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A Geometric Construction for the Associated Family of S–Isothermic CMC Surfaces

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Abstract

For discrete minimal and constant mean curvature surfaces, a complete theory that includes their associated families has only been known for the discrete isothermic discretization, and for special cases of s–isothermic minimal surfaces. For the s–isothermic discretization of cmc surfaces, the associated family was previously unknown. We present a geometric construction, applied to the elementary cubes of Christoffel pairs in the special case of edge–wise tangent vertex spheres, that extends many properties known from the corresponding minimal and the discrete isothermic cmc case. We identify characteristic properties of the resulting non–planar quadrilateral geometry and find numerical evidence that those are sufficient to construct the associated families of general s–isothermic cmc surfaces. We hope that the existence of those examples will help to develop a comprehensive surface theory for the s–isothermic world.

Contents

Chapter 1

Introduction

In classical differential geometry, surfaces of constant mean curvature, both non–zero and minimal, admit isothermal parametrizations — conformal parametrizations by curvature lines. These have been very successfully discretized as discrete isothermic surfaces ([BP96]) — maps with the local combinatorics of a square grid whose planar quadrilateral faces satisfy a cross–ratio condition that in particular makes them circular. Both smooth isothermal immersions and discrete isothermic surfaces can be characterized by a corresponding version of the *Moutard equation* in the classical linear model of Möbius geometry. If the discrete version of this characterization is taken from the light cone to space–like unit vectors, one arrives at the definition of *s–isothermic* surfaces ([BP99], [BHS06], [Hof10]). With the latter, the vertices are no longer just points, but spheres, and circularity changes to them having a common orthogonal circle, a common point or a common pair of points as they get larger. The original discrete isothermic surfaces can be seen as the limit case of the spheres shrinking to points. Among s–isothermic surfaces is the special case of s–conical ones ([BHKS15], [BH16]) which are a conformal version of, well, conical surfaces which are an interesting class in their own right ([LPW$^+$06]). Thus, s–isothermic surfaces can be seen as a class unifying different discretization approaches to parametrized surfaces. The best–studied version of s–isothermic surfaces is the special case in which the spheres are sized such that they are mutually tangent along edges. This was the definition first introduced and will be called s_1*–isothermic* here. It will also be the focus of the main construction in this work.

The notions of Christoffel duals and Darboux transforms directly carry over from the discrete isothermic to the s–isothermic case. They characterize minimal and cmc (by which we mean non–zero constant mean curvature) surfaces in the smooth and discrete isothermic case and were used for their definition in the s–isothermic cases ([BHKS15], [Hof10], [BH16]). In chapter 3, after giving an introduction to s–isothermic surfaces, we will provide some new calculations that facilitate the construction of examples of s–isothermic cmc surfaces in practice: A closer look at quadrilaterals satisfying the

Moutard equation and having a pair of parallel edges allows for the implementation of cmc–preserving Darboux transforms as well as the construction of rotationally symmetric general s–isothermic cmc surfaces. The latter construction had previously been extended from the discrete isothermic version only to the s_1–isothermic ([Hof10]) and s–conical ([BH16]) case.

Classically, isothermal minimal and cmc immersions come with a one–parameter family of deformations that preserve the mean curvature and first fundamental form. The parameter is an angle, constant over the surface, by which the parameter directions get rotated away from the principal curvature directions. This is the *associated family* of the respective surface.

For discrete isothermic minimal and cmc surfaces, a comprehensive theory, completely analogous to the classical smooth case, has been developed that contains the associated families ([BP96], [BP99]). All surfaces in these associated families have constant (vanishing or non–zero) mean curvature in the recently developed, very general discrete curvature theory of [HSW16].

In the s–isothermic world, associated families previously had only been constructed for minimal surfaces, and only in the s_1–isothermic ([BHS06]) and s–conical ([BHKS15]) special cases. For the latter, we provided the geometric construction of the associated family which we repeat in section 4.3. There, we also prove that this construction is consistent with the theory of discrete minimal surfaces developed independently by W. Y. Lam ([Lam16]).

For cmc surfaces, in this work (section 4.4) we present a new construction for the s_1–isothermic case which rotates edges around edge normals in analogy to the s_1–isothermic minimal case. Since in contrast to the minimal case these normals are not constant, a lot more work is needed.

In our construction, we identified characteristic *folding* and *bending* properties of the resulting non–planar faces — governed by a global parameter — that lead to the definitions in section 4.1. These properties can be stated for all s–isothermic cases, not just the s_1–isothermic one. For vanishing radii of the s–isothermic vertex spheres, our folding property reduces to the known geometry of the *equally–folded parallelogram cubes* of the associated family of discrete isothermic cmc surfaces.

In sections 4.2 and 4.3, we check that our bending property also holds for the previous s_1–isothermic and s–conical minimal cases.

In section 4.5, we present a numerical construction that indicates that our bending and folding properties are sufficient to construct the associated family for general s–isothermic cmc surfaces. Based on numerical evidence, we conjecture that this will provide a consistent construction with the properties one would expect for the associated family.

Applying the general numerical construction to s–conical cmc surfaces, we found a more direct geometric construction for this special case (section 4.6) which only relies on numerical optimization for a single parameter. Again, the numerical results lead us to conjecture that this construction will provide the surfaces of the associated family.

Chapter 2

Tools and Motivation

2.1 Conventions and Notation

The main objects of this work are discrete surfaces with quadrilateral faces. Since our considerations are local in nature, we will not concern ourselves with the large–scale topology of the domain and always speak of maps

$$f : \mathbb{Z}^k \to X,$$

where the term surface of course refers to $k = 2$, and X will mostly be some $\mathbb{R}^n$. The terms *edges* and *faces* will refer to the combinatorics of the standard cubical cell decompostion of $\mathbb{R}^k$ with $\mathbb{Z}^k$ as the vertices. The terms will equally be used for their images under the map. *Faces* will always be meant to be 2–dimensional, and the image of their adjacent vertices under a map f also called an *elementary quadrilateral*. In an abuse of notation, we will also write $\mathbb{Z}^k$ for the domain but mean "any reasonably behaved subset", which mostly means a simply connected pure k–dimensional subcomplex. In this way, pairs or collections of surfaces will often be written as maps defined on $\mathbb{Z}^3$ or $\mathbb{Z}^4$. The notion of elementary quadrilaterals will be extended to *elementary cubes* for the image of the vertices adjacent to a 3–cell.

Maps of this type will also be called *nets*. Given a net $f : \mathbb{Z}^k \to X$, by f we will not only denote the map itself, but also an arbitrary vertex $f(z_1, \dots, z_k)$. In this case, we will employ the *shift notation* to designate neighboring vertices:

$$f_i := f(z_1, \dots, z_i + 1, \dots, z_k), \quad f_{ij} := f(z_1, \dots, z_i + 1, \dots, z_j + 1, \dots, z_k)$$

etc. For example, (f, f_1, f_{12}, f_2) will be an elementary quadrilateral of f in cyclical ordering of the vertices.

Most of the theory will directly generalize from the combinatorics of the square grid to edge–bipartite quad–graphs, but do to the aforementioned local nature we will not explore this possibility.

A few more, minor notational conventions we will use are

- For points $a, b, c, d \in \mathbb{R}^3$, we will denote the dihedral angle of the planes spanned by a, b, c and a, b, d by $\measuredangle_{a,b}(c, d)$.
- For $x, y \in \mathbb{R}^n$, we will use the symbol $\overset{+}{\sim}$ as "proportional by a positive factor", i.e.

$$x \overset{+}{\sim} y \quad :\Leftrightarrow \quad \exists \lambda > 0 \text{ s.th. } x = \lambda y.$$

  The symbol "$\|$" will mean parallelity regardless of orientation.
- We will use the word "trapezoid" in the American meaning, i.e. for quadrilaterals with a pair of parallel edges.

2.2 The Classical Model for Möbius Geometry

The definition of s–isothermic surfaces is based on a characterization of isothermic surfaces in a classical linear model for the geometry of points and spheres in $\mathbb{R}^n$ which we will briefly introduce here. Our presentation is mainly based on [Hof09], similar introductions can be found in the papers on s–isothermic surfaces, e.g. [Hof10] and [BH16]. A more thorough treatment of this and related models can be found at the end of [BS08].

We equip $\mathbb{R}^{n+2}$ with the Minkowksi inner product

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n+1} x_i y_i$$

and call it $\mathbb{R}^{n+1,1}$. We will adopt the terminology of space–like, light–like etc. from relativity: vectors $v \neq 0$ (or the one–dimensional subspace they span) are space–like, if $\langle v, v \rangle > 0$, time–like, if $\langle v, v \rangle < 0$, and light–like or isotropic, if $\langle v, v \rangle = 0$. More generally, subspaces of dimension ≥ 2 are space–like if the inner product restricted to them is positive definite, time–like if it is indefinite, and light–like if it is degenerate. Points in $\mathbb{R}^n$ will be represented by one–dimensional light–like subspaces, and hyperspheres/–planes by space–like unit vectors.

Specifically, for a point $p \in \mathbb{R}^n$ we set

$$P := \left(\frac{1 + \|p\|^2}{2}, p\,, \frac{1 - \|p\|^2}{2} \right) \tag{2.1}$$

which satisfies $\langle P, P \rangle = 0$. We equivalently let P, any nonzero multiple or the subspace spanned by it represent our point p. Conversely, if we have any vector $0 \neq P \in \mathbb{R}^{n+1,1}$ with $\langle P, P \rangle = 0$, we can find the corresponding point in $\mathbb{R}^n$ by

$$p = \frac{1}{P_0 + P_{n+1}} \big(P_1, \ldots, P_n \big).$$

The point at infinity is represented by $(\lambda, 0, \ldots, 0, -\lambda)$.

Now consider a hypersphere s in $\mathbb{R}^n$ with center c and radius $r \neq 0$. The radius can be positive or negative, corresponding to the orientation of the sphere. Then we set

$$S := \frac{1}{2r} \Big(1 + (\|c\|^2 - r^2),\, 2c\,, 1 - (\|c\|^2 - r^2) \Big)$$

which satisfies $\langle S, S \rangle = 1$. A hyperplane $\{v \mid \langle v, n \rangle = d\} \subset$ with unit normal n is represented by

$$S := \big(d, n, -d \big).$$

Conversely, if we are given a space–like unit vector $S \in \mathbb{R}^{n+1,1}$, we can find the corresponding sphere by

$$r = \frac{1}{S_0 + S_{n+1}}, \quad c = r(S_1, \ldots, S_n).$$

If $r = \infty$, S represents a hyperplane $\{v \mid \langle v, n \rangle = d\} \subset$ with $n = (S_1, \ldots, S_n)$ and $d = S_0$. Note that rescaling S by -1 corresponds to reversing the orientation of the sphere/plane.

How these representations arise geometrically is illustrated in fig. 2.1.

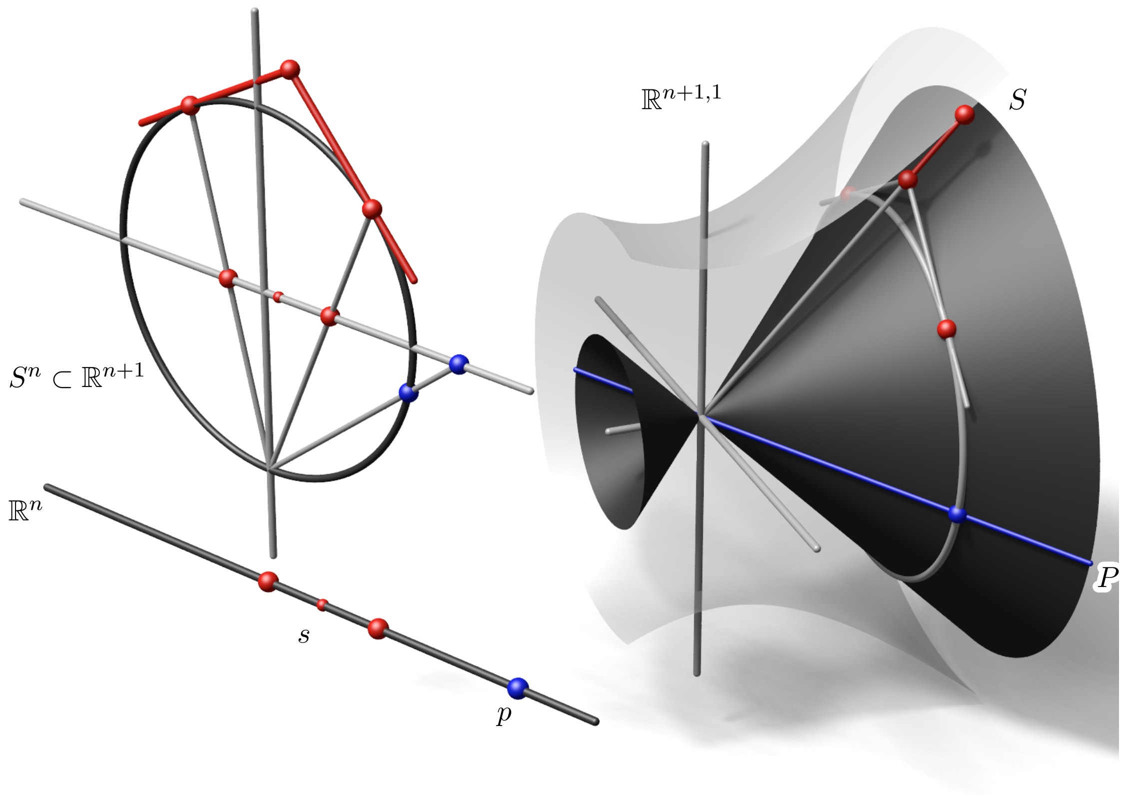

Figure 2.1: Geometry of the classical model. First, points and hyperspheres are mapped to $S^n \subset \mathbb{R}^{n+1}$ via stereographic projection (for our choice of identification, at the south pole). For the hyperspheres, we mark the tip of the cone tangent to S^n in the image of the hypersphere. Then, this $\mathbb{R}^{n+1}$ is embedded in $\mathbb{R}^{n+1,1}$ as the hyperplane $\{x_0 = 1\}$, making the S^n lie in the light cone. The image of the cone tip can be normalized to lie on the space–like unit hyperboloid, and the result represents the hypersphere.

The most prominent property of this model is that Möbius transformations — appliable to both points and spheres — become orthogonal maps (with respect to the Minkowski inner product): For a sphere (or plane) represented by $S\mathbb{R}^{n+1,1}$ the inversion on that sphere is given by

$$X \mapsto X - 2 \langle X, S \rangle S. \tag{2.2}$$

As an orthogonal map, this preserves both the light cone and the space–like unit hyperboloid, and consequently, the map can be applied to points and spheres alike. But since having constant mean curvature is not Möbius invariant, this will not be the most important property to us.

Some properties that are of interest to us are the following:

- For two points $p, q \in \mathbb{R}^n$ with their representatives P and Q scaled as in eq. (2.1),

$$\langle P, Q \rangle = -\tfrac{1}{2} \|p - q\|^2 . \tag{2.3}$$

- Incidence of points and spheres is characterized by orthogonality:

$$p \in s \subset \mathbb{R}^n \quad \Leftrightarrow \quad \langle P, S \rangle = 0.$$

- For two intersecting spheres, the inner product is the cosine of the intersection angle:

$$\cos \measuredangle(s_1, s_2) = \langle S_1, S_2 \rangle .$$

The latter property is related to a general geometric characterization of the inner product of two spheres: if s_1, s_2 are spheres with centers c_1, c_2 and radii r_1, r_2, and $d := \|c_2 - c_1\|$ is the distance of their centers, we have

$$\langle S_1, S_2 \rangle = \frac{r_1^2 + r_2^2 - d^2}{2 r_1 r_2}. \tag{2.4}$$

In particular, spheres intersect orthogonally iff their representatives in $\mathbb{R}^{n+1,1}$ are orthogonal.

From now on, we will always consider $n = 3$. We can represent further geometric objects in this model. Consider e.g. a circle $k \subset \mathbb{R}^3$. If s_1 and s_2 are spheres such that $k = s_1 \cap s_2$, the points p on the circle are characterized by $\langle P, S_1 \rangle = \langle P, S_2 \rangle = 0$, i.e. all the light–like directions in the orthogonal space $\operatorname{span}(S_1, S_2)^\perp$. Consequently, all space–like unit vectors in $\operatorname{span}(S_1, S_2)$ represent spheres containing the circle k. So we can represent circles as 2–dimensional space–like subspaces.

Similarly, three distinct mutually intersecting spheres span a 3–dimensional space–like subspace. Its orthogonal space is 2–dimensional time–like and contains two light–like directions: the two points of intersection of the three spheres.

If two spheres do not intersect, they span a time–like subspace, and its two light–like directions represent the points common to all spheres orthogonal to the original two. Two tangent spheres span a light–like subspace, and its light–like direction — also contained in the orthogonal space —, is the point of tangency — also contained in all spheres orthogonal to the original two. Figures 2.2 to 2.4 illustrate those situations.

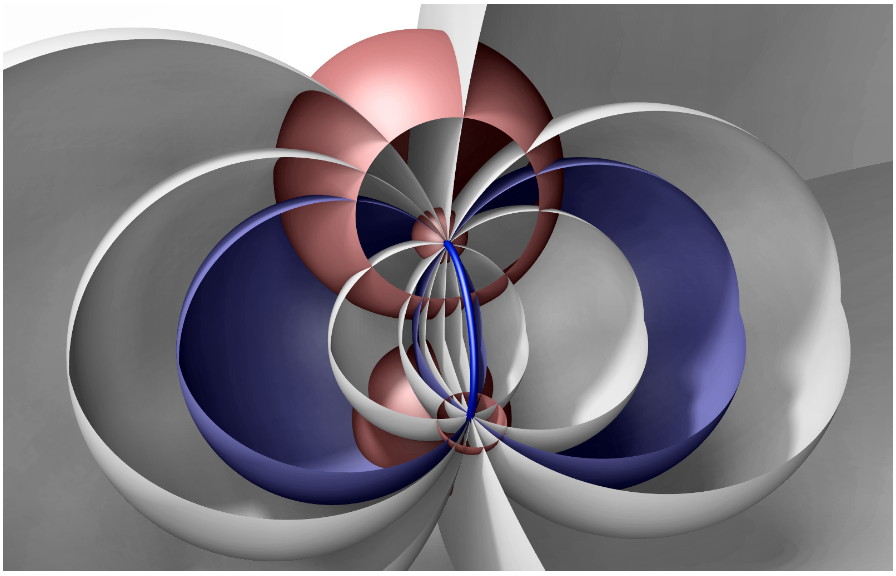

Figure 2.2: A circle (blue) represented by the two–dimensional space–like subspace spanned by the two blueish spheres. All grey spheres are linear combinations of those two and also contain the circle. Reddish spheres are in the orthogonal space and intersect the circle orthogonally.

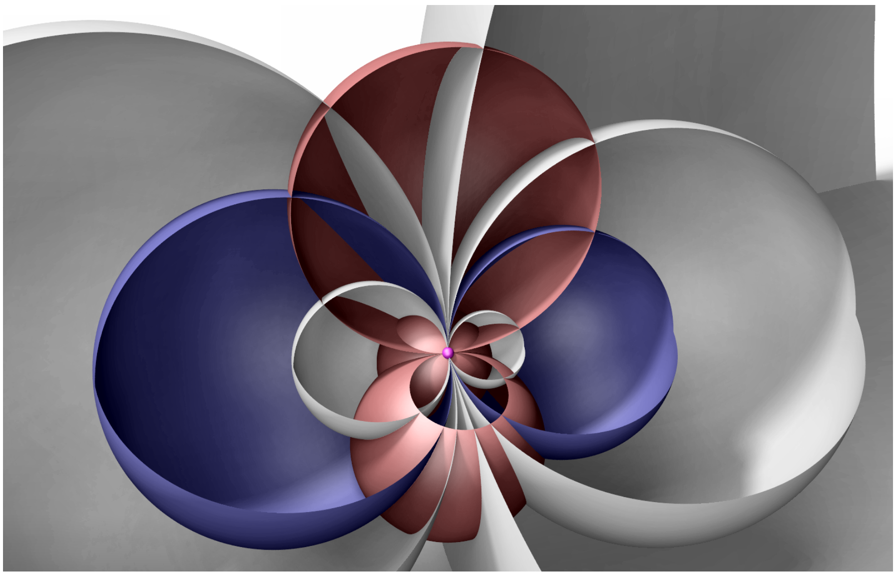

Figure 2.3: The subspace spanned by the two blueish spheres is light–like. It has a common light–like direction with its orthogonal space, representing the purple point. This point is the common point of tangency of the spheres in the subspace, and also contained in all orthogonal spheres.

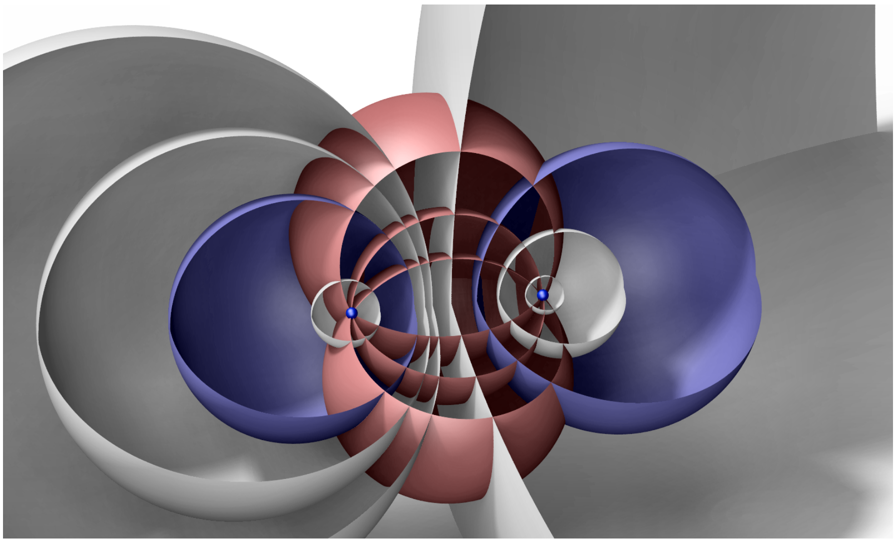

Figure 2.4: If the subspace spanned by the two blueish spheres is time–like, it contains two light–like directions, representing the blue points. Those points are common to all orthogonal spheres.

2.3 Smooth and Discrete Isothermic Surfaces

Here we briefly introduce classical isothermally parametrized (smooth) surfaces and their first structure–preserving discretization, discrete isothermic nets as defined in [BP96]. We present the characterizations that have motivated the definition of s–isothermic nets we will see in chapter 3. Again, we mainly follow [Hof09] and [BS08].

Definition 2.3.1. Let $U \subset \mathbb{R}$ be open and $f : U \to \mathbb{R}^n$ a smooth immersion. f is called *isothermal* if it is conformal (i.e. $f_x \perp f_y$ and $\|f_x\| = \|f_y\|$), and $f_{xy} \in \operatorname{span}(f_x, f_y)$.

Remark 2.3.1. *More generally, the immersion is still called isothermal if conformality is relaxed to orthogonality and*

$$\|f_x(x,y)\|^2 = \alpha(x)s(x,y)^2, \quad \|f_y(x,y)\|^2 = \beta(y)s(x,y)^2$$

for some functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}_+$ and a conformal factor $s : \mathbb{R}^2 \to \mathbb{R}_+$, i.e. f is conformal after independent reparametrization of the variables x and y.

Remark 2.3.2. *Orthogonality and $f_{xy} \in \operatorname{span}(f_x, f_y)$ mean first and second fundamental form are diagonal. Consequently, isothermality can equivalently be characterized as a conformal curvature line parametrization. (Possibly with the generalization of remark 2.3.1).*

In the smooth case, an *isothermic* surface is one that admits an isothermal reparametrization.

For the first discretization, it has been observed that a parametrized surface is isothermal (in the narrower conformal case) iff the quadrilaterals obtained as the second order Taylor approximation of $f(x \pm \epsilon, y \pm \epsilon)$ have cross–ratio -1 up to second order in ϵ. This led to

Definition 2.3.2. A planar quad mesh $f : \mathbb{Z}^2 \to \mathbb{R}^n$ is called *discrete isothermic* if all elementary quadrilaterals have

$$\operatorname{cr}(f, f_1, f_{12}, f_2) = -1,$$

or more generally

$$\operatorname{cr}\big(f(k,l), f(k+1,l), f(k+1,l+1), f(k,l+1)\big) = \frac{\alpha(k)}{\beta(l)} < 0$$

for some $\alpha, \beta : \mathbb{Z} \to \mathbb{R}$.

Remark 2.3.3. *In particular, the elementary quadrilaterals of a discrete isothermic net are circular. In general, circular nets can be seen as a discretization of curvature line parametrizations: the axes of adjacent circles intersect, i.e lie both in the plane of the* *segment connecting the two circle centers. If we interpret them as normals, this means that in going from circle center to circle center in a parameter direction, the change in normals is parallel to the parameter step.*

Now, we put the classical model from section 2.2 to use in characterizing both smooth isothermal and discrete isothermic surfaces: first, let $f : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be a smooth immersion and $F : U \to \mathbb{R}^{n+1,1}$ the pointwise representation in the light cone, in the scaling such that $F = (\cdot, f, \cdot)$.

Theorem 2.3.4. *f is isothermal with $\|f_x\|^2 = \alpha(x)s^2$, $\|f_y\|^2 = \beta(y)s^2$ iff*

$$\left(\frac{1}{s}F\right)_{xy} = \lambda \frac{1}{s}F$$

for some function λ.

This differential equation for $\frac{1}{s}F$ has a name:

Definition 2.3.3. A smooth map $f : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ satisfies the *Moutard equation* if

$$f_{xy} = \lambda f$$

for a function λ.

Now, if f is a quad net $\mathbb{Z}^2 \to \mathbb{R}^n$, it is natural to discretize the mixed derivative as a difference of edges:

$$\Delta_2 \Delta_1 f = (f_1 - f)_2 - (f_1 - f) = f_{12} + f - (f_1 + f_2).$$

Since this is an average over an elementary quad, we do the same for f on the right hand side and arrive at the natural discretization

$$f_{12} + f - (f_1 + f_2) = \mu(f + f_1 + f_2 + f_{12}),$$

or

$$f_{12} + f = \frac{1+\mu}{1-\mu}(f_1 + f_2)$$

of the Moutard equation, introduced by [NS98]. After rewriting the parameter, we get

Definition 2.3.4. A map $f : \mathbb{Z}^2 \to \mathbb{R}^n$ satisfies the *discrete Moutard equation* (with a plus sign) if

$$f_{12} + f = \lambda(f_1 + f_2)$$

for a function $\lambda : \mathbb{Z}^2 \to \mathbb{R} \setminus \{0\}$.

If λ is known, this determines f_{12} from f, f_1 and f_2. This situation happens for example if we restrict f to a quadric of constant length: Suppose our $\mathbb{R}^n$ is equipped with an inner product and

$$\langle f, f\rangle = \langle f_1, f_1\rangle = \langle f_2, f_2\rangle = c.$$

Then, if $\langle f_1, f_2\rangle \neq -c$, an f_{12} such that the Moutard equation holds and $\langle f_{12}, f_{12}\rangle = c$ is found via

$$\lambda = \frac{\langle f, f_1 + f_2\rangle}{c + \langle f_1, f_2\rangle}. \tag{2.5}$$

In particular, we can restrict the discrete Moutard equation to the light cone in $\mathbb{R}^{n+1,1}$ and state

Theorem 2.3.5. *$f : \mathbb{Z}^2 \to \mathbb{R}^n$ is discrete isothermic with*

$$\mathrm{cr}(f, f_1, f_{12}, f_2) = \frac{\alpha}{\beta} < 0$$

iff its appropriately scaled pointwise representation F in the light cone satisfies the discrete Moutard equation.

This completely analogous characterization of smooth and discrete isothermicity by Moutard equations motivates the definition of s–isothermic surfaces we will see in chapter 3.

Now for another important property of isothermic surfaces, smooth and discrete.

Theorem 2.3.6. *Let $f : U \to \mathbb{R}^n$ be smooth isothermal with $\|f_x\|^2 = \alpha(x)s^2$, $\|f_y\|^2 = \beta(y)s^2$. Then*

$$f_x^* = \alpha \frac{f_x}{\|f_x\|^2} = \frac{f_x}{s^2}, \quad f_y^* = -\beta \frac{f_y}{\|f_y\|^2} = -\frac{f_y}{s^2}$$

define another isothermal immersion f^* with $\|f_x\|^2 = \alpha(x)s^{-2}$, $\|f_y\|^2 = \beta(y)s^{-2}$.

The f^* is called the *Christoffel dual* or, for short, just *dual* of f. In complete analogy, we have

Theorem 2.3.7. *Let $f : \mathbb{Z}^2 \to \mathbb{R}^n$ be discrete isothermic with $\mathrm{cr}(f, f_1, f_{12}, f_2) = \frac{\alpha}{\beta} < 0$. Then*

$$f_1^* - f^* = \alpha \frac{f_1 - f}{\|f_1 - f\|^2}, \quad f_2^* - f^* = \beta \frac{f_2 - f}{\|f_2 - f\|^2}$$

define another discrete isothermic net.

This is also called the *Christoffel dual* of f. Note that the term “dual” refers to the property $f^{**} = f$ which holds (up to translation) in the smooth and discrete cases.

2.4 Parallel Surfaces, Curvature, and Duality

The importance of Christoffel duals in this context arises from the definition of curvatures for planar quad nets presented in this section. It has been introduced by [BPW10], and has become a standard notion in discrete surface theory as it unifies many earlier (and later) characterizations of discrete surfaces of constant curvature. Again, our brief introduction is mainly based on [Hof09].

The motivation for this notion of discrete curvature is the classical Steiner formula for curvatures of smooth surfaces, dating back to [Ste40]. For a smooth immersion $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ let $n : U \to \mathbb{R}^3$ be the unit normal map. Now consider the family of *offset* or *parallel* surfaces $f^t := f + tn$ with a real parameter t. For t small enough (and if necessary U with compact closure within the domain of f — we are only interested in local behavior), these are smooth immersions again. The Steiner formula calculates the surface area $A(f^t)$ of the offset surfaces relative to that of the original, $A(f)$:

Theorem 2.4.1 (Steiner formula)**.** *If f is a smooth immersion and f^t a smooth parallel surface,*

$$A(f^t) = A(f) + 2tH(f) + t^2 K(f),$$

where $H(f)$ and $K(f)$ are the integrals over mean and Gaussian curvature of f, respectively.

[Hof09] contains a proof in modern language.

Since discrete surfaces do not come so conveniently pre–equipped with a unit normal map, in order to get parallel surfaces, we have make some defintions:

Definition 2.4.1. A *line congruence net* is a planar quad net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ together with a map $l : \mathbb{Z}^2 \to \{\text{lines in } \mathbb{R}^3\}$ such that all $f \in l$ and all pairs l, l_1 and l, l_2 of adjacent lines are coplanar.

Now if we have a line congruence net f, we can choose an initial offset t by which to move an initial vertex f along the corresponding line l to a new vertex f^t. If we draw edges from the new vertex parallel to those of the original net, the coplanarity condition ensures they hit the line l_1, l_2 corresponding to the other vertex f_1, f_2 of the original edge, and we set this intersection point as the new vertex f_1^t, f_2^t. Planarity of faces guarantees that this process is consistent, and we get a new line congruence net f^t with the same lines l, and edges (and thus faces) parallel to the corresponding ones of f.

Definition 2.4.2. Given a line congruence net (f, l), we define its *normal map* $n : \mathbb{Z}^2 \to \mathbb{R}^3$ by the conditions that all $f + n \in l$, $f + n$ is edge–parallel to f, and by choice of an initial non–vanishing length of n at one vertex.

Now the process of constructing offset surfaces above can be written as $f^t = f + tn$. Note that that the parallelity condition for f and $f + n$ makes n itself a mesh edge–parallel to f.

On the other hand, if we are given a (distinct) pair of planar quad nets with parallel corresponding edges, we can define a line congruence (for both nets) by connecting corresponding vertices by lines.

Remark 2.4.2. *Similar to remark 2.3.3, if we interpret the lines as normals, line congruence nets can be seen as discretizations of curvature line parametrizations: coplanarity of adjacent normals means when moving in parameter direction, i.e. along edges, the change in normal is parallel to the edge (if given appropriate length as in the offset surface above). However, this only makes sense if the line directions can actually be justifiably considered normals, which is in no way guaranteed just from the definition.*

Now that we have a class of discrete surfaces that allow for a one–parameter family of offset surfaces, we can move on to a discrete version of the Steiner formula. Areas will obviously be calculated quad–wise, so we look into the area of families of edge–parallel quads. First of all, we note that for a given quadrilateral (which we will consider fixed in the following), the set $\mathcal{P}$ of all edge–parallel quads is a vector space (with vertex–wise addition and scalar multiplication).

The area $A(p)$ of quadrilaterals p is a quadratic form on $\mathcal{P}$, and the *mixed area* $A(p, q)$ of quads p and q is the corresponding symmetric bilinear form, i.e. $A(p, p) = A(p)$. In particular, for $p, q \in \mathcal{P}$ and $t \in \mathbb{R}$, we have

$$A(p + tq) = A(p) + 2tA(p, q) + t^2 A(q).$$

Thus, in the spirit of the Steiner formula, we can make the following

Definition 2.4.3. Let (f, l) be a line congruence net with normal map n. We define its mean curvatur H and Gaussian curvature K to satisfy

$$A(f + tn) = (1 + 2tH + t^2 K)A(f),$$

i.e. $H = \frac{A(f,n)}{A(f)}$ and $K = \frac{A(n)}{A(f)}$.

Note that the curvatures depend on the choice of scaling for n, but having constant curvature is well–defined from the line congruence alone. We see directly that $H = 0 \Leftrightarrow A(f, n) = 0$ and consequently call a line congruence net *minimal* if $A(f, n) = 0$ everywhere. On the other hand, for $H \neq 0$, we can rewrite

$$H = \frac{A(f, n)}{A(f)} \Leftrightarrow HA(f, f) - A(f, n) = 0 \Leftrightarrow A(f, f - \tfrac{1}{H}n) = 0$$

and see that we get a constant mean curvature H_0 iff we find a parallel surface $f^* = f - \frac{1}{H_0}n$ in constant “distance” with $A(f, f^*) = 0$ — once n is actually a reasonably constant–length normal map. In both cases, vanishing mixed area is crucial:

Definition 2.4.4. Two edge–parallel quadrilaterals p, q are *dual* to each other if

$$A(p, q) = 0.$$

Dual quadrilaterals are characterized by parallel non–corresponding diagonals, i.e. if $p = (p_0, p_1, p_2, p_3)$ and $q = (q_0, q_1, q_2, q_3)$ are edge–parallel, they are dual iff

$$p_2 - p_0 \parallel q_3 - q_1 \quad \text{and} \quad p_3 - p_1 \parallel q_2 - q_0.$$

Note that this implies that one corresponding pair of opposite edges is equally oriented in both quads, while the other pair have opposing directions, corresponding to a reversal of one parameter direction.

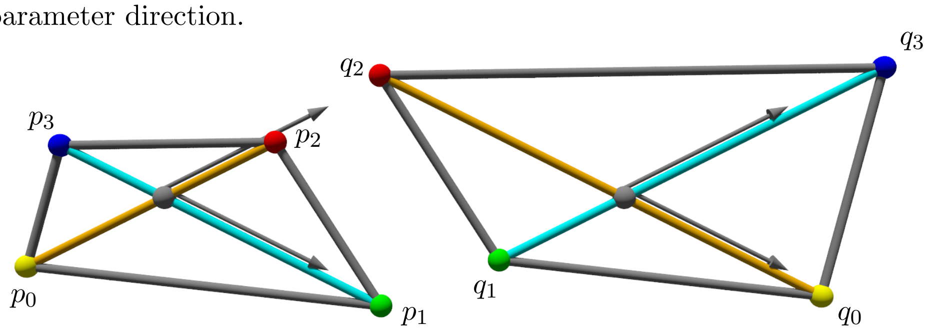

Figure 2.5: A pair of dual quadrilaterals.

Every quad has a dual unique up to scaling (and translation). But for a whole planar quad net, the scaling of one dual quad determines the scaling of the duals of adjacent faces, which may not be consistently possible. If it is, the net is called a (discrete) *Koenigs* net, and the dual net is called its *Christoffel dual.* This is of course not a coincidence: the dual for discrete isothermic nets from theorem 2.3.7 is an example of a dual in the mixed area sense. Note that also in the smooth case, not only isothermally parametrized surfaces have Christoffel duals, but a broader class, smooth Koenigs nets. Their duals arise by a similar inversion of parameter directions. Cf. e.g. [BS08, section 1.2].

2.5 Smooth and Discrete Minimal and CMC Surfaces, and Darboux Transforms

In the smooth world, minimal and cmc surfaces are of course defined as having constant mean curvature $H = 0$ or $H = H_0 \neq 0$, respectively. Minimal and cmc surfaces without umbilic points are isothermic and thus have Christoffel duals. The characterization of discrete minimal and cmc line congruence nets in section 2.4 has a direct analogue in the smooth case, where surfaces already come with a well–defined unit normal map:

Theorem 2.5.1. *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth immersion without umbilic points and $n : \mathbb{R}^2 \to S^2$ its Gauss map.*

- *If f is minimal, then f is Christoffel dual to n;*
- *if f has constant mean curvature $H_0 \neq 0$, then*

$$f + \tfrac{1}{H_0} n$$

is Christoffel dual to f, and also has constant mean curvature H_0.

Cf. [BS08, section 1.8].

So for minimal surfaces, one can start with an isothermic map to the unit sphere and find the minimal surface as its Christoffel dual. This also works in the discrete world: a discrete Koenigs n net that reasonably discretizes the unit sphere is automatically a reasonable discrete "unit" normal for itself and thus any parallel surface — like its Christoffel dual n^*. The line congruence defined by n on $f := n^*$ will be minimal. This has been done for discrete isothermic minimal surfaces in [BP96], where the Gauss map is discrete isothermic with vertices on the unit sphere, for s_1–isothermic (more on that in section 3.3.1) minimal surfaces in [BHS06], where the Gauss map has edges tangent to the unit sphere, and for s–conical (more on that in section 3.3.2) minimal surfaces in [BHKS15], where the Gauss map has faces tangent to the unit sphere.

Discrete isothermic cmc surfaces with a unit vertex normal map defining a line congruence — and thus a Christoffel dual cmc surface in constant vertex distance — have been introduced in [BP99]. For the s_1–isothermic cmc surfaces of [Hof10], the dual will have one constant edge distance per parameter direction, and for s–conical ones ([BH16]), a constant face offset. Both types will be visited in more detail in chapter 3 as special cases of s–isothermic cmc surfaces. In general, their parallel surfaces will not be in a specific constant distance in a direct geometrical sense. Instead, another characterization of isothermic cmc surfaces based on Darboux transforms will be used. We present the smooth (from [HP97]) and discrete isothermic (from [HHP96]) analogues here.

Darboux transformations for smooth isothermic surfaces were studied in classical differential geometry, c.f. [Dar99] and [Bla29]. We present a sketch of a geometric definition from [BS08] without going into details.

Definition 2.5.1. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^3$ be isothermal with the same functions α and β from remark 2.3.1. f and g are said to be a *Darboux pair* or *Darboux transforms* of each other if corresponding pairs of parameter lines envelop common 1–parameter families of circles.

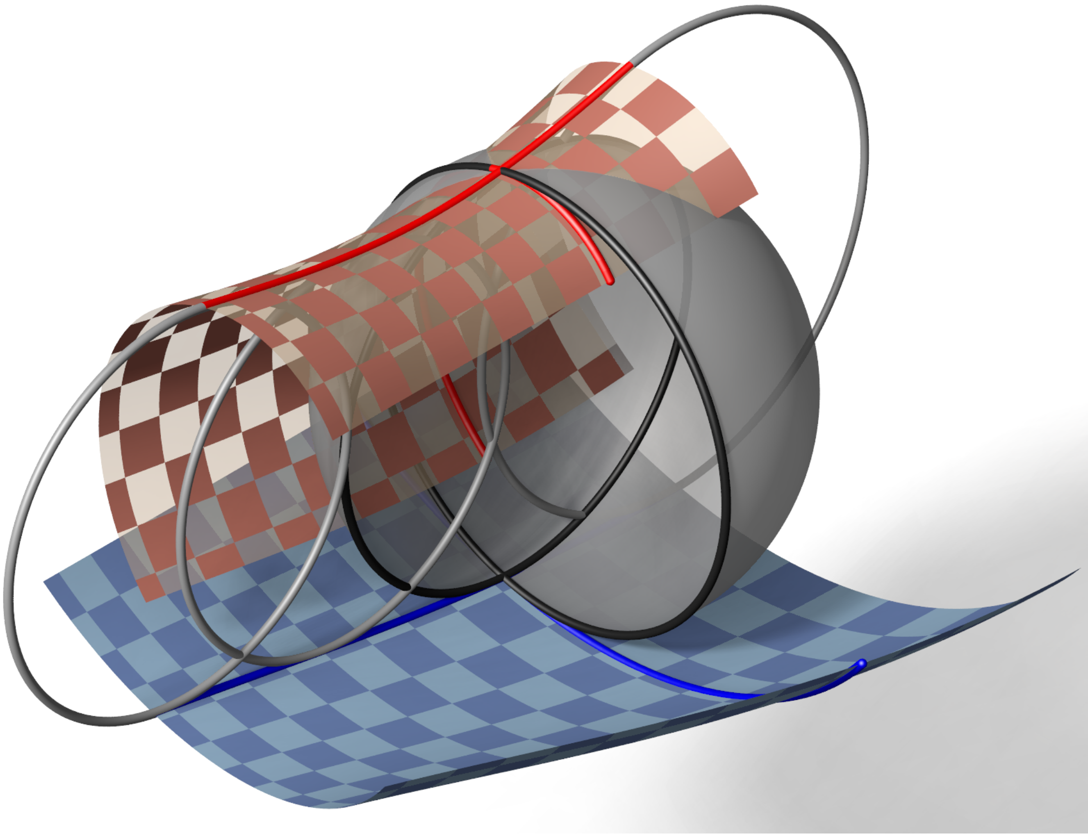

Figure 2.6: A Darboux pair of isothermic surfaces. Some of the circles enveloped by one pair of parameter lines and one of the spheres in the congruence enveloped by the two surfaces are shown.

The parameter line enveloping condition alone constitutes a *Ribaucour* transformation for orthogonally parametrized surfaces, the preservation of conformality up to independent reparametrizations makes it a Darboux transformation. A each point, the two orthogonally intersecting circles span a sphere to which both parameter lines are tangent, so the two surfaces envelop a sphere congruence. The per–coordinate families of circles can be seen as making the correspondence preserve direction of the curvature lines. A sketch of the situation is shown in fig. 2.6.

The characterization of Darboux transforms whose discrete version will later serve as a definition lives in the setting of theorem 2.3.4. Following [BS08], we first introduce the general notion of a Moutard transformation:

Definition 2.5.2. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^n$ satisfy the Moutard equation as in def. 2.3.3. f and g are called *Moutard transforms* of each other if

$$g_x + f_x = \mu(g - f), \quad g_y - f_y = \nu(g + f)$$

for some functions $\mu, \nu : \mathbb{R}^2 \to \mathbb{R}$ (or the same with the combination of coordinate directions and plus and minus signs interchanged).

If we interpret the step from f to g as a discrete third coordinate direction, these equations can be seen as semi–discrete Moutard equations.

Now, a Darboux transformation is just a Moutard transformation of representations in the light cone:

Theorem 2.5.2. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}^3$ be isothermal and $F, G : \mathbb{R}^2 \to \mathbb{R}^{4,1}$ their (appropriately scaled) representatives in the light cone. f and g are Darboux transforms of each other iff F and G are Moutard transforms of each other.*

We skip discussing the appropriate scaling since in the s–isothermic analogue, is will just be normalization.

The semi–discrete nature of the equations in def. 2.5.2 already suggests that in the discrete case, the situation for coordinate and transformation directions will be symmetric. Indeed, in [HHP96], a Darboux transform between to discrete isothermic nets f and g is characterized by analogous cross–ratio conditions for the elementary quadrilaterals (f, f_1, f_{12}, f_2), (f, f_1, g_1, g) and (f, f_2, g_2, g).

As in the semi–discrete case, in the fully discrete setting we will need Moutard equations with different signs for a characterization of Darboux transforms. Our discussion will follow [BS08].

Definition 2.5.3. A map $f : \mathbb{Z}^2 \to \mathbb{R}^n$ satisfies the *discrete Moutard equation* with a minus sign if

$$f_{12} - f = \lambda(f_2 - f_1)$$

for a function $\lambda : \mathbb{Z}^2 \to \mathbb{R}$.

For short, we will also call this the Moutard$-$ equation, and distinguish it from the previous Moutard equation with a plus sign as in def. 2.3.4 by referring to the latter as the Moutard$+$ equation. The Moutard$-$ equation is multidimensionally consistent, i.e. the definition can be extended to maps $\mathbb{Z}^m \to \mathbb{R}^n$, where for any selection of three coordinate directions, given seven vertices of an elementary cube satisfying the equation on the respective faces, there is a unique eighth such that the whole cube satisfies the equation.

To get the Moutard+ equation back into play, note that we can switch between signs in Moutard equations if we change the signs of f alternatingly in one of the involved coordinate directions: $(k,l) \mapsto f(k,l)$ satisfies the Moutard− equation iff $(k,l) \mapsto (-1)^l f(k,l)$ satisfies the Moutard+ equation. In particular, a 3D system with plus signs in two pairs of coordinate directions and a minus sign in the third will be consistent, and we can make

Definition 2.5.4. Let $f, g : \mathbb{Z}^2 \to \mathbb{R}^n$ satisfy the Moutard+ equation as in def. 2.3.4. f and g are called *Moutard transforms* of each other if

$$g_1 + f = \mu(g + f_1), \quad g_2 - f = \nu(g - f_2)$$

for appropriate values of μ and ν. (or the same with the combination of coordinate directions and plus and minus signs interchanged).

Since now coordinate and transformation directions are equivalent, we will often call $g =: f_3$ and the elementary cubes of a Moutard transformation *Moutard cubes.*

As with a plus sign, the Moutard− equation restricts to quadrics of constant length, and we can apply it to nets in the light cone.

Definition 2.5.5. Let $f, f_3 : \mathbb{Z}^2 \to \mathbb{R}^3$ be discrete isothermic and $F, F_3 : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$ their representations in the light cone. f and f_3 (or F and F_3) are called *Darboux transforms* of each other if F and F_3 are Moutard transforms of each other.

Quadrilaterals whose light cone representatives satisfy the Moutard± equation (or Moutard quads for short) are circular: the linear equations guarantee that F_{12} lies in the time–like subspace $\mathrm{span}(F, F_1, F_2)$, and its space–like two–dimensional orthogonal complement in $\mathbb{R}^{4,1}$ represents the common circle. Similarly, the whole Moutard cube lies in a four–dimensional time–like subspace and thus lies on a common sphere. This gives a discrete analogue of the classical geometric definition of Ribaucour and Darboux transforms of smooth surfaces as in def. 2.5.1. Figure 2.7 shows a sketch of the situation.

Now that we have smooth and discrete Darboux transforms, we can state the characterizations of cmc surfaces from [HP97] and [HHP96]:

Theorem 2.5.3. *An isothermal immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$ has constant mean curvature $H \neq 0$ iff its correctly scaled and placed Christoffel dual is a Darboux transform.*

Theorem 2.5.4. *An discrete isothermic net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ is a discrete cmc surface iff its correctly scaled and placed Christoffel dual is a Darboux transform.*

Remark 2.5.5. *In the discrete isothermic case, if we look at the original cmc condition of having a Christoffel dual in constant vertex distance from a Möbius geometric perspective, by eq. (2.3) it can be seen as the scalar product $\langle F, F^* \rangle$ of the representations in $\mathbb{R}^{4,1}$ of the vertices and their duals being constant.*

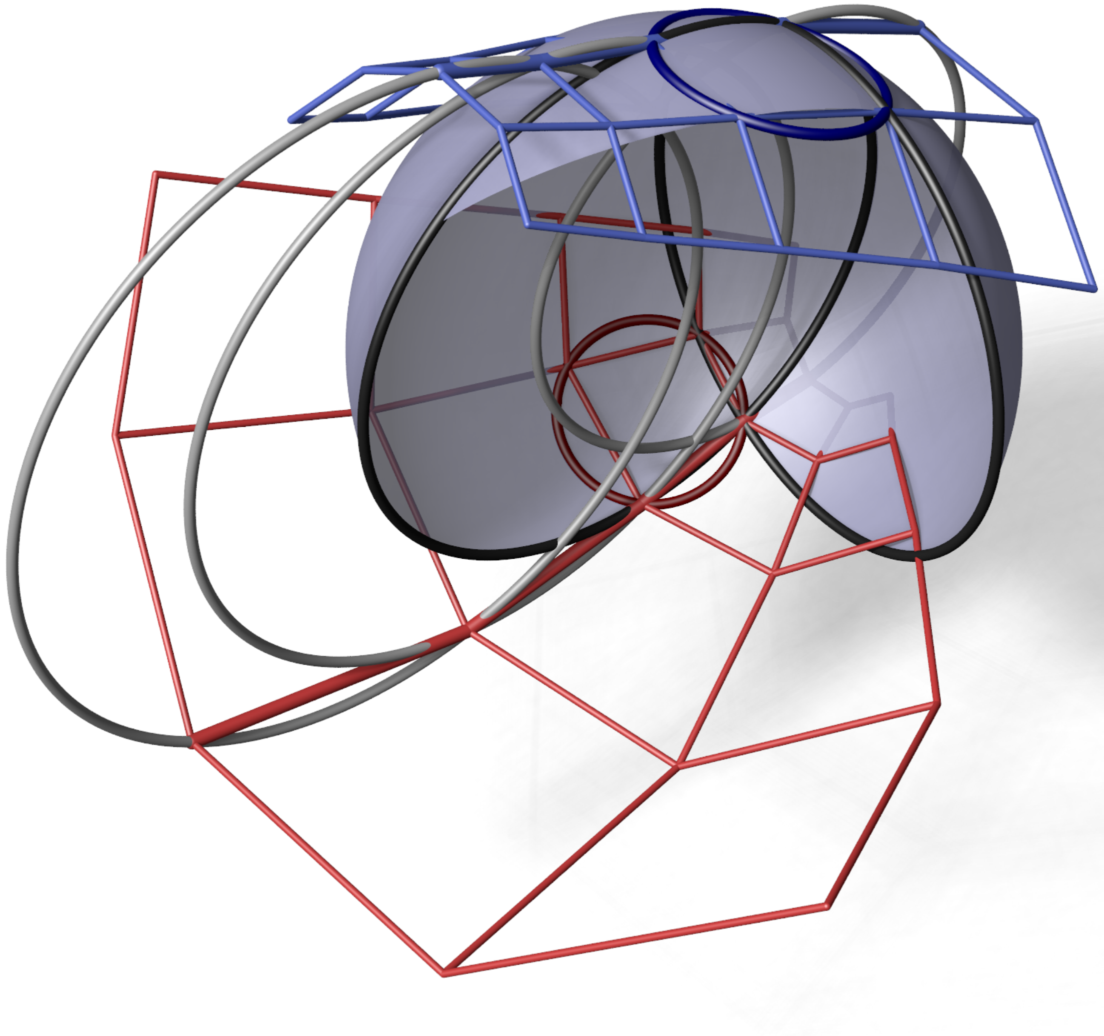

Figure 2.7: A Darboux pair of discrete isothermic surfaces. The family of circles enveloped by one pair of parameter lines is shown in grey. One sphere containing a pair of elementary quadrilaterals and their circles is partially shown.

2.6 Associated Families of Smooth and Discrete Minimal and CMC Surfaces

Minimal and cmc surfaces come in families parametrized by the circle S^1 sharing the same conformal metric, with the surfaces for 1 and -1 parametrized by curvature lines. Any structure preserving discretization of isothermally parametrized minimal and cmc surfaces should be part of such a family as well. Here we give a short introduction on how this associated family arises for smooth surfaces and their discrete isothermic counterparts, as well as hint at the existing s_1–isothermic and s–conical minimal cases. To geometrically construct s–isothermic analogues for cmc surfaces is the main topic of this work.

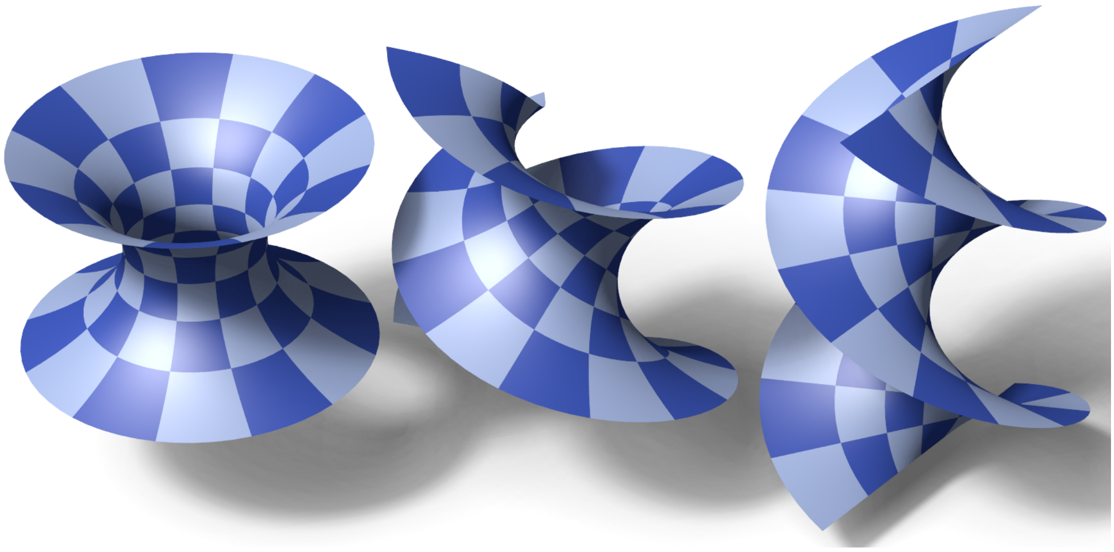

Figure 2.8: Members of the associated family of a smooth minimal surface: catenoid and helicoid (and an intermediate step). The the catenoid is curvature line parametrized, the helicoid asymptotically parametrized.

For smooth minimal surfaces, see e.g. [DHKW92] on which we base this short introduction. Without loss of generality, all smooth surfaces will be parametrized conformally. Then, for an immersion $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ the condition $H = 0$ is equivalent to

$$\Delta f = f_{xx} + f_{yy} = 0,$$

i.e. f is a minimal surface iff it is harmonic. This makes the Cauchy–Riemann equations with f as the real part integrable, i.e. locally there exists a function $g : U \to \mathbb{R}^3$ satisfying

$$g_x = -f_y, \quad g_y = f_x.$$

Thus, for U simply connected we can extend f to a holomorphic curve

$$F : \mathbb{C} \cong \mathbb{R}^2 \supset U \to \mathbb{C}^3, \quad F(x+iy) = f(x,y) + ig(x,y).$$

The imaginary part $g : U \to \mathbb{R}^3$ is a minimal surface again, called the *adjoint* or *conjugate* minimal surface for f. It has the same conformal metric as f, and geometrically, g arises by rotating the parameter directions by $\frac{\pi}{2}$ within the tangent spaces. In particular, f and g share the same Gauss map. Now, this rotation within tangent spaces can be done for arbitrary angles: for any angle $\vartheta \in \mathbb{R}$, the consider the holomorphic curve

$$e^{-i\vartheta} F : \mathbb{C} \to \mathbb{C}^3.$$

The maps

$$f^\vartheta := \operatorname{Re}\bigl(e^{-i\vartheta} F\bigr) = \cos\vartheta f + \sin\vartheta g : \mathbb{R}^2 \to \mathbb{R}^3$$

are again minimal surfaces with the same Gauss map and conformal metric. This one–parameter family of minimal surfaces is the *associated family* of f.

In the associated family of a minimal surface, if $\vartheta_2 - \vartheta_1 = \frac{\pi}{2}$, i.e. f^{ϑ_1} and f^{ϑ_2} are a conjugate pair, the curvature lines of f^{ϑ_1} are asymptotic lines of f^{ϑ_2} and vice versa. In particular, if we start with an isothermally parametrized minimal surface, its conjugate will be asymptotically parametrized.

The standard example of the catenoid and helicoid is illustrated in fig. 2.8.

The geometric process of rotating parameter directions around a fixed normal translates to the discrete cases as rotating edges around fixed edge normals which are constructed from the discrete Gauss map — the exact process depends on the type of discretization.

For discrete isothermic minimal surfaces $f : \mathbb{Z}^2 \to \mathbb{R}^3$, the Gauss map $n : \mathbb{Z}^2 \to S^2$ has vertices on the unit sphere, such that for adjacent vertices n, n_1 the average is normal to the edge: $n + n_1 \perp n_1 - n$, and since the Gauss map is Christoffel dual to the surface (as mentioned in section 2.5), $n + n_1$ is also a suitable edge normal for $f_1 - f$. This naturally holds for the second coordinate direction as well. Rotating all edges around these edge normals by a constant angle gives the associated family for discrete isothermic minimal surfaces. A concise overview can be found e.g. in [HSW16]. Similar processes for s_1–isothermic and s–conical minimal surfaces will be revisited in sections 4.2 and 4.3.

For surfaces of nonzero constant mean curvature, the description that was the basis for the discrete isothermic version ([BP99]) stems from [Bob94]. Based on those two we present this brief introduction. Again, we look at conformal immersions

$$f : \mathbb{C} \cong \mathbb{R}^2 \supset U \to \mathbb{R}^3 \subset \mathbb{C}^3.$$

Partial derivatives will be expressed in terms of the Wirtinger derivatives

$$\partial_z = \tfrac{1}{2}\bigl(\partial_x - i\partial_y\bigr), \quad \partial_{\bar z} = \tfrac{1}{2}\bigl(\partial_x + i\partial_y\bigr);$$

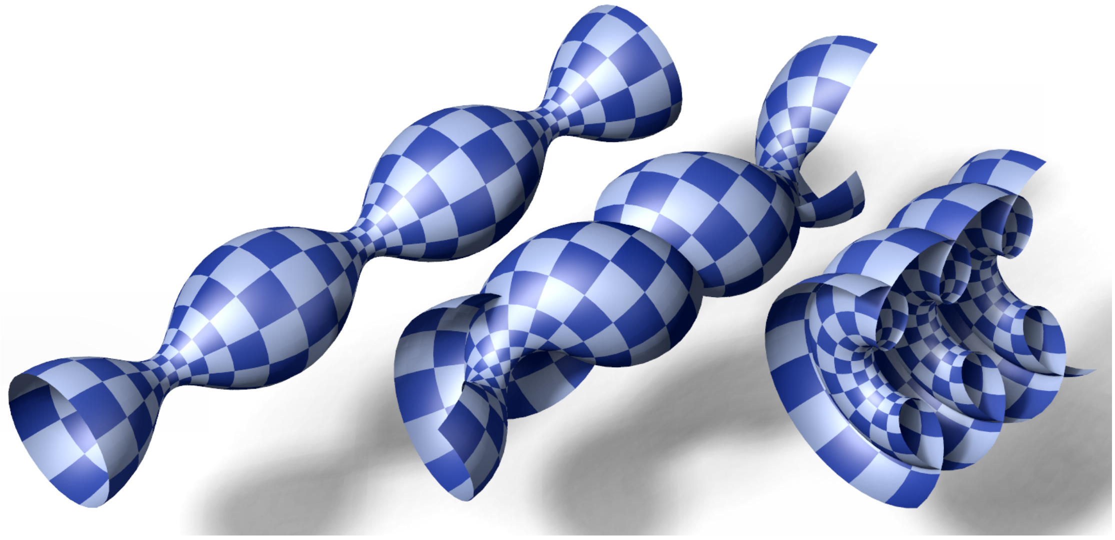

Figure 2.9: Sketch of members of the associated family of a smooth cmc surface: unduloid, 'twizzler' and nodoid. The former and latter are curvature line parametrized. Note that the picture was actually produced with our construction from section 4.4 although we do not have a convergence result. For pictures that are proven to approximate the smooth theory, see e.g. [Kil04], [Sch02], [BHHS].

and we let $\langle\cdot,\cdot\rangle : \mathbb{C}^3\times\mathbb{C}^3 \to \mathbb{C}$ denote the bilinear extension of the standard inner product on $\mathbb{R}^3$. The conformal first fundamental form is determined by a function $u : U \to \mathbb{R}$ with $\langle f_z, f_{\bar z}\rangle = \frac{1}{2}e^u$, and with n the Gauss map of f, the second fundamental form can be expressed in terms of

$$Q = \langle f_{zz}, n\rangle\,, \quad \langle f_{z\bar z}, n\rangle = \tfrac{1}{2}He^u,$$

where Q is the *Hopf differential* and H is the mean curvature. The Gauss–Codazzi equations, which are the compatibility conditions for the first and second fundamental form to define a surface, become

$$\begin{aligned} u_{z\bar z} + \tfrac{1}{2}H^2e^u - 2Q\bar{Q}e^{-u} &= 0,\\ Q_{\bar z} - \tfrac{1}{2}H_ze^u &= 0. \end{aligned}$$

Thus, for constant H, Q is holomorphic. If f does not have umbilic points (which are charaterized by $Q = 0$), by holomorphic reparametrization Q can be normalized to be constant. f is a curvature line parametrization (and thus isothermal) iff Q is real. Furthermore, the equations are invariant under multiplication of Q by a unit norm constant:

$$Q \rightsquigarrow Q^t = \lambda Q, \quad \text{where} \quad \lambda = e^{2it},\ t \in \mathbb{R}.$$

The one–parameter family of immersions with the same metric and mean curvature defined by the Q^t is the *associated family* of f. λ is called the *spectral parameter*.

Now we look at the *moving frame* for the immersion f — the transformation that maps the standard basis of $\mathbb{R}^3$ to (f_x, f_y, n) at each point of the parameter domain. Since f is conformal, this is a common rescaling of the first two vectors by $e^{\frac{u}{2}}$ and a rotation. The rotations will be described quaternionically, and the quaternions in turn by complex two–by–two matrices, i.e. we use the identifications

$$\mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}, \quad \mathbb{H} \cong \operatorname{span}_{\mathbb{R}}\{\mathrm{Id}, -i\sigma_1, -i\sigma_2, -i\sigma_3\} \subset \mathbb{C}^{2\times 2}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, the frame for f is a map $\Phi : U \to \mathbb{H}$ with

$$f_z = -ie^{\frac{u}{2}}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi,\ f_{\bar{z}} = -ie^{\frac{u}{2}}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi,\ n = \Phi^{-1}\Bbbk\Phi.$$

For f a cmc immersion as above (i.e. with conformal metric factor e^u, Hopf differential λQ and mean curvature H), the frame Φ is characterized by the differential equations

$$\Phi_z = U\Phi, \quad \Phi_{\bar{z}} = V\Phi$$

with

$$U = \begin{pmatrix} \frac{u_z}{2} & -\lambda Q e^{-\frac{u}{2}} \\ \frac{1}{2}He^{\frac{u}{2}} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -\frac{1}{2}He^{\frac{u}{2}} \\ \overline{\lambda Q}e^{-\frac{u}{2}} & \frac{u_{\bar{z}}}{2} \end{pmatrix}$$

and normalization $\det\Phi = e^{\frac{u}{2}}$. Once a frame solving these is known for all $\lambda = e^{2it}$, cmc immersions can be retrieved by a differentiation with respect to the spectral parameter instead of integrating the coordinate directions via the Sym–Bobenko formula:

$$f = -\tfrac{1}{H}\left(\Phi^{-1}\partial_t\Phi + n\right)$$

with Gauss map $n = \Phi^{-1}\Bbbk\Phi$ is a cmc immersion with metric factor e^u, mean curvature H and Hopf differential λQ.

In the whole associated family, the offset surface $f + \frac{1}{H}n$ also has constant mean curvature H — for the isothermal case $\lambda Q \in \mathbb{R}$ this is the Christoffel dual as mentioned in theorem 2.5.1.

A completely analogous description for discrete isothermic cmc surfaces was given in [BP99] and further studied in [HSW16]. We base our introduction mainly on the latter. Again, consider a moving frame rotating $\Bbbk$ into the normal, but now living on vertices:

$$\Phi : \mathbb{Z}^2 \to \mathbb{H}.$$

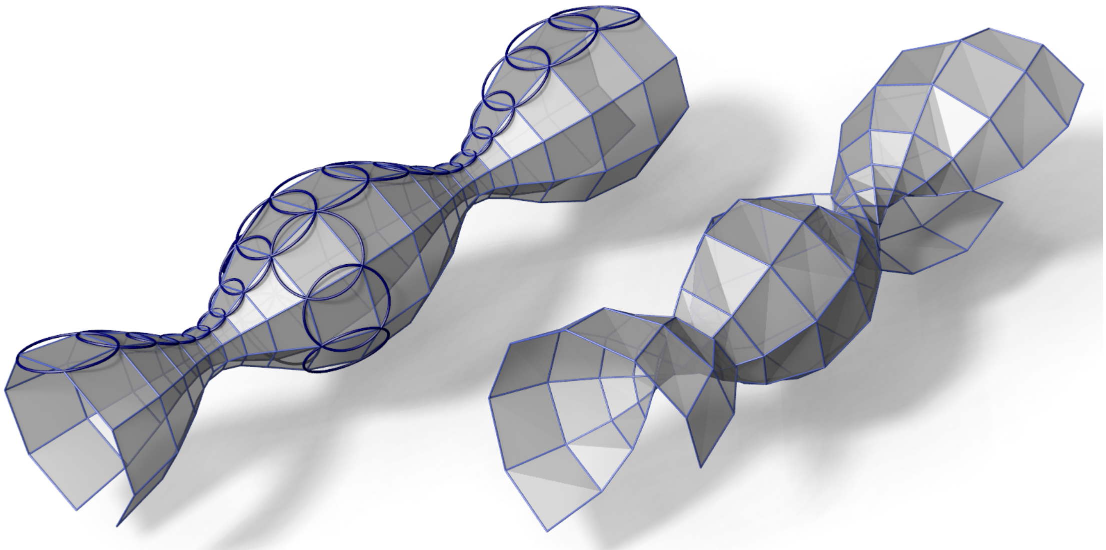

Figure 2.10: Sketch of a discrete isothermic unduloid and an intermediate member of its associated family. Note that the picture was actually produced with our construction from section 4.5 for very small spheres and therefore should only be considered a sketch.

Instead of partial derivatives, U and V now characterize coordinate shifts:

$$\Phi_1 = U\Phi, \quad \Phi_2 = V\Phi$$

with

$$U = \begin{pmatrix} a & -\lambda u - \frac{1}{\lambda u} \\ \frac{\lambda}{u} + \frac{u}{\lambda} & \bar{a} \end{pmatrix}, \quad V = \begin{pmatrix} b & -i\lambda + \frac{i}{\lambda v} \\ i\frac{\lambda}{v} - i\frac{v}{\lambda} & \bar{b} \end{pmatrix},$$

where the spectral parameter is now $\lambda = e^{i\alpha}$ for $\alpha \in \mathbb{R}$, a, b are complex–valued and u, v positive real–valued functions on vertices. U and V must satisfy the compatibility conditions

$$V_1 U = U_2 V, \quad \det U = \det U_2, \quad \det V = \det V_2.$$

Then the discrete Sym–Bobenko formula generates the surfaces $f : \mathbb{Z}^2 \to \operatorname{Im}\mathbb{H} = \mathbb{R}^3$ as

$$f = \operatorname{Im}\big(-\Phi^{-1}\partial_\alpha \Phi + \frac{1}{2}n\big),$$

where $n = \Phi^{-1}\Bbbk\Phi$ is the discrete Gauss map. The family parametrized by α is the *associated family* of a discrete isothermic cmc net. Here, $\lambda^2 \in \mathbb{R}$ characterizes the discrete isothermic case. Again, $f^* = f + n$ is also a discrete cmc net which is the Christoffel dual in the isothermic case.

For the geometry of the members of the associated family of a discrete isothermic cmc net, we look at the elementary cubes

$$(f, f_1, f_2, f_{12}, f^*, f_1^*, f_2^*, f_{12}^*)$$

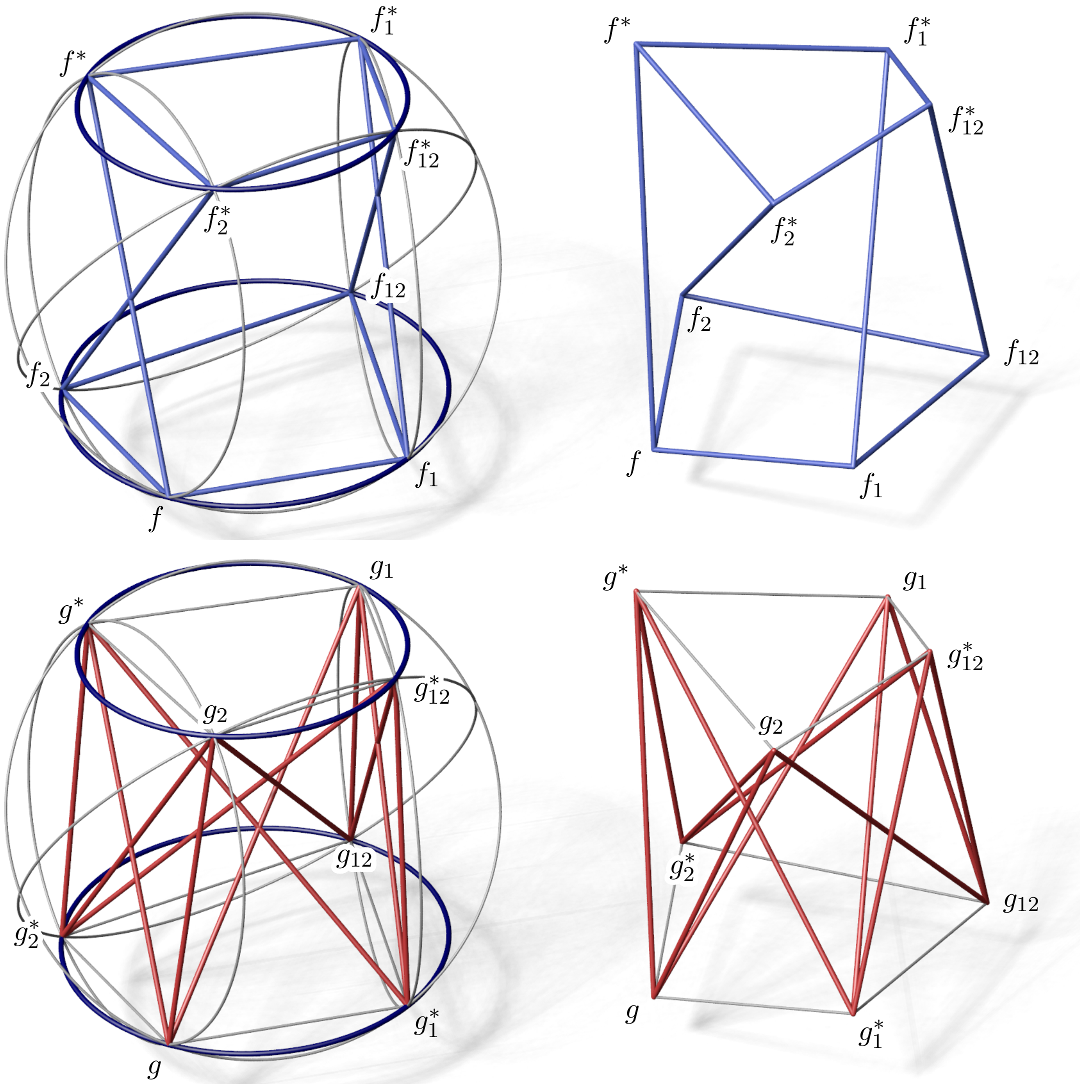

Figure 2.11: Sketch of a discrete isothermic cmc cube and a twisted version from the associated family. Original combinatorics at the top, interleaved version at the bottom. Note that the picture was actually produced with our construction from section 4.5 for very small spheres and therefore should only be considered a sketch.

formed by corresponding pairs of elementary quadrilaterals of f and its dual f^*. In the isothermic case, its faces are planar, and the side faces are trapezoids with vertices on a sphere. As such, they have diagonals of equal length, and by changing the combinatorics, the trapezoids become folded parallelograms — non–embedded planar quadrilaterals with two mutually opposite pairs of edges of equal length and parallel diagonals. This is done for the whole cube by defining the *interleaved* cube

$$g = f,\ g_1 = f_1^*,\ g_2 = f_2^*,\ g_{12} = f_{12},$$
$$g^* = f^*,\ g_1^* = f_1,\ g_2^* = f_2,\ g_{12}^* = f_{12}^*.$$

For an illustration, see fig. 2.11. Since the quadrilaterals of f and f^* are Christoffel dual in the planar case, they have parallel non–corresponding diagonals, and also the faces (g, g_1, g_2, g_{12}) and $(g^*, g_1^*, g_2^*, g_{12}^*)$ become planar folded parallelograms.

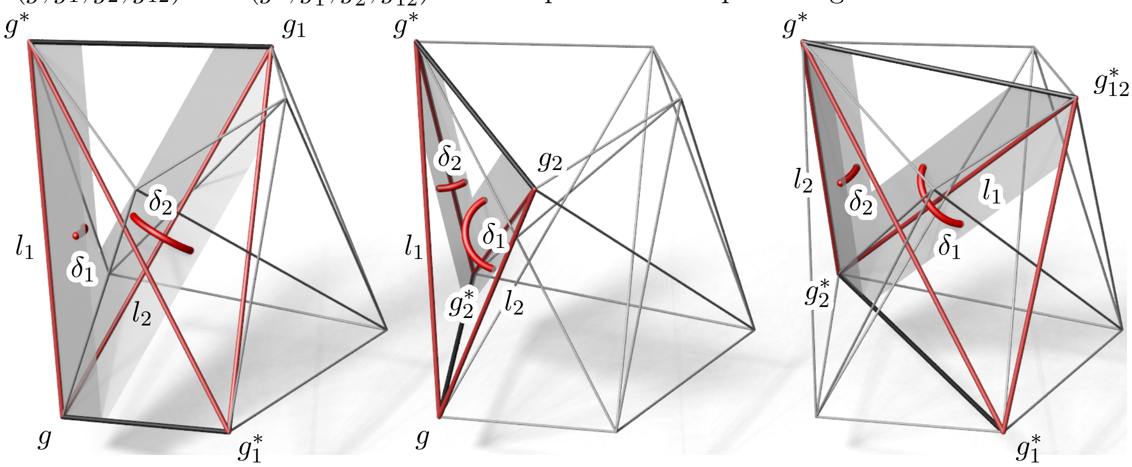

Figure 2.12: Dihedral angles in the side faces of the interleaved cube.

In the associated family, the constant vertex distance $\|f - f^*\| = \|g - g^*\| = \frac{1}{H}$ remains constant, and the other edges in each face of the interleaved cube still come in opposite pairs of equal length as well — the faces are still *skew parallelograms*. As such, they also have equal dihedral angles along the pairs of edges of equal length, say δ_1 and δ_2 along the edges of length l_1 and l_2. These skew parallelograms are characterized by a *folding parameter*

$$\sigma = \frac{\sin \delta_1}{l_1} = \frac{\sin \delta_2}{l_2}.$$

See also the sketch in fig. 2.12. In the interleaved cube of the associated family, this parameter is the same for all faces — they are *equally–folded parallelogram cubes*.

This property is 3D compatible in the sense that if a skew parallelogram (g, g_1, g_2, g_{12}) and an initial vertex g^* is given, they are part of a unique equally–folded parallelogram cube $(g, g_1, g_2, g_{12}, g^*, g_1^*, g_2^*, g_{12}^*)$ with equal vertex distances

$$\|g - g^*\| = \|g_1 - g_1^*\| = \|g_2 - g_2^*\| = \|g_{12} - g_{12}^*\|.$$

Any such cube will be discrete cmc in the curvature theory of [HSW16]. If $\lambda = e^{i\alpha}$ is the spectral parameter as above and f and $f^* = f + n$ are the pair of discrete cmc surfaces, i.e. H is normalized to be 1, the folding parameter σ of the cubes is related to the spectral parameter as

$$\sigma = \sin(2\alpha).$$

An adaptation of the equally–folded cube property to s–isothermic cmc cubes will be the characterizing property we introduce in section 4.1.

Chapter 3

S–Isothermic Surfaces

3.1 Introduction

S–isothermic surfaces first appeared in [BP99] as the special type we will here call s_1–isothermic: Quad nets with spheres centered at the vertices that touch along edges and have a common orthogonal circle through those points of tangency for each elementary quadrilateral. There, they were called *Schramm isothermic surfaces*, hinting at their relation to orthogonal circle patterns, studied by O. Schramm ([Sch97]). Minimal surfaces of this type — whose associated family we will revisit in section 4.2 — have been studied in [BHS06].

In [Hof10], the definition was extended to a more general class of discretizations of isothermally parametrized surfaces. They also contain a — somewhat degenerate — case in which the resulting surfaces are conical ([BHKS15], [BH16]). Conical nets are a discretization of curvature line parametrized surfaces interesting in their own right (see e.g. [LPW+06], where they were introduced).

The general notion of s–isothermic nets is in complete analogy to discrete isothermic nets: the characterizations by Moutard equations are simply taken from the light cone to the space–like unit hyperboloid in $\mathbb{R}^{4,1}$ and taken as definitions. In our presentation, we mainly follow [Hof10] and [BH16].

Definition 3.1.1. A map $s : \mathbb{Z}^2 \to \{\text{spheres in } \mathbb{R}^3\}$ is called *s–isothermic* if its representative $s : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$ in the classical model satisfies the discrete Moutard+ equation

$$s_{12} + s = \lambda(s_1 + s_2)$$

for some $\lambda \neq 0$ per elementary quadrilateral.

As noted in section 2.3 for discrete isothermic nets, this relies on the fact that the discrete Moutard equation restricts to quadrics of constant length — in this case, to space–like unit vectors $\langle s, s\rangle = 1$.

For well–behaved geometry, we want the radii of all spheres to be positive. For s–isothermic nets $s : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$, we will consider the net $c : \mathbb{Z}^2 \to \mathbb{R}^3$ of sphere centers as vertices, and (mostly) denote the radii by $r : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$.

Before we get some insight into the geometry of s–isothermic quadrilaterals, we note that we will simultaneously consider the discrete Moutard− equation

$$s_{12} - s = \lambda(s_2 - s_1)$$

for quadrilaterals of spheres represented in $\mathbb{R}^{4,1}$. We will often, for short, call quadrilaterals of spheres satisfying either or any of the Moutard equations *Moutard+*, *Moutard−* or *Moutard±* quads, respectively.

Proposition 3.1.1. *Let $s, s_1, s_{12}, s_2 \in \mathbb{R}^{4,1}$ be a quadrilateral of spheres satisfying a Moutard± equation. Then*

$$\langle s, s_1 \rangle = \langle s_2, s_{12} \rangle \quad \textit{and} \quad \langle s, s_2 \rangle = \langle s_1, s_{12} \rangle ,$$

i.e. scalar products along edges are equal on opposite edges.

The proof is a short calculation and can be found in [BH16] for Moutard+ quadrilaterals, it equally works for Moutard− with adjusted signs. For s–isothermic nets, this means that scalar products along edges in one coordinate direction are constant in the other. This is reminiscent of the independent reparametrizations that separate general smooth isothermal immersions from conformality. Indeed, the sphere radii of s–isothermic nets can be seen as a discrete version of the conformal metric factor s, and the scalar products as the effect of the independent reparametrization factors α and β in remark 2.3.1.

As in the discrete isothermic case, the linear Moutard± equations mean that the spheres in a quadrilateral are linearly dependent. But now, the causal type of the three–dimensional subspace S they span is not determined, and we discern three types of Moutard quadrilaterals of spheres:

Type 1: The four spheres share a common orthogonal circle if S is time–like. Then $S^\perp$ is space–like and represents a circle (as the grey and blue spheres in fig. 2.2).

Type 2: The four spheres share a pair of points if S is space–like. Then $S^\perp$ is time–like and contains two isotropic directions representing the two points (as the grey and blue spheres in fig. 2.4, where the red spheres would be in S).

Type 3: The four spheres intersect in exactly one point if S is light–like. Then $S^\perp$ is also light–like, and S and $S^\perp$ share a unique isotropic direction that represents the common point (as the red and grey/blue families of spheres in fig. 2.3).

As already mentioned, we give the special case that was first introduced a short name:

Definition 3.1.2. A Moutard+ quad of spheres s, s_1, s_{12}, s_2 (with positive radii) is called an s_1*–isothermic* quad if

$$\langle s, s_1\rangle = \langle s, s_2\rangle = -1,$$

i.e. the spheres are in exterior tangency along edges. An s–isothermic map $\mathbb{Z}^2 \to \mathbb{R}^{4,1}$ is called s_1*–isothermic* if all elementary quadrilaterals are s_1–isothermic.

In all cases, the four spheres are orthogonal to the plane containing the vertices. Figure 3.1 illustrates some examples.

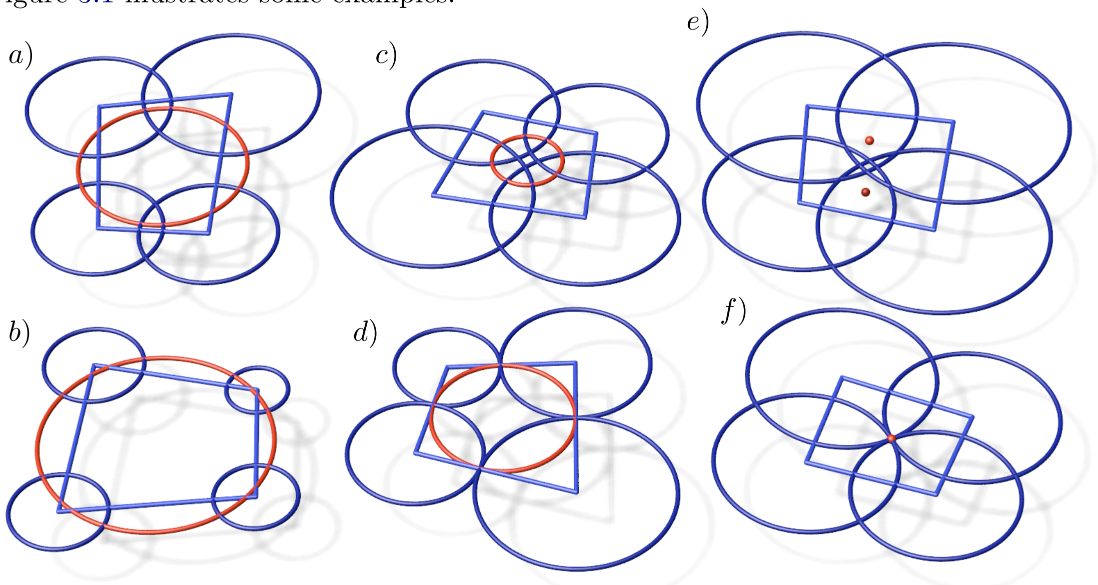

Figure 3.1: Moutard+ quads of different types. Spheres are shown as their intersection with the face plane. Orthogonal circles or common points in red. a) to d) are type 1, with d) the s_1–isothermic case. e) is type 2, and f) is type 3, the s–conical case.

For type 1 and 2, the converse of proposition 3.1.1 also holds ([BS08, Thm. 4.35]):

Theorem 3.1.2. *Let $s, s_1, s_{12}, s_2 \in \mathbb{R}^{4,1}$ be spheres such that $S = \operatorname{span}(s, s_1, s_{12}, s_2)$ is three–dimensional and the Minkowski scalar product restricted to S is non–degenerate. If*

$$\langle s, s_1\rangle = \langle s_2, s_{12}\rangle \quad \textit{and} \quad \langle s, s_2\rangle = \langle s_1, s_{12}\rangle\,,$$

the spheres form a Moutard± quad.

[BH16] provides characterizations of Moutard+ quads of type 3:

Theorem 3.1.3. *Let $s, s_1, s_{12}, s_2 \in \mathbb{R}^{4,1}$ be spheres. Then the following are equivalent:*

- *The spheres form a Moutard+ quad of type 3*
- *The spheres form a Moutard+ quad, and spheres intersect along edges with complementary angles, i.e. $\langle s, s_1\rangle = -\langle s, s_2\rangle$ and $|\langle s, s_1\rangle| \leq 1$*
- *$s_1 + s_2$ and $s + s_{12}$ are parallel isotropic vectors*
- *The diagonals of the quadrilateral formed by the sphere centers intersect in a point which lies on all four spheres.*

Slightly deviating from [BH16], in our context we will define s–conical just as the respective special case of s–isothermic:

Definition 3.1.3. A Moutard+ quad of spheres s, s_1, s_{12}, s_2 (with positive radii) of type 3 is also called an *s–conical* quad. An s–isothermic map $\mathbb{Z}^2 \to \mathbb{R}^{4,1}$ is called *s–conical* if all elementary quadrilaterals are s–conical.

S–isothermic nets again come with Christoffel duals:

Definition 3.1.4. Let $s : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$ be an s–isothermic net with sphere centers c and radii r. Then the net s^* with centers c^* and radii r^* defined by

$$r^* = \frac{1}{r},$$
$$c_1^* - c^* = \frac{c_1 - c}{rr_1},$$
$$c_2^* - c^* = -\frac{c_2 - c}{rr_2}$$

is called the *Christoffel dual*.

Note that the choice in which coordinate direction the edge orientation gets reversed is a convention; since we want to keep the radii positive, changing it is not simply a special case of a global rescaling but should be considered separately.

The Christoffel dual is an s–isothermic net of the same type, with $\langle s^*, s_1^*\rangle = \langle s, s_1\rangle$ and $\langle s^*, s_2^*\rangle = \langle s, s_2\rangle$. The elementary quadrilaterals formed by the sphere centers c, c^* in $\mathbb{R}^3$ are dual in the mixed area sense as in def. 2.4.4.

Remark 3.1.4. *If we let the sphere radii in an s–isothermic net get uniformly smaller, it is directly geometrically plausible that in the limit we get points on a common circle. Indeed, this limit can be made thorough if one considers spheres represented by space–like vectors normalized to general lengths, and one recovers discrete isothermic nets in the limit of vanishing radii. Cf. [BS08].*

3.2 Darboux Transforms and CMC Surfaces

Now, with Moutard± equations for quadrilaterals of spheres introduced, we can directly state the analogue of def. 2.5.4/def. 2.5.5 for s–isothermic surfaces:

Definition 3.2.1. Let $s, t : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$ be s–isothermic surfaces. s and t are called *Darboux transforms* of each other if the side faces of the elementary cubes satisfy Moutard equations

$$t_1 + s = \mu(t + s_1), \quad t_2 - s = \nu(t - s_2)$$

for some $\mu, \nu \neq 0$. (or the same with the combination of coordinate directions and plus and minus signs interchanged).

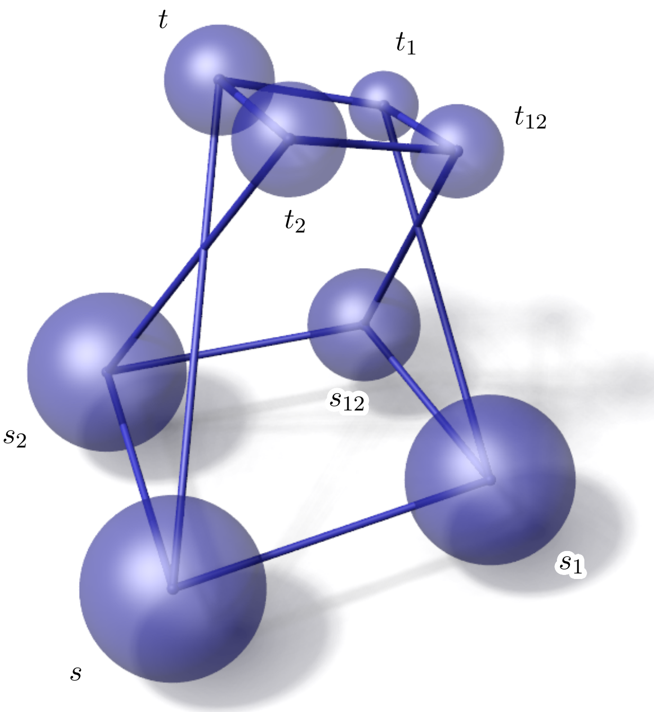

Figure 3.2: Elementary cube of an s–isothermic Darboux transformation. Moutard− side faces in the second parameter direction.

Remark 3.2.1. *While for a Moutard± quad of type 1 or 2, as in eq. (2.5) we get a uniquely determined*

$$\lambda = \frac{\langle s, s_1 \pm s_2 \rangle}{1 \pm \langle s_1, s_2 \rangle}$$

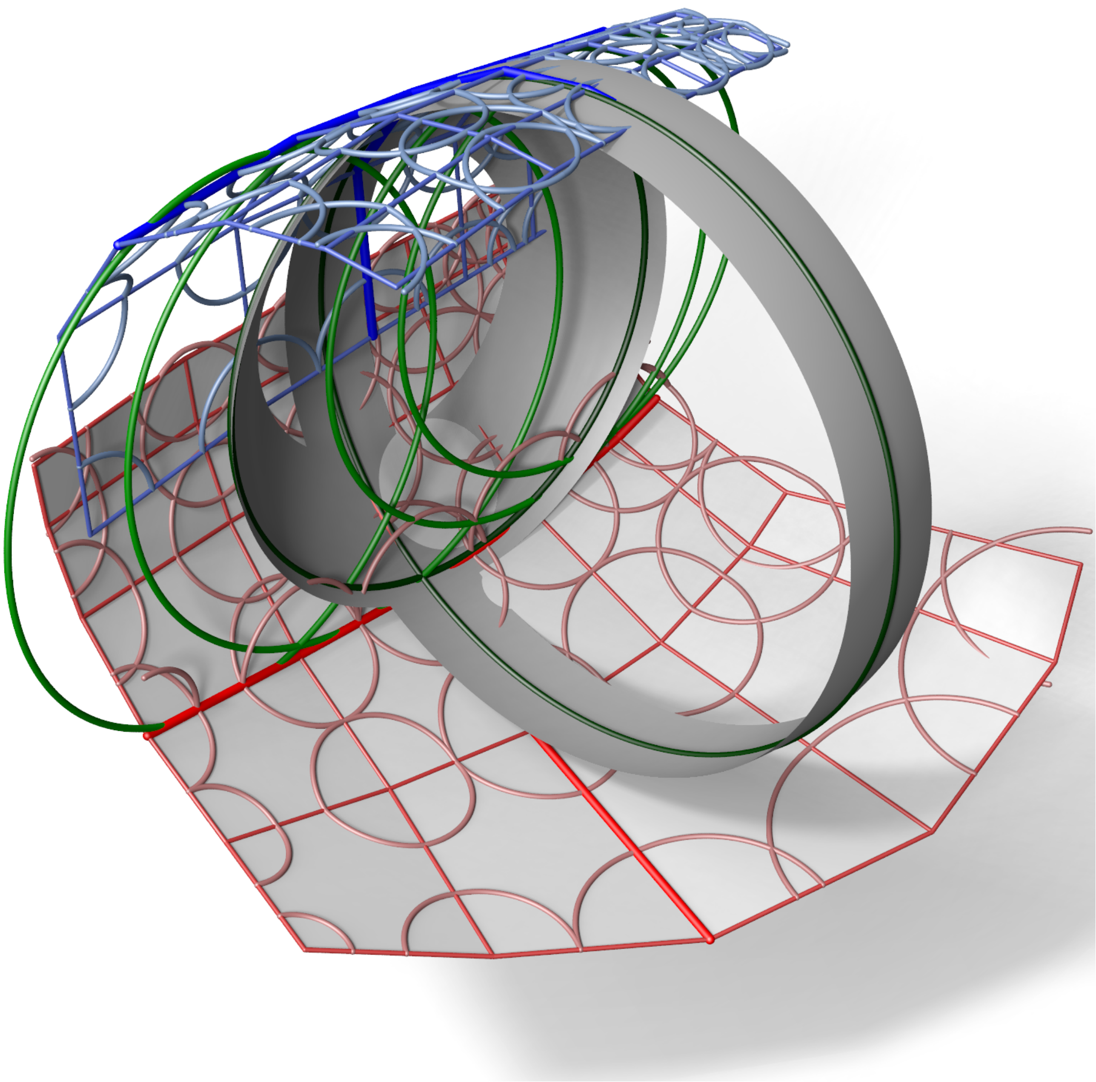

Figure 3.3: A Darboux pair of s–isothermic surfaces. The family of circles enveloped by one pair of parameter lines is shown in green. One sphere containing a pair of elementary quadrilaterals and their circles is partially shown. Spheres are shown as their intersections with the face planes, or along one parameter line also as their intersection with the vertical circle planes.

and therefore a unique s_{12}, for type 3 there is a choice for s_{12}: if s_1 and s_2 are tangent in a point on s, any sphere tangent to s in that point will be a viable s_{12} (cf. theorem 3.1.3). For that reason, we will only consider type 1 and 2 for the side faces of Darboux transformation cubes.

The Darboux transform of an s–isothermic net s is then uniquely defined by choice of one initial sphere of t (that respects remark 3.2.1).

We will call a cube of eight spheres that satisfy Moutard+ equations on two pairs of opposite side faces and Moutard− equations on the other pair a Moutard cube for short. Often, we will treat a Darboux transformation as a third coordinate direction and write s_3 instead of t.

Now, in analogy to theorems 2.5.3 and 2.5.4, we can make

Definition 3.2.2 ([Hof10])**.** An s–isothermic surface is a *cmc surface* if its Christoffel dual is a Darboux transform as well (after appropriate scaling and translation).

We will call the Moutard cubes formed by an elementary quadrilateral of an s–isothermic cmc surface and its corresponding dual *s–isothermic cmc cubes*. We want to learn more about their geometry, in particular how to construct them. For that end, we start by characterizing their side faces. They come in a Moutard+ and a Moutard− variety with analogous properties — in the following, we will treat them simultaneously by writing $\pm$ and $\mp$ signs with the understanding that the upper sign always refers to the Moutard+ case, and the lower one to the Moutard− case. Clearly, the side faces have a pair of parallel edges, and we give a name to this property:

Definition 3.2.3. We call a Moutard($\pm$) quadrilateral with sphere centers c, c_1, c_2, c_{12} a *Moutard($\pm$) trapezoid* if $c_1 - c \parallel c_{12} - c_2$.

Lemma 3.2.2. *Let $s, s_1, s_2, s_{12} \in \mathbb{R}^{4,1}$ be spheres with centers $c, c_1, c_2, c_{12} \in \mathbb{R}^3$ and radii $r, r_1, r_2, r_{12} > 0$ that satisfy a Moutard$\pm$ equation*

$$s_{12} = \lambda(s_2 \pm s_1) \mp s$$

such that $\langle s_1, s_2 \rangle \neq \mp 1$. We introduce the quantities (cf. fig. 3.4)

$$\alpha := r r_2, \quad e := \|c_1 - c\|, \ f := \|c_2 - c\|, \ d_{12} := \|c_1 - c_2\|, \quad \sigma := \measuredangle(c_1 - c, c_2 - c).$$

We assume non–degeneracy in the form of $\sin\sigma \neq 0$ and define a sphere $\widetilde{s}_{12}$ to have radius and center

$$\widetilde{r}_{12} := \frac{\alpha}{r_1}, \quad \widetilde{c}_{12} := c_2 \pm \alpha \frac{c_1 - c}{r r_1}.$$

Then the following are equivalent:

i) s, s_1, s_2, s_{12} form a Moutard trapezoid

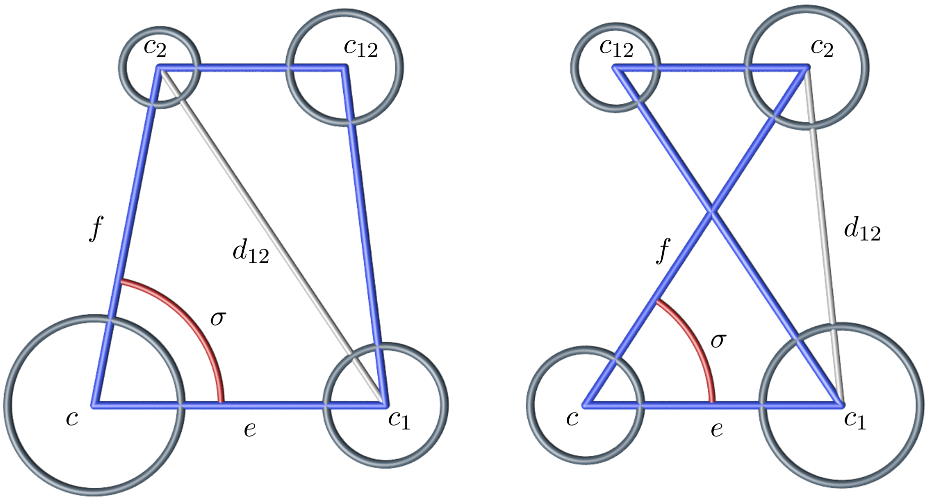

Figure 3.4: Moutard+ and Moutard− trapezoid illustrating the quantities in lemma 3.2.2.

ii) $s, s_1, s_2, \widetilde{s}_{12}$ *form a Moutard trapezoid*

iii) $s_{12} = \widetilde{s}_{12}$

iv) $c_{12} = \widetilde{c}_{12}$

v) $\lambda = \frac{r_1}{r}$

vi) $\lambda = \frac{r_2}{r_{12}}$

vii) $r_1^3 \pm r_2 r_1^2 + (f^2 - d_{12}^2 - r^2) r_1 \pm r_2 (e^2 - r^2) = 0$

viii) $2ef \cos\sigma = \big(1 \pm \frac{r_2}{r_1}\big)(r^2 - r_1^2) + \big(1 \mp \frac{r_2}{r_1}\big) e^2.$

Any of these imply

ix) $r_1 r_{12} = \alpha,$

x) $\langle s, s_2 \rangle = \langle s_1, \widetilde{s}_{12} \rangle \, .$

If $r_1 \neq r_2$*, each of the latter properties ix), x) is sufficient. In the Moutard+ case, ix) is sufficient. In the Moutard− case, x) is sufficient.*

Proof. All properties are translation invariant, so we assume $c = 0$. We recall that under our assumption, λ is uniquely determined to be

$$\lambda = \frac{\langle s, s_1 \pm s_2 \rangle}{1 \pm \langle s_1, s_2 \rangle}.$$

a) iii) $\Rightarrow$ i), ii), iv), ix) and iv) $\Rightarrow$ i) is immediate by definition, and ii) $\Rightarrow$ iii) follows from uniqueness of λ and thus the fourth sphere in the Moutard quad under our assumption. ii) $\Rightarrow$ x) is proposition 3.1.1.

b) Recall from section 2.2 that for any sphere $X = (X_0, \ldots, X_4) \in \mathbb{R}^{4,1}$ with radius R we have $\frac{1}{R} = X_0 + X_4$ and know from the Moutard equation:

$$\begin{aligned} \frac{1}{r_{12}} \pm \frac{1}{r} &= \lambda \left(\frac{1}{r_2} \pm \frac{1}{r_1} \right) \\ \Leftrightarrow \quad rr_1r_2 \pm r_1r_2r_{12} &= \lambda \left(rr_1r_{12} \pm rr_2r_{12} \right) \\ \Leftrightarrow \quad rr_1 \left(r_2 - \lambda r_{12} \right) &= \pm r_2r_{12} \left(\lambda r - r_1 \right). \end{aligned}$$

Therefore $\lambda r_{12} = r_2$ iff $\lambda r = r_1$, and we have v) $\Leftrightarrow$ vi).

If $r_1 r_{12} = \alpha$, the above equation becomes

$$\begin{aligned} r_1\alpha - \lambda r\alpha &= \pm \left(\lambda r_{12}\alpha - r_2\alpha \right) \\ \Leftrightarrow \quad rr_1 \left(r_1 - \lambda r \right) &= \pm (\lambda r\alpha - r_1\alpha) \\ \Leftrightarrow \quad \left(r_1 - \lambda r \right) \left(rr_1 \pm \alpha \right) &= 0 \\ \Leftrightarrow \quad \left(r_1 - \lambda r \right) \left(r_1 \pm r_2 \right) &= 0 \end{aligned}$$

In the Moutard+ case, since we assumed positive radii, we must have $\lambda r = r_1$. Equally, if $r_1 \neq r_2$ in the Moutard$-$ case. So under these conditions, ix) $\Rightarrow$ v).

c) We also recall from section 2.2 that for any sphere $X = (X_0, X_1, X_2, X_3, X_4) \in \mathbb{R}^{4,1}$ with radius R and center $(C_1, C_2, C_3) \in \mathbb{R}^3$ we have $(X_1, X_2, X_3) = \frac{1}{R}(C_1, C_2, C_3)$. The Moutard equation gives us, recalling we assumed $c = 0$,

$$\begin{aligned} c_{12} - c_2 &= r_{12} \left(\lambda \left(\frac{1}{r_2} c_2 \pm \frac{1}{r_1} c_1 \right) \mp \frac{1}{r} c \right) - c_2 \\ &= \left(\lambda \frac{r_{12}}{r_2} - 1 \right) c_2 \pm \lambda \frac{r_{12}}{r_1} c_1. \end{aligned}$$

So, since by assumption on σ $c_2 \nparallel c_1$, the edge $c_{12} - c_2$ is parallel to the edge c_1 iff $\lambda r_{12} = r_2$, and we have i) $\Leftrightarrow$ vi). In this case, we also see

$$c_{12} - c_2 = \pm \frac{r_2}{r_1} c_1 = \pm \frac{\alpha}{rr_1} c_1 = \pm \frac{\alpha}{rr_1} (c_1 - c) = \widetilde{c}_{12} - c_2,$$

and we have vi) $\Rightarrow$ iv).

d) If v) and vi) both hold, we get $\frac{r_1}{r} = \frac{r_2}{r_{12}}$ and thus ix). But we have already seen in b) that v) $\Leftrightarrow$ vi), so in particular we get e.g. vi) $\Rightarrow$ ix).

Now suppose vi) holds. From c) we already know that iv) follows, and together with ix) from above, which means $r_{12} = \widetilde{r}_{12}$, we get $s_{12} = \widetilde{s}_{12}$. Altogether, we see vi) $\Rightarrow$ iii).

e) For vii) $\Leftrightarrow$ viii) we just substitute $d_{12}^2 = e^2 + f^2 - 2ef\cos\sigma$:

$$
\begin{aligned}
& r_1^3 \pm r_2 r_1^2 + (f^2 - d_{12}^2 - r^2) r_1 \pm r_2(e^2 - r^2) = 0 \\
\Leftrightarrow \quad & r_1^3 \pm r_2 r_1^2 + (-e^2 + 2ef\cos\sigma - r^2) r_1 \pm r_2(e^2 - r^2) = 0 \\
\Leftrightarrow \quad & r_1^3 \pm r_2 r_1^2 + (-e^2 + 2ef\cos\sigma - r^2) r_1 \pm r_2(e^2 - r^2) = 0 \\
\Leftrightarrow \quad & r_1^2 \pm r_2 r_1 - e^2 - r^2 \pm \tfrac{r_2}{r_1}(e^2 - r^2) = -2ef\cos\sigma \\
\Leftrightarrow \quad & r^2\left(1 \pm \tfrac{r_2}{r_1}\right) - r_1^2\left(1 \pm \tfrac{r_2}{r_1}\right) + e^2\left(1 \mp \tfrac{r_2}{r_1}\right) = 2ef\cos\sigma
\end{aligned}
$$

f) For v) $\Leftrightarrow$ vii) we calculate

$$
\begin{aligned}
& r_1 = \lambda r \\
\Leftrightarrow \quad & r_1\left(1 \pm \langle s_1, s_2\rangle\right) = r\left(\langle s, s_1\rangle \pm \langle s, s_2\rangle\right) \\
\Leftrightarrow \quad & r_1\left(1 \pm \frac{r_1^2 + r_2^2 - d_{12}^2}{2r_1 r_2}\right) = r\left(\frac{r^2 + r_1^2 - e^2}{2rr_1} \pm \frac{r^2 + r_2^2 - f^2}{2rr_2}\right) \\
\Leftrightarrow \quad & 2r_1^2 r_2 \pm (r_1^3 + r_1 r_2^2 - r_1 d_{12}^2) = r^2 r_2 + r_1^2 r_2 - r_2 e^2 \pm (r^2 r_1 + r_1 r_2^2 - r_1 f^2) \\
\Leftrightarrow \quad & 0 = \pm r_1^3 + r_2 r_1^2 \pm (-d_{12}^2 - r^2 + f^2) r_1 - r^2 r_2 + r_2 e^2.
\end{aligned}
$$

g) For x) we introduce $\widetilde{f}_1 := \|\widetilde{c}_{12} - c_1\|$. We have

$$
\begin{aligned}
\widetilde{f}_1^2 &= f^2\sin^2\sigma + \left(f\cos\sigma \pm \frac{\alpha}{rr_1}e - e\right)^2 \\
&= f^2\sin^2\sigma + \left(f\cos\sigma \pm (\tfrac{r_2}{r_1} \mp 1)e\right)^2 \\
&= f^2 + (\tfrac{r_2}{r_1} \mp 1)^2 e^2 \pm 2ef\cos\sigma(\tfrac{r_2}{r_1} \mp 1).
\end{aligned}
$$

Now

$$
\begin{aligned}
& \langle s, s_2\rangle = \langle s_1, \widetilde{s}_{12}\rangle \\
\Leftrightarrow \quad & \frac{r^2 + r_2^2 - f^2}{2\alpha} = \frac{r_1^2 + \widetilde{r}_{12}^2 - \widetilde{f}_1^2}{2\alpha} \\
\Leftrightarrow \quad & r^2 + r_2^2 - r_1^2 - \frac{\alpha^2}{r_1^2} = f^2 - \widetilde{f}_1^2 \\
\Leftrightarrow \quad & r^2\left(1 - \frac{r_2^2}{r_1^2}\right) - r_1^2\left(1 - \frac{r_2^2}{r_1^2}\right) = -\left(\frac{r_2}{r_1} \mp 1\right)^2 e^2 \mp 2ef\cos\sigma\left(\frac{r_2}{r_1} \mp 1\right) \\
\Leftrightarrow \quad & 2ef\cos\sigma\left(\frac{r_2}{r_1} \mp 1\right) = \mp r^2\left(1 - \frac{r_2^2}{r_1^2}\right) \pm r_1^2\left(1 - \frac{r_2^2}{r_1^2}\right) \mp \left(\frac{r_2}{r_1} \mp 1\right)^2 e^2 \\
\Leftrightarrow \quad & 2ef\cos\sigma\left(\frac{r_2}{r_1} \mp 1\right) = \left(\pm r^2\left(\frac{r_2}{r_1} \pm 1\right) \mp r_1^2\left(\frac{r_2}{r_1} \pm 1\right) \mp e^2\left(\frac{r_2}{r_1} \mp 1\right)\right)\left(\frac{r_2}{r_1} \mp 1\right) \\
\Leftrightarrow \quad & 2ef\cos\sigma\left(\frac{r_2}{r_1} \mp 1\right) = \left(r^2\left(1 \pm \frac{r_2}{r_1}\right) - r_1^2\left(1 \pm \frac{r_2}{r_1}\right) + e^2\left(1 \mp \frac{r_2}{r_1}\right)\right)\left(\frac{r_2}{r_1} \mp 1\right),
\end{aligned}
$$

and x) $\Rightarrow$ viii) if $\left(\frac{r_2}{r_1} \mp 1\right) \neq 0$. By assumption of positive radii, this is always the case in the Moutard$-$ case. Otherwise, we have to assume $r_1 \neq r_2$.

Finally, we provide a diagram to help keep track of the proven implications:

$$
\begin{tikzcd}
iv) \arrow[r, Rightarrow, "a)"] & i) \arrow[d, Leftrightarrow, "c)"] & iii) \arrow[l, Rightarrow, "a)"'] \arrow[ll, Rightarrow, bend right, "a)"'] \arrow[r, Leftrightarrow, "a)"] \arrow[d, Rightarrow, "a)"] & ii) \arrow[d, Rightarrow, "a)"] \\
 & vi) \arrow[ul, Rightarrow, "c)"] \arrow[ur, Rightarrow, "d)"] \arrow[r, Rightarrow, "d)"] \arrow[d, Leftrightarrow, "b)"] & ix) \arrow[dl, Rightarrow, dashed, "b)"] & x) \arrow[d, Rightarrow, dashed, "g)"] \\
 & v) \arrow[r, Leftrightarrow, "f)"] & vii) \arrow[r, Leftrightarrow, "e)"] & viii)
\end{tikzcd}
$$

□

This allows us to relax the definition of s–isothermic cmc surfaces a bit: a pair of s–isothermic surfaces simultaneously being Christoffel duals and Darboux transforms of each other is property iii) in lemma 3.2.2 for all side faces of elementary cubes of the pair.

Proposition 3.2.3. *If in a Darboux pair s, s_3 of s–isothermic surfaces corresponding edges are parallel, they are Christoffel dual to each other — making s and s_3 a cmc pair.*

Proof. Parallelity of the corresponding edges means the connection side faces (s, s_1, s_3, s_{13}) and (s, s_2, s_3, s_{23}) are Moutard+ and Moutard$-$ trapezoids, respectively. Then property iii) in lemma 3.2.2 ensures that s_3 is already the dual, scaled by a factor of α (which is constant by property ix) in lemma 3.2.2). □

In view of remark 2.5.5 for discrete isothermic cmc surfaces, we can state

Theorem 3.2.4. *Generically, a Christoffel dual pair $s, s^* : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$ of s–isothermic surfaces is cmc iff $\langle s, s^* \rangle$ is constant.*

Proof. By generic we mean the situation that in each elementary side face formed by an edge and its dual, property x) in lemma 3.2.2 is sufficient to imply the first properties. Then, lemma 3.2.2 ensures that all elementary cubes are Darboux cubes, and the Christoffel dual is a Darboux transform. □

We will not discuss in more detail when non–generic situations will occur. One example is the cylinder, whose dual is itself rotates around its axis: any translation of the dual along the axis preserves the constant scalar product condition but not being Darboux transforms. Cf. also [BH16, Thm. 7], where such translated cases in the s–conical case are treated — the special geometry allowing constant face offset allows for a slightly wider definition of cmc surface, simply by mean curvature from the Steiner formula (as in def. 2.4.3).

Now, pairs of spheres s, s^* having constant scalar product $\langle s, s^* \rangle$ can not generally be interpreted as a sphere version of constant distance — a simple rescaling of both spheres and their distance preserves the scalar product. However, in the s–isothermic Christoffel dual case, the pairs also have a constant product of radii $rr^* = \alpha$. This prevents rescaling, and $\langle s, s^* \rangle$ can indeed be seen as a form of distance taking into account the sphere radii.

3.3 Geometry of Special Cases

In light the upcoming constructions for their associated families, we want to give the geometry of the elementary cubes of s_1–isothermic and s–conical cmc pairs a closer look, based on [Hof10] and [BH16], respectively.

3.3.1 S_1–Isothermic CMC Cubes

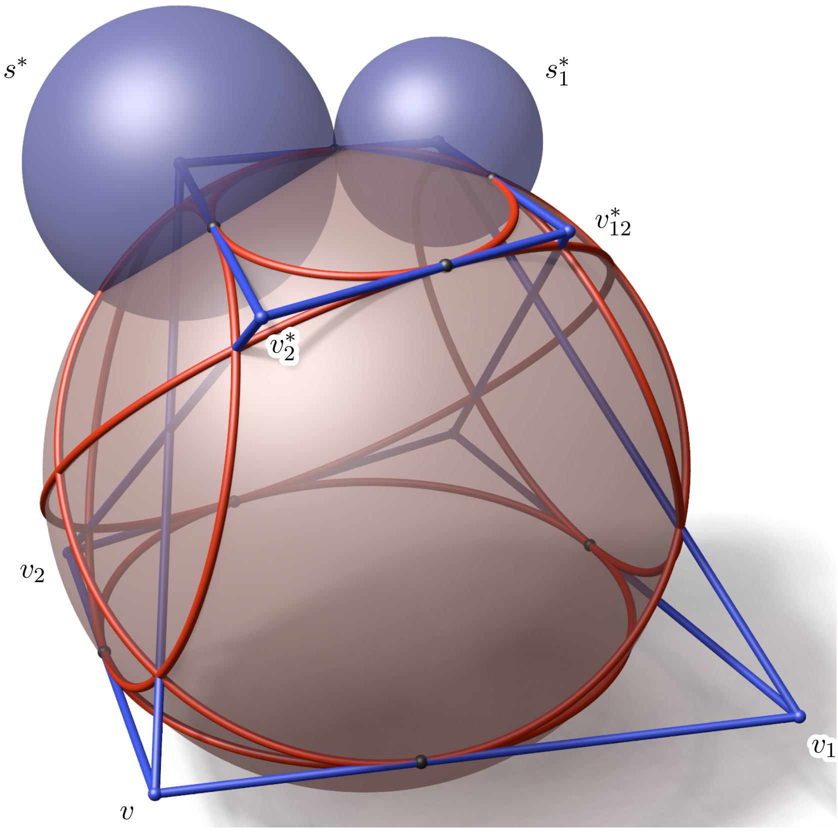

Figure 3.5: An elementary cube of an s_1–isothermic cmc pair. Vertices and edges as well as two of the spheres are shown in blue. Orthogonal circles of faces are the intersections of the face planes with the common orthogonal sphere of all eight spheres.

What we call s_1–isothermic according to def. 3.1.2 was the original definition of s–isothermic in [BP99]. Recall that in this case all adjacent spheres are exteriorly tangent along their connecting edge, with orthogonal circles of elementary quadrilaterals intersecting the spheres and touching the edges in these points of tangency. Since forming s–isothermic duals as well as Darboux transforms preserve scalar products along edges, any dual or Darboux transform of an s_1–isothermic surface is s_1–isothermic again.

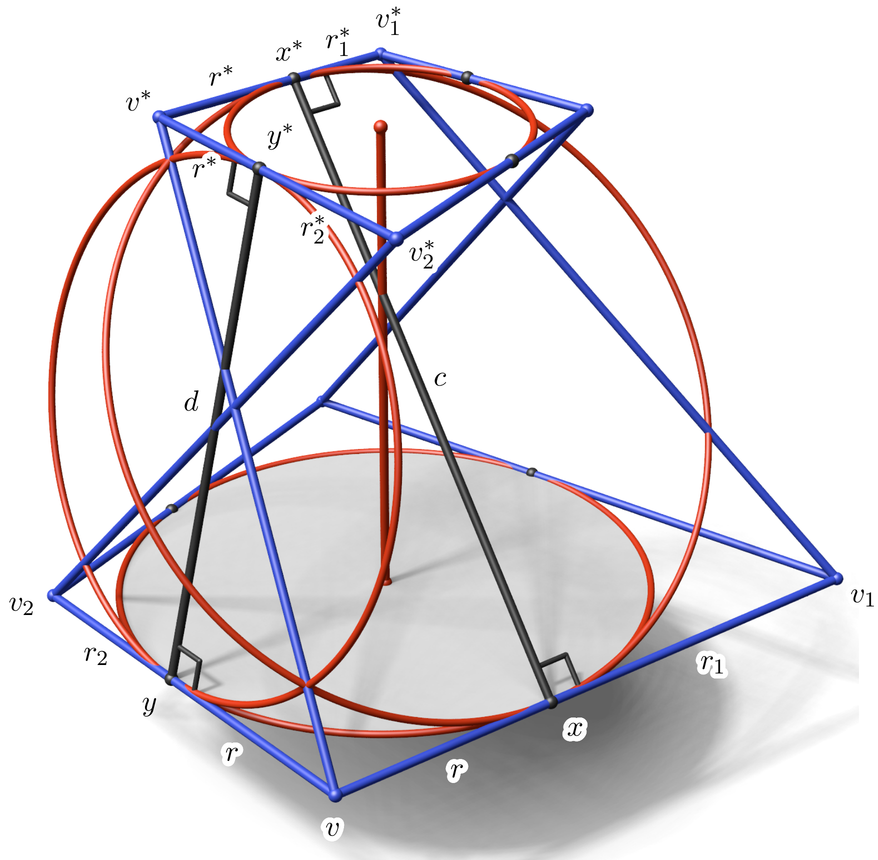

Figure 3.6: Now, the common axis of the orthogonal circles of the dual pair of top and bottom quadrilaterals is shown. For two of the side faces, the axis which perpendicularly connects the tangent points on the edges is shown.

Now consider a cmc pair of s_1–isothermic surfaces $s, s^* : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$. We look at an elementary cube $(s, s_1, s_2, s_{12}, s^*, s_1^*, s_2^*, s_{12}^*)$. Here, we will denote the sphere centers by v for vertices.

Forming a Moutard cube, all eight spheres lie in the 4–dimensional subspace spanned by s, s_1, s_2, s^* and have a common orthogonal sphere. The orthogonal circles of the side faces are the intersections of this orthogonal sphere with the face planes. In particular, the orthogonal circles of the top and bottom quadrilateral (the elementary quadrilaterals of the surfaces) are coaxial.

In the side faces, let us call the points of tangency of the spheres along the surface edges x and x^* along s, s_1 and s^*, s_1^* and y and y^* along s, s_2 and s^*, s_2^*. The orthogonal circles of the side faces are tangent to the edges in x and x^* and y and y^*, respectively. Therefore, the axes (x, x^*) and (y, y^*) connect the parallel edges perpendicularly. We denote the edge distances/orthogonal circle diameters by c and d. There are only two possible distances for parallel edges tangent to a pair of coaxial circles — each pair of opposite side faces shares the same. This makes the two edge distances c and d constant on the surface — a form of the dual cmc surface being a Christoffel dual in constant distance.

On the other hand, if we have a dual pair of s–isothermic quadrilaterals such that the connection of the sphere tangency points among each dual edge pair is perpendicular to the edges, we already have a cmc cube: the perpendicular connection is the diameter of an orthogonal circle, and by theorem 3.1.2 we only have to check that $\langle s, s^* \rangle = \langle s_1, s_1^* \rangle$ and $\langle s, s^* \rangle = \langle s_2, s_2^* \rangle$ for such quadrilaterals. By duality, all radii satisfy $r^* = \frac{\alpha}{r}$ etc. for a common factor α. Let us assume $v_1^* - v^* \overset{+}{\sim} v_1 - v$ and $v_2^* - v^* \overset{+}{\sim} -(v_2 - v)$ (recall from section 2.1 that by $\overset{+}{\sim}$ we mean oriented parallelity). Then, in the s, s_1 side face, we can calculate

$$\langle s, s^* \rangle = \frac{r^2 + r^{*2} - (c^2 + (r - r^*)^2)}{2\alpha} = \frac{c^2 + 2\alpha}{2\alpha},$$

$$\langle s_1, s_1^* \rangle = \frac{r_1^2 + r_1^{*2} - (c^2 + (r_1 - r_1^*)^2)}{2\alpha} = \frac{c^2 + 2\alpha}{2\alpha},$$

and analogously for the other side:

$$\langle s, s^* \rangle = \frac{r^2 + r^{*2} - (d^2 + (r + r^*)^2)}{2\alpha} = \frac{d^2 - 2\alpha}{2\alpha},$$

$$\langle s_2, s_2^* \rangle = \frac{r_2^2 + r_2^{*2} - (d^2 + (r_2 + r_2^*)^2)}{2\alpha} = \frac{d^2 - 2\alpha}{2\alpha}.$$

The same of course holds for the opposite side faces with the same geometry, and we have a Moutard cube of dual s–isothermic quadrilaterals.

Finally, let us recall the relation of the radii of the orthogonal circles of the dual pair of quadrilaterals to the sphere radii which will be important in our later constructions. Since the intersection points of the vertex spheres with the orthogonal circles of the elementary surface quadrilaterals are the tangent points of the edges with the orthogonal circle, the quadrilaterals formed by a vertex, its adjacent tangent points and the orthogonal circle center are kites, cf. fig. 3.7. Let the angle at the orthogonal circle center in the kite at v be called ρ. By the parallelity of edges and their duals, with orientation reversal in one coordinate, the corresponding angle in the kite at v^* is $\pi - \rho$. Thus we have

Lemma 3.3.1. *With r, r^* the radii of the spheres around v, v^* and R and R^* the radii of the orthogonal circle of the elementary quadrilateral and its dual,*

$$R^2(1 - \cos\rho) = r^2(1 + \cos\rho) \quad \textit{and} \quad R^{*2}(1 + \cos\rho) = r^{*2}(1 - \cos\rho),$$

and in particular

$$RR^* = rr^* = \alpha.$$

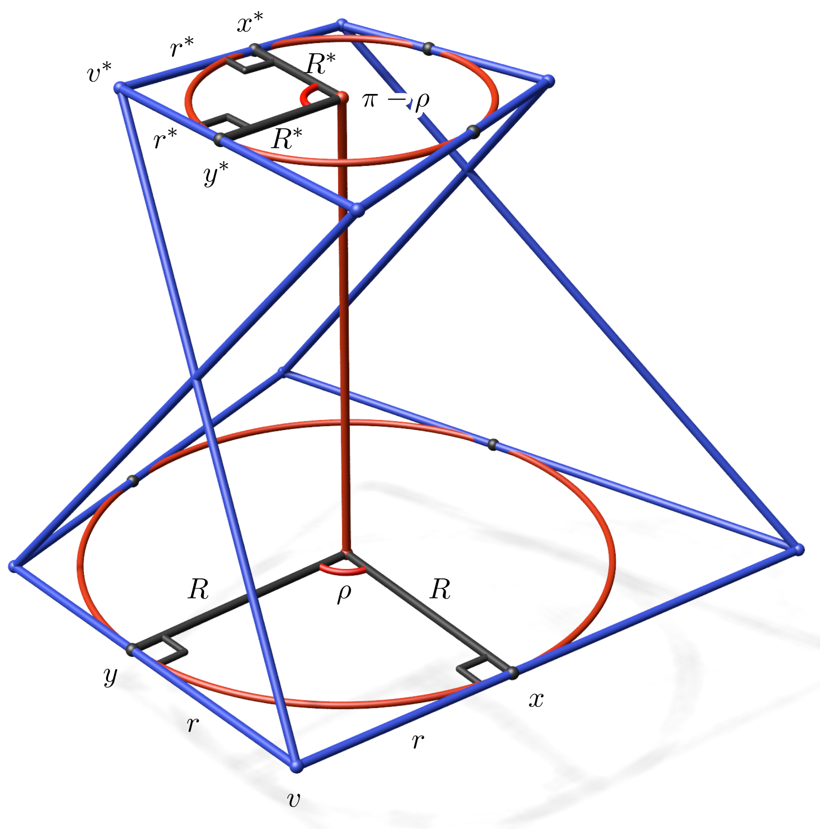

Figure 3.7: The kites at v and v^*

Proof. We have $r = R\tan\left(\frac{\rho}{2}\right)$ and $r^* = R^*\tan\left(\frac{\pi-\rho}{2}\right)$. The results follow with the half–angle formula for the tangent. □

3.3.2 S–Conical CMC Cubes

Again, we start by reminding ourselves that being s–conical can characterized by scalar products of adjacent spheres (cf. theorem 3.1.3) and thus is preserved under dualization and Darboux transforms.

As already mentioned, note that in [BH16] s–conical cmc surfaces are defined slightly broader than we do here with def. 3.2.2. We restrict ourselves to the case of Christoffel dual Darboux transforms as for the other types of s–isothermic surfaces. The eight spheres of such cubes again have a common orthogonal sphere, and the common point of the four spheres of each surface quadrilateral is the tangency point of the face planes to the orthogonal sphere. As these points are the intersection of the diagonals, in particular, the diagonals are tangent to the orthogonal sphere.

As points of tangency of parallel planes to the orthogonal sphere, the diagonal inter-

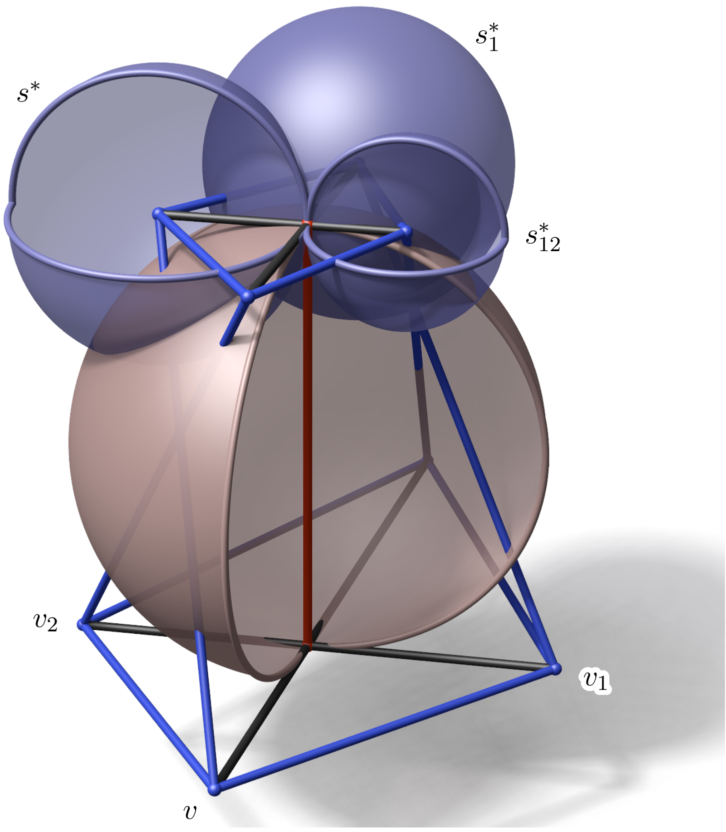

Figure 3.8: S–conical cmc cube with three spheres and the orthogonal sphere, cut open for visibility.

section points are connected orthogonally, and their distance h is the face offset between the dual pair of quadrilaterals. From [BH16] we know that h is constant in the surface.

In Christoffel dual quadrilaterals the diagonals were parallel to their non–corresponding one in the dual. We call their intersection angle σ.

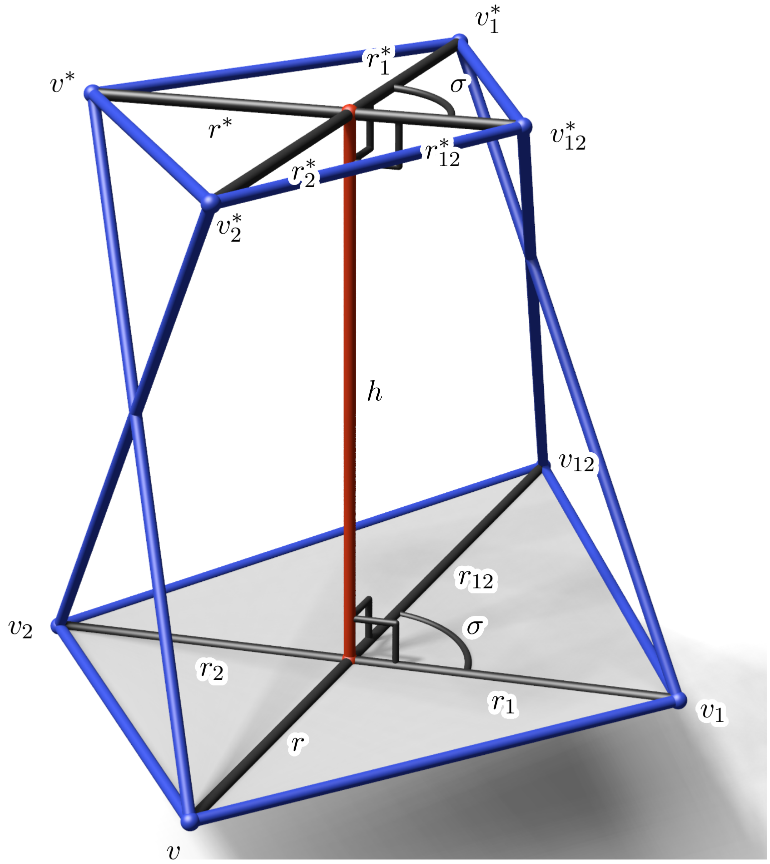

Figure 3.9: S–conical cmc cube, with the orthogonal connection of the diagonal intersection points.

3.4 Constructing Examples

The main purpose of our characterization in lemma 3.2.2 and the subsequent proposition 3.2.3 is to establish a well–stocked zoo of examples of s–isothermic cmc surfaces by simple, direct (and thus easily implementable) constructions. Since a complete classification is not our goal, we accept the exclusion of the degenerate cases in lemma 3.2.2 without investigating them further.

3.4.1 CMC Darboux Transforms

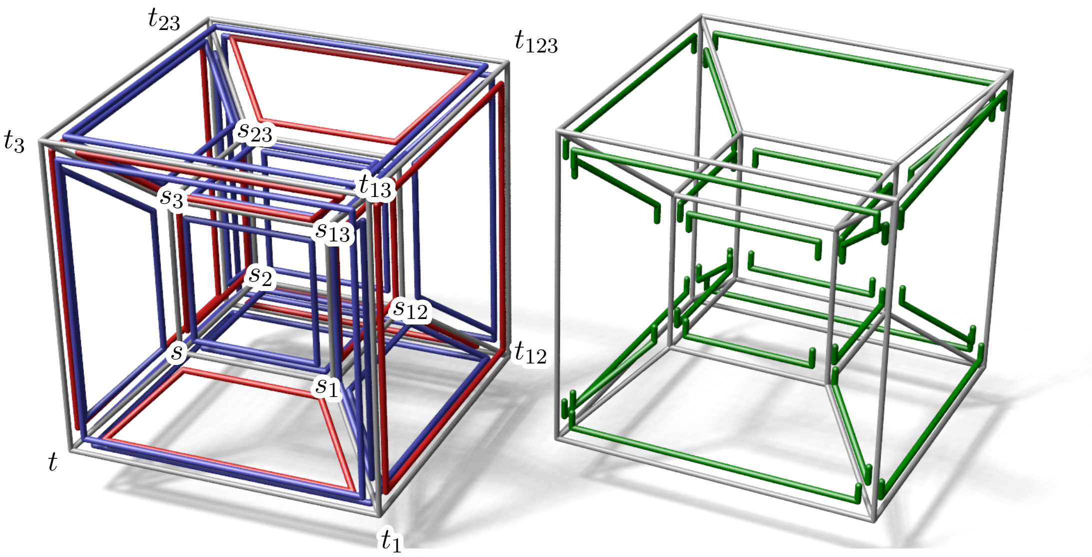

Figure 3.10: Combinatorical sketch of the 4D Moutard hypercube of the Darboux transformation of a cmc Moutard cube. To the left, a blue outline indicates a plus sign in the Moutard equation of the face, and a red one a minus sign. To the right, green bars with ends pointing towards each other within a face indicate parallel edges.

By multidimensional consistency of the Moutard equation (with correct signs), we can simultaneously Darboux transform the surfaces in a Darboux pair and get another Darboux pair.

In particular, by proposition 3.2.3 we can find new s–isothermic cmc surfaces as Darboux transforms of existing ones as soon as we can make sure that the pair of transforms of the original and its dual have parallel edges again. Since Moutard quadrilaterals are planar, if two adjacent faces of a Moutard cube are trapezoids, so are their opposite faces, since the two remaining faces lie in parallel planes. So if we start with a cmc Moutard cube $(s, s_1, s_2, s_3, \ldots)$ and construct the 4D Moutard hypercube of the cube and its Darboux transform $(t, t_1, t_2, t_3, \ldots)$, by proposition 3.2.3 (t) will be cmc again if

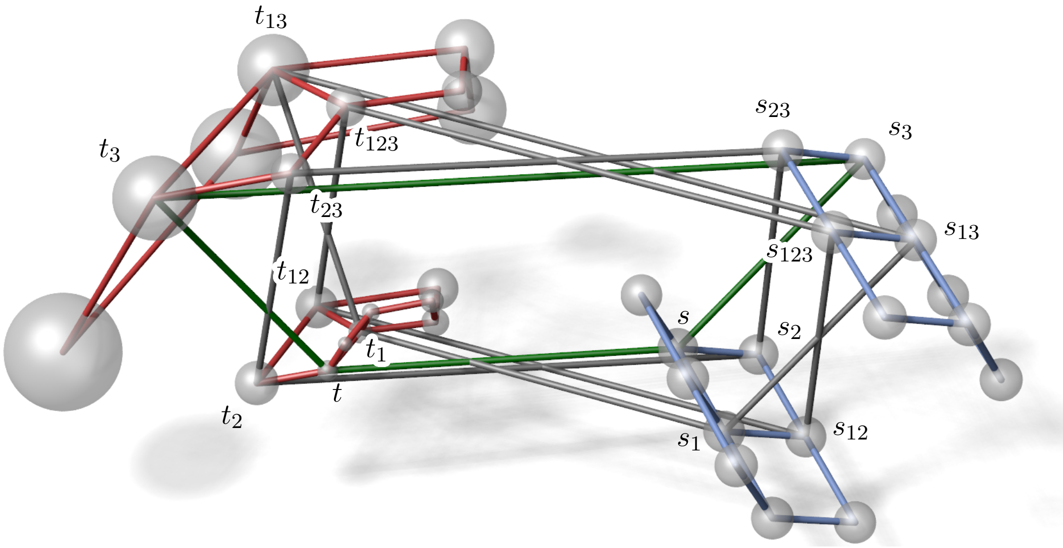

Figure 3.11: A small example of a cmc Darboux transform, with one of the hypercubes drawn. The trapezoid (s, t, s_3, t_3) is shown in green.

the quad (s, t, s_3, t_3) is a trapezoid.

The sketches in fig. 3.10 help keep track of the signs of Moutard equations (as noted in section 2.5, for multidimensional consistency the signs must multiply to -1 among the three face directions of each 3D cube) and parallelities.

The properties vii) or viii) in in lemma 3.2.2 tell us how to choose an initial t to make (s, t, s_3, t_3) a trapezoid.

In fig. 3.11, a Darboux transformation hypercube is shown as it appears geometrically.

Finally, in the following figs. 3.12 to 3.15 we present some pictures of s–isothermic cmc surfaces arising as Darboux transforms of simpler ones.

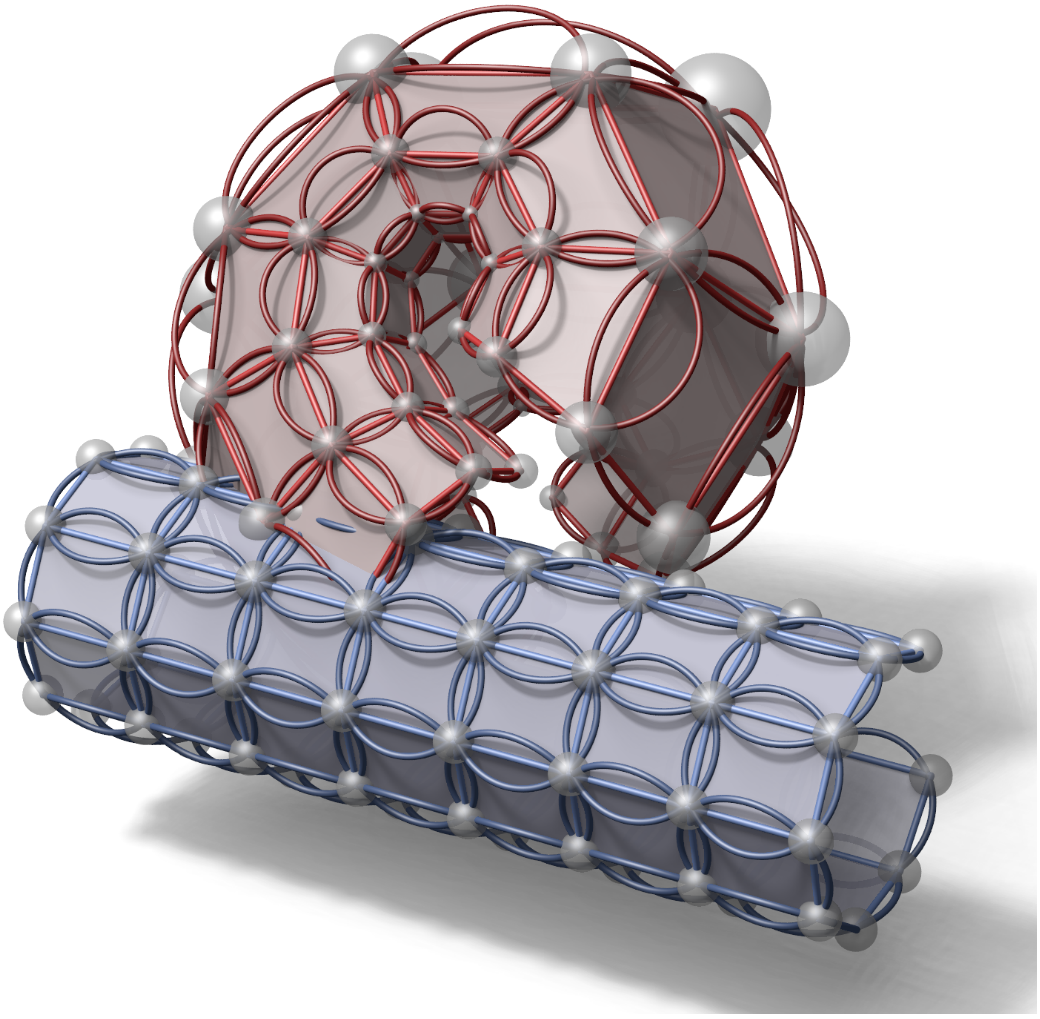

Figure 3.12: A cylindrical patch with small sphere radii and its cmc Darboux transform; edges, faces, spheres and orthogonal circles are shown.

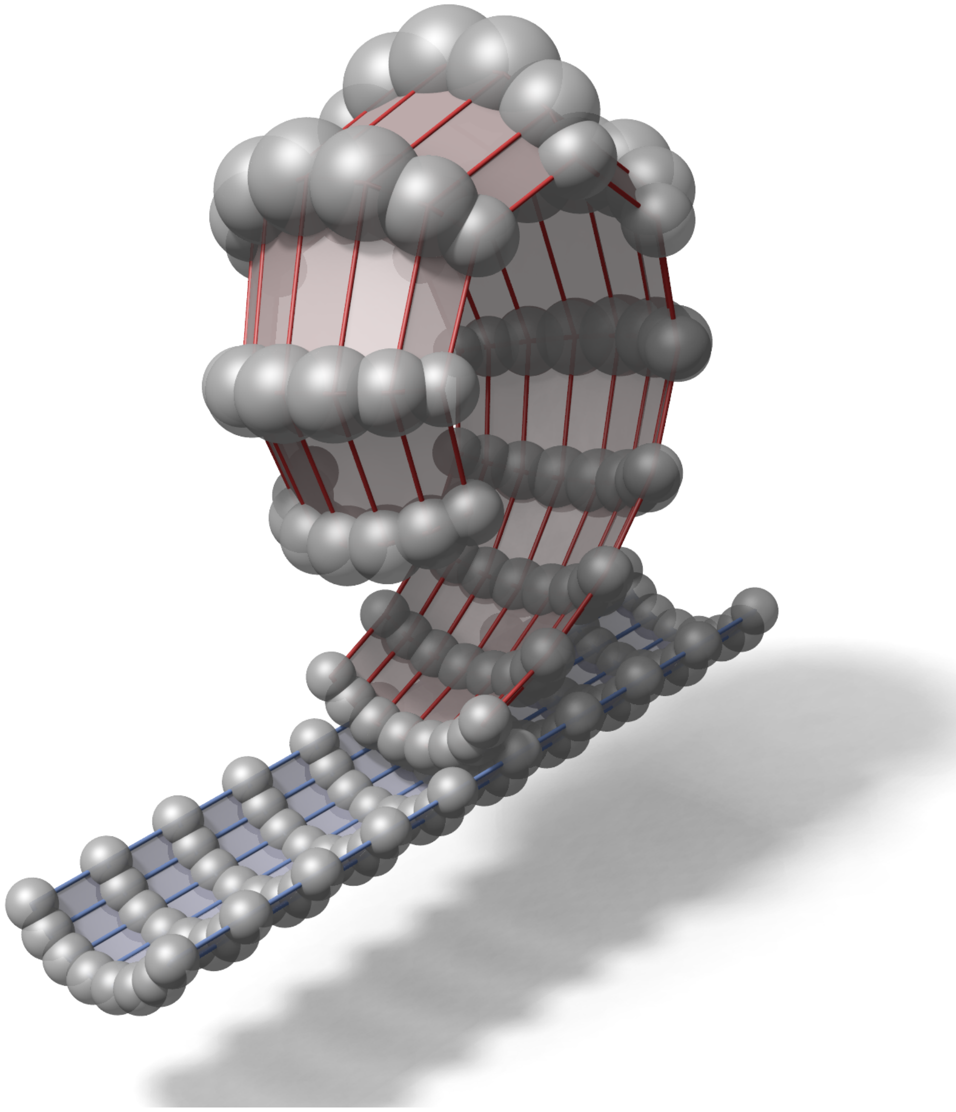

Figure 3.13: Again a cylindrical patch, but this time the spheres intersect in one coordinate direction. Edges, faces and spheres are shown.

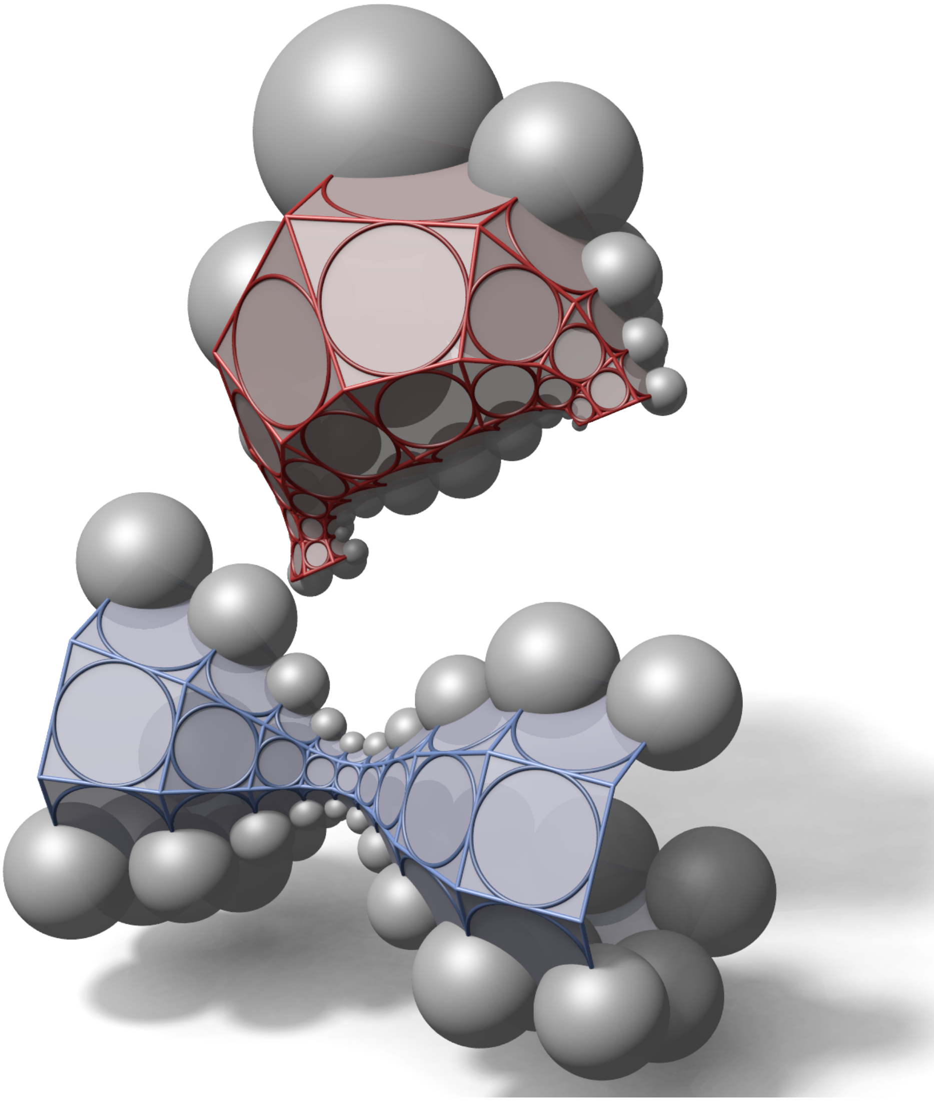

Figure 3.14: An s_1–isothermic Delaunay patch; edges, faces, some spheres and the orthogonal circles are shown. Note that for better visibility, the Darboux transform has been translated relative to the original surface.

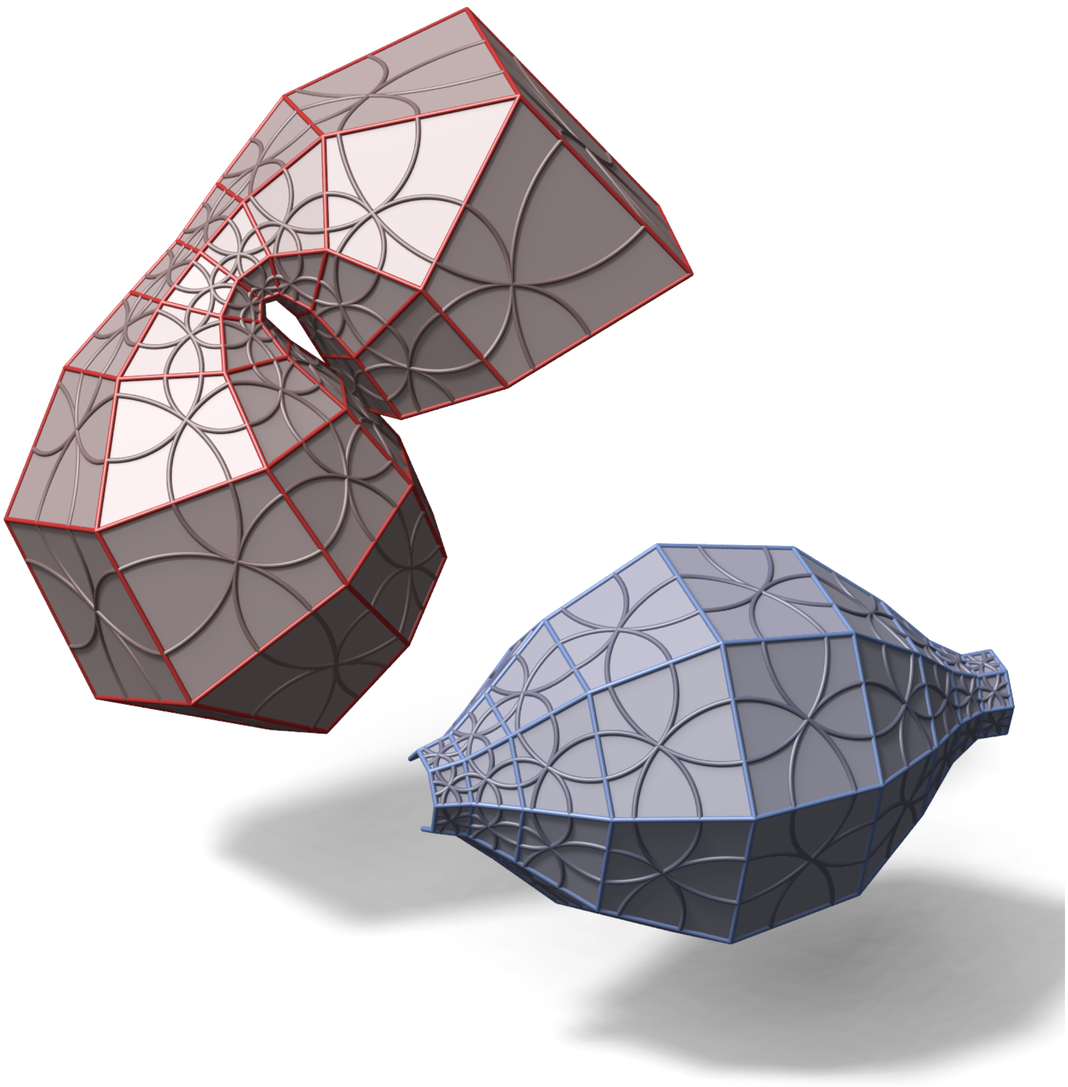

Figure 3.15: An s–conical Delaunay patch; edges, faces, and the intersection of the spheres with the faces are shown. Note that for better visibility, the Darboux transform has been translated relative to the original surface.

3.4.2 Rotationally Symmetric CMC Surfaces

Rotationally symmetric cmc surfaces have been studied classically by Delaunay ([Del41]), and discrete analogues have been developed in the discrete isothermic case in [Hof98], the s_1–isothermic case in [Hof10] and the s–conical case in [BH16]. In all cases, their meridian curves are found by tracing focal points of ellipses or hyperbolae (or discretizations of them) when rolling them along an axis.

Here, we use lemma 3.2.2 to find a direct construction for general s–isothermic surfaces.

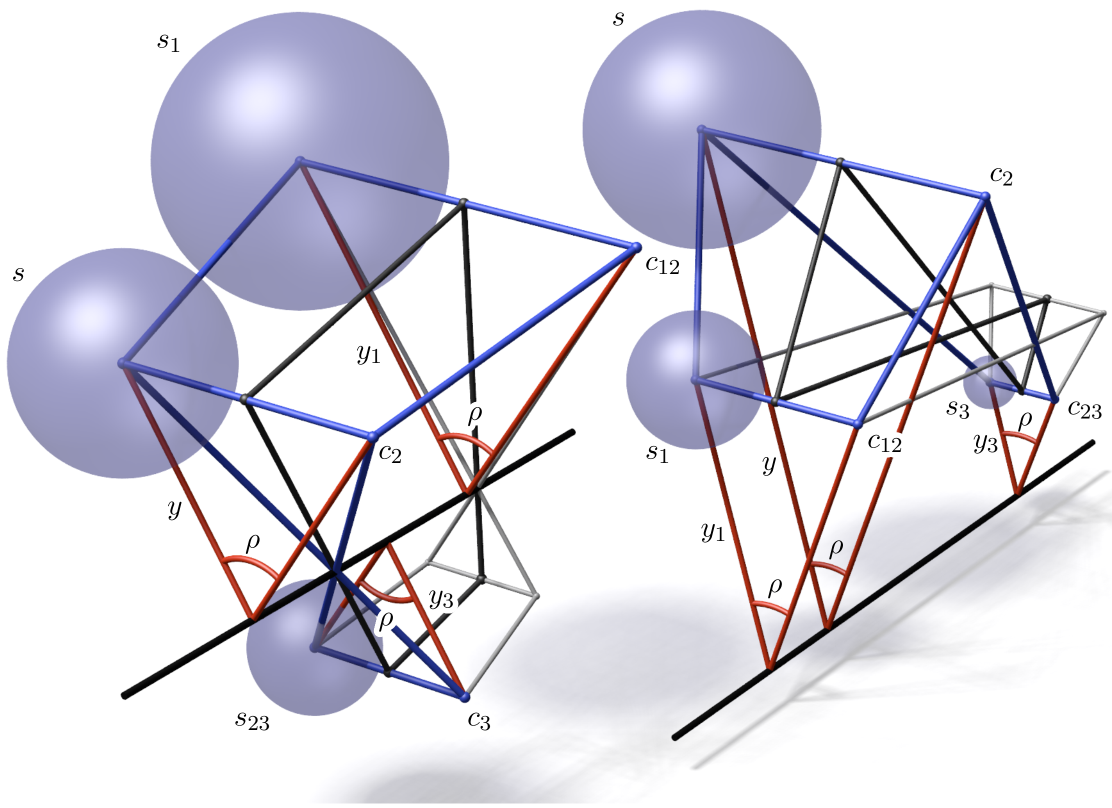

Figure 3.16: Elementary cubes of s–isothermic unduloid and nodoid. Surface quadrilaterals (in $(1,2)$–direction) and side faces of latitudinal edge pairs (in $(2,3)$–direction) are symmetrical Moutard trapezoids. Intersection of the symmetry plane with the cube in dark grey. Rotational axis in black.

Lemma 3.4.1. *Let $s, s_3 : \mathbb{Z} \to \mathbb{R}^{4,1}$ be planar curves of spheres, i.e. such that the centers $c = (x, y), c_3 = (x_3, y_3) : \mathbb{Z} \to \mathbb{R}^2 \subset \mathbb{R}^3$ lie in the (x, y)–plane, and let $s_2, s_{23} : \mathbb{Z} \to \mathbb{R}^{4,1}$ be s, s_3 rotated around the x–axis by an angle $0 < \rho < \pi$. (If y and y_3 have opposite signs, we want to exclude the case $\langle s_2, s_3 \rangle = 1$.)*

If there is a constant $\beta > 0$ such that, with $r, r_3 : \mathbb{Z} \to \mathbb{R}^+$ the radii of s, s_3,

$$\frac{|y|}{r} = \frac{|y_3|}{r_3} = \beta \quad \textit{for all spheres,}$$

the elementary quadrilaterals (s, s_1, s_2, s_{12}) and (s, s_2, s_3, s_{23}) are Moutard trapezoids.

Proof. For an illustration, see fig. 3.16. By symmetry, the quads have a pair of parallel edges whose length is proportional to the distance $|y|$ from the axis. So the claim boils down to symmetrical trapezoids being Moutard if the length of the parallel edges is proportional to the (equal) radii of the spheres at their ends.

So let, for now, (s, s_1, s_2, s_{12}) be any such quad with $e = \|c - c_1\| = \gamma r = \gamma r_1$ and $e_2 = \|c_2 - c_{12}\| = \gamma r_2 = \gamma r_{12}$ the lengths of the parallel edges. We can e.g. look at property vii) of lemma 3.2.2. In our situation, the cubic term becomes (with d the distance of the parallel edges)

$$\begin{aligned}
& r_1^3 \pm r_2 r_1^2 + (f^2 - d_{12}^2 - r^2) r_1 \pm r_2(e^2 - r^2) \\
= {} & \pm r_2 r^2 + \left((\tfrac{1}{2}e \mp \tfrac{1}{2}e_2)^2 + d^2 - ((\tfrac{1}{2}e \pm \tfrac{1}{2}e_2)^2 + d^2) \right) r \pm r_2(e^2 - r^2) \\
= {} & \mp e e_2 r \pm e^2 r_2 \\
= {} & \mp e(\gamma r r_1 - \gamma r r_2) = 0.
\end{aligned}$$

So we only need to check that our original symmetric s_{12} is the $\widetilde{s}_{12}$ of lemma 3.2.2. Since parallelity and the correct radius are given by symmetry, we check

$$e_2 = \gamma r_2 = r r_2 \frac{\gamma r}{r^2} = r r_2 \frac{e}{r r_1}.$$

The case not covered by lemma 3.2.2 nor excluded here is that of symmetric embedded quads with s_1 and s_2 (and thus s and s_{12}) in exterior tangency. They are s–conical quads (and thus Moutard trapezoids) as soon as the points of tangency coincide with the diagonal intersection point o. So let $a = \|c - o\| = \|c_1 - o\|$ and $b = \|c_2 - o\| = \|c_{12} - o\|$ be the diagonal segment lengths. They are proportional to the edge lengths e and e_2 and thus, by assumption, to the radii r and r_2. Together with the touching spheres (and the symmetry), we get

$$\frac{a}{b} = \frac{r}{r_2} \quad \text{and} \quad a + b = r + r_2,$$

resulting in $a = r$ and $b = r_2$ as desired. □

So if we have a pair of meridian curves s and s_3 with radii proportional to the distance to the axis, we get a cmc surface of revolution as soon as the elementary quadrilaterals (s, s_1, s_3, s_{13}) are Moutard trapezoids as well.

The elementary trapezoids from lemma 3.4.1, i.e. those with edges in rotational direction, are embedded iff the y–values of their centers have the same sign. Since we

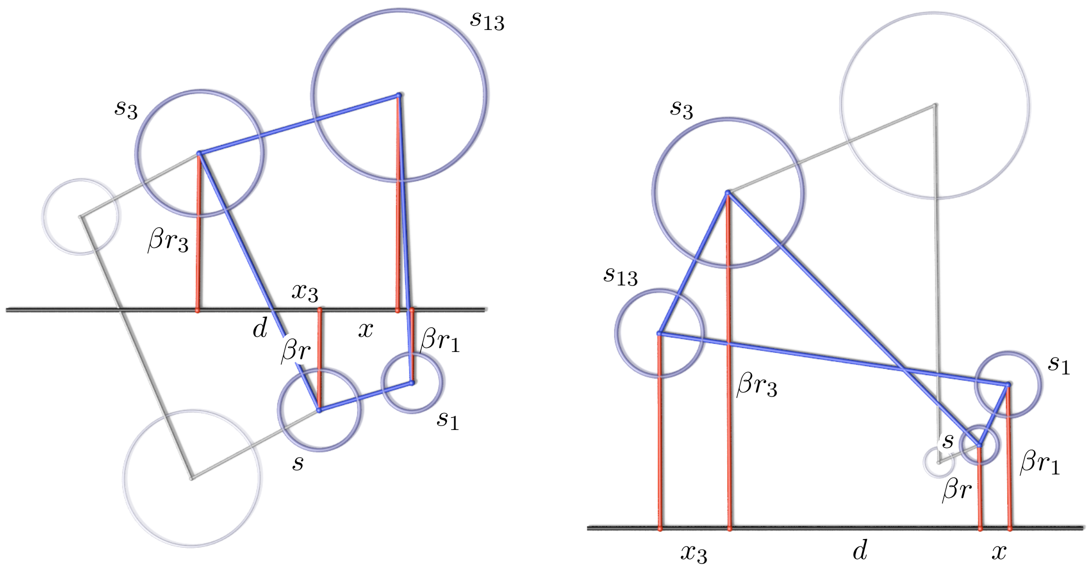

Figure 3.17: Elementary quadrilaterals of the pairs of meridian curves of an s–isothermic unduloid and nodoid. Rotational axis in black.

want the surface quadrilaterals (in $(1,2)$–direction) to be embedded, the two meridian curves can never cross the axis. In the case that the curves lie on opposite sides, the $(2,3)$–quads are non–embedded, i.e. Moutard−, and the $(1,3)$–quads of the meridian curve pair will have to be Moutard+: this is the unduloid case. Conversely, if both meridian curves lie on the same side of the axis, the $(1,3)$–quads of the meridian curve pair will have to be Moutard−, and we are in the nodoid case.

Now, we want to see how to construct suitable pairs of meridian curves. As in lemma 3.2.2, in the $\pm$ and $\mp$ signs, the top one will always refer to the Moutard+ and the bottom one to the Moutard− case.

Lemma 3.4.2. *Let s, s_3 be spheres with radii $r, r_3 > 0$ and centers*

$$c = (0, \mp\beta r),\ c_3 = (-d, \beta r_3) \in \mathbb{R}^2$$

for some $d \in \mathbb{R}$ and $0 < \beta < 1$. Let $\alpha := rr_3$. For some $x \in \mathbb{R}$, consider the spheres s_1 with some radius $r_1 > 0$ and center $c_1 = (x, \mp\beta r_1)$ and s_{13} with radius $r_{13} = \frac{\alpha}{r_1}$ and center $c_{13} = (-d \pm x_3, \beta r_{13})$, where $x_3 := \frac{\alpha}{rr_1}x$. Assume s_1 satisfies the assumption of unique λ in lemma 3.2.2. Then (s, s_1, s_3, s_{13}) is a Moutard$\pm$ trapezoid with $\langle s, s_1\rangle = \tau$ if, with $b := 1 - \beta^2$ and $t := \tau - \beta^2$,

- *$d = 0$ and*

$$r_1 = \frac{r^2 b \pm \alpha t}{rt \pm r_3 b},$$

$$x^2 = \big(r^2 + r_1^2\big)b - 2rr_1 t, \tag{3.1}$$

- *or* $d \neq 0$ *and*

$$\begin{aligned} & r_1^2\Big(r_3^2 b^2 + r^2 t^2 + b\big(\pm 2\alpha t - d^2\big)\Big) \\ & -2rr_1\Big(\pm\alpha\big(b^2+t^2\big) + t\big((r^2+r_3^2)b - d^2\big)\Big) \\ & +r^2\Big(r^2 b^2 + r_3^2 t^2 + b\big(\pm 2\alpha t - d^2\big)\Big) = 0, \end{aligned}$$

$$x = \frac{1}{d}\Big(\pm r_3\big(r_1 b - rt\big) + r\big(r_1 t - rb\big)\Big).$$

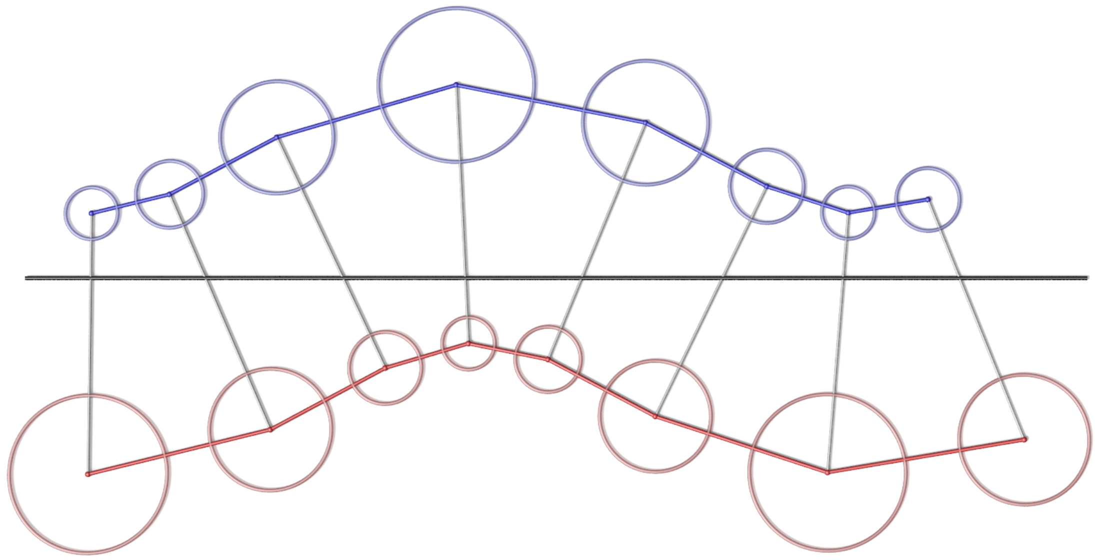

Figure 3.18: A dual pair of meridian curves for an s–isothermic unduloid. Rotational axis in black.

Proof. For an illustration of the geometry, see fig. 3.17. The construction of s_{13} is the same as $\widetilde{s}_{12}$ in lemma 3.2.2, so it suffices to show property vii) for s_1. In our case, the quantities there become

$$\begin{aligned} e^2 &= x^2 + \beta^2(r - r_1)^2, \\ f^2 &= d^2 + \beta^2(r \pm r_3)^2, \\ d_{12}^2 &= (d + x)^2 + \beta^2(r_1 \pm r_3)^2. \end{aligned}$$

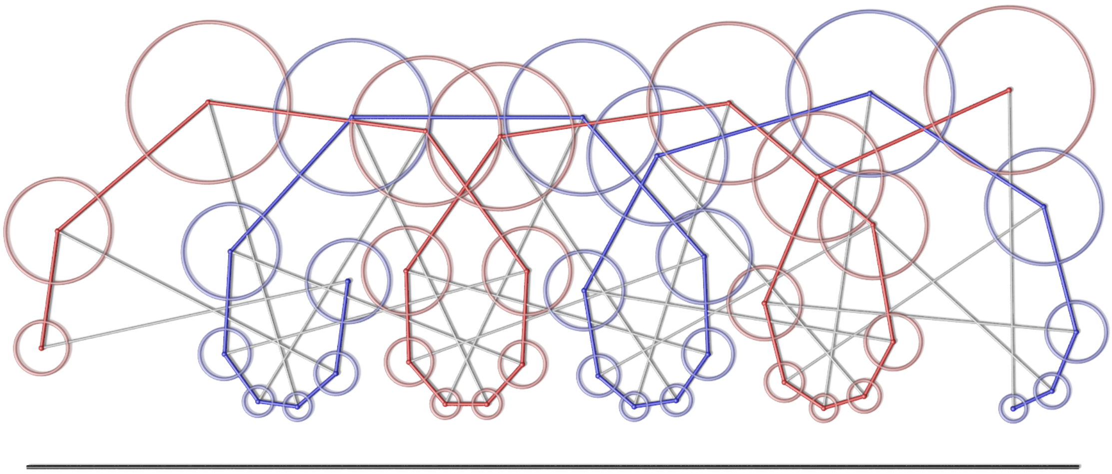

Figure 3.19: A dual pair of meridian curves for an s–isothermic nodoid. Rotational axis in black.

The term from property vii) we want to vanish is

$$
\begin{aligned}
&r_1^3 \pm r_3 r_1^2 + (f^2 - d_{12}^2 - r^2) r_1 \pm r_3 (e^2 - r^2) \\
&= r_1^3 \pm r_3 r_1^2 + \big(-2xd - x^2 + \beta^2 (r^2 \pm 2\alpha - r_1^2 \mp 2 r_1 r_3) - r^2\big) r_1 \\
&\quad \pm r_3 \big(x^2 + (\beta^2 - 1) r^2 - 2\beta^2 r r_1 + \beta^2 r_1^2\big) \\
&= b r_1^3 \pm b r_1^2 r_3 - x^2 (r_1 \mp r_3) - 2xd r_1 - b r^2 r_1 \mp b \alpha r \\
&= b (r_1^2 - r^2)(r_1 \pm r_3) - x^2 (r_1 \mp r_3) - 2xd r_1. \qquad (3.2)
\end{aligned}
$$

Now, first suppose $d = 0$ and the assumptions for that case hold. The term (3.2) becomes

$$
\begin{aligned}
&b(r_1^2 - r^2)(r_1 \pm r_3) - x^2 (r_1 \mp r_3) \\
&= b(r_1^2 - r^2)(r_1 \pm r_3) - \big((r^2 + r_1^2) b - 2 r r_1 t\big)(r_1 \mp r_3) \\
&= \pm 2 b r_1^2 r_3 - 2 b r^2 r_1 + 2 r r_1^2 t \mp 2 \alpha r_1 t \\
&= 2 r_1 \big(r_1 (rt \pm r_3 b) - (r^2 b \pm \alpha r)\big),
\end{aligned}
$$

which vanishes by the assumption on r_1.

If $d \neq 0$, again suppose r_1 and x satisfy the assumptions. This time, (3.2) becomes

$$
\begin{aligned}
&b(r_1^2 - r^2)(r_1 \pm r_3) - 2xd r_1 - x^2 (r_1 \mp r_3) \\
&= b(r_1^2 - r^2)(r_1 \pm r_3) - 2 r_1 \big(\pm r_3 (r_1 b - rt) + r(r_1 t - rb)\big) - x^2 (r_1 \mp r_3) \\
&= b\big(r_1^3 \pm r_1^2 r_3 - r^2 r_1 \mp r^2 r_3 \mp 2 r_1^2 r_3 + 2 r^2 r_1\big) - 2 t r r_1 (r_1 \mp r_3) - x^2 (r_1 \mp r_3)
\end{aligned}
$$

$$
\begin{aligned}
&= b\big(r_1^3 \mp r_1^2 r_3 + r^2 r_1 \mp r^2 r_3\big) - 2trr_1(r_1 \mp r_3) - x^2(r_1 \mp r_3) \\
&= (r_1 \mp r_3)\big(br^2 + br_1^2 - 2trr_1 - x^2\big),
\end{aligned}
$$

which again vanishes if (3.1) holds.

Now we square the assumption for x and get

$$
\begin{aligned}
x^2 d^2 &= r_3^2(r_1 b - rt)^2 + r^2(r_1 t - rb)^2 \pm 2\alpha(r_1 b - rt)(r_1 t - rb) \\
&= r_1^2(r_3^2 b^2 + r^2 t^2 \pm 2\alpha bt) - 2rr_1\big(r_3^2 bt + r^2 bt \pm \alpha(b^2 + t^2)\big) \\
&\quad + r^2(r_3^2 t^2 + r^2 b^2 \pm 2\alpha bt),
\end{aligned}
$$

telling us that our assumption on r_1 is just

$$
x^2 d^2 = \big(br_1^2 - 2trr_1 + br^2\big)d^2,
$$

which, since we had $d \neq 0$, is the same as (3.1).

And since, if (3.1) holds,

$$
\begin{aligned}
\langle s, s_1 \rangle &= \frac{1}{2rr_1}\big(r^2 + r_1^2 - e^2\big) \\
&= \frac{1}{2rr_1}\big(r^2(1 - b - \beta^2) + r_1^2(1 - b - \beta^2) + 2rr_1(t + \beta^2)\big) \\
&= \tau,
\end{aligned}
$$

we get the desired scalar product in both cases. □

Given an initial pair s, s_3, we can use this to iteratively construct a pair of meridian curves of Moutard trapezoids satisfying the prerequisites of lemma 3.4.1 with prescribed scalar products along the edges. Note that in the first step, the quadratic equations in lemma 3.4.2 give us a choice of direction, while in the subsequent steps one solution corresponds to the previous quadrilateral. Examples of the resulting meridian curves are shown in figs. 3.18, 3.19 and 3.22

The Moutard trapezoid properties from lemma 3.4.1 and lemma 3.4.2 together guarantee that we get an s–isothermic cmc surface by repeatedly rotating the meridian pair around the x axis. For completeness, if we also want to prescribe the rotational angle steps by the scalar product along edges in rotational direction, we note that they are connected by the simple geometric relation

Lemma 3.4.3. *Let s be a sphere with radius $r > 0$ and center c in distance βr from the x axis, and let s_2 be s rotated around said axis by an angle of $0 < \rho < \pi$. Then $\langle s, s_2 \rangle = \sigma$ if*

$$
\cos\rho = \frac{1}{\beta^2}\big(\sigma - 1\big) + 1.
$$

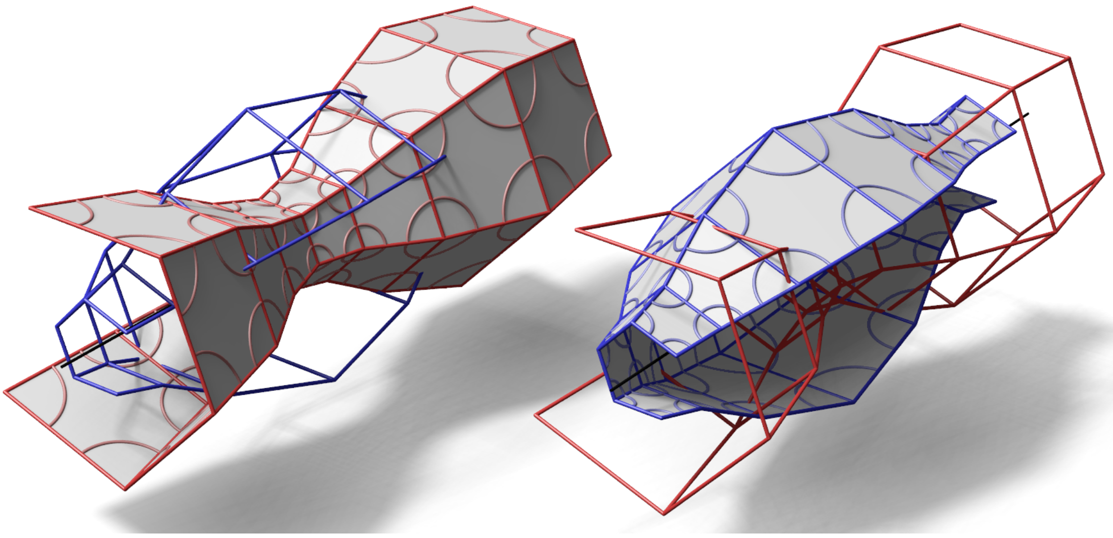

Figure 3.20: A patch of the s–isothermic unduloid with the meridian curves from fig. 3.18. Each surface of the dual pair is shown separately, with the respective other one shown as wireframe. Spheres are shown as their intersection with the faces.

Proof. We have

$$\|c_2 - c\|^2 = \left(2\beta r \sin\frac{\rho}{2}\right)^2 = 2\beta^2 r^2(1 - \cos\rho)$$

and thus

$$\sigma = \frac{1}{2r^2}\left(2r^2 - 2\beta^2 r^2(1 - \cos\rho)\right) = 1 + \beta^2(\cos\rho - 1).$$

□

A few examples are illustrated in figs. 3.20, 3.21 and 3.23.

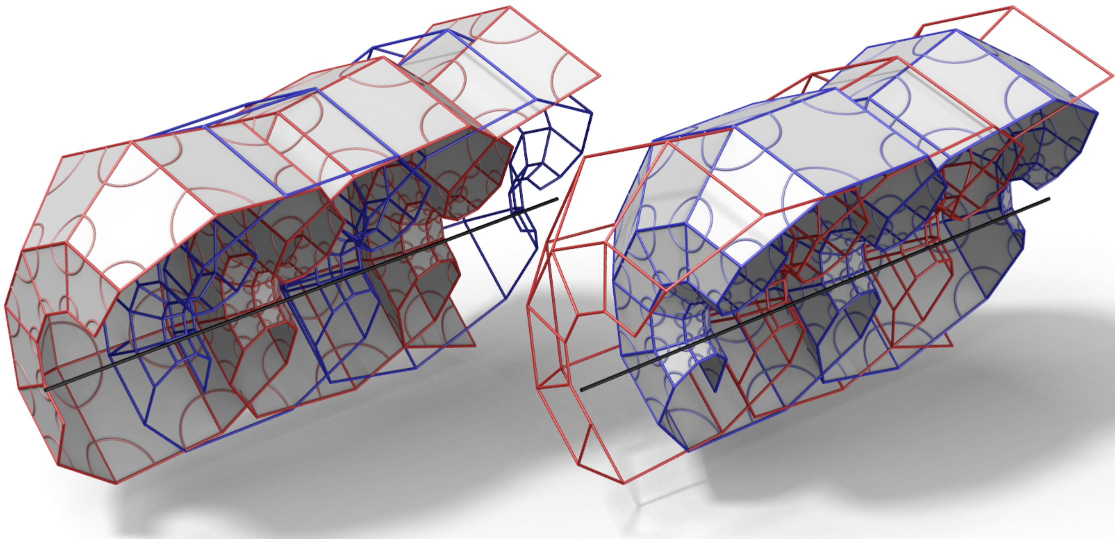

Figure 3.21: The same for the nodoid with the meridian curves from fig. 3.19.

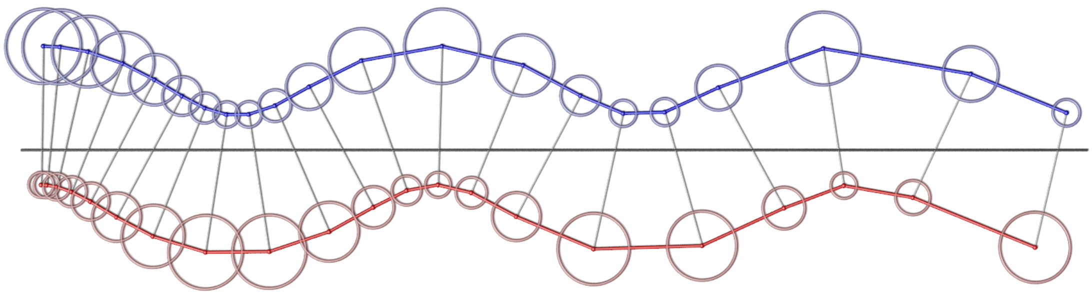

Figure 3.22: Another unduloid meridian curve. With the ratio of sphere radii and distance to the axis constant, prescribing the scalar products along the curve controls the coarseness of the discretization.

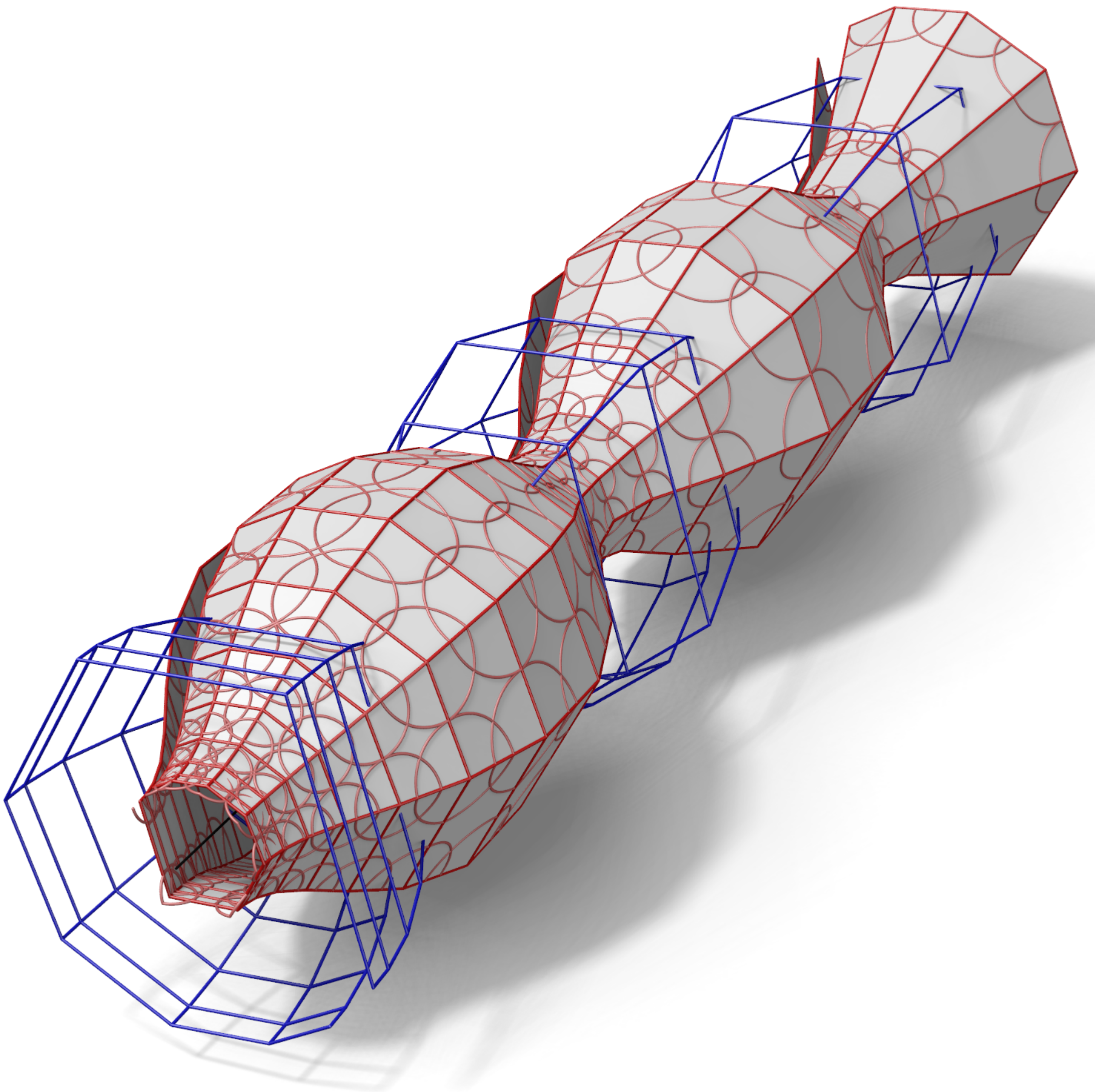

Figure 3.23: An unduloid with the meridian curve from fig. 3.22. Different rotational angle steps control the coarseness in rotational direction.

Chapter 4

Non–Planar Geometry

In this chapter, we will explore the properties of the non–planar quadrilaterals occurring in the surfaces in the associated families of the various types of s–isothermic minimal and cmc surfaces. We begin by defining a (or more specifically, two versions of a) condition on the dihedral angles along edges of non–planar quadrilaterals that will characterize these quadrilaterals.

We will then, in section 4.2, show that the quadrilaterals of the previously known (cf. [BHS06]) associated family of s_1–isothermic minimal surfaces satisfy (the first version of) this condition.

S–Conical minimal surfaces have been studied in [BHKS15], where we also provided a construction for the associated family. In section 4.3, we recapitulate this construction, and additionally show that its quadrilaterals satisfy our conditions as well.

In section 4.4, we develop a geometric construction for the associated family of s_1–isothermic cmc surfaces, which have been studied under the name *s–cmc surfaces* in [Hof10]. We will then show that this construction will meet the full extent of the conditions we define in section 4.1; indeed the idea for the conditions arose from the study of this construction.

We will proceed to conjecture, as indicated by evidence from numerical experiments, that our conditions are sufficient to construct associated families for general s–isothermic cmc surfaces. Finally we will apply the construction found in this way to s–conical cmc surfaces, which have been studied in [BH16]. Again backed by numerical evidence, for this case we propose a more direct geometric construction.

4.1 Bent and Folded Quadrilaterals

These will be the properties that characterize the non–planar quadrilaterals in the associated families of s–isothermic minimal and cmc surfaces. We first present formal definitions, and then a more direct formulation that is equivalent up to orientational considerations. In the end, we will mention how those properties can be considered an extension of the discrete isothermic case as mentioned in section 2.6.

Definition 4.1.1. Let (s, s_1, s_2, s_{12}) be a quadrilateral of spheres with centers

$$v, v_1, v_2, v_{12} \in \mathbb{R}^3$$

and radii

$$r, r_1, r_2, r_{12} > 0.$$

Let

$$e^1 := v_1 - v, \quad e_2^1 := v_{12} - v_2, \quad e^2 := v_2 - v, \quad e_1^2 := v_{12} - v_1$$

be the oriented edges and

$$n := \frac{e^1 \times e^2}{\|e^1 \times e^2\|}, \quad n_1 := \frac{e^1 \times e_1^2}{\|e^1 \times e_1^2\|}, \quad n_2 := \frac{e_2^1 \times e^2}{\|e_2^2 \times e^2\|}, \quad n_{12} := \frac{e_2^1 \times e_1^2}{\|e_2^1 \times e_1^2\|}$$

the unit corner normals. The quadrilateral is called *a–bent* for an $a \in \mathbb{R}$ if

$$\begin{aligned}
n \times n_1 &= \frac{-a}{\sqrt{(r^2 + a^2)(r_1^2 + a^2)}}\, e^1, \\
n_2 \times n_{12} &= \frac{-a}{\sqrt{(r_2^2 + a^2)(r_{12}^2 + a^2)}}\, e_2^1, \\
n \times n_2 &= \frac{a}{\sqrt{(r^2 + a^2)(r_2^2 + a^2)}}\, e^2, \\
n_1 \times n_{12} &= \frac{a}{\sqrt{(r_1^2 + a^2)(r_{12}^2 + a^2)}}\, e_1^2.
\end{aligned}$$

Definition 4.1.2. Let $(s, s_1, s_2, s_3, s_{12}, s_{13}, s_{23}, s_{123})$ be a cube of spheres. The cube is called *a–bent* for an $a \in \mathbb{R}$ if the s–quadrilaterals

$$(s, s_1, s_2, s_{12}), \quad (s_3, s_{13}, s_{23}, s_{123}), \quad (s, s_1, s_3, s_{13}) \quad \text{and } (s_2, s_{12}, s_{23}, s_{123})$$

are a–bent and

$$(s, s_2, s_3, s_{23}) \text{ and } (s_1, s_{12}, s_{13}, s_{123})$$

are $-a$–bent.

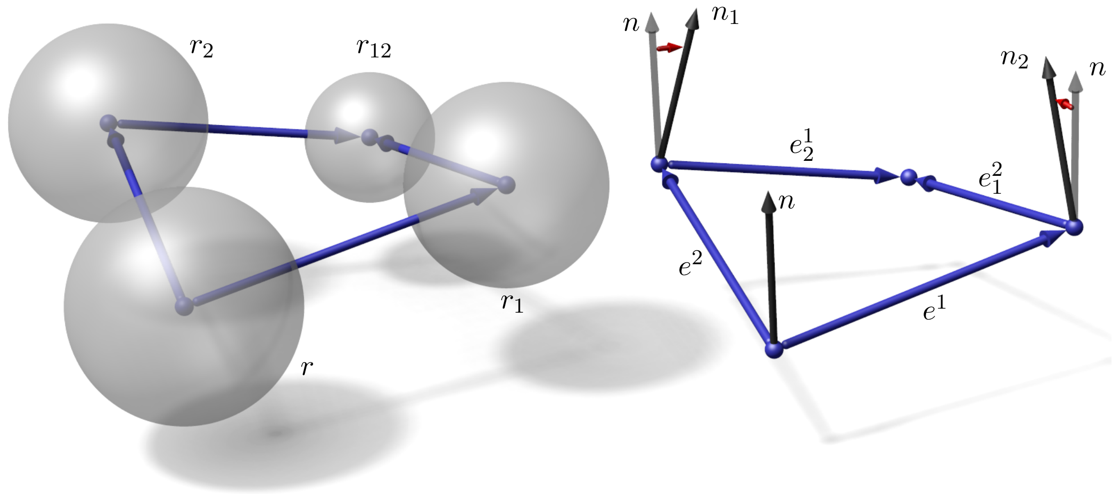

Figure 4.1: An example of an a–bent quadrilateral, with spheres and corner normals respectively. Note the opposite rotational directions around e^1 and e^2, corresponding to the alternating signs at a in def. 4.1.1.

Definition 4.1.3. Let (s, s_1, s_2, s_{12}) be a quadrilateral of spheres with centres

$$v, v_1, v_2, v_{12} \in \mathbb{R}^3$$

and radii

$$r, r_1, r_2, r_{12} > 0.$$

Let

$$e^1 := v_1 - v, \quad e_2^1 := v_{12} - v_2, \quad e^2 := v_2 - v, \quad e_1^2 := v_{12} - v_1$$

be the oriented edges and

$$n := \frac{e^1 \times e^2}{\|e^1 \times e^2\|}, \quad n_1 := \frac{e^1 \times e_1^2}{\|e^1 \times e_1^2\|}, \quad n_2 := \frac{e_2^1 \times e^2}{\|e_2^1 \times e^2\|}, \quad n_{12} := \frac{e_2^1 \times e_1^2}{\|e_2^1 \times e_1^2\|}$$

the unit corner normals. The quadrilateral is called *k–folded* for a $k \in \mathbb{R}$ if

$$\begin{aligned}
n \times n_1 &= \frac{k}{\sqrt{(1+k^2r^2)(1+k^2r_1^2)}}\, e^1, \\
n_2 \times n_{12} &= \frac{k}{\sqrt{(1+k^2r_2^2)(1+k^2r_{12}^2)}}\, e_2^1, \\
n \times n_2 &= \frac{-k}{\sqrt{(1+k^2r^2)(1+k^2r_2^2)}}\, e^2, \\
n_1 \times n_{12} &= \frac{-k}{\sqrt{(1+k^2r_1^2)(1+k^2r_{12}^2)}}\, e_1^2.
\end{aligned}$$

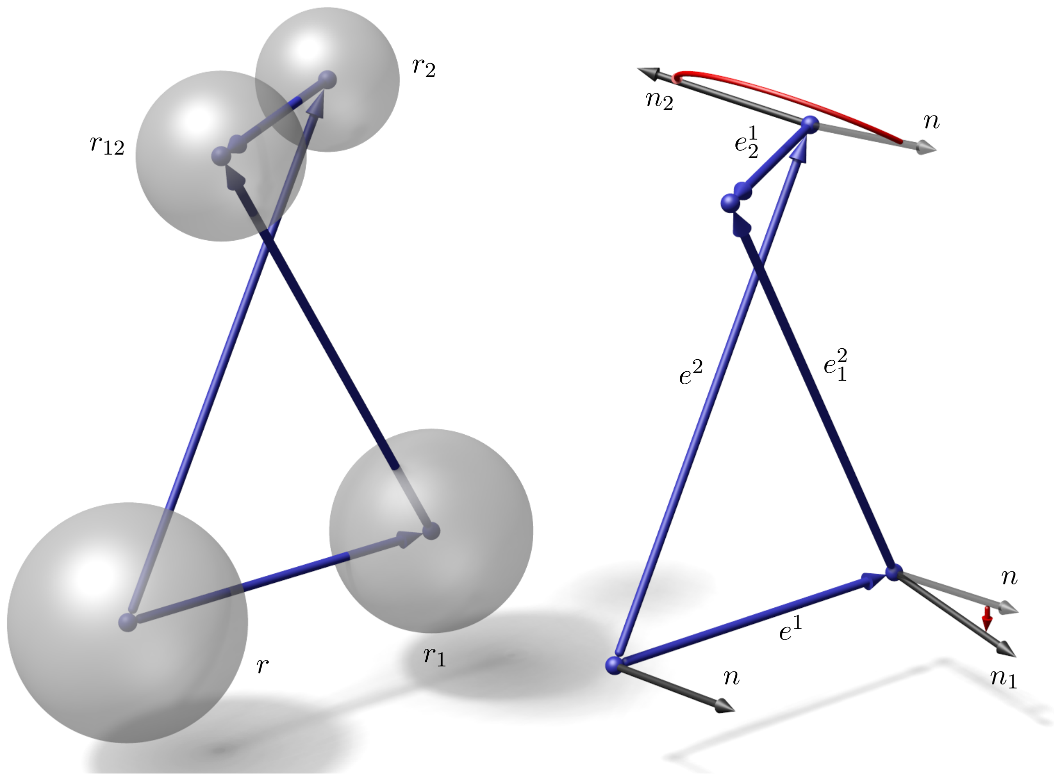

Figure 4.2: A second example of an a–bent quadrilateral.

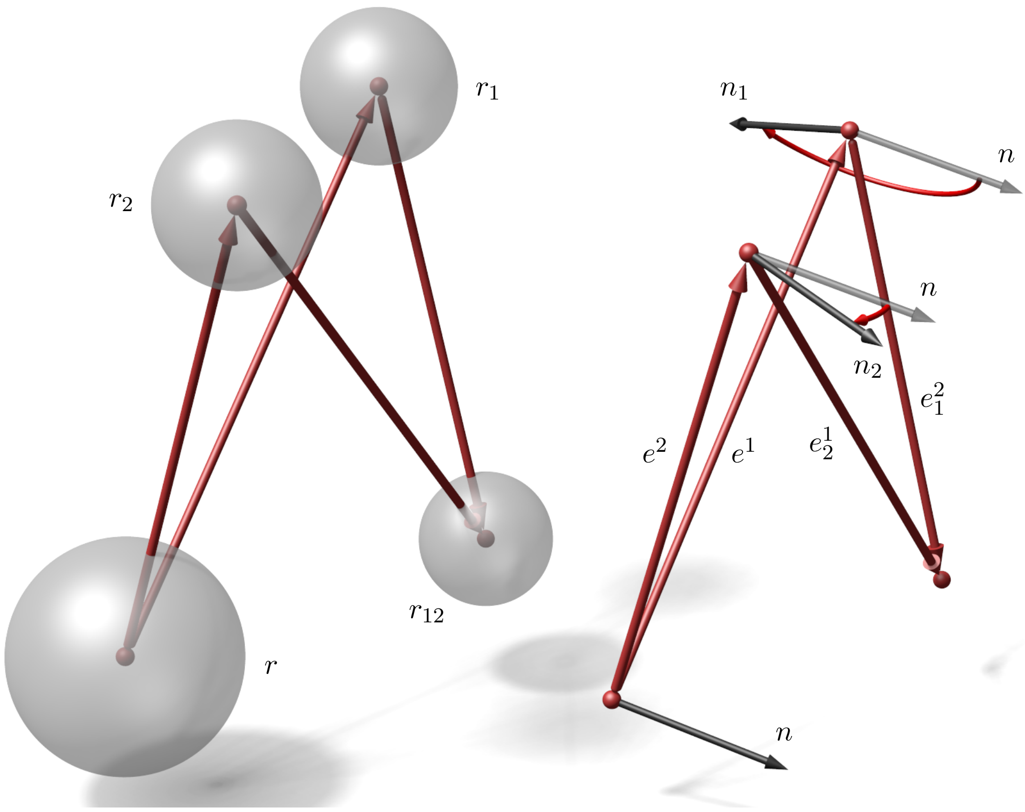

Figure 4.3: An example of a k–folded quadrilateral.

Definition 4.1.4. Let $(s, s_1, s_2, s_3, s_{12}, s_{13}, s_{23}, s_{123})$ be a cube of spheres. The cube is called *k–folded* for a $k \in \mathbb{R}$ if the s–quadrilaterals

$$(s, s_1, s_2, s_{12}), \quad (s_3, s_{13}, s_{23}, s_{123}), \quad (s, s_1, s_3, s_{13}) \quad \text{and } (s_2, s_{12}, s_{23}, s_{123})$$

are k–folded and

$$(s, s_2, s_3, s_{23}) \text{ and } (s_1, s_{12}, s_{13}, s_{123})$$

are $-k$–folded.

Remark 4.1.1. *With*

$$\delta^1 = \measuredangle(n, n_1) = \measuredangle_{v,v_1}(v_2, v_{12})$$

the dihedral angle along the edge e^1*, and of course analogously for the other edges, the equations in defs. 4.1.1 and 4.1.3 mean*

$$\frac{\sin \delta^1}{\|e^1\|} = \frac{-a}{\sqrt{(r^2 + a^2)(r_1^2 + a^2)}} \quad \textit{etc.}$$

or

$$\frac{\sin \delta^1}{\|e^1\|} = \frac{k}{\sqrt{(1 + k^2 r^2)(1 + k^2 r_1^2)}} \quad \textit{etc.}$$

respectively; the cross product notation makes sure we consider the correct sign (i.e. rotational direction) of the dihedral angle.

Remark 4.1.2. *Note that in the above definitions, the choice of sign is just a convention. Changing orientation of or cyclically relabelling a quadrilateral changes the sign, and consequently reversing one coordinate direction conserves foldedness and bentness.*

Remark 4.1.3. *Note that for $a = 0$ or $k = 0$, the definitions just mean the the quadrilateral is planar. For $k \neq 0$, being k–folded is the same as being $-\frac{1}{k}$–bent, but we keep the separate definition because we will suggestively use it for different views of the same cube.*

Remark 4.1.4. *Looking at the reformulation of k–foldedness in remark 4.1.1 we see that for vanishing sphere radii, we recover the property of equally–folded parallelogram cubes as mentioned in section 2.6, with k replacing the folding parameter σ. Our adaptation will appear for interleaved cubes just at the property for the associated family of discrete isothermic cmc surfaces did.*

4.2 S_1–Isothermic Minimal Surfaces

S_1–isothermic minimal surfaces have been introduced in [BHS06], including the associated family and a Weierstraß–type representation. Here we revisit the geometry of the elementary quadrilaterals in the associated family, as laid out in lemma 4 of [BHS06], to show they fit the definitions we proposed in chapter 4.

Recall that the minimal surface is Christoffel dual to the Gauß map, which in the s_1–isothermic case is edge–tangent to the unit sphere. The spheres sitting at the vertices of the Gauß map intersect the unit sphere orthogonally and are mutually tangent along edges, in the same point in which the edge touches the unit sphere.

We start out with an elementary quadrilateral of the Gauss map, i.e. a planar quadrilateral with edges tangent to the unit sphere. The intersection of the plane containing the quadrilateral and the unit sphere is the incircle of the quadrilateral. Let $p_0, \dots, p_3$ be the vertices of the quadrilateral, $l_i = p_{i+1} - p_i$ its edges with tangent points c_i to the sphere/incircle, and N the unit normal to the plane of the quadrilateral. For each vertex p_i, let

$$r_i = \|c_i - p_i\| = \|c_{i-1} - p_i\|$$

denote the tangent length, such that $\|l_i\| = r_i + r_{i+1}$. Assuming the sphere is centered at the origin, for each edge let l_i^φ be l_i rotated around c_i by an angle of φ; then

$$e_i^\varphi := \frac{(-1)^i}{r_i r_{i+1}} l_i^\varphi$$

are the edges of the corresponding quadrilateral in the associated family of the minimal surface. For $\varphi = 0$, it is just the dual planar quadrilateral.

Now to make our calculation, we want to introduce a common coordinate system for three consecutive edges. First recall the edge–wise ONBs introduced in the proof of [BHS06, lemma 4]:

$$v_i = \frac{l_i}{\|l_i\|}, \quad N, \quad w_i = N \times v_i.$$

In those ONBs, the edges were

$$e_i^\varphi = (-1)^i \Big(\frac{1}{r_i} + \frac{1}{r_{i+1}}\Big) v_i^\varphi,$$

where

$$v_i^\varphi = \cos\varphi v_i + \sin\varphi \cos\vartheta w_i + \sin\varphi \sin\vartheta N,$$

with ϑ the angle between N and c_i (which is the same for all edges).

Since N is the same for all edges, we only need to set up common coordinates for v_{i-1}, v_i, v_{i+1} and w_{i-1}, w_i, w_{i+1} within the orthogonal complement of N, i.e. the plane

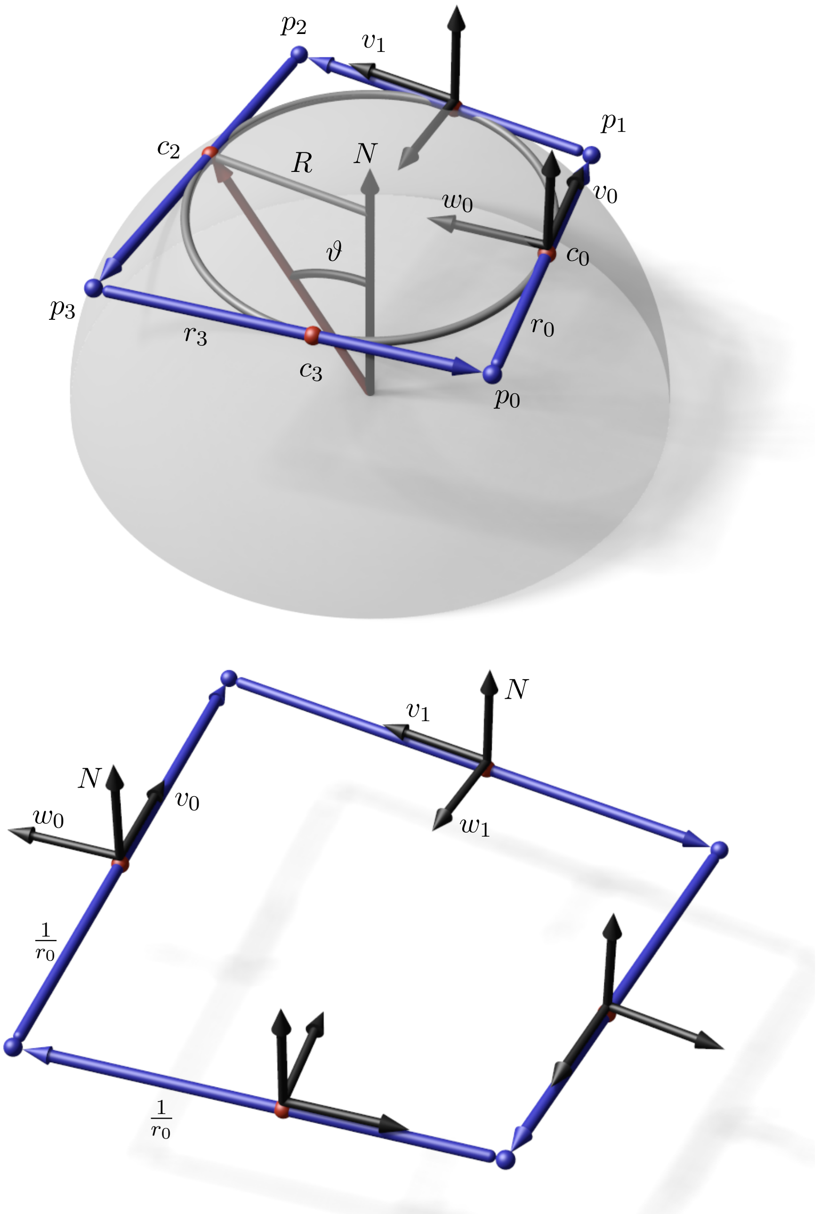

Figure 4.4: A quadrilateral of the Gauss map and its dual (lengths not to scale)

of the Gauss quadrilateral and its incircle. To that end, let $R = \sin\vartheta$ be the radius of the incircle. We set the centre of the incircle to be the origin, and

$$v_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

putting

$$c_i = \begin{pmatrix} 0 \\ -R \end{pmatrix}, \quad p_i = c_i - r_i v_i = \begin{pmatrix} -r_i \\ -R \end{pmatrix}, \quad p_{i+1} = c_i + r_{i+1} v_i = \begin{pmatrix} r_{i+1} \\ -R \end{pmatrix}.$$

Now we find c_{i-1} and c_{i+1} by mirroring c_i at p_i and p_{i+1}, respectively:

$$c_{i-1} = 2\frac{\langle c_i, p_i \rangle}{\|p_i\|^2} p_i - c_i = -2\frac{R^2}{r_i^2 + R^2} \begin{pmatrix} r_i \\ R \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix} = \frac{R}{r_i^2 + R^2} \begin{pmatrix} -2r_i R \\ r_i^2 - R^2 \end{pmatrix},$$

$$c_{i+1} = 2\frac{\langle c_i, p_{i+} \rangle}{\|p_{i+1}\|^2} p_{i+1} - c_i = 2\frac{R^2}{r_{i+1}^2 + R^2} \begin{pmatrix} r_{i+1} \\ -R \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix} = \frac{R}{r_{i+1}^2 + R^2} \begin{pmatrix} 2r_{i+1} R \\ r_{i+1}^2 - R^2 \end{pmatrix}.$$

The w_k are just the normalized negatives of the c_k, and the v_k perpendicular to the former (correctly oriented):

$$w_{i-1} = \frac{1}{r_i^2 + R^2} \begin{pmatrix} 2r_i R \\ R^2 - r_i^2 \end{pmatrix},$$

$$w_{i+1} = \frac{1}{r_{i+1}^2 + R^2} \begin{pmatrix} -2r_{i+1} R \\ R^2 - r_{i+1}^2 \end{pmatrix},$$

$$v_{i-1} = \frac{1}{r_i^2 + R^2} \begin{pmatrix} R^2 - r_i^2 \\ -2r_i R \end{pmatrix},$$

$$v_{i+1} = \frac{1}{r_{i+1}^2 + R^2} \begin{pmatrix} R^2 - r_{i+1}^2 \\ 2r_{i+1} R \end{pmatrix}.$$

We use these coordinates to prove

Proposition 4.2.1. *The elementary quadrilaterals in the associated family of an s–isothermic minimal surface, consisting of the edges*

$$e_i^\varphi := \frac{(-1)^i}{r_i r_{i+1}} l_i^\varphi$$

and spheres of radii $\frac{1}{r_i}$ *at the vertices* p_i*, are* $\sin\varphi$*–bent.*

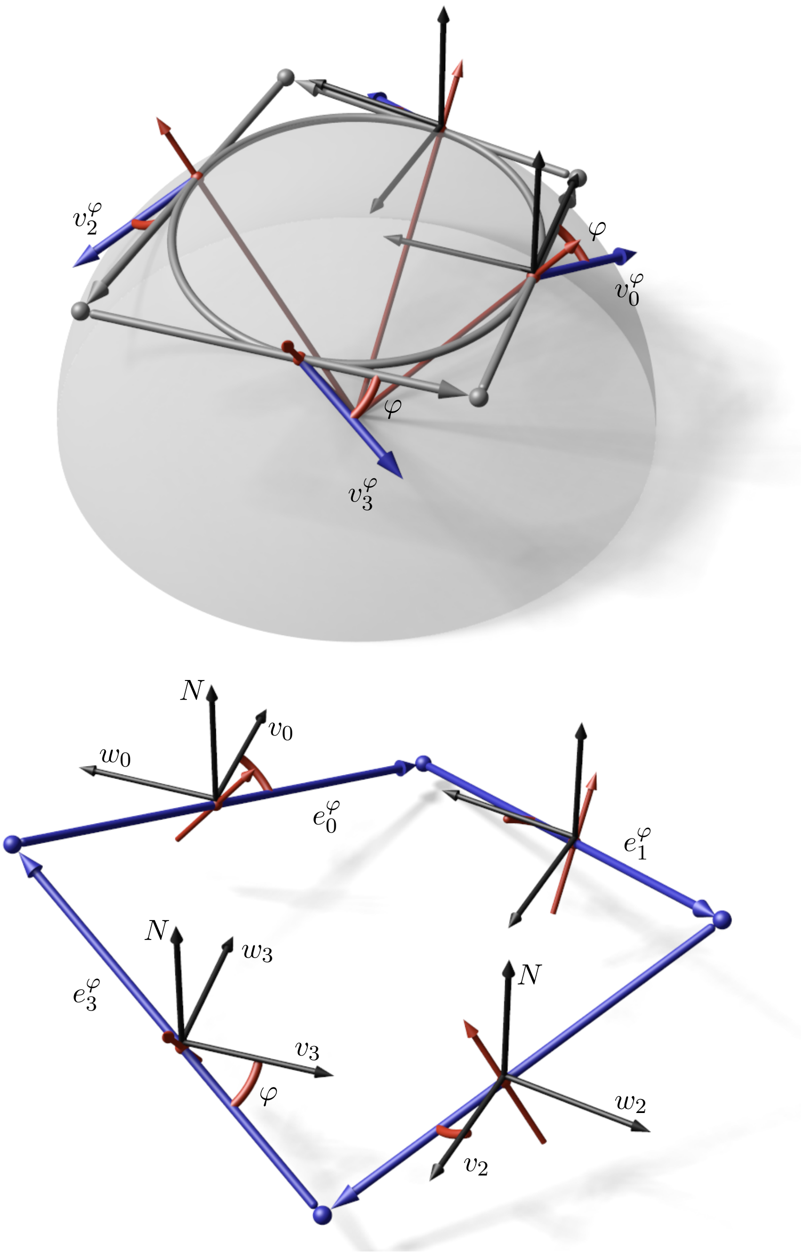

Figure 4.5: The rotated directions and edges (lengths not to scale)

Proof. We pick an edge e_i^φ, and look at it together with its adjacent edges e_{i-1}^φ and e_{i+1}^φ. In the coordinates introduced above, with the added third dimension

$$N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we get

$$v_i^\varphi = \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta \end{pmatrix},$$

$$v_{i-1}^\varphi = \frac{1}{r_i^2 + R^2} \begin{pmatrix} \cos\varphi(R^2 - r_i^2) + \sin\varphi\cos\vartheta(2r_iR) \\ \cos\varphi(-2r_iR) + \sin\varphi\cos\vartheta(R^2 - r_i^2) \\ \sin\varphi\sin\vartheta(r_i^2 + R^2) \end{pmatrix},$$

$$v_{i+1}^\varphi = \frac{1}{r_{i+1}^2 + R^2} \begin{pmatrix} \cos\varphi(R^2 - r_{i+1}^2) + \sin\varphi\cos\vartheta(-2r_{i+1}R) \\ \cos\varphi(2r_{i+1}R) + \sin\varphi\cos\vartheta(R^2 - r_{i+1}^2) \\ \sin\varphi\sin\vartheta(r_{i+1}^2 + R^2) \end{pmatrix}.$$

Now, for the corner normals n and n_1 of def. 4.1.1, we only need the direction of the edges. In the cross products, the signs of the two adjacent e_k^φ cancel; but note that the choice of signs in def. 4.1.1 assumed the opposite orientation of e_{i-1}^φ. So we calculate, on occasion invoking $\sin\vartheta = R$,

$$\begin{aligned}
n &\overset{+}{\sim} -v_i^\varphi \times v_{i-1}^\varphi \\
&\overset{+}{\sim} -\begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta \end{pmatrix} \times \begin{pmatrix} \cos\varphi(R^2 - r_i^2) + \sin\varphi\cos\vartheta(2r_iR) \\ \cos\varphi(-2r_iR) + \sin\varphi\cos\vartheta(R^2 - r_i^2) \\ \sin\varphi\sin\vartheta(r_i^2 + R^2) \end{pmatrix} \\
&= -\begin{pmatrix} \sin^2\varphi\cos\vartheta R(r_i^2 + R^2) + 2\sin\varphi\cos\varphi r_i R^2 - \sin^2\varphi\cos\vartheta R(R^2 - r_i^2) \\ -\sin\varphi\cos\varphi R(r_i^2 + R^2) + \sin\varphi\cos\varphi R(R^2 - r_i^2) + 2\sin^2\varphi\cos\vartheta r_i R^2 \\ -2\cos^2\varphi r_i R - 2\sin^2\varphi\cos^2\vartheta r_i R \end{pmatrix} \\
&= -\begin{pmatrix} \sin\varphi R(2\sin\varphi\cos\vartheta r_i^2 + 2\cos\varphi r_i R) \\ \sin\varphi R(-2\cos\varphi r_i^2 + 2\sin\varphi\cos\vartheta r_i R) \\ -2r_iR\big(\cos^2\varphi + \sin^2\varphi(1 - R^2)\big) \end{pmatrix} \\
&\overset{+}{\sim} \begin{pmatrix} -\sin\varphi(\sin\varphi\cos\vartheta r_i + \cos\varphi R) \\ \sin\varphi(\cos\varphi r_i - \sin\varphi\cos\vartheta R) \\ 1 - \sin^2\varphi R^2 \end{pmatrix} \\
&=: \widetilde{n}
\end{aligned}$$

and analogously

$$
\begin{aligned}
n_1 &\overset{\pm}{\sim} v_i^\varphi \times v_{i+1}^\varphi \\
&\overset{\pm}{\sim} \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta \end{pmatrix} \times \begin{pmatrix} \cos\varphi(R^2 - r_{i+1}^2) - \sin\varphi\cos\vartheta(2r_{i+1}R) \\ \cos\varphi(2r_{i+1}R) + \sin\varphi\cos\vartheta(R^2 - r_{i+1}^2) \\ \sin\varphi\sin\vartheta(r_{i+1}^2 + R^2) \end{pmatrix} \\
&= \begin{pmatrix} \sin^2\varphi\cos\vartheta R(r_{i+1}^2 + R^2) - 2\sin\varphi\cos\varphi r_{i+1}R^2 - \sin^2\varphi\cos\vartheta R(R^2 - r_{i+1}^2) \\ -\sin\varphi\cos\varphi R(r_{i+1}^2 + R^2) + \sin\varphi\cos\varphi R(R^2 - r_{i+1}^2) - 2\sin^2\varphi\cos\vartheta r_{i+1}R^2 \\ 2\cos^2\varphi r_{i+1}R + 2\sin^2\varphi\cos^2\vartheta r_{i+1}R \end{pmatrix} \\
&= \begin{pmatrix} \sin\varphi R(2\sin\varphi\cos\vartheta r_{i+1}^2 - 2\cos\varphi r_{i+1}R) \\ \sin\varphi R(-2\cos\varphi r_{i+1}^2 - 2\sin\varphi\cos\vartheta r_{i+1}R) \\ 2r_{i+1}R\big(\cos^2\varphi + \sin^2\varphi(1 - R^2)\big) \end{pmatrix} \\
&\overset{\pm}{\sim} \begin{pmatrix} \sin\varphi(\sin\varphi\cos\vartheta r_{i+1} - \cos\varphi R) \\ -\sin\varphi(\cos\varphi r_{i+1} + \sin\varphi\cos\vartheta R) \\ 1 - \sin^2\varphi R^2 \end{pmatrix} \\
&=: \widetilde{n}_1
\end{aligned}
$$

Their squared norms are

$$
\begin{aligned}
\|\widetilde{n}\|^2 &= \sin^2\varphi\big(\sin^2\varphi\cos^2\vartheta r_i^2 + \cos^2\varphi R^2 + 2\sin\varphi\cos\varphi\cos\vartheta r_i R \\
&\qquad + \cos^2\varphi r_i^2 + \sin^2\varphi\cos^2\vartheta R^2 - 2\sin\varphi\cos\varphi\cos\vartheta r_i R\big) \\
&\quad + (1 - \sin^2\varphi R^2)^2 \\
&= \sin^2\varphi(r_i^2 + R^2)\big(\sin^2\varphi(1 - R^2) + \cos^2\varphi\big) + (1 - \sin^2\varphi R^2)^2 \\
&= (1 - \sin^2\varphi R^2)\big(\sin^2\varphi(r_i^2 + R^2) + 1 - \sin^2\varphi R^2\big) \\
&= (1 - \sin^2\varphi R^2)(1 + \sin^2\varphi r_i^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\widetilde{n}_1\|^2 &= \sin^2\varphi\big(\sin^2\varphi\cos^2\vartheta r_{i+1}^2 + \cos^2\varphi R^2 - 2\sin\varphi\cos\varphi\cos\vartheta r_{i+1}R \\
&\qquad + \cos^2\varphi r_{i+1}^2 + \sin^2\varphi\cos^2\vartheta R^2 + 2\sin\varphi\cos\varphi\cos\vartheta r_{i+1}R\big) \\
&\quad + (1 - \sin^2\varphi R^2)^2 \\
&= (1 - \sin^2\varphi R^2)(1 + \sin^2\varphi r_{i+1}^2),
\end{aligned}
$$

and we have

$$
\widetilde{n} \times \widetilde{n}_1 = \begin{pmatrix} -\sin\varphi(\sin\varphi\cos\vartheta r_i + \cos\varphi R) \\ \sin\varphi(\cos\varphi r_i - \sin\varphi\cos\vartheta R) \\ 1 - \sin^2\varphi R^2 \end{pmatrix} \times \begin{pmatrix} \sin\varphi(\sin\varphi\cos\vartheta r_{i+1} - \cos\varphi R) \\ -\sin\varphi(\cos\varphi r_{i+1} + \sin\varphi\cos\vartheta R) \\ 1 - \sin^2\varphi R^2 \end{pmatrix}
$$

$$
\begin{aligned}
&= \begin{pmatrix} (1-\sin^2\varphi R^2)\sin\varphi(\cos\varphi r_i - \sin\varphi\cos\vartheta R + \cos\varphi r_{i+1} + \sin\varphi\cos\vartheta R) \\ (1-\sin^2\varphi R^2)\sin\varphi(\sin\varphi\cos\vartheta r_i + \cos\varphi R + \sin\varphi\cos\vartheta r_{i+1} - \cos\varphi R) \\ \sin^2\varphi\big(\sin^2\varphi\cos^2\vartheta(r_i R + r_{i+1}R) + \cos^2\varphi(r_{i+1}R + r_i R)\big) \end{pmatrix} \\
&= \sin\varphi(r_i + r_{i+1}) \begin{pmatrix} (1-\sin^2\varphi R^2)\cos\varphi \\ (1-\sin^2\varphi R^2)\sin\varphi\cos\vartheta \\ \sin\varphi R\big(\cos^2\varphi + \sin^2\varphi(1-R^2)\big) \end{pmatrix} \\
&= \sin\varphi(1-\sin^2\varphi R^2)(r_i + r_{i+1}) \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta \end{pmatrix} \\
&= \sin\varphi(1-\sin^2\varphi R^2)(r_i + r_{i+1}) v_i^\varphi.
\end{aligned}
$$

Together, we get

$$
\begin{aligned}
n \times n_1 &= \frac{1}{(1-\sin^2\varphi R^2)\sqrt{(1+\sin^2\varphi r_i^2)(1+\sin^2\varphi r_{i+1}^2)}} \widetilde{n} \times \widetilde{n}_1 \\
&= \frac{\sin\varphi}{\sqrt{(\frac{1}{r_i^2} + \sin^2\varphi)(\frac{1}{r_{i+1}^2} + \sin^2\varphi)}} \frac{r_i + r_{i+1}}{r_i r_{i+1}} v_i^\varphi \\
&= \frac{(-1)^i \sin\varphi}{\sqrt{(\frac{1}{r_i^2} + \sin^2\varphi)(\frac{1}{r_{i+1}^2} + \sin^2\varphi)}} e_i^\varphi.
\end{aligned}
$$

Recalling that def. 4.1.1 also requires the alternating sign, this proves $\sin\varphi$–bentness for the sphere quadrilateral. □

4.3 S–Conical Minimal Surfaces

Here we present the construction of an associated family for s–conical minimal surfaces that has already been our contribution to [BHKS15]. In addition, we show in proposition 4.3.13 that this construction satisfies our conditions from def. 4.1.1, and in section 4.3.3 that it is consistent with the more recent and more general construction independently developed by Wai Yeung Lam in [Lam16].

Again, the minimal surface is the Christoffel dual of its Gauss map, whose vertex spheres are orthogonal to the unit sphere. In the s–conical case, the four spheres of all elementary quadrilaterals have exactly one point in common, which for the Gauss map lies on the unit sphere and is the tangent point of the face plane to the unit sphere. We call such a discetization of the unit sphere an *s–conical Gauss map*. For details refer back to [BHKS15].

4.3.1 Construction of the Associated Family

In this section, let n always be an s–conical Gauss map with diagonal intersection angle σ and f its dual s–conical minimal surface. We always assume the scaling of f is such that the lengths of its diagonal segments — which are just the radii of the spheres centered at the vertices — are the inverse of the length of the respective segments in n.

Theorem and definition 4.3.1. Let n be an s–conical Gauss map and f its dual s–conical minimal surface on a simply connected quad–graph. Let $e = f_i - f$ be an edge of f, $g = n_i - n$ the corresponding edge of n, and N the face normal of an adjacent face. Let ψ be any angle. Then we can define

i) an *edge normal* E as the direction given by the closest point to the origin on g;

ii) the angle α formed by E and N (also cf. figure 4.6);

iii) a scaling factor

$$\lambda := \sqrt{1 + \sin^2 \psi \tan^2 \alpha};$$

iv) a rotation angle φ satisfying

$$\cos \varphi = \frac{1}{\lambda} \cos \psi, \quad \sin \varphi = \frac{1}{\lambda} \frac{\sin \psi}{\cos \alpha};$$

v) and a transformed edge e^{ψ} by rotating λe around E by φ.

Neither of these quantities depend on the choice of adjacent face. For each quadrilateral of f, its four transformed edges again close to a (in general non–planar) quadrilateral, and we get a transformed surface f^{ψ}. The family of these f^{ψ} is called the *associated family* of f.

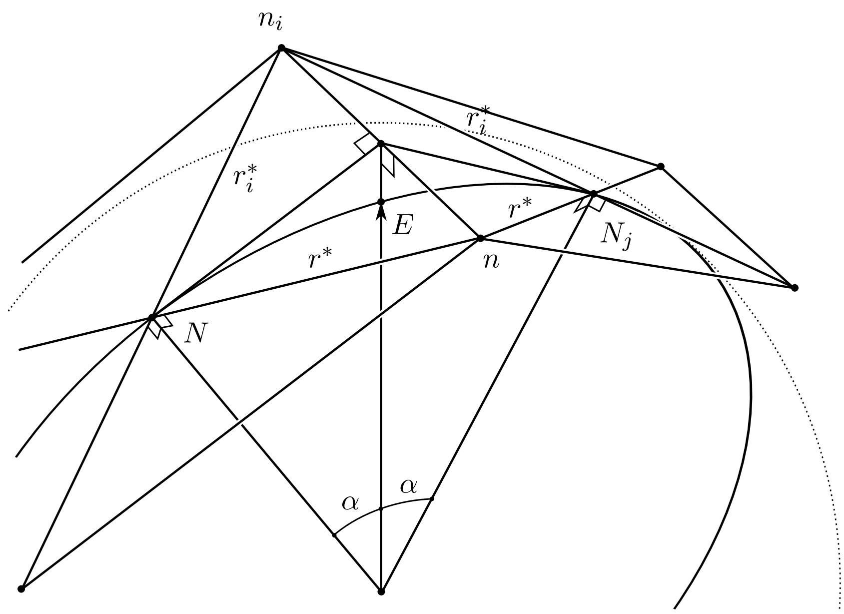

Figure 4.6: The edge normal

We collect the calculation steps needed for the proof in several lemmata.

Lemma 4.3.1. *None of the quantities E, α, λ, φ and e^ψ defined above depend on the choice of the face adjacent to g.*

Proof. The only definition directly involving this choice is the one of α, so this is the one we have to check. But the normals N, N_i of the adjacent faces are both the tangent points of planes containing g to the unit sphere. Therefore the situation is completely symmetric with respect to the plane containing g and the origin. In particular, E is the angle bisector of N and N_j, and α is just half the angle between the face normals. □

Now we look at the individual quadrilaterals.

Lemma 4.3.2. *Let P denote the plane of the adjacent face with normal N. Then the projection e_P^ψ of e^ψ into P is just e rotated around N by ψ.*

Proof. As depicted in figure 4.7, consider the spherical triangle formed by the normalizations of e, e^ψ and e_P^ψ. It has a right angle at e_P^ψ, and the angle at e is α. The length of its hypotenuse ee^ψ is φ. Denote the angle formed by e^ψ and e_P^ψ, i.e. the length of the edge $e^\psi e_P^\psi$ in the triangle, by ϑ. Identities from spherical trigonometry yield

$$\tan\vartheta = \sin\psi \tan\alpha$$

and therefore

$$\cos\vartheta = \frac{1}{\lambda};$$

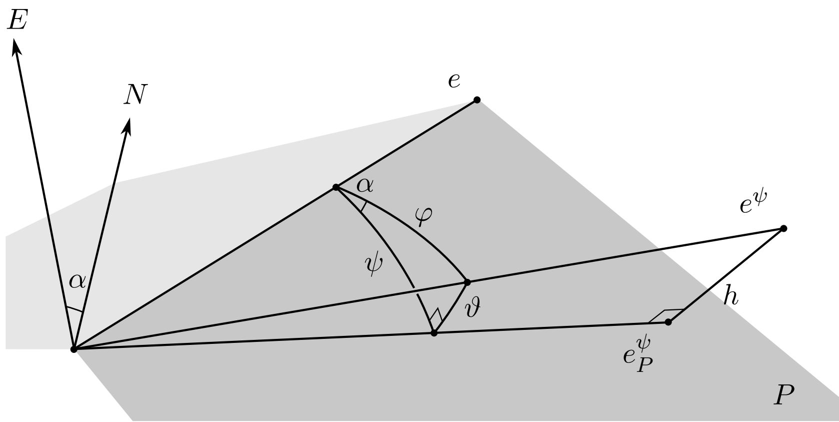

Figure 4.7: The rotated and rescaled edge and its projection into P

they further confirm that our choice of φ is the right one for the side ee_P^{ψ} to have length ψ.

Since the length of the projection satisfies $|e_P^{\psi}| = \cos\vartheta |e^{\psi}|$, our rescaling factor λ precisely conserves projected length. □

As we want the actual transformed edges e^{ψ} to form closed quadrilaterals, we still have to consider the — with respect to face planes — vertical component. We begin with its absolute value in order to avoid working with signed angles. First, we note that we know something about the lengths of the original edges:

Lemma 4.3.3. *The length of e is*

$$|e| = \cot\alpha \sin\sigma,$$

where σ is the (globally constant) diagonal intersection angle.

Proof. First consider the triangle (n, n_i, N) formed by the corresponding edge g and adjacent diagonal segments of the corresponding face in the Gauss map. The lengths of its diagonal segment are r^* and r_i^*. Its angle at N is σ (or $\pi - \sigma$, which would give the same result). The height connecting N and the edge g in this triangle is

$$\sin\sigma \frac{r^* r_i^*}{|g|} = \tan\alpha$$

since the point on g closest to N is just the edge normal E: both N and E lie in the plane orthogonal to g through the origin. So

$$\frac{|g|}{r^* r_i^*} = \cot\alpha \sin\sigma.$$

In our choice of scaling, this was precisely the length $|e|$ of the dual edge. □

Lemma 4.3.4. *The absolute height of e^ψ in N–direction is*

$$h = \sin\psi \sin\sigma.$$

Proof. As above, let ϑ be the angle formed by e^ψ and the face plane. By spherical trigonometry,

$$\sin\vartheta = \sin\varphi \sin\alpha,$$

and recalling the definition of φ and using lemma 4.3.3 we calculate

$$h = \sin\vartheta |e^\psi| = \sin\psi \sin\sigma.$$

$\square$

Lemma 4.3.5. *The sign of the N–component of e^ψ alternates around each face.*

Proof. For each edge of a quadrilateral, let p be the the point on g closest to the origin (such that $E = \frac{p}{||p||}$). Since $p - N$, g and N are orthogonal, we can identify our $\mathbb{R}^3$ with $\operatorname{Im}\mathbb{H}$ by orientation–preserving isometry such that

$$p - N = d\mathbb{i},\ e = l\mathbb{j},\ N = \mathbb{k} \text{ for some } d > 0, l \in \mathbb{R}.$$

Then, always up to positive factor, $E \overset{+}{\sim} d\mathbb{i} + \mathbb{k}$. Our transformed edge e^ψ was, up to positive rescaling, e rotated around E by φ. So in our identification, by quaternionic rotation we have

$$e^\psi \overset{+}{\sim} \left(s\cos\left(\frac{\varphi}{2}\right) + \sin\left(\frac{\varphi}{2}\right)(d\mathbb{i} + \mathbb{k})\right) l\mathbb{j} \left(\left(s\cos\left(\frac{\varphi}{2}\right) - \sin\left(\frac{\varphi}{2}\right)(d\mathbb{i} + \mathbb{k})\right)\right).$$

with some further positive factor $s > 0$. We calculate its $\mathbb{k}$–component and find it to be positively proportional to

$$l\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right).$$

From the definition of φ we see that φ is in the same quadrant as ψ, independently of $\alpha \in [0, \frac{\pi}{2})$. So the signs of $\cos\left(\frac{\varphi}{2}\right)$ and $\sin\left(\frac{\varphi}{2}\right)$ are the same for each edge of our quadrilateral.

Therefore our desired sign is just given by the orientation of $(p - N, e, N)$; since N is the same for all edges, by the orientation of $(p - N, e)$ within the face plane P. Let v_1, v_2 be the normalized diagonal directions; our quadrilateral is then $(a_1 v_1, a_2 v_2, a_3 v_1, a_4 v_2)$ for some $a_1, \ldots, a_4 \in \mathbb{R}\backslash\{0\}$ (with the diagonal intersection point N as the origin of P). The (non–rotated) dual quadrilateral is, up to global scaling, $(-\frac{1}{a_1}v_2, -\frac{1}{a_2}v_1, -\frac{1}{a_3}v_2, -\frac{1}{a_4}v_1)$. Now for each edge $g = a_n v_j - a_m v_i$ (with $i \neq j \in \{1, 2\}$), we find p to be given (within P) as

$$p = \frac{a_m a_n}{a_m^2 + a_n^2 - 2a_m a_n \langle v_i, v_j\rangle}\left((a_m - a_n\langle v_i, v_j\rangle)v_j + (a_n - a_m\langle v_i, v_j\rangle)v_i\right).$$

We calculate the desired orientation of $(p-N, e)$ (within P):

$$
\begin{aligned}
\det(p,e) &= \frac{a_m a_n}{a_m^2 + a_n^2 - 2a_m a_n \langle v_i, v_j\rangle} \\
&\quad \cdot \det\left((a_m - a_n\langle v_i, v_j\rangle)v_j + (a_n - a_m\langle v_i, v_j\rangle)v_i, \frac{1}{a_m}v_j - \frac{1}{a_n}v_i \right) \\
&= \frac{a_m a_n}{a_m^2 + a_n^2 - 2a_m a_n \langle v_i, v_j\rangle} \left(\frac{a_m - a_n\langle v_i, v_j\rangle}{a_n} + \frac{a_n - a_m\langle v_i, v_j\rangle}{a_m} \right) \det(v_i, v_j) \\
&= \det(v_i, v_j).
\end{aligned}
$$

Since (i,j) alternates between being $(1,2)$ and $(2,1)$ around our quadrilateral, the orientation indeed alternates, and so does the sign of $\langle e^{\psi}, N\rangle$. □

Proof of the claims in theorem and definition 4.3.1. By the lemmata 4.3.2 through 4.3.5, we see that for each quadrilateral, the transformed edges close to form a new (not necessarily planar) quadrilateral. By lemma 4.3.1, the edges of adjacent transformed quadrilaterals fit together just by translation, so they patch together around each vertex. By simple connectedness of the domain, they form a well–defined surface. □

4.3.2 Properties

We collect some important properties of f^{ψ} we have seen in the construction in

Corollary 4.3.6. *i) For each face of f, with P its plane, the projection of the corresponding face of f^{ψ} into P is (up to translation) just the original face rotated by ψ around the face normal N.*

ii) The diagonals of each face of f^{ψ} are parallel to their corresponding diagonal in f rotated around N by ψ and retain their length.

iii) The diagonals of each face of f^{ψ} have distance $h = \sin\psi \sin\sigma$ to each other.

In the classical theory of minimal surfaces, the metric and the Gauss map is preserved in the associated family. In the s–conical case, we can look at the spheres centered at the vertices and meeting in the diagonal intersection points and interpret their radii r as a metric parameter: a coefficient at each vertex such that lengths of diagonals are the sum of the values at the adjacent vertices. Surface normals come in three variants: vertex normals are the cone axes, their direction given by the vertices of the Gauss map n; face normals are obvious for the planar faces, and here their direction coincides with the diagonal intersection points of the Gauss map; edge normals E have been defined in theorem and definition 4.3.1 above.

From what we have seen, it is most natural to make the following definitions, giving names to the different types of normals and the metric coefficient in the associated family:

Definition 4.3.2. Let f^ψ be in the associated family of an s–conical minimal surface.

i) The *face normals* N^ψ are the directions orthogonal to both diagonals of the face. We choose their orientation w.r.t. directed diagonals to be the same as that of N w.r.t. the correspondingly directed diagonals of f.

ii) The angle bisector of two adjacent face normals is orthogonal to the common edge; we define it to be the *edge normal* E^ψ.

iii) For each face adjacent to a fixed vertex of f^ψ, consider the triangle formed by the edges containing the vertex. These triangles are tangent to a common cone with tip at the vertex; we define its axis to be the *vertex normal* direction n^ψ. Its orientation is again set to be the same as that of n w.r.t. the corresponding edges of f.

iv) On the two diagonals of each face we denote the points closest to the respective other diagonal by o^ψ and $\widetilde{o}^\psi$ (at the moment, in our notation we do not care for the attribution to the specific diagonals). By the definition of N^ψ above, $o^\psi - \widetilde{o}^\psi \parallel N^\psi$. Cf. figure 4.8.

v) On each outgoing diagonal of a fixed vertex of f^ψ, consider the point o^ψ (or $\widetilde{o}^\psi$). These points lie on a sphere centered at the vertex; we define its radius to be the *metric coefficient* r^ψ at the vertex.

The existence of the spheres allowing for the definition of the metric coefficients will be shown together with

Theorem 4.3.7. *The vertex, edge and face normals as well as the metric coefficients are preserved in the associated family:*

Let f be an s–conical minimal surface with n its dual s–conical Gauss map and f^ψ be in the associated family f.

i) $N^\psi = N$, where N is the face normal map of f and n.

ii) $E^\psi = E$, where E is the edge normal map of f and n as defined in theorem and definition 4.3.1.

iii) $n^\psi = n$

iv) $r^\psi = r$, where r are the sphere radii of f.

Proof. i) Follows directly from 4.3.6, part ii)).

ii) By definition, the edges of f^ψ are orthogonal to E, and by i)) the angle bisector of adjacent N^ψ is still E.

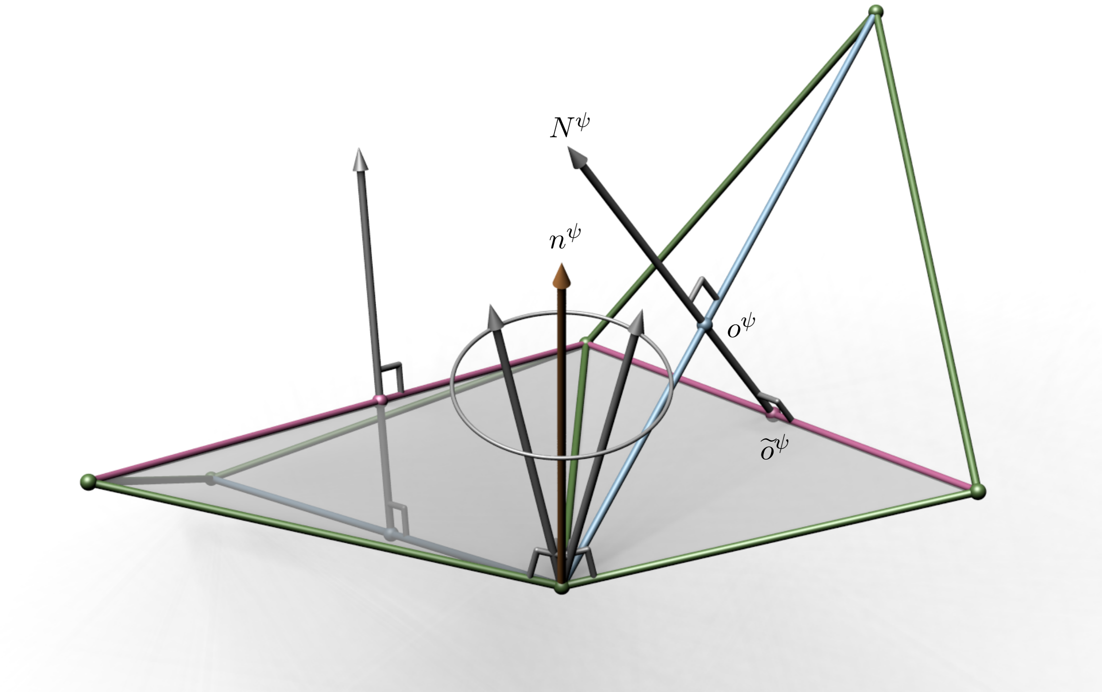

Figure 4.8: A vertex normal and two face normals in the associated family. The normals of the triangles adjacent to the front center vertex lie on a circle around the vertex normal. The face normals are orthogonal to both diagonals.

iii) Will follow from the next proposition and its corollary 4.3.9.

iv) By 4.3.6, part ii)), up to rotation around N, the diagonals projected into the plane P orthogonal to N are just the original diagonals of f. Since N is orthogonal to both diagonals of f^ψ, in the projection their closest points become the intersection point. So the distances from vertices to the closest points are the same as the distances of vertices and diagonal intersection points of f, where we knew the spheres existed.

□

Proposition 4.3.8. *Let n be an s–conical Gauss map and f its dual s–conical minimal surface. Then for each vertex f^ψ and adjacent face of f^ψ, the angle ν between the vertex normal n and the normal $\widetilde{N}$ of the plane spanned by the vertex and its two neighboring ones in the face satisfies*

$$\cos\nu = \sqrt{\cos^2\kappa + \sin^2\kappa \sin^2\psi},$$

where κ is the angle formed by n and the original face normal N.

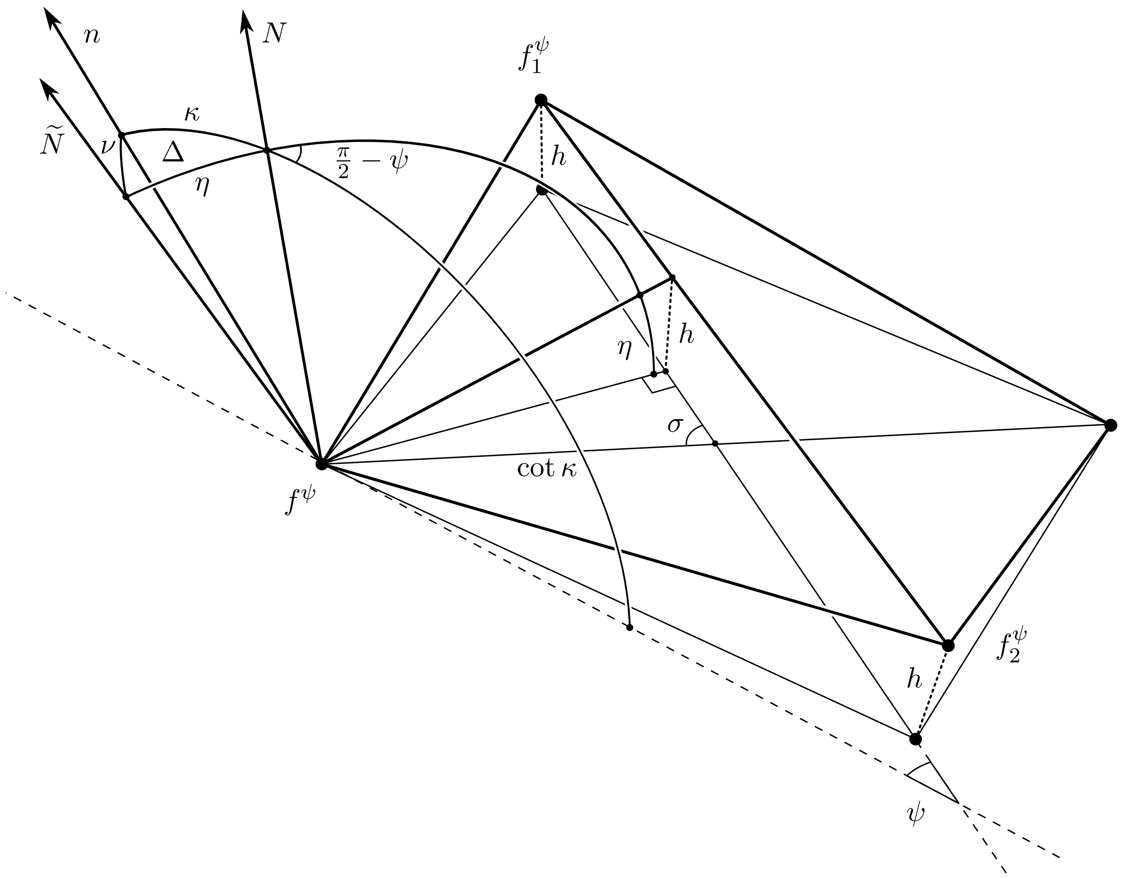

Figure 4.9: The configuration in proposition 4.3.8

Proof. For an overview of the situation, we provide figure 4.9. We look at the spherical triangle Δ formed by N, n and $\widetilde{N}$ with sides κ, the angle η formed by N and $\widetilde{N}$ and the desired ν. We can calculate ν from η and the angle at N.

Note that w.l.o.g. we consider, as depicted in figure 4.9, the case where $f_{\pm 1}^\psi$ and $f_{\pm 2}^\psi$ lie above f^ψ with respect to N. In the other case, both sides κ and η of Δ change orientation, leaving the angle at N unchanged.

Let Q be the original (planar) quadrilateral of f and P the plane parallel to Q sitting at our vertex f^ψ; by $Q^\psi = (f^\psi, f_1^\psi, f_{12}^\psi, f_2^\psi)$ we denote the (non–planar) quadrilateral of f^ψ (Since we are looking at just one fixed quadrilateral, we use this notation regardless of any actual grid directions, in particular, the same notation applies for a vertex with valency other than 4). By corollary 4.3.6, i)), the projection Q_P^ψ of Q^ψ into P is Q rotated around N by ψ.

By definition of s–conical Gauss maps, the projection of the cone axis n into P, indicated in figure 4.9 by a dashed line, is parallel to the diagonal nn_{12}, so by duality it is parallel to the diagonal $f_1 f_2$ of Q and consequently forms the angle ψ with the diagonal of Q_P^ψ not containing the vertex f^ψ. Since the plane through f^ψ spanned by the directions N and $\widetilde{N}$ is perpendicular to this diagonal, the angle at N in Δ is $\frac{\pi}{2} - \psi$.

The side η in this triangle is the angle between P and the triangle f^ψ, f_1^ψ and f_2^ψ; we can calculate it as follows: The length of the diagonal segment at f^ψ of Q_P^ψ is $\cot\kappa$, since the corresponding length in the Gauss map is $\tan\kappa$. The distance from f^ψ to the other diagonal of Q_P^ψ is therefore $\sin\sigma\cot\kappa$, so

$$h = \sin\sigma\cot\kappa\tan\eta;$$

but by lemma 4.3.4 we also know

$$h = \sin\psi\sin\sigma.$$

So we know

$$\tan\eta = \sin\psi\tan\kappa.$$

Now we can use trigonometric identities to calculate

$$\begin{aligned}
\cos\nu &= \cos\kappa\cos\eta + \sin\kappa\sin\eta\cos\left(\frac{\pi}{2} - \psi\right) \\
&= \cos\kappa\cos(\arctan(\sin\psi\tan\kappa)) + \sin\kappa\sin(\arctan(\sin\psi\tan\kappa))\sin\psi \\
&= \sqrt{\cos^2\kappa + \sin^2\kappa\sin^2\psi}.
\end{aligned}$$

□

The preceding result allows us to notice that, in a non–planar sense, the conicity property survives in the associated family:

Corollary 4.3.9. *For each vertex f^ψ of the transformed surface, the adjacent half–face triangles are still tangent to a common cone with tip at the vertex and axis direction n.*

Proof. Since f was conical, κ was constant around each vertex. Therefore ν is constant as well. □

For classical minimal surfaces, the member of the associated family rotated out of curvature line parametrization by $\psi = \frac{\pi}{2}$, called the *conjugate* minimal surface, is parametrized asymptotically. Recalling that a discretization of asymptotically parametrized surfaces is given by nets with planar vertex stars — called A–nets, see e.g. [BS08] —, we see that our discretization shares this property:

Corollary 4.3.10. *For $\psi = \frac{\pi}{2}$, the transformed surface f^ψ is a discrete asymptotic net.*

Proof. Proposition 4.3.8 yields $\cos\nu = 1$, and this precisely means planar vertex stars. $\square$

In the smooth as well as discrete isothermic and s–isothermic cases, the associated family arises just by rotating partial derivatives or edges around (edge) normals by a constant angle, so obviously each member of the family is the linear combination of the conjugate pair by the cosine and sine of that angle. In the s–conical case, we rescale and rotate by a varying angle, but in the result this is still true:

Proposition 4.3.11.

$$f^{\psi} = \cos\psi f + \sin\psi f^{\frac{\pi}{2}}.$$

Proof. We work edge–wise and use the notation of theorem and definition 4.3.1. We choose b with $|b| = |e|$ such that E, e, b form a positively oriented orthogonal basis. Then, by construction,

$$e^{\psi} = \lambda(\cos\varphi e + \sin\varphi b) = \cos\psi e + \sin\psi \frac{1}{\cos\alpha} b.$$

In particular, if we plug in $\psi = \frac{\pi}{2}$, we see that $\frac{1}{\cos\alpha} b$ is just $e^{\frac{\pi}{2}}$. $\square$

For use later on, we note that from the proof of proposition 4.3.8 we can calculate the distance of diagonals and vertices (of the same face): for each vertex of f^{ψ}, the diagonals not originating at the vertex of all adjacent faces have distance

$$d = \sqrt{h^2 + \cot^2\kappa \sin^2\sigma} = \sqrt{\sin^2\psi \sin^2\sigma + \cot^2\kappa \sin^2\sigma} = \sin\sigma \sqrt{\cot^2\kappa + \sin^2\psi} \quad (4.1)$$

to the vertex.

Another fact we note for later use is

Corollary 4.3.12. *For $\sigma = \frac{\pi}{2}$, the point on a diagonal closest to both vertices not contained in the diagonal is o^{ψ} (or $\widetilde{o}^{\psi}$, whichever lies on that particular diagonal).*

Proof. In this case, the other diagonal and the line through o^{ψ} and $\widetilde{o}^{\psi}$ with direction N span the plane perpendicular to the diagonal being considered, so in particular, the vertices in question as well as o^{ψ} and $\widetilde{o}^{\psi}$ are contained in that plane. $\square$

Now we can proceed to show how these quadrilaterals fit into our new characterizations from section 4.1. Note that the work already presented in [BHKS15] ends here, and this will be new to this publication.

Proposition 4.3.13. *The sphere quadrilaterals of f^{ψ} are $\sin\psi$–bent.*

Proof. Consider the vertices $f^\psi, f_1^\psi, f_2^\psi, f_{12}^\psi$ of a quadrilateral of f^ψ, with corresponding sphere radii r, r_1, r_2, r_{12}. By our previous findings, we can choose coordinates such that

$$f^\psi = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}, \quad f_1^\psi = \begin{pmatrix} r_1 \cos\sigma \\ r_1 \sin\sigma \\ h \end{pmatrix}, \quad f_2^\psi = \begin{pmatrix} -r_2 \cos\sigma \\ -r_2 \sin\sigma \\ h \end{pmatrix}, \quad f_{12}^\psi = \begin{pmatrix} -r_{12} \\ 0 \\ 0 \end{pmatrix},$$

where, as seen in lemma 4.3.4, $h = \sin\psi \sin\sigma$. In this model, we calculate the edges and corner normals, with notation as in def. 4.1.1:

$$e^1 = \begin{pmatrix} r_1 \cos\sigma - r \\ r_1 \sin\sigma \\ h \end{pmatrix}, \quad e_2^1 = \begin{pmatrix} r_2 \cos\sigma - r_{12} \\ r_2 \sin\sigma \\ -h \end{pmatrix},$$

$$e^2 = \begin{pmatrix} -r_2 \cos\sigma - r \\ -r_2 \sin\sigma \\ h \end{pmatrix}, \quad e_1^2 = \begin{pmatrix} -r_1 \cos\sigma - r_{12} \\ -r_1 \sin\sigma \\ -h \end{pmatrix};$$

$$n = \frac{1}{\sqrt{r^2 + \sin^2\psi}} \begin{pmatrix} \sin\psi \sin\sigma \\ -\sin\psi \cos\sigma \\ r \end{pmatrix}, \quad n_1 = \frac{1}{\sqrt{r_1^2 + \sin^2\psi}} \begin{pmatrix} 0 \\ -\sin\psi \\ r_1 \end{pmatrix},$$

$$n_2 = \frac{1}{\sqrt{r_2^2 + \sin^2\psi}} \begin{pmatrix} 0 \\ \sin\psi \\ r_2 \end{pmatrix}, \quad n_{12} = \frac{1}{\sqrt{r_{12}^2 + \sin^2\psi}} \begin{pmatrix} -\sin\psi \sin\sigma \\ \sin\psi \cos\sigma \\ r_{12} \end{pmatrix}.$$

Now we see

$$\begin{aligned} n \times n_1 &= \frac{1}{\sqrt{(r^2 + \sin^2\psi)(r_1^2 + \sin^2\psi)}} \begin{pmatrix} -r_1 \sin\psi \cos\sigma + r \sin\psi \\ -r_1 \sin\psi \sin\sigma \\ -\sin^2\psi \sin\sigma \end{pmatrix} \\ &= \frac{-\sin\psi}{\sqrt{(r^2 + \sin^2\psi)(r_1^2 + \sin^2\psi)}} e^1, \end{aligned}$$

and similarly

$$n_2 \times n_{12} = \frac{-\sin\psi}{\sqrt{(r_2^2 + \sin^2\psi)(r_{12}^2 + \sin^2\psi)}} e_2^1,$$

$$n \times n_2 = \frac{\sin\psi}{\sqrt{(r^2 + \sin^2\psi)(r_2^2 + \sin^2\psi)}} e^2,$$

$$n_1 \times n_{12} = \frac{\sin\psi}{\sqrt{(r_1^2 + \sin^2\psi)(r_{12}^2 + \sin^2\psi)}} e_1^2,$$

which gives the formulae from def. 4.1.1. □

4.3.3 Consistency With Lam's Theory

In [Lam16], Wai Yeung Lam independently introduced a broader class of discrete minimal surfaces which contains conical minimal surfaces as a special case. His work includes a construction of the associated family; here we show that our construction coincides with his.

Lam defines *A–minimal* and *C–minimal* surfaces, the former discretising asymptotically parametrised and the latter curvature line parametrised minimal surfaces. In his paper, he shows that conical minimal surfaces in the Steiner formula sense are C–minimal.

For the construction of the associated family, Lam first associates a C–minimal surface f^C to each A–minimal one f^A and vice versa, forming a conjugate pair. As in the smooth theory, the associated family f^ϑ arises as the appropriate linear combination:

$$f^\vartheta = \cos\vartheta f^A + \sin\vartheta f^C.$$

Note that we call the angle parameter ϑ instead of ψ since in our version, the angle $\psi = 0$ corresponds to the curvature line parametrisation and $\psi = \frac{\pi}{2}$ yields the asymptotic net.

To show that our associated family is the same, we first show that for an s–conical minimal surface f, the members f^0 and $f^{\frac{\pi}{2}}$ of our associated family form a conjugate pair in Lam's sense.

First, we recall Lam's definition of A–minimal surfaces.

Definition 4.3.3 ([Lam16])**.** Let f be a discrete surface with unit face normals N. For each edge $e = f_i - f$, let N and N_j be the normals of the two adjacent faces. f is called *A–minimal* if for every edge e

$$e \parallel N_j - N \quad \text{and} \quad e \perp N + N_j.$$

Lemma 4.3.14. *Let f be an s–conical minimal surface with s–conical Gauss map n. Then the conjugate surface $f^{\frac{\pi}{2}}$ in the associated family is A–minimal.*

Proof. We use the notation of theorem and definition 4.3.1.

Recall that by symmetry, e, E and $N_j - N$ are pairwise orthogonal. In proposition 4.3.11 we saw that $e^{\frac{\pi}{2}} \perp E$ and $e^{\frac{\pi}{2}} \perp e$, resulting in $e^{\frac{\pi}{2}} \parallel N_j - N$.

By the same symmetry, we had $N + N_j \parallel E$, so certainly $e^{\frac{\pi}{2}} \perp N + N_j$. □

In the next step, we have to show that Lam's construction of the conjugate C–minimal surface gives back our original surface f. Recall

Theorem and definition 4.3.4 ([Lam16])**.** Let e^A be an edge of an A–minimal surface f^A, and N, N_j the adjacent unit face normals. Then the corresponding edge of the conjugate C–minimal surface f^C is given by

$$e^C := N \times e^A = N_j \times e^A.$$

Lemma 4.3.15. *Let e be the edge of an s–conical minimal surface with adjacent face normals N, N_j. Then $N \times e^{\frac{\pi}{2}} = -e$.*

Proof. We have already seen that $e \perp e^{\frac{\pi}{2}}$, and of course $e \perp N$, so

$$N \times e^{\frac{\pi}{2}} \parallel e.$$

Since $E \perp e$ as well, N, E and $e^{\frac{\pi}{2}}$ lie in a common plane. With $E \perp e^{\frac{\pi}{2}}$, we have

$$|\sin \measuredangle(N, e^{\frac{\pi}{2}})| = |\cos \measuredangle(N, E)| = \cos \alpha,$$

so

$$|N \times e^{\frac{\pi}{2}}| = \cos \alpha |e^{\frac{\pi}{2}}| = |e|,$$

since we have already seen that for $\psi = \frac{\pi}{2}$ we have $\lambda = \frac{1}{\cos \alpha}$. (Note that in our construction we always had $\alpha \in [0, \frac{\pi}{2})$).

If we trace orientations, we see that in our construction $(E, e, e^{\frac{\pi}{2}})$ as well as $(E, e, e^{\frac{\pi}{2}})$ was positively oriented, whereas $(N, N \times e^{\frac{\pi}{2}}, e^{\frac{\pi}{2}})$ is negatively oriented. ☐

We conclude:

Theorem 4.3.16. *Let f be an s–conical minimal surface. Then*

$$(f^{\frac{\pi}{2}})^{\vartheta} = f^{\frac{\pi}{2}+\vartheta},$$

where the outer upper index on the left denotes Lam's construction.

Proof. Let $(f^{\frac{\pi}{2}})^C$ denote the C–minimal surface conjugate to the A–minimal surface $f^{\frac{\pi}{2}}$ by Lam's construction. Then by Lam's definition ([Lam16], definition 2.3.2) and lemma 4.3.15,

$$(f^{\frac{\pi}{2}})^{\vartheta} = \cos \vartheta f^{\frac{\pi}{2}} + \sin \vartheta (f^{\frac{\pi}{2}})^C = \cos \vartheta f^{\frac{\pi}{2}} - \sin \vartheta f,$$

and by proposition 4.3.11

$$f^{\frac{\pi}{2}+\vartheta} = \cos\left(\frac{\pi}{2} + \vartheta\right) f + \sin\left(\frac{\pi}{2} + \vartheta\right) f^{\frac{\pi}{2}} = -\sin \vartheta f + \cos \vartheta f^{\frac{\pi}{2}}.$$

☐

4.4 S_1–Isothermic CMC Surfaces

Here we develop a geometric construction for the associated family of the s_1–isothermic cmc surfaces introduced as "the narrow definition of s–isothermic" in [Hof10]. The construction is carried out for each cube, whose geometry was described in section 3.3.1. We will explore the properties of our construction, and will see that the elementary cubes will be αk–bent as in def. 4.1.2 for some global quantities α and k, and the *interleaved* cubes obtained by combinatorically interchanging the primal and dual for two diagonally opposite vertices will be k–folded in accordance with def. 4.1.4.

We will further conclude from the properties found in section 4.4.1 that our construction is compatible over all the cubes of a surface patch.

4.4.1 Cubewise Construction

We begin with an elementary cube of an s_1–isothermic cmc surface as described in section 3.3.1, but for this section we make some changes to our notation. The spheres and their duals will be denoted by S and S^*, and s and s^* will be their radii — the letters r will be given a different meaning shortly.

The four spheres S, S_1, S_{12}, S_2 of the elementary quadrilateral and $S^*, S_1^*, S_{12}^*, S_2^*$ of the dual quadrilateral have coaxial orthogonal circles of radius R and R^* and centers m and m^*, their planes distance $h = \|m - m^*\| \neq 0$. Recall from lemma 3.3.1 that for each of the pairs S, S^* of spheres and their dual, the radii s, s^* satisfy

$$s^* s = R^* R =: \alpha.$$

The vertices v, v^* of the quadrilateral and its dual are the centers of the spheres S, S^*. For any two neighboring spheres in a quadrilateral, the orthogonal circle intersects both spheres in their common point of contact. Let these points be called the *edge points*

$$x = S \cap S_1,\ x_2 = S_2 \cap S_{12} \text{ and } y = S \cap S_2,\ y_1 = S_1 \cap S_{12},$$

and respective starred versions for the dual quadrilateral. Let e, f etc. denote the edge touching the orthogonal circle tangentially in x, y etc., and let the edges be oriented cyclically, s.th.

$$e = v_1 - v,\ f_1 = v_{12} - v_1,\ e_2 = v_2 - v_{12},\ f = v - v_2,$$

and the same with stars for the dual. Then, wlog,

$$e \overset{+}{\sim} e^* \text{ and } f \overset{+}{\sim} -f^*.$$

The initial situation is illustrated in fig. 4.10.

Now, for twisting our cube into the associated family, we prescribe an arbitrary angle φ — which will turn out to be the angle between e and e^* — and build a new cube using the following quantities.

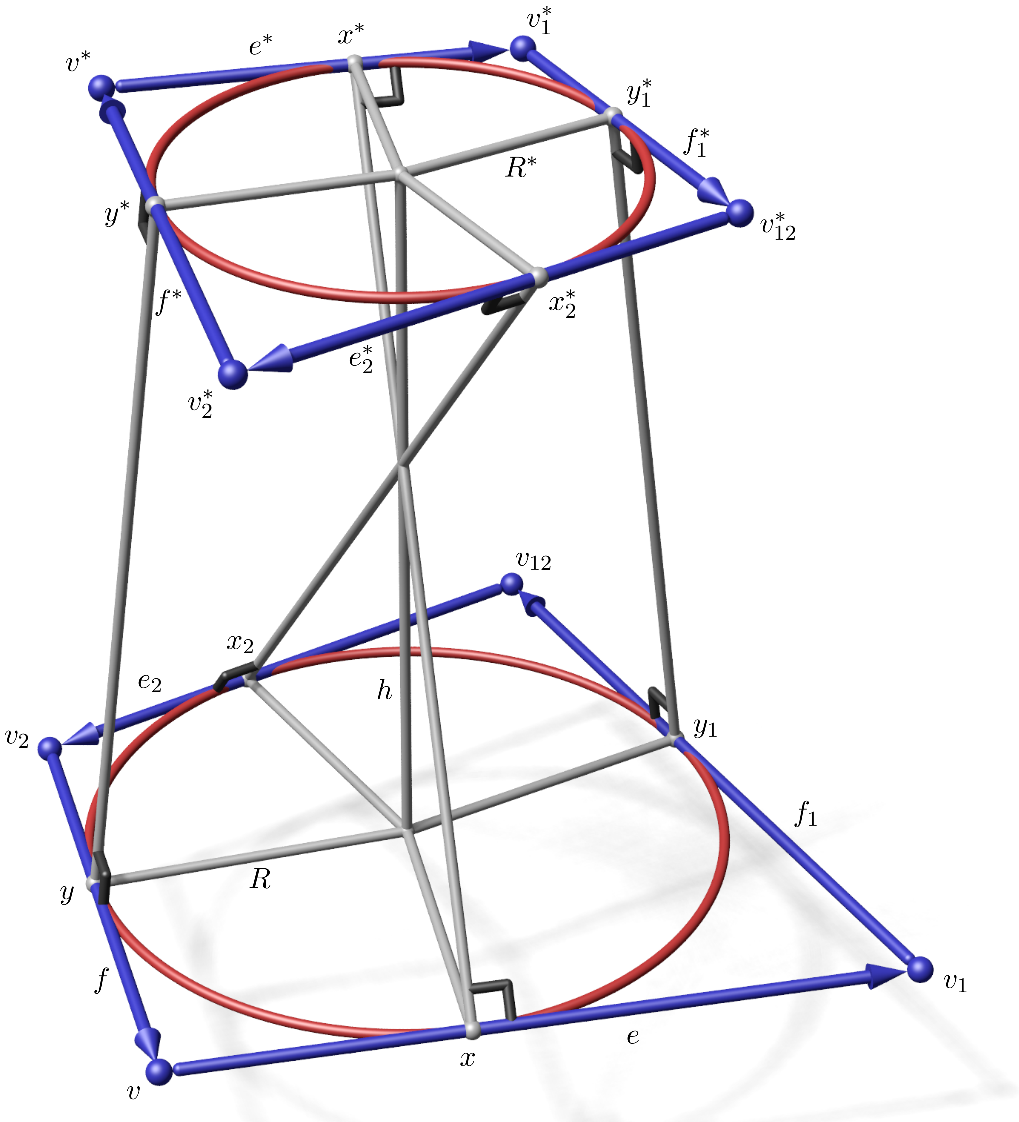

Figure 4.10: Original s_1–isothermic cube, with cyclically oriented edges.

Definition 4.4.1. Let φ be an arbitrary angle. We set

- c positive with
$$c^2 := h^2 + R^2 + R^{*2} + 2\alpha\cos\varphi,$$
-
$$k := \frac{\sin\varphi}{c},$$
- r and r^* positive with
$$r^2 := R^2 - \alpha^2 k^2 \text{ and } r^{*2} := R^{*2} - \alpha^2 k^2,$$
- η the angle with
$$\cos\eta = \frac{\alpha}{r^* r}(\cos\varphi + \alpha k^2) \text{ and } \sin\eta = \frac{\alpha}{r^* r} hk.$$

To see that we can make these definitions, we check the following facts:

Lemma 4.4.1. *For the quantities introduced above, $R^2 - \alpha^2 k^2 > 0$ and $R^{*2} - \alpha^2 k^2 > 0$.*

Proof. Since $R^2 - \alpha^2 k^2 = R^2(1 - R^{*2}\frac{\sin^2\varphi}{c^2})$, we have to show that $R^{*2}\sin^2\varphi < c^2$. Now, since $h \neq 0$,

$$\begin{aligned} & (R + R^*\cos\varphi)^2 > -h^2 \\ \iff & R^2 + 2R^*R\cos\varphi + R^{*2}\cos^2\varphi + h^2 > 0 \\ \iff & c^2 - R^{*2} + R^{*2}\cos^2\varphi > 0 \\ \iff & c^2 - R^{*2}\sin^2\varphi > 0. \end{aligned}$$

□

Lemma 4.4.2. *With the definitions above,*
$$\left(\frac{r^* r}{\alpha}\right)^2 = (\cos\varphi + \alpha k^2)^2 + (hk)^2.$$

Proof.
$$\begin{aligned} \left(\frac{r^* r}{\alpha}\right)^2 &= \frac{1}{\alpha^2}\Big((R^2 - \alpha^2 k^2)(R^{*2} - \alpha^2 k^2)\Big) \\ &= \frac{1}{\alpha^2}\Big(\alpha^2 - (R^2 + R^{*2})\alpha^2 k^2 + \alpha^4 k^4\Big) \\ &= 1 - k^2(c^2 - h^2 - 2\alpha\cos\varphi) + \alpha^2 k^4 \\ &= \cos^2\varphi + \sin^2\varphi + 2\alpha k^2\cos\varphi - c^2k^2 + h^2k^2 + \alpha^2 k^4 \\ &= (\cos^2\varphi + 2\alpha k^2\cos\varphi + \alpha^2 k^4) + h^2k^2 + \sin^2\varphi - c^2k^2 \\ &= (\cos\varphi + \alpha k^2)^2 + h^2k^2 + c^2k^2 - c^2k^2. \end{aligned}$$

□

Corollary 4.4.3. *η above is well–defined.*

Proof. The preceding lemma means that

$$\frac{\alpha}{r^*r}(\cos\varphi + \alpha k^2) \text{ and } \frac{\alpha}{r^*r}hk$$

define a point on the unit circle. □

Now we construct our new, twisted cube: Circle centers remain the same, but the circles get new radii r and r^*. Additionally, the edge points x^*, x_2^*, y^* and y_1, now on the smaller circle, get rotated around the common axis by η. All edge directions are adjusted such that they are still perpendicular to the connection of their edge point to the circle center and to the connection of the edge point and its dual. Edge segment lengths, i.e. distances from edge point to vertex, are preserved. The construction is illustrated in fig. 4.11.

Lemma 4.4.4. *For $\varphi = 0$, the cube is the same as the original.*

Proof. For $\varphi = 0$, $k = 0$ as well, so $r = R$ and $r^* = R^*$. It follows that $\cos\eta = \cos\varphi = 1$ and $\sin\eta = 0$. □

So our construction continuously deforms the original cube, and for brevity we will often suppress the dependence on φ in our notation.

For our calculations, for each pair of edge point and its dual we will introduce appropriate coordinates. The circle center and dual circle center will always sit at $m = 0$ and $m^* = (0, 0, h)$. For x (and the same for x_2), in the original $\varphi = 0$ case we want to have $x = (R, 0, 0)$ and $x^* = (-R^*, 0, h)$. For the twisted cube, this becomes

$$x(\varphi) = (r, 0, 0) \text{ and } x^*(\varphi) = (-r^* \cos\eta, -r^* \sin\eta, h).$$

For the y points, similarly we originally have $y = (R, 0, 0)$ and $y^* = (R^*, 0, h)$, and

$$y(\varphi) = (r, 0, 0) \text{ and } y^*(\varphi) = (r^* \cos\eta, r^* \sin\eta, h).$$

Lemma 4.4.5.

$$\|x(\varphi) - x^*(\varphi)\| = \|x_2(\varphi) - x_2^*(\varphi)\| = c.$$

Proof. The argument works for both x and x_2. In our coordinates,

$$\begin{aligned}
\|x(\varphi) - x^*(\varphi)\|^2 &= h^2 + (r + r^* \cos\eta)^2 + (r^* \sin\eta)^2 \\
&= h^2 + r^2 + r^{*2} + 2r^*r\cos\eta \\
&= h^2 + R^2 + R^{*2} - 2\alpha^2 k^2 + 2\alpha(\cos\varphi + \alpha k^2) \\
&= h^2 + R^2 + R^{*2} + 2\alpha\cos\varphi \\
&= c^2.
\end{aligned}$$

□

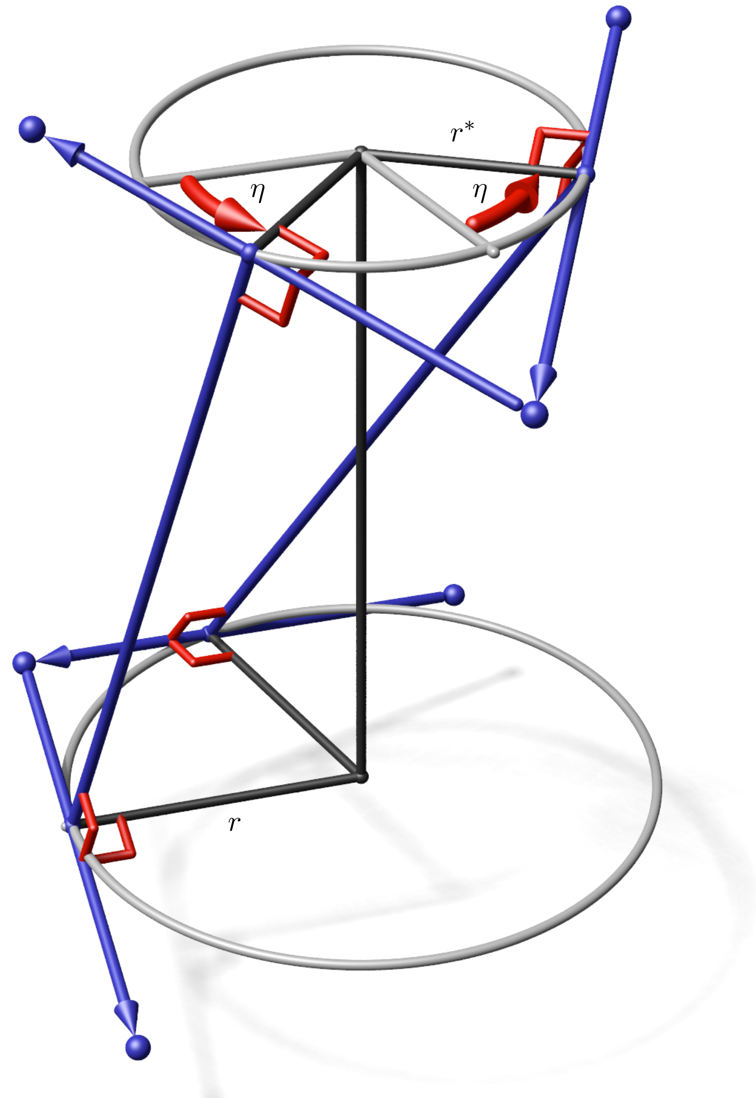

Figure 4.11: Construction of two of the new edge pairs.

In the light of this lemma, we will denote the distances of the other edge points by

$$d := \|y^* - y\|, \quad d_1 = \|y_1^* - y_1\|.$$

By symmetry, $d_1 = d$.

In the following, we will call the normalized edge directions $E = \frac{e}{\|e\|}$ and $F = \frac{f}{\|f\|}$ etc. Note that in our labelling

$$E \parallel (m - x) \times (x^* - x), \quad F \parallel (m - y) \times (y^* - y)$$

and

$$E^* \parallel (m^* - x^*) \times (x - x^*), \quad F^* \parallel (m^* - y^*) \times (y - y^*).$$

Lemma 4.4.6. *In the adapted coordinates for each edge pair,*

$$E = \frac{1}{hR}(0, rh, r^* r \sin\eta), \quad E^* = \frac{1}{hR^*}(-hr^* \sin\eta, hr^* \cos\eta, -r^* r \sin\eta)$$

and

$$F = \frac{1}{hR}(0, rh, -r^* r \sin\eta), \quad F^* = \frac{1}{hR^*}(hr^* \sin\eta, -hr^* \cos\eta, r^* r \sin\eta).$$

Proof. We just calculate the cross products above and normalize:

$$(m - x) \times (x^* - x) = \begin{pmatrix} -r \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -r^* \cos\eta - r \\ -r^* \sin\eta \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ rh \\ r^* r \sin\eta \end{pmatrix},$$

$$(m^* - x^*) \times (x - x^*) = \begin{pmatrix} r^* \cos\eta \\ r^* \sin\eta \\ 0 \end{pmatrix} \times \begin{pmatrix} r^* \cos\eta + r \\ r^* \sin\eta \\ -h \end{pmatrix} = \begin{pmatrix} -hr^* \sin\eta \\ hr^* \cos\eta \\ -r^* r \sin\eta \end{pmatrix},$$

$$(m - y) \times (y^* - y) = \begin{pmatrix} -r \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} r^* \cos\eta - r \\ r^* \sin\eta \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ rh \\ -r^* r \sin\eta \end{pmatrix},$$

$$(m^* - y^*) \times (y - y^*) = \begin{pmatrix} -r^* \cos\eta \\ -r^* \sin\eta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r^* \cos\eta + r \\ -r^* \sin\eta \\ -h \end{pmatrix} = \begin{pmatrix} hr^* \sin\eta \\ -hr^* \cos\eta \\ r^* r \sin\eta \end{pmatrix}.$$

The squared norms are

$$r^2 h^2 + (r^* r)^2 \sin^2\eta = h^2 r^2 + \alpha^2 k^2 h^2 = h^2 (r^2 + \alpha^2 k^2) = h^2 R^2,$$

$$r^{*2} h^2 + (r^* r)^2 \sin^2\eta = h^2 R^{*2},$$

and the same for F and F^*. $\square$

Lemma 4.4.7. *For our normalized edges,*

$$\begin{aligned}\langle E, E^* \rangle &= \cos\varphi, \\ E \times E^* &= k(x^* - x); \\ \langle F, F^* \rangle &= -\frac{r^* r}{\alpha}\cos\eta - \alpha k^2, \\ F \times F^* &= -k(y^* - y).\end{aligned}$$

Proof.

$$\begin{aligned}\langle E, E^* \rangle &= \frac{1}{h^2\alpha}\left(h^2 r^* r\cos\eta - (r^* r)^2\sin^2\eta\right) \\ &= \frac{1}{h^2\alpha}\left(h^2\alpha(\cos\varphi + \alpha k^2) - \alpha^2 k^2 h^2\right) \\ &= \cos\varphi,\end{aligned}$$

$$\begin{aligned}E \times E^* &= \frac{1}{h^2\alpha}\begin{pmatrix} 0 \\ rh \\ r^* r\sin\eta \end{pmatrix} \times \begin{pmatrix} -hr^*\sin\eta \\ hr^*\cos\eta \\ -r^* r\sin\eta \end{pmatrix} \\ &= \frac{1}{h^2\alpha}\begin{pmatrix} -hr^* r\sin\eta(r + r^*\cos\eta) \\ -hr^{*2} r\sin^2\eta \\ h^2 r^* r\sin\eta \end{pmatrix} \\ &= \frac{r^* r\sin\eta}{h\alpha}(x^* - x) \\ &= k(x^* - x);\end{aligned}$$

$$\begin{aligned}\langle F, F^* \rangle &= \frac{1}{h^2\alpha}\left(-h^2 r^* r\cos\eta - (r^* r)^2\sin^2\eta\right) \\ &= \frac{1}{h^2\alpha}\left(-h^2 r^* r\cos\eta - \alpha^2 k^2 h^2\right) \\ &= -\frac{r^* r}{\alpha}\cos\eta - \alpha k^2,\end{aligned}$$

$$\begin{aligned}F \times F^* &= \frac{1}{h^2\alpha}\begin{pmatrix} 0 \\ rh \\ -r^* r\sin\eta \end{pmatrix} \times \begin{pmatrix} hr^*\sin\eta \\ -hr^*\cos\eta \\ r^* r\sin\eta \end{pmatrix} \\ &= \frac{1}{h^2\alpha}\begin{pmatrix} hr^* r\sin\eta(r - r^*\cos\eta) \\ -hr^{*2} r\sin^2\eta \\ -h^2 r^* r\sin\eta \end{pmatrix} \\ &= -\frac{r^* r\sin\eta}{h\alpha}(y^* - y) \\ &= -k(y^* - y).\end{aligned}$$

$\square$

Corollary 4.4.8. *E^* is E rotated around $x^* - x$ by φ. F^* is F rotated around $y^* - y$ by ψ, where*

$$\cos\psi = -\frac{r^* r}{\alpha}\cos\eta - \alpha k^2,$$
$$\sin\psi = -kd.$$

Lemma 4.4.9. *The angles δ and δ^* by which the edges and dual edges get rotated out of the circle planes have the same absolute value for all edges of the quad and for the dual quad, respectively. Their sign alternates around the quads. In particular,*

$$|\sin\delta| = |R^* k| \text{ and } |\sin\delta^*| = |Rk|,$$

and

$$\cos\delta = \frac{r}{R} \text{ and } \cos\delta^* = \frac{r^*}{R^*}.$$

Proof. We already know the normalized edge directions in coordinates, and the sine of the respective angle is just the z–component. Plugging in the definition of $\sin\eta$ gives the desired formulae. Note that since $h \neq 0$, the rotated edges are never perpendicular to the circle planes, so the sine alone is sufficient to determine the angle (w.r.t a given orientation). $\square$

The angles in the corollaries above are depicted in fig. 4.12.

Corollary 4.4.10. *The rotated edges form closed (non–planar) quadrilaterals.*

Proof. The $\cos\delta$–formulae above together with the fact that the rotated edges are still perpendicular to the $x - m$ etc. show that the projections of the rotated edges into the circle planes are rescaled versions of the original quads (up to rotation by η within the plane for the dual). The alternating sign of the z–component guarantees that also the z–component of the new vertices is well–defined. $\square$

4.4.2 Properties

Lemma 4.4.11. *The vertical edge lengths $l := \|v - v^*\|$ are preserved under the rotation.*

Proof. The e–edges are perpendicular to the edge point connections $x - x^*$ with length c, so the vertex distance is

$$\begin{aligned}
||v - v^*||^2 &= c^2 + s^2 + s^{*2} - 2s^* s\cos\varphi \\
&= h^2 + R^2 + R^{*2} + 2R^* R\cos\varphi + s^2 + s^{*2} - 2s^* s\cos\varphi \\
&= h^2 + R^2 + R^{*2} + 2\alpha\cos\varphi + s^2 + s^{*2} - 2\alpha\cos\varphi \\
&= h^2 + R^2 + R^{*2} + s^2 + s^{*2}.
\end{aligned}$$

$\square$

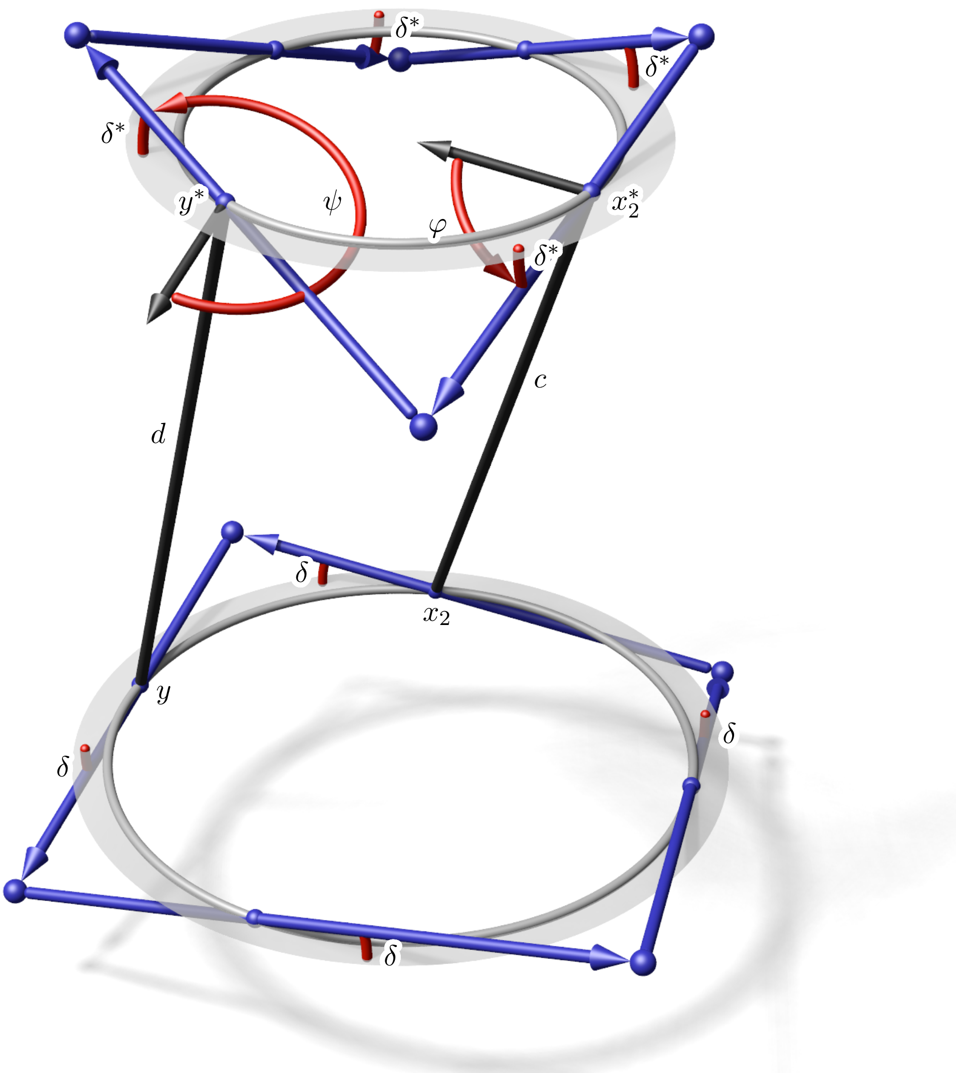

Figure 4.12: Angles φ and ψ in the edge pairs e_2, e_2^* and f, f^*; and angles δ and δ^*.

Now let ψ denote the angle formed by F and F^*. Note that for $\eta = \varphi = 0$, we have $\psi = \pi$.

Lemma 4.4.12. *The angle φ satisfies*

$$\cos\varphi = \frac{r^* r}{\alpha}\cos\eta - \alpha k^2.$$

Proof. We just complete the calculation from lemma 4.4.7, using the normalized edge directions from lemma 4.4.6 and the definition of η:

$$\begin{aligned}\cos\varphi = \langle E, E^*\rangle &= \frac{1}{h^2\alpha}(h^2 r^* r\cos\eta - (r^* r)^2\sin^2\eta)\\ &= \frac{r^* r}{\alpha}\cos\eta - \frac{(r^* r)^2}{h^2\alpha}\left(\frac{\alpha k h}{r^* r}\right)^2\\ &= \frac{r^* r}{\alpha}\cos\eta - \alpha k^2.\end{aligned}$$

□

Corollary 4.4.13. *The angles between edges and their duals for the two types of edges are related by*

$$\cos\varphi + \cos\psi = -2\alpha k^2.$$

Lemma 4.4.14. *For ψ, introduced in corollary 4.4.8, we have:*

$$\cos\eta = -\frac{\alpha}{r^* r}(\cos\psi + \alpha k^2),$$

$$d^2 = h^2 + R^2 + R^{*2} + 2\alpha\cos\psi.$$

Proof. We recall

$$\cos\psi = -\frac{r^* r}{\alpha}\cos\eta - \alpha k^2;$$

this immediately gives the first equation.

In our adapted coordinates, we had

$$y^* - y = \begin{pmatrix} r^*\cos\eta - r \\ r^*\sin\eta \\ h \end{pmatrix},$$

giving us

$$\begin{aligned}d^2 = \|y^* - y\|^2 &= r^{*2}\cos^2\eta + r^2 - 2r^* r\cos\eta + r^{*2}\sin^2\eta + h^2\\ &= h^* + r^2 + r^{*2} + 2\alpha(\cos\psi + \alpha k^2)\\ &= h^2 + R^2 - \alpha^2 k^2 + R^{*2} - \alpha^2 k^2 + 2\alpha\cos\psi + 2\alpha^2 k^2\\ &= h^2 + R^2 + R^{*2} + 2\alpha\cos\psi.\end{aligned}$$

□

Remark 4.4.15. *Suppose we define a new cube by cyclically relabelling the spheres/vertices:*

$$\widetilde{S} = S_1,\ \widetilde{S}_1 = S_{12},\ \widetilde{S}_2 = S,\ \widetilde{S}_{12} = S_2,$$

and the same for the duals etc. The angle formed by the new edges

$$\widetilde{e} = \widetilde{v}_1 - \widetilde{v} = v_{12} - v_1 = f_1$$

and

$$\widetilde{e}^* = \widetilde{v}_1^* - \widetilde{v}^* = v_{12}^* - v_1^* = f_1^*$$

becomes

$$\widetilde{\varphi} = \psi,$$

and

$$\widetilde{\psi} = \varphi$$

vice versa, as well as $\widetilde{c} = d$ and $\widetilde{d} = c$. By our construction and the previous calculations, if we set $\varphi = \pi$ in the original cube, we get $k = 0$, $\eta = \pi$ and consequently $\psi = 0$, so $\widetilde{e} \parallel \widetilde{e}^*$ and $\widetilde{f} \parallel -\widetilde{f}^*$, i.e. the $\widetilde{S}$ for $\widetilde{\varphi} = 0$ form an untwisted s_1–isothermic cmc cube by the considerations in section 3.3.1. So the coordinate directions in our cube are interchangeable, with the caveat that

$$\widetilde{k} = \frac{\sin\widetilde{\varphi}}{\widetilde{c}} = \frac{\sin\psi}{d} = -k.$$

For a visualization, see fig. 4.13.

Lemma 4.4.16. *The product cd of edge distances is constant in the associated family.*

Proof. In previous calculations we have already seen that

$$c^2 = h^2 + r^2 + r^{*2} + 2r^*r\cos\eta \quad \text{and} \quad d^2 = h^2 + r^2 + r^{*2} - 2r^*r\cos\eta.$$

We calculate $(cd)^2$:

$$\begin{aligned}
(cd)^2 &= (h^2 + r^2 + r^{*2} + 2r^*r\cos\eta)(h^2 + r^2 + r^{*2} - 2r^*r\cos\eta)\\
&= (h^2 + r^2 + r^{*2})^2 - 4(r^*r)^2\cos^2\eta\\
&= (h^2 + R^2 + R^{*2} - 2\alpha^2k^2)^2 - 4\alpha^2(\cos\varphi + \alpha k^2)^2\\
&= (h^2 + R^2 + R^{*2})^2 - 4\alpha^2k^2(h^2 + R^2 + R^{*2}) + 4\alpha^4k^4 - 4\alpha^4k^4 - 4\alpha^2\cos^2\varphi - 8\alpha^3k^2\cos\varphi\\
&= (h^2 + R^2 + R^{*2})^2 - 4\alpha^2k^2(h^2 + R^2 + R^{*2} + 2\alpha\cos\varphi) - 4\alpha^2\cos^2\varphi\\
&= (h^2 + R^2 + R^{*2})^2 - 4\alpha^2k^2c^2 - 4\alpha^2\cos^2\varphi\\
&= (h^2 + R^2 + R^{*2})^2 - 4\alpha^2(\sin^2\varphi + \cos^2\varphi)\\
&= (h^2 + R^2 + R^{*2})^2 - 4\alpha^2.
\end{aligned}$$

□

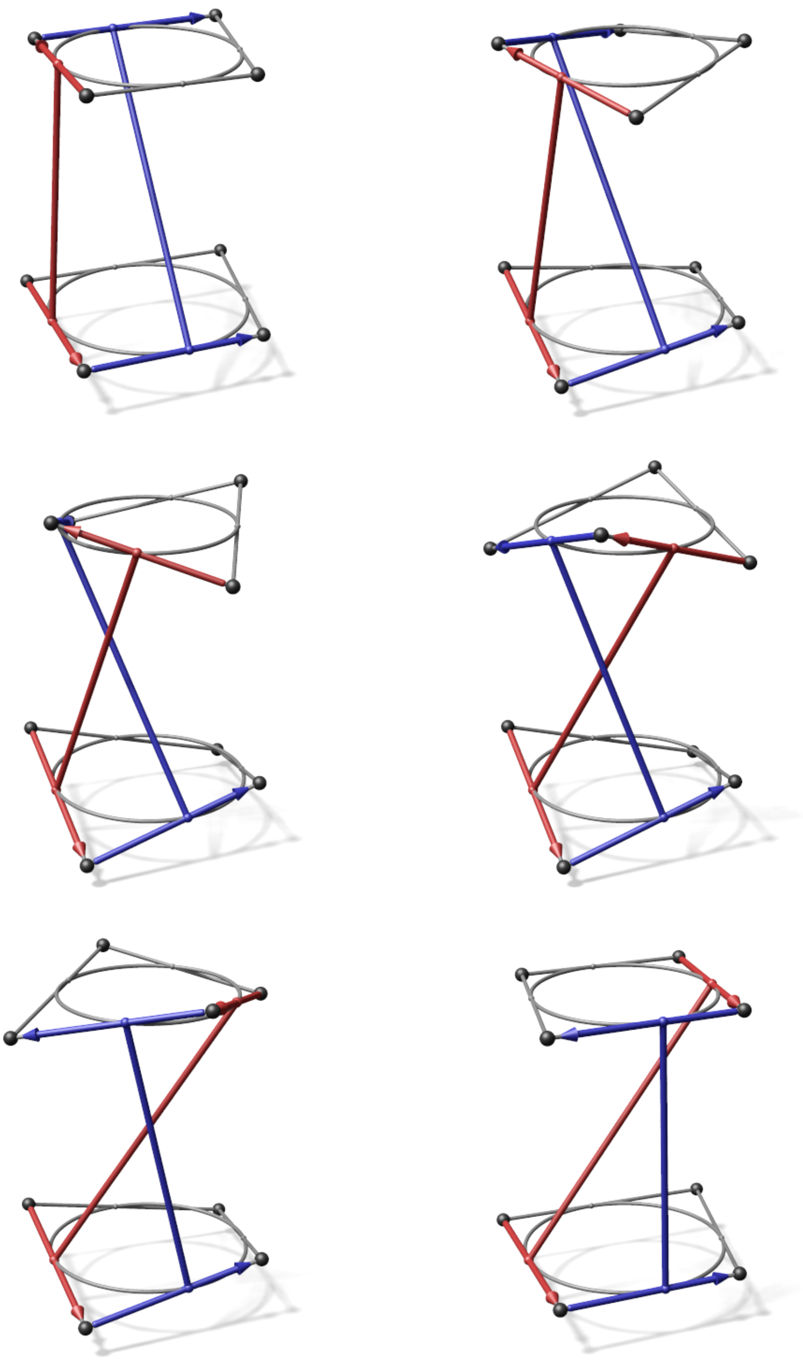

Figure 4.13: Interchanging of the two coordinate directions as mentioned in remark 4.4.15

Lemma 4.4.17. *For the angles φ and ψ, the product of cosines is*

$$\cos\varphi\cos\psi = k^2(h^2 + R^2 + R^{*2}) - 1.$$

Proof. We use the expressions for $\cos\varphi$ and $\cos\psi$ from lemma 4.4.12 and corollary 4.4.8:

$$\begin{aligned}
\cos\varphi\cos\psi &= \left(-\alpha k^2 + \frac{r^*r}{\alpha}\cos\eta\right)\left(-\alpha k^2 - \frac{r^*r}{\alpha}\cos\eta\right)\\
&= \alpha^2k^4 - \frac{(r^*r)^2}{\alpha^2}\cos^2\eta\\
&= \alpha^2k^4 - \frac{(r^*r)^2}{\alpha^2}(1-\sin^2\eta)\\
&= \alpha^2k^4 - \frac{(r^*r)^2}{\alpha^2} + k^2h^2\\
&= \alpha^2k^4 - \frac{1}{\alpha^2}(R^2-\alpha^2k^2)(R^{*2}-\alpha^2k^2) + k^2h^2\\
&= \alpha^2k^4 - \frac{1}{\alpha^2}(\alpha^2+\alpha^4k^4-\alpha^2k^2(R^2+R^{*2})) + k^2h^2\\
&= -1 + k^2(R^2+R^{*2}) + k^2h^2.
\end{aligned}$$

□

Now we want to work towards proving that the cubes constructed in this way, when coming from an s_1–isothermic cmc surface, again fit together to form a new surface. For this, we look at the dihedral angle of the planes spanned by a vertex normal and one of two adjacent edges, respectively. First, we choose a vertex, w.l.o.g. v.

Definition 4.4.2. We will denote the vertex unit normal at v by

$$N := \frac{v^*-v}{\|v^*-v\|} = \frac{1}{l}(v^*-v).$$

The outgoing edges at v are e and $-f$. We give names to the angles showing up in our calculations:

Definition 4.4.3. We denote the angles formed by the vertex normal and adjacent edges by

$$\sigma := \measuredangle(N,e) \quad \text{and} \quad \tau := \measuredangle(N,-f),$$

and the dihedral angle we want to calculate by

$$\xi := \measuredangle_{v,v^*}(v_1,v_2) = \measuredangle(\mathrm{span}(N,e),\mathrm{span}(N,-f)) = \measuredangle(E\times N, -F\times N).$$

For an illustration, cf. fig. 4.14.

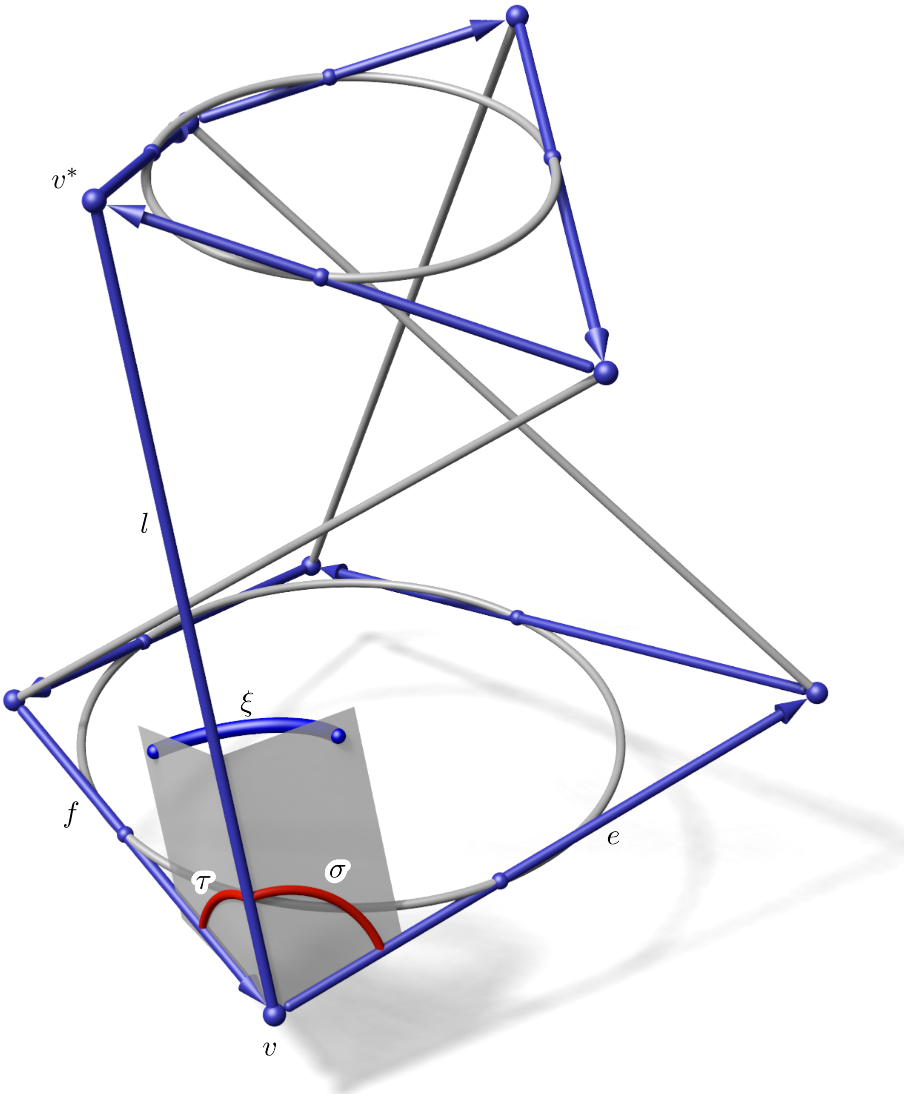

Figure 4.14: Length of and angles at a vertical edge, as they occur in lemma 4.4.11 and def. 4.4.3.

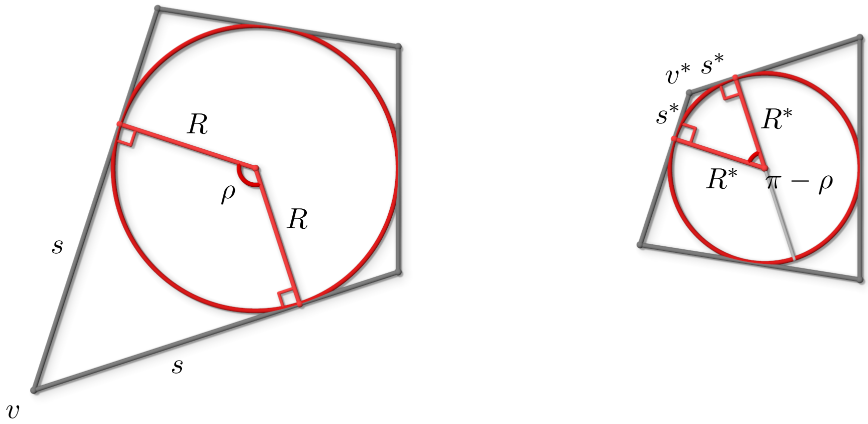

Figure 4.15: The kites at v and v^*, untwisted

Since E and F were the normalized edges, we have

$$\cos\xi = \frac{1}{\|E \times N\| \, \|-F \times N\|} \langle E \times N, -F \times N \rangle = \frac{1}{\sin\sigma \sin\tau} \langle E \times N, -F \times N \rangle,$$

with Lagrange's identity for the cross product and the fact that N was normalized this becomes

$$\cos\xi = \frac{1}{\sin\sigma \sin\tau} \left(\langle E, -F \rangle - \langle E, N \rangle \langle -F, N \rangle \right).$$

Now we collect the calculations needed for the individual terms in some lemmata.

First, we look at $\langle E, -F \rangle$. For that, we look closer at the kite at v, and along the way note some properties of the dual kite at v^* as well. We denote the central angle of the orthogonal kite at v by

$$\rho := \measuredangle(x - m, y - m).$$

From lemma 3.3.1 we recall (see also fig. 4.15)

$$R^2(1 - \cos\rho) = s^2(1 + \cos\rho) \quad \text{and} \quad R^{*2}(1 + \cos\rho) = s^{*2}(1 - \cos\rho).$$

Lemma 4.4.18. *Our two adjacent edge directions satisfy*

$$\langle E, -F \rangle = k^2 s^{*2} - \cos\rho \, (1 + k^2 s^{*2}).$$

Proof. Recall that the edges e, f get rotated around $x - m$, $y - m$ by the angle δ, such that after projection into the circle plane, the quad is just the original planar quad rescaled by $\cos\delta = \frac{r}{R}$. So we can decompose the normalized edges into the projected and perpendicular part:

$$\langle E, -F\rangle = \frac{r^2}{R^2}\langle E(0), -F(0)\rangle + \sin^2\delta.$$

But $\langle E(0), -F(0)\rangle$ is just $\cos(\pi-\rho)$ (cf. fig. 4.15), and using lemma 4.4.9 and lemma 3.3.1 we see

$$\begin{aligned}\langle E, -F\rangle &= -\frac{r^2}{R^2}\cos\rho + k^2R^{*2}\\ &= -\frac{R^2-\alpha^2k^2}{R^2}\cos\rho + k^2R^{*2}\\ &= -(1-R^{*2}k^2)\cos\rho + k^2R^{*2}\\ &= -\cos\rho + k^2R^{*2}(1+\cos\rho)\\ &= -\cos\rho + k^2s^{*2}(1-\cos\rho)\\ &= k^2s^{*2} - \cos\rho(1+k^2s^{*2}).\end{aligned}$$

□

Now we get to the remaining scalar products, $\langle E, N\rangle = \cos\sigma$ and $\langle -F, N\rangle = \cos\tau$.

Lemma 4.4.19. *For the vertex normals and edge directions we have*

$$\langle E, N\rangle = \frac{1}{l}(s - s^*\cos\varphi) \quad \textit{and} \quad \langle -F, N\rangle = \frac{1}{l}(s - s^*\cos\psi).$$

Proof. Since E and E^* are always perpendicular to $x^* - x$, we can calculate $v^* - v$ in the ONB with the first coordinate direction aligned with E and the third aligned with $x^* - x$ as in fig. 4.16:

$$v = \begin{pmatrix}-s\\0\\0\end{pmatrix}, \quad v^* = \begin{pmatrix}-s^*\cos\varphi\\-s^*\sin\varphi\\c\end{pmatrix},$$

and since E just becomes $(1,0,0)$, the result is immediate. We can do the same for $-F$, where ψ replaces φ (and d replaces c). □

Now the last remaining ingredients are $\sin\sigma$ and $\sin\tau$.

Lemma 4.4.20. *The angles σ and τ formed by vertex normals and edge directions satisfy*

$$\sin^2\sigma = \frac{c^2}{l^2}(1+k^2s^{*2}) \quad \textit{and} \quad \sin^2\tau = \frac{d^2}{l^2}(1+k^2s^{*2}).$$

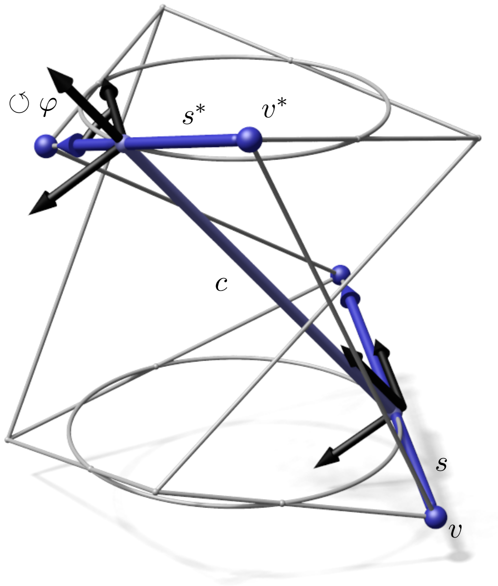

Figure 4.16: Coordinates for one side face.

Proof. Recall from the definition of c and lemma 4.4.14 that we have

$$c^2 := h^2 + R^2 + R^{*2} + 2\alpha \cos\varphi \quad \text{as well as} \quad d^2 := h^2 + R^2 + R^{*2} + 2\alpha \cos\psi,$$

and that by the proof of lemma 4.4.11, the vertical edge length was

$$l^2 = h^2 + R^2 + R^{*2} + s^2 + s^{*2}.$$

We use this in our results from lemma 4.4.19 above:

$$\begin{aligned}
\sin^2\sigma = 1 - \langle E, N\rangle^2 &= \frac{1}{l^2}\Big(l^2 - s^2 - s^{*2}\cos^2\varphi + 2\alpha\cos\varphi\Big) \\
&= \frac{1}{l^2}\Big(h^2 + R^2 + R^{*2} + s^{*2} - s^{*2}\cos^2\varphi + 2\alpha\cos\varphi\Big) \\
&= \frac{1}{l^2}\Big(c^2 + s^{*2}\sin^2\varphi\Big) \\
&= \frac{c^2}{l^2}\Big(1 + s^{*2}k^2\Big).
\end{aligned}$$

The calculation for $\sin^2\tau$ is the same with ψ. $\square$

Now we finally pick up our calculation of the cosine of the dihedral angle, plugging in the results from the preceding lemmata:

$$
\begin{aligned}
\cos\xi &= \frac{1}{\sin\sigma\sin\tau}\left(\langle E,-F\rangle - \langle E,N\rangle\langle -F,N\rangle\right)\\
&= \frac{l^2}{cd(1+k^2s^{*2})}\left(k^2s^{*2} - \cos\rho\,(1+k^2s^{*2}) - \frac{1}{l^2}(s-s^*\cos\varphi)(s-s^*\cos\psi)\right)\\
&= \frac{-l^2\cos\rho}{cd} + \frac{l^2}{cd(1+k^2s^{*2})}\left(k^2s^{*2} - \frac{1}{l^2}(s-s^*\cos\varphi)(s-s^*\cos\psi)\right)\\
&= \frac{-l^2\cos\rho}{cd} + \frac{l^2}{cd(1+k^2s^{*2})}\left(k^2s^{*2} - \frac{1}{l^2}(s^2+s^{*2}\cos\varphi\cos\psi - \alpha\cos\varphi - \alpha\cos\psi)\right).
\end{aligned}
$$

We use the results from corollary 4.4.13 and lemma 4.4.17 to continue with the big bracket:

$$
\begin{aligned}
&\left(k^2s^{*2} - \frac{1}{l^2}(s^2+s^{*2}\cos\varphi\cos\psi - \alpha\cos\varphi - \alpha\cos\psi)\right)\\
&= \left(k^2s^{*2} - \frac{1}{l^2}\left(s^2 + s^{*2}(k^2(h^2+R^2+R^{*2})-1) + 2\alpha^2k^2\right)\right)\\
&= \left(k^2s^{*2} - \frac{1}{l^2}\left(s^2 - s^{*2} + s^{*2}(k^2(l^2-s^2-s^{*2})) + 2\alpha^2k^2\right)\right)\\
&= \left(k^2s^{*2} - k^2s^{*2} - \frac{1}{l^2}\left(s^2 - s^{*2} + s^{*2}(k^2(-s^2-s^{*2})) + 2\alpha^2k^2\right)\right)\\
&= -\frac{1}{l^2}\left(s^2 - s^{*2} + s^{*2}(k^2(-s^2-s^{*2})) + 2\alpha^2k^2\right)\\
&= -\frac{1}{l^2}\left(s^2 - s^{*2} + k^2s^{*2}(-s^2-s^{*2}+2s^2)\right)\\
&= -\frac{1}{l^2}(s^2-s^{*2})(1+k^2s^{*2}).
\end{aligned}
$$

Plugging back in, we get

$$
\begin{aligned}
\cos\xi &= \frac{-l^2\cos\rho}{cd} - \frac{(s^2-s^{*2})(1+k^2s^{*2})}{cd(1+k^2s^{*2})}\\
&= -\frac{1}{cd}\left(l^2\cos\rho + s^2 - s^{*2}\right).
\end{aligned}
$$

We collect this result and provide an alternative formulation in

Proposition 4.4.21. *The dihedral angle $\xi = \measuredangle_{v,v^*}(v_1,v_2)$ of the planes spanned by the vertex normal at v and either of the adjacent edges satisfies*

$$
\cos\xi = \frac{1}{cd}\left(s^{*2} - s^2 - l^2\cos\rho\right) = \frac{1}{cd}\left(R^{*2} - R^2 - h^2\cos\rho\right).
$$

Proof. The first identity is the result of the calculation above, and for the second we again use lemma 3.3.1 and lemma 4.4.11:

$$\begin{aligned}
s^{*2} - s^2 - l^2 \cos\rho &= s^{*2} - s^2 - \cos\rho(h^2 + R^2 + R^{*2} + s^2 + s^{*2}) \\
&= s^{*2}(1 - \cos\rho) - s^2(1 + \cos\rho) - \cos\rho(h^2 + R^2 + R^{*2}) \\
&= R^{*2}(1 + \cos\rho) - R^2(1 - \cos\rho) - R^2\cos\rho - R^{*2}\cos\rho - h^2\cos\rho \\
&= R^{*2} - R^2 - h^2\cos\rho.
\end{aligned}$$

□

Corollary 4.4.22. *The dihedral angle ξ is constant under twisting of the cube.*

Proof. In particular recalling lemma 4.4.16, we see that all the terms in the formula for the cosine of ξ are constant. □

In the following, we will notationally consider the dual as the third parameter direction: $S_3 := S^*$ etc. We introduce a second combinatorial view on our cube:

$$T := S, \quad T_1 := S_{13}, \quad T_2 := S_{23}, \quad T_{12} := S_{12},$$

$$T_3 := S_3, \quad T_{13} := S_1, \quad T_{23} := S_2, \quad T_{123} := S_{123},$$

i.e. primal and dual sphere are interchanged at every other vertex. We call the cube formed by the spheres T the *interleaved* cube, and denote the sphere centres by w and the radii by t. The vertices and edges of the original and interleaved cube are depicted in fig. 4.17.

Proposition 4.4.23. *i) The twisted s_1–isothermic cmc cube, formed by the spheres S, is αk–bent.*

ii) The interleaved twisted s_1–isothermic cmc cube, formed by the spheres T, is k–folded.

Proof. We start with a side face, say (S, S_1, S_3, S_{13}), as seen in fig. 4.18. Recall from lemma 4.4.19 that for these faces, we had adapted coordinates such that

$$v = \begin{pmatrix} -s \\ 0 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} s_1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -s_3\cos\varphi \\ -s_3\sin\varphi \\ c \end{pmatrix}, \quad v_{13} = \begin{pmatrix} s_{13}\cos\varphi \\ s_{13}\sin\varphi \\ c \end{pmatrix}.$$

With notation as in def. 4.1.1, we calculate the edges

$$e^1 = v_1 - v = (s + s_1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^1_3 = v_{13} - v_3 = (s_3 + s_{13})\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix},$$

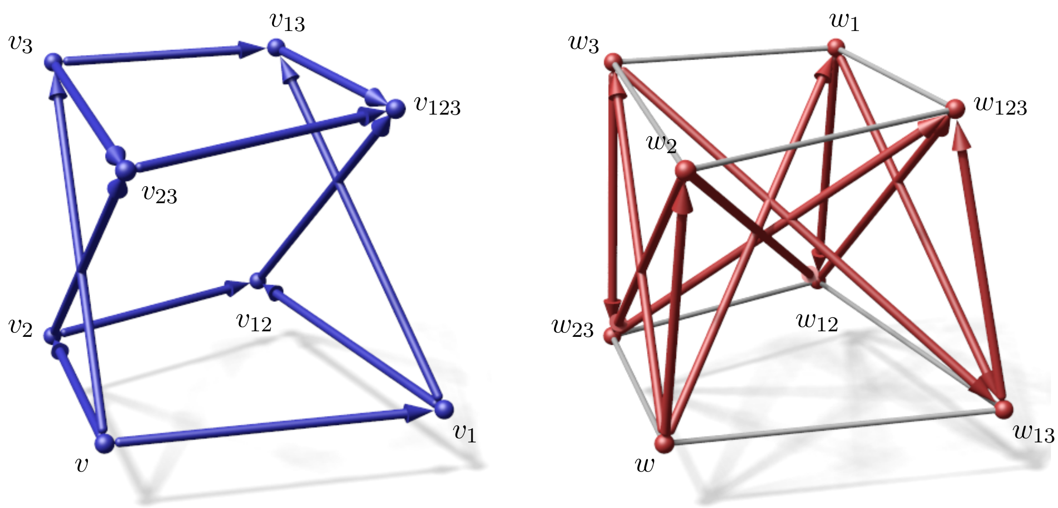

Figure 4.17: Original s_1–isothermic cube and interleaved cube, with edges oriented in coordinate direction.

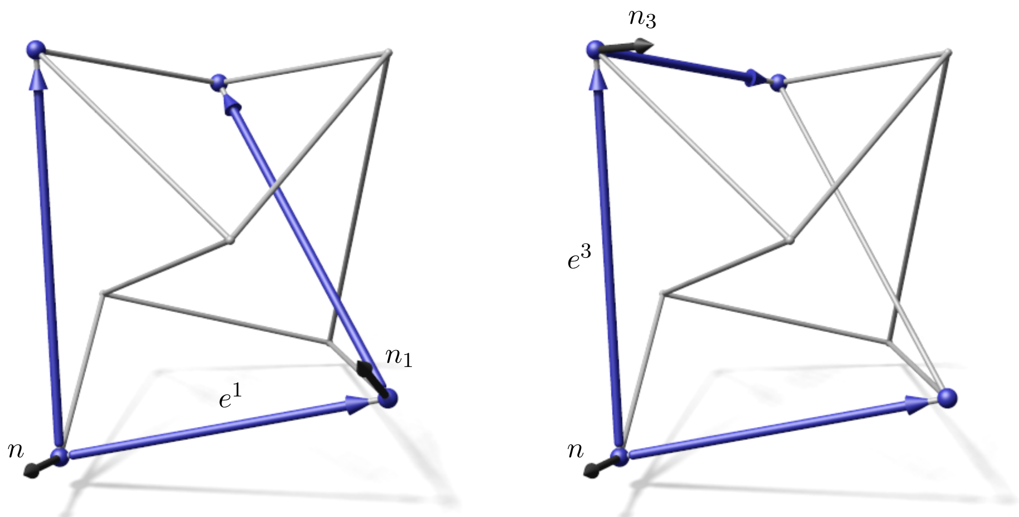

Figure 4.18: At the edges e^1 and e^3 of the face (S, S_1, S_3, S_{13}), with corner normals.

$$e^3 = v_3 - v = \begin{pmatrix} s - s_3\cos\varphi \\ -s_3\sin\varphi \\ c \end{pmatrix}, \quad e_1^3 = v_{13} - v_1 = \begin{pmatrix} s_{13}\cos\varphi - s_1 \\ s_{13}\sin\varphi \\ c \end{pmatrix}$$

and corner normals

$$n = \frac{1}{\sqrt{1+s_3^2k^2}}\begin{pmatrix} 0 \\ -1 \\ -s_3k \end{pmatrix}, \quad n_1 = \frac{1}{\sqrt{1+s_{13}^2k^2}}\begin{pmatrix} 0 \\ -1 \\ s_{13}k \end{pmatrix},$$

$$n_3 = \frac{1}{\sqrt{1+s^2k^2}}\begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ -sk \end{pmatrix}, \quad n_{13} = \frac{1}{\sqrt{1+s_1^2k^2}}\begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ s_1k \end{pmatrix};$$

Now, using that $ss_3 = s_1s_{13} = \alpha$, we see

$$\begin{aligned} n \times n_1 &= \frac{1}{\sqrt{(1+s_3^2k^2)(1+s_{13}^2k^2)}}\begin{pmatrix} -(s_3+s_{13})k \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{-(\alpha s_1 + \alpha s)k}{\sqrt{(s^2+\alpha^2k^2)(s_1^2+\alpha^2k^2)}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{-\alpha k}{\sqrt{(s^2+\alpha^2k^2)(s_1^2+\alpha^2k^2)}}\, e^1 \end{aligned}$$

and by the same calculation $n_3 \times n_{13} = \frac{-\alpha k}{\sqrt{(s_3^2+\alpha^2k^2)(s_{13}^2+\alpha^2k^2)}} e_3^1$. For the vertical edges, we calculate

$$n \times n_3 = \frac{1}{\sqrt{(1+s_3^2k^2)(1+s^2k^2)}}\begin{pmatrix} sk - s_3k\cos\varphi \\ -s_3k\sin\varphi \\ \sin\varphi \end{pmatrix} = \frac{k}{\sqrt{(1+s_3^2k^2)(1+s^2k^2)}}\, e^3,$$

and

$$\frac{k}{\sqrt{(1+s_3^2k^2)(1+s^2k^2)}} = \frac{ss_3k}{\sqrt{s^2(1+s_3^2k^2)s_3^2(1+s^2k^2)}} = \frac{\alpha k}{\sqrt{(s^2+\alpha^2k^2)(s_3^2+\alpha^2k^2)}};$$

analogously

$$n_1 \times n_{13} = \frac{k}{\sqrt{(1+s_{13}^2k^2)(1+s_1^2k^2)}} e_1^3 = \frac{\alpha k}{\sqrt{(s_1^2+\alpha^2k^2)(s_{13}^2+\alpha^2k^2)}} e_1^3.$$

This completes αk–bentness for the side face (S, S_1, S_3, S_{13}); we now turn to k–foldedness of the corresponding side face (T, T_1, T_3, T_{13}), cf. fig. 4.19. We denote the centres of the spheres T by w, their radii by t, the edges by f and the corner normals by m; we have

$$t = s, \quad t_1 = s_{13}, \quad t_3 = s_3, \quad t_{13} = s_1$$

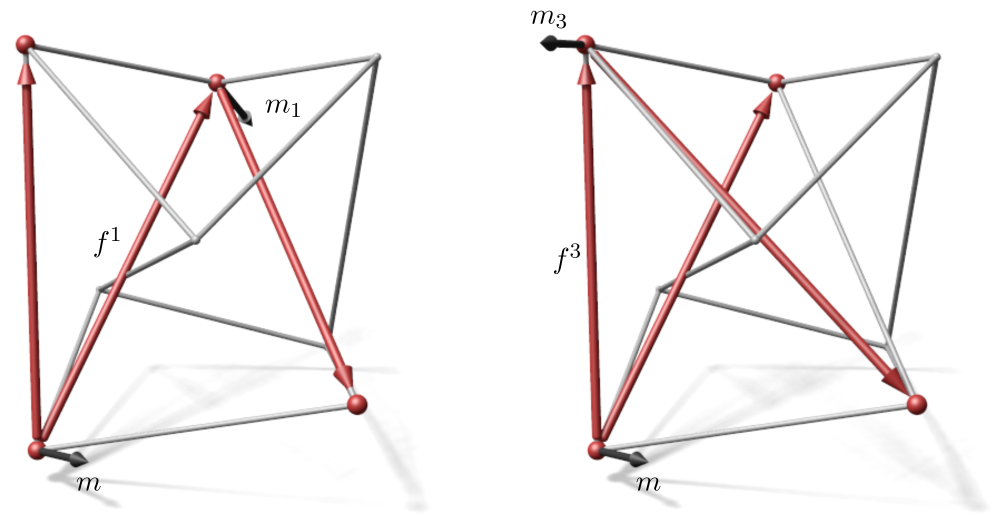

Figure 4.19: At the edges f^1 and f^3 of the face (T, T_1, T_3, T_{13}), with corner normals.

as well as

$$w = v, \quad w_1 = v_{13}, \quad w_3 = v_3, \quad w_{13} = v_1$$

and consequently

$$f^1 = w_1 - w = v_{13} - v = e_1^3 + e^1 = e_3^1 + e^3,$$

$$f_3^1 = w_{13} - w_3 = v_1 - v_3 = e^1 - e^3 = e_3^1 - e_1^3,$$

$$f^3 = w_3 - w = v_3 - v = e^3, \quad f_1^3 = w_{13} - w_1 = v_1 - v_{13} = -e_1^3.$$

We get

$$f^1 \times f^3 = (e_3^1 + e^3) \times e^3 = e_3^1 \times e^3, \text{ so } m = n_3,$$

$$f^1 \times f_1^3 = -(e_1^3 + e^1) \times e_1^3 = -e^1 \times e_1^3, \text{ so } m_1 = -n_1,$$

$$f_3^1 \times f^3 = (e^1 - e^3) \times e^3 = e^1 \times e^3, \text{ so } m_3 = n,$$

$$f_3^1 \times f_1^3 = -(e_3^1 - e_1^3) \times e_1^3 = -e_3^1 \times e_1^3, \text{ so } m_{13} = -n_{13}.$$

Now we can use our previous calculations to see that

$$m \times m_3 = n_3 \times n = \frac{-k}{\sqrt{(1+s_3^2k^2)(1+s^2k^2)}} e^3 = \frac{-k}{\sqrt{(1+t_3^2k^2)(1+t^2k^2)}} f^3,$$

$$m_1 \times m_{13} = (-n_1) \times (-n_{13}) = \frac{k}{\sqrt{(1+s_{13}^2k^2)(1+s_1^2k^2)}} e_1^3 = \frac{-k}{\sqrt{(1+t_1^2k^2)(1+t_{13}^2k^2)}} f_1^3.$$

For the edges in direction 1, we note

$$f^1 = v_{13} - v = \begin{pmatrix} s_{13}\cos\varphi + s \\ s_{13}\sin\varphi \\ c \end{pmatrix}, \quad f^1_3 = v_1 - v_3 = \begin{pmatrix} s_1 + s_3\cos\varphi \\ s_3\sin\varphi \\ -c \end{pmatrix}$$

and calculate

$$\begin{aligned} m \times m_1 = n_3 \times (-n_1) &= \frac{1}{\sqrt{(1+s^2k^2)(1+s_{13}^2k^2)}} \begin{pmatrix} sk + s_{13}k\cos\varphi \\ s_{13}k\sin\varphi \\ \sin\varphi \end{pmatrix} \\ &= \frac{k}{\sqrt{(1+t^2k^2)(1+t_1^2k^2)}} f^1, \end{aligned}$$

$$\begin{aligned} m_3 \times m_{13} = n \times (-n_{13}) &= \frac{1}{\sqrt{(1+s_3^2k^2)(1+s_1^2k^2)}} \begin{pmatrix} s_1k + s_3k\cos\varphi \\ s_3k\sin\varphi \\ -\sin\varphi \end{pmatrix} \\ &= \frac{k}{\sqrt{(1+t_3^2k^2)(1+t_{13}^2k^2)}} f^1_3. \end{aligned}$$

This concludes k–foldedness for the interleaved side face (T, T_1, T_3, T_{13}). For the other side faces, recall from remark 4.4.15 that cyclically relabelling our cube is the same as if we had worked with $-k$ instead of k. This shows $-\alpha k$–bentness and $-k$–foldedness for the other side faces, recalling that the reversing of direction 3 when switching to the opposite side face of the interleaved cube does not affect foldedness.

For the bottom face (S, S_1, S_2, S_3), we look at the edges individually, beginning with (S, S_1). In view of lemma 4.4.6 and lemma 4.4.9, we first introduce coordinates for the planar quadrilateral, and then rescale and add the vertical component to get a model for the relevant edge and the adjacent ones in the twisted cube, as depicted in fig. 4.20. We denote the planar coordinates of the untwisted case by the respective capital letters. We start with the centre of the orthogonal circle,

$$M = 0,$$

and the point of tangency of the edge e with the circle,

$$X = \begin{pmatrix} 0 \\ -R \end{pmatrix}.$$

The vertices then become

$$V = X - \begin{pmatrix} s \\ 0 \end{pmatrix} = \begin{pmatrix} -s \\ -R \end{pmatrix} \text{ and } V_1 = X + \begin{pmatrix} s_1 \\ 0 \end{pmatrix} = \begin{pmatrix} s_1 \\ -R \end{pmatrix}.$$

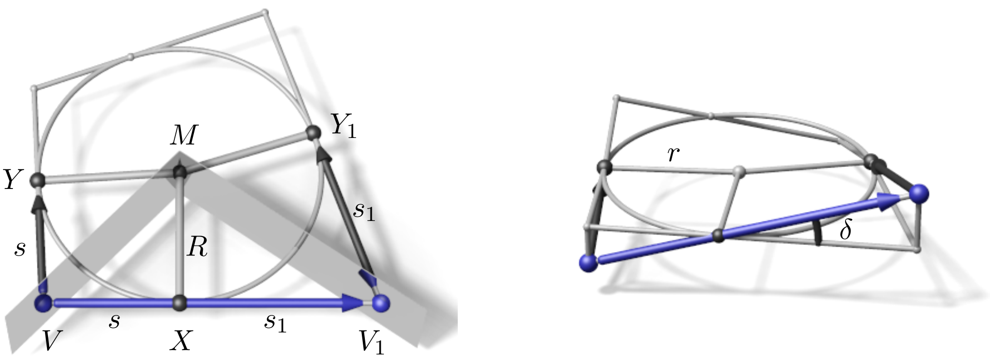

Figure 4.20: On the coordinates for the bottom face (S, S_1, S_2, S_3).

For the corner normals, we only need the directions of the adjacent edges, so it is sufficient to model their points of tangency, y and y_1. Since these are the other points of intersection of the orthogonal circle and the spheres around the vertices, we find them by mirroring X at $V - M = V$ and $V_1 - M = V_1$, respectively, and find

$$Y = 2\frac{\langle V, X\rangle}{\|V\|^2}V - X = \frac{2R^2}{R^2+s^2}\begin{pmatrix}-s\\-R\end{pmatrix} + \begin{pmatrix}0\\R\end{pmatrix} = \frac{-R}{R^2+s^2}\begin{pmatrix}2Rs\\R^2-s^2\end{pmatrix},$$

$$Y_1 = 2\frac{\langle V_1, X\rangle}{\|V_1\|^2}V_1 - X = \frac{2R^2}{R^2+s_1^2}\begin{pmatrix}s_1\\-R\end{pmatrix} + \begin{pmatrix}0\\R\end{pmatrix} = \frac{R}{R^2+s_1^2}\begin{pmatrix}2Rs_1\\-R^2+s_1^2\end{pmatrix}$$

and the edge segments

$$Y - V = \frac{-R}{R^2+s^2}\begin{pmatrix}2Rs\\R^2-s^2\end{pmatrix} + \begin{pmatrix}s\\R\end{pmatrix} = \frac{s}{R^2+s^2}\begin{pmatrix}s^2-R^2\\2Rs\end{pmatrix},$$

$$Y_1 - V_1 = \frac{R}{R^2+s_1^2}\begin{pmatrix}2Rs_1\\-R^2+s_1^2\end{pmatrix} + \begin{pmatrix}-s_1\\R\end{pmatrix} = \frac{s_1}{R^2+s_1^2}\begin{pmatrix}R^2-s_1^2\\2Rs_1\end{pmatrix}$$

with length s and s_1, respectively. Now, in the light of lemma 4.4.6, we choose the rotational direction of δ to satisfy $\operatorname{sgn}\sin\delta = \operatorname{sgn} k$. For the twisted cube, we rescale in the plane of the orthogonal circle by $\cos\delta = \frac{r}{R}$, and v and v_1 get a vertical component of $-s\sin\delta$ and $s_1\sin\delta$, respectively. Our edge becomes

$$e = (s+s_1)\begin{pmatrix}\cos\delta\\0\\\sin\delta\end{pmatrix},$$

and the adjacent ones

$$e^2 \overset{+}{\sim} \cos\delta \frac{s}{R^2+s^2}\begin{pmatrix} s^2-R^2 \\ 2Rs \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ \sin\delta \end{pmatrix} \overset{+}{\sim} \begin{pmatrix} \cos\delta(s^2-R^2) \\ 2\cos\delta Rs \\ \sin\delta(s^2+R^2) \end{pmatrix},$$

$$e_1^2 \overset{+}{\sim} \cos\delta \frac{s_1}{R^2+s_1^2}\begin{pmatrix} R^2-s_1^2 \\ 2Rs_1 \\ 0 \end{pmatrix} + s_1\begin{pmatrix} 0 \\ 0 \\ -\sin\delta \end{pmatrix} \overset{+}{\sim} \begin{pmatrix} \cos\delta(R^2-s_1^2) \\ 2\cos\delta Rs_1 \\ -\sin\delta(R^2+s_1^2) \end{pmatrix}.$$

Now we can calculate

$$n \overset{+}{\sim} e \times e^2 \overset{+}{\sim} \begin{pmatrix} \cos\delta \\ 0 \\ \sin\delta \end{pmatrix} \times \begin{pmatrix} \cos\delta(s^2-R^2) \\ 2\cos\delta Rs \\ \sin\delta(s^2+R^2) \end{pmatrix} \overset{+}{\sim} \begin{pmatrix} -s\sin\delta \\ -R\sin\delta \\ s\cos\delta \end{pmatrix},$$

$$n_1 \overset{+}{\sim} e \times e_1^2 \overset{+}{\sim} \begin{pmatrix} \cos\delta \\ 0 \\ \sin\delta \end{pmatrix} \times \begin{pmatrix} \cos\delta(R^2-s_1^2) \\ 2\cos\delta Rs_1 \\ \sin\delta(R^2+s_1^2) \end{pmatrix} \overset{+}{\sim} \begin{pmatrix} -s_1\sin\delta \\ R\sin\delta \\ s_1\cos\delta \end{pmatrix},$$

giving us, using lemma 4.4.9,

$$n = \frac{1}{\sqrt{s^2+R^2\sin^2\delta}}\begin{pmatrix} -s\sin\delta \\ -R\sin\delta \\ s\cos\delta \end{pmatrix} = \frac{1}{\sqrt{s^2+\alpha^2k^2}}\begin{pmatrix} -s\sin\delta \\ -R\sin\delta \\ s\cos\delta \end{pmatrix}$$

and

$$n_1 = \frac{1}{\sqrt{s_1^2+R^2\sin^2\delta}}\begin{pmatrix} -s_1\sin\delta \\ R\sin\delta \\ s_1\cos\delta \end{pmatrix} = \frac{1}{\sqrt{s_1^2+\alpha^2k^2}}\begin{pmatrix} -s_1\sin\delta \\ R\sin\delta \\ s_1\cos\delta \end{pmatrix}.$$

Finally, with

$$\begin{pmatrix} -s\sin\delta \\ -R\sin\delta \\ s\cos\delta \end{pmatrix} \times \begin{pmatrix} -s_1\sin\delta \\ R\sin\delta \\ s_1\cos\delta \end{pmatrix} = -\sin\delta R(s+s_1)\begin{pmatrix} \cos\delta \\ 0 \\ \sin\delta \end{pmatrix} = -\alpha k e,$$

we see that

$$n \times n_1 = \frac{-\alpha k}{\sqrt{(s^2+\alpha^2k^2)(s_1^2+\alpha^2k^2)}} e.$$

For the other edges of the quadrilateral, we again recall from remark 4.4.15 that the situation is symmetric with respect to cyclical relabelling, up to a change of the sign of k. So altogether, we get all four equations needed for αk–bentness. For the dual quadrilateral, $(S_3, S_{13}, S_{23}, S_{123})$, the calculation is analogous. Note however that apart from replacing radii with dual radii, the edges at y as well as the $x^* - m^*$ change

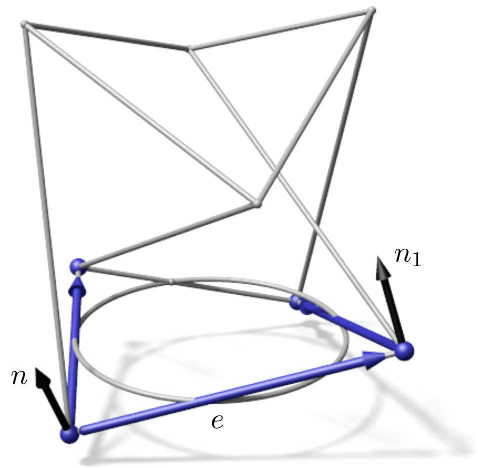

Figure 4.21: At the edge e of the face (S, S_1, S_2, S_{12}), with corner normals.

orientation in comparison to the primal quad. So in our coordinates, all edges get minus sign, leaving the corner normals and their cross products unchanged. The sign gets compensated for by the fact — cf. lemma 4.4.6 — that the z–component of the dual edges has the opposite sign, reversing the relation of $\operatorname{sgn}\sin\delta^*$ and $\operatorname{sgn} k$.

For the interleaved cube, we have to further extend our coordinate system. The calculations will get long and messy, so we move them into the appendix: We will prove k–foldedness of the sphere–quadrilateral (T, T_1, T_2, T_{12}) of the interleaved cube at the edge

$$f = f^1 = w_1 - w = v_{13} - v,$$

with adjacent edges

$$f^2 = w_2 - w = v_{23} - v \quad \text{and} \quad f_1^2 = w_{12} - w_1 = v_{12} - v_{13}$$

in appendix A. The equations for the other edges are again a consequence of symmetry as in remark 4.4.15.

□

4.4.3 Global Compatibility and Examples

With our findings in section 4.4.1, in particular corollary 4.4.22, we can prove that the construction not only works for a single cube, but simultaneously for all the cubes in an s_1–isothermic cmc surface, fitting together to form a new surface.

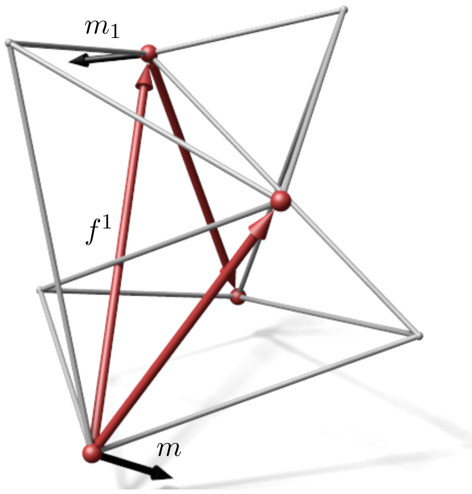

Figure 4.22: At the edge f^1 of the face (T, T_1, T_2, T_{12}), with corner normals.

Theorem 4.4.24. *For any global choice of φ or ψ, our construction applied to all the cubes in an s–isothermic cmc surface yields a well–defined new surface.*

Proof. Any two cubes adjacent along a pair of edge and dual edge fit together: the twisted edge pair is uniquely determine by φ and c (or analogously ψ and d). φ is chosen to be the same for both cubes, and since $R^*R = \alpha = \widetilde{R^*}\widetilde{R}$, we have

$$c^2 - \widetilde{c}^2 = h^2 + R^2 + R^{*2} - \widetilde{h}^2 - \widetilde{R}^2 - \widetilde{R^*}^2 = 0$$

since the cubes fit together in the untwisted case:

$$0 = c(0)^2 - \widetilde{c}(0)^2 = h^2 + R^2 + R^{*2} + 2\alpha - \widetilde{h}^2 - \widetilde{R}^2 - \widetilde{R^*}^2 - 2\alpha.$$

Here of course the tilded letters denote the corresponding quantities in the second cube.

What remains to show is that this process, carried out for all the adjacent cubes around a vertex, closes up. For this, consider any vertical edge connecting a vertex v and its dual v^*. For any edge adjacent to v, the angle formed by the edge and $v^* - v$ is determined as a quantity of the edge–and–dual pair. The only thing left to show is that projected into the plane perpendicular to $v^* - v$, the angles between adjacent edges close up. But these angles are precisely the dihedral angles ξ from proposition 4.4.21 – they close up in the untwisted case and are constant. □

We conclude the section with some examples.

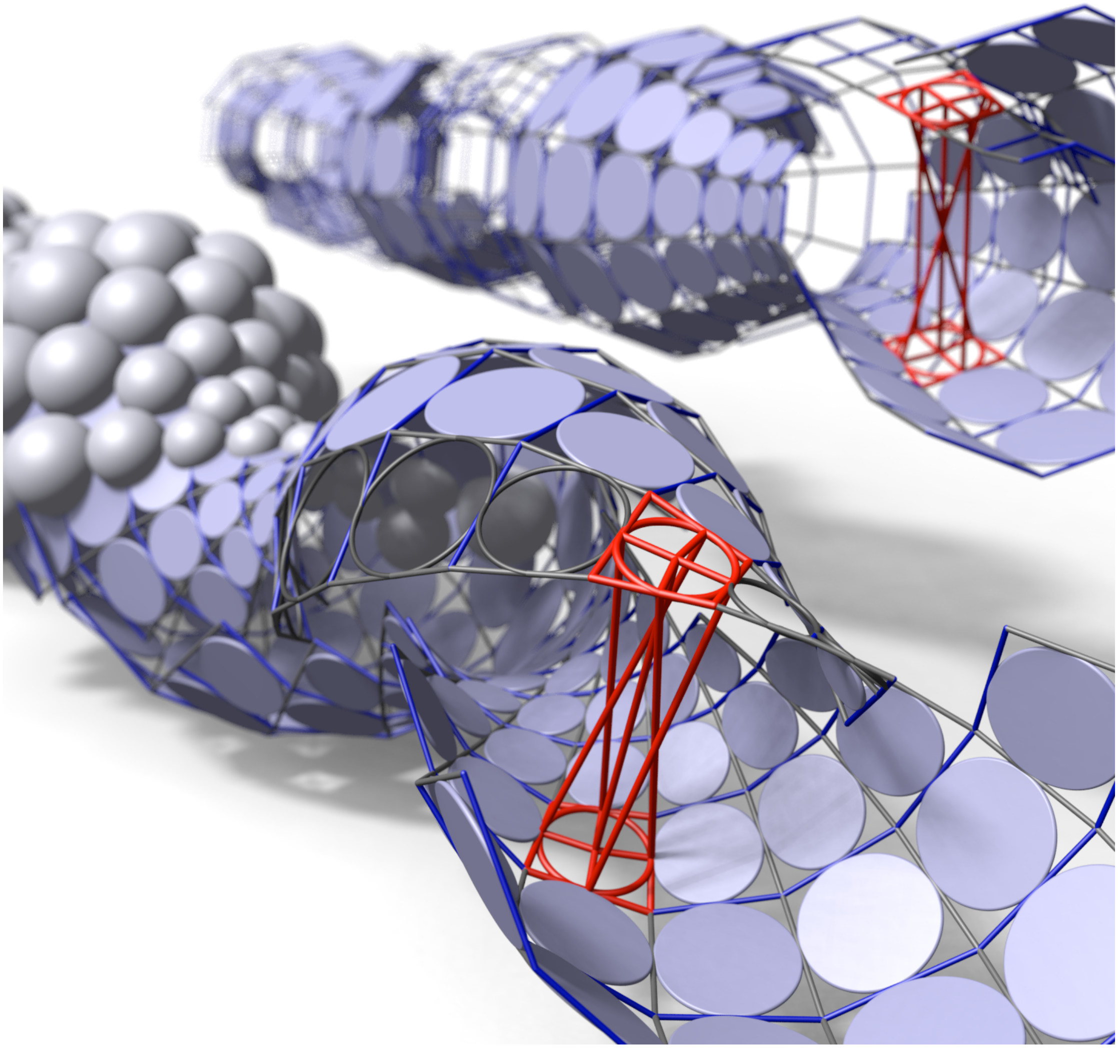

Figure 4.23: A patch of Delaunay surface and dual, with one of the cubes highlighted. Untwisted surface in the background.

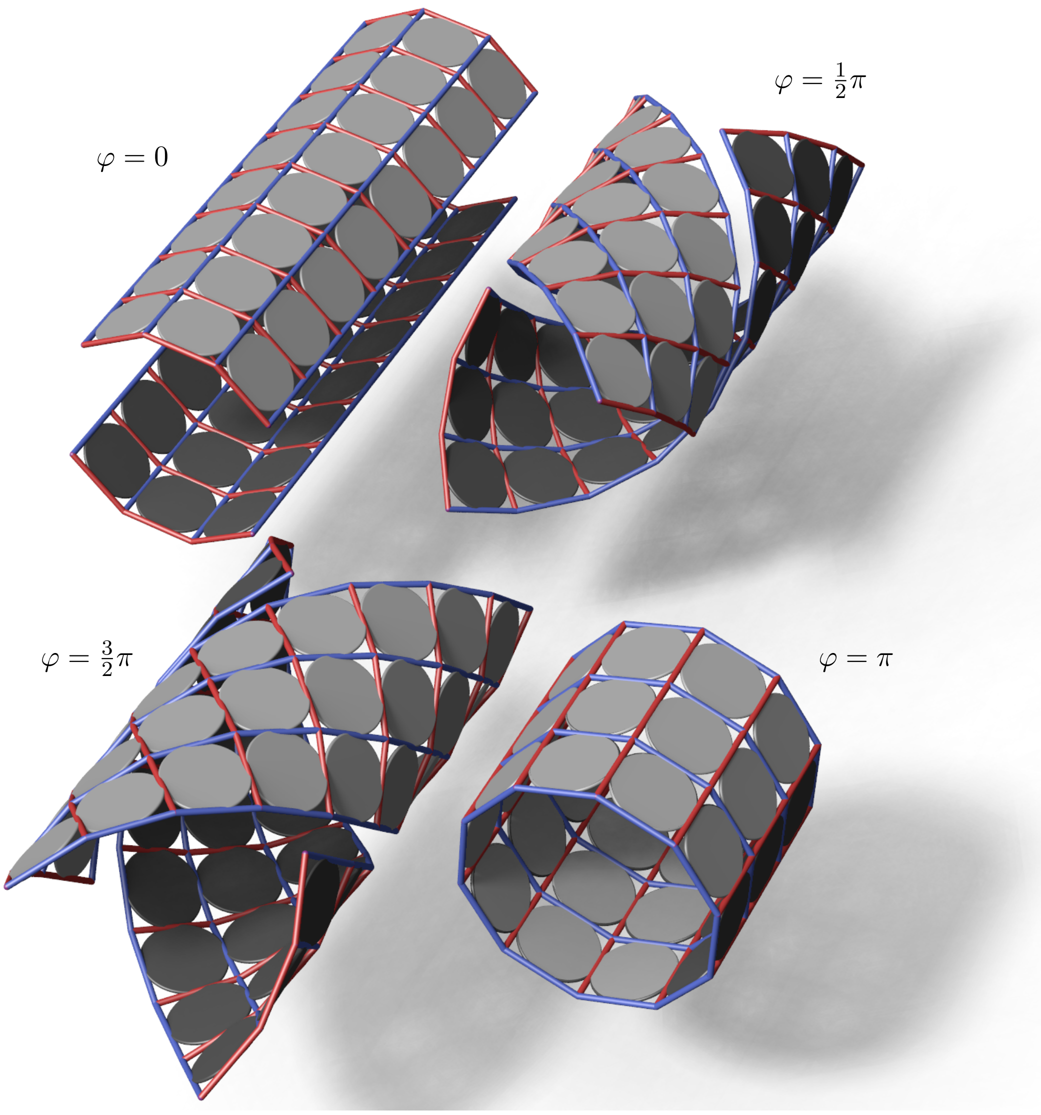

Figure 4.24: For the cylinder, the associated family is just a reparametrization. A dual pair of patches is shown. The edges in the two coordinate directions are drawn in red and blue respectively, illustrating their interchanging mentioned in remark 4.4.15 and fig. 4.13.

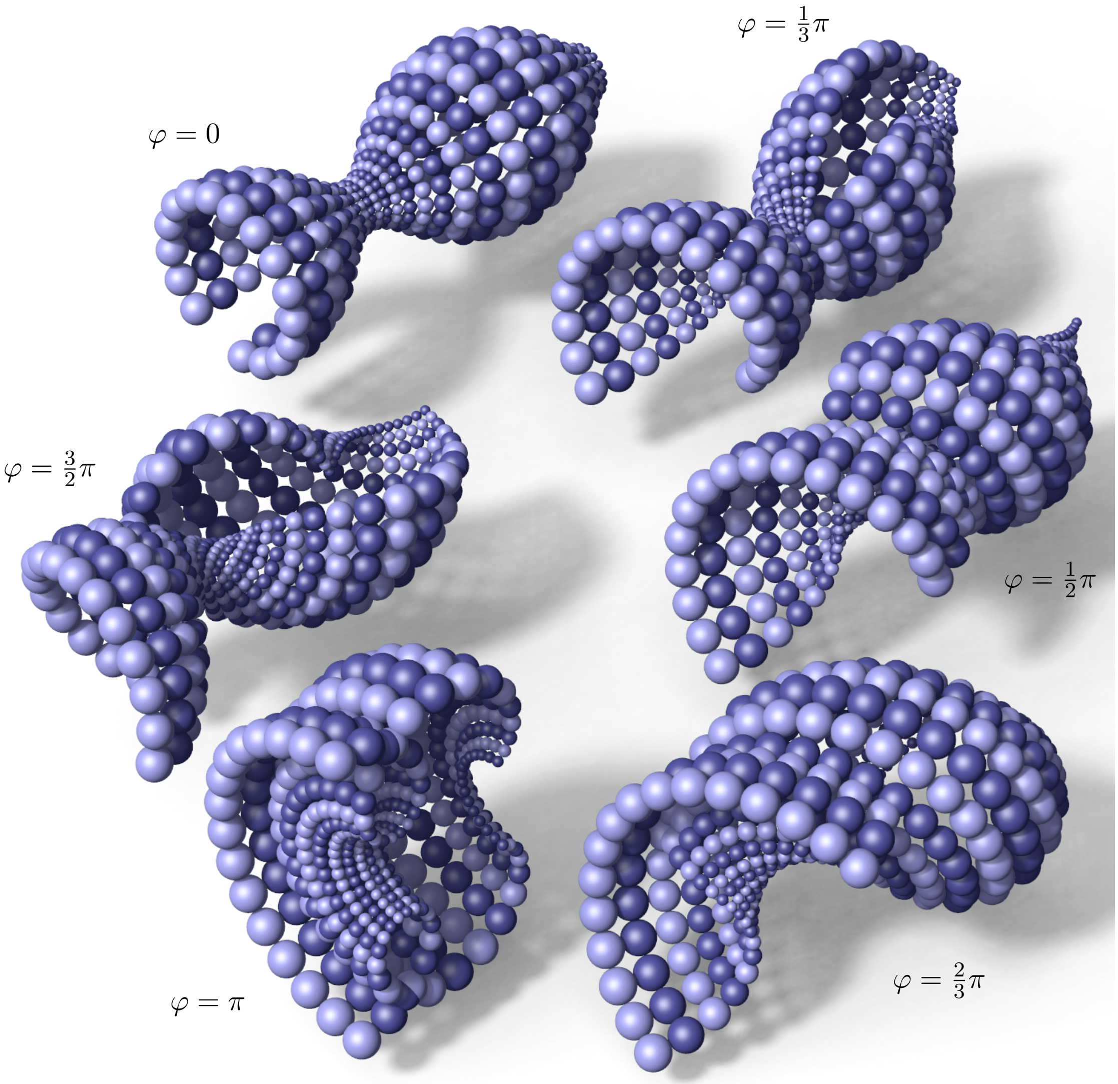

Figure 4.25: Another Delaunay surface. This time, only the vertex spheres of the primal surface are drawn. Note the unduloid shape at $\varphi = 0$ and the nodoid shape at $\varphi = \pi$.

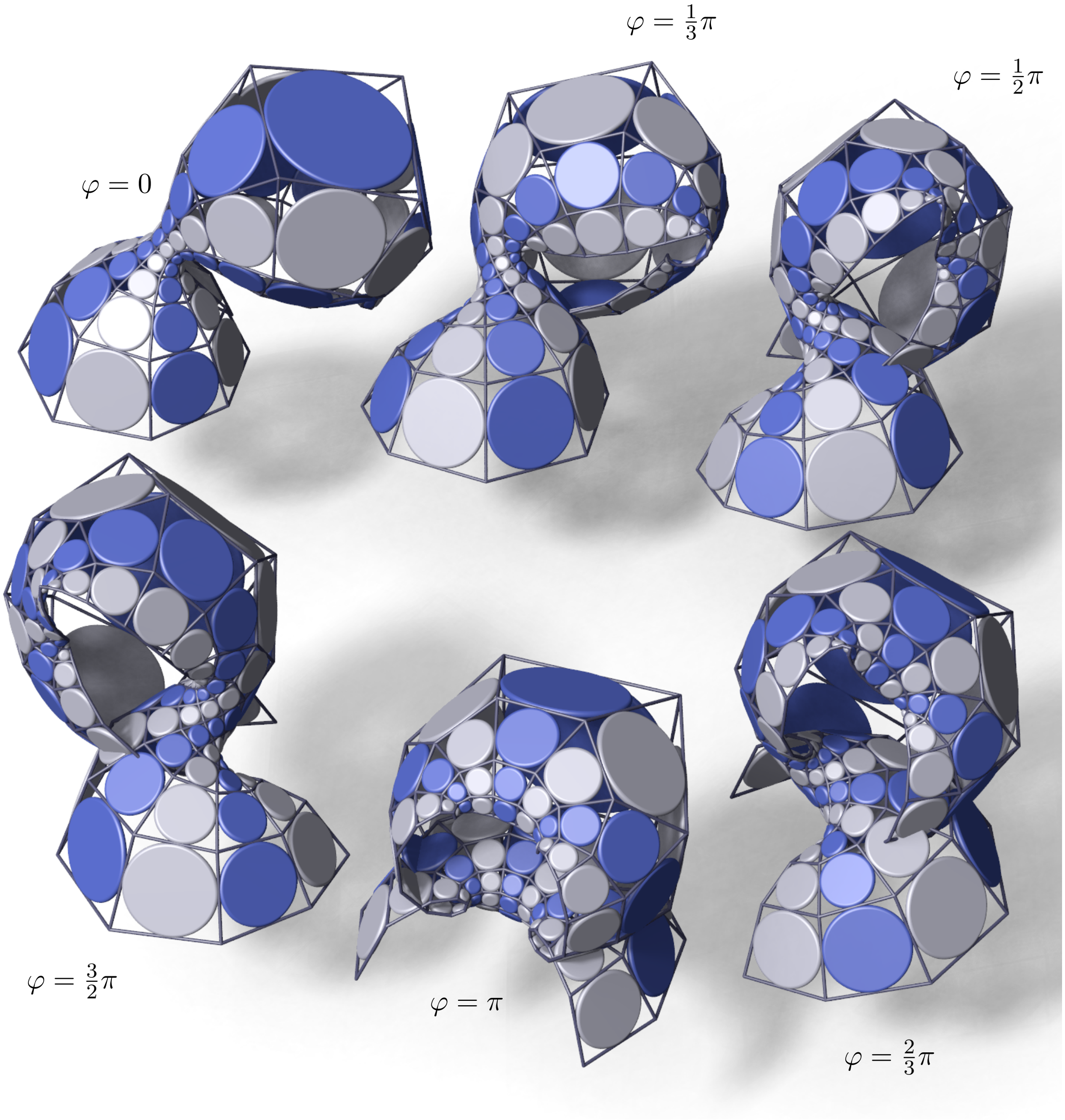

Figure 4.26: A Darboux transform of a Delaunay surface. This time, edges and face circles of the primal surface are drawn.

4.5 General S–Isothermic CMC Surfaces

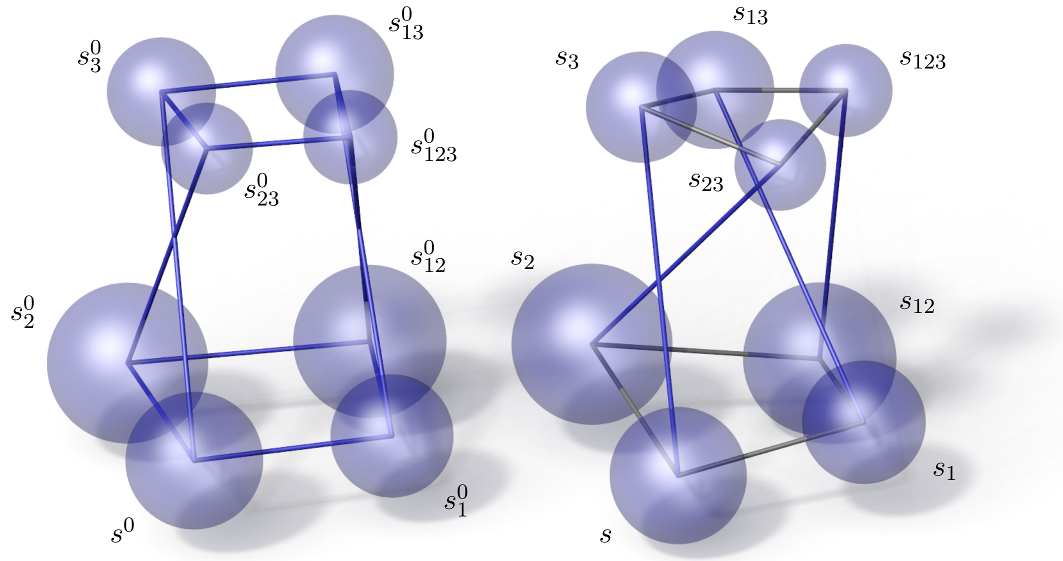

Figure 4.27: An original s–isothermic cmc cube and the twisted version we will construct in this section.

In this section, we will try to use our definitions from section 4.1 to construct surfaces in the associated family of general s–isothermic cmc surfaces. Our constructions are not yet proven to go through, but have been tested in numerical experiments and seem to hold as good as machine precision allows. We state our results as conjectures and hope they will be proven in future work.

Our construction will be cube–wise, i.e. we start with an elementary s–isothermic cmc cube with a primal quadrilateral of spheres $s^0, s_1^0, s_2^0, s_{12}^0$ and a dual one, $s_3^0, s_{13}^0, s_{23}^0, s_{123}^0$. We want to construct a corresponding twisted version $(s, s_1, s_2, s_{12}, s_3, s_{13}, s_{23}, s_{123})$. We choose a folding parameter $k \neq 0$ in a suitable interval: our indirect numerical construction does not yet allow for twisting beyond (or too close to) an angle of $\frac{\pi}{2}$ between any primal and dual edge.

We recall the desired properties of the twisted cube:

- The cube has to be αk–bent, with, as usual $\alpha = rr_3$;
- the combinatorically interleaved cube has to be k–folded;
- the sphere radii stay the same as in the original cube: $r = r^0$ etc.;
- the distances between each pair of primal and corresponding dual vertex stay the same as in the original: $\|c - c_3\| = \left\|c^0 - c_3^0\right\|$ etc.;

- the dihedral angles formed by adjacent side faces along each vertical edge connecting a pair of primal and dual vertices stay the same as in the original cube: $\measuredangle_{c,c_3}(c_1, c_2) = \measuredangle_{c^0,c_3^0}(c_1^0, c_2^0)$, $\measuredangle_{c,c_3}(c_{13}, c_{23}) = \measuredangle_{c^0,c_3^0}(c_{13}^0, c_{23}^0)$ etc.

The first step will be to construct the side faces of the cube, formed by edge and corresponding dual. It is illustrated in figs. 4.28 and 4.29.

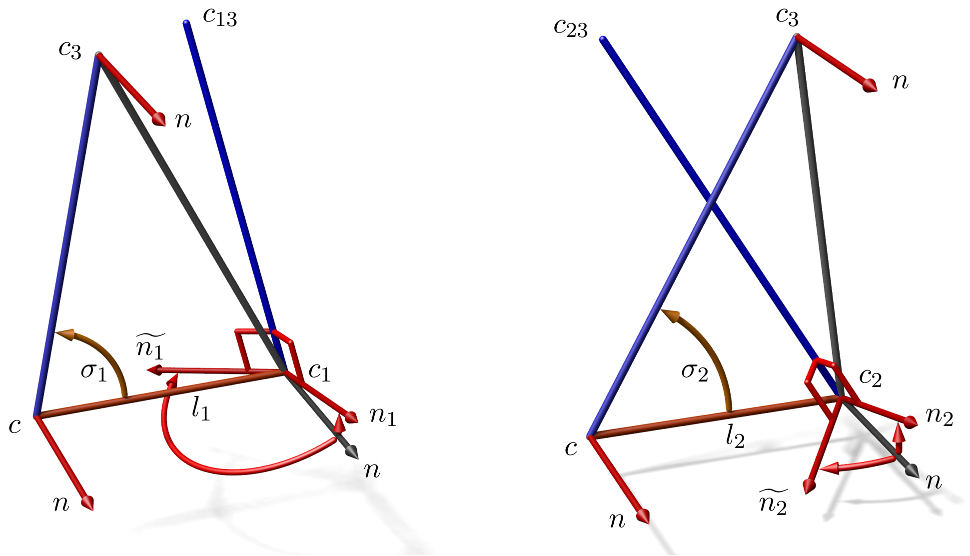

Figure 4.28: The first construction step for the side faces, giving c_{13} (or c_{23}) after choosing l_1 (or l_2) and σ_1 (or σ_2).

We know from the s–conical minimal case that the surface edge lengths $l_1 = \|c_1 - c\|$ and $l_2 = \|c_2 - c\|$ need not be preserved, so we keep them variable and build our quad for arbitrary values (but reasonably close to the original length). We will describe the construction for the face (s, s_1, s_3, s_{13}) which we assume to be the Moutard+ one in the original cube. The construction works the same for the other face.

After choosing a value for l_1, we introduce as another variable the angle $\sigma_1 = \measuredangle(c_1 - c, c_3 - c)$. Since we know the length $\|c_3 - c\|$, σ_1 determines the triangle (c, c_1, c_3). Let n be its normal. Since we know l_1 and the sphere radii, the condition of being αk–bent determines the rotation around the edge $c_1 - c$ that turns n into the normal n_1 of the plane containing c, c_1 and c_{13}. Similarly, $c_1 - c_3$ is an edge of the k–folded interleaved quad and allows us to rotate n into the normal $\widetilde{n_1}$ of the plane containing c_3, c_1 and c_{13}. Now, $c_{13} - c_1$ has to be orthogonal to n_1 and $\widetilde{n_1}$, and we know its length from the original cube. (For its direction, we can just assume it forms an acute angle with $c_3 - c$). This determines c_{13} as a function of σ_1.

Now, for each value of σ_1, we can compute the deviation of our quad from the desired properties: the vertical vertex distances (and sphere radii) are the original ones

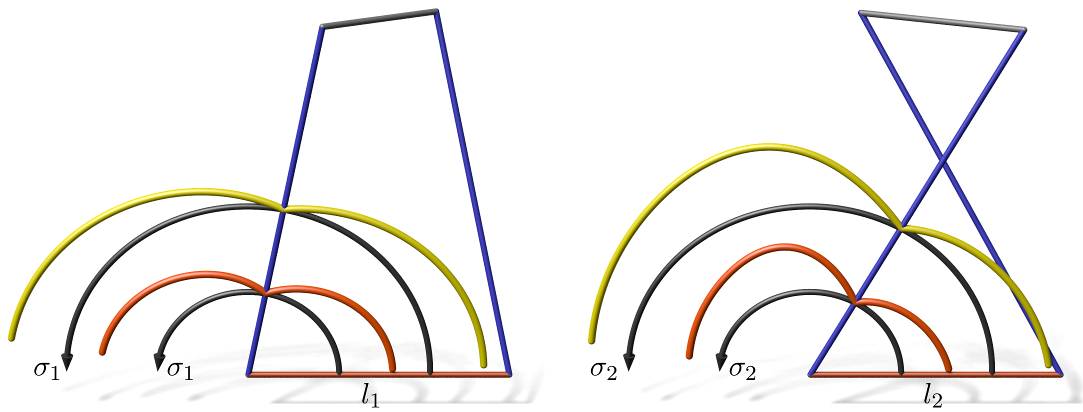

Figure 4.29: For the second step, we find the correct σ by numerical optimization. The yellow graph shows the deviation from the result being αk-bent, the orange one the deviation from the interleaved version being k–folded.

by construction, so just αk–bentness and k–foldedness remain to check.

Conjecture 4.5.1. *For any l_1 (or l_2) in a suitable neighborhood of the original edge length, there is a uniqe σ_1 (or σ_2) such that the above construction yields an αk–bent and interleavedly k–folded quadrilateral.*

This is backed by numerical calculations, as illustrated in fig. 4.29. We find the correct σ by numerical optimization.

To construct the whole cube, we first recall from the defs. 4.1.2 and 4.1.4 of bent and folded cubes that the bending/folding parameters of the side faces (s, s_1, s_3, s_{13}) and (s, s_2, s_3, s_{23}) need opposite signs. After choosing values for l_1 and l_2, we apply the above construction for k to (s, s_1, s_3, s_{13}) and for $-k$ to (s, s_2, s_3, s_{23}). We glue the two side faces along the edge c, c_3 such that the dihedral angle $\measuredangle_{c,c_3}(c_1, c_2)$ is the same as in the original cube. Note that being bent/folded for the same (up to sign) parameter ensures that this also holds for $\measuredangle_{c,c_3}(c_{13}, c_{23})$.

Now we can find c_{12} as the intersection point of three planes: as for the side faces, the condition that the bottom face (s, s_1, s_2, s_{12}) is αk–bent allows us to rotate the plane containing c, c_1 and c_2 (with normal say n) around $c_1 - c$ into the plane of c, c_1 and c_{12} (with normal n_1). For the second plane we rotate the plane containing c_1, c_{13} and c around $c_{13} - c_1$ to match the dihedral angle $\measuredangle_{c_1^0,c_{13}^0}(c^0, c_{12}^0)$ of the original cube to find the plane of c_1, c_{13} and c_{12}. For the third plane we do the same at the edge (c_2, c_{23}). Analogously, we find three planes intersecting in c_{123}. The construction is illustrated in fig. 4.30.

Altogether, we have found all vertices of a cube for each (reasonable) choice of l_1 and l_2. Now we have to adjust those edge lengths such that our cube satisfies the remaining

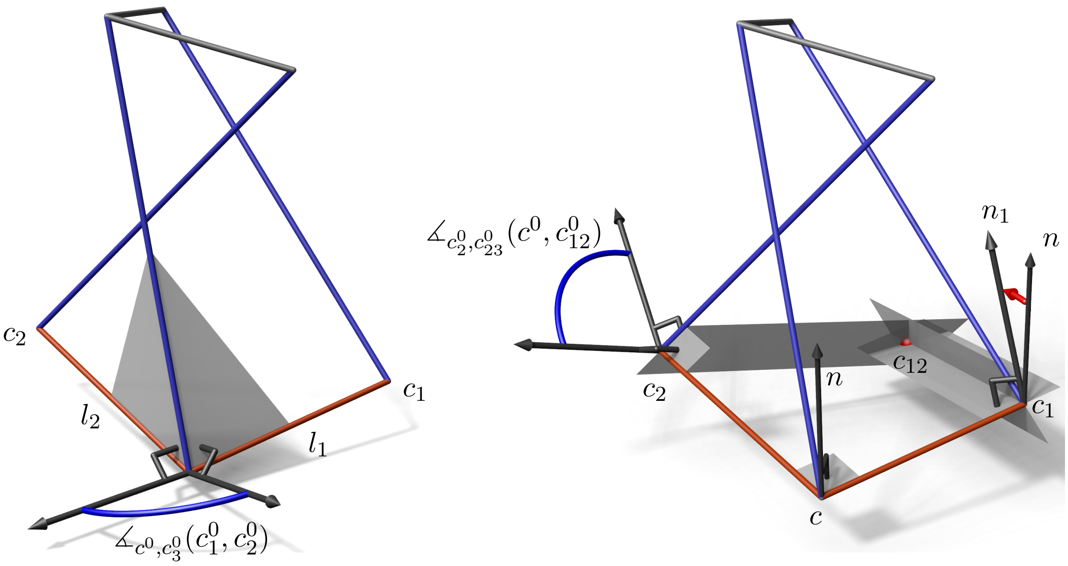

Figure 4.30: Assembly of the side faces and construction of c_{12}.

conditions:

- $\|c_{123} - c_{12}\| = \|c^0_{123} - c^0_{12}\|$,
- the cube being αk–bent,
- the interleaved cube being k–folded,
- and $\measuredangle_{c_{12},c_{123}}(c_1, c_2) = \measuredangle_{c^0_{12},c^0_{123}}(c^0_1, c^0_2)$, $\measuredangle_{c_{12},c_{123}}(c_{13}, c_{23}) = \measuredangle_{c^0_{12},c^0_{123}}(c^0_{13}, c^0_{23})$.

Conjecture 4.5.2. *There are unique values for l_1 and l_2 such that the above construction satisfies all conditions of a twisted s–isothermic cmc cube with folding parameter k.*

Again, this is based on numerical evidence, and we find the correct edge lengths by optimization. The dependence of the deviation from the desired properties on l_1 and l_2 is illustrated in figs. 4.31 to 4.35.

Before we can apply our construction to entire surfaces, we have to check that the cubes will fit together:

Conjecture 4.5.3. *When applied to adjacent cubes of a pair of dual s–isothermic cmc surfaces, the above construction yields the same lengths for common edges.*

In figs. 4.36 to 4.39 we show a few examples of our construction applied to Darboux transforms of cylinders.

Note that we cannot use the simplest example of the cylinder itself due to its highly symmetrical cubes: For the same radii, the original edge lengths l_1 and l_2 can take any

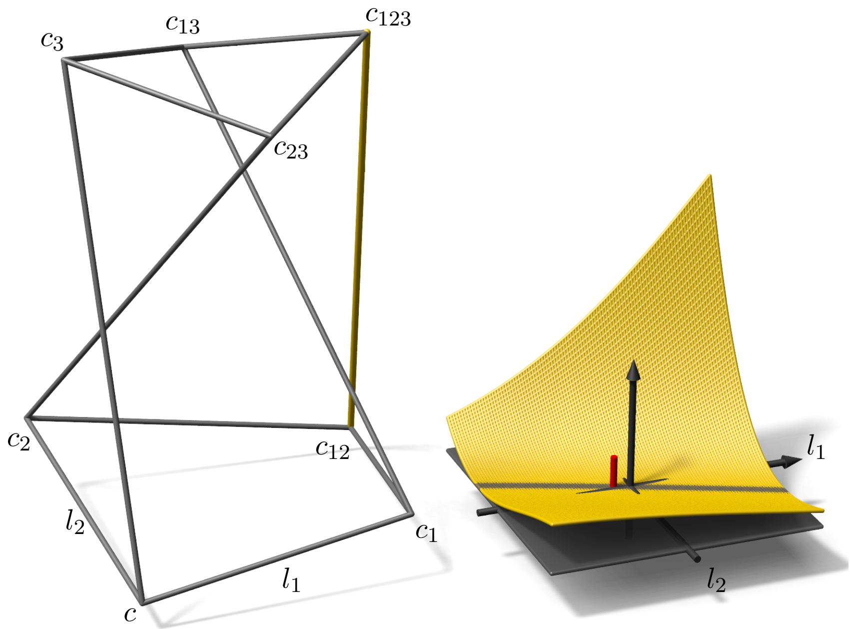

Figure 4.31: A graph of the (quadratic) deviation of $\|c_{123} - c_{12}\|$ from the original cube as a function of l_1 and l_2. The graph is centered at the original edge lengths, and covers a deviation of 80% – 120% of the original lengths. The minimum/zero is marked by the red peg.

value, and the construction would always yield an optimal cube without being able to find the one belonging to the specific original cube.

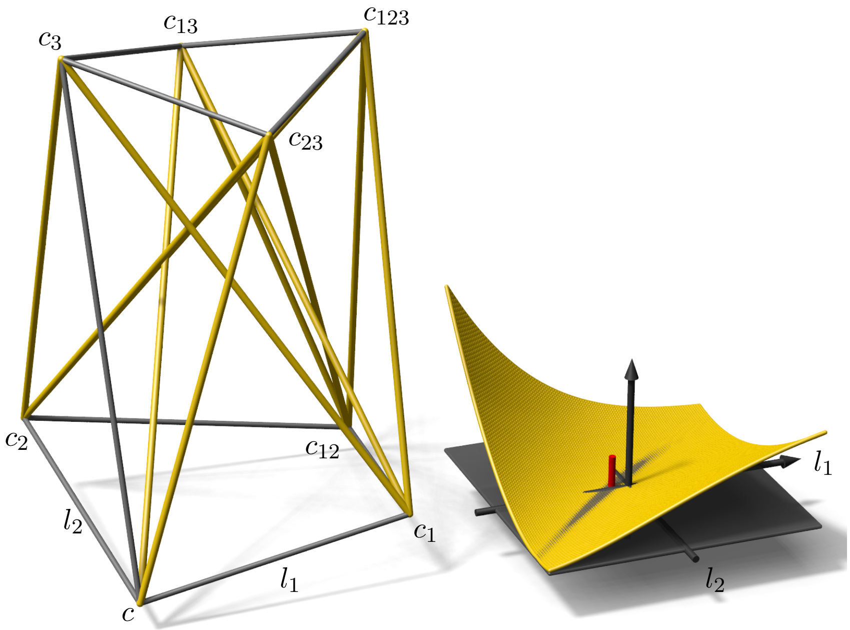

Figure 4.32: Deviation of the interleaved cube from being k–folded.

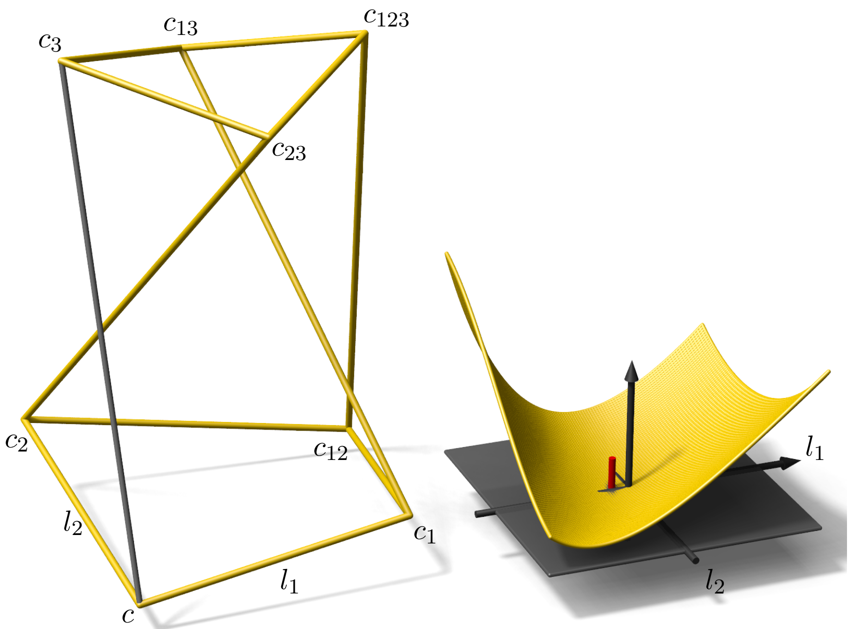

Figure 4.33: Deviation of the cube from being αk–bent.

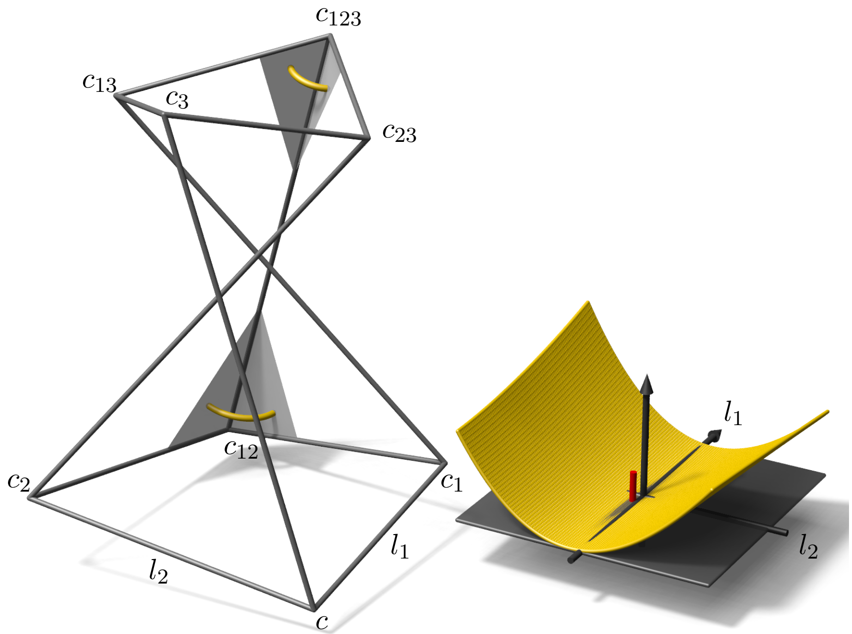

Figure 4.34: Deviation of the dihedral angles between side faces at the edge (c_{12}, c_{123}).

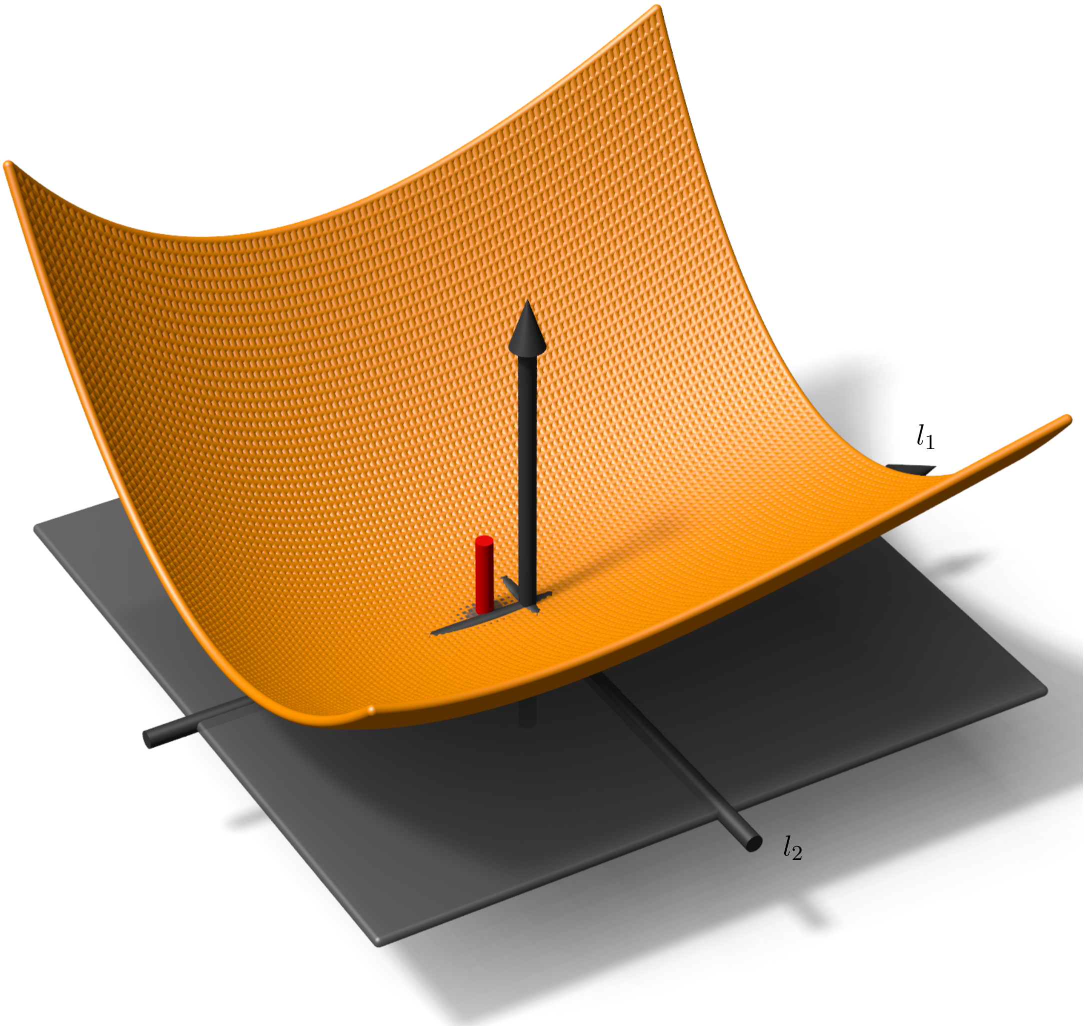

Figure 4.35: Aggregate deviation from all properties to check.

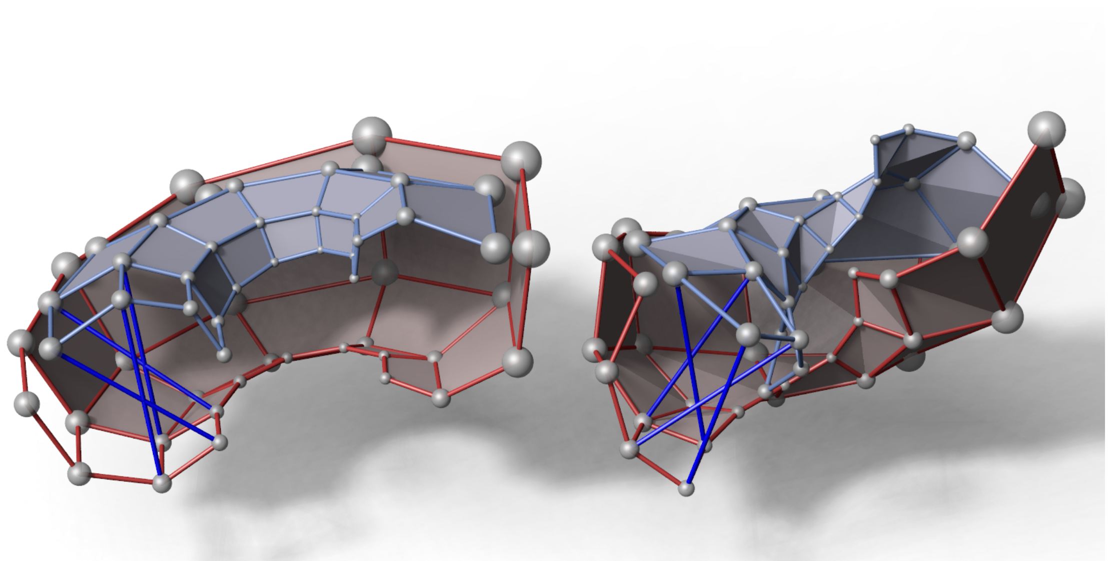

Figure 4.36: An example of primal and dual surface showing one of the cubes.

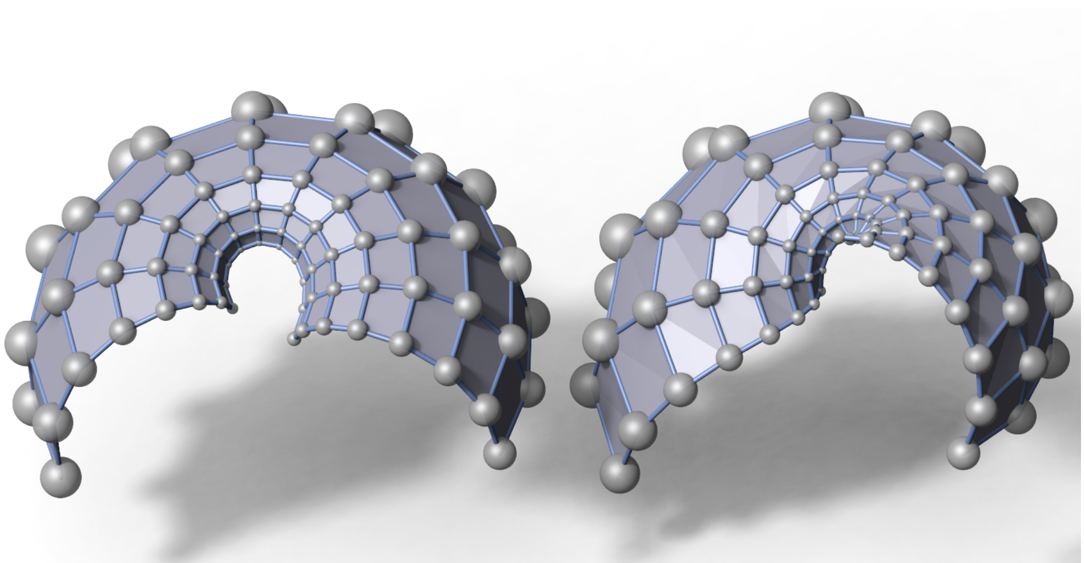

Figure 4.37: An example of a surface patch.

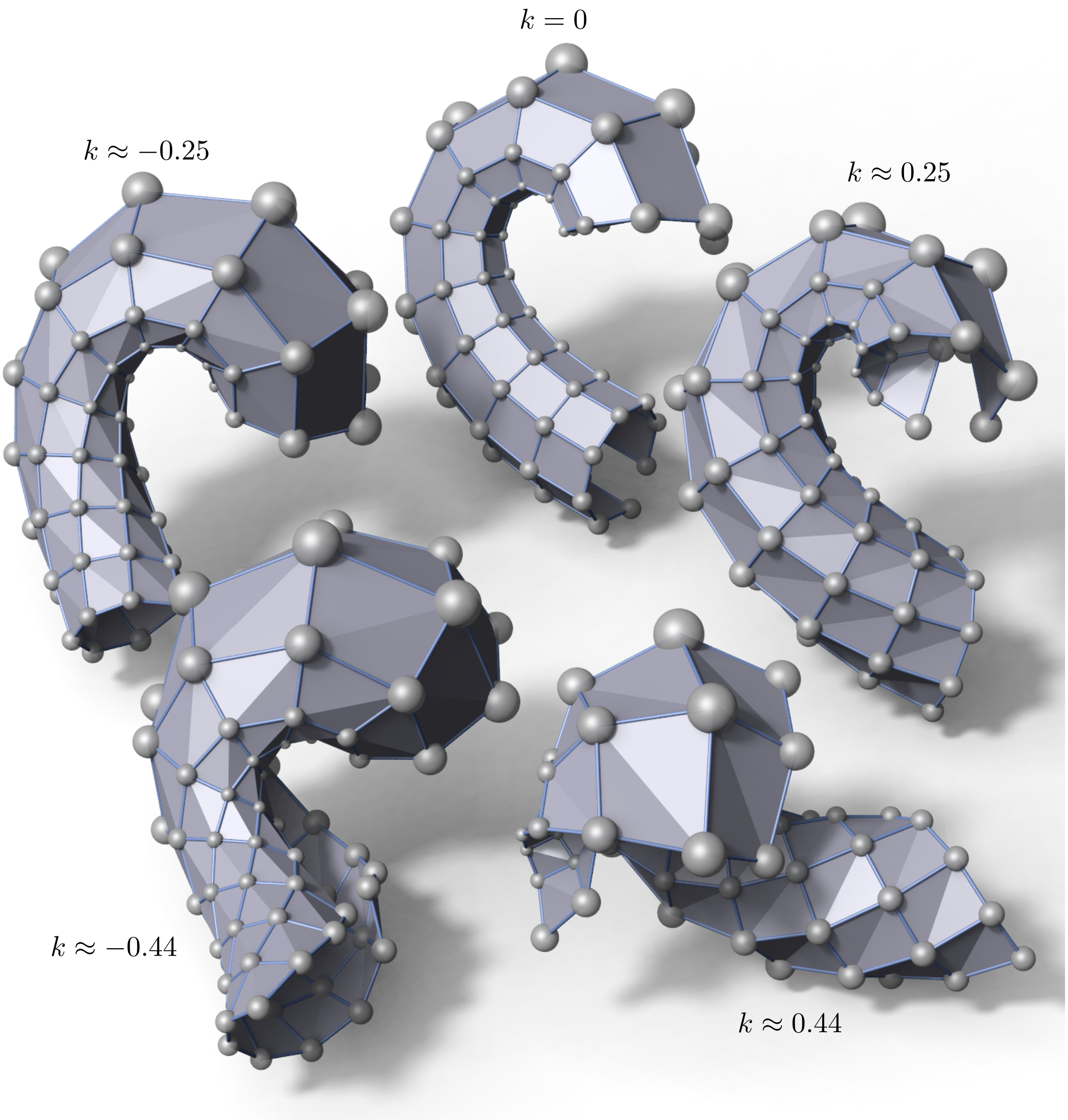

Figure 4.38: An example with various values of k.

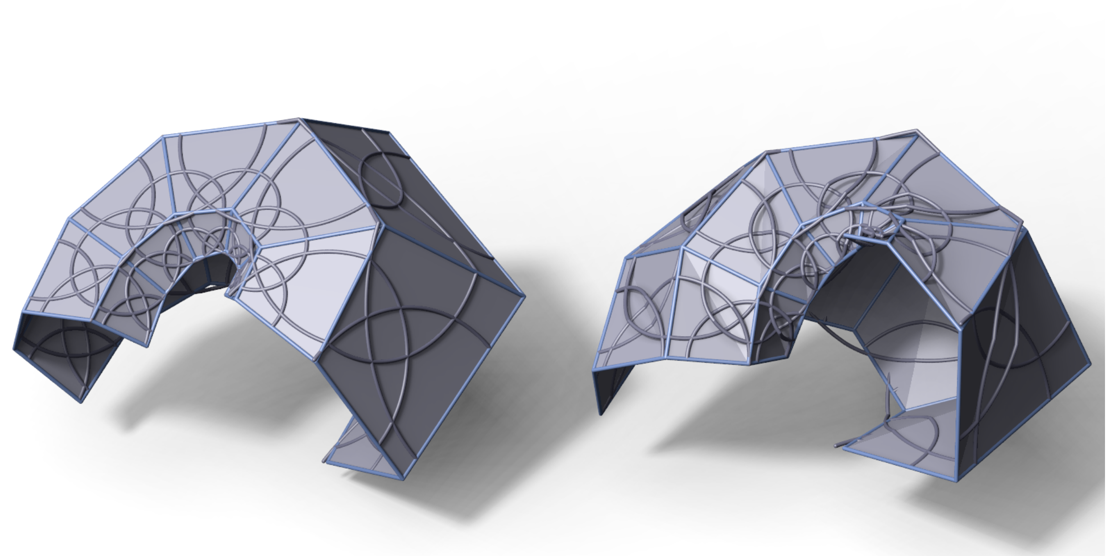

Figure 4.39: An example of a surface patch with larger sphere radii. Spheres are shown as their intersection with face planes.

4.6 S–Conical CMC Surfaces

Here we apply the construction from section 4.5 to the elementary cubes of s–conical cmc surfaces (cf. section 3.3.2). In the numerical results, we make a few observations that guide us to a more direct construction, which in turn allows us to twist s–conical cmc surfaces by an arbitrary angle. Again, most of the construction is only backed by numerical evidence, so the results remain conjectures for now. However, given the construction presented here is much more direct than the general one, we expect it to be more straightforward to actually prove.

We begin by observing the special properties of the cube constructed in section 4.5 when applied to an s–conical cmc cube.

Conjecture 4.6.1. *In the twisted version of an s–conical cmc cube, constructed as in section 4.5, the following properties hold:*

- *The diagonals (c, c_{12}), (c_1, c_2), (c_3, c_{123}) and (c_{13}, c_{23}) all intersect a common axis orthogonally.*
- *These diagonals retain their original length, and the tangent point of the spheres at their ends is the intersection point with the axis from above.*
- *The noncorresponding pairs of diagonals (c, c_{12}), (c_{13}, c_{23}) and (c_1, c_2), (c_3, c_{123}) (which are parallel in the original cube) form the same angle*

  $$\measuredangle(c_{12} - c, c_{13} - c_{23}) = \measuredangle(c_{123} - c_3, c_1 - c_2) =: \psi$$

  and are separated by the same distance h.
- *Consequently, the pairs of diagonals (c, c_{12}), (c_1, c_2) and (c_3, c_{123}), (c_{13}, c_{23}) of the primal and dual quadrilateral form the same angle*

  $$\measuredangle(c_{12} - c, c_1 - c_2) = \measuredangle(c_{123} - c_3, c_{13} - c_{23}) =: \sigma$$

  and are separated by the same distance d.

From now on, we prescribe ψ as the twisting parameter of our construction. As soon as we know h, d and σ, the properties above will yield a unique cube. In our next step, we want to find out more about the relationship of these three quantities.

First we note that conjecture 4.6.1 implies that the sphere quadrilateral (s_3, s_{123}, s_2, s_1) is of the same type as the side faces of an s_1–isothermic cmc cube (cf. section 4.4). From the calculations there, we know that the corresponding interleaved quadrilateral (s_3, s_1, s_2, s_{123}) is k–folded for

$$k = \frac{\sin\psi}{h}.$$

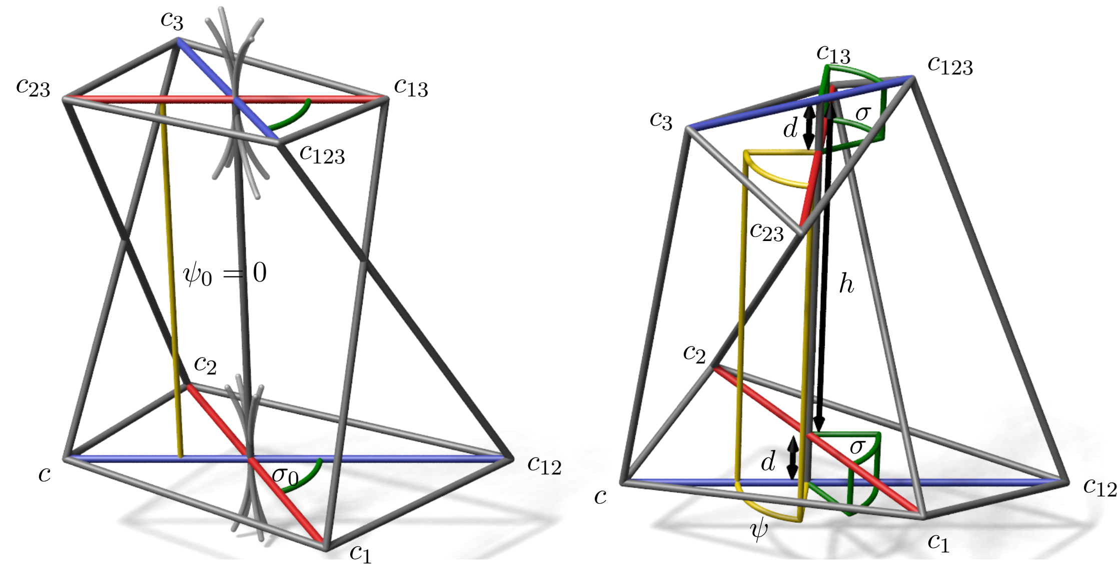

Figure 4.40: Original s–conical cmc cube and twisted version.

But this quadrilateral is the top (or bottom) face of the interleaved version of our s–conical cube, so this k must be the folding parameter of our cube.

Similarly, we look at the sphere quadrilateral (s, s_1, s_2, s_{12}) and notice that it is of the same type as the quadrilaterals of s–conical minimal surfaces (cf. section 4.3). From the calculations in proposition 4.3.13, together with lemma 4.3.4, we can conclude that our quadrilateral is a–bent for

$$a = \frac{d}{\sin\sigma}.$$

As a face of our twisted cube, it must be αk–bent (with, as always, $\alpha = rr_3$). Put together, we know that

$$k = \frac{\sin\psi}{h} = \frac{d}{\alpha\sin\sigma},$$

or

$$dh = \alpha\sin\sigma\sin\psi. \tag{4.2}$$

These observations are illustrated in fig. 4.41.

We further exploit that in our twisted cube, the vertical edge length $l = \|c_3 - c\|$ is supposed to be constant. Given the geometry described in conjecture 4.6.1, we can calculate

$$\begin{aligned} l^2 &= r^2 + r_3^2 - 2rr_3\cos(\psi - \sigma) + (d+h)^2 \\ &= r^2 + r_3^2 - 2\alpha(\cos\psi\cos\sigma + \sin\psi\sin\sigma) + d^2 + h^2 + 2dh \\ &= r^2 + r_3^2 - 2\alpha\cos\psi\cos\sigma + d^2 + h^2. \end{aligned}$$

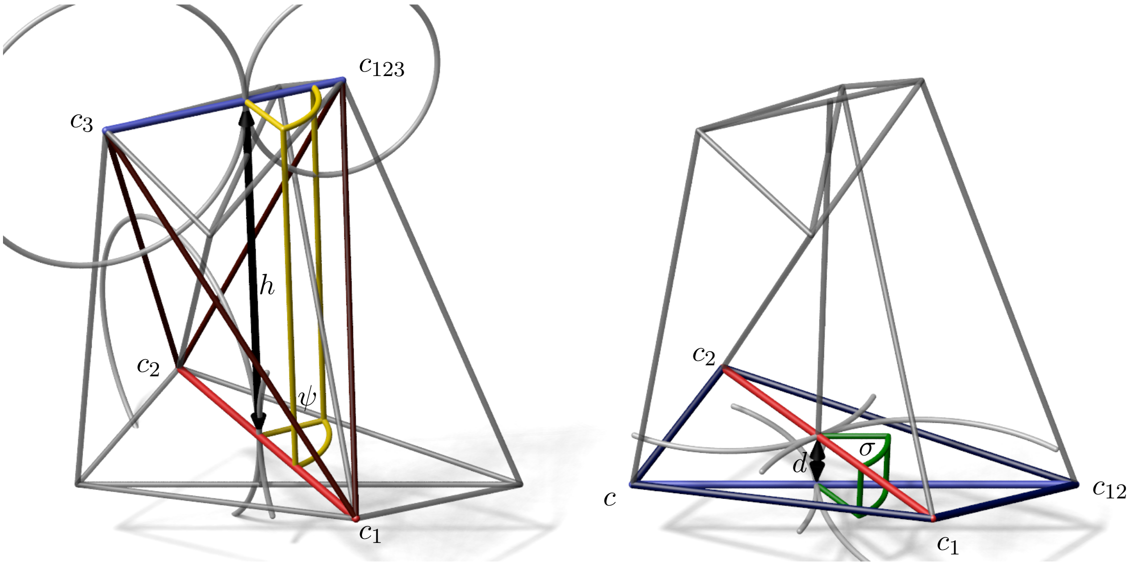

Figure 4.41: The quadrilaterals (s_3, s_1, s_2, s_{123}) and (s, s_1, s_2, s_{12}) in an s–conical cmc cube.

Using eq. (4.2) once more to eliminate d, this gives us the equation

$$h^4 + (r^2 + r_3^2 - l^2 - 2\alpha \cos\psi \cos\sigma)h^2 + \alpha^2 \sin^2\sigma \sin^2\psi = 0.$$

With l known from the original cube, for a given ψ we can choose a σ and solve for h^2, also determining d. Among the solutions for h^2, numerical evidence suggests there is an obvious candidate much closer to the original h_0 than the other.

To find the correct value for our last parameter σ, we recall that in our twisted cube, the dihedral angles $\measuredangle_{c,c_3}(c_1, c_2)$ etc. are supposed to be constant as well. For now, we numerically minimize the deviation of these dihedral angles from the ones in the untwisted cube over the variable σ.

Conjecture 4.6.2. *Given an (untwisted) s–conical cmc cube and an angle ψ, there is a unique σ such that the construction outlined above yields the corresponding twisted cmc cube with folding parameter $k = \frac{\sin\psi}{h}$.*

This is again backed by numerical experiments; an illustration is given in fig. 4.42.

Now we apply this construction simultaneously to the cubes of a surface. We see

Conjecture 4.6.3. *If the construction above is applied to all cubes of a common dual pair of s–conical cmc surfaces for the same value of ψ, the resulting σ, h and d are the same for all cubes as well. Adjacent twisted cubes fit together.*

This allows us to twist whole surfaces, and we propose the result as a construction for the associated family of s–conical cmc surfaces.

Figures 4.43 to 4.45 illustrate some examples.

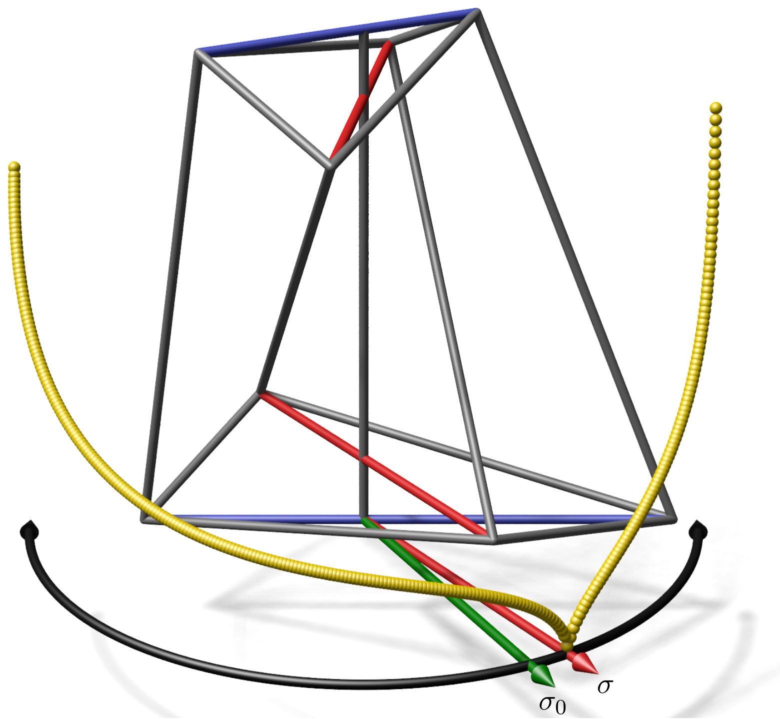

Figure 4.42: A plot of the aggregate deviation of dihedral angles along the vertical edges as a function of σ.

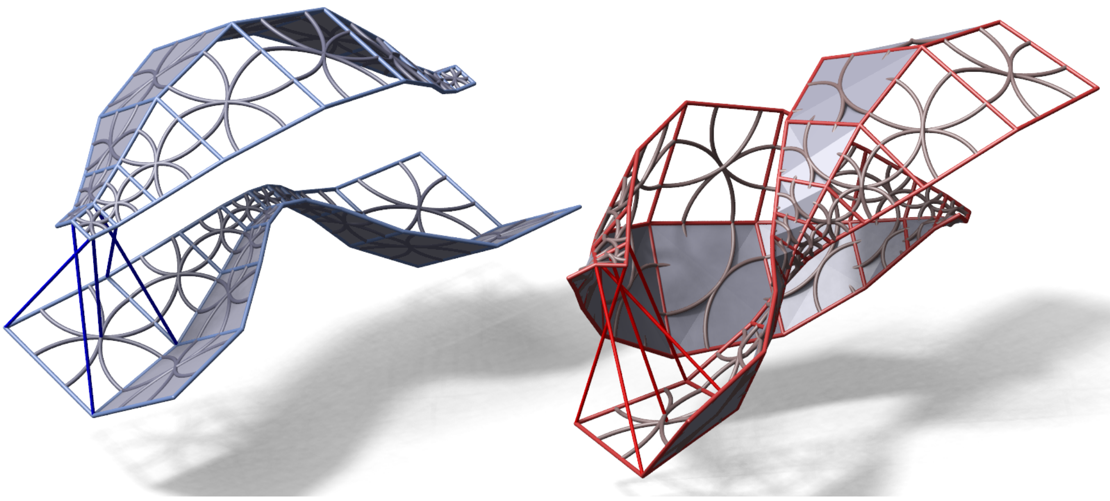

Figure 4.43: A strip of s–conical Delaunay surface with its dual and the twisted version for $\psi = \frac{\pi}{3}$. The highlighted vertical edges and axes indicate one of the cubes.

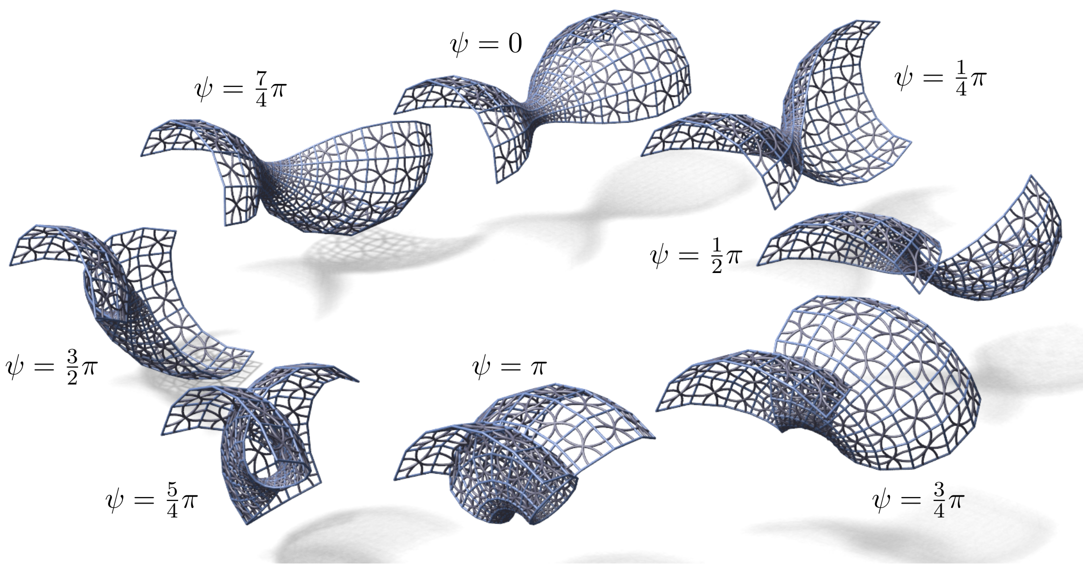

Figure 4.44: Another example of an s–conical Delaunay surface for various values of ψ.

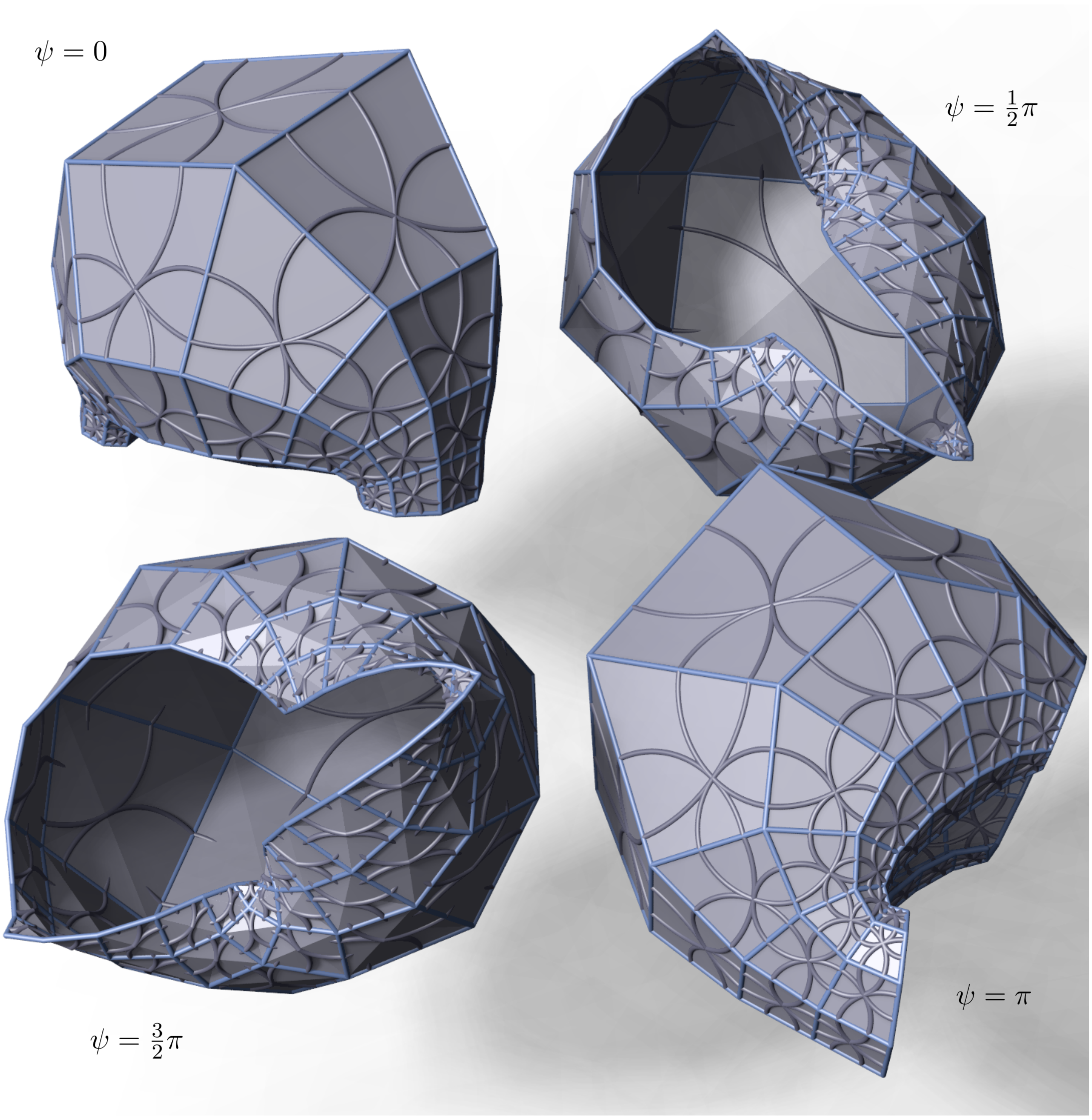

Figure 4.45: A Darboux transform of an s–conical Delaunay surface for some values of ψ.

Chapter 5

Conclusion

For the first time, we have explored the previously unknown associated family of s–isothermic cmc surfaces. Since the existence of the associated family is an innate property of cmc surfaces, any structure–preserving discretization must allow for an associated family as well. With our findings, we confirm that the s_1–isothermic discretization is a full–fledged theory no less powerful than the discrete isothermic one, and provide solid numerical evidence that this extends to the full generality of s–isothermic discretizations.

We are confident that the constructions we propose will be confirmed once a complete theory with a moving frame description and Sym–Bobenko formula for s–isothermic cmc surfaces is found, with a spectral parameter closely related to the global parameter k of our bent and folded quadrilaterals and cubes. We hope that the availability of examples, albeit only numerical in the general case, will greatly help the effort to find such a theory.

On a more general note, we hope our examples will contribute to the development of a general curvature theory that extends the one from [HSW16] to include the entire class of s–isothermic discretizations and their associated families. Such a theory will hopefully confirm that our constructions preserve the surfaces' constant mean curvature throughout the associated family.

Meanwhile, near–future work could find a more elegant proof of proposition 4.4.23 and take at least the s–conical construction from a numerics–based conjecture to a thorough level in analogy to the s_1–isothermic case.

In the general case, we expect both the folded and bent cube properties to form a 3D–compatible system reminiscent of the equally–folded parallelogram cubes appearing for discrete isothermic cmc surfaces. Exploring these systems should lead to a better understanding of non–planar geometry derived from the s–isothermic world.

In lieu of a concluding statement, in fig. 5.1 we present an illustration of an example containing various constructions of this work: a general Delaunay surface from section 3.4.2 in which some quadrilaterals are of s_1–isothermic or s–conical type, all twisted into the associated family simultaneously.

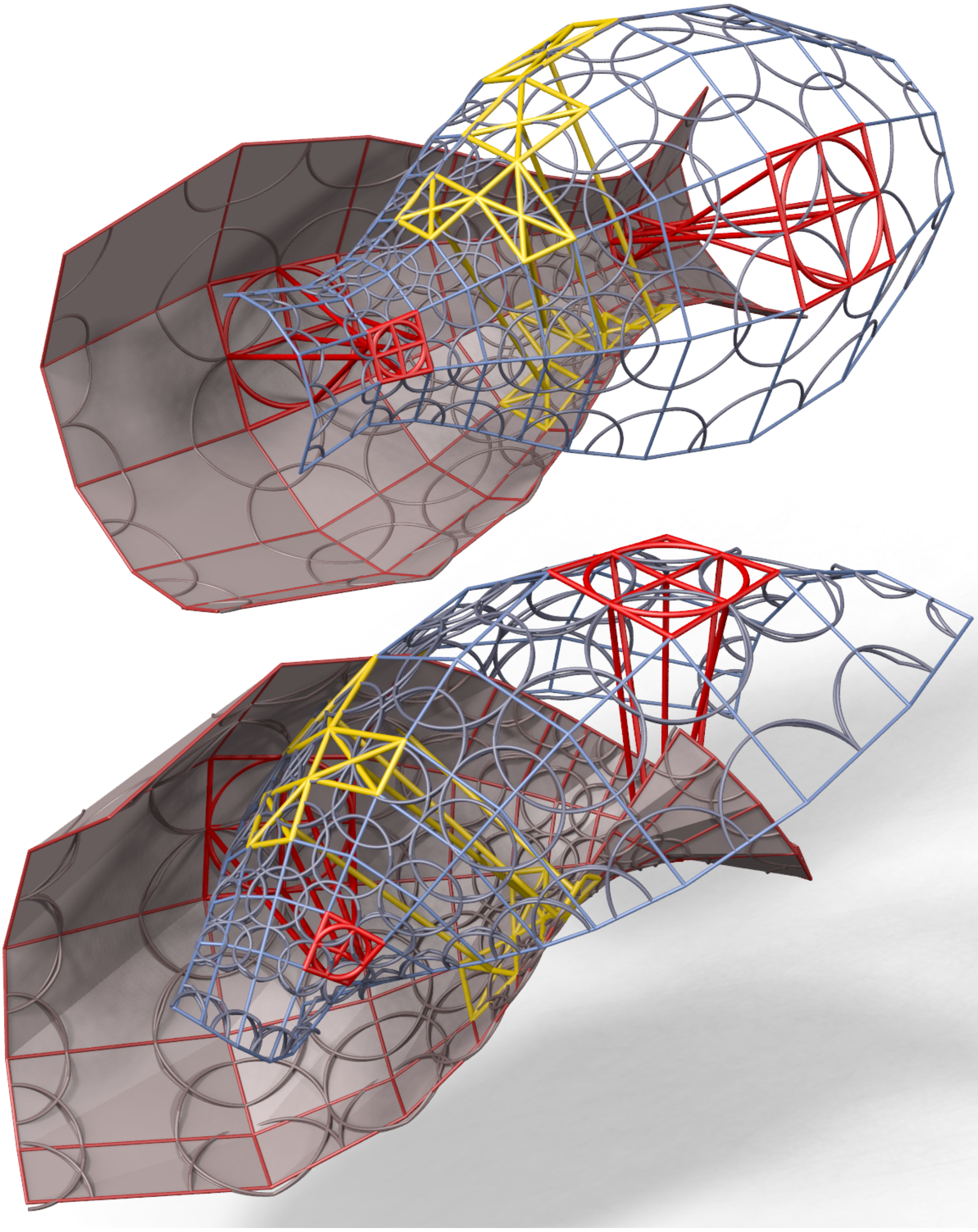

Figure 5.1: An unduloid with various edge scalar products and a member of the associated family (with $k \approx 0.18$). Occurring s_1–isothermic cubes are drawn in red, s–conical ones in yellow.

Appendix A

Calculations for Proposition 4.4.23

In the following, we consider an s_1–isothermic cmc cube, with notation for all quantities as in section 4.4.1. We pick up the situation of proposition 4.4.23 to complete the calculations.

First, we introduce abbreviations for frequently occurring terms:

Definition A.0.1.

$$A_1 := \alpha + s_{12}s_{13}, \quad B_1 := s_{13}R^2 - \alpha s_{12}, \quad C_1 := s_{13}^2R^2 + \alpha^2;$$

$$A_2 := \alpha + ss_{23}, \quad B_2 := s_{23}R^2 - \alpha s, \quad C_2 := R^2 + s^2;$$

$$A_3 := \alpha + ss_{13}, \quad B_3 := s_{13}R^2 - \alpha s, \quad C_3 := s_{23}^2R^2 + \alpha^2;$$

$$A_4 := \alpha + s_{12}s_{23}, \quad B_4 := s_{23}R^2 - \alpha s_{12}, \quad C_4 := R^2 + s_{12}^2;$$

and

$$D := h^2 + R^2 + R^{*2}.$$

Since in the untwisted case the planar quadrilateral v, v_1, v_2, v_{12} has edges tangent to the orthogonal circle, we can use facts about tangential quadrilaterals to derive some relations that we will need to simplify our calculations:

Lemma A.0.1.

$$A_1B_2 + A_2B_1 = 0 \quad \textit{and} \quad A_3B_4 + A_4B_3 = 0,$$

$$B_4C_1C_2 + B_3(R^2A_1A_2 - B_1B_2) = 0,$$

$$A_4C_1C_2 - A_3(R^2A_1A_2 - B_1B_2) = 0;$$

$$A_1B_3 = \frac{s_{12}s_{13} - ss_{23}}{s_{13} + s_{23}}C_1 \quad \textit{and} \quad A_3B_1 = \frac{ss_{13} - s_{12}s_{23}}{s_{13} + s_{23}}C_1,$$

$$A_1B_3 + A_3B_1 = (s + s_{12})\frac{s_{13} - s_{23}}{s_{13} + s_{23}}C_1;$$

$$A_2B_3 = \alpha\frac{s_{12}s_{13} - ss_{23}}{s + s_{12}}C_2 \quad and \quad A_3B_2 = \alpha\frac{s_{12}s_{23} - ss_{13}}{s + s_{12}}C_2,$$

$$A_2B_3 + A_3B_2 = \alpha(s_{13} + s_{23})\frac{s_{12} - s}{s + s_{12}}C_2.$$

Proof. In a tangential quadrilateral, the radius of the incircle can be calculated from the tangent lengths — in our case the sphere radii — as

$$R^2 = \frac{ss_1s_2 + ss_1s_{12} + ss_2s_{12} + s_1s_2s_{12}}{s + s_1 + s_2 + s_{12}},$$

cf. eg. [Jos10]. After we replace the appropriate sphere radii with the dual radii we want to consider, this reformulates to

$$\begin{aligned} & R^2\left(s + \frac{\alpha}{s_{13}} + \frac{\alpha}{s_{23}} + s_{12}\right) = s\frac{\alpha^2}{s_{13}s_{23}} + s\frac{\alpha}{s_{13}}s_{12} + s\frac{\alpha}{s_{23}}s_{12} + \frac{\alpha^2}{s_{13}s_{23}}s_{12} \\ \Leftrightarrow \quad & R^2(ss_{13}s_{23} + \alpha s_{23} + \alpha s_{13} + s_{12}s_{13}s_{23}) = \alpha(\alpha s + ss_{23}s_{12} + ss_{13}s_{12} + \alpha s_{12}). \end{aligned}$$

We regroup this in three ways:

i)

$$\begin{aligned} & R^2\left(s_{13}(\alpha + ss_{23}) + s_{23}(\alpha + s_{12}s_{13})\right) = \alpha\left(s_{12}(\alpha + ss_{23}) + s(\alpha + s_{12}s_{13})\right) \\ \Leftrightarrow \quad & R^2\left(s_{13}A_2 + s_{23}A_1\right) = \alpha\left(s_{12}A_2 + sA_1\right) \\ \Leftrightarrow \quad & A_1(s_{23}R^2 - \alpha s) = A_2(\alpha s_{12} - s_{13}R^2) \\ \Leftrightarrow \quad & A_1B_2 = -A_2B_1, \end{aligned}$$

giving the first identity;

ii)

$$\begin{aligned} & R^2\left(s_{13}(\alpha + s_{12}s_{23}) + s_{23}(\alpha + ss_{13})\right) = \alpha\left(s_{12}(\alpha + ss_{13}) + s(\alpha + s_{12}s_{23})\right) \\ \Leftrightarrow \quad & R^2\left(s_{13}A_4 + s_{23}A_3\right) = \alpha\left(s_{12}A_3 + sA_4\right) \\ \Leftrightarrow \quad & A_3(s_{23}R^2 - \alpha s_{12}) = A_4(\alpha s - s_{13}R^2) \\ \Leftrightarrow \quad & A_3B_4 = -A_4B_3, \end{aligned}$$

giving the second identity;

iii) and as

$$\begin{aligned} & s_{13}s_{23}R^2(s + s_{12}) + \alpha R^2(s_{13} + s_{23}) = \alpha ss_{12}(s_{13} + s_{23}) + \alpha^2(s + s_{12}) \\ \Leftrightarrow \quad & (s + s_{12})(s_{13}s_{23}R^2 - \alpha^2) = \alpha(s_{13} + s_{23})(ss_{12} - R^2). \end{aligned} \tag{A.1}$$

For the third identity, we expand and simplify the term in question:

$$
\begin{aligned}
&B_4C_1C_2 + B_3(R^2A_1A_2 - B_1B_2)\\
&= (s_{23}R^2 - \alpha s_{12})(s_{13}^2R^2 + \alpha^2)(R^2 + s^2)\\
&\quad + (s_{13}R^2 - \alpha s)\left(R^2(\alpha + s_{12}s_{13})(\alpha + ss_{23}) - (s_{13}R^2 - \alpha s_{12})(s_{23}R^2 - \alpha s)\right)\\
&= (s_{23}R^2 - \alpha s_{12})(s_{13}^2R^4 + R^2(\alpha^2 + s^2s_{13}^2) + \alpha^2s^2)\\
&\quad + (s_{13}R^2 - \alpha s)\Big(R^2\big(\alpha^2 + \alpha(s_{12}s_{13} + ss_{23}) + ss_{12}s_{13}s_{23}\big)\\
&\qquad - s_{13}s_{23}R^4 + \alpha R^2(ss_{13} + s_{12}s_{23}) - \alpha^2ss_{12}\Big)\\
&= s_{13}^2s_{23}R^6 + \alpha^2s_{23}R^4 + s^2s_{13}^2s_{23}R^4 + \alpha^2s^2s_{23}R^2\\
&\quad - \alpha s_{12}s_{13}^2R^4 - \alpha^3s_{12}R^2 - \alpha s^2s_{12}s_{13}^2R^2 - \alpha^3s^2s_{12}\\
&\quad + \alpha^2s_{13}R^4 + \alpha s_{12}s_{13}^2R^4 + \alpha ss_{13}s_{23}R^4 + ss_{12}s_{13}^2s_{23}R^4\\
&\quad - s_{13}^2s_{23}R^6 + \alpha ss_{13}^2R^4 + \alpha s_{12}s_{13}s_{23}R^4 - \alpha^2ss_{12}s_{13}R^2\\
&\quad - \alpha^3sR^2 - \alpha^2ss_{12}s_{13}R^2 - \alpha^2s^2s_{23}R^2 - \alpha s^2s_{12}s_{13}s_{23}R^2\\
&\quad + \alpha ss_{13}s_{23}R^4 - \alpha^2s^2s_{13}R^2 - \alpha^2ss_{12}s_{23}R^2 + \alpha^3s^2s_{12}\\
&= \alpha^2R^4(s_{13} + s_{23}) + ss_{13}^2s_{23}R^4(s + s_{12}) - \alpha^3R^2(s + s_{12})\\
&\quad - \alpha s^2s_{12}s_{13}R^2(s_{13} + s_{23}) + \alpha s_{13}s_{23}R^4(s + s_{12}) + \alpha ss_{13}R^4(s_{13} + s_{23})\\
&\quad - \alpha^2ss_{13}R^2(s + s_{12}) - \alpha^2ss_{12}R^2(s_{13} + s_{23})\\
&= (s + s_{12})(ss_{13}^2s_{23}R^4 - \alpha^3R^2 + \alpha s_{13}s_{23}R^4 - \alpha^2ss_{13}R^2)\\
&\quad + (s_{13} + s_{23})(\alpha^2R^4 - \alpha s^2s_{12}s_{13}R^2 + \alpha ss_{13}R^4 - \alpha^2ss_{12}R^2)\\
&= R^2(s + s_{12})(s_{13}s_{23}R^2(ss_{13} + \alpha) - \alpha^2(\alpha + ss_{13}))\\
&\quad + \alpha R^2(s_{13} + s_{23})(R^2(\alpha + ss_{13}) - ss_{12}(ss_{13} + \alpha))\\
&= R^2A_3\left((s + s_{12})(s_{13}s_{23}R^2 - \alpha^2) + \alpha(s_{13} + s_{23})(R^2 - ss_{12})\right),
\end{aligned}
$$

which vanishes by eq. (A.1).

Similarly, we calculate the next identity:

$$
\begin{aligned}
&A_4C_1C_2 - A_3(R^2A_1A_2 - B_1B_2)\\
&= (\alpha + s_{12}s_{23})(s_13^2R^2 + \alpha^2)(R^2 + s^2)\\
&\quad - R^2(\alpha + ss_{13})(\alpha + s_{12}s_{13})(\alpha + ss_{23})\\
&\quad + (\alpha + ss_{13})(s_{13}R^2 - \alpha s_{12})(s_{23}R^2 - \alpha s)\\
&= \alpha s_{13}^2R^4 + \alpha^3R^2 + s_{12}s_{13}^3s_{23}R^4 + \alpha^2s_{12}s_{23}R^2 + \alpha s^2s_{13}^2R^2 + \alpha^3s^2\\
&\quad + s^2s_{12}s_{13}^2s_{23}R^2 + \alpha^2s^2s_{12}s_{23} - \alpha^3R^2 - \alpha^2ss_{23}R^2 - \alpha^2s_{12}s_{13}R^2\\
&\quad - \alpha ss_{12}s_{13}s_{23}R^2 - \alpha^2ss_{13}R^2 - \alpha s^2s_{13}s_{23}R^2 - \alpha ss_{12}s_{13}^2R^2 - s^2s_{12}s_{13}^2s_{23}R^2\\
&\quad + \alpha s_{13}s_{23}R^4 - \alpha^2ss_{13}R^2 - \alpha^2s_{12}s_{23}R^2 + \alpha^3ss_{12} + ss_{13}^2s_{23}R^4\\
&\quad - \alpha s^2s_{13}^2R^2 - \alpha ss_{12}s_{13}s_{23}R^2 + \alpha^2s^2s_{12}s_{13}
\end{aligned}
$$

$$
\begin{aligned}
&= (s_{13}R^2 - \alpha s)\big(\alpha s_{13}R^2 + s_{12}s_{13}s_{23}R^2 - \alpha^2 s - \alpha s s_{12}s_{23} + \alpha s_{23}R^2 - \alpha^2 s_{12} \\
&\quad + s s_{13}s_{23}R^2 - \alpha s s_{12}s_{13}\big) \\
&= B_3\big((s_{13}+s_{23})\alpha R^2 - (s+s_{12})\alpha^2 - \alpha s s_{12}(s_{13}+s_{23}) + s_{13}s_{23}R^2(s+s_{12})\big) \\
&= B_3\big(\alpha(s_{13}+s_{23})(R^2 - s s_{12}) + (s+s_{12})(s_{13}s_{23}R^2 - \alpha^2)\big),
\end{aligned}
$$

which again vanishes by eq. (A.1).

For the identities about A_1B_3 and A_3B_1, we see from eq. (A.1) that

$$
\alpha(R^2 - s s_{12}) = \frac{s+s_{12}}{s_{13}+s_{23}}(\alpha^2 - s_{13}s_{23}R^2)
$$

and calculate

$$
\begin{aligned}
A_1B_3 &= (\alpha + s_{12}s_{13})(s_{13}R^2 - \alpha s) \\
&= \alpha s_{13}(R^2 - s s_{12}) + s_{12}s_{13}^2R^2 - \alpha^2 s \\
&= \frac{1}{s_{13}+s_{23}}\big(s_{13}(s+s_{12})(\alpha^2 - s_{13}s_{23}R^2) + (s_{13}+s_{23})(s_{12}s_{13}^2R^2 - \alpha^2 s)\big) \\
&= \frac{1}{s_{13}+s_{23}}\big(\alpha^2(s s_{13} + s_{12}s_{13} - s s_{13} - s s_{23}) \\
&\quad + s_{13}^2R^2(-s s_{23} - s_{12}s_{23} + s_{12}s_{13} + s_{12}s_{23})\big) \\
&= \frac{1}{s_{13}+s_{23}}C_1(s_{12}s_{13} - s s_{23});
\end{aligned}
$$

the calculation for A_3B_1 is the same with s and s_{12} interchanged. Their sum is quickly calculated as

$$
\begin{aligned}
A_1B_3 + A_3B_1 &= \frac{C_1}{s_{13}+s_{23}}\big(s_{12}s_{13} - s s_{23} + s s_{13} - s_{12}s_{23}\big) \\
&= \frac{C_1}{s_{13}+s_{23}}(s+s_{12})(s_{13}-s_{23}).
\end{aligned}
$$

Analogously for A_2B_3 and A_3B_2: we reformulate eq. (A.1) as

$$
s_{13}s_{23}R^2 - \alpha^2 = \alpha\frac{s_{13}+s_{23}}{s+s_{12}}(s s_{12} - R^2)
$$

and calculate

$$
\begin{aligned}
A_2B_3 &= (\alpha + s s_{23})(s_{13}R^2 - \alpha s) \\
&= s(s_{13}s_{23}R^2 - \alpha^2) + \alpha(s_{13}R^2 - s^2 s_{23}) \\
&= \frac{1}{s+s_{12}}\big(\alpha s(s_{13}+s_{23})(s s_{12} - R^2) + \alpha(s+s_{12})(s_{13}R^2 - s^2 s_{23})\big) \\
&= \frac{\alpha}{s+s_{12}}\big(R^2(-s s_{13} - s s_{23} + s s_{13} + s_{12}s_{13}) \\
&\quad + s^2(s_{12}s_{13} + s_{12}s_{23} - s s_{23} - s_{12}s_{23})\big) \\
&= \frac{\alpha}{s+s_{12}}C_2(s_{12}s_{13} - s s_{23});
\end{aligned}
$$

again A_3B_2 is the same with with s_{13} and s_{23} interchanged. Finally, their sum is

$$\begin{aligned} A_2B_3 + A_3B_2 &= \frac{\alpha C_2}{s+s_{12}}\big(s_{12}s_{13} - ss_{23} + s_{12}s_{23} - ss_{13}\big) \\ &= \frac{\alpha C_2}{s+s_{12}}(s_{12}-s)(s_{13}+s_{23}). \end{aligned}$$

□

Now we want to show k–foldedness of the sphere–quadrilateral (T, T_1, T_2, T_{12}) of the interleaved cube. We will perform the calculations for the edge

$$f = f^1 = w_1 - w = v_{13} - v,$$

with adjacent edges

$$f^2 = w_2 - w = v_{23} - v \quad \text{and} \quad f_1^2 = w_{12} - w_1 = v_{12} - v_{13}.$$

We will have to show that

$$\frac{1}{\|f\times f^2\|\,\|f\times f_1^2\|}\big(f\times f^2\big)\times\big(f\times f_1^2\big) = \frac{k}{\sqrt{(1+s^2k^2)(1+s_{13}^2k^2)}}f.$$

Since by the properties of the cross product

$$\big(f\times f^2\big)\times\big(f\times f_1^2\big) = \det(f,f^2,f_1^2)f = \langle f, f^2\times f_1^2\rangle f,$$

this is equivalent to

$$\langle f, f^2\times f_1^2\rangle\sqrt{(1+s^2k^2)(1+s_{13}^2k^2)} = k\,\big\|f\times f^2\big\|\,\big\|f\times f_1^2\big\|\,.$$

To calculate this quantities, we need the coordinates of v, v_{12}, v_{13} and v_{23}.

As in proposition 4.4.23, we denote the planar coordinates of the untwisted case by capital letters. For the primal quadrilateral (S, S_1, S_2, S_{12}), we already have found

$$V = \begin{pmatrix}-s\\-R\end{pmatrix} \quad \text{and} \quad V_1 = \begin{pmatrix}s_1\\-R\end{pmatrix},$$

and we can extend the adjacent edges to find V_2 and V_{12}:

$$V_2 = V + \frac{s+s_2}{s}(Y-V) = \frac{1}{R^2+s^2}\begin{pmatrix}-2sR^2+s^2s_2-s_2R^2\\ R(-R^2+s^2+2ss_2)\end{pmatrix},$$

$$V_{12} = V_1 + \frac{s_1+s_{12}}{s_1}(Y_1-V_1) = \frac{1}{R^2+s_1^2}\begin{pmatrix}2s_1R^2-s_1^2s_{12}+s_{12}R^2\\ R(-R^2+s_1^2+2s_1s_{12})\end{pmatrix}.$$

For the dual quadrilateral $(S_3, S_{13}, S_{23}, S_{123})$, recall that for the edge in question,

$$V_1 - V \overset{+}{\sim} V_{13} - V_3$$

and

$$V_{23} - V_3 \overset{+}{\sim} -V_2 + V, \quad V_{123} - V_{13} \overset{+}{\sim} -V_{12} + V_1.$$

We get

$$V_3 = \begin{pmatrix} -s_3 \\ R^* \end{pmatrix}, \quad V_{13} = \begin{pmatrix} s_{13} \\ R^* \end{pmatrix}$$

and

$$V_{23} = V_3 + \frac{s_3 + s_{23}}{s}(V - Y) = \frac{1}{R^2 + s^2} \begin{pmatrix} -2\alpha s - s^2 s_{23} + s_{23} R^2 \\ -\alpha R + s^2 R^* - 2 s s_{23} R \end{pmatrix},$$

$$V_{123} = V_{13} + \frac{s_{13} + s_{123}}{s}(V_1 - Y_1) = \frac{1}{R^2 + s_1^2} \begin{pmatrix} 2\alpha s_1 + s_1^2 s_{123} - s_{123} R^2 \\ -\alpha R + s_1^2 R^* - 2 s_1 s_{123} R \end{pmatrix}.$$

For the final coordinates in our model, we have to rescale the planar components by $\cos\delta$ and $\cos\delta^*$, rotate the dual quadrilateral by η, and add the vertical components. We will denote the horizontal parts of the vertex coordinates by $\overline{v} \in \mathbb{R}^3$ etc, and the vertical component with respect to the circle planes by $v' \in \mathbb{R}^3$ etc. The rotation by η around the z–axis will be D_η, described by the matrix

$$D_\eta = \begin{pmatrix} \cos\eta & -\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, let

$$h' = m^* - m = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}.$$

We get

$$\overline{v} = \cos\delta \begin{pmatrix} V \\ 0 \end{pmatrix}, \quad \overline{v_1} = \cos\delta \begin{pmatrix} V_1 \\ 0 \end{pmatrix},$$

$$\overline{v_2} = \cos\delta \begin{pmatrix} V_2 \\ 0 \end{pmatrix}, \quad \overline{v_{12}} = \cos\delta \begin{pmatrix} V_{12} \\ 0 \end{pmatrix},$$

$$\overline{v_3} = \cos\delta^* D_\eta \begin{pmatrix} V_3 \\ 0 \end{pmatrix}, \quad \overline{v_{13}} = \cos\delta^* D_\eta \begin{pmatrix} V_{13} \\ 0 \end{pmatrix},$$

$$\overline{v_{23}} = \cos\delta^* D_\eta \begin{pmatrix} V_{23} \\ 0 \end{pmatrix}, \quad \overline{v_{123}} = \cos\delta^* D_\eta \begin{pmatrix} V_{123} \\ 0 \end{pmatrix}$$

and the vertical components

$$v' = -\sin\delta s e_z, \quad v_1' = \sin\delta s_1 e_z,$$

$$v_2' = \sin\delta s_2 e_z, \quad v_{12}' = -\sin\delta s_{12} e_z,$$

$$v_3' = -\sin\delta^* s_3 e_z, \quad v_{13}' = \sin\delta^* s_{13} e_z,$$

$$v_{23}' = \sin\delta^* s_{23} e_z, \quad v_{123}' = -\sin\delta^* s_{123} e_z,$$

with e_z the unit vector in z–direction, and with lemma 4.4.9 and lemma 4.4.6 for choosing the correct sign, $\sin\delta = R^* k$ and $\sin\delta^* = -Rk$. With these components, the complete vertex coordinates are

$$v = \overline{v} + v', \quad v_1 = \overline{v_1} + v_1',$$

$$v_2 = \overline{v_2} + v_2', \quad v_{12} = \overline{v_{12}} + v_{12}',$$

$$v_3 = \overline{v_3} + v_3' + h', \quad v_{13} = \overline{v_{13}} + v_{13}' + h',$$

$$v_{23} = \overline{v_{23}} + v_{23}' + h', \quad v_{123} = \overline{v_{123}} + v_{123}' + h'.$$

Put together, we get

$$\begin{aligned} v &= \overline{v} + v' \\ &= \cos\delta \begin{pmatrix} V \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\sin\delta s \end{pmatrix} \\ &= \frac{r}{R} \begin{pmatrix} -s \\ -R \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -ksR^* \end{pmatrix} \\ &= \frac{-1}{R} \begin{pmatrix} rs \\ rR \\ \alpha ks \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
v_{12} &= \overline{v_{12}} + v'_{12} \\
&= \cos\delta \begin{pmatrix} V_{12} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\sin\delta s_{12} \end{pmatrix} \\
&= \frac{r}{R(R^2+s_1^2)} \begin{pmatrix} 2s_1R^2 - s_1^2 s_{12} + s_{12}R^2 \\ R(-R^2+s_1^2+2s_1s_{12}) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -ks_{12}R^* \end{pmatrix} \\
&= \frac{r}{R(R^2+\frac{\alpha^2}{s_{13}^2})} \begin{pmatrix} 2\frac{\alpha}{s_{13}}R^2 - \frac{\alpha^2}{s_{13}^2}s_{12} + s_{12}R^2 \\ R(-R^2+\frac{\alpha^2}{s_{13}^2}+2\frac{\alpha}{s_{13}}s_{12}) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -ks_{12}\frac{\alpha}{R} \end{pmatrix} \\
&= \frac{r}{R(s_{13}^2R^2+\alpha^2)} \begin{pmatrix} 2\alpha s_{13}R^2 - \alpha^2 s_{12} + s_{12}s_{13}^2R^2 \\ R(-s_{13}^2R^2+\alpha^2+2\alpha s_{13}s_{12}) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -ks_{12}\frac{\alpha}{R} \end{pmatrix} \\
&= \frac{1}{RC_1} \begin{pmatrix} r\left(s_{13}R^2(\alpha+s_{12}s_{13}) + \alpha(s_{13}R^2-\alpha s_{12})\right) \\ rR\left(\alpha(\alpha+s_{12}s_{13}) + s_{13}(\alpha s_{12}-s_{13}R^2)\right) \\ -\alpha ks_{12}C_1 \end{pmatrix} \\
&= \frac{1}{RC_1} \begin{pmatrix} r\left(s_{13}R^2A_1 + \alpha B_1\right) \\ rR\left(\alpha A_1 - s_{13}B_1\right) \\ -\alpha ks_{12}C_1 \end{pmatrix}.
\end{aligned}$$

For the remaining two vertices we need the rotation

$$D_\eta = \begin{pmatrix} \cos\eta & -\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{\alpha}{r^*r} \begin{pmatrix} H & -hk & 0 \\ hk & H & 0 \\ 0 & 0 & \frac{r^*r}{\alpha} \end{pmatrix}$$

with $H := \frac{r^*r}{\alpha}\cos\eta = \cos\varphi + \alpha k^2$. Note that

$$H^2 = \left(\frac{r^*r}{\alpha}\right)^2 (1-\sin^2\eta) = \left(\frac{r^*r}{\alpha}\right)^2 - h^2k^2.$$

Now

$$\begin{aligned}
v_{13} &= \overline{v_{13}} + v'_{13} + h' \\
&= \cos\delta^* D_\eta \begin{pmatrix} V_{13} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sin\delta^* s_{13} + h \end{pmatrix} \\
&= \frac{\alpha}{rR^*} \begin{pmatrix} H & -hk & 0 \\ hk & H & 0 \\ 0 & 0 & \frac{r^*r}{\alpha} \end{pmatrix} \begin{pmatrix} s_{13} \\ R^* \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h - ks_{13}R \end{pmatrix}
\end{aligned}$$

$$= \frac{1}{r}\begin{pmatrix} Hs_{13}R - \alpha kh \\ khs_{13}R + H\alpha \\ r(h - ks_{13}R) \end{pmatrix},$$

$$\begin{aligned}
v_{23} &= \overline{v_{23}} + v_{23}' + h' \\
&= \cos\delta^* D_\eta \begin{pmatrix} V_{23} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sin\delta^* s_{23} + h \end{pmatrix} \\
&= \frac{R}{r}\begin{pmatrix} H & -hk & 0 \\ hk & H & 0 \\ 0 & 0 & \frac{r^*r}{\alpha} \end{pmatrix} \frac{1}{R^2+s^2}\begin{pmatrix} -2\alpha s - s^2 s_{23} + s_{23}R^2 \\ -\alpha R + s^2 R^* - 2ss_{23}R \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h - ks_{23}R \end{pmatrix} \\
&= \frac{1}{rC_2}\begin{pmatrix} H & -hk & 0 \\ hk & H & 0 \\ 0 & 0 & \frac{r^*r}{\alpha} \end{pmatrix} \begin{pmatrix} R\left(s_{23}R^2 - \alpha s - s(\alpha + ss_{23})\right) \\ -R^2(\alpha + ss_{23}) - s(s_{23}R^2 - \alpha s) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h - ks_{23}R \end{pmatrix} \\
&= \frac{1}{rC_2}\begin{pmatrix} H & -hk & 0 \\ hk & H & 0 \\ 0 & 0 & \frac{r^*r}{\alpha} \end{pmatrix} \begin{pmatrix} R\left(B_2 - sA_2\right) \\ -R^2A_2 - sB_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h - ks_{23}R \end{pmatrix} \\
&= \frac{1}{rC_2}\begin{pmatrix} HR\left(B_2 - sA_2\right) + kh(R^2A_2 + sB_2) \\ khR\left(B_2 - sA_2\right) - H(R^2A_2 + sB_2) \\ rC_2(h - ks_{23}R) \end{pmatrix}.
\end{aligned}$$

Our edges are

$$\begin{aligned}
f &= \frac{1}{rR}\begin{pmatrix} Hs_{13}R^2 - \alpha khR + r^2 s \\ khs_{13}R^2 + H\alpha R + r^2 R \\ r(hR + k(\alpha s - s_{13}R^2)) \end{pmatrix} \\
&= \frac{1}{rR}\begin{pmatrix} Hs_{13}R^2 - \alpha khR + r^2 s \\ khs_{13}R^2 + H\alpha R + r^2 R \\ r(hR - kB_3) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
f^2 &= \frac{1}{rRC_2}\begin{pmatrix} HR^2(B_2 - sA_2) + khR(R^2A_2 + sB_2) + r^2 sC_2 \\ khR^2(B_2 - sA_2) - HR(R^2A_2 + sB_2) + r^2RC_2 \\ rRC_2(h - ks_{23}R) + r\alpha ksC_2 \end{pmatrix} \\
&= \frac{1}{rRC_2}\begin{pmatrix} HR^2(B_2 - sA_2) + khR(R^2A_2 + sB_2) + r^2 sC_2 \\ khR^2(B_2 - sA_2) - HR(R^2A_2 + sB_2) + r^2RC_2 \\ rC_2(hR - kB_2) \end{pmatrix},
\end{aligned}$$

$$f_1^2 = \frac{1}{rRC_1}\begin{pmatrix} r^2(s_{13}R^2A_1 + \alpha B_1) - RC_1(Hs_{13}R - \alpha kh) \\ r^2R(\alpha A_1 - s_{13}B_1) - RC_1(khs_{13}R + H\alpha) \\ rC_1(-\alpha ks_{12} - R(h - ks_{13}R)) \end{pmatrix}$$
$$= \frac{1}{rRC_1}\begin{pmatrix} \alpha khRC_1 - Hs_{13}R^2C_1 + r^2(s_{13}R^2A_1 + \alpha B_1) \\ -khs_{13}R^2C_1 - H\alpha RC_1 + r^2R(\alpha A_1 - s_{13}B_1) \\ rC_1(kB_1 - hR) \end{pmatrix}.$$

Now we proceed to $f^2 \times f_1^2$, component by component:

$$\begin{aligned}
(f^2 \times f_1^2)_x &= \frac{1}{r^2R^2C_1C_2}\Big(\big(khR^2(B_2 - sA_2) - HR(R^2A_2 + sB_2) + r^2RC_2\big)\big(rC_1(kB_1 - hR)\big) \\
&\quad - \big(rC_2(hR - kB_2)\big)\big(-khs_{13}R^2C_1 - H\alpha RC_1 + r^2R(\alpha A_1 - s_{13}B_1)\big)\Big) \\
&= \frac{1}{rRC_1C_2}\Big(\big(khR(B_2 - sA_2) - H(R^2A_2 + sB_2) + r^2C_2\big)\big(C_1(kB_1 - hR)\big) \\
&\quad + \big(C_2(hR - kB_2)\big)\big(khs_{13}RC_1 + H\alpha C_1 + r^2(s_{13}B_1 - \alpha A_1)\big)\Big) \\
&= \frac{1}{rRC_1C_2}\Big(k^2hRC_1\big(B_1(B_2 - sA_2) - s_{13}B_2C_2\big) \\
&\quad + kHC_1\big(-B_1(R^2A_2 + sB_2) - \alpha B_2C_2\big) \\
&\quad + k\big(-h^2R^2C_1(B_2 - sA_2) + r^2B_1C_1C_2 - r^2B_2C_2(s_{13}B_1 - \alpha A_1) + h^2R^2s_{13}C_1C_2\big) \\
&\quad + HhRC_1\big(R^2A_2 + sB_2 + \alpha C_2\big) \\
&\quad + r^2hRC_2\big(-C_1 + s_{13}B_1 - \alpha A_1\big)\Big) \\
&= \frac{1}{rRC_1C_2}\Big(k^2hRC_1\big(B_1B_2 - sA_2B_1 - s_{13}B_2C_2\big) \\
&\quad - kHC_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&\quad + k\big(h^2R^2C_1(s_{13}C_2 - B_2 + sA_2) + r^2C_2(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2)\big) \\
&\quad + HhRC_1\big(R^2A_2 + sB_2 + \alpha C_2\big) \\
&\quad - r^2hRC_2\big(C_1 - s_{13}B_1 + \alpha A_1\big)\Big),
\end{aligned}$$

$$\begin{aligned}
(f^2 \times f_1^2)_y &= \frac{1}{r^2R^2C_1C_2}\Big(-\big(HR^2(B_2 - sA_2) + khR(R^2A_2 + sB_2) + r^2sC_2\big)\big(rC_1(kB_1 - hR)\big) \\
&\quad + \big(rC_2(hR - kB_2)\big)\big(\alpha khRC_1 - Hs_{13}R^2C_1 + r^2(s_{13}R^2A_1 + \alpha B_1)\big)\Big) \\
&= \frac{1}{rR^2C_1C_2}\Big(\big(HR^2(B_2 - sA_2) + khR(R^2A_2 + sB_2) + r^2sC_2\big)\big(C_1(hR - kB_1)\big) \\
&\quad + \big(C_2(hR - kB_2)\big)\big(\alpha khRC_1 - Hs_{13}R^2C_1 + r^2(s_{13}R^2A_1 + \alpha B_1)\big)\Big) \\
&= \frac{1}{rR^2C_1C_2}\Big(k^2hRC_1\big(-B_1(R^2A_2 + sB_2) - \alpha B_2C_2\big)
\end{aligned}$$

$$
\begin{aligned}
&+ kHR^2C_1\big(-B_1(B_2 - sA_2) + s_{13}B_2C_2\big)\\
&+ k\big(h^2R^2C_1(R^2A_2 + sB_2) - r^2sB_1C_1C_2 + h^2R^2C_1\alpha C_2 - r^2B_2C_2(s_{13}R^2A_1 + \alpha B_1)\big)\\
&+ HhR^3C_1\big(B_2 - sA_2 - s_{13}C_2\big)\\
&+ r^2hRC_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1)\Big)\\
= &\frac{1}{rR^2C_1C_2}\Big(-k^2hRC_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\\
&+ kHR^2C_1\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big)\\
&+ k\big(h^2R^2C_1(R^2A_2 + sB_2 + \alpha C_2) - r^2C_2(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2)\big)\\
&+ HhR^3C_1\big(B_2 - sA_2 - s_{13}C_2\big)\\
&+ r^2hRC_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1)\Big);
\end{aligned}
$$

in the last component we will expand H^2 and use lemma A.0.1:

$$
\begin{aligned}
(f^2 \times f_1^2)_z = &\frac{1}{r^2R^2C_1C_2}\Big(\big(HR^2(B_2 - sA_2) + khR(R^2A_2 + sB_2) + r^2sC_2\big)\\
&\big(-khs_{13}R^2C_1 - H\alpha RC_1 + r^2R(\alpha A_1 - s_{13}B_1)\big)\\
&- \big(khR^2(B_2 - sA_2) - HR(R^2A_2 + sB_2) + r^2RC_2\big)\\
&\big(\alpha khRC_1 - Hs_{13}R^2C_1 + r^2(s_{13}R^2A_1 + \alpha B_1)\big)\Big)\\
= &\frac{1}{r^2RC_1C_2}\Big(\big(HR^2(B_2 - sA_2) + khR(R^2A_2 + sB_2) + r^2sC_2\big)\\
&\big(-khs_{13}RC_1 - H\alpha C_1 + r^2(\alpha A_1 - s_{13}B_1)\big)\\
&- \big(khR(B_2 - sA_2) - H(R^2A_2 + sB_2) + r^2C_2\big)\\
&\big(\alpha khRC_1 - Hs_{13}R^2C_1 + r^2(s_{13}R^2A_1 + \alpha B_1)\big)\Big)\\
= &\frac{1}{r^2RC_1C_2}\Big(k^2h^2R^2C_1\big(-s_{13}(R^2A_2 + sB_2) - \alpha(B_2 - sA_2)\big)\\
&+ kHhRC_1\big(-s_{13}R^2(B_2 - sA_2) - \alpha(R^2A_2 + sB_2)\\
&\quad + s_{13}R^2(B_2 - sA_2) + \alpha(R^2A_2 + sB_2)\big)\\
&+ H^2R^2C_1\big(-\alpha(B_2 - sA_2) - s_{13}(R^2A_2 + sB_2)\big)\\
&+ khr^2R\big((R^2A_2 + sB_2)(\alpha A_1 - s_{13}B_1) - ss_{13}C_1C_2\\
&\quad - (B_2 - sA_2)(s_{13}R^2A_1 + \alpha B_1) - \alpha C_1C_2\big)\\
&+ Hr^2\big(R^2(B_2 - sA_2)(\alpha A_1 - s_{13}B_1) - \alpha sC_1C_2\\
&\quad + (R^2A_2 + sB_2)(s_{13}R^2A_1 + \alpha B_1) + s_{13}R^2C_1C_2\big)\\
&+ r^4C_2\big(s(\alpha A_1 - s_{13}B_1) - s_{13}R^2A_1 - \alpha B_1)\Big)\\
= &\frac{1}{r^2RC_1C_2}\Big(-(\frac{r^*r}{\alpha})^2R^2C_1\big(B_2(\alpha + ss_{13}) + A_2(s_{13}R^2 - \alpha s)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ khr^2R\big(-C_1C_2(\alpha + ss_{13}) + R^2A_1A_2(\alpha + ss_{13}) + A_1B_2(\alpha s - s_{13}R^2)\big) \\
&\quad + A_2B_1(-s_{13}R^2 + \alpha s) + B_1B_2(-ss_{13} - \alpha) \\
&+ Hr^2\big(C_1C_2(s_{13}R^2 - \alpha s) + R^2A_1A_2(-\alpha s + s_{13}R^2) + R^2A_1B_2(\alpha + ss_{13}) \\
&\quad + R^2A_2B_1(ss_{13} + \alpha) + B_1B_2(-s_{13}R^2 + \alpha s)\big) \\
&+ r^4C_2\big(A_1(\alpha s - s_{13}R^2) - B_1(ss_{13} + \alpha)\big)\Big) \\
= &\frac{1}{RC_1C_2}\Big(-\big(\frac{r^*}{\alpha}\big)^2R^2C_1\big(A_3B_2 + A_2B_3\big) \\
&+ khR\big(A_3(-C_1C_2 + R^2A_1A_2 - B_1B_2) - B_3(A_1B_2 + A_2B_1)\big) \\
&+ H\big(B_3(C_1C_2 + R^2A_1A_2 - B_1B_2) + R^2A_3(A_1B_2 + A_2B_1)\big) \\
&- r^2C_2\big(A_1B_3 + B_1A_3\big)\Big) \\
= &\frac{1}{RC_1C_2}\Big(khRA_3\big(-C_1C_2 + R^2A_1A_2 - B_1B_2\big) \\
&+ HB_3\big(C_1C_2 + R^2A_1A_2 - B_1B_2\big) \\
&- \big(\frac{r^*}{\alpha}\big)^2R^2C_1\big(A_3B_2 + A_2B_3\big) - r^2C_2\big(A_1B_3 + B_1A_3\big)\Big).
\end{aligned}
$$

We first write out the summands of $\langle f, f^2 \times f_1^2\rangle$ separately:

$$
\begin{aligned}
f_x\left(f^2 \times f_1^2\right)_x = &\frac{1}{r^2R^2C_1C_2}\Big(-k\alpha hR + Hs_{13}R^2 + r^2s\Big) \\
&\Big(k^2hRC_1\big(B_1B_2 - sA_2B_1 - s_{13}B_2C_2\big) \\
&- kHC_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&+ k\big(h^2R^2C_1(s_{13}C_2 - B_2 + sA_2) + r^2C_2(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2)\big) \\
&+ HhRC_1\big(R^2A_2 + sB_2 + \alpha C_2\big) \\
&- r^2hRC_2\big(C_1 - s_{13}B_1 + \alpha A_1\big)\Big) \\
= &\frac{1}{r^2R^2C_1C_2}\Big(-k^3\alpha h^2R^2C_1\big(B_1B_2 - sA_2B_1 - s_{13}B_2C_2\big) \\
&+ k^2HhRC_1\big(\alpha(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2) + s_{13}R^2(B_1B_2 - sA_2B_1 - s_{13}B_2C_2)\big) \\
&- kH^2s_{13}R^2C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&+ k^2hR\Big(-\alpha\big(h^2R^2C_1(s_{13}C_2 - B_2 + sA_2) + r^2C_2(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2)\big) \\
&\quad + r^2sC_1\big(B_1B_2 - sA_2B_1 - s_{13}B_2C_2\big)\Big) \\
&+ kH\Big(-\alpha h^2R^2C_1\big(R^2A_2 + sB_2 + \alpha C_2\big) - r^2sC_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&\quad + s_{13}R^2\big(h^2R^2C_1(s_{13}C_2 - B_2 + sA_2) + r^2C_2(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2)\big)\Big) \\
&+ H^2s_{13}hR^3C_1\Big(R^2A_2 + sB_2 + \alpha C_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ kr^2\Big(\alpha h^2R^2C_2\big(C_1 - s_{13}B_1 + \alpha A_1\big) + s\big(h^2R^2C_1(s_{13}C_2 - B_2 + sA_2)\\
&\quad + r^2C_2(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2)\big)\Big)\\
&+ Hr^2hR\Big(-s_{13}R^2C_2\big(C_1 - s_{13}B_1 + \alpha A_1\big) + sC_1\big(R^2A_2 + sB_2 + \alpha C_2\big)\Big)\\
&- r^4hsRC_2\big(C_1 - s_{13}B_1 + \alpha A_1\big)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
f_y\left(f^2 \times f_1^2\right)_y = {}& \frac{1}{r^2R^2C_1C_2}\Big(khs_{13}R + H\alpha + r^2\Big)\\
&\Big(-k^2hRC_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\\
&+ kHR^2C_1\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big)\\
&+ k\big(h^2R^2C_1(R^2A_2 + sB_2 + \alpha C_2) - r^2C_2(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2)\big)\\
&+ HhR^3C_1\big(B_2 - sA_2 - s_{13}C_2\big)\\
&+ r^2hRC_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big)\Big)\\
= {}& \frac{1}{r^2R^2C_1C_2}\Bigg(-k^3h^2R^2s_{13}C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\\
&+ k^2HhRC_1\Big(s_{13}R^2\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big) - \alpha\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\Big)\\
&+ kH^2\alpha R^2C_1\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big)\\
&+ k^2hR\Big(s_{13}h^2R^2C_1\big(R^2A_2 + sB_2 + \alpha C_2\big)\\
&\quad - s_{13}r^2C_2\big(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2\big) - r^2C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\Big)\\
&+ kH\Big(h^2s_{13}R^4C_1\big(B_2 - sA_2 - s_{13}C_2\big) + \alpha h^2R^2C_1\big(R^2A_2 + sB_2 + \alpha C_2\big)\\
&\quad - \alpha r^2C_2\big(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2\big) + r^2R^2C_1\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big)\Big)\\
&+ H^2\alpha hR^3C_1\big(B_2 - sA_2 - s_{13}C_2\big)\\
&+ kr^2\Big(h^2s_{13}R^2C_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big) + h^2R^2C_1\big(R^2A_2 + sB_2 + \alpha C_2\big)\\
&\quad - r^2C_2\big(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2\big)\Big)\\
&+ Hr^2hR\Big(\alpha C_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big) + R^2C_1\big(B_2 - sA_2 - s_{13}C_2\big)\Big)\\
&+ r^4hRC_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
f_z\left(f^2 \times f_1^2\right)_z = {}& \frac{1}{R^2C_1C_2}\Big(-kB_3 + hR\Big)\\
&\Big(khRA_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ HB_3\big(R^2A_1A_2 - B_1B_2 + C_1C_2\big)\\
&- \big(\frac{r^*}{\alpha}\big)^2 R^2C_1\big(A_3B_2 + A_2B_3\big) - r^2C_2\big(A_1B_3 + A_3B_1\big)\Big)\\
= \frac{1}{R^2C_1C_2}&\Big(-k^2hRA_3B_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)\\
&- kHB_3^2\big(R^2A_1A_2 - B_1B_2 + C_1C_2\big)\\
&+ k\Big(\big(\frac{r^*}{\alpha}\big)^2 R^2B_3C_1\big(A_3B_2 + A_2B_3\big) + r^2B_3C_2\big(A_1B_3 + A_3B_1\big)\\
&\quad + h^2R^2A_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)\Big)\\
&+ HhRB_3\big(R^2A_1A_2 - B_1B_2 + C_1C_2\big)\\
&- \big(\frac{r^*}{\alpha}\big)^2 hR^3C_1\big(A_3B_2 + A_2B_3\big) - hr^2RC_2\big(A_1B_3 + A_3B_1\big)\Big).
\end{aligned}
$$

Now, we sum up the coefficients at the various powers of k and H, expanding $H^2 = \big(\frac{r^*r}{\alpha}\big)^2 - h^2k^2$, which redistributes the coefficients at kH^2 to k and k^3, and those at H^2 to k^2 and the zeroth power. We get for the coefficient at

k^3:

$$
\begin{aligned}
\frac{1}{r^2R^2C_1C_2}\Big(&- \alpha h^2R^2C_1\big(B_1B_2 - sA_2B1 - s_{13}B_2C_2\big)\\
&+ s_{13}h^2R^2C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\\
&- s_{13}h^2R^2C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\\
&- \alpha h^2R^2C_1\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big)\Big)\\
= 0,&
\end{aligned}
$$

k^2H:

$$
\begin{aligned}
\frac{hRC_1}{r^2R^2C_1C_2}\Big(&\alpha\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\\
&+ s_{13}R^2\big(B_1B_2 - sA_2B_1 - s_{13}B_2C_2\big)\\
&+ s_{13}R^2\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big)\\
&- \alpha\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big)\Big)\\
= 0,&
\end{aligned}
$$

k^2:

$$
\begin{aligned}
&\frac{hR}{r^2R^2C_1C_2}\Big(-\alpha h^2R^2C_1\big(s_{13}C_2 - B_2 + sA_2\big) \\
&\qquad -\alpha r^2C_2\big(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2\big) \\
&\qquad + r^2sC_1\big(B_1B_2 - sA_2B_1 - s_{13}B_2C_2\big) \\
&\qquad - s_{13}h^2R^2C_1\big(R^2A_2 + sB_2 + \alpha C_2\big) \\
&\qquad + s_{13}h^2R^2C_1\big(R^2A_2 + sB_2 + \alpha C_2\big) \\
&\qquad - s_{13}r^2C_2\big(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2\big) \\
&\qquad - r^2C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&\qquad - \alpha h^2R^2C_1\big(B_2 - sA_2 - s_{13}C_2\big) \\
&\qquad - r^2A_3B_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)\Big) \\
&= \frac{h}{RC_1C_2}\Big(-\alpha C_2\big(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2\big) \\
&\qquad + sC_1\big(B_1B_2 - sA_2B_1 - s_{13}B_2C_2\big) \\
&\qquad - s_{13}C_2\big(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2\big) \\
&\qquad - C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&\qquad - A_3B_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)\Big) \\
&= \frac{h}{RC_1C_2}\Big(-A_1B_2C_2\big(\alpha^2 + s_{13}^2R^2\big) - A_2B_1C_1\big(s^2 + R^2\big) \\
&\qquad + B_1B_2\big(C_2(\alpha s_{13} - \alpha s_{13}) + C_1(s - s)\big) \\
&\qquad - B_1C_1C_2\big(\alpha + ss_{13}\big) - B_2C_2C_1\big(ss_{13} + \alpha\big) \\
&\qquad - A_3B_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)\Big) \\
&= \frac{h}{RC_1C_2}\Big(-C_1C_2\big(A_1B_2 + A_2B_1\big) \\
&\qquad - A_3C_1C_2\big(B_1 + B_2 - B_3\big) \\
&\qquad - A_3B_3\big(R^2A_1A_2 - B_1B_2\big)\Big) \\
&= 0,
\end{aligned}
$$

where we used

$$B_1 + B_2 - B_3 = (s_{13} + s_{23} - s_{13})R^2 - \alpha(s_{12} + s - s) = B_4$$

and lemma A.0.1 in the last step,

kH:

$$\begin{aligned}
&\frac{1}{r^2R^2C_1C_2}\Big(-\alpha h^2R^2C_1\big(R^2A_2+sB_2+\alpha C_2\big)\\
&\qquad -r^2sC_1\big(R^2A_2B_1+sB_1B_2+\alpha B_2C_2\big)\\
&\qquad +s_{13}h^2R^4C_1\big(s_{13}C_2-B_2+sA_2\big)\\
&\qquad +s_{13}r^2R^2C_2\big(B_1C_1-s_{13}B_1B_2+\alpha A_1B_2\big)\\
&\qquad +s_{13}h^2R^4C_1\big(B_2-sA_2-s_{13}C_2\big)\\
&\qquad +\alpha h^2R^2C_1\big(R^2A_2+sB_2+\alpha C_2\big)\\
&\qquad -\alpha r^2C_2\big(sB_1C_1+s_{13}R^2A_1B_2+\alpha B_1B_2\big)\\
&\qquad +r^2R^2C_1\big(sA_2B_1-B_1B_2+s_{13}B_2C_2\big)\\
&\qquad -r^2B_3^2\big(R^2A_1A_2-B_1B_2+C_1C_2\big)\Big)\\
&=\frac{1}{R^2C_1C_2}\Big(-sC_1\big(R^2A_2B_1+sB_1B_2+\alpha B_2C_2\big)\\
&\qquad +s_{13}R^2C_2\big(B_1C_1-s_{13}B_1B_2+\alpha A_1B_2\big)\\
&\qquad -\alpha C_2\big(sB_1C_1+s_{13}R^2A_1B_2+\alpha B_1B_2\big)\\
&\qquad +R^2C_1\big(sA_2B_1-B_1B_2+s_{13}B_2C_2\big)\\
&\qquad -B_3^2\big(R^2A_1A_2-B_1B_2+C_1C_2\big)\Big)\\
&=\frac{1}{R^2C_1C_2}\Big(A_1B_2C_2\big(\alpha s_{13}R^2-\alpha s_{13}R^2\big)+A_2B_1C_1\big(-sR^2+sR^2\big)\\
&\qquad +B_1B_2\big(C_1(-s^2-R^2)+C_2(-s_{13}^2R^2-\alpha^2)+B_3^2\big)\\
&\qquad +B_1C_1C_2\big(s_{13}R^2-\alpha s\big)\\
&\qquad +B_2C_2C_1\big(-\alpha s+s_{13}R^2\big)\\
&\qquad -B_3^2\big(R^2A_1A_2+C_1C_2\big)\Big)\\
&=\frac{1}{R^2C_1C_2}\Big(C_1C_2\big(B_1B_3+B_2B_3-B_1B_2-B_3^2\big)\\
&\qquad +B_1B_2\big(B_3^2-C_1C_2\big)\\
&\qquad -R^2B_3^2A_1A_2\Big)\\
&=\frac{1}{R^2C_1C_2}\Big(C_1C_2\big(B_3B_4-B_1B_2\big)\\
&\qquad +B_3^2\big(B_1B_2-R^2A_1A_2\big)\\
&\qquad -B_1B_2C_1C_2\Big)\\
&=\frac{1}{R^2C_1C_2}\Big(-2B_1B_2C_1C_2\\
&\qquad +B_3\big(B_4C_1C_2+B_3(B_1B_2-R^2A_1A_2)\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{R^2 C_1 C_2}\Big(-2B_1B_2C_1C_2 \\
&\qquad + 2B_3B_4C_1C_2\Big) \\
&= \frac{2}{R^2}\big(B_3B_4 - B_1B_2\big) \\
&= 2\alpha(s - s_{12})(s_{13} - s_{23}),
\end{aligned}
$$

where we used $B_4C_1C_2 = -B_3(R^2A_1A_2 - B_1B_2)$ from lemma A.0.1 and straightforwardly expanded $B_3B_4 - B_1B_2$;

k:

$$
\begin{aligned}
\frac{1}{R^2C_1C_2}\Big(&\alpha h^2R^2C_2\big(C_1 - s_{13}B_1 + \alpha A_1\big) \\
&+ sh^2R^2C_1\big(s_{13}C_2 - B_2 + sA_2\big) \\
&+ r^2sC_2\big(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2\big) \\
&- \big(\frac{r^*}{\alpha}\big)^2 s_{13}R^2C_1\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&+ h^2s_{13}R^2C_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big) \\
&+ h^2R^2C_1\big(R^2A_2 + sB_2 + \alpha C_2\big) \\
&- r^2C_2\big(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2\big) \\
&+ \big(\frac{r^*}{\alpha}\big)^2 \alpha R^2C_1\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big) \\
&+ \big(\frac{r^*}{\alpha}\big)^2 R^2B_3C_1\big(A_3B_2 + A_2B_3\big) \\
&+ r^2B_3C_2\big(A_1B_3 + A_3B_1\big) \\
&+ h^2R^2A_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{R^2C_1C_2}\Big(h^2R^2\Big(\alpha C_2\big(C_1 - s_{13}B_1 + \alpha A_1\big) \\
&\qquad\qquad + sC_1\big(s_{13}C_2 - B_2 + sA_2\big) \\
&\qquad\qquad + s_{13}C_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big) \\
&\qquad\qquad + C_1\big(R^2A_2 + sB_2 + \alpha C_2\big) \\
&\qquad\qquad + A_3\big(R^2A_1A_2 - B_1B_2 - C_1C_2\big)\Big) \\
&\qquad + r^2C_2\Big(s\big(B_1C_1 - s_{13}B_1B_2 + \alpha A_1B_2\big) \\
&\qquad\qquad - \big(sB_1C_1 + s_{13}R^2A_1B_2 + \alpha B_1B_2\big) \\
&\qquad\qquad + B_3\big(A_1B_3 + A_3B_1\big)\Big) \\
&\qquad + \big(\frac{r^*}{R^*}\big)^2C_1\Big(- s_{13}\big(R^2A_2B_1 + sB_1B_2 + \alpha B_2C_2\big) \\
&\qquad\qquad + \alpha\big(sA_2B_1 - B_1B_2 + s_{13}B_2C_2\big) \\
&\qquad\qquad + B_3\big(A_3B_2 + A_2B_3\big)\Big)\Big) \\
&= \frac{1}{R^2C_1C_2}\Big(h^2R^2\Big(C_1C_2\big(A_3 + A_3 - A_3\big) \\
&\qquad\qquad + B_1C_2\big(-\alpha s_{13} + \alpha s_{13}\big) + B_2C_1\big(-s + s\big) \\
&\qquad\qquad + A_1C_2\big(\alpha^2 + s_{13}^2R^2\big) + A_2C_1\big(s^2 + R^2\big) \\
&\qquad\qquad + A_3\big(R^2A_1A_2 - B_1B_2\big)\Big) \\
&\qquad + r^2C_2\Big(- B_1B_2A_3 - A_1B_2B_3 + B_3\big(A_1B_3 + A_3B_1\big)\Big) \\
&\qquad + \big(\frac{r^*}{R^*}\big)^2C_1\Big(- A_2B_1B_3 - B_1B_2A_3 + B_3\big(A_3B_2 + A_2B_3\big) \\
&\qquad\qquad + B_2C_2\big(-\alpha s_{13} + \alpha s_{13}\big)\Big)\Big) \\
&= \frac{1}{R^2C_1C_2}\Big(h^2R^2\Big(C_1C_2\big(A_1 + A_2 + A_3 + A_4\big)\Big) \\
&\qquad + r^2C_2\Big(\big(B_3 - B_2\big)\big(A_1B_3 + A_3B_1\big)\Big) \\
&\qquad + \big(\frac{r^*}{R^*}\big)^2C_1\Big(\big(B_3 - B_1\big)\big(A_3B_2 + A_2B_3\big)\Big)\Big),
\end{aligned}
$$

where we used lemma A.0.1 to get the A_4 in the last step;

H:

$$\begin{aligned}
&\frac{h}{RC_1C_2}\Big(- s_{13}R^2C_2\big(C_1 - s_{13}B_1 + \alpha A_1\big) + sC_1\big(R^2A_2 + sB_2 + \alpha C_2\big)\\
&\qquad + \alpha C_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big) + R^2C_1\big(B_2 - sA_2 - s_{13}C_2\big)\\
&\qquad + B_3\big(R^2A_1A_2 - B_1B_2 + C_1C_2\big)\Big)\\
&= \frac{h}{RC_1C_2}\Big(C_1C_2\big(-s_{13}R^2 + \alpha s + \alpha s - s_{13}R^2 + B_3\big)\\
&\qquad + B_1C_2\big(s_{13}^2R^2 + \alpha^2\big) + B_2C_1\big(s^2 + R^2\big)\\
&\qquad + A_1C_2\big(-\alpha s_{13}R^2 + \alpha s_{13}R^2\big) + A_2C_1\big(sR^2 - sR^2\big)\\
&\qquad + B_3\big(R^2A_1A_2 - B_1B_2\big)\Big)\\
&= \frac{h}{R}\big(B_1 + B_2 - B_3 - B_4\big)\\
&= \frac{h}{R}\big(s_{13}R^2 - \alpha s_{12} + s_{23}R^2 - \alpha s - s_{13}R^2 + \alpha s - s_{23}R^2 + \alpha s_{12}\big)\\
&= 0,
\end{aligned}$$

again using lemma A.0.1 for B_4;

k^0H^0:

$$\begin{aligned}
&\frac{h}{RC_1C_2}\Big(- r^2sC_2\big(C_1 - s_{13}B_1 + \alpha A_1\big)\\
&\qquad + \big(\frac{r^*}{R^*}\big)^2 s_{13}C_1\Big(R^2A_2 + sB_2 + \alpha C_2\Big)\\
&\qquad + r^2C_2\big(sC_1 + s_{13}R^2A_1 + \alpha B_1\big)\\
&\qquad + \big(\frac{r^*}{R^*}\big)^2 \alpha C_1\big(B_2 - sA_2 - s_{13}C_2\big)\\
&\qquad - \big(\frac{r^*}{R^*}\big)^2 C_1\big(A_3B_2 + A_2B_3\big) - r^2C_2\big(A_1B_3 + A_3B_1\big)\Big)\\
&= \frac{h}{RC_1C_2}\Big(r^2C_2\Big(C_1\big(-s + s\big) + B_1\big(ss_{13} + \alpha - A_3\big) + A_1\big(-\alpha s + s_{13}R^2 - B_3\big)\Big)\\
&\qquad + \big(\frac{r^*}{R^*}\big)^2 C_1\Big(A_2\big(s_{13}R^2 - \alpha s - B_3\big) + B_2\big(ss_{13} + \alpha - A_3\big) + C_2\big(\alpha s_{13} - \alpha s_{13}\big)\Big)\Big)\\
&= 0.
\end{aligned}$$

Now we can collect the only non–vanishing terms and find

$$
\begin{aligned}
\langle f, f^2 \times f_1^2\rangle = k\Big(& H\big(2\alpha(s-s_{12})(s_{13}-s_{23})\big) \\
&+ h^2\big(A_1 + A_2 + A_3 + A_4\big) \\
&+ \frac{r^2}{R^2C_1}\big(B_3 - B_2\big)\big(A_1B_3 + A_3B_1\big) \\
&+ \frac{r^{*2}}{\alpha^2C_2}\big(B_3 - B_1\big)\big(A_3B_2 + A_2B_3\big)\Big).
\end{aligned}
$$

We reformulate the result by expanding $r^2 = R^2 - \alpha^2k^2$, $r^{*2} = R^{*2} - \alpha^2k^2$ and $H = \cos\varphi + \alpha k^2$:

$$
\begin{aligned}
\langle f, f^2 \times f_1^2\rangle = k\Big(& \cos\varphi\big(2\alpha(s-s_{12})(s_{13}-s_{23})\big) \\
&+k^2\big(2\alpha^2(s-s_{12})(s_{13}-s_{23})\big) \\
&-k^2\frac{\alpha^2}{R^2C_1}\big(B_3 - B_2\big)\big(A_1B_3 + A_3B_1\big) \\
&-k^2\frac{1}{C_2}\big(B_3 - B_1\big)\big(A_3B_2 + A_2B_3\big) \\
&+h^2\big(A_1 + A_2 + A_3 + A_4\big) \\
&+\frac{1}{C_1}\big(B_3 - B_2\big)\big(A_1B_3 + A_3B_1\big) \\
&+\frac{1}{R^2C_2}\big(B_3 - B_1\big)\big(A_3B_2 + A_2B_3\big)\Big) \\
= 2\alpha k&\cos\varphi(s-s_{12})(s_{13}-s_{23}) \\
&+k^3\Big(2\alpha^2(s-s_{12})(s_{13}-s_{23}) \\
&\quad - \frac{\alpha^2}{C_1}(s_{13}-s_{23})\big(A_1B_3 + A_3B_1\big) \\
&\quad + \frac{\alpha}{C_2}(s-s_{12})\big(A_3B_2 + A_2B_3\big)\Big) \\
&+k\Big(h^2\big(A_1 + A_2 + A_3 + A_4\big) \\
&\quad + \frac{1}{C_1}\big(B_3 - B_2\big)\big(A_1B_3 + A_3B_1\big) \\
&\quad + \frac{1}{R^2C_2}\big(B_3 - B_1\big)\big(A_3B_2 + A_2B_3\big)\Big);
\end{aligned}
$$

the term at k^3 can be rewritten as

$$
\begin{aligned}
\alpha\Big(&\frac{s-s_{12}}{C_2}\big((s_{13}-s_{23})\alpha C_2 + A_3B_2 + A_2B_3\big) \\
&+\frac{\alpha(s_{13}-s_{23})}{C_1}\big((s-s_{12})C_1 - A_1B_3 - A_3B_1\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha\Big(\frac{s-s_{12}}{C_2}\big(\alpha(s_{13}-s_{23})(R^2+s^2)+(\alpha+ss_{13})(s_{23}R^2-\alpha s)\\
&\qquad +(\alpha+ss_{23})(s_{13}R^2-\alpha s)\big)\\
&\quad +\frac{\alpha(s_{13}-s_{23})}{C_1}\big((s-s_{12})(s_{13}^2R^2+\alpha^2)-(\alpha+s_{12}s_{13})(s_{13}R^2-\alpha s)\\
&\qquad -(\alpha+ss_{13})(s_{13}R^2-\alpha s_{12})\big)\Big)\\
&= \alpha\Big(\frac{s-s_{12}}{C_2}\big(R^2(\alpha s_{13}-\alpha s_{23}+\alpha s_{23}+\alpha s_{13}+2ss_{13}s_{23})\\
&\qquad +\alpha(s^2s_{13}-s^2s_{23}-\alpha s-s^2s_{13}-\alpha s-s^2s_{23})\big)\\
&\quad +\frac{\alpha(s_{13}-s_{23})}{C_1}\big(s_{13}R^2(ss_{13}-s_{12}s_{13}-\alpha-s_{12}s_{13}-\alpha-ss_{13})\\
&\qquad +\alpha(\alpha s-\alpha s_{12}+\alpha s+ss_{12}s_{13}+\alpha s_{12}+ss_{12}s_{13})\big)\Big)\\
&= \alpha\Big(\frac{s-s_{12}}{C_2}\big(2s_{13}R^2A_2-2\alpha sA_2\big)\\
&\quad +\frac{\alpha(s_{13}-s_{23})}{C_1}\big(-2s_{13}R^2A_1+2\alpha sA_1\big)\Big)\\
&= \alpha\Big(2\frac{s-s_{12}}{C_2}A_2B_3-2\frac{\alpha(s_{13}-s_{23})}{C_1}A_1B_3\Big)\\
= {}& 2\alpha B_3\Big(\frac{1}{C_2}(s-s_{12})A_2-\frac{\alpha}{C_1}(s_{13}-s_{23})A_1\Big),
\end{aligned}
$$

leaving us with

$$
\begin{aligned}
\langle f, f^2\times f_1^2\rangle &= 2\alpha k^3B_3\Big(\frac{1}{C_2}(s-s_{12})A_2-\frac{\alpha}{C_1}(s_{13}-s_{23})A_1\Big)\\
&\quad +2\alpha k\cos\varphi(s-s_{12})(s_{13}-s_{23})\\
&\quad +k\Big(h^2\big(A_1+A_2+A_3+A_4\big)\\
&\qquad +\frac{1}{C_1}\big(B_3-B_2\big)\big(A_1B_3+A_3B_1\big)\\
&\qquad +\frac{1}{R^2C_2}\big(B_3-B_1\big)\big(A_3B_2+A_2B_3\big)\Big).
\end{aligned}
$$

Let us now turn to $\|f\times f^2\|$ and $\|f\times f_1^2\|$. We have

$$\left\|f\times f^2\right\|^2=\|f\|^2\left\|f^2\right\|^2-\left\langle f,f^2\right\rangle^2$$

and

$$\left\|f\times f_1^2\right\|^2=\|f\|^2\left\|f_1^2\right\|^2-\left\langle f,f_1^2\right\rangle^2,$$

so we first calculate $\|f\|^2$, $\|f^2\|^2$, $\|f_1^2\|^2$, $\langle f,f^2\rangle$ and $\langle f,f_1^2\rangle$. Like in lemma 4.4.11, the squared edge lengths can be calculated in the simpler coordinates of the individual side faces:

$$\begin{aligned}\|f\|^2 = \|v_{13} - v\|^2 &= c^2 + s^2 + s_{13}^2 + 2ss_{13}\cos\varphi \\ &= h^2 + R^2 + R^{*2} + s^2 + s_{13}^2 + 2(\alpha + ss_{13})\cos\varphi \\ &= D + s^2 + s_{13}^2 + 2\cos\varphi A_3;\end{aligned}$$

for the other two edges, we recall lemma 4.4.14 about d and ψ:

$$\begin{aligned}\left\|f^2\right\|^2 = \|v_{23} - v\|^2 &= d^2 + s^2 + s_{23}^2 + 2ss_{23}\cos\psi \\ &= h^2 + R^2 + R^{*2} + s^2 + s_{23}^2 + 2(\alpha + ss_{23})\cos\psi \\ &= D + s^2 + s_{23}^2 + 2\cos\psi A_2;\end{aligned}$$

$$\begin{aligned}\left\|f_1^2\right\|^2 = \|v_{12} - v_{13}\|^2 &= d^2 + s_{12}^2 + s_{13}^2 + 2s_{12}s_{13}\cos\psi \\ &= h^2 + R^2 + R^{*2} + s_{12}^2 + s_{13}^2 + 2(\alpha + s_{12}s_{13})\cos\psi \\ &= D + s_{12}^2 + s_{13}^2 + 2\cos\psi A_1.\end{aligned}$$

For the scalar products, we have to return to the coordinates for the entire cube:

$$\begin{aligned}\left\langle f, f^2\right\rangle = \frac{1}{r^2R^2C_2}\Big(&\big(Hs_{13}R^2 - \alpha khR + r^2s\big)\big(HR^2(B_2 - sA_2) + khR(R^2A_2 + sB_2) + r^2sC_2\big) \\ &+ \big(khs_{13}R^2 + H\alpha R + r^2R\big)\big(khR^2(B_2 - sA_2) - HR(R^2A_2 + sB_2) + r^2RC_2\big) \\ &+ r^2C_2\big(hR - kB_3\big)\big(hR - kB_2\big)\Big) \\ = \frac{1}{r^2R^2C_2}\Big(&k^2\big(-\alpha h^2R^2(R^2A_2 + sB_2) + h^2s_{13}R^4(B_2 - sA_2) + r^2B_2B_3C_2\big) \\ &+kH\big(hs_{13}R^3(R^2A_2 + sB_2) - \alpha hR^3(B_2 - sA_2) \\ &\quad - hs_{13}R^3(R^2A_2 + sB_2) + \alpha hR^3(B_2 - sA_2)\big) \\ &+H^2\big(s_{13}R^4(B_2 - sA_2) - \alpha R^2(R^2A_2 + sB_2)\big) \\ &+k\big(-\alpha hr^2sRC_2 + hr^2sR(R^2A_2 + sB_2) + hr^2s_{13}R^3C_2 \\ &\quad + hr^2R^3(B_2 - sA_2) - hr^2RC_2(B_2 + B_3)\big) \\ &+H\big(r^2ss_{13}R^2C_2 + r^2sR^2(B_2 - sA_2) + \alpha r^2R^2C_2 - r^2R^2(R^2A_2 + sB_2)\big) \\ &+r^4s^2C_2 + r^4R^2C_2 + r^2h^2R^2C_2\Big) \\ = \frac{1}{r^2R^2C_2}\Big(&k^2\Big(h^2R^2\big(R^2A_2(-\alpha - ss_{13}) + B_2(-\alpha s + s_{13}R^2)\big) + r^2B_2B_3C_2\Big) \\ &+H^2R^2\big(R^2A_2(-ss_{13} - \alpha) + B_2(s_{13}R^2 - \alpha s)\big) \\ &+khr^2R\big(A_2(sR^2 - sR^2) + B_2(s^2 + R^2) + C_2(-\alpha s + s_{13}R^2 - B_2 - B_3)\big) \\ &+Hr^2R^2\big(A_2(-s^2 - R^2) + B_2(s - s) + C_2(ss_{13} + \alpha)\big)\end{aligned}$$

$$
\begin{aligned}
&\qquad +r^2\big(r^2C_2(s^2+R^2)+h^2R^2C_2\big)\Big)\\
&=\frac{1}{r^2R^2C_2}\Big(k^2\big(h^2R^2(-R^2A_2A_3+B_2B_3)+r^2B_2B_3C_2\big)\\
&\qquad +\big((\frac{r^*r}{\alpha})^2-h^2k^2\big)R^2\big(-R^2A_2A_3+B_2B_3\big)\\
&\qquad +khr^2R\big(B_2C_2+C_2(B_3-B_2-B_3)\big)\\
&\qquad +Hr^2R^2C_2\big(A_3-A_2)\big)\\
&\qquad +r^2C_2\big(r^2C_2+h^2R^2\big)\Big)\\
&=\frac{1}{R^2}\Big(k^2B_2B_3+\frac{1}{C_2}(\frac{r^*}{R^*})^2\big(B_2B_3-R^2A_2A_3\big)+HR^2(A_3-A_2)+r^2C_2+h^2R^2\Big);
\end{aligned}
$$

we plug in the definitions $r^2=R^2-\alpha^2k^2$ and $r^{*2}=R^{*2}-\alpha^2k^2$ and recall $H=\cos\varphi+\alpha k^2$:

$$
\begin{aligned}
\langle f,f^2\rangle &=\frac{1}{R^2}\Big(k^2B_2B_3+\frac{1}{C_2}\big(1-R^2k^2\big)\big(B_2B_3-R^2A_2A_3\big)\\
&\quad+\big(\cos\varphi+\alpha k^2\big)R^2\big(A_3-A_2\big)+\big(R^2-\alpha^2k^2\big)C_2+h^2R^2\Big)\\
&=\frac{1}{R^2C_2}\Big(k^2\big(B_2B_3C_2-R^2B_2B_3+R^4A_2A_3-\alpha^2C_2^2+\alpha R^2(A_3-A_2)C_2\big)\\
&\quad+\cos\varphi R^2C_2(A_3-A_2)+B_2B_3-R^2A_2A_3+R^2C_2^2+h^2R^2C_2\Big)\\
&=\frac{1}{R^2C_2}\Big(k^2\big(s^2(s_{23}R^2-\alpha s)B_3+(\alpha R^4+ss_{13}R^4-\alpha R^4-\alpha s^2R^2)A_2\\
&\qquad+\alpha C_2(-\alpha R^2-\alpha s^2+\alpha R^2+ss_{13}R^2)\big)\\
&\quad+\cos\varphi R^2C_2(A_3-A_2)+B_2B_3-R^2A_2A_3+R^2C_2^2+h^2R^2C_2\Big)\\
&=\frac{1}{R^2C_2}\Big(k^2B_3\big(s^2s_{23}R^2-\alpha s^3+sR^2A_2+\alpha sC_2\big)\\
&\quad+\cos\varphi R^2C_2s(s_{13}-s_{23})\\
&\quad+B_2B_3-R^2A_2A_3+R^2C_2^2+h^2R^2C_2\Big)\\
&=\frac{1}{R^2C_2}\Big(k^2sR^2B_3\big(ss_{23}+A_2+\alpha\big)\\
&\quad+\cos\varphi R^2C_2s(s_{13}-s_{23})\\
&\quad+B_2B_3-R^2A_2A_3+R^2C_2^2+h^2R^2C_2\Big)\\
&=2k^2\frac{s}{C_2}A_2B_3+\cos\varphi s(s_{13}-s_{23})+\frac{1}{R^2C_2}\big(B_2B_3-R^2A_2A_3\big)+C_2+h^2;
\end{aligned}
$$

$$
\begin{aligned}
\langle f, f_1^2\rangle = \frac{1}{r^2R^2C_1}\Big(&\big(Hs_{13}R^2-\alpha khR+r^2s\big)\big(\alpha khRC_1-Hs_{13}R^2C_1+r^2(s_{13}R^2A_1+\alpha B_1)\big)\\
&+\big(khs_{13}R^2+H\alpha R+r^2R\big)\big(-khs_{13}R^2C_1-H\alpha RC_1+r^2R(\alpha A_1-s_{13}B_1)\big)\\
&+r^2C_1\big(hR-kB_3\big)\big(kB_1-hR\big)\Big)\\
= \frac{1}{r^2R^2C_1}\Big(&k^2\big(-\alpha^2h^2R^2C_1-h^2s_{13}^2R^4C_1-r^2B_1B_3C_1\big)\\
&+kH\big(2\alpha hs_{13}R^3C_1-2\alpha hs_{13}R^3C_1\big)\\
&+H^2\big(-s_{13}^2R^4C_1-\alpha^2R^2C_1\big)\\
&+k\big(-\alpha hr^2R(s_{13}R^2A_1+\alpha B_1)+\alpha hr^2sRC_1+hr^2s_{13}R^3(\alpha A_1-s_{13}B_1)\\
&\qquad-hr^2s_{13}R^3C_1+hr^2RB_1C_1+hr^2RB_3C_1\big)\\
&+H\big(r^2s_{13}R^2(s_{13}R^2A_1+\alpha B_1)-r^2ss_{13}R^2C_1\\
&\qquad+\alpha r^2R^2(\alpha A_1-s_{13}B_1)-\alpha r^2R^2C_1\big)\\
&+r^4s(s_{13}R^2A_1+\alpha B_1)+r^4R^2(\alpha A_1-s_{13}B_1)-r^2h^2R^2C_1\Big)\\
= \frac{1}{r^2R^2C_1}\Big(&-k^2\big(h^2R^2C_1^2+r^2B_1B_3C_1\big)\\
&-\big((\frac{r^*r}{\alpha})^2-h^2k^2\big)R^2C_1^2\\
&+khr^2R\big(A_1(-\alpha s_{13}R^2+\alpha s_{13}R^2)+B_1(-\alpha^2-s_{13}^2R^2)\\
&\qquad+C_1(\alpha s-s_{13}R^2+B_1+B_3)\big)\\
&+Hr^2R^2\big(A_1(s_{13}^2R^2+\alpha^2)+B_1(\alpha s_{13}-\alpha s_{13})-C_1(ss_{13}+\alpha)\big)\\
&+r^2\big(r^2R^2A_1(ss_{13}+\alpha)+r^2B_1(\alpha s-s_{13}R^2)-h^2R^2C_1\big)\Big)\\
= \frac{1}{r^2R^2C_1}\Big(&-k^2r^2B_1B_3C_1\\
&-(\frac{r^*}{R^*})^2r^2C_1^2\\
&+khr^2R\big(-B_1C_1+C_1(-B_3+B_1+B_3)\big)\\
&+Hr^2R^2\big(A_1C_1-A_3C_1\big)\\
&+r^4\big(R^2A_1A_3-B_1B_3\big)-h^2r^2R^2C_1\Big)\\
= \frac{1}{R^2}\Big(&-k^2B_1B_3-(\frac{r^*}{R^*})^2C_1+HR^2\big(A_1-A_3\big)+\frac{r^2}{C_1}\big(R^2A_1A_3-B_1B_3\big)-h^2R^2\Big);
\end{aligned}
$$

again we expand r^2, r^{*2} and H:

$$
\begin{aligned}
\langle f, f_1^2\rangle = \frac{1}{R^2}\Big(&-k^2B_1B_3-\big(1-R^2k^2\big)C_1+\frac{1}{C_1}\big(R^2-\alpha^2k^2\big)\big(R^2A_1A_3-B_1B_3\big)\\
&+\big(\cos\varphi+\alpha k^2\big)R^2\big(A_1-A_3\big)-h^2R^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{R^2C_1}\Big(k^2\big(-B_1B_3C_1 + R^2C_1^2 - \alpha^2R^2A_1A_3 + \alpha^2B_1B_3 + \alpha R^2(A_1 - A_3)C_1\big) \\
&\quad + \cos\varphi R^2C_1s_{13}(s_{12} - s) \\
&\quad + R^2\big(R^2A_1A_3 - B_1B_3\big) - C_1^2 - h^2R^2C_1\Big) \\
&= \frac{1}{R^2C_1}\Big(k^2\big(A_1(\alpha s_{13}^2R^4 + \alpha^3R^2 - \alpha^3R^2 - \alpha^2 ss_{13}R^2) + B_1B_3(\alpha^2 - s_{13}^2R^2 - \alpha^2) \\
&\qquad + R^2C_1(s_{13}^2R^2 + \alpha^2 - \alpha^2 - \alpha ss_{13})\big) \\
&\quad + \cos\varphi R^2C_1s_{13}(s_{12} - s) \\
&\quad + R^2\big(R^2A_1A_3 - B_1B_3\big) - C_1^2 - h^2R^2C_1\Big) \\
&= \frac{1}{R^2C_1}\Big(k^2\big(\alpha s_{13}R^2A_1B_3 - s_{13}^2R^2B_1B_3 + s_{13}R^2C_1B_3\big) \\
&\quad + \cos\varphi R^2C_1s_{13}(s_{12} - s) \\
&\quad + R^2\big(R^2A_1A_3 - B_1B_3\big) - C_1^2 - h^2R^2C_1\Big) \\
&= \frac{1}{R^2C_1}\Big(k^2s_{13}R^2\big(\alpha A_1B_3 + B_3(C_1 - s_{13}B_1)\big) \\
&\quad + \cos\varphi R^2C_1s_{13}(s_{12} - s) \\
&\quad + R^2\big(R^2A_1A_3 - B_1B_3\big) - C_1^2 - h^2R^2C_1\Big) \\
&= \frac{1}{R^2C_1}\Big(k^2s_{13}R^2\big(\alpha A_1B_3 + B_3(\alpha^2 + \alpha s_{12}s_{13})\big) \\
&\quad + \cos\varphi R^2C_1s_{13}(s_{12} - s) \\
&\quad + R^2\big(R^2A_1A_3 - B_1B_3\big) - C_1^2 - h^2R^2C_1\Big) \\
&= 2k^2\frac{\alpha s_{13}}{C_1}A_1B_3 + \cos\varphi s_{13}(s_{12} - s) + \frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - R^{*2} - h^2.
\end{aligned}
$$

Now, for $\|f \times f^2\|$, we calculate $\|f\|^2\,\|f^2\|^2$, remembering from corollary 4.4.13 and lemma 4.4.17 that $\cos\psi = -\cos\varphi - 2\alpha k^2$ and $\cos\varphi\cos\psi = k^2D - 1$:

$$
\begin{aligned}
\|f\|^2\,\|f^2\|^2 &= \big(2\cos\varphi A_3 + s^2 + s_{13}^2 + D\big)\big(2\cos\psi A_2 + s^2 + s_{23}^2 + D\big) \\
&= 4\big(k^2D - 1\big)A_2A_3 \\
&\quad + 2\cos\varphi A_3\big(s^2 + s_{23}^2 + D\big) - 2\big(\cos\varphi + 2\alpha k^2\big)A_2\big(s^2 + s_{13}^2 + D\big) \\
&\quad + \big(s^2 + s_{13}^2 + D\big)\big(s^2 + s_{23}^2 + D\big) \\
&= 4k^2\big(D(A_2A_3 - \alpha A_2) - \alpha(s^2 + s_{13}^2)A_2\big) \\
&\quad + 2\cos\varphi\big((D + s^2)(A_3 - A_2) + s_{23}^2A_3 - s_{13}^2A_2\big) \\
&\quad + \big(s^2 + s_{13}^2 + D\big)\big(s^2 + s_{23}^2 + D\big) - 4A_2A_3 \\
&= 4k^2A_2\big(ss_{13}D - \alpha(s^2 + s_{13}^2)\big) \\
&\quad + 2\cos\varphi\big((D + s^2)s(s_{13} - s_{23}) + \alpha(s_{23}^2 - s_{13}^2) + ss_{13}s_{23}(s_{23} - s_{13})\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \big(s^2 + s_{13}^2 + D\big)\big(s^2 + s_{23}^2 + D\big) - 4A_2A_3 \\
&= 4k^2 A_2\big(ss_{13}D - \alpha(s^2 + s_{13}^2)\big) \\
&\quad + 2\cos\varphi(s_{23} - s_{13})\big(-s(D + s^2) + \alpha(s_{23} + s_{13}) + ss_{13}s_{23}\big) \\
&\quad + \big(s^2 + s_{13}^2 + D\big)\big(s^2 + s_{23}^2 + D\big) - 4A_2A_3.
\end{aligned}
$$

For $\langle f, f^2\rangle^2$, we first rewrite the constant part:

$$
\begin{aligned}
&\frac{1}{R^2C_2}\big(B_2B_3 - R^2A_2A_3\big) + C_2 + h^2 \\
&= \frac{1}{R^2C_2}\big(B_2B_3 - R^2A_2A_3\big) + s^2 + D - R^{*2} \\
&= \frac{1}{R^2C_2}\big(B_2B_3 - R^2A_2A_3 - \alpha^2C_2\big) + s^2 + D \\
&= \frac{1}{R^2C_2}\big(s_{23}R^2B_3 - \alpha ss_{13}R^2 + \alpha^2s^2 - R^2A_2A_3 - \alpha^2s^2 - \alpha^2R^2\big) + s^2 + D \\
&= \frac{1}{C_2}\big(s_{23}B_3 - \alpha(ss_{13} + \alpha) - A_2A_3\big) + s^2 + D \\
&= \frac{1}{C_2}\big(s_{23}B_3 - (\alpha + A_2)A_3\big) + s^2 + D,
\end{aligned}
$$

with that and

$$
\cos^2\varphi = 1 - \sin^2\varphi = 1 - c^2k^2 = 1 - k^2(D + 2\alpha\cos\varphi),
$$

we get

$$
\begin{aligned}
\langle f, f^2\rangle^2 &= \Big(2k^2\frac{s}{C_2}A_2B_3 + \cos\varphi s(s_{13} - s_{23}) + \frac{1}{C_2}\big(s_{23}B_3 - (\alpha + A_2)A_3\big) + s^2 + D\Big)^2 \\
&= 4k^4\frac{s^2}{C_2^2}A_2^2B_3^2 \\
&\quad + \cos^2\varphi s^2(s_{13} - s_{23})^2 \\
&\quad + 4k^2\cos\varphi\frac{s^2}{C_2}(s_{13} - s_{23})A_2B_3 \\
&\quad + 4k^2\frac{s}{C_2}A_2B_3\Big(\frac{1}{C_2}\big(s_{23}B_3 - (\alpha + A_2)A_3\big) + s^2 + D\Big) \\
&\quad + 2\cos\varphi s(s_{13} - s_{23})\Big(\frac{1}{C_2}\big(s_{23}B_3 - (\alpha + A_2)A_3\big) + s^2 + D\Big) \\
&\quad + \Big(\frac{1}{C_2}\big(s_{23}B_3 - (\alpha + A_2)A_3\big) + s^2 + D\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&= 4k^4 \frac{s^2}{C_2^2} A_2^2 B_3^2 \\
&\quad + 2k^2 \cos\varphi \big(\frac{2s^2}{C_2} (s_{13} - s_{23}) A_2 B_3 - \alpha s^2 (s_{13} - s_{23})^2 \big) \\
&\quad + k^2 \bigg(\frac{4s}{C_2} A_2 B_3 \Big(\frac{1}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + s^2 + D \Big) - s^2 (s_{13} - s_{23})^2 D \bigg) \\
&\quad + 2 \cos\varphi s (s_{13} - s_{23}) \Big(\frac{1}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + s^2 + D \Big) \\
&\quad + \Big(\frac{1}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + s^2 + D \Big)^2 + s^2 (s_{13} - s_{23})^2
\end{aligned}
$$

Put together, we have

$$
\begin{aligned}
\left\| f \times f^2 \right\|^2 &= -4k^4 \frac{s^2}{C_2^2} A_2^2 B_3^2 \\
&\quad - 2k^2 \cos\varphi s^2 (s_{13} - s_{23}) \big(\frac{2}{C_2} A_2 B_3 - \alpha (s_{13} - s_{23}) \big) \\
&\quad + k^2 \bigg(s^2 (s_{13} - s_{23})^2 D + 4 A_2 \big(s s_{13} D - \alpha (s^2 + s_{13}^2) \big) \\
&\qquad - \frac{4s}{C_2} A_2 B_3 \Big(\frac{1}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + s^2 + D \Big) \bigg) \\
&\quad + 2 \cos\varphi (s_{23} - s_{13}) \bigg(-s (D + s^2) + \alpha (s_{23} + s_{13}) + s s_{13} s_{23} \\
&\qquad + s \Big(\frac{1}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + s^2 + D \Big) \bigg) \\
&\quad + \big(s^2 + s_{13}^2 + D \big) \big(s^2 + s_{23}^2 + D \big) - 4 A_2 A_3 \\
&\qquad - \Big(\frac{1}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + s^2 + D \Big)^2 - s^2 (s_{13} - s_{23})^2 .
\end{aligned}
$$

We simplify the individual terms. For the one at $k^2 \cos\varphi$,

$$
\begin{aligned}
&\frac{2}{C_2} A_2 B_3 - \alpha (s_{13} - s_{23}) \\
&= \frac{1}{C_2} \big(A_2 B_3 + (\alpha + s s_{23})(s_{13} R^2 - \alpha s) - \alpha (s_{13} - s_{23})(R^2 + s^2) \big) \\
&= \frac{1}{C_2} \big(A_2 B_3 + \alpha s_{13} R^2 - \alpha^2 s + s s_{13} s_{23} R^2 - \alpha s^2 s_{23} - \alpha s_{13} R^2 - \alpha s^2 s_{13} + \alpha s_{23} R^2 + \alpha s^2 s_{23} \big) \\
&= \frac{1}{C_2} \big(A_2 B_3 - \alpha^2 s + s s_{13} s_{23} R^2 - \alpha s^2 s_{13} + \alpha s_{23} R^2 \big) \\
&= \frac{1}{C_2} \big(A_2 B_3 + s_{23} R^2 (s s_{13} + \alpha) - \alpha s (\alpha + s s_{13}) \big) \\
&= \frac{1}{C_2} \big(A_2 B_3 + A_3 B_2 \big);
\end{aligned}
$$

equally for the term at $\cos\varphi$:

$$
\begin{aligned}
&- s(D+s^2) + \alpha(s_{23}+s_{13}) + ss_{13}s_{23} + s\Big(\frac{1}{C_2}\big(s_{23}B_3 - (\alpha+A_2)A_3\big) + s^2 + D\Big)\\
&= \alpha(s_{23}+s_{13}) + ss_{13}s_{23} + \frac{s}{C_2}\big(s_{23}B_3 - (\alpha+A_2)A_3\big)\\
&= \frac{1}{C_2}\Big(ss_{23}B_3 - \alpha sA_3 - sA_2A_3 + \alpha(s_{23}+s_{13})(R^2+s^2) + ss_{13}s_{23}(R^2+s^2)\Big)\\
&= \frac{1}{C_2}\Big(ss_{23}B_3 - \alpha sA_3 - s\big(\alpha^2 + \alpha s(s_{13}+s_{23}) + s^2s_{13}s_{23}\big)\\
&\quad + \alpha(s_{23}+s_{13})(R^2+s^2) + ss_{13}s_{23}(R^2+s^2)\Big)\\
&= \frac{1}{C_2}\Big(ss_{23}B_3 - \alpha sA_3 - \alpha^2 s + \alpha(s_{23}+s_{13})R^2 + ss_{13}s_{23}R^2\Big)\\
&= \frac{1}{C_2}\Big(ss_{23}B_3 - \alpha sA_3 + s_{23}R^2(\alpha+ss_{13}) + \alpha(s_{13}R^2 - \alpha s)\Big)\\
&= \frac{1}{C_2}\Big(A_2B_3 + A_3B_2\Big).
\end{aligned}
$$

The term at k^2 can be rewritten as

$$
\begin{aligned}
&s^2(s_{13}-s_{23})^2D + 4A_2\big(ss_{13}D - \alpha(s^2+s_{13}^2)\big)\\
&\quad - \frac{4s}{C_2}A_2B_3\Big(\frac{1}{C_2}\big(s_{23}B_3 - (\alpha+A_2)A_3\big) + s^2 + D\Big)\\
&= s^2(s_{13}-s_{23})^2D + \frac{4s}{C_2}A_2D\big(s_{13}C_2 - B_3\big)\\
&\quad - \frac{4}{C_2^2}A_2\Big(\alpha(s^2+s_{13}^2)C_2^2 + ss_{23}B_3^2 - s(\alpha+A_2)A_3B_3 + s^3B_3C_2\Big)\\
&= s^2(s_{13}-s_{23})^2D + \frac{4s}{C_2}A_2D\big(s_{13}R^2 + s^2s_{13} - s_{13}R^2 + \alpha s\big)\\
&\quad - \frac{4}{C_2^2}A_2\Big(ss_{23}B_3^2 + \alpha B_3^2 - \alpha B_3^2 + s^2C_2(\alpha C_2 + sB_3) + \alpha s_{13}^2C_2^2 - s(\alpha+A_2)A_3B_3\Big)\\
&= s^2(s_{13}-s_{23})^2D + \frac{4s^2}{C_2}A_2A_3D - \frac{4}{C_2^2}A_2^2B_3^2\\
&\quad - \frac{4}{C_2^2}A_2\Big(s^2C_2(\alpha R^2 + \alpha s^2 + ss_{13}R^2 - \alpha s^2) + \alpha s_{13}^2C_2^2 - \alpha B_3^2 - s(\alpha+A_2)A_3B_3\Big)\\
&= s^2(s_{13}-s_{23})^2D + \frac{4s^2}{C_2}A_2A_3D - \frac{4}{C_2^2}A_2^2B_3^2\\
&\quad - \frac{4}{C_2^2}A_2\Big(s^2R^2A_3C_2 - s(\alpha+A_2)A_3B_3\\
&\qquad + \alpha\big(s_{13}^2(s^4+R^4+2s^2R^2) - s_{13}^2R^4 - \alpha^2s^2 + 2\alpha ss_{13}R^2\big)\Big)\\
&= s^2(s_{13}-s_{23})^2D + \frac{4s^2}{C_2}A_2A_3D - \frac{4}{C_2^2}A_2^2B_3^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{4}{C_2^2} A_2 \Big(s^2 R^2 A_3 C_2 - s(\alpha + A_2) A_3 B_3 + \alpha \big(s^2 (s^2 s_{13}^2 - \alpha^2) + 2 s s_{13} R^2 (\alpha + s s_{13}) \big) \Big) \\
&= s^2 (s_{13} - s_{23})^2 D + \frac{4s^2}{C_2} A_2 A_3 D - \frac{4}{C_2^2} A_2^2 B_3^2 \\
&\quad - \frac{4}{C_2^2} A_2 A_3 \Big(s^2 R^2 C_2 - s(\alpha + A_2) B_3 + \alpha \big(s^2 (s s_{13} - \alpha) + 2 s s_{13} R^2 \big) \Big) \\
&= s^2 (s_{13} - s_{23})^2 D + \frac{4s^2}{C_2} A_2 A_3 D - \frac{4}{C_2^2} A_2^2 B_3^2 \\
&\quad - \frac{4}{C_2^2} A_2 A_3 \Big(s^2 R^2 C_2 - s(\alpha + A_2) B_3 + \alpha \big(s s_{13} (s^2 + R^2) + s (s_{13} R^2 - \alpha s) \big) \Big) \\
&= s^2 (s_{13} - s_{23})^2 D + \frac{4s^2}{C_2} A_2 A_3 D - \frac{4}{C_2^2} A_2^2 B_3^2 \\
&\quad - \frac{4}{C_2^2} A_2 A_3 \Big(s^2 R^2 C_2 - s A_2 B_3 + \alpha s^3 s_{13} + \alpha s s_{13} R^2 \big) \Big) \\
&= s^2 (s_{13} - s_{23})^2 D + \frac{4s^2}{C_2} A_2 A_3 D - \frac{4}{C_2^2} A_2^2 B_3^2 \\
&\quad - \frac{4}{C_2^2} A_2 A_3 \Big(s^2 R^2 C_2 - s(\alpha + s s_{23})(s_{13} R^2 - \alpha s) + \alpha s^3 s_{13} + \alpha s s_{13} R^2 \Big) \\
&= s^2 (s_{13} - s_{23})^2 D + \frac{4s^2}{C_2} A_2 A_3 D - \frac{4}{C_2^2} A_2^2 B_3^2 \\
&\quad - \frac{4}{C_2^2} A_2 A_3 \Big(s^2 R^2 C_2 + \alpha^2 s^2 - s^2 s_{13} s_{23} R^2 + \alpha s^3 s_{23} + \alpha s^3 s_{13} \Big) \\
&= s^2 (s_{13} - s_{23})^2 D + \frac{4s^2}{C_2} A_2 A_3 D - \frac{4}{C_2^2} A_2^2 B_3^2 \\
&\quad - \frac{4s^2}{C_2^2} A_2 A_3 \Big(R^2 (C_2 - s_{13} s_{23}) + \alpha^2 + \alpha s (s_{13} + s_{23}) \Big),
\end{aligned}
$$

and the constant one is

$$
\begin{aligned}
&\big(s^2 + s_{13}^2 + D \big) \big(s^2 + s_{23}^2 + D \big) - 4 A_2 A_3 \\
&\quad - \Big(\frac{1}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + s^2 + D \Big)^2 - s^2 (s_{13} - s_{23})^2 \\
&= (D + s^2)^2 + (D + s^2)(s_{13}^2 + s_{23}^2) + s_{13}^2 s_{23}^2 - s^2 (s_{13} - s_{23})^2 - 4 A_2 A_3 \\
&\quad - (D + s^2)^2 - \frac{2}{C_2} (D + s^2) \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) - \frac{1}{C_2^2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big)^2 \\
&= D (s_{13} - s_{23})^2 + 2 D s_{13} s_{23} - \frac{2D}{C_2} \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) \\
&\quad + 2 s^2 s_{13} s_{23} + s_{13}^2 s_{23}^2 - 4 A_2 A_3 \\
&\quad - \frac{1}{C_2^2} \Big(2 s^2 C_2 \big(s_{23} B_3 - (\alpha + A_2) A_3 \big) + \big(s_{23} B_3 - (\alpha + A_2) A_3 \big)^2 \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= D(s_{13}-s_{23})^2 + \frac{2D}{C_2}\Big(s_{13}s_{23}C_2 - s_{23}B_3 + (\alpha + A_2)A_3\Big) \\
&\quad + \frac{1}{C_2^2}\Big(\big(s_{13}^2 s_{23}^2 - 4\alpha^2 - 4\alpha s(s_{13}+s_{23}) - 2s^2 s_{13}s_{23}\big)C_2^2 \\
&\quad - 2s^2C_2\big(s_{23}B_3 - (\alpha + A_2)A_3\big) - \big(s_{23}B_3 - (\alpha + A_2)A_3\big)^2\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{2D}{C_2}\Big(ss_{23}(ss_{13}+\alpha) + (\alpha + A_2)A_3\Big) \\
&\quad + \frac{1}{C_2^2}\Big(\big(-4\alpha^2 - 4\alpha s(s_{13}+s_{23})\big)C_2^2 \\
&\quad - 2s^2C_2\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3\big) + s_{13}^2 s_{23}^2 C_2^2 - \big(s_{23}B_3 - (\alpha + A_2)A_3\big)^2\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{2D}{C_2}\Big(2A_2A_3\Big) \\
&\quad + \frac{1}{C_2^2}\Big(\big(-4\alpha^2 - 4\alpha s(s_{13}+s_{23})\big)C_2^2 - 2s^2C_2\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3\big) \\
&\quad + \big(s_{13}s_{23}C_2 - s_{23}B_3 + (\alpha + A_2)A_3\big)\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3\big)\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\quad + \frac{1}{C_2^2}\Big(\big(-4\alpha^2 - 4\alpha s(s_{13}+s_{23})\big)C_2^2 \\
&\quad + 2\big(A_2A_3 - s^2C_2\big)\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3\big)\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\quad + \frac{2}{C_2^2}\Big(A_2A_3\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3\big) + s^2A_2A_3C_2 \\
&\quad - \big(2\alpha^2 + 2\alpha s(s_{13}+s_{23})\big)C_2^2 - s^2C_2\big(s_{13}s_{23}C_2 + s_{23}B_3 - \alpha A_3\big)\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\quad + \frac{2}{C_2^2}\Bigg(A_2A_3\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3 + s^2C_2\big) \\
&\quad - C_2\Big(2\alpha^2s^2 + 2\alpha^2R^2 + 2\alpha s^3(s_{13}+s_{23}) + 2\alpha s(s_{13}+s_{23})R^2 \\
&\qquad + s^4s_{13}s_{23} + s^2s_{13}s_{23}R^2 + s^2s_{13}s_{23}R^2 - \alpha s^3s_{23} - \alpha s^2A_3\Big)\Bigg) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\quad + \frac{2}{C_2^2}\Bigg(A_2A_3\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3 + s^2C_2\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - C_2\Big(2R^2A_2A_3 + \alpha^2 s^2 + \alpha s^3(s_{13}+s_{23}) + s^4 s_{13}s_{23}\Big)\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\quad + \frac{2}{C_2^2}\Big(A_2A_3\big(s_{13}s_{23}C_2 + s_{23}B_3 - (\alpha + A_2)A_3 - 2R^2C_2\big)\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\quad + \frac{2}{C_2^2}A_2A_3\Big(-2R^2C_2 + s_{13}s_{23}R^2 + s^2 s_{13}s_{23} + s_{13}s_{23}R^2 - \alpha s s_{23} \\
&\qquad - \alpha^2 - \alpha s s_{13} - \alpha^2 - \alpha s(s_{13}+s_{23}) - s^2 s_{13}s_{23}\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\quad + \frac{2}{C_2^2}A_2A_3\Big(-2R^2C_2 + 2s_{13}s_{23}R^2 - 2\alpha^2 - 2\alpha s(s_{13}+s_{23})\Big) \\
&= D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D - \frac{4}{C_2^2}A_2A_3\Big(R^2\big(C_2 - s_{13}s_{23}\big) + \alpha^2 + \alpha s(s_{13}+s_{23})\Big).
\end{aligned}
$$

Back to $\left\|f \times f^2\right\|^2$, we see

$$
\begin{aligned}
\left\|f \times f^2\right\|^2 &= -4k^4\frac{s^2}{C_2^2}A_2^2B_3^2 \\
&\quad + 2k^2\cos\varphi s^2(s_{23}-s_{13})\frac{1}{C_2}\big(A_2B_3 + A_3B_2\big) \\
&\quad + k^2\Bigg(s^2(s_{13}-s_{23})^2D + \frac{4s^2}{C_2}A_2A_3D - \frac{4}{C_2^2}A_2^2B_3^2 \\
&\qquad - \frac{4s^2}{C_2^2}A_2A_3\Big(R^2(C_2 - s_{13}s_{23}) + \alpha^2 + \alpha s(s_{13}+s_{23})\Big)\Bigg) \\
&\quad + 2\cos\varphi(s_{23}-s_{13})\frac{1}{C_2}\big(A_2B_3 + A_3B_2\big) \\
&\quad + D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D \\
&\qquad - \frac{4}{C_2^2}A_2A_3\Big(R^2\big(C_2 - s_{13}s_{23}\big) + \alpha^2 + \alpha s(s_{13}+s_{23})\Big) \\
&= -\big(1+s^2k^2\big)\frac{4k^2}{C_2^2}A_2^2B_3^2 \\
&\quad + 2\big(1+s^2k^2\big)\cos\varphi(s_{23}-s_{13})\frac{1}{C_2}\big(A_2B_3 + A_3B_2\big) \\
&\quad + (1+s^2k^2)\Bigg(D(s_{13}-s_{23})^2 + \frac{4}{C_2}A_2A_3D
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{4}{C_2^2} A_2 A_3 \Big(R^2 \big(C_2 - s_{13} s_{23} \big) + \alpha^2 + \alpha s (s_{13} + s_{23}) \Big) \Big) \\
&= \big(1 + s^2 k^2\big) \frac{1}{C_2^2} \Big(-4k^2 A_2^2 B_3^2 + 2 \cos\varphi (s_{23} - s_{13}) C_2 \big(A_2 B_3 + A_3 B_2 \big) \\
&\quad + (s_{13} - s_{23})^2 C_2^2 D + 4 A_2 A_3 \Big(C_2 D - R^2 \big(C_2 - s_{13} s_{23} \big) - \alpha^2 - \alpha s (s_{13} + s_{23}) \Big) \Big) \\
&= \big(1 + s^2 k^2\big) \frac{1}{C_2^2} \Big(-4k^2 A_2^2 B_3^2 + 2 \cos\varphi (s_{23} - s_{13}) C_2 \big(A_2 B_3 + A_3 B_2 \big) \\
&\quad + (s_{13} - s_{23})^2 C_2^2 D \\
&\quad + 4 A_2 A_3 \Big(C_2 (D - R^2) + s_{13} s_{23} R^2 + s^2 s_{13} s_{23} - \alpha^2 - \alpha s (s_{13} + s_{23}) - s^2 s_{13} s_{23} \Big) \Big) \\
&= \big(1 + s^2 k^2\big) \frac{1}{C_2^2} \Big(-4k^2 A_2^2 B_3^2 + 2 \cos\varphi (s_{23} - s_{13}) C_2 \big(A_2 B_3 + A_3 B_2 \big) \\
&\quad + (s_{13} - s_{23})^2 C_2^2 D \\
&\quad + 4 A_2 A_3 \Big(C_2 (D - R^2) + s_{13} s_{23} C_2 - A_2 A_3 \Big) \Big) \\
&= \big(1 + s^2 k^2\big) \frac{1}{C_2^2} \Big(-4k^2 A_2^2 B_3^2 + 2 \cos\varphi (s_{23} - s_{13}) C_2 \big(A_2 B_3 + A_3 B_2 \big) \\
&\quad + (s_{13} - s_{23})^2 C_2^2 D - 4 A_2 A_3 \Big(A_2 A_3 + (R^2 - s_{13} s_{23} - D) C_2 \Big) \Big).
\end{aligned}
$$

Now that we have split off $1 + s^2 k^2$, we bring the last term of result back into a more convenient form. Since

$$
\begin{aligned}
R^2 A_2 A_3 + B_2 B_3 &= R^2 \big(\alpha^2 + \alpha s (s_{13} + s_{23}) + s^2 s_{13} s_{23} \big) \\
&\quad + s_{13} s_{23} R^4 - \alpha s (s_{13} + s_{23}) R^2 + \alpha^2 s^2 \\
&= \alpha^2 (R^2 + s^2) + s_{13} s_{23} R^2 (s^2 + R^2) \\
&= (\alpha^2 + s_{13} s_{23} R^2) C_2,
\end{aligned}
$$

we get

$$
\begin{aligned}
&A_2 A_3 + (R^2 - s_{13} s_{23} - D) C_2 \\
&= -\frac{1}{R^2} B_2 B_3 + \big(R^{*2} + s_{13} s_{23} + R^2 - s_{13} s_{23} - h^2 - R^2 - R^{*2} \big) C_2 \\
&= -\frac{1}{R^2} B_2 B_3 - h^2 C_2,
\end{aligned}
$$

and our result becomes

$$
\begin{aligned}
\big\| f \times f^2 \big\|^2 = \big(1 + s^2 k^2\big) \Big(&-4 \frac{k^2}{C_2^2} A_2^2 B_3^2 + 2 \cos\varphi (s_{23} - s_{13}) \frac{1}{C_2} \big(A_2 B_3 + A_3 B_2 \big) \\
&+ (s_{13} - s_{23})^2 D + 4 A_2 A_3 \frac{h^2}{C_2} + \frac{4}{R^2 C_2^2} A_2 A_3 B_2 B_3 \Big).
\end{aligned}
$$

We do the same for $\|f \times f_1^2\|$; first we calculate $\|f\|^2 \|f_1^2\|^2$:

$$
\begin{aligned}
\|f\|^2 \|f_1^2\|^2 &= \big(2\cos\varphi A_3 + s^2 + s_{13}^2 + D\big)\big(2\cos\psi A_1 + s_{12}^2 + s_{13}^2 + D\big) \\
&= 4\big(k^2 D - 1\big) A_1 A_3 \\
&\quad + 2\cos\varphi A_3\big(s_{12}^2 + s_{13}^2 + D\big) - 2\big(\cos\varphi + 2\alpha k^2\big) A_1\big(s^2 + s_{13}^2 + D\big) \\
&\quad + \big(s^2 + s_{13}^2 + D\big)\big(s_{12}^2 + s_{13}^2 + D\big) \\
&= 4k^2\big(D(A_1A_3 - \alpha A_1) - \alpha(s^2 + s_{13}^2)A_1\big) \\
&\quad + 2\cos\varphi\big((D + s_{13}^2)(A_3 - A_1) + s_{12}^2 A_3 - s^2 A_1\big) \\
&\quad + \big(s^2 + s_{13}^2 + D\big)\big(s_{12}^2 + s_{13}^2 + D\big) - 4A_1A_3 \\
&= 4k^2 A_1\big(s s_{13} D - \alpha(s^2 + s_{13}^2)\big) \\
&\quad + 2\cos\varphi\big((D + s_{13}^2)s_{13}(s - s_{12}) + \alpha(s_{12}^2 - s^2) + s s_{12} s_{13}(s_{12} - s)\big) \\
&\quad + \big(s^2 + s_{13}^2 + D\big)\big(s_{12}^2 + s_{13}^2 + D\big) - 4A_1A_3 \\
&= 4k^2 A_1\big(s s_{13} D - \alpha(s^2 + s_{13}^2)\big) \\
&\quad + 2\cos\varphi(s_{12} - s)\big(-s_{13}(D + s_{13}^2) + \alpha(s + s_{12}) + s s_{12} s_{13}\big) \\
&\quad + \big(s^2 + s_{13}^2 + D\big)\big(s_{12}^2 + s_{13}^2 + D\big) - 4A_1A_3.
\end{aligned}
$$

Next, we square $\langle f, f_1^2\rangle$, again using $\cos^2\varphi = 1 - k^2(D + 2\alpha\cos\varphi)$:

$$
\begin{aligned}
\langle f, f_1^2\rangle^2 &= \Big(2k^2\frac{\alpha s_{13}}{C_1}A_1B_3 + \cos\varphi s_{13}(s_{12} - s) + \frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - R^{*2} - h^2\Big)^2 \\
&= 4k^4\frac{\alpha^2 s_{13}^2}{C_1^2}A_1^2B_3^2 \\
&\quad + \cos^2\varphi s_{13}^2(s_{12} - s)^2 \\
&\quad + 4k^2\cos\varphi\frac{\alpha s_{13}^2}{C_1}(s_{12} - s)A_1B_3 \\
&\quad + 4k^2\frac{\alpha s_{13}}{C_1}A_1B_3\Big(\frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - D + R^2\Big) \\
&\quad + 2\cos\varphi s_{13}(s_{12} - s)\Big(\frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - D + R^2\Big) \\
&\quad + \Big(\frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - D + R^2\Big)^2 \\
&= 4k^4\frac{\alpha^2 s_{13}^2}{C_1^2}A_1^2B_3^2 \\
&\quad + 2\alpha k^2\cos\varphi s_{13}^2(s_{12} - s)\Big(\frac{2}{C_1}A_1B_3 + s - s_{12}\Big) \\
&\quad + k^2\bigg(4\frac{\alpha s_{13}}{C_1}A_1B_3\Big(\frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - D + R^2\Big) - s_{13}^2(s_{12} - s)^2 D\bigg) \\
&\quad + 2\cos\varphi s_{13}(s_{12} - s)\Big(\frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - D + R^2\Big)
\end{aligned}
$$

$$+\Big(\frac{1}{C_1}\big(R^2A_1A_3-B_1B_3\big)-s_{13}^2-D+R^2\Big)^2+s_{13}^2(s_{12}-s)^2.$$

As before, we assemble

$$\begin{aligned}\|f\times f_1^2\|^2 &= -4k^4\frac{\alpha^2 s_{13}^2}{C_1^2}A_1^2B_3^2\\ &\quad -2\alpha k^2\cos\varphi s_{13}^2(s_{12}-s)\Big(\frac{2}{C_1}A_1B_3+s-s_{12}\Big)\\ &\quad +k^2\Bigg(4A_1\big(ss_{13}D-\alpha(s^2+s_{13}^2)\big)+s_{13}^2(s_{12}-s)^2D\\ &\qquad -4\frac{\alpha s_{13}}{C_1}A_1B_3\Big(\frac{1}{C_1}\big(R^2A_1A_3-B_1B_3\big)-s_{13}^2-D+R^2\Big)\Bigg)\\ &\quad +2\cos\varphi(s_{12}-s)\Bigg(-s_{13}(D+s_{13}^2)+\alpha(s+s_{12})+ss_{12}s_{13}\\ &\qquad -s_{13}\Big(\frac{1}{C_1}\big(R^2A_1A_3-B_1B_3\big)-s_{13}^2-D+R^2\Big)\Bigg)\\ &\quad +\big(s^2+s_{13}^2+D\big)\big(s_{12}^2+s_{13}^2+D\big)-4A_1A_3-s_{13}^2(s_{12}-s)^2\\ &\qquad -\Big(\frac{1}{C_1}\big(R^2A_1A_3-B_1B_3\big)-s_{13}^2-D+R^2\Big)^2\end{aligned}$$

and simplify the individual terms: at $k^2\cos\varphi$,

$$\begin{aligned}&\frac{2}{C_1}A_1B_3+s-s_{12}\\ &=\frac{1}{C_1}\big(A_1B_3+(\alpha+s_{12}s_{13})(s_{13}R^2-\alpha s)+(s-s_{12})(s_{13}^2R^2+\alpha^2)\big)\\ &=\frac{1}{C_1}\big(A_1B_3+s_{13}R^2(\alpha+s_{12}s_{13}+ss_{13}-s_{12}s_{13})+\alpha(-\alpha s-ss_{12}s_{13}+\alpha s-\alpha s_{12})\big)\\ &=\frac{1}{C_1}\big(A_1B_3+s_{13}R^2A_3-\alpha s_{12}(ss_{13}+\alpha)\big)\\ &=\frac{1}{C_1}\big(A_1B_3+A_3B_1\big);\end{aligned}$$

at $\cos\varphi$,

$$\begin{aligned}&-s_{13}(D+s_{13}^2)+\alpha(s+s_{12})+ss_{12}s_{13}-s_{13}\Big(\frac{1}{C_1}\big(R^2A_1A_3-B_1B_3\big)-s_{13}^2-D+R^2\Big)\\ &=\alpha(s+s_{12})+ss_{12}s_{13}-s_{13}\Big(\frac{1}{C_1}\big(R^2A_1A_3-B_1B_3\big)+R^2\Big)\\ &=\frac{1}{C_1}\Big(\alpha(s+s_{12})(s_{13}^2R^2+\alpha^2)+ss_{12}s_{13}(s_{13}^2R^2+\alpha^2)-s_{13}R^2(s_{13}^2R^2+\alpha^2)\\ &\qquad -s_{13}R^2\big(\alpha^2+\alpha s_{13}(s+s_{12})+ss_{12}s_{13}^2\big)+s_{13}\big(s_{13}^2R^4-\alpha s_{13}R^2(s+s_{12})+\alpha^2ss_{12}\big)\Big)\\ &=\frac{1}{C_1}\Big(s_{13}R^2\big(\alpha s_{13}(s+s_{12})+ss_{12}s_{13}^2-\alpha^2-\alpha^2-\alpha s_{13}(s+s_{12})-ss_{12}s_{13}^2-\alpha s_{13}(s+s_{12})\big)\Big)\end{aligned}$$

$$
\begin{aligned}
&- s_{13}^3R^4 + s_{13}^3R^4 + \alpha^2\big(\alpha(s+s_{12}) + ss_{12}s_{13} + ss_{12}s_{13})\Big) \\
&= \frac{\alpha}{C_1}\Big(s_{13}R^2\big(-\alpha - ss_{13} - \alpha - s_{12}s_{13}\big) + \alpha s\big(\alpha + s_{12}s_{13}\big) + \alpha s_{12}\big(\alpha + ss_{13}\big)\Big) \\
&= -\frac{\alpha}{C_1}\Big(A_1B_3 + A_3B_1\Big);
\end{aligned}
$$

the constant term is

$$
\begin{aligned}
&\big(s^2 + s_{13}^2 + D\big)\big(s_{12}^2 + s_{13}^2 + D\big) - 4A_1A_3 - s_{13}^2(s_{12} - s)^2 \\
&\quad - \Big(\frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - D + R^2\Big)^2 \\
&= (s_{13}^2 + D)^2 + (s_{13}^2 + D)(s^2 + s_{12}^2) + s^2s_{12}^2 - 4A_1A_3 - s_{13}^2(s_{12} - s)^2 \\
&\quad - (s_{13}^2 + D)^2 + (s_{13}^2 + D)\frac{2}{C_1}\big(R^2A_1A_3 - B_1B_3 + R^2C_1\big) - \frac{1}{C_1^2}\big(R^2A_1A_3 - B_1B_3\big)^2 \\
&\quad - \frac{2}{C_1}\big(R^4A_1A_3 - R^2B_1B_3\big) - R^4 \\
&= D\Big((s - s_{12})^2 + 2ss_{12} + \frac{2}{C_1}\big(R^2A_1A_3 - B_1B_3 + R^2C_1\big)\Big) \\
&\quad + s_{13}^2\big(s^2 + s_{12}^2 - (s_{12} - s)^2\big) + s^2s_{12}^2 - 4A_1A_3 - R^4 \\
&\quad + \frac{1}{C_1^2}\Big(2s_{13}^2C_1\big(R^2A_1A_3 - B_1B_3 + R^2C_1\big) - 2R^2C_1\big(R^2A_1A_3 - B_1B_3\big) \\
&\qquad - \big(R^2A_1A_3 - B_1B_3\big)^2\Big) \\
&= (s - s_{12})^2D + \frac{2D}{C_1}\big(R^2A_1A_3 - B_1B_3 + R^2C_1 + ss_{12}C_1\big) \\
&\quad + \frac{1}{C_1^2}\Big(A_1A_3\big(-4C_1^2 + 2s_{13}^2R^2C_1 - 2R^4C_1 - R^4A_1A_3 + 2R^2B_1B_3\big) \\
&\qquad + C_1^2\big(2ss_{12}s_{13}^2 + s^2s_{12}^2 - R^4 + 2s_{13}^2R^2\big) + B_1B_3\big(-2s_{13}^2C_1 + 2R^2C_1 - B_1B_3\big)\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
&- B_1B_3 + R^2C_1 + ss_{12}C_1 \\
&= -s_{13}^2R^4 + \alpha s_{13}R^2(s + s_{12}) - \alpha^2ss_{12} + s_{13}^2R^4 + \alpha^2R^2 + ss_{12}s_{13}^2R^2 + \alpha^2ss_{12} \\
&= R^2\big(\alpha^2 + \alpha s_{13}(s + s_{12}) + ss_{12}s_{13}^2\big) \\
&= R^2A_1A_3
\end{aligned}
$$

and, using that,

$$
\begin{aligned}
&C_1^2\big(2ss_{12}s_{13}^2 + s^2s_{12}^2 - R^4 + 2s_{13}^2R^2\big) + B_1B_3\big(-2s_{13}^2C_1 + 2R^2C_1 - B_1B_3\big) \\
&= C_1\Big(2s_{13}^2\big(ss_{12}C_1 + R^2C_1\big) + s^2s_{12}^2C_1 - R^4C_1\Big) \\
&\quad + B_1B_3\Big(-B_1B_3 + R^2C_1 + ss_{12}C_1 - ss_{12}C_1 + R^2C_1 - 2s_{13}^2C_1\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= C_1\Big(2s_{13}^2\big(ss_{12}C_1 + R^2C_1 - B_1B_3\big) + ss_{12}\big(ss_{12}C_1 - B_1B_3\big)\\
&\quad - R^2\big(R^2C_1 - B_1B_3\big)\Big) + R^2A_1A_3B_1B_3\\
&= C_1\Big(2s_{13}^2R^2A_1A_3 + ss_{12}\big(ss_{12}C_1 - B_1B_3 + R^2C_1\big) - ss_{12}R^2C_1\\
&\quad - R^2\big(R^2C_1 - B_1B_3 + ss_{12}C_1\big) + ss_{12}R^2C_1\Big) + R^2A_1A_3B_1B_3\\
&= R^2A_1A_3\Big(2s_{13}^2C_1 + ss_{12}C_1 - R^2C_1 + B_1B_3\Big)\\
&= R^2A_1A_3\Big(B_1B_3 - R^2C_1 - ss_{12}C_1 + 2ss_{12}C_1 + 2s_{13}^2C_1\Big)\\
&= R^2A_1A_3\Big(-R^2A_1A_3 + 2\big(ss_{12} + s_{13}^2\big)C_1\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&- 4C_1^2 + 2s_{13}^2R^2C_1 - 2R^4C_1 - R^4A_1A_3 + 2R^2B_1B_3\\
&= 2R^2\big(B_1B_3 - R^2C_1 - ss_{12}C_1\big) + 2ss_{12}R^2C_1 + 2s_{13}^2R^2C_1 - 4C_1^2 - R^4A_1A_3\\
&= -3R^4A_1A_3 + 2(ss_{12} + s_{13}^2)R^2C_1 - 4C_1^2.
\end{aligned}
$$

Plugging these results back into the constant term of $\left\|f \times f_1^2\right\|^2$, we get

$$
\begin{aligned}
&(s - s_{12})^2D + \frac{2D}{C_1}\big(R^2A_1A_3 - B_1B_3 + R^2C_1 + ss_{12}C_1\big)\\
&\quad + \frac{1}{C_1^2}\Big(A_1A_3\big(-4C_1^2 + 2s_{13}^2R^2C_1 - 2R^4C_1 - R^4A_1A_3 + 2R^2B_1B_3\big)\\
&\qquad + C_1^2\big(2ss_{12}s_{13}^2 + s^2s_{12}^2 - R^4 + 2s_{13}^2R^2\big) + B_1B_3\big(-2s_{13}^2C_1 + 2R^2C_1 - B_1B_3\big)\Big)\\
&= (s - s_{12})^2D + \frac{4D}{C_1}R^2A_1A_3\\
&\quad + \frac{1}{C_1^2}\Big(A_1A_3\big(-3R^4A_1A_3 + 2(ss_{12} + s_{13}^2)R^2C_1 - 4C_1^2\big)\\
&\qquad + R^2A_1A_3\big(-R^2A_1A_3 + 2(ss_{12} + s_{13}^2)C_1\big)\Big)\\
&= (s - s_{12})^2D + \frac{4D}{C_1}R^2A_1A_3\\
&\quad + \frac{4}{C_1^2}\Big(A_1A_3\big(-R^4A_1A_3 + (ss_{12} + s_{13}^2)R^2C_1 - C_1^2\big)\Big)\\
&= (s - s_{12})^2D + \frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3 + ss_{12}R^2 - \alpha^2\Big).
\end{aligned}
$$

Finally, the term at k^2 is

$$
\begin{aligned}
&4A_1\big(ss_{13}D - \alpha(s^2 + s_{13}^2)\big) + s_{13}^2(s_{12} - s)^2D\\
&\quad - 4\frac{\alpha s_{13}}{C_1}A_1B_3\Big(\frac{1}{C_1}\big(R^2A_1A_3 - B_1B_3\big) - s_{13}^2 - D + R^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= s_{13}^2(s-s_{12})^2D \\
&\quad + \frac{4}{C_1}s_{13}A_1D\big(sC_1+\alpha B_3\big) \\
&\quad - \frac{4}{C_1^2}\alpha A_1\Big((s^2+s_{13}^2)C_1^2 + s_{13}B_3\big(R^2A_1A_3 - B_1B_3 - s_{13}^2C_1 + R^2C_1\big)\Big) \\
&= s_{13}^2(s-s_{12})^2D \\
&\quad + \frac{4}{C_1}s_{13}A_1D\big(ss_{13}^2R^2 + \alpha^2 s + \alpha s_{13}R^2 - \alpha^2 s\big) \\
&\quad - \frac{4}{C_1^2}A_1\Big(s_{13}R^2A_1A_3\big(\alpha s_{13}R^2 - \alpha^2 s\big) + \alpha B_3^2\big(-s_{13}^2R^2 + \alpha s_{12}s_{13}\big) \\
&\qquad + \alpha C_1\big((s^2+s_{13}^2)C_1 - s_{13}^3B_3 + s_{13}R^2B_3\big)\Big) \\
&= s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}R^2A_1A_3D \\
&\quad - \frac{4}{C_1^2}A_1\Big(s_{13}R^2A_1A_3\big((A_3-ss_{13})s_{13}R^2 - \alpha^2 s\big) + \alpha B_3^2\big(-s_{13}^2R^2 + \alpha(A_1-\alpha)\big) \\
&\qquad + \alpha C_1\big((s^2+s_{13}^2)C_1 - s_{13}^3B_3 + s_{13}R^2B_3\big)\Big) \\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}R^2A_1A_3D - s_{13}^2\frac{4}{C_1^2}R^4A_1^2A_3^2 \\
&\quad - \frac{4}{C_1^2}A_1\Big(s_{13}R^2A_1A_3\big(-ss_{13}^2R^2 - \alpha^2 s\big) + \alpha B_3^2\big(-s_{13}^2R^2 - \alpha^2\big) \\
&\qquad + \alpha C_1\big((s^2+s_{13}^2)C_1 - s_{13}^3B_3 + s_{13}R^2B_3\big)\Big) \\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3\Big) \\
&\quad - \frac{4}{C_1}A_1\Big(-ss_{13}R^2A_1A_3 - \alpha B_3^2 + \alpha(s^2+s_{13}^2)C_1 - \alpha s_{13}^3B_3 + \alpha s_{13}R^2B_3\Big) \\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3\Big) \\
&\quad - \frac{4}{C_1}A_1\Big(-ss_{12}s_{13}^2R^2A_3 + \alpha\big(-ss_{13}R^2A_3 + B_3(-B_3 - s_{13}^3 + s_{13}R^2) + (s^2+s_{13}^2)C_1\big)\Big) \\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3\Big) \\
&\quad - \frac{4}{C_1}A_1\Big(-ss_{12}s_{13}^2R^2A_3 \\
&\qquad + \alpha\big(-ss_{13}R^2A_3 + (s_{13}R^2 - \alpha s)(\alpha s - s_{13}^3) + (s^2+s_{13}^2)(s_{13}^2R^2 + \alpha^2)\big)\Big) \\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3\Big) \\
&\quad - \frac{4}{C_1}A_1\Big(-ss_{12}s_{13}^2R^2A_3
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \alpha\big(-ss_{13}R^2A_3 + \alpha s(s_{13}R^2 + s_{13}^3) + s^2s_{13}^2R^2 + \alpha^2 s_{13}^2)\Big)\\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3\Big)\\
&\quad - \frac{4}{C_1}A_1\Big(-ss_{12}s_{13}^2R^2A_3\\
&\quad + \alpha\big(-ss_{13}R^2A_3 + ss_{13}R^2(\alpha + ss_{13}) + \alpha s_{13}^2(ss_{13}+\alpha)\big)\Big)\\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3\Big)\\
&\quad - \frac{4}{C_1}A_1\Big(-ss_{12}s_{13}^2R^2A_3 + \alpha^2 s_{13}^2A_3\Big)\\
&= -\frac{4\alpha^2}{C_1^2}A_1^2B_3^2 + s_{13}^2(s-s_{12})^2D + s_{13}^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3 + ss_{12}R^2 - \alpha^2\Big).
\end{aligned}
$$

Returning to $\|f \times f_1^2\|^2$ with these reformulated terms, we see

$$
\begin{aligned}
\|f \times f_1^2\|^2 &= -4s_{13}^2k^4\frac{\alpha^2}{C_1^2}A_1^2B_3^2\\
&\quad - 2\alpha s_{13}^2k^2\cos\varphi(s_{12}-s)\frac{1}{C_1}\big(A_1B_3 + A_3B_1\big)\\
&\quad - 4k^2\frac{\alpha^2}{C_1^2}A_1^2B_3^2\\
&\quad + s_{13}^2k^2(s-s_{12})^2D + s_{13}^2k^2\frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3 + ss_{12}R^2 - \alpha^2\Big)\\
&\quad - 2\alpha\cos\varphi(s_{12}-s)\frac{1}{C_1}\big(A_1B_3 + A_3B_1\big)\\
&\quad + (s-s_{12})^2D + \frac{4}{C_1}A_1A_3\Big(R^2D - \frac{1}{C_1}R^4A_1A_3 + ss_{12}R^2 - \alpha^2\Big)\\
&= \big(1 + s_{13}^2k^2\big)\frac{1}{C_1^2}\Big(-4k^2\alpha^2A_1^2B_3^2 + 2\alpha\cos\varphi(s-s_{12})C_1\big(A_1B_3 + A_3B_1\big)\\
&\qquad + (s-s_{12})^2C_1^2D - 4A_1A_3\big(R^4A_1A_3 - (ss_{12}R^2 - \alpha^2 + R^2D)C_1\big)\Big).
\end{aligned}
$$

Again, we change the last terms back by remembering

$$
R^2A_1A_3 = -B_1B_3 + (R^2 + ss_{12})C_1,
$$

giving us

$$
\begin{aligned}
&R^4A_1A_3 - (ss_{12}R^2 - \alpha^2 + R^2D)C_1\\
&= -R^2B_1B_3 + \big(R^4 + ss_{12}R^2 - ss_{12}R^2 + \alpha^2 - R^2h^2 - R^4 - \alpha^2\big)C_1\\
&= -R^2B_1B_3 - h^2R^2C_1,
\end{aligned}
$$

and find

$$\left\|f \times f_1^2\right\|^2 = \left(1 + s_{13}^2 k^2\right)\Big(-4\frac{k^2}{C_1^2}\alpha^2 A_1^2 B_3^2 + 2\alpha\cos\varphi(s - s_{12})\frac{1}{C_1}\left(A_1B_3 + A_3B_1\right) + (s - s_{12})^2 D + \frac{4}{C_1}R^2 A_1 A_3 h^2 + \frac{4}{C_1^2}R^2 A_1A_3B_1B_3\Big).$$

Let us now collect the results found so far:

$$\begin{aligned}\langle f, f^2 \times f_1^2\rangle = k\Big(&2\alpha k^2 B_3\Big(\frac{1}{C_2}(s - s_{12})A_2 - \frac{\alpha}{C_1}(s_{13} - s_{23})A_1\Big)\\ &+ 2\alpha\cos\varphi(s - s_{12})(s_{13} - s_{23})\\ &+ h^2\left(A_1 + A_2 + A_3 + A_4\right)\\ &+ \frac{1}{C_1}\left(B_3 - B_2\right)\left(A_1B_3 + A_3B_1\right)\\ &+ \frac{1}{R^2C_2}\left(B_3 - B_1\right)\left(A_3B_2 + A_2B_3\right)\Big),\end{aligned}$$

$$\left\|f \times f^2\right\|^2 = \left(1 + s^2k^2\right)\Big(-4\frac{k^2}{C_2^2}A_2^2B_3^2 + 2\cos\varphi(s_{23} - s_{13})\frac{1}{C_2}\left(A_2B_3 + A_3B_2\right) + (s_{13} - s_{23})^2 D + 4A_2A_3\frac{h^2}{C_2} + \frac{4}{R^2C_2^2}A_2A_3B_2B_3\Big),$$

$$\left\|f \times f_1^2\right\|^2 = \left(1 + s_{13}^2 k^2\right)\Big(-4\frac{k^2}{C_1^2}\alpha^2 A_1^2 B_3^2 + 2\alpha\cos\varphi(s - s_{12})\frac{1}{C_1}\left(A_1B_3 + A_3B_1\right) + (s - s_{12})^2 D + \frac{4}{C_1}R^2 A_1 A_3 h^2 + \frac{4}{C_1^2}R^2 A_1A_3B_1B_3\Big).$$

With the identities from lemma A.0.1 and the quick calculations

$$\begin{aligned}&(s - s_{12})(s_{13} + s_{23}) - (s + s_{12})(s_{13} - s_{23})\\ &\quad= ss_{13} + ss_{23} - s_{12}s_{13} - s_{12}s_{23} - ss_{13} + ss_{23} - s_{12}s_{13} + s_{12}s_{23}\\ &\quad= 2(ss_{23} - s_{12}s_{13}),\end{aligned}$$

$$\begin{aligned}&(s - s_{12})(s_{13} + s_{23}) + (s + s_{12})(s_{13} - s_{23})\\ &\quad= ss_{13} + ss_{23} - s_{12}s_{13} - s_{12}s_{23} + ss_{13} - ss_{23} + s_{12}s_{13} - s_{12}s_{23}\\ &\quad= 2(ss_{13} - s_{12}s_{23}),\end{aligned}$$

these can be further simplified to

$$
\begin{aligned}
\langle f, f^2 \times f_1^2 \rangle = k\Big(& 2\alpha^2 k^2 (s_{12}s_{13} - ss_{23})\Big(\frac{s - s_{12}}{s + s_{12}} - \frac{s_{13} - s_{23}}{s_{13} + s_{23}}\Big) \\
& + 2\alpha \cos\varphi (s - s_{12})(s_{13} - s_{23}) \\
& + h^2\big(4\alpha + (s + s_{12})(s_{13} + s_{23})\big) \\
& + (s + s_{12})\frac{(s_{13} - s_{23})^2}{s_{13} + s_{23}} R^2 + (s_{13} + s_{23})\frac{(s_{12} - s)^2}{s + s_{12}} R^{*2}\Big) \\
= {} & \frac{k}{(s + s_{12})(s_{13} + s_{23})} \\
& \Big(2\alpha^2 k^2 (s_{12}s_{13} - ss_{23})\big((s - s_{12})(s_{13} + s_{23}) - (s + s_{12})(s_{13} - s_{23})\big) \\
& + 2\alpha \cos\varphi (s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2) \\
& + 4\alpha (s + s_{12})(s_{13} + s_{23}) h^2 + (s + s_{12})^2 (s_{13} + s_{23})^2 h^2 \\
& + (s + s_{12})^2 (s_{13} - s_{23})^2 R^2 + (s_{13} + s_{23})^2 (s_{12} - s)^2 R^{*2}\Big) \\
= {} & \frac{k}{(s + s_{12})(s_{13} + s_{23})}\Big(-4\alpha^2 k^2 (ss_{23} - s_{12}s_{13})^2 \\
& + 2\alpha \cos\varphi (s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2) \\
& + 4\alpha (s + s_{12})(s_{13} + s_{23}) h^2 + (s + s_{12})^2 (s_{13} + s_{23})^2 h^2 \\
& + (s + s_{12})^2 (s_{13} - s_{23})^2 R^2 + (s_{13} + s_{23})^2 (s_{12} - s)^2 R^{*2}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\|f \times f^2\|^2 = (1 + s^2 k^2)\Big(& -4\alpha^2 k^2 \frac{(ss_{23} - s_{12}s_{13})^2}{(s + s_{12})^2} \\
& + 2\alpha \cos\varphi (s_{13}^2 - s_{23}^2)\frac{s - s_{12}}{s + s_{12}} \\
& + (s_{13} - s_{23})^2 D + 4h^2 \frac{1}{C_2} A_2 A_3 \\
& + \frac{4R^{*2}}{(s + s_{12})^2}(ss_{23} - s_{12}s_{13})(ss_{13} - s_{12}s_{23})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1+s^2k^2}{(s+s_{12})^2}\Bigg(-4\alpha^2k^2(ss_{23}-s_{12}s_{13})^2 \\
&\quad + 2\alpha\cos\varphi(s^2-s_{12}^2)(s_{13}^2-s_{23}^2) \\
&\quad + h^2(s+s_{12})^2\big((s_{13}-s_{23})^2+\frac{4}{C_2}A_2A_3\big) \\
&\quad + R^2(s+s_{12})^2(s_{13}-s_{23})^2 \\
&\quad + R^{*2}\Big((s+s_{12})^2(s_{13}-s_{23})^2 \\
&\qquad + \big((s-s_{12})(s_{13}+s_{23})-(s+s_{12})(s_{13}-s_{23})\big) \\
&\qquad \big((s-s_{12})(s_{13}+s_{23})+(s+s_{12})(s_{13}-s_{23})\big)\Big)\Bigg) \\
&= \frac{1+s^2k^2}{(s+s_{12})^2}\Bigg(-4\alpha^2k^2(ss_{23}-s_{12}s_{13})^2 \\
&\quad + 2\alpha\cos\varphi(s^2-s_{12}^2)(s_{13}^2-s_{23}^2) \\
&\quad + h^2(s+s_{12})^2\big((s_{13}-s_{23})^2+\frac{4}{C_2}A_2A_3\big) \\
&\quad + R^2(s+s_{12})^2(s_{13}-s_{23})^2 \\
&\quad + R^{*2}(s-s_{12})^2(s_{13}+s_{23})^2\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\left\|f\times f_1^2\right\|^2 &= \big(1+s_{13}^2k^2\big)\Bigg(-4\alpha^2k^2\frac{(ss_{23}-s_{12}s_{13})^2}{(s_{13}+s_{23})^2} \\
&\quad + 2\alpha\cos\varphi(s^2-s_{12}^2)\frac{s_{13}-s_{23}}{s_{13}+s_{23}} \\
&\quad + (s-s_{12})^2D+\frac{4}{C_1}R^2A_1A_3h^2 \\
&\quad - \frac{4R^2}{(s_{13}+s_{23})^2}(ss_{23}-s_{12}s_{13})(ss_{13}-s_{12}s_{23})\Bigg) \\
&= \frac{1+s_{13}^2k^2}{(s_{13}+s_{23})^2}\Bigg(-4\alpha^2k^2(ss_{23}-s_{12}s_{13})^2 \\
&\quad + 2\alpha\cos\varphi(s^2-s_{12}^2)(s_{13}^2-s_{23}^2) \\
&\quad + h^2(s_{13}+s_{23})^2\big((s-s_{12})^2+\frac{4}{C_1}R^2A_1A_3\big) \\
&\quad + R^2\Big((s-s_{12})^2(s_{13}+s_{23})^2 \\
&\qquad - \big((s-s_{12})(s_{13}+s_{23})-(s+s_{12})(s_{13}-s_{23})\big) \\
&\qquad \big((s-s_{12})(s_{13}+s_{23})+(s+s_{12})(s_{13}-s_{23})\big)\Big) \\
&\quad + R^{*2}(s-s_{12})^2(s_{13}+s_{23})^2\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1+s_{13}^2k^2}{(s_{13}+s_{23})^2}\Big(-4\alpha^2k^2(ss_{23}-s_{12}s_{13})^2 \\
&\quad + 2\alpha\cos\varphi(s^2-s_{12}^2)(s_{13}^2-s_{23}^2) \\
&\quad + h^2(s_{13}+s_{23})^2\big((s-s_{12})^2+\frac{4}{C_1}R^2A_1A_3\big) \\
&\quad + R^2(s+s_{12})^2(s_{13}-s_{23})^2 \\
&\quad + R^{*2}(s-s_{12})^2(s_{13}+s_{23})^2\Big).
\end{aligned}
$$

Comparing these three results, we see we can introduce an abbreviation

$$
\begin{aligned}
\widetilde{Q} &:= -4\alpha^2k^2(ss_{23}-s_{12}s_{13})^2 + 2\alpha\cos\varphi(s^2-s_{12}^2)(s_{13}^2-s_{23}^2) \\
&\quad + R^2(s+s_{12})^2(s_{13}-s_{23})^2 + R^{*2}(s-s_{12})^2(s_{13}+s_{23})^2\big)
\end{aligned}
$$

and write

$$
\begin{aligned}
\langle f, f^2\times f_1^2\rangle &= \frac{k}{(s+s_{12})(s_{13}+s_{23})}\Big(\widetilde{Q}+h^2(s+s_{12})(s_{13}+s_{23})\big(4\alpha+(s+s_{12})(s_{13}+s_{23})\big)\Big), \\
\big\|f\times f^2\big\|^2 &= \frac{1+s^2k^2}{(s+s_{12})^2}\Big(\widetilde{Q}+h^2(s+s_{12})^2\big((s_{13}-s_{23})^2+\frac{4}{C_2}A_2A_3\big)\Big), \\
\big\|f\times f_1^2\big\|^2 &= \frac{1+s_{13}^2k^2}{(s_{13}+s_{23})^2}\Big(\widetilde{Q}+h^2(s_{13}+s_{23})^2\big((s-s_{12})^2+\frac{4}{C_1}R^2A_1A_3\big)\Big).
\end{aligned}
$$

Using eq. (A.1) from the proof of lemma A.0.1, we further simplify

$$
\begin{aligned}
&(s+s_{12})\big((s_{13}-s_{23})^2+\frac{4}{C_2}A_2A_3\big) \\
&\quad = (s+s_{12})\Big((s_{13}+s_{23})^2+\frac{4}{C_2}\big(A_2A_3-s_{13}s_{23}C_2\big)\Big) \\
&\quad = (s+s_{12})\Big((s_{13}+s_{23})^2+\frac{4}{C_2}\big(\alpha^2+\alpha s(s_{13}+s_{23})+s^2s_{13}s_{23}-s^2s_{13}s_{23}-s_{13}s_{23}R^2\big)\Big) \\
&\quad = (s+s_{12})(s_{13}+s_{23})^2+\frac{4}{C_2}\big((s+s_{12})(\alpha^2-s_{13}s_{23}R^2)+\alpha s(s+s_{12})(s_{13}+s_{23})\big) \\
&\quad = (s+s_{12})(s_{13}+s_{23})^2+\frac{4}{C_2}\alpha(s_{13}+s_{23})\big(R^2-ss_{12}+s(s+s_{12})\big) \\
&\quad = (s_{13}+s_{23})\big((s+s_{12})(s_{13}+s_{23})+4\alpha\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&(s_{13}+s_{23})\big((s-s_{12})^2+\frac{4}{C_1}R^2A_1A_3\big) \\
&\quad = (s_{13}+s_{23})\Big((s+s_{12})^2+\frac{4}{C_1}\big(R^2A_1A_3-ss_{12}C_1\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= (s_{13}+s_{23})\Big((s+s_{12})^2 \\
&\quad + \frac{4}{C_1}\big(\alpha^2R^2 + \alpha s_{13}(s+s_{12})R^2 + ss_{12}s_{13}^2R^2 - ss_{12}s_{13}^2R^2 - \alpha^2 ss_{12}\big)\Big) \\
&= (s+s_{12})^2(s_{13}+s_{23}) + \frac{4}{C_1}\big(\alpha^2(s_{13}+s_{23})(R^2-ss_{12}) + \alpha s_{13}(s+s_{12})(s_{13}+s_{23})R^2\big) \\
&= (s+s_{12})^2(s_{13}+s_{23}) + \frac{4}{C_1}\alpha(s+s_{12})\big(\alpha^2 - s_{13}s_{23}R^2 + s_{13}(s_{13}+s_{23})R^2\big) \\
&= (s+s_{12})\big((s+s_{12})(s_{13}+s_{23}) + 4\alpha\big);
\end{aligned}
$$

we see that

$$
\begin{aligned}
\left\|f \times f^2\right\|^2 &= \frac{1+s^2k^2}{(s+s_{12})^2}\Big(\widetilde{Q} + h^2(s+s_{12})(s_{13}+s_{23})\big(4\alpha + (s+s_{12})(s_{13}+s_{23})\big)\Big), \\
\left\|f \times f_1^2\right\|^2 &= \frac{1+s_{13}^2k^2}{(s_{13}+s_{23})^2}\Big(\widetilde{Q} + h^2(s+s_{12})(s_{13}+s_{23})\big(4\alpha + (s+s_{12})(s_{13}+s_{23})\big)\Big).
\end{aligned}
$$

and introduce

$$
Q := \widetilde{Q} + h^2(s+s_{12})(s_{13}+s_{23})\big(4\alpha + (s+s_{12})(s_{13}+s_{23})\big).
$$

Now, finally,

$$
\begin{aligned}
\left\langle f, f^2 \times f_1^2\right\rangle &= \frac{k}{(s+s_{12})(s_{13}+s_{23})}Q, \\
\left\|f \times f^2\right\|^2 &= \frac{1+s^2k^2}{(s+s_{12})^2}Q, \\
\left\|f \times f_1^2\right\|^2 &= \frac{1+s_{13}^2k^2}{(s_{13}+s_{23})^2}Q,
\end{aligned}
$$

and to prove

$$
\left\langle f, f^2 \times f_1^2\right\rangle \sqrt{(1+s^2k^2)(1+s_{13}^2k^2)} = k\left\|f \times f^2\right\|\left\|f \times f_1^2\right\|,
$$

all that is left to show is $Q \geq 0$. Clearly,

$$
h^2(s+s_{12})(s_{13}+s_{23})\big(4\alpha + (s+s_{12})(s_{13}+s_{23})\big) > 0,
$$

so we look to $\widetilde{Q}$. We have

$$\begin{aligned}
\widetilde{Q} &= -\alpha^2 k^2 \big(2(ss_{23} - s_{12}s_{13})\big)^2 + 2\alpha\cos\varphi(s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2) \\
&\quad + R^2(s+s_{12})^2(s_{13}-s_{23})^2 + R^{*2}(s-s_{12})^2(s_{13}+s_{23})^2\big) \\
&= -\alpha^2 k^2 \big((s-s_{12})(s_{13}+s_{23}) - (s+s_{12})(s_{13}-s_{23})\big)^2 \\
&\quad + 2\alpha\cos\varphi(s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2) \\
&\quad + R^2(s+s_{12})^2(s_{13}-s_{23})^2 + R^{*2}(s-s_{12})^2(s_{13}+s_{23})^2\big) \\
&= (s+s_{12})^2(s_{13}-s_{23})^2\big(R^2 - \alpha^2 k^2\big) \\
&\quad + (s-s_{12})^2(s_{13}+s_{23})^2\big(R^{*2} - \alpha^2 k^2\big) \\
&\quad + 2\alpha(s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2)\big(\cos\varphi + \alpha k^2\big) \\
&= (s+s_{12})^2(s_{13}-s_{23})^2 r^2 \\
&\quad + (s-s_{12})^2(s_{13}+s_{23})^2 r^{*2} \\
&\quad + 2(s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2) r^* r \cos\eta \\
&= r^2(s+s_{12})^2(s_{13}-s_{23})^2 \\
&\quad + r^{*2}(s-s_{12})^2(s_{13}+s_{23})^2 \\
&\quad \pm 2r^* r \left|(s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2)\cos\eta\right| \\
&\geq r^2(s+s_{12})^2(s_{13}-s_{23})^2 \\
&\quad + r^{*2}(s-s_{12})^2(s_{13}+s_{23})^2 \\
&\quad - 2r^* r \left|(s^2 - s_{12}^2)(s_{13}^2 - s_{23}^2)\right| \\
&= \big(r(s+s_{12})\,|s_{13}-s_{23}| - r^*\,|s-s_{12}|\,(s_{13}+s_{23})\big)^2 \\
&\geq 0.
\end{aligned}$$

This concludes the proof of k–foldedness at the edge $f = w_1 - w$ of the sphere–quadrilateral (T, T_1, T_2, T_{12}) in our model.

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