^d(\mathbb{K})$ with $|F_1| \leq |F_2|$, $m \in \mathbb{N}$ with $m \geq 2|F_1|$, and let $S_1, \ldots, S_m \in \mathcal{S}^d$ be different. Then $\sum_{i=1}^{m} ||X_{S_i} F_1 - X_{S_i} F_2||_1 < 2|F_1|$ implies $F_1 = F_2$.*

Proof. Clearly $\sum_{i=1}^{m} ||X_{S_i} F_1 - X_{S_i} F_2||_1 \geq m(|F_2| - |F_1|)$. Thus, $\sum_{i=1}^{m} ||X_{S_i} F_1 - X_{S_i} F_2||_1 < 2|F_1|$ implies $|F_1| = |F_2|$, and the assertion follows from Corollary 2.3. ∎

Next we give a stable version of a theorem of Gardner and Gritzmann [7] for the set $\mathcal{C}^d$ of *convex lattice sets*, i.e., of sets $F \in \mathcal{F}^d$ with $F = \mathrm{conv}(F \cap \mathbb{Z}^d)$.

COROLLARY 2.5. *Let F_1, $F_2 \in \mathcal{C}^d$ with $|F_1| = |F_2|$.*

(i) *There are sets $\{S_1, S_2, S_3, S_4\} \subseteq \mathcal{L}^d$ of four lines such that $\sum_{i=1}^{4} ||X_{S_i} F_1 - X_{S_i} F_2||_1 < 6$ implies $F_1 = F_2$.*

(ii) *For any set $\{S_1, \ldots, S_m\} \subseteq \mathcal{L}^d$ of $m \geq 7$ coplanar lattice lines, $\sum_{i=1}^{m} ||X_{S_i} F_1 - X_{S_i} F_2||_1 < 2(m-1)$ implies $F_1 = F_2$.*

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Proof. By Theorem 2.1, F_1 and F_2 are tomographically equivalent in both parts of the statement; hence the assertion follows from the uniqueness theorems of [7]. ∎

Note that this theorem also holds for the somewhat more general class of *Q-convex lattice sets* because they are uniquely determined by the same sets of lattice lines as the convex lattice sets (see [5]).

Let us now turn to results on some algorithmic tasks related to stability and instability in discrete tomography. We concentrate on the case of finite lattice sets whose X-rays are taken in lattice directions. Thus, let $S_1, \ldots, S_m \in \mathcal{L}^d$. Proofs of the following statements will be given in section 4.

We begin with two examples of algorithmic consequences of Theorem 2.1, "noisy extensions" of known complexity results. It has been shown in [8] that the two problems

Consistency$_{\mathcal{F}^d}(S_1, \ldots, S_m)$
Input: For $i = 1, \ldots, m$ data functions $f_i : \mathcal{A}_{\mathbb{Z}}(S_i) \to \mathbb{N}_0$ with finite support.
Question: Does there exist a finite lattice set $F \in \mathcal{F}^d$ such that $X_{S_i}F = f_i$ for $i = 1, \ldots, m$?

and

Uniqueness$_{\mathcal{F}^d}(S_1, \ldots, S_m)$
Input: A set $F_1 \in \mathcal{F}^d$.
Question: Does there exist a set $F_2 \in \mathcal{F}^d$ with $F_1 \neq F_2$ such that $X_{S_i}F_1 = X_{S_i}F_2$ for $i = 1, \ldots, m$?

can be solved in polynomial time for $m \leq 2$ but are $\mathbb{NP}$-complete for $m \geq 3$.

With the aid of Theorem 2.1 these results can be extended as follows.

Corollary 2.6. *Let $S_1, \ldots, S_m \in \mathcal{L}^d$ be different. The two problems*

X-Ray-Correction$_{\mathcal{F}^d}(S_1, \ldots, S_m)$
Input: *For every $i = 1, \ldots, m$ a data function $f_i : \mathcal{A}_{\mathbb{Z}}(S_i) \to \mathbb{N}_0$ with finite support.*
Question: *Does there exist a finite lattice set $F \in \mathcal{F}^d$ with $\sum_{i=1}^m ||X_{S_i}F - f_i||_1 \leq m - 1$?*

and

Similar-Solution$_{\mathcal{F}^d}(S_1, \ldots, S_m)$
Input: *A finite lattice set $F_1 \in \mathcal{F}^d$.*
Question: *Does there exist a finite lattice set $F_2 \in \mathcal{F}^d$ with $|F_1| = |F_2|$ and $F_1 \neq F_2$ such that $\sum_{i=1}^m ||X_{S_i}F_1 - X_{S_i}F_2||_1 \leq 2m - 3$?*

are in $\mathbb{P}$ for $m \leq 2$ but are $\mathbb{NP}$-complete for $m \geq 3$.

Note that X-Ray-Correction$_{\mathcal{F}^d}(S_1, \ldots, S_m)$ can also be formulated as the task to decide, for given data functions $f_i : \mathcal{A}_{\mathbb{Z}}(S_i) \to \mathbb{N}_0$ $(i = 1, \ldots, m)$ with finite support, whether there exist "corrected" data functions $g_i : \mathcal{A}_{\mathbb{Z}}(S_i) \to \mathbb{N}_0$ $(i = 1, \ldots, m)$ with finite support that are consistent and do not differ from the given functions by more than a total of $m - 1$. Corollary 2.6 shows that this form of measurement correction is just as hard as checking consistency.

If the data is noisy it seems natural to try to find a finite lattice set that fits the measurements best. This task is studied in the following theorem.

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THEOREM 2.7. *Let $S_1,\dots,S_m \in \mathcal{L}^d$ be different. The problem*

NEAREST-SOLUTION$_{\mathcal{F}^d}(S_1,\dots,S_m)$
Input: *For every $i=1,\dots,m$, a data function $f_i : \mathcal{A}_{\mathbb{Z}}(S_i) \to \mathbb{N}_0$ with finite support.*
Task: *Determine a set $F^* \in \mathcal{F}^d$ such that*
$\sum_{i=1}^m ||X_{S_i}F^* - f_i||_1 = \min_{F\in\mathcal{F}^d} \sum_{i=1}^m ||X_{S_i}F - f_i||_1$

is in $\mathbb{P}$ for $m \le 2$ but is $\mathbb{NP}$-hard for $m \ge 3$.

From the $\mathbb{NP}$-hardness of CONSISTENCY$_{\mathcal{F}^d}(S_1,\dots,S_m)$ the statement for $m \ge 3$ follows easily. In fact, for a given instance $(f_1,\dots,f_m)$ of CONSISTENCY$_{\mathcal{F}^d}(S_1,\dots,S_m)$ let F^* denote a solution of NEAREST-SOLUTION$_{\mathcal{F}^d}(S_1,\dots,S_m)$ for the input $(f_1,\dots,f_m)$. Then $(f_1,\dots,f_m)$ is a yes-instance of CONSISTENCY$_{\mathcal{F}^d}(S_1,\dots,S_m)$ if and only if $X_{S_i}F^* = f_i$ for all $i=1,\dots,m$. However, the proof of the polynomial-time solvability in the case $m=2$ is more involved and will be given in section 4.

3. Proof of the main stability result. Note first that it is enough to prove Theorem 2.1 for $\mathbb{K} = \mathbb{F}$. The proof will be based on four lemmas. The first lemma is a simple combinatorial observation.

LEMMA 3.1. *Let $S \in \mathcal{S}^d$ and let $f, g : \mathcal{A}_{\mathbb{F}}(S) \to \mathbb{N}_0$ be data functions with finite support. Further, set $\mathcal{A}^+ = \{T \in \mathcal{A}_{\mathbb{F}}(S) : f(T) - g(T) > 0\}$ and $\mathcal{A}^- = \{T \in \mathcal{A}_{\mathbb{F}}(S) : f(T) - g(T) < 0\}$. Then*

$$||f-g||_1 = 2\sum_{T\in\mathcal{A}^+}\big(f(T)-g(T)\big) - \|f\|_1 + \|g\|_1.$$

In particular, when $\|f\|_1 = \|g\|_1$ the number $||f-g||_1$ is even.

Proof. Since

$$\sum_{T\in\mathcal{A}_{\mathbb{F}}(S)} (f(T)-g(T)) = \sum_{T\in\mathcal{A}_{\mathbb{F}}(S)} f(T) - \sum_{T\in\mathcal{A}_{\mathbb{F}}(S)} g(T) = \|f\|_1 - \|g\|_1,$$

we have

$$\begin{aligned}
||f-g||_1 &= \sum_{T\in\mathcal{A}_{\mathbb{F}}(S)} |f(T)-g(T)| = \sum_{T\in\mathcal{A}^+}(f(T)-g(T)) - \sum_{T\in\mathcal{A}^-}(f(T)-g(T))\\
&= \sum_{T\in\mathcal{A}^+}(f(T)-g(T)) - \sum_{T\in\mathcal{A}^-}(f(T)-g(T)) + \sum_{T\in\mathcal{A}^+}(f(T)-g(T))\\
&\quad + \sum_{T\in\mathcal{A}^-}(f(T)-g(T)) - \|f\|_1 + \|g\|_1\\
&= 2\sum_{T\in\mathcal{A}^+}(f(T)-g(T)) - \|f\|_1 + \|g\|_1. \qquad \square
\end{aligned}$$

In the present section we will apply Lemma 3.1 to the X-rays of sets $F_1, F_2 \in \mathcal{F}^d(\mathbb{F})$, i.e., to $f = X_S F_1$ and $g = X_S F_2$.

The next lemma is geometric in nature and will enable us to reduce the proof of Theorem 2.1 to the planar case.

LEMMA 3.2. *Let $d \ge 3$, $S_1,\dots,S_m \in \mathcal{S}^d$ be different and $F_1, F_2 \in \mathcal{F}^d(\mathbb{F})$. Then there exists a surjective linear map $\varphi : \mathbb{F}^d \to \mathbb{F}^2$ with the following properties.*

(i) *$\varphi(S_1),\dots,\varphi(S_m)$ are different lines in $\mathcal{S}^2$.*

(ii) *If $i \in \{1,\dots,m\}$ and $a, b \in F_1 \cup F_2$ satisfy $\varphi(b) \in \varphi(a) + \varphi(S_i)$, then $b \in a + S_i$.*

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Proof. In order to satisfy the two properties the kernel $\ker(\varphi)$ will be chosen complementary to any plane spanned by two of the m lines, and also complementary to any plane spanned by one of the lines $S_1, \ldots, S_m$ and a line generated by the difference of two of the vectors of $F_1 \cup F_2$. Let us denote the set of these exceptional planes by $\mathcal{P}$. Each of the planes $P \in \mathcal{P}$ can be described as the set of solutions of a homogeneous $(d-2) \times d$ system of linear equations; let A_P denote a corresponding coefficient matrix. Now, let $\pi_1, \ldots, \pi_{2d}$ be different primes. Further, for $x \in \mathbb{F}$ let $B(x)$ be the $2 \times d$ matrix with row vectors $(x^{\pi_1}, x^{\pi_2}, \ldots, x^{\pi_d})$ and $(x^{\pi_{d+1}}, x^{\pi_{d+2}}, \ldots, x^{\pi_{2d}})$, and let $H(x)$ be the solution space of the corresponding homogeneous $2 \times d$ system. Then for each $P \in \mathcal{P}$ the determinant of the matrix composed of A_P and $B(x)$ is a nontrivial polynomial in x. (In fact, the coefficients are $(d-2) \times (d-2)$ subdeterminants of A_P, and by the choice of the exponents of x in $B(x)$ there is generically no cancellation.) Hence for all sufficiently large integers x, $H(x)$ is complementary to each plane $P \in \mathcal{P}$. Now taking a fixed such vector x, we define φ by choosing an arbitrary basis of $H(x)$, extend it to a basis of $\mathbb{F}^d$, and specify that φ maps the basis vectors of $H(x)$ to 0 and the remaining two to the standard basis vectors of $\mathbb{F}^2$. Then $\ker(\varphi) = H(x)$, whence φ has the desired properties. ∎

Note that a linear mapping φ with the properties of Lemma 3.2 is necessarily injective on $F_1 \cup F_2$.

The following two lemmas are more algebraic in nature. The next contains a well-known result on the elementary part of the Prouhet–Tarry–Escott Problem on solutions of a specific power system of polynomial equations. As a service to the reader we still outline the proof. For a survey on the Prouhet–Tarry–Escott Problem see [3] or [4].

LEMMA 3.3. *Let $x_1, \ldots, x_q, y_1, \ldots, y_q \in \mathbb{F}$ such that*

$$\sum_{i=1}^{q} x_i^j = \sum_{i=1}^{q} y_i^j$$

for $j = 1, \ldots, q$. Then the multisets $\{x_1, \ldots, x_q\}$ and $\{y_1, \ldots, y_q\}$ coincide.

Proof. We show that $x_1, \ldots, x_q$ and $y_1, \ldots, y_q$ are the roots of the same polynomial of degree q.

For $i = 1, \ldots, q$ let $p_i, s_i \in \mathbb{F}[X_1, \ldots, X_q]$ be defined by

$$p_i = X_1^i + X_2^i + \cdots + X_q^i, \qquad s_i = \sum_{1 \le k_1 < \cdots < k_i \le q} X_{k_1} \cdots X_{k_i}.$$

The polynomials p_i and s_i are the well-known *power sums* and *elementary symmetric functions* of the indeterminates $X_1, \ldots, X_q$, respectively. Clearly, for the indeterminates $X_1, \ldots, X_q, Y$ we have

$$\prod_{i=1}^{q} (Y - X_i) = Y^q - s_1 Y^{q-1} + s_2 Y^{q-2} + \cdots + (-1)^q s_q.$$

Using the *Newton identities* (see, e.g., [15]) it follows inductively that for $i = 1, \ldots, q$

$$s_i \in \mathbb{F}[p_1, \ldots, p_q].$$

Since by assumption

$$p_i(x_1, \ldots, x_q) = p_i(y_1, \ldots, y_q) \quad \text{for } i = 1, \ldots, q,$$

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this implies

$$s_i(x_1, \ldots, x_q) = s_i(y_1, \ldots, y_q) \quad \text{for } i = 1, \ldots, q.$$

Consequently,

$$\prod_{i=1}^{q}(Y - x_i) = \sum_{i=0}^{q}(-1)^i Y^{q-i} s_i(x_1, \ldots, x_q) = \prod_{i=1}^{q}(Y - y_i);$$

i.e., the two polynomials $\prod_{i=1}^{q}(Y - x_i)$ and $\prod_{i=1}^{q}(Y - y_i)$ in $\mathbb{F}[Y]$ are identical. Hence $x_1, \ldots, x_q$ is just a permutation of $y_1, \ldots, y_q$. ∎

LEMMA 3.4. *Let $k \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_{k+1}, \tau_1, \ldots, \tau_{k+1} \in \mathbb{F}$ such that $S_i = \operatorname{lin}\left\{(\sigma_i, \tau_i)^T\right\} \in \mathcal{S}^2$, $i = 1, \ldots, k+1$, are different. Then*

$$(\tau_1 X - \sigma_1 Y)^k, \ldots, (\tau_{k+1} X - \sigma_{k+1} Y)^k \in \mathbb{F}[X, Y]$$

form a basis of the $\mathbb{F}$-vector space V_k that is generated by the $k+1$ binomials Y^k, $X^1 Y^{k-1}, \ldots, X^{k-1} Y^1$, $X^k \in \mathbb{F}[X, Y]$.

Proof. Every polynomial $(\tau_i X - \sigma_i Y)^k$ can be expressed in terms of its coefficient vector

$$\left(\binom{k}{0}\tau_i^0(-\sigma_i)^k, \ldots, \binom{k}{k}\tau_i^k(-\sigma_i)^0\right)$$

with respect to the binomial basis $\{Y^k, X^1 Y^{k-1}, \ldots, X^{k-1} Y^1, X^k\}$. Thus, we have to show only that these $k+1$ vectors are linearly independent, i.e., that the matrix

$$C = \left(\binom{k}{j-1}(\tau_i)^{j-1}(-\sigma_i)^{k-j+1}\right)_{i,j=1,\ldots,k+1} \in \mathbb{F}^{(k+1)\times(k+1)}$$

is nonsingular.

Suppose first that $\sigma_1 \cdots \sigma_{k+1} \neq 0$. By setting $\rho_i = -\sigma_i^{-1}\tau_i$, and by denoting the Vandermonde matrix $(\rho_i^{j-1})_{i,j=1,\ldots,k+1}$ by C', we obtain

$$\det(C) = \det(C') \cdot \prod_{i=1}^{k+1}\binom{k}{i-1}(-\sigma_i)^k = \prod_{i>j}(\rho_i - \rho_j) \cdot \prod_{i=1}^{k+1}\binom{k}{i-1}(-\sigma_i)^k.$$

Thus, if $\det(C) = 0$, then there exist indices i_0, j_0 in $\{1, \ldots, k+1\}$ with $i_0 \neq j_0$ but $\rho_{i_0} = \rho_{j_0}$. This means that $\sigma_{i_0}^{-1}\tau_{i_0} = \sigma_{j_0}^{-1}\tau_{j_0}$, whence $S_{i_0} = S_{j_0}$, contrary to the assumption. Therefore $\det(C) \neq 0$.

Now suppose that one of the σ_i is zero. Without loss of generality we may assume that $\sigma_1 = 0$. Note that then $\sigma_i \neq 0$ for $i > 1$. The first row of C is now a nonzero multiple of $(0, \ldots, 0, 1)$. By developing $\det(C)$ with respect to the first row, we see that the same argument as in the first case applies again. ∎

Now we are ready to prove our main stability result.

Proof of Theorem 2.1. Let $F_1, F_2 \in \mathcal{F}^d(\mathbb{F})$ with $|F_1| = |F_2|$ and $0 < \sum_{i=1}^{m} ||X_{S_i}F_1 - X_{S_i}F_2||_1 < 2(m-1)$. By Lemma 3.1, this implies that $m \geq 3$.

Suppose first that the error involves more than one direction; i.e., $X_{S_i}F_1 \neq X_{S_i}F_2$ for at least two different indices i_1 and i_2. By Lemma 3.1, $\|X_{S_i}F_1 - X_{S_i}F_2\|_1 \geq 2$ for $i = i_1, i_2$. Therefore, ignoring S_{i_1}, the sets F_1 and F_2 provide a counterexample already for $m-1$ directions. Hence we may in the following assume that $X_{S_i}F_1 =$

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$X_{S_i}F_2$ for $i = 1, \dots, m-1$; i.e., the error occurs only for S_m. Similarly, we may assume that the error is exactly $2(m-2)$.

Next, we reduce the statement to the planar case. Let $d \geq 3$ and suppose that $F_1, F_2 \in \mathcal{F}^d(\mathbb{F})$ with $|F_1| = |F_2|$ and $0 < \sum_{i=1}^m ||X_{S_i}F_1 - X_{S_i}F_2||_1 < 2(m-1)$. Let φ be a linear mapping according to Lemma 3.2, and set $F_j' = \varphi(F_j)$ for $j = 1, 2$ and $S_i' = \varphi(S_i)$ for $i = 1, \dots, m$. Then $F_1', F_2' \in \mathcal{F}^2(\mathbb{F})$, $|F_1'| = |F_2'|$, $S_1', \dots, S_m' \in \mathcal{S}^2$ are different, and $X_{S_i'}F_j' = X_{S_i}F_j$ for $i = 1, \dots, m$ and $j = 1, 2$. Hence we obtain a counterexample already in dimension 2.

Finally we turn to the planar case. So, in the following let $d = 2$. The n points of F_1 and F_2 will be denoted by $(x_1, y_1), \dots, (x_n, y_n)$ and $(x_1', y_1'), \dots, (x_n', y_n')$, respectively.

Let $\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_m \in \mathbb{F}$ be such that $S_i = \text{lin}\left\{(\sigma_i, \tau_i)^T\right\}$ for $i = 1, \dots, m$. By Lemma 3.4 we know that for $k = 1, \dots, m-2$

$$(\tau_1 X - \sigma_1 Y)^k, \dots, (\tau_{k+1} X - \sigma_{k+1} Y)^k$$

form a basis of the $\mathbb{F}$-vector space V_k generated by the binomials $Y^k, X^1Y^{k-1}, \dots, X^{k-1}Y^1, X^k$. Since, of course, $(\tau_m X - \sigma_m Y)^k \in V_k$, there are coefficients $\alpha_{1,k}, \dots, \alpha_{m-1,k} \in \mathbb{F}$ such that

$$(\tau_m X - \sigma_m Y)^k = \sum_{i=1}^{m-1} \alpha_{i,k} (\tau_i X - \sigma_i Y)^k.$$

For every line T parallel to any of the lines $S_1, \dots, S_{m-1}$ we have $|F_1 \cap T| = |F_2 \cap T|$. Hence, as multisets the projections of F_1 and F_2 parallel to S_i (on any line complementary to S_i) coincide for $i = 1, \dots, m-1$. Thus

$$\{(\tau_i x_1 - \sigma_i y_1), \dots, (\tau_i x_n - \sigma_i y_n)\} = \{(\tau_i x_1' - \sigma_i y_1'), \dots, (\tau_i x_n' - \sigma_i y_n')\}$$

for $i = 1, \dots, m-1$. As a consequence we have

$$\begin{aligned}
&\sum_{j=1}^{n} \left((\tau_m x_j - \sigma_m y_j)^k - (\tau_m x_j' - \sigma_m y_j')^k\right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{m-1} \alpha_{i,k} \left((\tau_i x_j - \sigma_i y_j)^k - (\tau_i x_j' - \sigma_i y_j')^k\right) = 0
\end{aligned}$$

for each $k = 1, \dots, m-2$.

Now we define the multiset differences

$$A = \{(\tau_m x_1 - \sigma_m y_1), \dots, (\tau_m x_n - \sigma_m y_n)\} \setminus \{(\tau_m x_1' - \sigma_m y_1'), \dots, (\tau_m x_n' - \sigma_m y_n')\}$$

and

$$B = \{(\tau_m x_1' - \sigma_m y_1'), \dots, (\tau_m x_n' - \sigma_m y_n')\} \setminus \{(\tau_m x_1 - \sigma_m y_1), \dots, (\tau_m x_n - \sigma_m y_n)\}.$$

Note that $|A|$ and $|B|$ count the positive excess of F_1 over F_2 and of F_2 over F_1, respectively, on lines parallel to S_m. To be more precise, let $\mathcal{A}^+ = \{T \in \mathcal{A}_{\mathbb{F}}(S_m) : X_{S_m}F_1(T) - X_{S_m}F_2(T) > 0\}$ and $\mathcal{A}^- = \{T \in \mathcal{A}_{\mathbb{F}}(S_m) : X_{S_m}F_1(T) - X_{S_m}F_2(T) < 0\}$. Then with the aid of Lemma 3.1

$$|A| = \sum_{T \in \mathcal{A}^+} \left(X_{S_m}F_1(T) - X_{S_m}F_2(T)\right) = \frac{1}{2}||X_{S_m}F_1 - X_{S_m}F_2||_1;$$

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similarly,

$$|B| = \sum_{T\in\mathcal{A}^-} \big(X_{S_m}F_2(T) - X_{S_m}F_1(T)\big) = \frac{1}{2}||X_{S_m}F_1 - X_{S_m}F_2||_1.$$

Hence

$$|A| = |B| = m-2$$

and thus, particularly, $A\neq B$. Using the notation $A=\{a_1,\dots,a_q\}$ and $B=\{b_1,\dots,b_q\}$ with $q=m-2$, we have for each $k=1,\dots,q$

$$\sum_{j=1}^{n} \big((\tau_m x_j - \sigma_m y_j)^k - (\tau_m x_j' - \sigma_m y_j')^k\big) = \sum_{j=1}^{q} a_j^k - \sum_{j=1}^{q} b_j^k = 0,$$

a contradiction to Lemma 3.3. This completes the proof of Theorem 2.1. ◻

4. Proofs of the algorithmic results. In the following we give the proofs for the algorithmic results stated in section 2. We begin with the membership of X-Ray-Correction$_{\mathcal{F}^d}(S_1,\dots,S_m)$ and Similar-Solution$_{\mathcal{F}^d}(S_1,\dots,S_m)$ in the class $\mathbb{NP}$. Given an instance $(f_1,\dots,f_m)$ or F_1, respectively, one would, of course, like to use as a certificate a corresponding set F or F_2, respectively. If the set is available and polynomial in the encoding length, the conditions can be checked efficiently. Let us call a set F *support consistent* if for each of the m directions the support of the X-ray $X_{S_i}F$ is a subset of the support of the data function f_i, i.e.,

$$\{T\in\mathcal{A}_{\mathbb{Z}}(S_i) : X_{S_i}F(T)\neq 0\} \subset \mathcal{T}_i \quad \text{for } i=1,\dots,m,$$

where

$$\mathcal{T}_i = \{T\in\mathcal{A}_{\mathbb{Z}}(S_i) \;:\; f_i(T)\neq 0\} \quad \text{for } i=1,\dots,m.$$

In fact, every support consistent solution is a subset of the *grid*

$$G = \mathbb{Z}^d \cap \bigcap_{i=1}^{m} \bigcup_{T\in\mathcal{T}_i} T,$$

and G contains only polynomially many points $v_1,\dots,v_k$ of polynomially bounded size.

Since, in general, errors are allowed we cannot restrict ourselves to support consistent solutions. But then not every solution must consist of lattice points whose binary size is bounded by a polynomial in the input. The next lemma shows, however, that there always exist solutions of polynomial size.

Lemma 4.1. *Let $\gamma\in\mathbb{N}$ be a constant. Further, for $i=1,\dots,m$ let $f_i:\mathcal{A}_{\mathbb{Z}}(S_i)\to\mathbb{N}_0$ be a data function with finite support, and let $F\in\mathcal{F}^d$ be such that*

$$\sum_{i=1}^{m} ||X_{S_i}F - f_i||_1 \le \gamma \sum_{i=1}^{m} ||f_i||_1.$$

Then there exists a finite lattice set $F^*\in\mathcal{F}^d$ of binary size that is bounded by a polynomial in the binary size of $(f_1,\dots,f_m)$ with

$$|F| = |F^*| \qquad \textit{and} \qquad \sum_{i=1}^{m} ||X_{S_i}F^* - f_i||_1 = \sum_{i=1}^{m} ||X_{S_i}F - f_i||_1 \quad \textit{for } i=1,\dots,m.$$

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Proof. Without loss of generality we may assume that the grid G contains the origin. Now, for $v_1, v_2 \in G$ and $i, j = 1, \dots, m$ with $i \neq j$, the point of intersection of the two lines $v_1 + S_i$ and $v_2 + S_j$ has binary size that is bounded by a polynomial in the binary size of $(f_1, \dots, f_m)$. Hence there is a constant λ of polynomial size such that $\lambda[-1, 1]^d$ contains all such intersections and such that for every $v \in G$ and $i = 1, \dots, m$ the line $v + S_i$ contains at least two lattice points of $\lambda[-1, 1]^d$. Let

$$\mathcal{T} = G + \{S_1, \dots, S_m\}, \qquad k = \max\left\{m\lambda, \gamma \sum_{i=1}^{m} ||f_i||_1\right\}$$

and

$$W = (1 + k)\lambda[-1, 1]^d, \qquad C = W \setminus \left(\lambda[-1, 1]^d\right).$$

Then each line $v + S_i$ with $v \in G$ intersects the annulus C in at least $2k$ lattice points. Now, if $q \in F \setminus W$, then there is at most one line in $\mathcal{T}$ that passes through q. We will successively replace the points of $F \setminus W$ by points in C. Let us deal first with those points of $F \setminus W$ which are met by one of the X-ray lines in $\mathcal{T}$. We replace such points q one by one by the lattice point of C closest to q on that line with smallest ℓ_∞ norm among all such points which have not previously been inserted. By the choice of k there are always enough points of C on each line.

After having handled all such points we replace all points $q \in F \setminus W$ that are not met by any of the X-ray lines by a set of points of the same cardinality on the boundary of W that is disjoint from any line in $\mathcal{T}$. An elementary lattice point count shows that by the choice of k a set of appropriate cardinality always exists. This way we obtain a finite lattice set F^* with $|F| = |F^*|$. By construction, the X-ray images of F and F^* coincide on each line of $\mathcal{T}$. Also the total sums for F and F^* on all other lines are the same. This proves the assertion. ∎

It follows now directly from Lemma 4.1 that X-Ray-Correction$_{\mathcal{F}^d}(S_1, \dots, S_m)$ and Similar-Solution$_{\mathcal{F}^d}(S_1, \dots, S_m)$ are indeed in $\mathbb{NP}$.

For $m = 2$ the result of Lemma 4.1 can be sharpened. It is not just possible to avoid points "too far out" but it suffices to consider only instances and solutions "with no empty line in between." To be precise, we call a data function $f : \mathcal{A}_{\mathbb{Z}^d}(S) \to \mathbb{N}_0$ *consecutive* if for $v_1, v_2, v_3 \in \mathbb{Z}^d$ it is true that $f(v_2 + S) \neq 0$ whenever $f(v_1 + S) \neq 0$, $f(v_3 + S) \neq 0$, and $v_2 + S \subset \operatorname{conv}(v_1 + S) \cup (v_3 + S)$. Further, an m-tuple $(f_1, \dots, f_m)$ of data functions with respect to $S_1, \dots, S_m$ is called *consecutive* if $f_1, \dots, f_m$ are consecutive. Similarly, a finite lattice set F is called *consecutive* if and only if $(X_{S_1}F, \dots, X_{S_m}F)$ is consecutive. It is clear that for $m = 2$ we can always replace a given instance of any of our problems by an equivalent consecutive one.

Now we can give the proof of Corollary 2.6.

Proof of Corollary 2.6. Let first $m \geq 3$ and let us begin with X-Ray-Correction$_{\mathcal{F}^d}$ $(S_1, \dots, S_m)$.

Let $(f_1, \dots, f_m)$ be an instance of Consistency$_{\mathcal{F}^d}(S_1, \dots, S_m)$. Then $(f_1, \dots, f_m)$ is also an instance of X-Ray-Correction$_{\mathcal{F}^d}(S_1, \dots, S_m)$. Suppose first that no set $F \in \mathcal{F}^d$ exists with $\sum_{i=1}^m ||X_{S_i}F - f_i||_1 \leq m - 1$. Then, of course, $(f_1, \dots, f_m)$ is a no-instance of Consistency$_{\mathcal{F}^d}(S_1, \dots, S_m)$.

Thus, suppose there is a set $F \in \mathcal{F}^d$ with $\sum_{i=1}^m ||X_{S_i}F - f_i||_1 \leq m - 1$. Let $||f_1|| = \cdots = ||f_m||$. In polynomial time we can construct a line $T^* \in \mathcal{A}_{\mathbb{Z}}(S_1)$ with

$$T^* \cap \bigcup_{T \in \mathcal{T}_i} T \cap \bigcup_{T \in \mathcal{T}_j} T = \emptyset \qquad \text{for all } i \neq j.$$

Now let $f_1^*(T) = f_1(T)$ for $T \in \mathcal{A}_{\mathbb{Z}}(S_1) \setminus \{T^*\}$ and $f_1^*(T^*) = m-1$. Then, clearly, $(f_1^*, f_2, \ldots, f_m)$ is a yes-instance of X-Ray-Correction$_{\mathcal{F}^d}(S_1,\ldots,S_m)$ if and only if $(f_1, f_2, \ldots, f_m)$ is a yes-instance of Consistency$_{\mathcal{F}^d}(S_1,\ldots,S_m)$. The result, therefore, is that Consistency$_{\mathcal{F}^d}(S_1,\ldots,S_m)$ reduces polynomially to X-Ray-Correction$_{\mathcal{F}^d}(S_1,\ldots,S_m)$. Since by [8] the former is $\mathbb{NP}$-hard, so is the latter.

Next, let F_1 be an instance of Uniqueness$_{\mathcal{F}^d}(S_1,\ldots,S_m)$. Of course, F_1 is also an instance of Similar-Solution$_{\mathcal{F}^d}(S_1,\ldots,S_m)$. Let $F_2 \in \mathcal{F}^d$ with $|F_1| = |F_2|$ and $\sum_{i=1}^m ||X_{S_i}F_1 - X_{S_i}F_2||_1 < 2(m-1)$. Then by Theorem 2.1, F_2 is tomographically equivalent to F_1. Hence F_1 is a yes-instance of Uniqueness$_{\mathcal{F}^d}(S_1,\ldots,S_m)$ if and only if F_1 is a yes-instance of Similar-Solution$_{\mathcal{F}^d}(S_1,\ldots,S_m)$. Since Uniqueness$_{\mathcal{F}^d}(S_1,\ldots,S_m)$ is $\mathbb{NP}$-hard by [8] this concludes the proof for $m \geq 3$.

The case $m=1$ is trivial, so let $m=2$. The fact that Similar-Solution$_{\mathcal{F}^d}(S_1,S_2)$ is in $\mathbb{P}$ follows in conjunction with Theorem 2.1 directly from the polynomial-time solvability of Uniqueness$_{\mathcal{F}^d}(S_1,S_2)$.

Now let (f_1,f_2) be an instance of X-Ray-Correction$_{\mathcal{F}^d}(S_1,S_2)$. Without loss of generality let (f_1,f_2) be consecutive. Clearly, (f_1,f_2) is a yes-instance if and only if there exist consecutive and consistent functions $g_i : \mathcal{A}_{\mathbb{Z}}(S_i) \to \mathbb{N}_0$ $i=1,2$ with $\sum_{i=1}^2 ||g_i - f_i||_1 \leq 1$. On the one hand, there are at most $\|f_1\|_1 + \|f_2\|_1 + 1$ many different choices of pairs (g_1,g_2) of such functions; hence all such pairs can be enumerated in polynomial time. On the other hand, for each choice (g_1,g_2) it can be checked in polynomial-time whether it is a yes-instance of Consistency$_{\mathcal{F}^d}$ (S_1,S_2). $\square$

Finally we will show that Nearest-Solution$_{\mathcal{F}^d}(S_1,S_2)$ can be solved in polynomial time. (Again, the case $m=1$ is trivial.)

Proof of the polynomial-time solvability of Nearest-Solution$_{\mathcal{F}^d}(S_1,S_2)$. Let (f_1,f_2) be an instance of Nearest-Solution$_{\mathcal{F}^d}(S_1,S_2)$. Without loss of generality we may assume that (f_1,f_2) is consecutive. Also, since the empty set is a feasible solution with error $\|f_1\|_1 + \|f_2\|_1$, we know that there is always a solution within the grid G' that is obtained from G by adding for $i=1,2$ to the support of f_i the next $\|f_1\|_1 + \|f_2\|_1$ lattice lines parallel to S_i and taking all intersections of any two of the extended two sets of parallel lines. Then G' contains at most $(2\|f_1\|_1 + \|f_2\|_1)(\|f_1\|_1 + 2\|f_2\|_1)$ lattice points which can all be determined in polynomial time. Let $N = |G'|$, and let M denote the number of different lines parallel to S_1 or S_2 that meet G. The points of G' will be the candidate points among which we will choose a solution.

Further, an optimal solution has at most $2\max\{\|f_1\|_1, \|f_2\|_1\}$ points. Therefore it suffices to solve at most that many instances with the same data but the additional constraint that the solution F has cardinality γ.

Let $F \in \mathcal{F}^d$ with $|F| = \gamma$. Then we have by Lemma 3.1

$$
\begin{aligned}
&||X_{S_1}F - f_1||_1 + ||X_{S_2}F - f_2||_1 \\
&\quad = 2\sum_{T\in\mathcal{A}_1^+}\big(X_{S_1}F(T) - f_1(T)\big) - |F| + \|f_1\|_1 + 2\sum_{T\in\mathcal{A}_2^+}\big(X_{S_2}F(T) - f_2(T)\big) - |F| + \|f_2\|_1 \\
&\quad = 2\left(\sum_{T\in\mathcal{A}_1^+}\big(X_{S_1}F(T) - f_1(T)\big) + \sum_{T\in\mathcal{A}_2^+}\big(X_{S_2}F(T) - f_2(T)\big)\right) - 2\gamma + \|f_1\|_1 + \|f_2\|_1,
\end{aligned}
$$

where $\mathcal{A}_i^+ = \{T \in \mathcal{A}_{\mathbb{Z}}(S_i) : X_{S_i}F(T) - f_i(T) > 0\}$ for $i=1,2$.

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Hence it suffices to find a finite lattice set F with $|F| = \gamma$ that minimizes the sum of the excess of $X_{S_i}F(T)$ over $f_i(T)$.

Introducing one 0-1-variable for each candidate point of G', taking the incidence matrix $A \in \{0,1\}^{M\times N}$ whose rows correspond to the X-ray lines and whose columns correspond to the candidate points, collecting the X-ray data in a right-hand $b \in \mathbb{N}_0^M$, and using the notation $\mathbb{1}$ for a vector of ones of appropriate size, we can formulate this task as an integer linear programming problem.

$$\begin{array}{rl} & \mathbb{1}^T y \to \min \\ \text{s.t.} & Ax \le b + y \\ & \mathbb{1}^T x = \gamma \\ & x \in \{0,1\}^N, y \in \mathbb{N}_0^M. \end{array}$$

Its linear programming relaxation can then be stated as the task to find a real vector solving

$$\begin{array}{rl} & \mathbb{1}^T y \to \min \\ \text{s.t.} & C\begin{pmatrix} x \\ y \end{pmatrix} \le c, \end{array}$$

where

$$C = \begin{pmatrix} A^T & \mathbb{1} & -\mathbb{1} & -I_N & I_N & 0 \\ -I_M & 0 & 0 & 0 & 0 & -I_M \end{pmatrix}^T \quad \text{and} \quad c = (b, \gamma, -\gamma, 0, \mathbb{1}, 0)^T,$$

and where I_M and I_N denote the appropriately sized unit matrices.

We show that C is totally unimodular. Clearly it suffices to show that the submatrix

$$B = \begin{pmatrix} A \\ \mathbb{1}^T \end{pmatrix}$$

is totally unimodular. But this follows from the fact that each collection of rows from B can be split into two parts such that the difference of the sums of the rows in the first and in the second part is a vector with coefficients in $\{-1, 0, 1\}$ (see [18]). This is trivial if the collection does not involve the last row of B since the rows of A can be partitioned into two sets that correspond to the two directions and each column of A contains exactly two entries 1, one corresponding to S_1 and one corresponding to S_2. If, on the other hand, the last row is involved, take it as one part of the partition.

One can now use any polynomial-time linear programming algorithm to solve the task. $\square$

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