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Fundamental Properties of Phirotopes –

Duality, Chirotopality, Realisability, Euclideaneness

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[1] You may rest assured that further acts of appreciation and declarations of love will follow when not the whole scientific world interested in phirotopes is watching.

Contents

1. Why Phirotopes are Interesting

Oriented matroids are one of the most fundamental subject areas in discrete and combinatorial geometry. They abstract the concepts of linear dependency, relative position, and orientation in real settings. A natural mathematical advancement is to extend these structures to complex numbers. This led to the invention of “phirotopes”, that is complex oriented matroids. This thesis is dedicated to the research of phirotopes.

As the field of complex numbers is not an ordered field, complex oriented matroids are a structure that differs from real oriented matroids in some key areas. In particular, their realisability theories are significantly different. In other aspects, however, complex and real oriented matroids behave very similarly. Thus, we will encounter various definitions, concepts, and theorems that look familiar.

We assume that the reader is familiar with the theory of matroids and (real) oriented matroids, especially with chirotopes. There is plenty of very good literature on these topics that one might want to consult, if this is not the case.

The invention of *matroids* is attributed to Whitney and his work on linear dependencies [Whi35]. Since then, there has been a lot of intensive research in the field of matroids. The early work on matroid theory was outlined in 1986 by Kung [Kun86]. To familiarise oneself with matroid theory the books of Welsh [Wel76], Oxley [Oxl92], and Läuchli [Läu98] might be good starting points.

Oriented matroids were introduced by several researchers independently and almost simultaneously. They were all working on different approaches (cf. Las Vergnas [LV75], Bland [Bla77], and Folkmann and Lawrence [FL78]). The definitive book on oriented matroids, which summarises the state of the art of that time, was written by Björner et al. in 1993 [BLSWZ93]. A survey that allows for a quick overview of oriented matroids was written by Richter-Gebert and Ziegler (cf. [RGZ04]), and for a short introduction that contains all important concepts we refer the reader to the first chapter of [RG92a].

Compared to matroids and oriented matroids, the concept of complex oriented matroids is relatively young. It was introduced in 2003 by Below, Krummeck, and Richter-Gebert [BKR03] and continued by Delucchi [Del03], Anderson and Delucchi [AD12] and Baker and Bowler [BB17].

So far, only few applications of phirotopes have been discovered. However, the areas of application are substantial. Oriented matroids are already known to be important tools in some parts of modern physics, such as loop quantum gravity, gauge fields theory, and quantum mechanics (cf. [BR10], [Nie04], [Nie05], [Nie05], [NL10], [Nie11]). Additionally, phirotopes seem to have an application in the super p-branes formalism and the qubit theory (cf. [Nie14]).

There is hope that the research presented here may pave the way for exciting findings in natural sciences. Until then, we think that the theory of phirotopes is itself very rich and deserves to be researched for its own sake. Parts of this theory will be presented in the course of this thesis.

This thesis is structured as follows:

- In Chapter 2, we will introduce phirotopes. We will give the basic definitions and discuss different approaches to complex oriented matroids. We compare phirotopes to chirotopes and point out differences and similarities so as to integrate phirotopes into the knowledge of oriented matroids and point out connecting factors. Furthermore, the concept of duality is examined closely, as it provides powerful tools that will be used in subsequent chapters.
- In Chapter 3, we will deal with reorientations and chirotopality. We will extend the existing theory to phirotopes of arbitrary rank and to non-uniform phirotopes. The main result of this chapter is the characterisation of chirotopality depicted in the Theorem 3.31. While the result is a more or less straightforward generalisation of the uniform rank-2 case, the corresponding proof turned out to be unexpectedly involved.
- In Chapter 4, we extend the existing theory of realisability of phirotopes to arbitrary rank and non-uniform phirotopes. We will show that the five-point condition can be generalised to higher ranks. Furthermore, we will see that only uniform minors need to be considered in order to decide the realisability of a phirotope.
- In Chapter 5, we will use phirotopes as a new coordinate system. With these coordinates we seek to carry out Euclidean geometry. We will translate several Euclidean properties into the phirotopal language and prove some Euclidean theorems with them.
- In Chapter 6, we present our findings of incidence theorems. In oriented matroid theory, one popular tool to “bend” the lines which are subjects of incidence theorems in such a way that the configuration at hand becomes non-realisable. We will show that similar perturbations of the theorem of Pappus and Desargues in a complex setting are impossible. For phirotopes, these incidence theorems are always true.
- In Chapter 7, we will discuss some open problems and point out possible topics for future work.

In order to enhance the readability and clarity and to make reading as enjoyable as possible, coloured boxes were added to structure the text:

Definitions are highlighted by teal boxes, ...

..., further notation is highlighted by smaller and lighter teal boxes, ...

..., theorems and lemmas come in purple boxes, ...

..., conjectures are bordered in violet, ...

..., and examples are marked by yellow boxes.

2. Axioms and Duality

In this chapter we will introduce complex oriented matroids. We will summarise the status quo of the research on this topic. Particular attention will be paid to the concepts of duality and realisability, as those will be treated extensively in the next chapters.

2.1. Axiom systems

There are many cryptomorphic ways to define oriented matroids. Three axiomatisations that are often used define oriented matroids via chirotopes, circuits, or cocircuits. In [BLSWZ93] the equivalence of these different axiom systems is shown. The situation for complex oriented matroids is similar. In [BKR03] complex oriented matroids are defined as phirotopes – the analogue of chirotopes. One can also define complex oriented matroids as complex circuits. The latter were introduced in [Del03] and [BB17]. The equivalence of so called F-circuits of a strong F-matroid and phirotopes is shown in [BB17].

In this chapter, the definition of phirotopes will be examined and compared to the definition of chirotopes. This will give a first insight into the structure of complex oriented matroids and the basic paradigmatic idea behind them. We will then see different ways to generalise oriented matroids.

Phirotopes

To avoid ambiguities, we first introduce some notation:

$\mathcal{S}^k$ is the k-sphere.

$[n] := \{1, \dots, n\}$.

Omitting the element λ_i in the sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ is denoted by

$$\lambda_{\setminus i} := (\lambda_1, \dots, \widehat{\lambda_i}, \dots \lambda_n) := (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n).$$

For two sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\tau = (\tau_1, \dots, \tau_k)$, their concatenation is $\lambda\tau := (\lambda, \tau) := (\lambda_1, \dots, \lambda_n, \tau_1, \dots, \tau_k)$.

S_k is the symmetric group of degree k.

I_k is the $k \times k$ identity matrix.

The following definition originates from [BKR03] and corresponds to the definition of strong Grassmann-Plücker functions in [BB17].

Definition 2.1 (Phirotope)
Let $\mathcal{E} \subset \mathbb{N}$ be a finite index set. The mapping $\varphi : \mathcal{E}^d \to \mathcal{S}^1 \cup \{0\} \subset \mathbb{C}$ is called a *rank-d phirotope* on the *index set* $\mathcal{E}$, if

(φ0) it is non-zero, meaning:
$\varphi \not\equiv 0$,

(φ1) it is alternating, meaning:
$\varphi(\lambda_{\pi(1)}, \dots, \lambda_{\pi(d)}) = \operatorname{sign}(\pi) \cdot \varphi(\lambda)$ for all $\lambda \in \mathcal{E}^d$ and all $\pi \in S_d$, and

(φ2) it does not obviously violate the Grassmann-Plücker relations, meaning:
for all sequences $\lambda \in \mathcal{E}^{d-1}$, $\tau \in \mathcal{E}^{d+1}$ there are $r_1, \dots, r_{d+1} \in \mathbb{R}^+$ such that

$$\sum_{i}^{d+1} r_i \cdot \varphi(\lambda\tau_i) \cdot \varphi(\tau_{\setminus i}) = 0.$$

Grassmann-Plücker relations are formulae that hold true for determinants. To shorten the following formulae, for $X_1, \dots, X-d \in \mathbb{K}^d$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ set

$$[X_1, \dots, X_d] := \det(X_1, \dots, X_d)$$

We further introduce the notation

$$(P_1, \dots, P_{d-1} \,|\, Q_1, \dots, Q_{d+1}) := \sum_{i=1}^{d+1} [P_1, \dots, P_{d-1}, Q_i] \cdot [Q_1, \dots, \widehat{Q_i}, \dots, Q_{d+1}] = 0.$$

to denote the Grassmann-Plücker relation on the elements $P_1, \dots, P_{d-1}$ and $Q_1, \dots, Q_{d+1}$. With the notion that a mapping should "not obviously violate" the Grassmann-Plücker relations, we mean, loosely speaking, that the behaviour of phirotopes should not differ too much from the behaviour of determinants of a point configuration. In other words, the abstract concept of phirotopes should not differ to much from the realisable situation. A nice visualisation of the non-violation requirement is given in [BKR03] and reproduced here: We visualise every summand of the Grassmann-Plücker relation as a vector in $\mathbb{R}^2$. Then, the Grassmann-Plücker relation is not obviously violated, if the origin is contained in the *interior* of the convex hull of the three vectors, see Figure 2.1.

The definition of phirotopes is very similar to the well-known definition of chirotopes. In fact, the two definitions only differ in the range of the respective mapping:

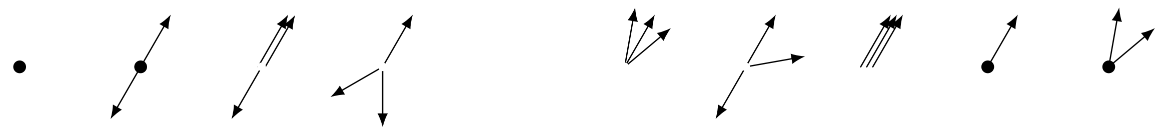


Figure 2.1.: A visualisation of configurations that (do not) violate the Grassmann-Plücker relations. The first four configurations contain the origin in the interior of their convex hulls and, thus, represent configurations that do not obviously violate the Grassmann-Plücker relations. The last five configurations do violate the Grassmann-Plücker relations.

Definition 2.2 (Chirotope)
Let $\mathcal{E} \subset \mathbb{N}$ be a finite index set. The mapping $\chi : \mathcal{E}^d \to \mathcal{S}^0 \cup \{0\} \subset \mathbb{R}$ is called a *rank-d chirotope* on the *index set* $\mathcal{E}$, if

(χ0) it is non-zero, meaning:
$\chi \not\equiv 0$,

(χ1) it is alternating, meaning:
$\chi(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(d)}) = \operatorname{sign}(\pi) \cdot \chi(\lambda)$ for all $\lambda \in \mathcal{E}^d$ and all $\pi \in S_d$, and

(χ2) it does not obviously violate the Grassmann-Plücker relations, meaning:
for all sequences $\lambda \in \mathcal{E}^{d-1}$, $\tau \in \mathcal{E}^{d+1}$ there are $r_1, \ldots, r_{d+1} \in \mathbb{R}^+$ such that

$$\sum_{i}^{d+1} r_i \cdot \chi(\lambda\tau_i) \cdot \chi(\tau_{\setminus i}) = 0.$$

In the above definitions, the domain of chirotopes and phirotopes is $\mathcal{E}^d$. Properties (φ1) and (χ1) ensure that sequences in which indices appear multiple times are mapped to zero. Furthermore, it is enough to know the values of the phirotope or chirotope on the subset $\Lambda(\mathcal{E}, d)$ that contains all ordered d-tuples:

$$\Lambda(\mathcal{E}, d) := \{(\lambda_1, \ldots, \lambda_d) \in \mathcal{E}^d \mid \lambda_1 < \cdots < \lambda_d\}.$$

With the help of properties (φ1) and (χ1) all other values can be reconstructed.

Note that every chirotope is a phirotope whose image is purely real. Likewise, every phirotope whose image is purely real is a chirotope. (We will see later that even if the image of a phirotope is not purely real, the phirotope might nevertheless be a “chirotopal phirotope”, that is a reoriented chirotope.) Thus, whenever we make statements about general phirotopes, they are also valid for chirotopes. Although their definitions look similar, the theories of chirotopes and phirotopes have some considerable differences. The most prominent of these differences lies in their realisability. With this we mean the following:

Some phirotopes stem from vector configurations. They give a good intuition of what a phirotope actually is: Much like chirotopes that originate from a (real) vector configuration encode the combinatorics of this vector configuration by assigning to each d-tuple the sign of its determinant, phirotopes coming from a complex vector configuration depict the combinatorics of this configuration. A complex analogue for the sign function is given by the phase function:

Definition 2.3 (Phase function)
The function $\omega : \mathbb{C} \to \mathcal{S}^1 \cup \{0\}$ with

$$\omega(z) = \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$$

is called *phase function*.

Definition 2.4 (Phirotope of vector configuration)
Let $V = (V_1, \dots, V_n) \in \mathbb{C}^{d \times n}$ be a configuration of finitely many vectors that linearly span $\mathbb{C}^d$. The *phirotope of* V is the map

$$\varphi_V : [n]^d \to \mathcal{S}^1 \cup \{0\}, \quad (\lambda_1, \dots, \lambda_d) \mapsto \omega\left(\det(V_{\lambda_1}, \dots, V_{\lambda_d})\right).$$

The dimension d of the ambient space is called the *rank* of the phirotope.

It has been shown in [Tro13] that phirotopes of vector configurations are phirotopes as defined in Definition 2.1: It is easy to see that ($\varphi 0$) and ($\varphi 1$) are satisfied. As every vector configuration fulfils the Grassmann-Plücker relations, ($\varphi 2$) is satisfied as well.

Definition 2.5 (Realisable phirotopes)
A rank-d phirotope on $\mathcal{E} = [n]$ is called *realisable*, if there is a vector configuration $V \in \mathbb{C}^{d \times n}$ such that

$$\varphi_V = \varphi.$$

The vector configuration V is then called a *realisation* of φ.

Compared to chirotopes that stem from real vector configurations and are discrete in nature, phirotopes contain substantially more information. For each d-tuple of points, chirotopes only "remember" the relative position of the points while phirotopes also contain continuous information about angles. This additional information is the reason for the differences in realisability that we will encounter in Chapter 4. There, the question of whether or not a phirotope is realisable will be discussed thoroughly. For now, we will focus on properties that realisable phirotopes have.

Lemma 2.6
Let a finite complex vector configuration $V \in \mathbb{C}^{d\times n}$ be given and let $V' \subset \mathbb{C}^{d\times n}$ be a vector configuration obtained from applying a projective transformation $T \in \mathbb{C}^{d\times d}$, $\det(T) \neq 0$, to V. Then

$$\varphi_{V'} = \omega(\det(T)) \cdot \varphi_V.$$

Proof. For every minor $X \in \mathbb{C}^{d\times d}$ of V we have

$$\varphi_{V'}(T \cdot X) = \omega\left(\det(T \cdot X)\right) = \omega(\det(T)) \cdot \omega(\det(X)) = \omega(\det(T)) \cdot \varphi_V(X). \qquad \square$$

Note that projective transformations with a positive real determinant do not alter the phirotope at all. Also, multiplying an element of the vector configuration with a real positive scalar has no effect on the phirotope. Because of this, we will understand the realisations of rank-d phirotopes to be vector configurations with elements in the oriented projective space $(\mathbb{C}^d \setminus \{0\})/\mathbb{R}^+$. The definition of the *real* oriented projective space can for example be found in [Sto91]. In that book, the real oriented projective space of dimension ν is denoted by $\mathbf{T}_\nu$, we will denote it by $(\mathbb{R}^d \setminus \{0\})/\mathbb{R}^+$:

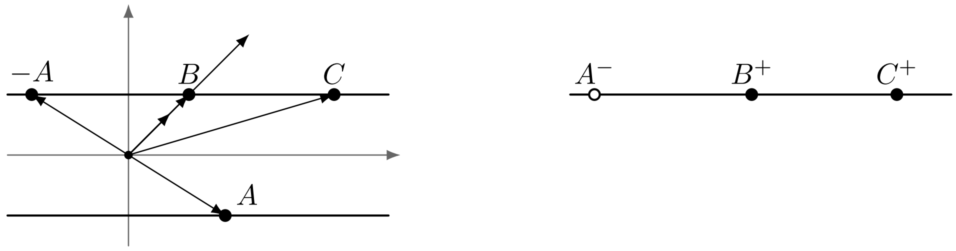
Definition 2.7 (Real oriented projective space)
The *real oriented projective space* of dimension $d-1$, which is denoted by $(\mathbb{R}^d \setminus \{0\})/\mathbb{R}^+$, consists of all non-zero vectors of $\mathbb{R}^d$, where two vectors are identified, if multiplying one of them by a real positive scalar will yield the other.

The definition of this complex oriented projective space is completely analogous to the real case:

Definition 2.8 (Complex oriented projective space)
The *complex oriented projective space* of dimension $d-1$, which is denoted by $(\mathbb{C}^d \setminus \{0\})/\mathbb{R}^+$, consists of all non-zero vectors of $\mathbb{C}^d$, where two vectors are identified, if multiplying one of them by a real positive scalar will yield the other.

Points in oriented projective space are rays emanating from the origin, which is itself not part of the rays. Multiplying by a positive scalar such that the last coordinate is an element in $\mathcal{S}^0$ (in the real case) or $\mathcal{S}^1$ (in the complex case) yields the *standard embedding* of the oriented projective space. This is particularly useful for visualising the point configurations.

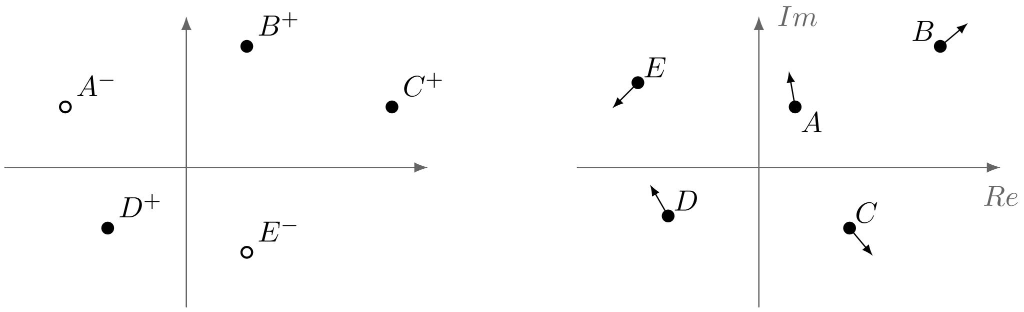
Figure 2.2 shows how the real oriented projective space of rank 2 is visualised. In every ray we choose the point whose last coordinate is 1 or -1. For point B some further representatives are shown. The one that lies on the line $y = 1$ is the representative in the standard embedding. For all points whose last coordinate is -1, like A, we draw the point $-A$ and label it as a negative point. In our visualisation, we attach to each point the element in $\mathcal{S}^0$ that is contained in its last



(a) The representatives of points with last coordinates in $\mathcal{S}^0$ ... (b) ... will yield positive and negative points, respectively.

Figure 2.2.: The real oriented projective space of dimension 1 (rank 2)

coordinate in the standard embedding: The points are labelled with + or −, respectively. For the complex case, we are going to do the same thing. Here, we attach an arrow that indicates the corresponding element in $\mathcal{S}^1$, see Figure 2.3. In order to compute the phirotope of an affine



(a) The real oriented projective space of dimension 2 (rank 3) (b) The complex oriented projective space of dimension 1 (rank 2)

Figure 2.3.: Comparison of real and complex oriented projective spaces

d-dimensional point configuration $V \in \mathbb{C}^{d\times n}$, we append a (complex) 1 to every vector in V and compute the rank-$(d+1)$ phirotope of this configuration.

The zero vector is not part of any projective space. In some cases, however, we will allow it to be part of the realisation (see Section 4.2). But firstly, we are going to focus on "uniform" phirotopes.

Definition 2.9 (Uniformity)
A phirotope is called *uniform*, if 0 is not contained in its image.

The zero vector cannot be part of any realisation of a uniform phirotope. Thus, realisable uniform phirotopes can be realised in projective space.

The notion of "affine representative" will be used in many theorems to come.

Definition 2.10 (Affine representative, phase)
Let $P \in \mathbb{C}^d$ with $P_d \neq 0$ be the homogeneous coordinates of a point. Write P as

$$P = r_P \cdot \omega_P \cdot \begin{pmatrix} | \\ p \\ | \\ 1 \end{pmatrix}, \quad \omega_P \in \mathcal{S}^1,\ r_P \in \mathbb{R}^+,\ p \in \mathbb{C}^{d-1}.$$

We call ω_P the *phase* of P, the value r_P its *radius*, and p its *affine representative.*
If $P_d = 0$, the *phase* of P is the phase $\omega_P := \omega(P_k)$ of the last non-zero entry P_k, $k < d$, of P.

When working over $(\mathbb{C}^d \setminus \{0\})/\mathbb{R}^+$ points are represented by equivalence classes of vectors that have the same affine representative and the same phase. Their radii may vary. Strict usage of notation would require to distinguish points of $(\mathbb{C}^d \setminus \{0\})/\mathbb{R}^+$ from vectors of $\mathbb{C}^d$ and to denote, for example, the latter by capital letters ($P \in \mathbb{C}^d$) and the former by equivalence classes ($[P] \in (\mathbb{C}^d \setminus \{0\})/\mathbb{R}^+$). For the sake of simplicity, we will denote both of them by P, whenever no confusion can arise. The following Lemma 2.11, which will be used extensively, is taken from [BKR03] where it is introduced as Lemma 2.3.

Lemma 2.11 (Freedom of choice of $d+1$ points)
Let φ be a uniform realisable rank-d phirotope on $[n]$ with $n > d$. For any choice of affine representatives $p_1, \dots, p_{d+1} \in \mathbb{C}^{d-1}$ in general position there is a realisation $V = (P_1, \dots, P_n)$ of φ such that

$$P_k = r_{P_k} \omega_{P_k} \begin{pmatrix} p_k \\ 1 \end{pmatrix}$$

where $r_{P_k} \in \mathbb{R}^+$ and $\omega_{P_k} \in \mathcal{S}^1$ for all $k \in [d+1]$.

This means that we can choose the position of $d+1$ points in a realisation of a phirotope. The phases of the corresponding points will then be fixed.

Altering the phases of points of a vector configuration will change the phirotope of this configuration. However, the new mapping obtained in this way will again be a realisable phirotope. This modification of phases is called "reorientation":

Definition 2.12 (Reorientation of a phirotope)
Let φ be a rank-d phirotope on $\mathcal{E} = [n]$ and $\varrho \in (\mathcal{S}^1)^n$ be a vector of phases. The map

$$\varphi^\varrho : \mathcal{E}^d \to \mathcal{S}^1 \cup \{0\}, \quad (\lambda_1, \dots, \lambda_d) \mapsto \varrho_{\lambda_1} \cdot \dots \cdot \varrho_{\lambda_d} \cdot \varphi(\lambda_1, \dots, \lambda_d)$$

is called a *reorientation* of φ with the *reorientation vector* ϱ.

There are two very important lemmas regarding reorientations.

Lemma 2.13 (Reorientations are phirotopes)
Any reorientation of a given phirotope is again a phirotope.

Lemma 2.14 (Reorientation preserves realisability)
A phirotope is realisable if and only if any reorientation of this phirotope is realisable.

The proof of the Lemma 2.14 can be found in [BKR03]. The proof of the Lemma 2.13 works analogously to the proof of the same statement for chirotopes. The latter can, for example, be found in [BLSWZ93].

When Below, Krummeck and Richter-Gebert gave their definition of complex oriented matroids, one of their main goals was to define them in such a way that these complex oriented matroids would support a reorientation theory that is similar to the one of real oriented matroids. This means that reorientations of phirotopes should be phirotopes and preserve realisability, so the Lemmas 2.13 and 2.14 should hold true. Furthermore, the realisations of oriented matroids lie in oriented projective spaces: multiplying the coordinates of one point with a real positive scalar does not change the sign or phase of the determinant. Thus, reorientations – that means multiplying the coordinates of a point of the realisation with any scalar – are the next natural generalisation.

Ziegler came up with a definition for complex oriented matroids that is different from the one presented here and that does not allow reorientations (cf. [Zie93]). Ziegler's complex oriented matroids have a discrete range: They can take values in the set $\{0, -1, +1, i, j\}$, depending on the sign of the real and imaginary part of the determinant. We will see later that realisations of phirotopes as introduced in [BKR03] are rigid. This is due to the fact that in our definition the range of phirotopes is $\mathcal{S}^1$. Thus, we have substantially more information – and substantially less freedom in choosing points in a realisation. As (real) oriented matroids also have a discrete range and (in general) non-rigid realisations, one could argue that this discreteness is a desirable property. It is still an open problem whether there is a way of defining complex oriented matroids in such a way that their image is discrete and that they allow reorientations nevertheless.

Generalisations of complex matroids

Defining them over the complex numbers is not the only way to generalise oriented matroids. Phirotopes can also be understood as "matroids with coefficients" in the sense of Dress (cf. [Dre86]), and Dress and Wenzel (cf. [DW88], [DW89], [DW91], and [DW92]). A different generalisation was given recently by Baker and Bowler, who showed that phirotopes are strong matroids over the phased hyperfield (cf. [BB17]). While the latter contains proofs that show the correspondence of matroids over hyperfields and phirotopes, the work of Dress and Wenzel lacks such proofs, not least because their work preceded the invention of phirotopes. Therefore, proofs are given here.

Firstly, we will show that phirotopes are matroids with coefficients in the fuzzy ring $\mathbb{C}//\mathbb{R}^+$. To this end, the definitions of fuzzy rings, the spaces $\mathbb{C}/\mathbb{R}^+$ and $\mathbb{C}//\mathbb{R}^+$, and matroids with coefficients are given here. They are taken from [Dre86] and [DW91].

Definition 2.15 (Fuzzy ring)
A *fuzzy ring* $\mathbb{K} = (K; +, \cdot; \varepsilon, K_0)$ consists of a set K together with two compositions $+ : K \times K \to K;\ (\kappa, \lambda) \mapsto \kappa + \lambda$ and $\cdot : K \times K \to K,\ (\kappa, \lambda) \mapsto \kappa \cdot \lambda$, a specified element $\varepsilon \in K$ and a specified subset $K_0 \subseteq K$ such that the following holds true:

(FR0) $(K, +)$ and $(K, \cdot)$ are Abelian subgroups with neutral elements 0 and 1, respectively,

(FR1) $0 \cdot \kappa = 0$ for all $\kappa \in K$,

(FR2) $\alpha \cdot (\kappa_1 + \kappa_2) = \alpha \cdot \kappa_1 + \alpha \cdot \kappa_2$ for all $\kappa_1, \kappa_2 \in K$ and $\alpha \in \dot{K} := \{\alpha \in K \mid 1 \in \alpha \cdot K\}$,

(FR3) $\varepsilon^2 = 1$,

(FR4) $K_0 + K_0 \subseteq K_0,\ K \cdot K_0 \subseteq K_0,\ 0 \in K_0,\ 1 \notin K_0$,

(FR5) for $\alpha \in \dot{K}$ one has $1 + \alpha \in K_0$ if and only if $\alpha = \varepsilon$,

(FR6) $\kappa_1, \kappa_2, \lambda_1, \lambda_2 \in K$ and $\kappa_1 + \lambda_1,\ \kappa_2 + \lambda_2 \in K_0$ implies
$\kappa_1 \cdot \kappa_2 + \varepsilon \cdot \lambda_1 \cdot \lambda_2 \in K_0$,

(FR7) $\kappa, \lambda, \kappa_1, \kappa_2, \in K$ and $\kappa + \lambda(\kappa_1 + \kappa_2) \in K_0$ implies $\kappa + \lambda\kappa_1 + \lambda\kappa_2 \in K_0$.

The set K_0 can be understood as the "fuzziness" of K or, more casually speaking, some sort of "not knowing". Intuitively, K_0 could for example contain those elements of which we do not know whether they are positive or negative. Multiplying such elements with any element in K again yields an element whose sign we do not know (compare (FR4)).

Definition 2.16 (The quotient space $\mathbb{C}/\mathbb{R}^+$)
Let $\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$ be the set of all subsets of $\mathbb{C}$ that are invariant under (component-by-component) multiplication with $\mathbb{R}^+$:

$$\mathscr{P}(\mathbb{C})^{\mathbb{R}^+} := \{T \subseteq \mathbb{C} \mid \mathbb{R}^+ \cdot T = T\}$$

Next, let the two binary operations $\oplus$ and $\odot$ on this set be given by

$$\oplus : \left(\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}\right)^2 \to \left(\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}\right),\quad (T_1, T_2) \mapsto \{t_1 + t_2 \mid t_i \in T_i\},$$
$$\odot : \left(\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}\right)^2 \to \left(\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}\right),\quad (T_1, T_2) \mapsto \{t_1 \cdot t_2 \mid t_i \in T_i\}.$$

The neutral elements with respect to these operations are

$$0_\oplus = \{0\} \qquad \text{and} \qquad 1_\odot = \mathbb{R}^+.$$

Furthermore, the set $\mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+}$ shall be defined as

$$\mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+} := \left\{T \in \mathscr{P}(\mathbb{C})^{\mathbb{R}^+} \mid 0 \in T\right\} \subseteq \mathscr{P}(\mathbb{C})^{\mathbb{R}^+}.$$

Then, $\mathbb{C}/\mathbb{R}^+$ is the quotient structure

$$\mathbb{C}/\mathbb{R}^+ := \left(\mathscr{P}(\mathbb{C})^{\mathbb{R}^+} ; \oplus, \odot ; \mathbb{R}^- , \mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+}\right).$$

Note that in the definition of $\mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+}$, the element "0", that is part of every subset of $\mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+}$, is the neutral element of $(\mathbb{C}, +)$. We will use the notation $0_\oplus$ and $1_\odot$ whenever we refer to the neutral elements of $\left(\mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+}, \oplus\right)$ and $\left(\mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+}, \odot\right)$, respectively, and use 0 and 1 for $0 \in \mathbb{C}$ and $1 \in \mathbb{C}$, respectively. If one visualises $\mathbb{C}$ – as is customary – as a modified Cartesian plane where the real part of a complex number is represented by a displacement along the x-axis and the imaginary part along the y-axis, then the set $\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$ consists of all unions of convex cones of this Cartesian plane. This especially includes all rays in the Cartesian plane that initiate at the origin. Here, it does not matter whether or not the origin is included in the cones, as both the cones with and those without the origin are part of $\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$. In Figure 2.4, the highlighted area is an example of an element of $\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$.

For $\mathbb{C}/\mathbb{R}^+$ to be well-defined we need to ensure that $\oplus$ and $\odot$ really map to $\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$. Let $T_1, T_2 \in \mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$ be two $\mathbb{R}^+$-invariant subsets. Then $(T_1 \oplus T_2)$ is also $\mathbb{R}^+$-invariant:

$$\begin{aligned}
\mathbb{R}^+ \cdot (T_1 \oplus T_2) &= \mathbb{R}^+ \cdot \{t_1 + t_2 \mid t_i \in T_i\} \\
&= \{\mathbb{R}^+ \cdot t_1 + \mathbb{R}^+ \cdot t_2 \mid t_i \in T_i\} \\
&= (\mathbb{R}^+ \cdot T_1) \oplus (\mathbb{R}^+ \cdot T_2) \\
&= T_1 \oplus T_2.
\end{aligned}$$

The same holds true for $(T_1 \odot T_2)$:

$$\begin{aligned}
\mathbb{R}^+ \cdot (T_1 \odot T_2) &= \mathbb{R}^+ \cdot \{t_1 \cdot t_2 \mid t_i \in T_i\} \\
&= \{\mathbb{R}^+ \cdot t_1 \cdot t_2 \mid t_i \in T_i\} \\
&= (\mathbb{R}^+ \cdot T_1) \odot T_2 \\
&= T_1 \odot T_2.
\end{aligned}$$

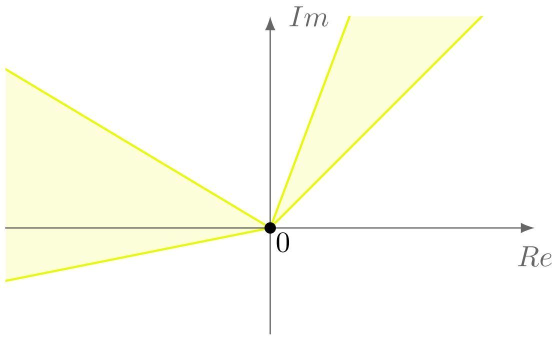


Figure 2.4.: An example of an element of $\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$.

Definition 2.17 (The space $\mathbb{C}//\mathbb{R}^+$)
Let $\mathbb{C}/\mathbb{R}^+$ be given as in Definition 2.16 and let $\mathcal{L}$ be the smallest subset of $\mathscr{P}(\mathbb{C})^{\mathbb{R}^+}$ containing $c \cdot \mathbb{R}^+$ for all $c \in \mathbb{C}$ such that $\mathcal{L} \oplus \mathcal{L} \subseteq \mathcal{L}$ and $\mathcal{L} \odot \mathcal{L} \subseteq \mathcal{L}$. With $\mathcal{L}_0 := \mathcal{L} \cap \mathscr{P}(\mathbb{C})_0^{\mathbb{R}^+}$, we define

$$\mathbb{C}//\mathbb{R}^+ := (\mathcal{L}\,; \oplus, \odot\,; \mathbb{R}^-\,, \mathcal{L}_0)\,.$$

Note that the set $\mathcal{L}$ contains the set $\{0\}$, all rays initiating at and including the origin, and all rays initiating at but not including the origin. Furthermore, $\mathcal{L}_0 = \{T \in \mathcal{L} \mid 0 \in T\}$, and thus $\mathcal{L}_0$ contains the set $\{0\}$ and all rays initiating at and including the origin:

$$\mathcal{L}_0 = \Big\{\{\alpha \cdot c \mid \alpha \in \mathbb{R}_0^+\} \mid c \in \mathbb{C}\Big\}\,.$$

Lemma 2.18
The space $\mathbb{C}//\mathbb{R}^+$ is a fuzzy ring.

Proof. We verify the claim by considering the Definition 2.15 of fuzzy rings.

- $(\mathcal{L}, \oplus)$ and $(\mathcal{L}, \odot)$ are Abelian semigroups, as the associativity and the commutativity are inherited from $(\mathbb{C}, +)$ and $(\mathbb{C}, \cdot)$, respectively. They are furthermore closed under $\oplus$ and $\odot$ by definition. $\Rightarrow$ (FR0)
- It holds true that $0_\oplus \odot T = \{0 \cdot t \mid t \in T\} = 0_\oplus$ for all $T \in \mathcal{L}$. $\Rightarrow$ (FR1)
- Consider the set

  $$\dot{\mathcal{L}} := \{T \in \mathcal{L} \mid 1_\odot \in (T \odot \mathcal{L})\} = \Big\{T \in \mathcal{L} \mid \mathbb{R}^+ \in (T \odot \mathcal{L})\Big\} = \mathcal{L} \setminus \{\{0\}\}.$$

  Note that $\odot$ is to be understood component wise here, that means:

  $$T \odot \mathcal{L} = \{T \odot L \mid L \in \mathcal{L}\}.$$

For every $R \in \dot{\mathcal{L}}$ and all $T_1, T_2 \in \mathcal{L}$ it holds true that

$$\begin{aligned} R \odot (T_1 \oplus T_2) &= R \odot \{t_1 + t_2 \mid t_i \in T_i\} \\ &= \{r \cdot (t_1 + t_2) \mid r \in R,\, t_i \in T_i\} \\ &= \{r \cdot t_1 + r \cdot t_2) \mid r \in R,\, t_i \in T_i\} \\ &= R \odot T_1 \oplus R \odot T_2. \end{aligned}$$

This proves (FR2).

- It holds true that $(\mathbb{R}^-)^2 = \mathbb{R}^- \odot \mathbb{R}^- = \mathbb{R}^+ = 1_\odot$. $\Rightarrow$ (FR3)
- For any $K_1, K_2 \in \mathcal{L}_0$ it holds true that $0 \in K_1$ and $0 \in K_2$ and, thus, $0 \in K_1 \oplus K_2$. This yields $\mathcal{L}_0 \oplus \mathcal{L}_0 \subseteq \mathcal{L}_0$.
  For any $L \in \mathcal{L}$ and $K \in \mathcal{L}_0$ it holds true that $0 \in (L \odot K)$ as $0 \in K$. This yields $\mathcal{L} \odot \mathcal{L}_0 \subseteq \mathcal{L}_0$.
  Furthermore, it holds true that $0_\oplus \in \mathcal{L}_0$, and $1_\odot \notin \mathcal{L}_0$ as $0 \notin \mathbb{R}^+$.
  With this we obtain (FR4).
- Certainly, $0 \in (\mathbb{R}^+ \oplus \mathbb{R}^-)$. Conversely, for $0 \in (\mathbb{R}^+ \oplus L)$ to hold true for some $L \in \dot{\mathcal{L}}$, this L has to contain the additive inverse to some element of $\mathbb{R}^+$. Thus, $L = \mathbb{R}^-$. $\Rightarrow$ (FR5)
- Let $L_1, L_2, N_1, N_2 \in \mathcal{L}$ be given with $L_1 \oplus N_1, L_2 \oplus N_2 \in \mathcal{L}_0$. Then there are $\ell_1 \in L_1$, $\ell_2 \in L_2$, $n_1 \in N_1$, and $n_2 \in N_2$ such that

  $$\ell_1 + n_1 = 0, \qquad \text{and} \qquad \ell_2 + n_2 = 0.$$

  Then, $\ell_1 \cdot \ell_2 - n_1 \cdot n_2 = 0$ and, thus, $(L_1 \odot L_2 \oplus \mathbb{R}^- \odot N_1 \odot N_2) \in \mathcal{L}_0$.
  This yields (FR6).
- To see that (FR7) holds true, consider $L, N, L_1, L_2 \in \mathcal{L}$ and $\ell \in L$, $\ell_1 \in L_1$, $\ell_2 \in L_2$, and $n \in N$ such that $0 = \ell + n \cdot (\ell_1 + \ell_2)$. This implies

  $$0 = \ell + n \cdot \ell_1 + n \cdot \ell_2$$

  and, thus, $L \oplus N \odot (L_1 \oplus L_2) \in \mathcal{L}_0$ implies $L \oplus N \odot L_1 \oplus N \odot L_2 \in \mathcal{L}_0$. This yields (FR7).

□

We will show that phirotopes are "Grassmann-Plücker maps" as defined in [DW91, Definition 4.1] and use [DW91, Theorem 4.1], which states that Grassmann-Plücker maps define matroids with coefficients, to show that phirotopes are indeed matroids with coefficients. For the sake of convenience, we give the above-mentioned definition and theorem here (note that we have changed some of the notation so that it is in line with the notation we have used so far):

Definition 2.19 (Grassmann-Plücker map of degree m)
Assume E is a set and K is a fuzzy ring. For $m \in \mathbb{N}$ a map $b : E^m \to \dot{K} \cup \{0\}$ is called a *Grassmann-Plücker map of degree* m, if the following conditions are satisfied:

(GP0) There exist $e_1, \ldots, e_m \in E$ with $b(e_1., \ldots e_m) \neq 0$.

(GP1) b is ϵ-alternating; this means, for $e_1, \ldots, e_m \in E$ and every odd permutation $\tau \in S_m$ we have:

$$b(e_{\tau(1)}, \ldots, e_{\tau(m)}) = \epsilon \cdot b(e_1, \ldots, e_m)$$

and in case $|\{e_1 \ldots, e_m\}| < m$ we have $b(e_1, \ldots, e_m) = 0$.

(GP2) For all $e_0, \ldots, e_m, f_2, f_m \in E$ we have

$$\sum_{i=0}^{m} \epsilon^i \cdot b(e_0, \ldots, \widehat{e_i}, \ldots, e_m) \cdot b(e_i, f_2, \ldots, f_m) \in K_0. \tag{2.1}$$

The relations (2.1) are called the Grassmann-Plücker relations.

By setting $E = \mathcal{E}$, $K = \mathbb{C}//\mathbb{R}^+$, $b = \varphi$, we see that (GP0) corresponds to (φ0), (GP1) to (φ1), and (GP2) to (φ2). Having a set of $\mathbb{R}^+$-invariant subsets as the range of the mappings accounts for the fact that in phirotopes we only work over $\mathcal{S}^1 \cup \{0\}$, that means with the phase of the determinants, or, in other words, we choose the representatives of length 1 for every element in $\mathbb{C}//\mathbb{R}^+$.

In order to state [DW91, Theorem 4.1], we also need to give [DW91, Definition 4.2]. This, in essence, contains the translation of phirotopes to circuits $\mathfrak{R}_b$ and cocircuits $\mathfrak{R}_{b^*}$:

Definition 2.20
For a Grassmann-Plücker map $b : E^m \to \dot{K} \cup \{0\}$ we define

$$\mathfrak{R}_b := \Bigg\{ r \in K^E \mid r \not\equiv 0 \text{ and there exist pairwise distinct } e_0 \ldots, e_m \in E$$
$$\text{and some } \alpha \in \dot{K} \text{ such that}$$
$$r(x) = \begin{cases} 0, & \text{for } x \notin \{e_0, \ldots, e_m\} \\ \alpha \cdot \epsilon^i \cdot b(e_0, \ldots, \widehat{e_i}, \ldots, e_m), & \text{for } x = e_i \end{cases} \Bigg\}$$

and

$$\mathfrak{R}_{b^*} := \{ s \in K^E \mid s \not\equiv 0 \text{ and there exist } f_2, \ldots, f_m \in E \text{ and some } \alpha \in \dot{K}$$
$$\text{such that } s(x) = \alpha \cdot b(x, f_2, \ldots, f_m) \text{ for all } x \in E \}.$$

Now, we give an adapted version of [DW91, Theorem 4.1]:

Lemma 2.21
Assume $b : E^m \to \dot{K} \cup \{0\}$ is a Grassmann-Plücker map. Then $(E, \mathfrak{R}_b)$ is a matroid (of finite type) with coefficients in the fuzzy ring K.

With this knowledge at hand, we can use the results of Dress, and Dress and Wenzel [Dre86], [DW88], [DW91]. Primarily, we will use their results regarding duality.

Another generalisation of complex matroids is given by Baker and Bowler in the form of weak and strong matroids over hyperfields [BB17]. A hyperfield is a field where the addition might be multivalued. The following definition is taken from [BB17].

Definition 2.22 (Phased hyperfield)
The *phased hyperfield* $\mathbb{P}$ is given by the set $\mathcal{S}^1 \cup \{0\}$ together with the usual multiplication of $\mathbb{C}$ and the hypersum, which for $x, y \neq 0$ is given by

$$x \boxplus y := \begin{cases} \{-x, 0, x\}, & \text{if } x = -y, \\ \left\{ \dfrac{\alpha x + \beta y}{|\alpha x + \beta y|} \;\middle|\; \alpha, \beta \in \mathbb{R}^+ \right\}, & \text{else.} \end{cases}$$

Note that the multivalued addition corresponds to the fuzziness of the fuzzy rings. For the detailed definitions of $\mathbb{P}$-circuits, strong and weak $\mathbb{P}$-matroids and proofs of the corresponding equivalences, we refer the interested reader to [BB17]. The results there generalise the work of Anderson and Delucchi [AD12] on phirotopes but also rectify a fault that was made in [AD12] and copied in the previous version of the paper of Baker ([Bak16]).

In [AD12] the authors try to prove that phirotopes are equivalent to phased circuits in the sense of [Del03]. However, in [BB17] it is shown that their proof is not correct and that two notions of Grassmann-Pücker functions as well as two notions of phased circuits are needed to properly show the equivalence of the definitions:
Strong Grassmann-Plücker functions are phirotope as we defined them. Baker and Bowler prove that these are equivalent to strong $\mathbb{P}$-matroids in terms of strong $\mathbb{P}$-circuits, see [BB17].

Weak Grassmann-Plücker functions are maps that also satisfy the phirotope axioms ($\varphi 0$) and ($\varphi 1$) but satisfy only a weaker version of ($\varphi 2$): Only the *three-term* Grassmann-Plücker relations are not obviously violated by a weak Grassmann-Plücker function:

($\varphi 2$)': For all sequences $\lambda \in \mathcal{E}^{d-1}$, $\tau \in \mathcal{E}^{d+1}$ such that $|\tau \setminus \lambda| = 3$ there are $r_1, \ldots, r_{d+1} \in \mathbb{R}^+$ such that

$$\sum_{i}^{d+1} r_i \cdot \varphi(\lambda \tau_i) \cdot \varphi(\tau_{\setminus i}) = 0.$$

Weak Grassmann-Plücker functions are equivalent to weak $\mathbb{P}$-matroids in terms of weak $\mathbb{P}$-circuits, see [BB17].

In [BB17], there is also an example that shows that while every strong Grassmann-Plücker function is also a weak Grassmann-Plücker function, the reverse it not true.

Both the work of Anderson and Delucchi [AD12] and of Baker and Bowler [BB17] contain useful theorems that we will make use of. For example those that concern duality.

2.2. Duality

In this section we will discuss the perception of duality. Firstly, the duality of chirotopes will be reviewed shortly as the construction of dual phirotopes works similarly. We will then construct a realisation of the dual phirotope starting from a configuration of the primal.

Dual chirotopes

To construct dual chirotopes, we follow the instructions in [BLSWZ93, p. 135]. We use their notation but extend their concept to handling the case of an index appearing repeatedly in the input.

Definition 2.23 (Dual chirotope)
Given a rank-d chirotope χ on $\mathcal{E} = [n]$. Its *dual* is the rank-$(n-d)$ chirotope χ^* on $\mathcal{E}$ given by

$$\begin{aligned} \chi^* : \mathcal{E}^{n-d} &\longrightarrow \mathcal{S}^0 \cup \{0\} \\ (x_1, \ldots, x_{n-d}) &\longmapsto \begin{cases} 0, \text{ if } |\{x_1, \ldots, x_{n-d}\}| < n-d, \\ \chi(x'_1, \ldots, x'_d) \cdot \operatorname{sign}(x_1, \ldots, x_{n-d}, x'_1, \ldots, x'_d), \text{ else,} \end{cases} \end{aligned}$$

where $(x'_1, \ldots, x'_d)$ is a permutation of the elements in $\mathcal{E} \setminus \{x_1, \ldots, x_{n-d}\}$.

It is noted in [BLSWZ93] that the choice of the permutation does not affect the outcome of the construction.

Example 2.24 The dual of a rank 3 chirotope χ on $\mathcal{E} = \{1, 2, 3, 4, 5\}$ is given by

$$\begin{aligned} \chi^* : \mathcal{E}^2 &\longrightarrow \{-1, 0, +1\} \\ \chi^*(12) &= \chi(345) \cdot \operatorname{sign}(12345) = +\chi(345), \\ \chi^*(13) &= \chi(245) \cdot \operatorname{sign}(13245) = -\chi(245) \\ &\ldots \\ \chi^*(45) &= \chi(123) \cdot \operatorname{sign}(45123) = +\chi(123). \end{aligned}$$

As dualising changes the rank of a chirotope but leaves other properties unchanged, it is often used in proofs to reduce the rank of the chirotope at hand. An example of a property that is left unchanged is the realisability. The following result can for example be found in [Ric89].

Lemma 2.25 (Dualising preserves realisability)
A chirotope is realisable if and only if its dual is realisable.

The goal is to achieve a similarly strong notion of duality for phirotopes.

Dual phirotopes

Starting from a phirotope φ, we construct a new map φ^* similar to how we constructed the dual chirotope. We will then check that this map φ^* is indeed a phirotope and exhibits properties that we expect a dual to have. Furthermore, we will examine a construction of the vector configuration of the dual phirotope that is known from chirotope theory and we will show that it also works for phirotopes.

Definition 2.26 (Dual phirotope)
Given a rank-d phirotope φ on $\mathcal{E} = [n]$. Its *dual* is the rank-$(n-d)$ phirotope φ^* given by

$$\begin{aligned} \varphi^* : \mathcal{E}^{n-d} &\longrightarrow \mathcal{S}^1 \cup \{0\} \\ (x_1, \ldots, x_{n-d}) &\longmapsto \begin{cases} 0, & \text{if } |\{x_1, \ldots, x_{n-d}\}| < n-d, \\ \varphi(x'_1, \ldots, x'_d) \cdot \operatorname{sign}(x_1, \ldots, x_{n-d}, x'_1, \ldots, x'_d), & \text{else,} \end{cases} \end{aligned}$$

where $(x'_1, \ldots, x'_d)$ is a permutation of the elements in $\mathcal{E} \setminus \{x_1, \ldots, x_{n-d}\}$.

This definition differs from that given in [AD12] in the sense that the authors there use $\varphi(x'_1, \ldots, x'_d)^{-1}$ instead of $\varphi(x'_1, \ldots, x'_d)$ in the above statement. Our definition not only follows [BB17] and [DW91] but also builds on a different – and in our opinion more natural – understanding of “orthogonality”. The dual complex oriented matroid should (as the dual real oriented matroid does as well) contain those circuits that are perpendicular to the circuits of the primal phirotope. Anderson and Delucchi use $\varphi(\cdot)^{-1}$ to be able to use the hermitian product of two complex vectors to test for orthogonality. In our understanding, orthogonality is a concept that should stem from Grassmann-Plücker relations, as those are already valid (or in our case not violated) on an abstract combinatorial level. Thus, we can speak about orthogonality without referring to (maybe non-existent) realisations. We illustrate our understanding of orthogonality with an example:

Consider the real rank-3 point configuration given in Figure 2.5. It can be shown that

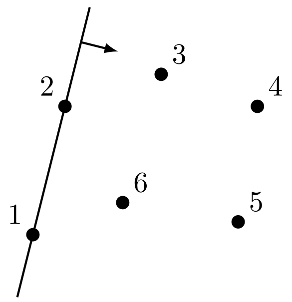


Figure 2.5.: An example of a real rank-3 point configuration that contains a circuit and a cocircuit that both contain the elements 3, 4, 5 and 6 in their support.

$$[4,5,6]\cdot 3-[3,5,6]\cdot 4+[3,4,6]\cdot 5-[3,4,5]\cdot 6=0,$$

(for details see [RG11]) which gives rise to the circuit

$$C=(0,0,+[4,5,6],-[3,5,6],+[3,4,6],-[3,4,5]).$$

The line spanned by 1 and 2 gives rise to the cocircuit

$$D=(0,0,+[1,2,3],+[1,2,4],+[1,2,5],+[1,2,6]).$$

Cocircuits of the primal chirotope are circuits of the dual chirotope, so C and D should be orthogonal, and we see that their product is a Grassmann-Plücker relation. With this, the Grassmann-Plücker relation ensuring the orthogonality of the above circuit and cocircuit is $(1,2\,|\,3,4,5,6)=0$. This, of course, holds true for real valued C and D as well as for complex valued circuits and cocircuits.

This relation between orthogonality and Grassmann-Plücker relations is the reason we do not want to introduce complex conjugation to dualisation. Thus, we define orthogonality as follows:

Definition 2.27 (Orthogonality)
Two vectors $C,D\in\mathbb{C}^d$ are called *orthogonal*, if

$$\langle C,D\rangle:=\sum_{i=1}^{d}(C_i\cdot D_i)=0.$$

Although we use a different notion of orthogonality and, thus, dual phirotopes, the proofs from [AD12] apply (apart from the complex conjugation even literally) to our framework. In most cases, dualisation is used twice in the proofs and thus complex conjugation is applied twice as well and cancels.

Lemma 2.28
The mapping φ^* defined in Definition 2.26 is a phirotope.

The proof of this lemma is contained in the proof of the Lemma 6.2 in [BB17].

There are further properties that a dualisation should satisfy. One of them is given by the following lemma:

Lemma 2.29
Let φ be a rank d phirotope on $[n]$. Up to a global multiplication with (-1), dualising twice will yield the original phirotope:

$$(\varphi^*)^* = (-1)^{(n-d)d} \cdot \varphi.$$

Proof. The only thing that is not obvious from the Definition 2.26 of a dual phirotope is the correctness of the sign. With the notation of the Definition 2.26 and the abbreviation $t = n - d$, we obtain for any $(x_1, \ldots, x_d) \in [n]^d$:

$$\begin{aligned}(\varphi^*)^*(x_1, \ldots, x_d) &= \varphi^*(x'_1, \ldots, x'_t) \cdot \operatorname{sign}(x_1, \ldots, x_d, x'_1, \ldots, x'_t)\\ &= \varphi^*(x'_1, \ldots, x'_t) \cdot (-1)^{td} \cdot \operatorname{sign}(x'_1, \ldots, x'_t, x_1, \ldots, x_d)\\ &= \varphi(x_1, \ldots, x_d) \cdot (-1)^{td} \cdot \operatorname{sign}(x'_1, \ldots, x'_t, x_1, \ldots, x_d)^2\\ &= (-1)^{(n-d)d} \cdot \varphi(x_1, \ldots, x_d)\end{aligned}$$

□

As with chirotopes, the dual realisation of a phirotope can be constructed from the primal configuration. The analogue of the following lemma exists for oriented matroids as well. Its proof in the complex setting can be found in [AD12].

Lemma 2.30
If a phirotope φ is realised by a vector configuration that spans the space W, then the dual phirotope φ^* is realised by a vector configuration that spans the space $W^\perp$.

For the formulation of the next lemma we need the notion of basis of a phirotope.

Definition 2.31 (Basis of a phirotope)
Let φ be a rank-d phirotope. Any set $\{\lambda_1, \ldots, \lambda_d\} \subseteq \mathcal{E}$ such that $\varphi(\lambda_1, \ldots, \lambda_d) \neq 0$ is called *basis* of φ.

The next lemma is an extension of the Theorem 2.2.8 of [Oxl92], where an analogue claim for matroids is proved. It gives specific instructions on how to calculate the realisation of a dual phirotope.

Lemma 2.32 (Construction of dual realisable phirotopes)
Let $\varphi : \mathcal{E}^d \to \mathcal{S}^1 \cup \{0\}$ be a realisable phirotope on $\mathcal{E} = [n]$ such that $[d]$ is a basis of φ. Let a realisation of φ be given by the columns of the matrix $[I_d|D]$. A vector configuration of the dual phirotope φ^* is then given by $[-D^T|I_{n-d}]$.

As the proof of this lemma is completely analogous to the real setting and the corresponding statement for (oriented) matroids, we refer the interested reader to [Oxl92] (for matroids) or [Ric89] (for oriented matroids).

The realisations of most phirotopes cannot be converted to the form $[I_d|D]$. Because of the Lemma 2.11, we can choose the affine representatives of d points as unit vectors, but their phases will most likely not all be equal to 1. To achieve that the phases are 1, we need to reorient the phirotope. Although reorientation changes the phirotope, we know that reorientation preserves realisability, cf. Lemma 2.14. Therefore, the last Lemma 2.32 together with the Lemmas 2.14 and 2.29 justifies the following corollary.

Corollary 2.33 (Dualising preserves realisability)
A phirotope is realisable, if and only if its dual is realisable.

3. Reorientation and Chirotopality

We have seen in the course of the Definition 2.2 that every chirotope is a phirotope. However, the realisability theory for chirotopes and the one of phirotopes that are not chirotopes differ substantially. For analysing the realisability of phirotopes, we therefore need to be able to speak about the set of phirotopes that are not chirotopes. The main goal of this chapter is to establish the notion of "chirotopality". This is the property of a phirotope being a (possibly reoriented) chirotope. We analyse this for uniform and not uniform phirotopes separately.

3.1. Uniform Phirotopes

This section is subdivided into the theory for rank-2 phirotopes, which is mainly a review of what has already been established in [BKR03], and the theory for phirotopes of higher ranks.

Rank 2

We review cross ratio phases as introduced in [BKR03].

Definition 3.1 (Cross ratio phases)
Let φ be a rank-2 phirotope on $\mathcal{E}$. Given four elements $a, b, c, d \in \mathcal{E}$ the value

$$\mathrm{cr}_{\varphi}\,(a, b \,|\, c, d) := \frac{\varphi(a, c)\, \varphi(b, d)}{\varphi(a, d)\, \varphi(b, c)}$$

is called the *cross ratio phase* of a, b, c and d.

In the realisable case, the cross ratio phase of four indices is equal to the phase of the cross ratio of the corresponding points:

$$\mathrm{cr}_{\varphi}\,(a, b \,|\, c, d) = \omega\,(\mathrm{cr}(A, B|C, D)) \tag{3.1}$$

if the labels a, b, c, and d from $\mathcal{E}$ correspond to the points A, B, C, and D in $\mathbb{CP}^1$, respectively.

There are some further properties of the cross ratio phase that we will use throughout this dissertation. Most of them are contained in [BKR03, Lemma 2.4], which is given here:

Lemma 3.2
Let φ be a uniform phirotope in rank 2.

1. For all permutations $\pi \in S_4(a,b,c,d)$ is holds true that

$$\mathrm{cr}_{\varphi}\,(a,b\,|\,c,d) \in \mathbb{R} \quad \Leftrightarrow \quad \mathrm{cr}_{\varphi}\,(\pi(A),\pi(B)\,|\,\pi(C),\pi(D)) \in \mathbb{R}.$$

2. If $\mathrm{cr}_{\varphi}\,(a,b\,|\,c,d) \notin \mathbb{R}$, then $\mathrm{cr}(A,B|C,D)$ is determined by φ for all realisations of φ where A, B, C, and D are the points corresponding to the indices a, b, c, and d, respectively.

3. All cross ratio phases of φ are purely real, if and only if there is a reorientation of φ that is a chirotope, that means all reoriented phirotope values are in $\{-1,+1\}$.

4. Out of the set of the cross ratio phases on five indices there is either no real value, one real value, or they are all real.

5. If it holds true that $\mathrm{cr}_{\varphi}\,(a,b\,|\,c,d) \notin \mathbb{R}$, then for each choice of two elements $k,l \in \{a,b,c,d\}$ the alternating function φ' defined as

$$\varphi'(x,y) = \begin{cases} -\varphi(x,y), & \text{if } \{x,y\} = \{k,l\} \\ \varphi(x,y), & \text{if } \{x,y\} \neq \{k,l\} \end{cases}$$

is not a phirotope.

With the notion of cross ratio phase at hand we can characterise the class of phirotopes that are chirotopes. This definition was also given in [BKR03].

Definition 3.3 (Chirotopality according to [BKR03])
A uniform rank-2 phirotope is called *chirotopal*, if all cross ratio phases are real. If there exists at least one non-real cross ratio phase, the phirotope is called *non-chirotopal*.

Part 3 of the Lemma 3.2 states that chirotopal phirotopes are reoriented chirotopes. We use this characterisation as the definition in order to generalise the concept of chirotopality to higher ranks and non-uniform phirotopes. Thus, the definition of chirotopality used in this thesis is the following:

Definition 3.4 (Chirotopality)
A phirotope φ on $[n]$ is called *chirotopal*, if there exists a reorientation vector $\varrho \in (\mathcal{S}^1)^n$ such that φ^{ϱ} is a chirotope. Otherwise it is called *non-chirotopal*.

Higher Ranks

In this section, we want to find a characterisation of chirotopality for uniform phirotopes of arbitrary rank. To this end, we generalise the concept of cross ratio phases to higher ranks. This will be done with the help of "contractions". Contraction is a common tool used in (oriented) matroid theory and has been extended to phirotopes in [AD12]. For the proofs of several lemmas we will also need to reduce the number of indices of the phirotope at hand and generate a new phirotope this way. This allows us to execute proofs by induction on the number of indices in the domain of a phirotope. The tool to do this is called "deletion". As with contractions, deletions are widely used in (oriented) matroid theory and have been applied to phirotopes in [AD12] and [BB17]. Readers familiar with oriented matroid theory can easily skip the next one and a half pages that are dedicated to deletion and contraction and continue reading with the Definition 3.11.

For a proper notion of contraction and deletion, the concept of "φ-independence" is introduced as in [AD12]. In this source, one will also find the definitions of contraction and deletion given below. They are designed in such a way that they apply to non-uniform phirotopes.

Definition 3.5 (φ-independence)
Let φ be a rank-d phirotope on $\mathcal{E}$. A subset $\{e_1, \ldots, e_k\} \subset \mathcal{E}$ is called *φ-independent*, if there is a sequence $\lambda \in \Lambda(\mathcal{E}, d-k)$ such that $\varphi(e_1, \ldots, e_k, \lambda) \neq 0$. Otherwise, it is called *φ-dependent*.

Note that the bases of a phirotope are the maximal φ-independent sets within the index set.

Definition 3.6 (Contraction)
Let φ be a rank-d phirotope on $\mathcal{E}$ and $A \subset \mathcal{E}$ a subset of indices. Furthermore, let $(a_1, \ldots, a_k) \in \Lambda(A, k)$ be the ordered sequence containing the elements of a maximal φ-independent set of A. The phirotope

$$\begin{aligned} \varphi_{/A} : (\mathcal{E} \setminus A)^{d-k} &\longrightarrow \mathcal{S}^1 \cup \{0\} \\ \lambda &\longmapsto \varphi(a_1, \ldots, a_k, \lambda) \end{aligned}$$

is called the *contraction of φ at A*.
The phirotope $\varphi_{/A}$ is independent of the choice of $(a_1, \ldots, a_k)$, up to a global multiplication by a constant $c \in \mathcal{S}^1$ (see [AD12]).

The proof that the contraction really is a phirotope can also be found in [AD12].

Definition 3.7 (Deletion)
Let φ be a rank-d phirotope on $\mathcal{E}$ and $A \subset \mathcal{E}$ a subset of indices. Choose a minimal subset $\{a_1, \dots, a_k\} \subset A$ such that $(\mathcal{E} \setminus A) \cup \{a_1, \dots, a_k\}$ contains a basis of φ. The phirotope

$$\begin{aligned} \varphi_{\setminus A} : (\mathcal{E} \setminus A)^{d-k} &\longrightarrow \mathcal{S}^1 \cup \{0\} \\ \lambda &\longmapsto \varphi(a_1, \dots, a_k, \lambda) \end{aligned}$$

is called the *deletion of A from φ*.
The phirotope $\varphi_{\setminus A}$ is independent of the choice of $(a_1, \dots, a_k)$, up to a global multiplication by a constant $c \in \mathcal{S}^1$ (see [AD12]).

Note that the minimal subset $\{a_1, \dots a_k\} \subset A$, which needs to be chosen when constructing the deletion of A, may well be the empty subset. The proof that the deletion really is a phirotope can also be found in [AD12].

Deletions and contractions lead to new phirotopes. The following definition of "minors" enables us to talk about the structure that is left after deleting and contracting. In that sense, minors are exactly what a reader familiar with matroid theory would expect them to be.

Definition 3.8 (Minor)
Let φ be a phirotope on $\mathcal{E}$. A phirotope that is obtained from φ by a series of deletions and contractions is called *minor of φ*.

The following lemma, which again is in complete analogy with the concepts of deletion and contraction of oriented matroids, is also proved in [AD12]:

Lemma 3.9 (Contraction and deletion are dual concepts)
For any phirotope φ and all subsets A of indices of this phirotope it holds true that

$$(\varphi_{\setminus A})^* = (\varphi^*)_{/A}.$$

Lemma 3.10 (Deletion and contraction preserve realisability)
Let φ be a realisable phirotope on $\mathcal{E}$ and $A \subset \mathcal{E}$ a subset of indices. Then, $\varphi_{/A}$ and $\varphi_{\setminus A}$ are realisable.

Proof. If φ is realisable, then omit from the realisation of φ all points corresponding to indices in A. This yields a realisation of $\varphi_{\setminus A}$, hence $\varphi_{\setminus A}$ is realisable. To prove that $\varphi_{/A}$ is realisable as well, use Lemma 3.9 and Corollary 2.33:

$$\begin{aligned}
&\varphi \text{ is realisable,} && \Rightarrow \varphi^* \text{ is realisable,}\\
\Rightarrow\ &(\varphi^*)_{\setminus A} \text{ is realisable,} && \Rightarrow ((\varphi^*)_{\setminus A})^* \text{ is realisable,}\\
\Rightarrow\ &((\varphi^*)^*)_{/A} \text{ is realisable,} && \Rightarrow \varphi_{/A} \text{ is realisable.}
\end{aligned}$$

□

With this, we can define cross ratio phases in arbitrary ranks. To this end, we return to uniform phirotopes.

Definition 3.11 (Cross ratio phase in higher rank for uniform phirotopes)
Let φ be a uniform rank-d phirotope on $\mathcal{E}$. Furthermore let $a, b, c, d \in \mathcal{E}$ be four indices, and let $F \subset \mathcal{E}$ be a subset of $d-2$ indices of $\mathcal{E}$. The *cross ratio phase of* a, b, c, *and* d *seen from* F is defined as

$$\mathrm{cr}_\varphi\,(a, b \,|\, c, d)_F := \frac{\varphi(F, a, c)\,\varphi(F, b, d)}{\varphi(F, a, d)\,\varphi(F, b, c)},$$

where $\varphi(F, x, y) := \varphi(f_1, \dots f_{d-2}, x, y)$ with $\{f_1, \dots f_{d-1}\} = F$.

Corollary 3.12
For a uniform phirotope φ on $\mathcal{E}$ and four indices $a, b, c, d \in \mathcal{E}$ the cross ratio phase seen from a φ-independent set $F \subset \mathcal{E}$ is the same as the cross ratio phase of the four indices a, b, c, d in the contraction $\varphi_{/F}$:

$$\mathrm{cr}_\varphi\,(a, b \,|\, c, d)_F = \mathrm{cr}_{\varphi_{/F}}\,(a, b \,|\, c, d)$$

With all the terms and definitions in place, we can characterise chirotopality for phirotopes of arbitrary rank.

Lemma 3.13 (Characterisation of chirotopality of uniform phirotopes)
Let φ be a uniform rank-d phirotope on $\mathcal{E} = [n]$ with $d \geq 2$ and $n > d + 1$. The phirotope φ is chirotopal, if and only if for all $F \subset \mathcal{E}$ with $|F| = d - 2$ it holds true that

$$\mathrm{cr}_\varphi\,(a, b \,|\, c, d)_F \in \mathbb{R} \quad \forall a, b, c, d \in \mathcal{E}.$$

Moreover, rank-d phirotopes with $n = d$ or $n = d + 1$ always allow such a reorientation and are also called *chirotopal*.

The proof of this lemma is going to be rather technical and involved. This is somewhat surprising as the proof of the corresponding statement in rank 2 does not require more than ten lines (cf. [BKR03, Lemma 2.4 (3)]). The central point of this proof is to find for each reoriented

phirotope value a cross ratio that equals this phirotope value. The cross ratio is real by assumption and, thus, the reoriented phirotope value is real as well. In rank 2, this is relatively easy. One chooses the reorientation vector in such a way that applying the reorientation will multiply the phirotope value with other phirotope values to create a cross ratio. In higher ranks and on large index sets, however, this amounts to multiplying the phirotope value with large chains of fractions that will provide not only one but many cross ratios. As the size of the image of a rank d phirotope on n elements is $\binom{n}{d}$, another difficulty is to check that *all* of these values are real after the reorientation.
Before proving the Lemma 3.13, we will provide a formula that comprises the above mentioned chains of cross ratios.

Lemma 3.14
Let φ be a uniform phirotope on $\mathcal{E}$ and let $\alpha \in \Lambda(\mathcal{E}, k)$, $\beta \in \Lambda(\mathcal{E}, l)$, and $\gamma \in \Lambda(\mathcal{E}, l)$ be three sequences such that $k + l = d - 1$ and $|\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l, \gamma_1, \dots, \gamma_l\}| = k + 2l$. For any $x, y \in \mathcal{E} \setminus \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l, \gamma_1, \dots, \gamma_l\}$ the following holds true:

$$\frac{\varphi(\alpha, \beta, x)}{\varphi(\alpha, \beta, y)} \cdot \frac{\varphi(\alpha, \gamma, y)}{\varphi(\alpha, \gamma, x)} = \prod_{i=1}^{l} \mathrm{cr}_\varphi \left(\beta_i, \gamma_i \,|\, x, y\right)_{F_i} \tag{3.2}$$

with $F_i = \{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_{i-1}, \beta_{i+1}, \beta_l\}$.

In order to spare the reader chasing indices in the proof, we provide an example that demonstrates how this lemma works.

Example 3.15 Let φ be a uniform rank 5 phirotope on $\mathcal{E} = \{1, \cdots, 7, x, y\}$. Furthermore, let $k = 1$, $l = 3$ and $\alpha = (1)$, $\beta = (2, 3, 4)$, $\gamma = (5, 6, 7)$. Then, we obtain

$$\begin{aligned}
&\frac{\varphi(1,2,3,4,x)}{\varphi(1,2,3,4,y)} \cdot \frac{\varphi(1,5,6,7,y)}{\varphi(1,5,6,7,x)} \\
&= \frac{\varphi(1,2,3,4,x)}{\varphi(1,2,3,4,y)} \cdot \frac{\varphi(1,5,6,4,y) \cdot \varphi(1,5,6,4,x)}{\varphi(1,5,6,4,x) \cdot \varphi(1,5,6,4,y)} \cdot \frac{\varphi(1,5,6,7,y)}{\varphi(1,5,6,7,x)} \\
&= \frac{\varphi(1,2,3,4,x)}{\varphi(1,2,3,4,y)} \cdot \frac{\varphi(1,5,6,4,y)}{\varphi(1,5,6,4,x)} \cdot \mathrm{cr}_\varphi \left(4, 7 \,|\, x, y\right)_{\{1,5,6\}} \\
&= \frac{\varphi(1,2,3,4,x)}{\varphi(1,2,3,4,y)} \cdot \frac{\varphi(1,5,3,4,y) \cdot \varphi(1,5,3,4,x)}{\varphi(1,5,3,4,x) \cdot \varphi(1,5,3,4,y)} \cdot \frac{\varphi(1,5,6,4,y)}{\varphi(1,5,6,4,x)} \cdot \mathrm{cr}_\varphi \left(4, 7 \,|\, x, y\right)_{\{1,5,6\}} \\
&= \frac{\varphi(1,2,3,4,x)}{\varphi(1,2,3,4,y)} \cdot \frac{\varphi(1,5,3,4,y)}{\varphi(1,5,3,4,x)} \cdot \mathrm{cr}_\varphi \left(3, 6 \,|\, x, y\right)_{\{1,5,4\}} \cdot \mathrm{cr}_\varphi \left(4, 7 \,|\, x, y\right)_{\{1,5,6\}} \\
&= \mathrm{cr}_\varphi \left(2, 5 \,|\, x, y\right)_{\{1,3,4\}} \cdot \mathrm{cr}_\varphi \left(3, 6 \,|\, x, y\right)_{\{1,5,4\}} \cdot \mathrm{cr}_\varphi \left(4, 7 \,|\, x, y\right)_{\{1,5,6\}}
\end{aligned}$$

Proof of Lemma 3.14. We expand the fraction by the terms that give the cross ratio phases:

$$
\begin{aligned}
&\frac{\varphi(\alpha,\beta,x)}{\varphi(\alpha,\beta,y)}\cdot\frac{\varphi(\alpha,\gamma,y)}{\varphi(\alpha,\gamma,x)}\\
=&\frac{\varphi(\alpha,\beta,x)}{\varphi(\alpha,\beta,y)}\cdot\frac{\varphi(\alpha,\gamma,y)}{\varphi(\alpha,\gamma,x)}\cdot\frac{\varphi(\alpha,\gamma_1,\dots,\gamma_{l-1},\beta_l,x)}{\varphi(\alpha,\gamma_1,\dots,\gamma_{l-1},\beta_l,y)}\cdot\frac{\varphi(\alpha,\gamma_1,\dots,\gamma_{l-1},\beta_l,y)}{\varphi(\alpha,\gamma_1,\dots,\gamma_{l-1},\beta_l,x)}\\
=&\operatorname{cr}_\varphi\left(\beta_l,\gamma_l\,|\,x,y\right)_{F_l}\cdot\frac{\varphi(\alpha,\beta,x)}{\varphi(\alpha,\beta,y)}\cdot\frac{\varphi(\alpha,\gamma_1,\dots,\gamma_{l-1},\beta_l,y)}{\varphi(\alpha,\gamma_1,\dots,\gamma_{l-1},\beta_l,x)}
\end{aligned}
$$

With the same technique we substitute $\gamma_{l-1},\gamma_{l-2},\dots,\gamma_2$ one by one and obtain:

$$
\begin{aligned}
\dots=&\prod_{i=2}^{l}\operatorname{cr}_\varphi\left(\beta_i,\gamma_i\,|\,x,y\right)_{F_i}\cdot\frac{\varphi(\alpha,\beta,x)}{\varphi(\alpha,\beta,y)}\cdot\frac{\varphi(\alpha,\gamma_1,\beta_2,\dots,\beta_l,y)}{\varphi(\alpha,\gamma_1,\beta_2,\dots,\beta_l,x)}\\
=&\prod_{i=1}^{l}\operatorname{cr}_\varphi\left(\beta_i,\gamma_i\,|\,x,y\right)_{F_i}.
\end{aligned}
$$

□

Proof of Lemma 3.13. Firstly, we will show that every rank-d phirotope on d or $d+1$ elements allows a reorientation ϱ such that φ^ϱ is a chirotope. In the case of $\mathcal{E}=[d]$, there is only one phirotope value, namely $\varphi(1,\dots,d)$. As phase vector ϱ we can for example choose

$$\varrho=\left(\tfrac{1}{\varphi(1,\dots,d)},\ 1,\ \dots,\ 1\right).$$

With this, it holds true that $\varphi^\varrho(1,\dots,d)=1$ and thus φ^ϱ is a chirotope.

In the case of $\mathcal{E}=[d+1]$, there are $d+1$ different phirotope values: Let $\lambda\in\Lambda(\mathcal{E},d+1)$ be the ordered sequence of all indices in $\mathcal{E}$. Then the $d+1$ phirotope values are each the image of a sequence that contains all but one of the indices, namely the values $\varphi(\lambda_{\setminus i})$ for $i\in[d+1]$. We choose our phase vector as follows:

$$\varrho_1=\sqrt[d]{\frac{\varphi(\lambda_{\setminus 1})^{d-1}}{\varphi(\lambda_{\setminus 2})\cdot\ldots\cdot\varphi(\lambda_{\setminus d+1})}},\qquad\text{and}\qquad \varrho_i=\varrho_1\cdot\frac{\varphi(\lambda_{\setminus i})}{\varphi(\lambda_{\setminus 1})},\ \ \forall i\in[d+1]\setminus 1.$$

It is easily checked that all phirotope values of φ^ϱ are equal to 1 and therefore φ^ϱ is a chirotope.

Now for the main part and the phirotopes on more than $d+1$ indices. Let φ^ϱ be a reorientation of φ that is a chirotope. Then all phase values of φ^ϱ are contained in $\{-1,0,+1\}$. The cross ratio phase seen from F is invariant under reorientation. Thus, it follows that $\operatorname{cr}_\varphi\left(a,b\,|\,c,d\right)_F\in\mathbb{R}$ for all $a,b,c,d\in\mathcal{E}$ and all $F\subset\mathcal{E}$, $|F|=d-2$.

We will show the other direction using induction on the size of $\mathcal{E}$. A similar proof for rank $d=3$ was done in [Tro13]. We show the claim for arbitrary rank d here. To this end, let all cross ratio phases seen from any $d-2$ subset of $\mathcal{E}$ be real. For the base case $n=d+2$ let $\lambda(1,\dots,d+1)$ be the ordered sequence of the first $d+1$ indices in $\mathcal{E}$. We choose the phase vector

$\varrho \in (\mathcal{S}^1)^{d+2}$ as follows:

$$\varrho_1 = \sqrt[d]{\frac{\varphi(\lambda_{\setminus 1})^{d-1}}{\varphi(\lambda_{\setminus 2}) \cdot \ldots \cdot \varphi(\lambda_{\setminus d+1})}}, \tag{3.3}$$

$$\varrho_i = \varrho_1 \cdot \frac{\varphi(\lambda_{\setminus i})}{\varphi(\lambda_{\setminus 1})}, \quad \forall i \in [d+1]\setminus\{1\}, \tag{3.4}$$

$$\varrho_{d+2} = \varrho_1 \cdot \frac{\varphi(1,\ldots,d)}{\varphi(2,\ldots,d,d+2)}. \tag{3.5}$$

Note that we have chose the first $d+1$ entries of the phase vector as we did in the case with $d+1$ indices. With this, we already know that

$$\varphi^{\varrho}(\lambda) \in \mathbb{R} \quad \forall \lambda \in \Lambda([d+1], d).$$

We still have to check all phirotope values that contain the index $d+2$. Exemplarily, we check $\varphi^{\varrho}(1,\ldots,d-1,d+2)$:

$$\begin{aligned}
&\varphi^{\varrho}(1,\ldots,d-1,d+2) = \\
&\quad = \left(\prod_{i=1}^{d-1} \varrho_i\right) \cdot \varrho_{d+2} \cdot \varphi(1,\ldots,d-1,d+2) \\
&\quad = \varrho_1^d \cdot \left(\prod_{i=2}^{d-1} \frac{\varphi(\lambda_{\setminus i})}{\varphi(\lambda_{\setminus 1})}\right) \cdot \frac{\varphi(1,\ldots,d)}{\varphi(2,\ldots,d,d+2)} \cdot \varphi(1,\ldots,d-1,d+2) \\
&\quad = \frac{\varphi(\lambda_{\setminus 1})}{\varphi(\lambda_{\setminus d}) \cdot \varphi(\lambda_{\setminus d+1})} \cdot \frac{\varphi(1,\ldots,d)}{\varphi(2,\ldots,d,d+2)} \cdot \varphi(1,\ldots,d-1,d+2) \\
&\quad = \frac{\varphi(\lambda_{\setminus 1}) \cdot \varphi(1,\ldots,d-1,d+2)}{\varphi(\lambda_{\setminus d}) \cdot \varphi(2,\ldots,d,d+2)}
\end{aligned}$$

With $F = \{2,\ldots,d-1\}$ and $\alpha = (2,\ldots,d-1)$ the ordered sequence that contains the elements of F we continue the calculations above:

$$\begin{aligned}
\varphi^{\varrho}(1,\ldots,d-1,d+2) &= \frac{\varphi(\lambda_{\setminus 1}) \cdot \varphi(1,\ldots,d-1,d+2)}{\varphi(\lambda_{\setminus d}) \cdot \varphi(2,\ldots,d,d+2)} \\
&= \frac{\varphi(\alpha,d,d+1) \cdot \varphi(1,\alpha,d+2)}{\varphi(1,\alpha,d+1) \cdot \varphi(\alpha,d,d+2)} \\
&= \mathrm{cr}_{\varphi}\left(1,d \,|\, d+2,d+1\right)_F \quad \in \mathbb{R}
\end{aligned}$$

This cross ratio phase is real by assumption. All other phirotope values can be checked to be real by similar calculations. This completes the base case of our induction on n.

We will show that the phase vector $\varrho \in (\mathcal{S}^1)^n$ whose first $d+1$ entries are given by the assignments (3.3) and (3.4) and whose j-th entry is given by

$$\varrho_j = \varrho_1 \cdot \frac{\varphi(1,\dots,d)}{\varphi(2,\dots,d,j)}$$

for all $j \in \{d+2,\dots,n\}$ will yield a phirotope when used to reorient φ.

For the induction step, we consider the deletion $\varphi_{\setminus\{n\}}$. From the induction hypothesis we know that for this phirotope we find a phase vector $\tilde{\varrho} \in (\mathcal{S}^1)^{n-1}$ such that $\left(\varphi_{\setminus\{n\}}\right)^{\tilde{\varrho}}$ is a chirotope. Let ϱ be the phase vector given by

$$\begin{aligned}\varrho_i &= \tilde{\varrho}, \quad \forall i \in [n-1],\\ \varrho_n &= \varrho_1 \cdot \frac{\varphi(1,\dots,d)}{\varphi(2,\dots,d,n)}.\end{aligned}$$

This yields

$$\varphi^{\varrho}(\lambda) \in \mathbb{R}, \qquad \forall \lambda \in \Lambda(\mathcal{E} \setminus \{n\}, d).$$

Consider the phirotope values of the form $\varphi^{\varrho}(\lambda, n)$ where $\lambda \in \Lambda(\mathcal{E} \setminus \{n\}, d-1)$. We split λ into parts smaller and larger than d, so let $r \in \{0,\dots,d-1\}$ such that

$$\begin{aligned}&\lambda_i \in [d] && \forall i \in [r],\\ &\lambda_i \geq d+1 && \forall i \in \{r+1,\dots,d-1\}.\end{aligned}$$

Let $\mu \in \Lambda([d], d-r)$ be the ordered sequence such that

$$\{\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_{d-r}\} = [d].$$

Note that the index 1, which plays a special role here as ϱ_1 has a different form than all other entries of the reorientation vector, is either contained in λ or μ and depending on that, w.l.o.g., we set $\lambda_1 = 1$ or $\mu_1 = 1$. We now want to find some index m not contained in λ. This index m will be used to substitute ϱ_n by phases that we already know.

Start with the case in which there is an index $m \in \{d+1,\dots,n-1\} \setminus \{\lambda_1,\dots,\lambda_{d-1}\}$. For this index it holds true that

$$\begin{aligned}\varrho_n &= \varrho_1 \cdot \frac{\varphi(1,\dots,d)}{\varphi(2,\dots,d,n)}\\ &= \varrho_1 \cdot \frac{\varphi(1,\dots,d)}{\varphi(2,\dots,d,n)} \cdot \frac{\varphi(2,\dots,d,m)}{\varphi(2,\dots,d,m)}\\ &= \varrho_m \cdot \frac{\varphi(2,\dots,d,m)}{\varphi(2,\dots,d,n)}\end{aligned} \tag{3.6}$$

Before examining the phirotope values of indices containing n, we first provide a notational shorthand that will make the remainder of the proof more readable.

By $\star_{\mathbb{R}}$ we denote any real number (so for example $\star_{\mathbb{R}} \cdot (-5) = \star_{\mathbb{R}}$, as multiplying any real number by (-5) will again yield a real number).

Then:

$$\begin{aligned}
\varphi^{\varrho}(\lambda, n) &= \varrho_{\lambda_1} \cdot \ldots \cdot \varrho_{\lambda_{d-1}} \cdot \varrho_n \cdot \varphi(\lambda, n) \\
&= \varrho_{\lambda_1} \cdot \ldots \cdot \varrho_{\lambda_{d-1}} \cdot \varrho_m \cdot \frac{\varphi(2, \ldots, d, m)}{\varphi(2, \ldots, d, n)} \cdot \varphi(\lambda, n) \\
&= \varrho_{\lambda_1} \cdot \ldots \cdot \varrho_{\lambda_{d-1}} \cdot \varrho_m \cdot \frac{\varphi(2, \ldots, d, m)}{\varphi(2, \ldots, d, n)} \cdot \varphi(\lambda, n) \cdot \frac{\varphi(\lambda, m)}{\varphi(\lambda, m)} \\
&= \star_{\mathbb{R}} \cdot \frac{\varphi(2, \ldots, d, m)}{\varphi(2, \ldots, d, n)} \cdot \frac{\varphi(\lambda, n)}{\varphi(\lambda, m)}
\end{aligned}$$

In the last line we used the induction hypothesis. We further rewrite this in terms of λ and μ:

$$= \star_{\mathbb{R}} \cdot \frac{\varphi(1, \lambda_2, \ldots, \lambda_r, \mu_2, \ldots, \mu_{d-r}, m)}{\varphi(1, \lambda_2, \ldots, \lambda_r, \mu_2, \ldots, \mu_{d-r}, n)} \cdot \frac{\varphi(\lambda, n)}{\varphi(\lambda, m)}$$

With the Lemma 3.14 we conclude that this is indeed real.

In the case where $\{d+1, \ldots, n-1\} \setminus \{\lambda_1, \ldots, \lambda_{d-1}\} = \emptyset$, we choose $m \in [d] \setminus \{\lambda_1, \ldots, \lambda_r\}$. For this index we have

$$\begin{aligned}
\varrho_n &= \varrho_1 \cdot \frac{\varphi(1, \ldots, d)}{\varphi(2, \ldots, d, n)} \\
&= \varrho_1 \cdot \frac{\varphi(1, \ldots, d)}{\varphi(2, \ldots, d, n)} \cdot \frac{\varphi(1, \ldots, \widehat{m}, \ldots, d+1)}{\varphi(1, \ldots, \widehat{m}, \ldots, d+1)} \cdot \frac{\varphi(2, \ldots, d+1)}{\varphi(2, \ldots, d+1)} \\
&= \varrho_m \cdot \frac{\varphi(1, \ldots, d)}{\varphi(2, \ldots, d, n)} \cdot \frac{\varphi(2, \ldots, d+1)}{\varphi(1, \ldots, \widehat{m}, \ldots, d+1)} \qquad (3.7)
\end{aligned}$$

We use (3.7) and the induction hypothesis to examine $\varphi^{\varrho}(\lambda, n)$:

$$\begin{aligned}
\varphi^{\varrho}(\lambda, n) &= \varrho_{\lambda_1} \cdot \ldots \cdot \varrho_{\lambda_{d-1}} \cdot \varrho_n \cdot \varphi(\lambda, n) \\
&= \star_{\mathbb{R}} \cdot \frac{\varphi(1, \ldots, d)}{\varphi(2, \ldots, d, n)} \cdot \frac{\varphi(2, \ldots, d+1)}{\varphi(1, \ldots, \widehat{m}, \ldots, d+1)} \cdot \frac{\varphi(\lambda, n)}{\varphi(\lambda, m)}
\end{aligned}$$

With $F = \{2, \ldots, \widehat{m}, \ldots, d\}$ we obtain:

$$\begin{aligned}
\ldots &= \star_{\mathbb{R}} \cdot \frac{\varphi(\lambda, n)}{\varphi(\lambda, m)} \cdot \frac{\varphi(2, \ldots, d+1)}{\varphi(2, \ldots, \widehat{m}, \ldots, d+1, n)} \cdot \frac{\varphi(1, \ldots, d) \cdot \varphi(2, \ldots, \widehat{m}, \ldots, d+1, n)}{\varphi(1, \ldots, \widehat{m}, \ldots, d+1) \cdot \varphi(2, \ldots, d, n)} \\
&= \star_{\mathbb{R}} \cdot \frac{\varphi(\lambda, n)}{\varphi(\lambda, m)} \cdot \frac{\varphi(2, \ldots, d+1)}{\varphi(2, \ldots, \widehat{m}, \ldots, d+1, n)} \cdot \operatorname{cr}_{\varphi}(1, n \,|\, m, d+1)_F \\
&= \star_{\mathbb{R}} \cdot \frac{\varphi(\lambda, n)}{\varphi(\lambda, m)} \cdot \frac{\varphi(2, \ldots, d+1)}{\varphi(2, \ldots, \widehat{m}, \ldots, d+1, n)}
\end{aligned}$$

Finally, we use the Lemma 3.14 again to show that the term is real. ☐

The next two Lemmas 3.16 and 3.17 will help in the subsequent proofs. The Lemma 3.16 is an extension of the Lemma 3.2, part 2, to higher ranks.

Lemma 3.16 (Cross ratios are determined by cross ratio phases)
Let φ be a uniform rank-d phirotope on $\mathcal{E}$ and $F \subset \mathcal{E}$, $|F| = d - 2$. If $\mathrm{cr}_\varphi\,(a, b\,|\,c, d)_F \notin \mathbb{R}$, then $\mathrm{cr}(A, B|C, D)_G$ is determined by φ for all realisations of φ, where a, b, c, d are realised by A, B, C, and D, respectively and the indices in F are realised by the points of a set $G \subset \mathbb{C}^d$.

Proof. From phirotope axiom ($\varphi 2$), we know that there are $r_1,\ r_2,\ r_3 \in \mathbb{R}^+$ such that

$$r_1 \cdot \varphi(F, a, b)\varphi(F, c, d) - r_2 \cdot \varphi(F, a, c)\varphi(F, b, d) + r_3 \cdot \varphi(F, a, d)\varphi(F, b, c) = 0$$

As φ is uniform, dividing by $r_3 \cdot \varphi(F, a, d)\varphi(F, b, c)$ and setting $\widetilde{r_1} = \frac{r_1}{r_3}$ and $\widetilde{r_2} = \frac{r_2}{r_3}$ yields

$$-\widetilde{r_1}\mathrm{cr}_\varphi\,(a, c\,|\,b, d)_F - \widetilde{r_2}\mathrm{cr}_\varphi\,(a, b\,|\,c, d)_F + 1 = 0 \tag{3.8}$$

Thus, from $\mathrm{cr}_\varphi\,(a, c\,|\,b, d)_F \notin \mathbb{R}^+$ it follows that $\mathrm{cr}_\varphi\,(a, b\,|\,c, d)_F \notin \mathbb{R}^+$. With this, the real and imaginary part of Equation (3.8) each give rise to one real equation determining constraints on the $\widetilde{r_1},\ \widetilde{r_2}$. The equations are independent and, thus, they have a unique solution. As it holds true that the phases of cross ratios are the cross ratios of phases, we know that

$$\mathrm{cr}_\varphi\,(a, c\,|\,b, d)_F = \frac{\mathrm{cr}(A, C|B, D)_G}{|\mathrm{cr}(A, C|B, D)_G|} \quad \text{and} \tag{3.9}$$

$$\mathrm{cr}_\varphi\,(a, b\,|\,c, d)_F = \frac{\mathrm{cr}(A, B|C, D)_G}{|\mathrm{cr}(A, B|C, D)_G|}. \tag{3.10}$$

Thus, $\widetilde{r_1} = |\mathrm{cr}(A, C|B, D)|$ and $\widetilde{r_2} = |\mathrm{cr}(A, B|C, D)|$ is the solution to Equation (3.8). With the help of (3.10), we can now reconstruct $\mathrm{cr}(A, B|C, D)_G$. ☐

We will carry out the reconstruction of the cross ratio in detail in the Section 5.1. A useful property of chirotopality is given by the next lemma.

Lemma 3.17 (Duality preserves chirotopality)
A phirotope is chirotopal if and only if its dual is chirotopal.

Proof. Let φ be a rank-d phirotope on $\mathcal{E} = [n]$. Its dual is the rank-$(n-d)$ phirotope φ^* on $\mathcal{E}$. Let $a, b, c, d \in \mathcal{E}$ and $F \subset \mathcal{E} \setminus \{a, b, c, d\}$, $|F| = (n-d) - 2$. Furthermore, let F^* be the $(d-2)$-element subset of $\mathcal{E}$ given by $F^* := \mathcal{E} \setminus (F \cup \{a, b, c, d\})$. Consider the cross ratio phase

$$\mathrm{cr}_{\varphi^*}\left(a, b \,|\, c, d\right)_F = \frac{\varphi^*(a, c, F) \cdot \varphi^*(b, d, F)}{\varphi^*(a, d, F) \cdot \varphi^*(b, c, F)} = \frac{\varphi(b, d, F^*) \cdot \varphi(a, c, F^*)}{\varphi(b, c, F^*) \cdot \varphi(a, d, F^*)} = \mathrm{cr}_{\varphi}\left(a, b \,|\, c, d\right)_{F^*}.$$

As each cross ratio phase of φ^* translates to a cross ratio phase of φ and vice versa, the claim is proven. □

3.2. Phirotopes with zeros

The term "non-uniform" sometimes refers to the general case in which no assumptions are made on the uniformity of a phirotope. This especially *includes* the uniform case. As we have already dealt with the uniform case, we want to examine phirotopes with at least one zero in their images. We will then talk about phirotopes that are "not uniform" and make it very clear at every one of those points that we mean phirotopes with at least one zero in their images.

This is the first time that we encounter phirotopes with zeros in their images. Therefore, we start by explaining some basic properties and definitions.

Fundamentals

Depending on the constellation of the zeros in the image of the phirotope, there are different implications for the realisation.

Example 3.18 Consider the phirotope $\varphi : \{a, b, c\}^2 \to \mathcal{S}^1 \cup \{0\}$ with

$$\varphi(a, b) = 0, \qquad \varphi(b, c) = 0, \qquad \varphi(a, c) = \mathrm{i}.$$

If there were vectors A, B, C realising φ, then the affine representatives of A and B would have to be linearly dependent, due to $\varphi(a, b) = 0$. This means that the affine representatives are either the same or one of them is the zero vector. The same holds true for the affine representatives of B and C. But $\varphi(a, c) \neq 0$ and therefore the affine representatives of A and C must not be equal. The only way this can be realised is by setting $B = \begin{pmatrix}0, & 0\end{pmatrix}^T$.

In this example, the index b is called a "loop", a term that is borrowed from matroid theory, where it refers to the same thing.

Definition 3.19 (Loop)
Let φ be a rank-d phirotope on $\mathcal{E}$. An index $\ell \in \mathcal{E}$ is called *loop*, if for all $\lambda \in \Lambda(\mathcal{E}, d-1)$ it holds true that

$$\varphi(\lambda, \ell) = 0.$$

A loop is an element ℓ such that its realisation P_ℓ is linearly dependent. Thus, P_ℓ has to be the zero vector. This leads to the following corollary:

Corollary 3.20 (Loops prohibit realisability in projective space)
If a phirotope contains a loop, it has no realisation in projective space.

In Section 2.1, we stated that the realisation of a uniform rank-d phirotope is a subset of the complex oriented projective space $(\mathbb{C}^d \smallsetminus \{0\})/\mathbb{R}^+$. The reason was that multiplying an element with a positive real scalar does not change the phirotope. This is still true for elements of uniform phirotopes. Nevertheless, loops have to be realised as the zero vector. Thus, we will understand the realisations of non-uniform rank-d phirotopes to lie in the space $\mathbb{C}^d/\mathbb{R}^+$, which is the natural extension of the Definition 2.16 to higher dimensions. This is no projective space. But working over this space makes loops easy to handle:

Lemma 3.21
Let φ be a phirotope on $\mathcal{E}$ and let $\ell \in \mathcal{E}$ be a loop of φ. Then φ is realisable, if and only if $\varphi_{\setminus\{\ell\}}$ is realisable.

Proof. Let φ be realisable. Then Lemma 3.10 guarantees that $\varphi_{\setminus\{\ell\}}$ is also realisable.
Let $\varphi_{\setminus\{\ell\}}$ be realisable and $(P_1, \ldots, P_{|\mathcal{E}|-1})$ be a realisation of $\varphi_{\setminus\{\ell\}}$. Then, the vector configuration $(0, P_1, \ldots, P_{|\mathcal{E}|-1})$ is a realisation of φ, where the zero vector realises the index ℓ. $\square$

There are other constellations of zeros in the images of non-uniform phirotopes.

Example 3.22 Consider the phirotope $\varphi : \{a, b, c\}^2 \to \mathcal{S}^1 \cup \{0\}$ with

$$\varphi(a, b) = 0, \qquad \varphi(b, c) = 1, \qquad \varphi(a, c) = \mathrm{i}.$$

There is no loop in this phirotope and thus the affine representatives of A and B realising a and b, respectively, have to be the same.

The consequence that the affine representatives of A and B in Example 3.22 need to be equal arises from them forming something that is referred to as a "parallel element". This term is also borrowed from matroid theory.

Definition 3.23 (Parallel element)
Let φ be a rank-d phirotope on $\mathcal{E}$. A pair of indices $\{a,b\} \subset \mathcal{E}$ is called *parallel element*, if neither a nor b is a loop and for all $\lambda \in \Lambda(\mathcal{E} \setminus \{a,b\}, d-2)$ it holds true that

$$\varphi(a,b,\lambda) = 0.$$

If the phirotope at hand is realisable, parallel elements of this phirotope will be realised with the same affine representatives. The only other possibility of realising them would be to have them lie on a common subspace of rank less than d with every other $(d-2)$-element subset of points. But this would mean that there was a subspace of rank less than d that contained all points and thus $\varphi \equiv 0$. Also, note that in rank 2 the sequence λ is the empty sequence and any pair of indices mapped to zero forms a parallel element.

Loops, parallel elements, and non-bases in general make the realisations of phirotopes even more rigid. While in the uniform case we were able to choose the first $d+1$ points of the realisation of a realisable phirotope freely (cf. Lemma 2.11), this is no longer true for phirotopes containing parallel elements. The following example will illustrate this.

Example 3.24 Consider a rank-3 phirotope on $\{a,b,c,d\}$ where a and b form a parallel pair. W.l.o.g., the affine representatives p_A, p_B, p_C, p_D of the points of a realisation of the phirotope have coordinates

$$p_A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad p_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad p_C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad p_D = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Choosing the affine representatives for all four points in general position will not yield a realisation of the phirotope, as p_A and p_B need to be the same.

In most of our theorems regarding the realisability of phirotopes that contain zeros in their images, we will exclude configurations that are "too degenerate", like the one in Example 3.24. To this end, we will postulate an analogon of the axiom for projective spaces that requires that in every projective space of rank d there have to be at least $d+1$ points in general position (cf. for example [RG11, Definition 2.1]). The term "near-pencil" is again borrowed from matroid theory.

Definition 3.25 (Near-pencil)
A rank-d point configuration of n distinct points is called a *near-pencil*, if all but one point lie on a common hyperplane.
A rank-d phirotope is called a *near-pencil*, if every minor on $d+1$ indices is not uniform (contains at least one zero in its image).

Excluding near-pencils, we can reformulate the Lemma 2.11, which allowed us to choose the first $d+1$ affine representatives of a realisation freely, as follows.

Lemma 3.26 (Freedom of choice of the first $d+1$ affine representatives in a non-uniform setting)
Let φ be a realisable rank-d phirotope on $[n]$, $n > d$, that is not a near-pencil. Let the restriction $\varphi|_{[d+1]}$ be uniform. For any choice of affine representatives $p_1, \ldots, p_{d+1} \in \mathbb{C}^{d-1}$ in general position there is a realisation $V = (P_1, \ldots P_{d+1}, P_{d+2}, \ldots, P_n)$ of φ such that

$$P_k = r_k \omega_k \begin{pmatrix} p_k \\ 1 \end{pmatrix}$$

where $r_k \in \mathbb{R}^+$ and $\omega_k \in \mathcal{S}^1$ for all $k \in [d+1]$.

Proof of Lemma 3.26. Let $(Q_1, \ldots, Q_n)$ be a realisation of φ. As in the proof of [BKR03, Lemma 2.3], we will construct a projective transformation M that leaves φ invariant and maps $Q_1, \ldots Q_{d+1}$ to $P_1, \ldots, P_{d+1}$. The system $MQ_k = P_k$ for all $k \in [d+1]$ gives rise to $d \cdot (d+1)$ linear equations in the d^2 entries of the matrix M and the $d+1$ unknowns $(r_k \cdot \omega_k) \in \mathbb{C}$. The points $Q_1, \ldots, Q_d$ are in general position, thus the linear equations are linearly independent and $\det(M) \neq 0$. A complex projective transformation in rank d is fixed by $d+1$ pairs of points and their corresponding images. It is also possible to choose the matrix for the transformation out of $SL(d, \mathbb{C})$. This choice will leave the phirotope unchanged. $\square$

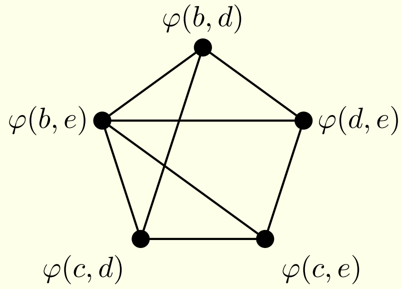
Note that in rank 2, a phirotope is a near-pencil, if and only if in all realisations every point coincides with one of two points. This case is relatively straightforward. In higher ranks, however, the situation becomes more complicated and excluding near-pencils in theorems guarantees that there is a projective basis that can be used as the starting point for constructing realisations. These constructions are often done by carefully moving from one basis to the next one. Basis graphs are a tool that assists in determining which basis should be considered next.

Basis Graphs

For a uniform rank-d phirotope, any d-tuple of indices that does not contain an index twice is a basis. If the phirotope is not uniform, however, this is not the case. The following sections are therefore dedicated to the study of bases in general and to "basis graphs". A definition of basis graphs can for example be found in the work of Maurer, who has worked extensively on matroid basis graphs (cf. [Mau73a] and [Mau73b]).

Definition 3.27 (Basis graphs of phirotopes)
The basis graph G of a phirotope φ is the graph whose vertices are the bases of φ and in which two vertices are adjacent, if and only if the bases of these vertices differ on exactly one index.

Example 3.28 Consider the rank-2 phirotope on $\mathcal{E} = \{a, b, c, d, e\}$ in which a is a loop and b and c are parallel elements. Apart from these exceptions, all pairs shall be mapped to a non-zero value. Thus, the basis graph of the phirotope looks as follows:



Note that as a is a loop, the index does not occur in any base and thus in no vertex of the basis graph.

We will mostly use basis graphs to keep track of our bases at hand and to find quadruples of phirotope values that form cross ratio phases. For further information on basis graphs of matroids we refer the reader to the work of Holzmann and Harary (who showed that for every edge in a basis graph there is one Hamiltonian path containing the edge and one Hamiltonian path not containing the edge [HH72]), Holzmann et al. (who proved that two matroids are equivalent, if and only if their basis graphs are isomorphic [HNT73]), and Liu (who gave bounds for the connectivity of basis graphs [Liu88]) to name just a few.

If the induced subgraph of four vertices in a graph is a cycle containing all four points, we will call this an *empty quadrilateral*.

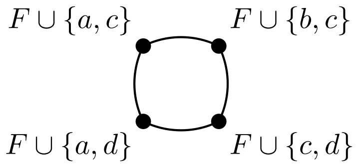


Figure 3.1.: The subgraph of the basis graph that corresponds to a cross ratio phase is an empty quadrilateral.

Lemma 3.29 (Cross ratio phases correspond to empty quadrilaterals)
Consider the basis graph of a phirotope φ. The induced subgraph of four vertices in the basis graph is an empty quadrilateral, if and only if the phirotope values corresponding to the vertices form a cross ratio phase of bases.

Proof. Let φ be a rank-d phirotope on $\mathcal{E}$. The structure of cross ratio phases makes it clear that every cross ratio phase of bases of φ corresponds to an empty quadrilateral in the basis graph.

Now, consider an empty quadrilateral in the basis graph. W.l.o.g., let two of its adjacent vertices be labelled with the bases $F \cup \{a, c\}$ and $F \cup \{b, c\}$, where $F \subset \mathcal{E}$, $|F| = d - 2$. The second vertex adjacent to $F \cup \{a, c\}$ is not adjacent to $F \cup \{b, c\}$ and, thus, it is w.l.o.g. labelled with $F \cup \{a, d\}$. The last vertex in the quadrilateral is adjacent to both $F \cup \{a, d\}$ and $F \cup \{b, c\}$ and may therefore carry one of the following labels:

$$F \cup \{a, b\}, \qquad F \cup \{a, c\}, \qquad F \cup \{c, d\}, \qquad F \cup \{b, d\}.$$

As the first three would also be adjacent to $F \cup \{a, c\}$ or even coincide with it, the label necessarily is $F \cup \{b, d\}$. This completes the cross ratio phase $\mathrm{cr}_\varphi\,(a, c \,|\, b, c)_F$. The Figure 3.1 illustrates this. □

The next lemma, which we adapted to phirotopes, was devised and proved by Brualdi [Bru69, Theorem 2].

Lemma 3.30 (Exchange of elements in bases)
If B_1 and B_2 are bases of a phirotope and $e \in B_1 \setminus B_2$, then there is an $f \in B_2 \setminus B_1$ such that both

$$(B_1 \setminus \{e\}) \cup \{f\} \qquad \text{and} \qquad (B_2 \setminus \{f\}) \cup \{e\}$$

are bases of this phirotope.

The proof in which we are going to use basis graphs is the one of the following lemma, which characterises chirotopality for phirotopes that are not uniform. This characterisation of chirotopality will be crucial in the analysis of realisability of phirotopes that contain zeros in their images.

Theorem 3.31 (Chirotopality of not uniform phirotopes in rank d)
Let φ be a rank-d phirotope on $\mathcal{E} = [n]$ with $d \geq 2$ and $n \geq d+2$ that is not a near-pencil and has at least one zero in its image. Let M be the set of all quintuples (F, a, b, c, d) with $F \subset \mathcal{E}$ and $a, b, c, d \in \mathcal{E}$ such that $|F| = d - 2$ and none of the following phirotope values is zero:

$$\varphi(a, c, F), \qquad \varphi(a, d, F), \qquad \varphi(b, c, F), \qquad \varphi(b, d, F).$$

There is a phase vector $\varrho \in (\mathcal{S}^1)^n$ such that the reorientation φ^ϱ is a chirotope, if and only if it holds true that

$$\mathrm{cr}_\varphi\,(a, b \,|\, c, d)_F \in \mathbb{R} \quad \forall (F, a, b, c, d) \in M.$$

Moreover, rank-d phirotopes with $n = d$ or $n = d + 1$ always allow such a reorientation.

The proof of the corresponding Lemma 3.13 in the uniform case was already quite technical. If the phirotope may now have zeros in its image, the situation becomes even more complicated and, thus, the next proof will be even more involved. In the proof of the Lemma 3.13 the reorientation vector provides fractions of phirotope values that yield chains of cross ratios. The main difficulty we have to overcome here is that these fractions might contain zeros. Therefore, we need to find a way to move step by step and without encountering zeros from one cross ratio to the next. This is done with the help of basis graphs.

To simplify the proof, we provide the following definition:

Definition 3.32 (Hyperplane)
Let φ be a rank-d phirotope on $\mathcal{E}$. A set of indices $F \subset \mathcal{E}$ is called a *hyperplane*, if the maximal φ-independent subset of F contains exactly $d - 1$ elements.

Proof of Theorem 3.31. If the phirotope contains a loop, the corresponding element of the reorientation vector can be arbitrary. It does not have consequences for the chirotopality of the phirotope. We will therefore assume that the phirotope does not contain loops.

The "only if" part of the proof is trivial for any rank. The other direction will be carried out in more detail.

Firstly, note that phirotopes on d indices are always uniform and chirotopal according to the Lemma 3.13.

In the case of $\mathcal{E} = [d+1]$, there is at least one d-tuple that is not mapped to zero. W.l.o.g., let $\varphi(\lambda_{\setminus 1}) \neq 0$ with $\lambda = (1, \ldots, d+1)$. Let ϑ be the product of all non-zero phirotope values in

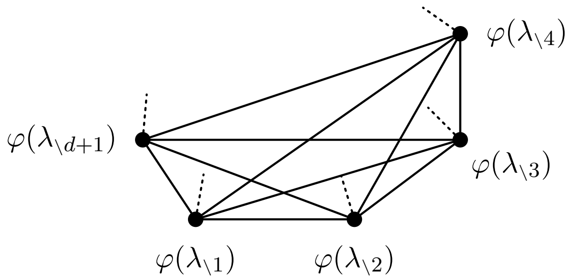


Figure 3.2.: Here, $\lambda = (1, \ldots, d+1)$. As $\varphi|_{[d+1]}$ is uniform, the basis graph of $\varphi|_{[d+1]}$ is a complete graph.

$\{\varphi(\lambda_{\setminus 2}), \ldots, \varphi(\lambda_{\setminus d+1})\}$:

$$\vartheta := \prod_{\substack{i \in \{2,\ldots,d+1\}: \\ \varphi(\lambda_{\setminus i}) \neq 0}} \varphi(\lambda_{\setminus i}).$$

As straightforward calculations show, using the reorientation vector ϱ given as follows will yield a chirotope φ^{ϱ}:

$$\varrho_1 = \sqrt[d]{\frac{\varphi(\lambda_{\setminus 1})^{d-1}}{\vartheta}},$$

$$\varrho_i = \begin{cases} \varrho_1 \cdot \dfrac{\varphi(\lambda_{\setminus i})}{\varphi(\lambda_{\setminus 1})}, & \text{if } \varphi(\lambda_{\setminus i}) \neq 0, \\ \varrho_1 \cdot \dfrac{1}{\varphi(\lambda_{\setminus 1})}, & \text{else,} \end{cases} \qquad \text{for all } i \in [d+1] \setminus \{1\}.$$

Proving the claim for $|\mathcal{E}| \geq d+2$ works by induction on $|\mathcal{E}|$. Start with the base case $|\mathcal{E}| = d+2$. As we required that the phirotope is not a near-pencil, there is a minor on $d+1$ indices that is uniform. This minor will serve as a projective basis throughout the rest of this proof. Let the indices of the minor be named 1, …, $d+1$. The basis graph of the minor is depicted in Figure 3.2. All cross ratio phases are real by assumption and, thus, the minor $\varphi|_{\{1,\ldots,d+1\}}$ is chirotopal. From now on, we assume that the phirotope is reoriented in a way that all phirotope values of $\varphi|_{\{1,\ldots,d+1\}}$ are real. Thus, assume that $\varrho_1 = \ldots = \varrho_{d+1} = 1$. This saves us from carrying the reorientations of these indices through all calculations to come.

The reorientation of the next index $(d+2)$ is determined by one basis of the phirotope that contains $d+2$. Such a basis exists because $d+2$ is not a loop. W.l.o.g., let this basis be given by $\{1, \ldots, d-1, d+2\}$. In Figure 3.3, the vertex $\{1, \ldots, d-1, d+2\}$ is added to the basis graph. The reorientation ϱ_{d+2} of $d+2$ is then given by

$$\varrho_{d+2} = \pm \frac{1}{\varphi(1, \ldots, d-1, d+2)}$$

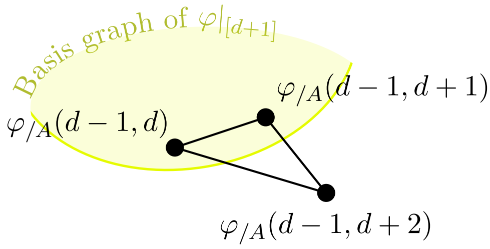


Figure 3.3.: In this figure (and in the proof) $A = \{1,\dots,d-2\}$. The first basis that is not contained in the basis graph of $\varphi|_{[d+1]}$ is connected with some nodes of this basis graph.

because

$$\varphi^{\varrho}(1,\dots,d-1,d+2) = \varrho_{d+2} \cdot \varphi(1,\dots,d-1,d+2) = \pm 1.$$

Choosing a minus over a plus (or the other way around) only changes the sign of the value 1 but has no effect on the chirotopality.

We need to show that all other values of the reoriented phirotope that contain the index $d+2$ are also real. To make the remainder of this proof more readable, from here on we refer to the reoriented phirotope φ^{ϱ} by φ.

Firstly, consider any basis $\{\lambda, d+2\}$ of φ with $\lambda \in \Lambda([d+1], d-1)$ such that the bases $\{\lambda, d+2\}$ and $\{1,\dots,d-1,d+2\}$ differ by one element. W.l.o.g., let $\{1,\dots,d-2,d,d+2\}$ be such a basis. To see that $\varphi(1,\dots,d-2,d,d+2)$ is real, we consider the cross ratio phase

$$\operatorname{cr}_{\varphi}(d-1,d\,|\,d+2,d+1)_A = \frac{\varphi_{/A}(d-1,d+2)\cdot\varphi_{/A}(d,d+1)}{\varphi_{/A}(d-1,d+1)\cdot\varphi_{/A}(d,d+2)}$$

where $A = \{1,\dots,d-2\}$. Note that reorientation does not change the value of the cross ratio phase, which real by assumption. As $\varphi_{/A}(d-1,d+2) \in \mathbb{R}_{\setminus 0}$, $\varphi_{/A}(d,d+1) \in \mathbb{R}_{\setminus 0}$, and $\varphi_{/A}(d-1,d+1) \in \mathbb{R}_{\setminus 0}$, it follows that $\varphi_{/A}(d,d+2) \in \mathbb{R}_{\setminus 0}$. The situation in the basis graph is depicted in Figure 3.4.

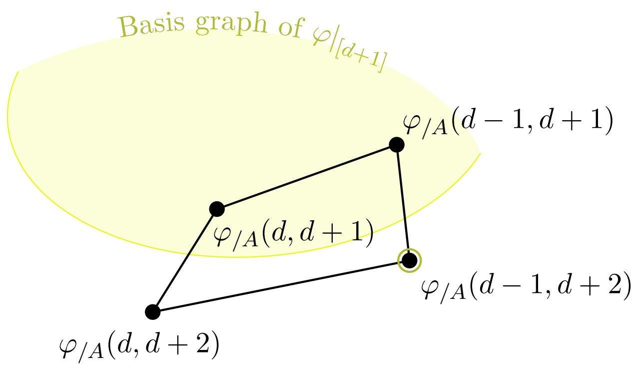


Figure 3.4.: The empty quadrilateral corresponds to $\operatorname{cr}_{\varphi}(d-1,d\,|\,d+2,d+1)_A$. The vertex $\varphi_{/A}(d-1,d+2)$ that was used to determine the reorientation of the index $d+2$ is highlighted.

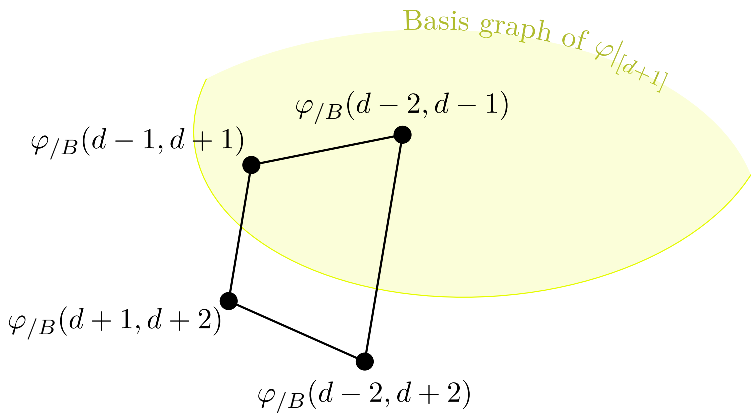


Figure 3.5.: In this figure (and in the proof) $B = \{1,\ldots,d-3,d\}$. Consider the two cross ratio phases $\mathrm{cr}_\varphi\,(d+1,d-2\,|\,d+2,d-1)_B$ and $\mathrm{cr}_\varphi\,(d+1,d-1\,|\,d+2,d-2)_B$. Both of them correspond to the empty quadrilateral.

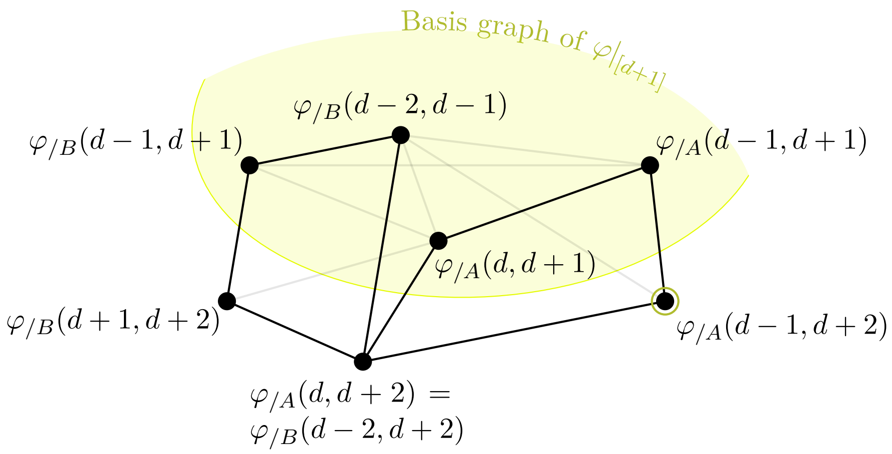


Figure 3.6.: Part of the basis graph of φ. The two black empty quadrilaterals correspond to the cross ratio phases used in the proof. Note that the vertex $\varphi_{/A}(d,d+2)$ is used in both cross ratio phases.

Secondly, consider any basis $\{\lambda, d+2\}$ of φ with $\lambda \in \Lambda([d+1], d-1)$ such that the bases $\{\lambda, d+2\}$ and $\{1,\ldots,d-1,d+2\}$ differ by *two* elements. W.l.o.g., let $\{1,\ldots,d-3,d,d+1,d+2\}$ be such a basis. By applying the Lemma 3.30 to the two bases $\{1,\ldots,d-1,d+2\}$ and $\{1,\ldots,d-3,d,d+1,d+2\}$ to exchange the index $d+1$, we obtain that either $\{1,\ldots,d-3,d-2,d,d+2\}$ or $\{1,\ldots,d-3,d-1,d,d+2\}$ is a basis. Both of them have real values in the reoriented phirotope because they differ from $\{1,\ldots,d-1,d+2\}$ by only one element. Then, the cross ratio phase $\mathrm{cr}_\varphi\,(d+1,d-2\,|\,d+2,d-1)_B$ or the cross ratio phase $\mathrm{cr}_\varphi\,(d+1,d-1\,|\,d+2,d-2)_B$ (both with $B = \{1,\ldots,d-3,d\}$) will yield the conclusion. This is depicted in the Figure 3.5 and in the Figure 3.6 the complete subgraph induced by all vertices used in the proof is shown.

The induction step works similarly. Again, we look for empty quadrilaterals in the basis graph that share a vertex:

Assume that we are given a phirotope φ and a reorientation vector ϱ such that $\varphi^\varrho|_{[k]}$ is a chirotope. For the sake of convenience, we again consider φ to be the reoriented phirotope. Thus, $\varrho_1 = \cdots = \varrho_k = 1$. W.l.o.g., let again $\varphi|_{[d+1]}$ be uniform. The reorientation of the next index (w.l.o.g. $k+2$) is determined by one basis of the phirotope that only contains $k+2$ and indices in $[d+1]$. Such a basis exists because $k+2$ is not a loop and, thus, there is a basis containing $k+2$. According to the Lemma 3.30 we can then find a basis containing only the index $k+2$ and elements from $[d+1]$. W.l.o.g., let this basis be given by $\{1,\dots,d-1,k+2\}$. The reorientation ϱ_{k+2} of $k+2$ is then given by

$$\varrho_{k+2} = \pm\frac{1}{\varphi(1,\dots,d-1,k+2)}.$$

It is left to show that all other phirotope values containing the index $k+2$ are now also real. We will do this with the help of an induction on the number of indices by which the bases containing $k+2$ differ from the basis $\{1,\dots,d-1,k+2\}$. For the base case, consider any basis that differs from $\{1,\dots,d-1,k+2\}$ on one index. W.l.o.g., let $\{1,\dots,d-2,x,k+2\}$ be such a basis. Consider the contraction $\varphi_{/A}$ with $A = \{1,\dots,d-2\}$. Note that $(\varphi_{/A})|_{[d+1]}$ is a uniform rank-2 phirotope on the indices $d-1$, d, $d+1$ and, thus, it is no near-pencil. Furthermore, it holds true that $\varphi(1,\dots,d-1,k+2) \neq 0$ and, thus, $\varphi_{/A}(d-1,k+2) \neq 0$. We need to find some index $y \in \{d,d+1\}$ such that both $\{1,\dots,d-1,y\}$ and $\{1,\dots,d-2,x,y\}$ are bases of φ. In different words, we are looking for an index y such that $\varphi_{/A}(d-1,y) \neq 0$ and $\varphi_{/A}(x,y) \neq 0$. As $\varphi_{/A}$ is no near-pencil, there are at least two hyperplanes containing all points. In this situation, the index $k+2$ lies on one hyperplane and the index $d-1$ and x on the other one. As $k+2 \notin [d+1]$, there is an index $y \in \{d,d+1\}$ on the same hyperplane as $k+2$. With this index y, we obtain $\varphi_{/A}(d-1,y) \neq 0$ and $\varphi_{/A}(x,y) \neq 0$. Then,

$$\frac{\varphi^{\varrho}_{/A}(d-1,k+2)\cdot\varphi^{\varrho}_{/A}(x,y)}{\varphi^{\varrho}_{/A}(d-1,y)\cdot\varphi^{\varrho}_{/A}(x,k+2)},$$

which is real by assumption, apart from $\varphi^{\varrho}_{/A}(x,k+2)$ contains only real values and, thus, $\varphi^{\varrho}_{/A}(x,k+2)$ is real as well.

For the induction step, assume that all bases of φ that contain the index $d+2$ and differ from $\{1,\dots,d-1,k+2\}$ by m indices yield a real value in the reoriented phirotope. For every basis containing $d+2$ and differing from $\{1,\dots,d-1,k+2\}$ by $m+1$ indices, we need to find a cross ratio phase that guarantees that the reoriented phirotope is real on this basis. W.l.o.g., consider the basis $\{1,\dots,d-m-2,z_1,\dots,z_{m+1},k+2\}$ with $z_i \notin [d+1]$. This especially means that the set $\{1,\dots,d-m-2,z_1,\dots,z_{m+1}\}$ is φ-independent. Applying the Lemma 3.30 to $\{1,\dots,d-m-2,z_1,\dots,z_{m+1},k+2\}$ and $\{1,\dots,d\}$ on the index z_{m+1} provides us with an index $a \in \{d-m-1,\dots,d\}$ such that $\{1,\dots,d-m-2,a,z_1,\dots,z_m,k+2\}$ is a basis of the phirotope. By induction hypothesis, the value of the reoriented phirotope on this basis is real.

For $B = \{1, \dots, d-m-2, z_1, \dots z_m\}$ consider the contraction $\varphi_{/B}$. We want to find a cross ratio phase (not containing a zero) that contains $\varphi_{/B}(z_{m+1}, k+2)$ and $\varphi_{/B}(a, k+2)$. For this, we need an index $y \in \{d-m-1, \dots, d+1\}$ such that $\varphi_{/B}(z_{m+1}, y) \neq 0$ and $\varphi_{/B}(a, y) \neq 0$.

Since both $\{1, \dots, d-m-2, z_1, \dots, z_{m+1}\}$ and $[d+1]$ are φ-independent, the number of possible assignments for the index y is at least

$$|\{d-m-1, \dots, d+1\}| - |\{z_1, \dots, z_m\}| = 3.$$

The cross ratio phase $\mathrm{cr}_{\varphi^{\varrho}}\,(z_{m+1}, a \,|\, k+2, y)_B$ is real by assumption and this guarantees that $\varphi^{\varrho}_{/B}(z_{m+1}, k+2)$ is real as well. This concludes the induction on m as well as the induction on k. $\square$

Note that if the phirotope is uniform, the characterisation of chirotopality given in the Theorem 3.31 reduces to the one of the uniform case given in the Lemma 3.13.

Now that we have established the notion of chirotopality for uniform and non-uniform phirotopes of arbitrary rank, we can use the characterisations in our analysis of the realisability problem.

4. Realisability and Rigidity

In general, deciding the realisability of real oriented matroids is NP-hard. The proof of this statement relies on Mnëv's Universality Theorem (cf. [Mnë88] as well as [RG95]). The proof of the Universality Theorem provides a polynomial reduction of the realisability problem of oriented matroids to the existential theory of the reals (cf. [BLSWZ93, Chapter 8.6] for the result and a sketch of the proof). The NP-hardness is also shown directly in [Sho91]. However, for chirotopes with relatively few elements there are some general results regarding realisability. The following lemma is an only slightly modified version of the Corollary 8.3.3 from [BLSWZ93].

Lemma 4.1 (Realisability of chirotopes)
All chirotopes χ of rank d on $\mathcal{E}$ are realisable, if and only if

- $d \leq 2$,
- $d = 3,\text{ and } |\mathcal{E}| \leq 8$,
- $d = 4,\text{ and } |\mathcal{E}| \leq 7$,
- $d = 5,\text{ and } |\mathcal{E}| \leq 8$, or
- $d \geq 6,\text{ and } |\mathcal{E}| \leq d + 2$.

In all other cases, there exist non-realisable uniform chirotopes.

The situation for non-chirotopal phirotopes is substantially different. We will start this chapter by analysing the realisability problem for uniform phirotopes and later turn to the more technical non-uniform case.

4.1. Uniform Phirotopes

Realisability for complex matroids has been studied in [BKR03] and in [Tro13]. In [BKR03], only uniform rank-2 phirotopes where examined. In [Tro13], the realisability of uniform rank-3 phirotopes was analysed. The following subsection basically reviews what is already known for the realisability of complex matroids in rank 2. It also introduces some basic concepts. The subsequent sections will then extend the realisability to higher ranks and non-uniform phirotopes.

Rank 2

The following results (summed up in the Lemmas 4.2 and 4.3) regarding the realisability of non-chirotopal phirotopes were published and proven in [BKR03].

Lemma 4.2 (Realisability and rigidity of small uniform rank-2 phirotopes)
Uniform rank-2 phirotopes on three or four points are always realisable.
Moreover, if the realisation of the first three points of a realisable, non-chirotopal phirotope is known, then the fourth point is determined as well.

We adapt and extend the notation of [BKR03] for squared phases

$$[\![\lambda]\!] := \varphi(\lambda)^2$$
$$[\![\lambda]\!]_{/A} := \left(\varphi_{/A}(\lambda)\right)^2$$

Lemma 4.3 (Realisability of large uniform rank-2 phirotopes)
Let φ be a non-chirotopal uniform rank-2 phirotope on $\mathcal{E} = [n]$ with $n \geq 5$. It is realisable, if and only if for each subset $\{a, b, c, d, e\} \subset \mathcal{E}$ the following is true:

$$\sum_{\substack{\pi \in S_4(a,b,c,d)\\ \pi(a)<\pi(d)}} \operatorname{sign}(\pi)[\![\pi(a), \pi(b)]\!][\![\pi(b), \pi(c)]\!][\![\pi(c), \pi(d)]\!][\![\pi(d), e]\!][\![e, \pi(a)]\!] = 0.$$

This algebraic relation is called the *five-point condition* in rank 2. The realisation is then unique up to Möbius transformations with real determinant.

Note that according to the Lemma 4.1, all rank-2 chirotopes are realisable. They trivially satisfy the five-point condition (all squared chirotopes values are equal to 1). But their realisations are not unique. In contrast, the realisations of non-chirotopal phirotopes are rigid and it is easy to decide whether they exist. This is in stark contrast to chirotopes where each point of a realisation can be moved within the cell bounded by the lines spanned respectively by two other points of the realisation. This movement will not alter the chirotope. Also, as said before, deciding the realisability of chirotopes is NP-hard.

Examples of realisable uniform phirotopes can easily be constructed by starting with a realisation and then computing the corresponding phirotope. An example of a non-realisable phirotope is given in the Appendix A.

The five-point condition gives cause to several investigations. Its structure, for example, is worth a closer inspection. In the way it is depicted in the Lemma 4.3, the five-point condition

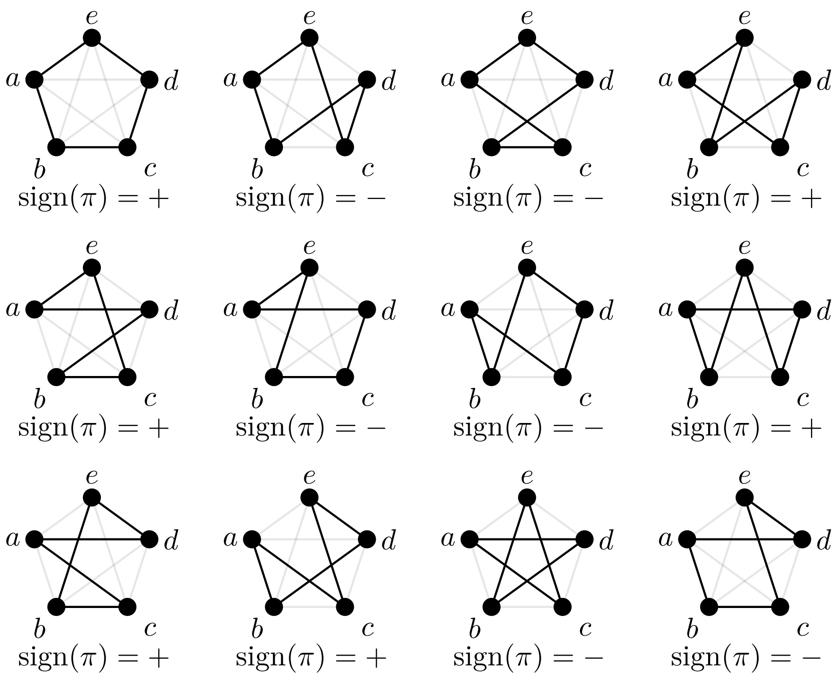


Figure 4.1.: All Hamilton cycles that correspond to the permutations considered in the five-point condition. The sign of the permutation is -1, if and only if the number of crossings in the Hamilton cycle is odd.

contains twelve summands, each of which consists of five squared phirotope values:

$$\begin{aligned}
&+ [\![a,b]\!][\![b,c]\!][\![c,d]\!][\![d,e]\!][\![e,a]\!] - [\![a,b]\!][\![b,d]\!][\![d,c]\!][\![c,e]\!][\![e,a]\!]\\
&- [\![a,c]\!][\![c,b]\!][\![b,d]\!][\![d,e]\!][\![e,a]\!] + [\![a,c]\!][\![c,d]\!][\![d,b]\!][\![b,e]\!][\![e,a]\!]\\
&+ [\![a,d]\!][\![d,b]\!][\![b,c]\!][\![c,e]\!][\![e,a]\!] - [\![a,d]\!][\![d,c]\!][\![c,b]\!][\![b,e]\!][\![e,a]\!]\\
&- [\![b,a]\!][\![a,c]\!][\![c,d]\!][\![d,e]\!][\![e,b]\!] + [\![b,a]\!][\![a,d]\!][\![d,c]\!][\![c,e]\!][\![e,b]\!]\\
&+ [\![b,c]\!][\![c,a]\!][\![a,d]\!][\![d,e]\!][\![e,b]\!] + [\![c,a]\!][\![a,b]\!][\![b,d]\!][\![d,e]\!][\![e,c]\!]\\
&- [\![c,a]\!][\![a,d]\!][\![d,b]\!][\![b,e]\!][\![e,c]\!] - [\![c,b]\!][\![b,a]\!][\![a,d]\!][\![d,e]\!][\![e,c]\!] = 0.
\end{aligned} \tag{4.1}$$

This is an already condensed version of the five-point condition. The statement of the Lemma 4.3 still holds true, if we sum over all permutations (not only those where $\pi(a) < \pi(d)$) and permute the index e as well. Thus, we can also sum over all permutations in $S_5(a,b,c,d,e)$. However, there will then be several terms that cancel or are identical.

As suggested by Richter-Gebert, the structure of the five-point condition can be illustrated by a Hamilton cycle in the complete graph on five vertices K_5 [RG14]. Each permutation stands for another Hamilton cycle. The order in which the indices appear in the summands of the five-point condition determines the cycle. The restrictions made on the permutations (not permuting the index e and having $\pi(a) < \pi(d)$) correspond to fixing one point as the starting point of the Hamilton cycle and considering only one of the two directions of traversing the cycle. The latter allows us to consider only undirected cycles. In the Figure 4.1, all Hamilton cycles corresponding to summands of the condensed five-point condition are illustrated. The parity of the number of crossings in these cycles indicates the sign of the permutation.[1]

Some further examinations of the structure of the five-point condition are carried out in the Section 4.3. For a rank-2 phirotope on n elements, we have $\binom{n}{5}$ sums of which we have to check if they are equal to zero. We will show that there are redundancies and we do not have to check all of them.

In the Section 4.2, we will see how the five-point condition can be extended to non-uniform phirotopes. And in the following section, we will examine what this formula looks like for uniform phirotopes in higher ranks.

Higher Ranks

The last subsection gave a method for checking the realisability of uniform rank-2 phirotopes. Often, we are dealing with phirotopes of higher rank. Their realisability will be examined in this section.

A first result regarding realisability follows from the Lemma 2.11, which states that the first $d+1$ points of a realisation of a phirotope can be chosen freely.

Corollary 4.4 (Realisability of small uniform rank-d phirotopes)
Every uniform rank-d phirotope on d or $d+1$ indices is realisable.

After the first $d+1$ points are chosen in general position, however, the rest of the realisation is determined. This rigidity is already known from the rank-2 case.

Lemma 4.5
Let φ be a uniform rank-d phirotope on $\mathcal{E}$ and $F \subset \mathcal{E}$, $|F| = d-2$. For five indices a, b, c, d, $e \in \mathcal{E} \setminus F$, consider the set of cross ratio phases $\{\mathrm{cr}_\varphi(a, b \,|\, c, d)_F, \mathrm{cr}_\varphi(a, b \,|\, c, e)_F, \mathrm{cr}_\varphi(a, b \,|\, d, e)_F, \mathrm{cr}_\varphi(a, c \,|\, d, e)_F, \mathrm{cr}_\varphi(b, c \,|\, d, e)_F\}$. It either contains no real value, one real value, or only real values.

[1] One can easily check that the sign of the permutations can be read off the number of crossings. We were, however, unable to find a theorem that would guarantee a similar statement for permutations of the S_n and Hamilton cycles on K_n for further values of n. We know that the statement can only be true for $n \equiv 1 \mod 4$: If n is even, then cyclic permutations can lead to a change of sign (but the number of crossings in the Hamilton path will not change) and if $n \equiv 3 \mod 4$, then reversing the string in the second row in Cauchy's two line notation will also change the sign (but the cycles will only be traversed in the other direction which does not have any effect on the number of crossings).

Conjecture. For $n \equiv 1 \mod 4$, the parity of the number of crossings in the graphs generate as described above equals the parity of the corresponding permutations in the S_n.

Proof. To show this, assume that two cross ratio phases are real, w.l.o.g. $\mathrm{cr}_\varphi\,(a,b\,|\,c,d)_F \in \mathbb{R}$ and $\mathrm{cr}_\varphi\,(a,b\,|\,c,e)_F \in \mathbb{R}$. Then also $\mathrm{cr}_\varphi\,(a,b\,|\,d,e)_F \in \mathbb{R}$, as

$$\mathrm{cr}_\varphi\,(a,b\,|\,d,e)_F = \frac{\mathrm{cr}_\varphi\,(a,b\,|\,c,e)_F}{\mathrm{cr}_\varphi\,(a,b\,|\,c,d)_F}.$$

Let $F = \{f_1,\ldots,f_{d-2}\}$. Then, from the non-violation of the Grassmann-Plücker relation $(a,f_1,\ldots,f_{d-2}\,|\,c,b,d,f_1,\ldots,f_{d-2})$ it follows that

$$\mathrm{cr}_\varphi\,(a,b\,|\,c,d)_F \in \mathbb{R} \quad\Leftrightarrow\quad \mathrm{cr}_\varphi\,(a,c\,|\,b,d)_F \in \mathbb{R}.$$

By multiplying cross ratio phases obtained in this way, one can show that all cross ratio phases on the indices a, b, c, d, and e seen from F are real. $\square$

Lemma 4.6 (Rigidity of realisations of phirotopes)
Let φ be a realisable non-chirotopal uniform rank-d phirotope on $[n]$ and let the first $d+1$ indices be realised by the points $P_1,\ldots,P_{d+1} \in (\mathbb{C}^d \setminus \{0\})/\mathbb{R}^+$. This determines the position and phase of the all further points of the realisation.

Proof. As φ is non-chirotopal, there are at least $d+2$ indices such that their cross ratio phase is not real. Also, there is a non-real cross ratio phase containing the first $d+1$ indices because otherwise we could conclude with the Lemma 4.5 that φ is chirotopal. W.l.o.g., let this be the cross ratio phase $\mathrm{cr}_\varphi\,(d-1,d\,|\,d+1,d+2)_{[d-1]} \notin \mathbb{R}$. We want to determine the affine representative and the phase of the point $P \in (\mathbb{C}^d \setminus \{0\})/\mathbb{R}^+$ that realises the index $d+2$. Let $G = \{P_1,\ldots,P_{d-2}\}$ be the set of the first $d-2$ points of the realisation of φ. According to the Lemma 3.16, the cross ratio $\gamma = \mathrm{cr}(P_{d-1},P_d|P_{d+1},P)_G$ is then determined by φ. As the realisations P_{d-1}, P_d, and P_{d+1} of the indices $d-1$, d, and $d+1$ are known, then because of

$$\Big[[P_{d-1},P_{d+1},G]\cdot P_d - \gamma\cdot[P_d,P_{d+1},G]\cdot P_{d-1},P,G\Big] = 0$$

the realisation of P is known up to its phase. The phase of P can then be determined by any phirotope value of $\varphi|_{[d+2]}$ that contains the index $d+2$.

As $\mathrm{cr}_\varphi\,(d-1,d\,|\,d+1,d+2)_{[d-1]} \notin \mathbb{R}$, according to the Lemma 4.5, w.l.o.g. we obtain that $\mathrm{cr}_\varphi\,(d-1,d\,|\,d+1,k)_{[d-1]} \notin \mathbb{R}$ for $k \in \{d+2,\ldots,n\}$. Thus, all other points K of the realisation are fixed by the same argument that we used on P. $\square$

Theorem 4.7 (Realisability of uniform rank-d phirotopes)
Let φ be a uniform non-chirotopal rank-d phirotope on $\mathcal{E} = [n]$ with $d > 2$ and $n \geq d+2$. It is realisable, if and only if for all $(d+2)$-element subsets $F \subset \mathcal{E}$ the restriction $\varphi|_F$ is realisable.

Proof. It is clear that if φ is realisable, then all restrictions are realisable: Use the realisation of φ and omit all points but those corresponding to the indices in F.

The other direction is shown by induction on $|\mathcal{E}|$. To this end, let $n = d + 2$. Then $F = \mathcal{E}$ and the statement is trivial. This concludes the base case.

For the induction step, consider $\varphi_{\setminus\{n\}}$ and any $(d+2)$-element subset $F \subset \mathcal{E}$ such that $n \notin F$. It holds true that $\left(\varphi_{\setminus\{n\}}\right)|_F$ is realisable as $\left(\varphi_{\setminus\{n\}}\right)|_F = \varphi|_F$ is realisable by assumption. With the induction hypothesis, it follows that $\varphi_{\setminus\{n\}}$ is realisable. For the realisation of $\varphi_{\setminus\{n\}}$ we choose the first $d+1$ points $P_1, \ldots, P_{d+1}$ freely according to the Lemma 2.11. Their phases are then determined and the remaining points are fixed according to the Lemma 4.6 to $P_{d+2}, \ldots, P_{n-1}$, as φ is non-chirotopal and uniform. As $n \geq d+3$, we at least fixed one point and especially we fixed the point P_{n-1} that corresponds to the index $n-1$. We now examine $\varphi_{\setminus\{n-1\}}$. It is realisable with the same argument we used for the realisability of $\varphi_{\setminus\{n\}}$. We choose the phases and affine representatives of the points corresponding to the first $d+1$ indices exactly as we chose them for $\varphi_{\setminus\{n\}}$, namely $P_1, \ldots, P_{d+1}$. This places all points that were fixed before to the same points as before, namely to $P_{d+2}, \ldots, P_{n-2}$ and additionally fixes P_n, the point corresponding to index n.

The claim is that $(P_1, \ldots, P_n)$ is a realisation of φ. Clearly, we have

- $\forall(\lambda_1, \ldots, \lambda_d) \in \Lambda(\mathcal{E} \setminus \{n-1, n\}, d)$:
$$\varphi(\lambda_1, \ldots, \lambda_d) = \omega\left(\det(P_{\lambda_1}, \ldots, P_{\lambda_d})\right)$$
- $\forall(\lambda_1, \ldots, \lambda_{d-1}) \in \Lambda(\mathcal{E} \setminus \{n-1, n\}, d-1)$
$$\varphi(\lambda_1, \ldots, \lambda_{d-1}, n-1) = \omega\left(\det(P_{\lambda_1}, \ldots, P_{\lambda_{d-1}}, P_{n-1})\right)$$
- $\forall(\lambda_1, \ldots, \lambda_{d-1}) \in \Lambda(\mathcal{E} \setminus \{n-1, n\}, d-1)$:
$$\varphi(\lambda_1, \ldots, \lambda_{d-1}, n) = \omega\left(\det(P_{\lambda_1}, \ldots, P_{\lambda_{d-1}}, P_n)\right)$$

It remains to examine the phirotope values $\varphi(\lambda, n-1, n)$ for all sequences $\lambda \in \Lambda(\mathcal{E} \setminus \{n-1, n\}, d-2)$. To this end, consider the index $m_1 \in \mathcal{E} \setminus \{n-1, n\}$. In the realisation of the deletion $\varphi_{\setminus\{m_1\}}$ we choose $d+1$ points as they where chosen before and obtain by this the vector configuration $(P_1, \ldots, \widehat{P_{m_1}}, \ldots, P_{n-1}, P_n)$. This additionally guarantees

- $\forall(\lambda_1, \ldots, \lambda_{d-2}) \in \Lambda(\mathcal{E} \setminus \{n-1, n, m_1\}, d-2)$:
$$\varphi(\lambda_1, \ldots, \lambda_{d-2}, n-1, n) = \omega\left(\det(P_{\lambda_1}, \ldots, P_{\lambda_{d-2}}, P_{n-1}, P_n)\right).$$

To examine index sequences containing $n-1$, n, and m_1, we need to study a further deletion $\varphi_{\setminus\{m_2\}}$ with $m_2 \notin \{m, n, n-1\}$. And yet another deletion is needed to examine index sequences containing $n-1$, n, m_1, m_2. In the end, we consider $d+1$ deletions to cover all index sequences. This concludes the proof. □

In the subsequent sections, we will work on an analogon of the five-point condition in rank d. For this, we will need the following lemma.

Lemma 4.8 (Realisability of phirotopes on $d+2$ indices)
For $d \geq 3$ let φ be a non-chirotopal uniform rank-d phirotope on $\mathcal{E} = [d+2]$. It is realisable, if and only if for all $a, b, c, d, e \in \mathcal{E}$, $|\{a, b, c, d, e\}| = 5$, and with $A = \mathcal{E} \setminus \{a, b, c, d, e\}$ it holds true that

$$\sum_{\substack{\pi \in S_4(a,b,c,d) \\ \pi(a) < \pi(d)}} \Big(\mathrm{sign}(\pi) \cdot [\![\pi(a), \pi(b), \pi(c)]\!]_{/A} \cdot [\![\pi(b), \pi(c), \pi(d)]\!]_{/A} \cdot \\ [\![\pi(c), \pi(d), e]\!]_{/A} \cdot [\![\pi(d), e, \pi(a)]\!]_{/A} \cdot [\![e, \pi(a), \pi(b)]\!]_{/A} \Big) = 0.$$

This algebraic relation is called the *five-point condition* in rank d.

Proof. The phirotope φ is realisable, if and only if its dual φ^* is realisable, cf. Corollary 2.33. The phirotope φ^* is a uniform non-chirotopal rank-2 phirotope, cf. Lemma 3.17. Thus, according to the Lemma 4.3, it is realisable, if and only if for all $a, b, c, d, e \in \mathcal{E}$ it holds true that

$$\sum_{\substack{\pi \in S_4(a,b,c,d) \\ \pi(a) < \pi(d)}} \Big(\mathrm{sign}(\pi) \cdot \varphi^*(\pi(a), \pi(b))^2 \cdot \varphi^*(\pi(b), \pi(c))^2 \cdot \\ \varphi^*(\pi(c), \pi(d))^2 \cdot \varphi^*(\pi(d), e)^2 \cdot \varphi^*(e, \pi(a))^2 \Big) = 0$$

Exemplarily, we examine one of the terms. Let $\alpha \in \Lambda(A, d-3)$ be the ordered sequence of all indices in $\mathcal{E} \setminus \{a, b, c, d, e\}$.

$$\begin{aligned} \varphi^*(\pi(a), \pi(b))^2 &= \big(\varphi(\alpha, \pi(c), \pi(d), e) \cdot \mathrm{sign}(\pi(a), \pi(b), \alpha, \pi(c), \pi(d), e)\big)^2 \\ &= \varphi_{/A}(\pi(c), \pi(d), e)^2 \end{aligned}$$

Translating all other terms analogously will yield the claim. $\square$

Corollary 4.9 (Five-point condition in rank d)
For $d \geq 3$ let φ be a non-chirotopal uniform rank-d phirotope on $\mathcal{E} = [n]$, $n \geq d+2$. It is realisable, if and only if for every $n-d-2$ element subset $F \subset \mathcal{E}$ the deletion $\varphi_{\setminus F}$ satisfies the five-point condition in rank d.

Proof. The corollary is a direct consequence of the Theorem 4.7 and the Lemma 4.8. $\square$

Another formulation of the Corollary 4.9 is given in the following corollary.

Corollary 4.10
For $d \geq 3$ let φ be a non-chirotopal uniform rank-d phirotope on $\mathcal{E} = [n]$, $n \geq d+2$. It is realisable, if and only if for each $d-3$ element subset $A \subset E$ and all $a, b, c, d, e \in \mathcal{E}$ it holds true that

$$\sum_{\substack{\pi \in S_4(a,b,c,d) \\ \pi(a) < \pi(d)}} \Big(\operatorname{sign}(\pi) \cdot [\![\pi(a), \pi(b), \pi(c)]\!]_{/A} \cdot [\![\pi(b), \pi(c), \pi(d)]\!]_{/A} \cdot \\ [\![\pi(c), \pi(d), e]\!]_{/A} \cdot [\![\pi(d), e, \pi(a)]\!]_{/A} \cdot [\![e, \pi(a), \pi(b)]\!]_{/A} \Big) = 0.$$

Proof. In this corollary, the same sums are considered as in the Corollary 4.9. □

So the Corollary 4.9 and the Corollary 4.10 are basically the same. The creation of five-point conditions, however, might be more convenient according to the formulation of the Corollary 4.10, as the Corollary 4.10 supersedes the consideration of deletions. Another formulation that is substantially different from these two is obtained by considering the duals of the phirotopes at hand. It reads as follows:

Corollary 4.11
For $d \geq 3$ let φ be a non-chirotopal uniform rank-d phirotope on $\mathcal{E} = [n]$, $n \geq d+3$. It is realisable, if and only if for each $(d-2)$-element subset $A \subset E$ and all $a, b, c, d, e \in \mathcal{E}$ it holds true that

$$\sum_{\substack{\pi \in S_4(a,b,c,d) \\ \pi(a) < \pi(d)}} \Big(\operatorname{sign}(\pi) \cdot [\![\pi(a), \pi(b)]\!]_{/A} \cdot [\![\pi(b), \pi(c)]\!]_{/A} \cdot \\ [\![\pi(c), \pi(d)]\!]_{/A} \cdot [\![\pi(d), e]\!]_{/A} \cdot [\![e, \pi(a)]\!]_{/A} \Big) = 0.$$

Proof. For the proof, we translate the condition given in the Corollary 4.11 to the five-point condition for the dual phirotope. Exemplarily, for $A \subset \mathcal{E}$, $|A| = d-2$ and $a, b, c, d, e \in \mathcal{E} \setminus A$ consider $B = \mathcal{E} \setminus (A \cup \{a, b, c, d, e\})$. Then $|B| = n - d - 3$ and

$$\varphi_{/A}(d, e) = (\varphi^*)_{/B}(a, b, c)$$

and thus – with similar translations of the other terms –

$$\varphi_{/A}(d,e)^2 \cdot \varphi_{/A}(a,e)^2 \cdot \varphi_{/A}(a,b)^2 \cdot \varphi_{/A}(b,c)^2 \cdot \varphi_{/A}(c,d)^2 = 0 \tag{4.2}$$

$$\Leftrightarrow (\varphi^*)_{/B}(a,b,c)^2 \cdot (\varphi^*)_{/B}(b,c,d)^2 \cdot (\varphi^*)_{/B}(c,d,e)^2 \cdot \tag{4.3}$$

$$(\varphi^*)_{/B}(d,e,a)^2 \cdot (\varphi^*)_{/B}(e,a,b)^2 = 0. \tag{4.4}$$

All other summands translate similarly to summands of the five-point condition of the dual phirotope. Choosing different subsets A and choosing five indices a, b, c, d, e (not in A) will in total yield

$$\binom{n}{d-2} \cdot \binom{n-d-2}{5}$$

equations of the five-point condition that are not trivially true (as would be the case if the indices are also allowed to be elements in A). A short calculation shows that this equals the number of equations of the five-point condition for the dual phirotope that are not trivially true. This number is given by

$$\binom{n}{n-d-3} \cdot \binom{n-d}{5}.$$

As the translation in the Equation (4.4) is one-to-one, the five-point condition for the dual phirotope is satisfied if and only if the primal phirotope satisfies the formula given in the Corollary 4.11. □

4.2. Phirotopes with zeros

In this chapter, we will examine the realisability of phirotopes that contain at least one zero in their images. Furthermore, we will analyse the rigidity of the realisations. We will also examine realisations of not uniform phirotopes of arbitrary rank. We start our investigations of the realisability of not uniform phirotopes by considering rank-2 phirotopes.

Rank 2

Firstly, we will take care of the “small” phirotopes. For clarity’s sake, we will treat the cases of three, four and five indices separately.

Lemma 4.12 (Realisability of not uniform rank-2 phirotopes on three indices)
A rank-2 phirotope on three indices which is not uniform (i.e. contains at least one zero in its image) is always realisable.

Proof. According to the Theorem 3.31, a rank-2 phirotope on three indices is chirotopal. All rank-2 chirotopes are realisable (cf. Lemma 4.1). □

To simplify the proofs for phirotopes on four and five elements, we characterise the realisability of them with the help of the following lemma.

Lemma 4.13 (Characterisation of the realisability of not uniform rank-2 phirotopes on four or five indices)
Let $\mathcal{E} = [n]$ be an index set with $4 \leq n \leq 5$. A rank-2 phirotope φ on $\mathcal{E}$ with

$$\varphi(x, y) = 0 \Leftrightarrow \{x, y\} = \{a, b\}$$

is realisable, if and only if it holds true that

$$\frac{\varphi(a, x)}{\varphi(b, x)} = \frac{\varphi(a, y)}{\varphi(b, y)}. \tag{4.5}$$

for all $x, y \in \mathcal{E} \setminus \{a, b\}$.

This lemma states that the elements a and b forming a parallel element should behave similarly. More precisely, assume that there is a realisation in which a is realised by A and b by B. Then the quotients of the form $\frac{\varphi(a,x)}{\varphi(b,x)}$ all need to have the same value, namely $\frac{\omega_A}{\omega_B}$. One way of expressing this is to say that the relative positions of the other points seen from A and B are the same, except that they both measure comparatively to their own phase ω_A and ω_B, respectively.

Proof of Lemma 4.13. Firstly, we show that the Condition (4.5) is necessary for the realisability of φ. For this, let φ be realised by $V \subset \mathbb{C}^2/\mathbb{R}^+$. Assume, w.l.o.g., $\varphi(a, b) = 0$ and a and b are to be realised by $A \in V$ and $B \in V$, respectively. Then, the affine representatives p_A and p_B of A and B are the same. Additionally,

$$\varphi(a, x) = \omega_A \cdot \omega_X \cdot \omega(p_A - p_X)$$
$$\varphi(b, x) = \omega_B \cdot \omega_X \cdot \omega(p_B - p_X)$$

for all $x \in \mathcal{E}$ with corresponding realisation $X \in V$, affine representative $p_X \in \mathbb{C}$, and phase ω_X. It follows that

$$\frac{\varphi(a, x)}{\varphi(b, x)} = \frac{\omega_A}{\omega_B} \quad \forall x \in \mathcal{E} \setminus \{a, b\},$$

which implies the claim.

For the sufficiency of condition (4.5), consider $\varphi_{\setminus\{b\}}$ and let $A, P_3, \ldots, P_n$ be the realisations of a, and $3, \ldots, n$, respectively. These can be found according to the Lemma 4.2. Let B be given by

$$p_B = p_A, \quad \text{and} \quad \omega_B = \omega_A \cdot \frac{\varphi(b, 3)}{\varphi(a, 3)},$$

where p_A and p_B are the affine representatives of A and B, respectively. Then, the vector configuration $(A, B, P_3, \ldots, P_n)$ realises φ. To verify this, consider the following calculations.

$$\begin{aligned}
\omega([A,B]) &= \omega_A \cdot \omega_B \cdot \omega(p_A - p_B) = 0 = \varphi(a,b), \\
\omega([B,P_3]) &= \omega_B \cdot \omega_3 \cdot \omega(p_B - p_3) \\
&= \omega_A \cdot \frac{\varphi(b,3)}{\varphi(a,3)} \cdot \omega_3 \cdot \omega(p_A - p_3) \\
&= \omega_A \cdot \frac{\varphi(b,3)}{\varphi(a,3)} \cdot \omega_3 \cdot \frac{\varphi(a,3)}{\omega_A \cdot \omega_3} \\
&= \varphi(b,3)
\end{aligned}$$

In the second to last line, we used that a and 3 are realised by A and P_3, and thus $\varphi(a,3) = \omega_A \cdot \omega_3 \cdot \omega(p_A - p_3)$. Verifying $\omega([B,P_4]) = \varphi(b,4)$ (and $\omega([B,P_5]) = \varphi(b,5)$, if $|\mathcal{E}| = 5$) works analogously by using condition (4.5) in order to replace ω_B by $\omega_A \cdot \frac{\varphi(b,4)}{\varphi(a,4)}$ (and $\omega_A \cdot \frac{\varphi(b,5)}{\varphi(a,5)}$, respectively). □

The following lemma illustrates a consequence of having zeros on the Grassmann-Plücker relations. As we will use it in rank 2 and 3 later, we will formulate it for both ranks right away.

Lemma 4.14

Let φ be a rank-2 phirotope on $[n]$. If for some $a, b \in \mathcal{E}$ it holds true that $\varphi(a,b) = 0$, then

$$\varphi(a,c)\varphi(b,d) = \varphi(a,d)\varphi(b,c)$$

for all $c, d \in \mathcal{E}$.

Let φ be a rank-3 phirotope on $[n]$. If for some $a, b, c \in \mathcal{E}$ it holds true that $\varphi(a,b,c) = 0$, then

$$\varphi(a,b,d)\varphi(a,c,e) = \varphi(a,b,e)\varphi(a,c,d)$$

for all $d, e \in \mathcal{E}$.

Proof. According to axiom ($\varphi 2$), the Grassmann-Plücker relation $(a \mid b, c, d)$ must not be obviously violated: As $\varphi(a,b) = 0$, there are $r_1, r_2 \in \mathbb{R}^+$ such that

$$r_1 \cdot \varphi(a,c) \cdot \varphi(b,d) - r_2 \cdot \varphi(a,d) \cdot \varphi(b,c) = 0.$$

As the phirotope values are contained in $\mathcal{S}^1$, we obtain that

$$r_1 = r_2 = 1.$$

With this, the claim follows for rank 2. The rank-3 case works analogously. □

Lemma 4.15 (Realisability of not uniform rank-2 phirotopes on four indices)
A rank-2 phirotope on $\mathcal{E} = \{a, b, c, d\}$ which is not uniform (i.e. contains at least one zero in its image) is always realisable.

Proof. If φ contains a loop (w.l.o.g. index d), realise $\varphi_{\setminus\{d\}}$ according to the Lemma 4.12. This realisation is complemented by the zero vector, which is to realise d, to form a realisation of φ.

Now assume that φ does not contain a loop. We split the proof into parts, according to the number of zeros in the image of φ.

Exactly one zero. W.l.o.g., let $\varphi(a, b) = 0$. With the Lemma 4.14, it follows that

$$\frac{\varphi(a,c)}{\varphi(b,c)} = \frac{\varphi(a,d)}{\varphi(b,d)},$$

which is the Condition (4.5).

Exactly two zeros. There is no loop. As $\varphi(a, b) = 0$ and $\varphi(a, c) = 0$ would imply that $\varphi(a, d) = 0$ or $\varphi(b, c) = 0$ (the Grassmann-Plücker relation $(a \,|\, b, c, d)$ must not be violated), the only possibility of having two zeros in the image of φ is up to renaming of the elements:

$$\varphi(a, b) = 0, \quad \varphi(c, d) = 0.$$

All other pairs are mapped to non-zero values. Choosing the affine representatives according to $p_A = p_B$ and $p_C = p_D$ but otherwise arbitrarily, we have four variables $\omega_A, \omega_B, \omega_C, \omega_D$ left that we can choose in a way that the following four equations are satisfied:

$$\begin{aligned}
\varphi(a, c) &= \omega_A \cdot \omega_C \cdot \omega(p_A - p_C),\\
\varphi(a, d) &= \omega_A \cdot \omega_D \cdot \omega(p_A - p_D),\\
\varphi(b, c) &= \omega_B \cdot \omega_C \cdot \omega(p_B - p_C),\\
\varphi(b, d) &= \omega_B \cdot \omega_D \cdot \omega(p_B - p_D).
\end{aligned}$$

Exactly three zeros. For this case, let $\varphi(a, b) = \varphi(a, c) = \varphi(b, c) = 0$. We choose the affine representatives p_A and p_D arbitrarily, and $p_B = p_C = p_A$. With this, we have four variables ω_A, ω_B, ω_C, ω_D left that we can choose in a way that the following three equations are satisfied:

$$\begin{aligned}
\varphi(a, d) &= \omega_A \cdot \omega_D \cdot \omega(p_A - p_D),\\
\varphi(b, d) &= \omega_B \cdot \omega_D \cdot \omega(p_B - p_D),\\
\varphi(c, d) &= \omega_C \cdot \omega_D \cdot \omega(p_C - p_D).
\end{aligned}$$

More than three zeros are not possible, as in this case the absence of loops would demand that $\varphi \equiv 0$. $\square$

Lemma 4.16

A rank-2 phirotope on $\mathcal{E} = \{a, b, c, d, e\}$ that is not uniform (i.e. contains at least one zero in its image) is always realisable.

Proof. If φ contains a loop (w.l.o.g. index e), realise $\varphi_{\setminus\{e\}}$ according to Lemma 4.15. This realisation is complemented by the zero vector, which is to realise e, to form a realisation of φ.

Now, assume that φ has no loop. Again, we split the proof into parts according to the number of zeros in the image of φ.

Exactly one zero. Let φ be a rank-2 phirotope on $\mathcal{E} = \{a, b, c, d, e\}$, where, w.l.o.g., $\varphi(a, b) = 0$. Due to the Lemma 4.14, it holds true that

$$\frac{\varphi(a,c)}{\varphi(b,c)} = \frac{\varphi(a,d)}{\varphi(b,d)} = \frac{\varphi(a,e)}{\varphi(b,e)},$$

which is exactly the Condition (4.5) for φ.

More than one zero. Let φ be a rank-2 phirotope on $\mathcal{E} = \{a, b, c, d, e\}$ with at least two zeros in its image. W.l.o.g., let $\varphi(a, b) = 0$ be one of the phirotope values that is mapped to zero. Consider $\varphi_{\setminus\{a\}}$. It is realisable due to the Lemma 4.15. Let the realisation be given by B, C, D, $E \in \mathbb{C}^2/\mathbb{R}^+$. As φ does not contain a loop, there is an index $i \in \{c, d, e\}$ such that $\varphi(a, i) \neq 0$ and $\varphi(b, i) \neq 0$:
Assume there is no index $i \in \{c, d, e\}$ such that $\varphi(a, i) \neq 0$ and $\varphi(b, i) \neq 0$. W.l.o.g., the phirotope has the following non-bases: $\varphi(a, e) = 0$, $\varphi(a, d) = 0$, and $\varphi(b, c) = 0$. As a must not be a loop, the non-violation of the Grassmann-Plücker relation $(a \,|\, b, e, c)$ yields that $\varphi(b, c) = 0$. Then, the non-violation of $(b \,|\, c, e, d)$ yields a contradiction to the requirement that b and e must not be loops either. Thus, let $i \in \{c, d, e\}$ be an index such that $\varphi(a, i) \neq 0$ and $\varphi(b, i) \neq 0$.

Let $A \in \mathbb{C}^2/\mathbb{R}^+$ be given via its affine representative $p_A = p_B$ that equals the affine representative of B and the phase

$$\omega_A = \omega_B \cdot \frac{\varphi(a,i)}{\varphi(b,i)}. \tag{4.6}$$

If there is another index $j \in \{b, c, d, e\}$, $j \neq i$, such that $\varphi(a, j) \neq 0$, then due to the non-violation of the Grassmann-Plücker relations we have

$$\frac{\varphi(a,i)}{\varphi(b,i)} = \frac{\varphi(a,j)}{\varphi(b,j)}.$$

With this substitution for the last term of (4.6), checking that (A, B, C, D, E) is a realisation of φ is a straightforward calculation. $\square$

Lemma 4.17 (Necessity of the five-point condition)
The five-point condition is necessary for the realisability of rank-2 phirotopes.

Proof. As the uniform case was dealt with in Lemma 4.3, we will here focus on the case that the image of φ contains at least one zero.

If a phirotope is realisable, then every minor on five indices is realisable. Therefore, it suffices to show that each equation of the five-point condition is necessary for the realisability of such a minor.

With φ we denote a realisable phirotope on $\mathcal{E} = \{a, b, c, d, e\}$. Let a realisation of φ be given by $V = (A, B, C, D, E) \subset \mathbb{C}^2/\mathbb{R}^+$. Furthermore, let φ be not uniform, that means there is at least one pair of indices mapped to zero.
We start with exactly one value being zero. W.l.o.g., let $\varphi(a, b) = 0$. This reduces the five-point condition to

$$\begin{aligned}
&-[\![a,c]\!][\![c,b]\!][\![b,d]\!][\![d,e]\!][\![e,a]\!] + [\![b,c]\!][\![c,a]\!][\![a,d]\!][\![d,e]\!][\![e,b]\!]\\
&-[\![c,a]\!][\![a,d]\!][\![d,b]\!][\![b,e]\!][\![e,c]\!] + [\![a,d]\!][\![d,b]\!][\![b,c]\!][\![c,e]\!][\![e,a]\!]\\
&-[\![a,d]\!][\![d,c]\!][\![c,b]\!][\![b,e]\!][\![e,a]\!] + [\![a,c]\!][\![c,d]\!][\![d,b]\!][\![b,e]\!][\![e,a]\!] = 0.
\end{aligned}$$

As $[\![x,y]\!] = [\![y,x]\!]$ for all x, y, this can be factorized as follows:

$$\begin{aligned}
&[\![a,c]\!][\![b,c]\!][\![d,e]\!] \cdot ([\![a,d]\!][\![b,e]\!] - [\![a,e]\!][\![b,d]\!])\\
+&[\![a,d]\!][\![b,d]\!][\![c,e]\!] \cdot ([\![a,e]\!][\![b,c]\!] - [\![a,c]\!][\![b,e]\!])\\
+&[\![a,e]\!][\![b,e]\!][\![c,d]\!] \cdot ([\![a,c]\!][\![b,d]\!] - [\![a,d]\!][\![b,c]\!]) = 0
\end{aligned} \tag{4.7}$$

With the Lemma 4.14, we see that all the terms in parentheses vanish. Thus, the five-point condition always holds true for not uniform minors. Note that this is in line with the Lemma 4.16 which states that a rank-2 phirotopes on five indices that contains at least one zero in its image is always realisable. ☐

Theorem 4.18 (Realisability for non-uniform rank-2 phirotopes)
A rank-2 phirotope on $\mathcal{E} = [n]$ with $n \geq 6$ is realisable, if and only if it satisfies the five-point condition.

Proof. Remember that rank-2 chirotopes are always realisable and always satisfy the five-point condition.

The necessity of the five-point condition is shown in the Lemma 4.17. Furthermore, the sufficiency in the uniform case is shown in Lemma 4.3. We still have to show that the realisations of all uniform minors fit together and that the elements that are part of a non-basis of φ fit into the realisation. Again, this will be a rather technical proof.

Consider phirotopes having at least one zero in their image. Let $\{\ell_1, \ldots, \ell_s\}$ be the set of loops of φ. Firstly, we will realise $\varphi_{\backslash\{\ell_1,\ldots,\ell_s\}}$ and then insert the loops to complete the realisation of φ afterwards. To simplify the notation, we use the shorthand:

$$\psi := \varphi_{\backslash\{\ell_1,\ldots,\ell_s\}} \qquad \text{and} \qquad \mathcal{F} := \mathcal{E} \smallsetminus \{\ell_1, \ldots, \ell_s\}.$$

Choose any $m_1 \in \mathcal{F}$ such that there is an $x \in \mathcal{F} \smallsetminus \{m_1\}$ such that

$$\psi(m_1, x) = 0$$

If $\psi_{\backslash\{m_1\}}$ is not uniform, choose $m_2 \in \mathcal{F} \smallsetminus \{m_1\}$ such that there is an $x \in \mathcal{F} \smallsetminus \{m_1, m_2\}$ such that

$$\psi_{\backslash\{m_1\}}(m_2, x) = 0.$$

We iterate this until $\psi_{\backslash\{m_1,\ldots,m_k\}}$ is either a uniform phirotope or a non-uniform phirotope on five indices. In both bases $\psi_{\backslash\{m_1,\ldots,m_k\}}$ is realisable, see Lemmas 4.3 and 4.16. Let this realisation be given by $(P_1, \ldots, P_{n-k-s})$. Step by step, we are going to extend this realisation to a realisation of the phirotopes $\psi_{\backslash\{m_1,\ldots,m_{k-1}\}}, \ldots, \psi_{\backslash\{m_1\}}$, and ψ, respectively.

For each $m_i \in \{m_1, \ldots, m_k\}$ choose an $\widetilde{m_i} \in \mathcal{F} \smallsetminus \{m_1, \ldots, m_k\}$ such that it holds true that $\varphi(m_i, \widetilde{m_i}) = 0$. These exist because ψ does not contain a loop: In general, if there are no loops, then from $\varphi(a, b) = 0$ and $\varphi(a, c) = 0$ and with the non-violation of the Grassmann-Plücker relations $(a \,|\, b, c, x)$ for all remaining indices x, it follows that $\varphi(b, c) = 0$. Note that it might be the case that $|\{\widetilde{m_1}, \ldots, \widetilde{m_k}\}| < k$. It is not possible that there is one index i such that $\psi(i, x) = 0$ for all $x \in \mathcal{F}$ as ψ does not contain loops. Thus, for the element m_k there is an index $e_k \in \mathcal{F} \smallsetminus \{m_1, \ldots, m_{k-1}\}$ such that $\psi(m_k, e_k) \neq 0$. The absence of loops also guarantees that $\psi(\widetilde{m_k}, e_k) \neq 0$. Let Q_k be the point whose affine representative q_k is the same as the affine representative p_k of $P_{\widetilde{m_k}}$ and whose phase is given by

$$\omega_{Q_k} = \omega_k \cdot \frac{\psi(m_k, e_k)}{\psi(\widetilde{m_k}, e_k)},$$

where ω_k is the phase of $P_{\widetilde{m_k}}$.
Then, $(P_1, \ldots, P_{n-k-s}, Q_k)$ realises the phirotope $\psi_{\backslash\{m_1,\ldots,m_{k-1}\}}$ on $\mathcal{F}_k = \mathcal{F} \smallsetminus \{m_1, \ldots, m_{k-1}\}$. To see this, check that $\omega(\det(Q_k, P_j)) = \psi(m_k, j)$ for all $j \in \mathcal{F}_k$:
Firstly, consider an arbitrary index $j \in \mathcal{F}_k$ such that $\psi(m_k, j) \neq 0$. Let its realisation P_j have

phase ω_j and affine representative p_j. Then,

$$\begin{aligned}\omega(\det(Q_k, P_j)) &= \omega_j \cdot \omega_{Q_k} \cdot \omega(q_k - p_j) \\ &= \omega_j \cdot \omega_k \cdot \frac{\psi(m_k, e_k)}{\psi(\widetilde{m_k}, e_k)} \cdot \omega(q_k - p_j).\end{aligned}$$

With the Lemma 4.14 and $\psi(m_k, \widetilde{m_k}) = 0$ it holds true that

$$\psi(m_k, j) \cdot \psi(\widetilde{m_k}, e_k) = \psi(m_k, e_k) \cdot \psi(\widetilde{m_k}, j).$$

Thus,

$$\begin{aligned}\omega(\det(Q_k, P_j)) &= \omega_j \cdot \omega_k \cdot \frac{\psi(m_k, j)}{\psi(\widetilde{m_k}, j)} \cdot \omega(q_k - p_j) \\ &= \omega_j \cdot \omega_k \cdot \frac{\psi(m_k, j)}{\psi(\widetilde{m_k}, j)} \cdot \omega(p_k - p_j) \\ &= \psi(m_k, j).\end{aligned}$$

For all indices $j \in \mathcal{F}_k$ such that $\psi(m_k, j) = 0$ it holds true that $\psi(\widetilde{m_k}, j) = 0$ as ψ does not contain loops and thus the affine representatives q_k, p_k and p_j are all the same. Thus,

$$\omega(\det(Q_k, P_j)) = 0 = \psi(m_k, j).$$

Adding the points $Q_1, \ldots, Q_{k-1}$ to realise the indices $m_1, \ldots, m_{k-1}$ works analogously. It remains to check that for any two $Q_i, Q_j \in \{Q_1, \ldots, Q_k\}$ it holds true that $\omega(\det(Q_i, Q_j)) = \psi(m_i, m_j)$. Firstly, consider the case in which $\psi(m_i, m_j) \neq 0$:

$$\omega(\det(Q_i, Q_j)) = \omega_i \cdot \omega_j \cdot \frac{\psi(m_i, e_i)}{\psi(\widetilde{m_i}, e_i)} \cdot \frac{\psi(m_j, e_j)}{\psi(\widetilde{m_j}, e_j)} \cdot \omega(p_i - p_j)$$

The non-violation of the Grassmann-Plücker relations yields

$$\frac{\psi(m_i, e_i)}{\psi(\widetilde{m_i}, e_i)} = \frac{\psi(m_i, m_j)}{\psi(\widetilde{m_i}, m_j)}, \qquad \frac{\psi(m_j, e_j)}{\psi(\widetilde{m_j}, e_j)} = \frac{\psi(m_j, m_i)}{\psi(\widetilde{m_j}, m_i)}.$$

Hence,

$$\omega(\det(Q_i, Q_j)) = \omega_i \cdot \omega_j \cdot \frac{\psi(m_i, m_j)}{\psi(\widetilde{m_i}, m_j)} \cdot \frac{\psi(m_j, m_i)}{\psi(\widetilde{m_j}, m_i)} \cdot \omega(p_i - p_j).$$

Again, axiom $(\varphi 2)$ yields

$$\psi(\widetilde{m_i}, m_j) \cdot \psi(\widetilde{m_j}, m_i) = \psi(\widetilde{m_i}, \widetilde{m_j}) \cdot \psi(m_j, m_i),$$

which leads to

$$\begin{aligned}\omega(\det(Q_i, Q_j)) &= \omega_i \cdot \omega_j \cdot \frac{\psi(m_i, m_j) \cdot \psi(m_j, m_i)}{\psi(\widetilde{m}_i, \widetilde{m}_j) \cdot \psi(m_j, m_i)} \cdot \omega(p_i - p_j) \\ &= \omega_i \cdot \omega_j \cdot \frac{\psi(m_i, m_j)}{\psi(\widetilde{m}_i, \widetilde{m}_j)} \cdot \omega(p_i - p_j) \\ &= \psi(m_i, m_j)\end{aligned}$$

If, however, $\psi(m_i, m_j) = 0$, then (due to the construction of m_i and m_j) there is an index $m \in \mathcal{F} \setminus \{m_1, \ldots, m_k\}$ such that $\psi(m_i, m) = 0$ and $\psi(m_j, m) = 0$. The index m is realised by a point $P_{\widetilde{m}}$ with affine representative p_m that equals both the affine representative of Q_i and Q_j. With this, ψ is realised on $\mathcal{F}$.

A realisation of φ on $\mathcal{E}$ is obtained by realising all indices $\ell_1, \ldots, \ell_s$ by the zero vector and choosing arbitrary phases. This concludes the proof. □

Higher Ranks

The rank-2 case provides a basis for the examination of the realisability of not uniform phirotopes of arbitrary rank. The first result regards phirotopes on relatively small index sets.

Lemma 4.19
Every rank-d phirotope on d or $d+1$ indices is realisable.

Proof. Due to the phirotope axiom (φ0), every phirotope contains at least one basis. Thus, a phirotope on d indices is uniform and realisable due to the Lemma 4.4.

Consider a phirotope φ on $d+1$ indices. If it is not a near-pencil, it is realisable due to the Lemma 4.4. The following proof includes the near-pencil case.

W.l.o.g., let $\{1, \ldots, d\}$ be a basis of φ. Choose a realisation of $\varphi|_{\{1,\ldots,d\}}$ according to the Lemma 2.11. As $\{1, \ldots, d\}$ is a basis of φ, the corresponding affine representatives linearly span $\mathbb{C}^d$. Therefore, the affine representative $p_{d+1} \in \mathbb{C}^d$ of the last point is a linear combination of the d affine representatives $p_1, \ldots, p_d$:

$$p_{d+1} = \sum_{i=1}^{d} \alpha_i \cdot p_i.$$

This yields $d+1$ unknowns: all parameters α_i and the phase ω_{d+1}. The phirotope values containing the index $d+1$ yield d equations in these unknowns. Thus, we can find an affine representative p_{d+1} and a phase ω_{d+1} such that all conditions are met. □

Lemma 4.20 (Rigidity in the non-uniform case)
Let φ be a realisable non-chirotopal rank-d phirotope that is not a near-pencil. If the coordinates of $d+1$ points in general position of the realisation are fixed, then the positions and phases of all other points are fixed as well.

Proof. The phirotope is not a near-pencil. Thus, w.l.o.g., $\varphi|_{\{1,\ldots d+1\}}$ is uniform. W.l.o.g., the first $d+1$ points are realised by the unit vectors and the vector $\begin{pmatrix}1, & \ldots, & 1\end{pmatrix}^T$, otherwise apply a reorientation and a projective transformation. We show that the position and phase of the next point, that is the one realising the index $d+2$, are determined. Consider its coordinates $\begin{pmatrix}x_1, & \ldots, & x_{d-1}, & 1\end{pmatrix}^T$. According to Lemma 2.32, the dual phirotope $(\varphi|_{[d+2]})^*$ is realised by the columns of

$$\begin{pmatrix} -1 & \ldots & -1 & -1 & 1 & 0 \\ -x_1 & \ldots & -x_{d-1} & -1 & 0 & 1 \end{pmatrix}$$

Thus, we prove that if three points in rank 2 are fixed to $A = \begin{pmatrix}1\\0\end{pmatrix}$, $B = \begin{pmatrix}0\\1\end{pmatrix}$, and $C = \begin{pmatrix}1\\1\end{pmatrix}$, then the coordinates of the fourth point D are determined:
If among the indices a, b, c, and d corresponding to the four points there is no φ-dependent pair, then D is determined according to the Lemma 4.6, as the minor $\varphi|_{\{a,b,c,d\}}$ is uniform and non-chirotopal. If there exists one index $z \in \{a, b, c\}$ such that $\varphi(z, d) = 0$, then the coordinates of D are the same as that of the point corresponding to z. If there are two indices $z, y \in \{a, b, c, d\}$, $z \neq y$, such that $\varphi(z, d) = 0 = \varphi(y, d)$, then either $D = \begin{pmatrix}0\\0\end{pmatrix}$ or the affine representatives of the points corresponding to z and y are the same – which is a contradiction to them being in general position.
In all cases, the position of D is determined. Thus, all values $x_1, \ldots, x_{d-1}$ are determined and, hence, the affine representative of the point corresponding to the index $d+2$ in the primal phirotope are determined. The phase is then determined, for example, by $\varphi(1, \ldots, d-1, d+2)$.

With this technique, we can find coordinates for each point of the realisation. As we assumed that the phirotope is realisable, there can be no contradictions within the newly added points. This is guaranteed by the five-point condition. $\square$

Theorem 4.21 (Five-point condition for phirotopes on $d+2$ elements)
Let φ be a rank-d phirotope on $[d+2]$ that is not a near-pencil. It is realisable, if and only if it satisfies the five-point condition in rank d.

Proof. Note that rank-d chirotopes on $d+2$ indices are always realisable (cf. Lemma 4.1) and always satisfy the five-point condition.

The Corollary 2.33 states that φ is realisable, if and only if its dual φ^* is realisable. The dual φ^* is a non-chirotopal rank-2 phirotope. According to the Theorem 4.18, φ^* is realisable, if and only if it satisfies the five-point condition. In the proof of the Lemma 4.8, we have already seen how the five-point condition for the dual translates to a five-point condition in rank d for the primal phirotope. $\square$

Theorem 4.22 (Realisability of rank-d phirotopes)
Let φ be a non-chirotopal rank-d phirotope on $\mathcal{E} = [n]$ with $n \geq d+2$ that is not a near-pencil. It is realisable, if and only if for all $d+2$ element subsets $F \subset \mathcal{E}$ the restriction $\varphi|_F$ is realisable.

Proof. It is clear that if φ is realisable, then all deletions are realisable as deletion preserves realisability, cf. Lemma 3.10.

The other direction is shown by induction on n. To this end, let $n = d+2$. Then $F = \mathcal{E}$ and the statement is trivial.

For the induction step, consider a non-chirotopal rank-d phirotope on $\mathcal{E} = [n]$ that is not a near-pencil. There are indices $a, b, c, d \in \mathcal{E}$ and a set $A \subset \mathcal{E}$, $|A| = d-2$ such that

$$\varphi(a,c,A) \neq 0, \qquad \varphi(b,d,A) \neq 0, \qquad \varphi(a,d,A) \neq 0, \qquad \varphi(b,c,A) \neq 0,$$
$$\text{and } \operatorname{cr}_\varphi(ab \mid c,d)_A \notin \mathbb{R}.$$

Choose an index x not in $\{a,b,c,d\} \cup A$ and consider $\varphi_{\setminus\{x\}}$. This phirotope is also non-chirotopal and not a near-pencil. For every $d+2$ element subset $F \subset \mathcal{E}$ with $x \notin F$, it holds true that $\varphi|_F$ is realisable and, thus, the restriction $\left(\varphi_{\setminus\{x\}}\right)|_F$ is realisable. By induction hypothesis, it follows that $\varphi_{\setminus\{x\}}$ is realisable. W.l.o.g., let $[d+1]$ be a basis of φ and let a realisation of this basis be given by $P_1, \dots, P_{d+1} \in \mathbb{C}^d/\mathbb{R}^+$. According to the Lemma 4.20, the realisations of all other points are fixed, as the phirotope is non-chirotopal. Also, the realisation of the index x is determined by the restriction $\varphi|_{\{1,\dots,d+1,x\}}$.

What is left to show is that the realisation X of the index x conforms with the whole realisation. All phirotope values of the form $\varphi(\lambda, x, a)$ where $\lambda \in \Lambda([d+1], d-2)$ and $a \in \mathcal{E} \setminus ([d+1] \cup \{x\})$ conform with the realisation, as $\varphi|_{\{\lambda_1,\dots,\lambda_{d-2},x,a\}}$ is a realisable rank-2 phirotope and, thus, satisfies the five-point condition: The phirotope of the vector configuration given by the realisation also satisfies the five-point condition. Thus, the two five-point conditions agree on all but one terms and hence

$$\varphi(\lambda, x, a) = \omega\big(\det(P_{\lambda_1}, \ldots, P_{\lambda_{d-2}}, X, A)\big),$$

where A is the realisation of a. We iterate this to show that for any index $b \in \mathcal{E} \setminus ([d+1] \cup \{x, y\})$ realised by B we have

$$\varphi(\lambda, x, a, b) = \omega\big(\det(P_{\lambda_1}, \ldots, P_{\lambda_{d-3}}, X, A, B)\big)$$

for all $\lambda \in \Lambda([d+1], d-3)$. This is the case because for every λ the term $\varphi(\lambda, x, a, b)$ is the only term on which the five-point condition of $\varphi|_{\{\lambda_1, \ldots, \lambda_{d+3}, x, a, b\}}$ and that of the vector configuration $(P_{\lambda_1}, \ldots, P_{\lambda_{d-3}}, X, A, B)$ do not already agree. We repeat this pattern to iteratively guarantee that all remaining phirotope values conform with the realisation. $\square$

Next, we want to analyse the effects of having zeros in the images of a rank-3 phirotope on the corresponding five-point conditions. To this end, consider a rank-3 phirotope φ on $\mathcal{E} = \{a, b, c, d, e\}$. If $\varphi(a, b, c) = 0$, the five-point condition reads as follows:

$$\begin{aligned}
0 = &+ [\![a,b,c]\!][\![b,c,d]\!][\![c,d,e]\!][\![d,e,a]\!][\![e,a,b]\!] - [\![a,b,d]\!][\![b,d,c]\!][\![d,c,e]\!][\![c,e,a]\!][\![e,a,b]\!] \\
&- [\![a,c,b]\!][\![c,b,d]\!][\![b,d,e]\!][\![d,e,a]\!][\![e,a,c]\!] + [\![a,c,d]\!][\![c,d,b]\!][\![d,b,e]\!][\![b,e,a]\!][\![e,a,c]\!] \\
&+ [\![a,d,b]\!][\![d,b,c]\!][\![b,c,e]\!][\![c,e,a]\!][\![e,a,d]\!] - [\![a,d,c]\!][\![d,c,b]\!][\![c,b,e]\!][\![b,e,a]\!][\![e,a,d]\!] \\
&- [\![b,a,c]\!][\![a,c,d]\!][\![c,d,e]\!][\![d,e,b]\!][\![e,b,a]\!] + [\![b,a,d]\!][\![a,d,c]\!][\![d,c,e]\!][\![c,e,b]\!][\![e,b,a]\!] \\
&+ [\![b,c,a]\!][\![c,a,d]\!][\![a,d,e]\!][\![d,e,b]\!][\![e,b,c]\!] + [\![c,a,b]\!][\![a,b,d]\!][\![b,d,e]\!][\![d,e,c]\!][\![e,c,a]\!] \\
&- [\![c,a,d]\!][\![a,d,b]\!][\![d,b,e]\!][\![b,e,c]\!][\![e,c,a]\!] - [\![c,b,a]\!][\![b,a,d]\!][\![a,d,e]\!][\![d,e,c]\!][\![e,c,b]\!] \\
= &- [\![a,b,d]\!][\![b,d,c]\!][\![d,c,e]\!][\![c,e,a]\!][\![e,a,b]\!] + [\![a,c,d]\!][\![c,d,b]\!][\![d,b,e]\!][\![b,e,a]\!][\![e,a,c]\!] \\
&+ [\![a,d,b]\!][\![d,b,c]\!][\![b,c,e]\!][\![c,e,a]\!][\![e,a,d]\!] - [\![a,d,c]\!][\![d,c,b]\!][\![c,b,e]\!][\![b,e,a]\!][\![e,a,d]\!] \\
&+ [\![b,a,d]\!][\![a,d,c]\!][\![d,c,e]\!][\![c,e,b]\!][\![e,b,a]\!] - [\![c,a,d]\!][\![a,d,b]\!][\![d,b,e]\!][\![b,e,c]\!][\![e,c,a]\!]
\end{aligned}$$

This factorises as follows:

$$\begin{aligned}
0 = &+ [\![a,b,d]\!][\![c,d,e]\!][\![e,a,b]\!] \cdot \big([\![a,d,c]\!][\![c,e,b]\!] - [\![b,d,c]\!][\![c,e,a]\!]\big) \\
&+ [\![a,c,d]\!][\![d,b,e]\!][\![e,a,c]\!] \cdot \big([\![a,d,b]\!][\![b,e,a]\!] - [\![a,d,b]\!][\![b,e,c]\!]\big) \\
&+ [\![d,b,c]\!][\![b,c,e]\!][\![e,a,d]\!] \cdot \big([\![a,d,b]\!][\![c,e,a]\!] - [\![a,d,c]\!][\![b,e,a]\!]\big)
\end{aligned} \tag{4.8}$$

The terms in parentheses yield zeros as they stem from the Grassmann-Plücker relations $(c, a \mid c, b, d, e)$, $(b, c \mid b, a, d, e)$, and $(a, b \mid a, c, d, e)$, respectively: With the Lemma 4.14 and $\varphi(a, b, c) = 0$ it holds true that

$$\begin{aligned}
&(c, a \mid c, b, d, e) && \Rightarrow \varphi(c, a, d)\varphi(c, b, e) = \varphi(c, a, e)\varphi(c, b, d),\\
&(b, c \mid b, a, d, e) && \Rightarrow \varphi(b, c, d)\varphi(b, a, e) = \varphi(b, c, e)\varphi(b, a, d),\\
&(a, b \mid a, c, d, e) && \Rightarrow \varphi(a, b, d)\varphi(a, c, e) = \varphi(a, b, e)\varphi(a, c, d).
\end{aligned}$$

Rearranging indices and squaring both sides of each equation shows that the Equation 4.8 holds true trivially. In fact, this statement can be generalised to higher ranks, which leads to the following theorem:

Theorem 4.23 (Realisability of non-uniform phirotopes)
A non-chirotopal rank-d phirotope is realisable, if and only if all its uniform rank-d minors are realisable.

Proof. Let φ be a non-chirotopal uniform rank-d phirotope on $[n]$. It is realisable, if and only if for all $(d-3)$-element subsets $A \subset [n]$ and all $a, b, c, d, e \in \mathcal{E}$, $|\{a, b, c, d, e\}| = 5$ it holds true that

$$\sum_{\substack{\pi \in S_4(a,b,c,d)\\ \pi(a) < \pi(d)}} \Big(\operatorname{sign}(\pi) \cdot [\![\pi(a), \pi(b), \pi(c)]\!]_{/A} \cdot [\![\pi(b), \pi(c), \pi(d)]\!]_{/A} \cdot \\ [\![\pi(c), \pi(d), e]\!]_{/A} \cdot [\![\pi(d), e, \pi(a)]\!]_{/A} \cdot [\![e, \pi(a), \pi(b)]\!]_{/A} \Big) = 0.$$

W.l.o.g., let a, b, c, and A such that $\varphi_{/A}(a, b, c) = 0$. Then, the five-point condition reads as follows:

$$\begin{aligned}
0 = &+ [\![a, b, d]\!]_{/A}[\![c, d, e]\!]_{/A}[\![e, a, b]\!]_{/A} \cdot \Big([\![a, d, c]\!]_{/A}[\![c, e, b]\!]_{/A} - [\![b, d, c]\!]_{/A}[\![c, e, a]\!]_{/A} \Big)\\
&+ [\![a, c, d]\!]_{/A}[\![d, b, e]\!]_{/A}[\![e, a, c]\!]_{/A} \cdot \Big([\![a, d, b]\!]_{/A}[\![b, e, a]\!]_{/A} - [\![a, d, b]\!]_{/A}[\![b, e, c]\!]_{/A} \Big)\\
&+ [\![d, b, c]\!]_{/A}[\![b, c, e]\!]_{/A}[\![e, a, d]\!]_{/A} \cdot \Big([\![a, d, b]\!]_{/A}[\![c, e, a]\!]_{/A} - [\![a, d, c]\!]_{/A}[\![b, e, a]\!]_{/A} \Big).
\end{aligned} \tag{4.9}$$

The Grassmann-Plücker relations that will yield that the terms in parenthesis are zero are $(A, c, a \mid A, c, b, d, e)$, $(A, b, c \mid A, b, a, d, e)$, and $(A, a, b \mid A, a, c, d, e)$, where when we write A we actually mean the sequence of all elements of A. The rest works analogously to the explanations above. $\square$

4.3. On the structure of the five-point condition

The five-point condition has an interesting structure that is worth a closer examination. We have already seen some of its properties in the Section 4.1 where we highlighted the possibility of illustrating the structure of the five-point condition with the help of Hamilton cycles. In the following, we will see that there are redundancies in the five-point condition and how the five-point condition in rank 2 can be written with the help of determinants. We have already learned that zeros in the image of the phirotope cause the five-point condition to automatically evaluate to zero. Therefore, we will examine uniform phirotopes in this section.

Consider a uniform rank-2 phirotope φ on six elements $\mathcal{E} = \{1, \ldots, 6\}$. According to the Lemma 4.3, it is realisable, if and only if for each five-element subset the five-point condition holds true. There are six of such five-element subsets and each of them contains all but one indices. In the following, we denote by Υ_k the five-point condition that does *not* contain the index k. With $\Upsilon_6 = 0$ we can realise $\varphi|_{[5]}$. For this we choose the points P_1, P_2, and P_3 that realise the indices 1, 2, and 3 in general position. The position of the points P_4 and P_5 that realise 4 and 5 are then determined. In the same way, Υ_5 yields the position of (again) P_4 and P_6, if we choose the same P_1, P_2, and P_3 to realise 1, 2, and 3. Now the only thing that might still forbid the realisability of the whole phirotope is the value of $\varphi(5, 6)$. The corresponding points are already determined and for the phirotope to be realisable, $\varphi(5, 6) = \omega(\det(P_5, P_6))$ has to hold true. This is for example ensured by $\Upsilon_4 = 0$.

Note that the realisability of φ was decided using only the five-point conditions Υ_4, Υ_5, and Υ_6. Thus, $\Upsilon_1 = 0$, $\Upsilon_2 = 0$, and $\Upsilon_3 = 0$ have to follow from $\Upsilon_4 = 0$, $\Upsilon_5 = 0$, and $\Upsilon_6 = 0$. This can be proven as follows.

Consider Υ_4, Υ_5, and Υ_6. The term $[\![4, 5]\!]$ is only contained in formula Υ_6, $[\![4, 6]\!]$ only in Υ_5, and $[\![5, 6]\!]$ only in Υ_4. We write

$$\begin{aligned} \Upsilon_6 &= A \cdot [\![4, 5]\!] - B, \\ \Upsilon_5 &= C \cdot [\![4, 6]\!] - D, \\ \Upsilon_4 &= E \cdot [\![5, 6]\!] - F, \end{aligned}$$

where $A, \ldots, F$ stand for polynomials of squared phirotope values that complete the five-point conditions and do not contain $[\![4, 5]\!]$, $[\![4, 6]\!]$, or $[\![5, 6]\!]$. The terms can also be depicted as (parts of) Hamilton paths as introduced in the Chapter 4.1. Recall that an edge between the vertices labelled with a and b corresponds to $[\![a, b]\!]$. This means that A and B are graphs on the vertices in $\{1, 2, 3, 4, 5\}$. B corresponds to all Hamilton cycles not containing the edge $(4, 5)$, while A contains all (undirected) Hamilton paths from 5 to 4 or of length four, cf. Figures 4.2 and 4.3.

The five-point condition Υ_3 contains all three terms $[\![4, 5]\!]$, $[\![4, 6]\!]$, and $[\![5, 6]\!]$. Each summand

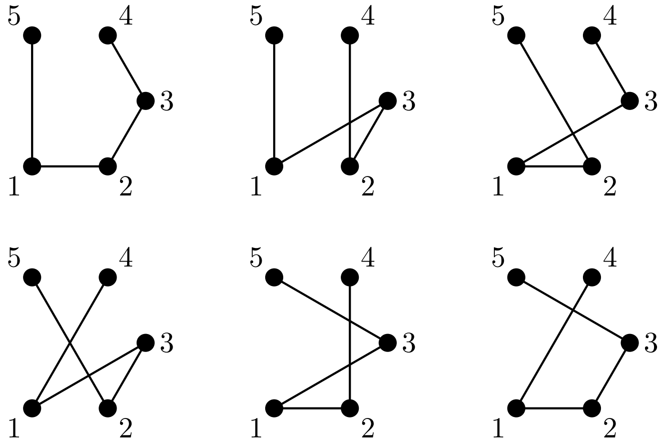


Figure 4.2.: The Hamilton paths that correspond to the term A.

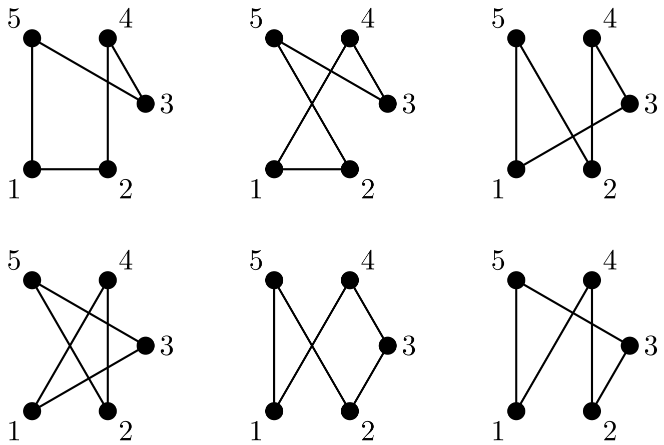


Figure 4.3.: The Hamilton cycles that correspond to the term B.

of Υ_3 contains either one or two of the terms $\llbracket 4,5 \rrbracket$, $\llbracket 4,6 \rrbracket$, and $\llbracket 5,6 \rrbracket$ and thus, we can write it as

$$\begin{aligned}\Upsilon_3 = &+ U \cdot \llbracket 4,5 \rrbracket + V \cdot \llbracket 4,6 \rrbracket + W \cdot \llbracket 5,6 \rrbracket \\ &+ X \cdot \llbracket 4,5 \rrbracket \cdot \llbracket 4,6 \rrbracket + Y \cdot \llbracket 4,5 \rrbracket \cdot \llbracket 5,6 \rrbracket + Z \cdot \llbracket 4,6 \rrbracket \cdot \llbracket 5,6 \rrbracket,\end{aligned} \tag{4.10}$$

where $U, \dots, Z$ are the polynomials in squared phases that complete Υ_3 and do not contain $\llbracket 4,5 \rrbracket$, $\llbracket 4,6 \rrbracket$, or $\llbracket 5,6 \rrbracket$. To prove that $\Upsilon_3 = 0$ follows from $\Upsilon_4 = 0$, $\Upsilon_5 = 0$, and $\Upsilon_6 = 0$, we find factors $\alpha, \dots, \eta$ such that

$$\begin{aligned}0 = \;& \alpha \cdot \Upsilon_4 + \beta \cdot \Upsilon_5 + \gamma \cdot \Upsilon_6 \\ &+ \delta \cdot \Upsilon_4 \cdot \Upsilon_5 + \epsilon \cdot \Upsilon_4 \cdot \Upsilon_6 + \zeta \cdot \Upsilon_5 \cdot \Upsilon_6 \\ &+ \eta \cdot \Upsilon_3.\end{aligned} \tag{4.11}$$

One possible solution is given by

$$\begin{aligned}
\alpha &= ACW + BCY + ADZ,\\
\beta &= AEV + BEX + AFZ,\\
\gamma &= CEU + DEX + FCY,\\
\delta &= AZ,\\
\epsilon &= CY,\\
\zeta &= EX,\\
\eta &= -ACE.
\end{aligned}$$

With this, all terms containing $[\![4,5]\!]$, $[\![4,6]\!]$, and $[\![5,6]\!]$ cancel:

$$\begin{aligned}
&\alpha \cdot \Upsilon_4 + \beta \cdot \Upsilon_5 + \gamma \cdot \Upsilon_6\\
&+ \delta \cdot \Upsilon_4 \cdot \Upsilon_5 + \epsilon \cdot \Upsilon_4 \cdot \Upsilon_6 + \zeta \cdot \Upsilon_5 \cdot \Upsilon_6\\
&+ \eta \cdot \Upsilon_3\\
=& - ACFW - ADEV - ADFZ - BDEX - BCEU - BCFY\\
=& \, 0. \qquad (4.12)
\end{aligned}$$

The last line can be checked with the help of a computer algebra system, cf. Appendix B.

Another way of showing the dependence of the five-point conditions is transposing $\Upsilon_4 = 0$, $\Upsilon_5 = 0$, and $\Upsilon_6 = 0$ as follows

$$[\![4,5]\!] = \frac{B}{A}, \qquad [\![4,6]\!] = \frac{D}{C}, \qquad [\![5,6]\!] = \frac{F}{E}, \qquad (4.13)$$

which can be done as φ was assumed to be uniform. Inserting these into Υ_1, Υ_2, and Υ_3 will each yield zero (cf. Appendix C).

All in all, this finding fits the theoretical contemplation at the beginning of this section. If three five-point conditions evaluate to zero, this is indeed sufficient for the realisability of a uniform, non-chirotopal rank-2 phirotope on five indices.

The result can be generalised to phirotopes with larger index sets. Consider a uniform, non-chirotopal rank-2 phirotope on $[n]$. Here, we have $\binom{n}{5}$ five-point conditions, so the number of five-point conditions is of quintic order in n. For three elements $a, b, c \in [n]$, we choose their affine representatives to be the projective basis that determines the position of all other points. Then, according to the considerations above, each pair of indices $x, y \in [n] \setminus \{a, b, c\}$ has to be part of at least one five-point condition. Thus, we only need $\binom{n-3}{2}$ five-point conditions, namely those on $\{a, b, c, x, y\}$. This number is of quadratic order in n.

Note that the argument we have used can only be applied to rank 2. In rank 2 the five-point condition is a formula on five elements and thus has two elements more than a projective basis, which consists of three elements. So in rank 2, the phirotope value of every pair of indices appears in a five-point condition together with the indices of the chosen projective basis. In arbitrary rank d, the five-point conditions are formulae on $d+2$ elements. This number differs from the number of elements in a projective basis, which contains $d+1$ points, by only one element.

Another interesting structural property of the five-point condition is that it can be written in terms of determinants. To see this, consider a five-point condition in which again one bracket is factored out. For example, consider again $\Upsilon_6 = A \cdot [\![4,5]\!] - B$. It holds true that

$$\begin{aligned} A =& + [\![1,2]\!][\![2,3]\!][\![3,4]\!][\![5,1]\!] - [\![1,3]\!][\![3,2]\!][\![2,4]\!][\![5,1]\!] - [\![2,1]\!][\![1,3]\!][\![3,4]\!][\![5,2]\!] \\ &+ [\![2,3]\!][\![3,1]\!][\![1,4]\!][\![5,2]\!] + [\![3,1]\!][\![1,2]\!][\![2,4]\!][\![5,3]\!] - [\![3,2]\!][\![2,1]\!][\![1,4]\!][\![5,3]\!] \\ =& \det \begin{pmatrix} [\![1,2]\!][\![1,3]\!] & [\![1,4]\!] & [\![1,5]\!] \\ [\![2,1]\!][\![2,3]\!] & [\![2,4]\!] & [\![2,5]\!] \\ [\![3,1]\!][\![3,2]\!] & [\![3,4]\!] & [\![3,5]\!] \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} B =& - [\![1,2]\!][\![2,4]\!][\![4,3]\!][\![3,5]\!][\![5,1]\!] + [\![1,3]\!][\![3,4]\!][\![4,2]\!][\![2,5]\!][\![5,1]\!] \\ &+ [\![1,4]\!][\![4,2]\!][\![2,3]\!][\![3,5]\!][\![5,1]\!] - [\![1,4]\!][\![4,3]\!][\![3,2]\!][\![2,5]\!][\![5,1]\!] \\ &+ [\![2,1]\!][\![1,4]\!][\![4,3]\!][\![3,5]\!][\![5,2]\!] - [\![3,1]\!][\![1,4]\!][\![4,2]\!][\![2,5]\!][\![5,3]\!] \\ =& \det \begin{pmatrix} [\![1,2]\!] & [\![1,4]\!][\![2,4]\!] & [\![1,5]\!][\![2,5]\!] \\ [\![1,3]\!] & [\![1,4]\!][\![3,4]\!] & [\![1,5]\!][\![3,5]\!] \\ [\![2,3]\!] & [\![2,4]\!][\![3,4]\!] & [\![2,5]\!][\![3,5]\!] \end{pmatrix}. \end{aligned}$$

We want to identify in which cases the value of the term $\varphi(4,5)$ is irrelevant for the realisability of the phirotope. This can only be the case, if the phirotope contains a zero in its image or is chirotopal. Otherwise, the value of the term $\varphi(4,5)$ cannot be irrelevant. Changing it will also change the value of the five-point condition. We want to assume that the phirotope $\varphi|_{[5]}$ is uniform because the Theorem 4.23 guarantees that the five-point condition Υ_6 holds true, if the phirotope contains a zero in its image. Thus, we deal with the chirotopal cases:

- *φ is chirotopal.*
  We will show that in this case the last column of the determinant of the term A is a multiple of the second one. By assumption, none of the phirotope values equals zero, thus

$$\mathrm{cr}_\varphi(1,2\,|\,4,5) = \pm 1 \qquad \Rightarrow \qquad [\![1,4]\!]\cdot[\![2,5]\!] = [\![1,5]\!]\cdot[\![2,4]\!],$$

$$\begin{aligned} \mathrm{cr}_\varphi(1,3\,|\,4,5) = \pm 1 \qquad &\Rightarrow \qquad [\![1,4]\!]\cdot[\![3,5]\!] = [\![1,5]\!]\cdot[\![3,4]\!], \\ \mathrm{cr}_\varphi(2,3\,|\,4,5) = \pm 1 \qquad &\Rightarrow \qquad [\![2,4]\!]\cdot[\![3,5]\!] = [\![2,5]\!]\cdot[\![3,4]\!]. \end{aligned}$$

By applying Laplace expansion to A we obtain $A = 0$.

Note that, as we are dealing with rank-2 phirotopes, chirotopality implies that the phirotope is realisable. And indeed, if all cross ratios are real, the last two columns of the matrix in B will also be equal. Thus, $B = 0$ and the five-point condition is satisfied.

- $\varphi|_{\{1,2,3,4\}}$ *is chirotopal.*
  For symmetry reasons, this case is analogous to the case in which $\varphi|_{\{1,2,3,5\}}$ is chirotopal. If the minor $\varphi|_{\{1,2,3,4\}}$ is chirotopal, then the cross ratio phase $\mathrm{cr}_\varphi(1,2\,|\,3,4)$ and the cross ratio phases of all permutations of the four indices are real. As none of the involved terms equals zero by assumption, this yields:

$$\begin{aligned} [\![1,3]\!][\![2,4]\!] &= [\![2,3]\!][\![1,4]\!] \\ [\![1,2]\!][\![3,4]\!] &= [\![2,3]\!][\![1,4]\!] \\ [\![1,2]\!][\![3,4]\!] &= [\![1,3]\!][\![2,4]\!] \end{aligned}$$

  By multiplying the first equation with $[\![1,2]\!]$, the second with $[\![1,3]\!]$ and the third with $[\![2,3]\!]$, we obtain that all 2-by-2-subdeterminants that consist of entries of the first two columns vanish:

$$\begin{vmatrix} [\![1,2]\!][\![1,3]\!] & [\![1,4]\!] \\ [\![1,2]\!][\![2,3]\!] & [\![2,4]\!] \end{vmatrix} = 0,$$

$$\begin{vmatrix} [\![1,2]\!][\![1,3]\!] & [\![1,4]\!] \\ [\![1,3]\!][\![2,3]\!] & [\![3,4]\!] \end{vmatrix} = 0,$$

$$\begin{vmatrix} [\![1,2]\!][\![2,3]\!] & [\![2,4]\!] \\ [\![1,3]\!][\![2,3]\!] & [\![3,4]\!] \end{vmatrix} = 0.$$

  Thus, $A = 0$. Note that the same calculations apply to B, meaning that if $\varphi|_{\{1,2,3,4\}}$ is chirotopal, the subdeterminants of the first two columns of B also vanish. This implies that if $\varphi|_{\{1,2,3,4\}}$ is chirotopal, φ is always realisable, regardless of the values $\varphi(1,5)$, $\varphi(2,5)$, $\varphi(3,5)$, and $\varphi(4,5)$.

So, we have seen that chirotopality – even of only a minor – will result in the vanishing of the terms A and B.

5. Towards Euclidean Geometry

Consider a realisable non-chirotopal rank-d phirotope. Up to choosing a projective basis (that is choosing $d+1$ points in general position freely), its realisation is rigid (cf. Lemma 4.20). This means that we can understand the phirotope values as some kind of novel coordinates of the point configuration of its realisation. In this chapter, we are going to examine the possibilities of carrying out Euclidean geometry in the plane in these new coordinates.

Firstly, we need to find the right subclass of phirotopes that can be understood as coordinates of a Euclidean point configuration. To this end, remember that in order to carry out Euclidean geometry in $\mathbb{RP}^2$, the points

$$\mathtt{I} = \begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathtt{J} = \begin{pmatrix} \mathrm{i} \\ 1 \\ 0 \end{pmatrix}$$

are added to $\mathbb{RP}^2$. With this, properties like the co-circularity of four points can be expressed and angles can be measured. For details, see [RG11] or [RGO09]. We want to characterise the appearance of phirotopes that stem from rank-3 vector configurations which, apart from $\mathtt{I}$ and $\mathtt{J}$, only contain real points. We will see that this can be done with the help of cross ratios.

In the following, we will make use of characterisations of Euclidean concepts by means of bracket polynomials. These characterisations are explained in detail in [RG11] and [RGO09].

5.1. Euclidean phirotopes

Consider a vector configuration $V = (P_1, \dots, P_n, \mathtt{I}, \mathtt{J})$, with $P_1, \dots, P_n \in \mathbb{RP}^2$. This yields a phirotope φ on the index set $\mathcal{E} = \{1, \dots, n, i, j\}$, where index k corresponds to the point P_k for $k \in [n]$, and i and j correspond to $\mathtt{I}$ and $\mathtt{J}$, respectively. Phirotopes that can be realised by real points together with $\mathtt{I}$ and $\mathtt{J}$ are the main focus of this chapter.

Definition 5.1 (Euclidean phirotope)
A realisable non-chirotopal rank-3 phirotope φ is called *Euclidean*, if it is a reorientation of a phirotope that can be realised by a vector configuration $V = (P_1, \dots, P_n, \mathtt{I}, \mathtt{J})$, where $P_1, \dots, P_n \in \mathbb{RP}^2$.

We chose the term “Euclidean phirotopes”, as it reflects the fact that these phirotopes describe a Euclidean situation. Up until now, however, it is still an open question if our concept of Euclideaness coincides with that of Edmonds and Mandel, who introduced Euclideaness for (real) oriented matroids (cf. [EM82]).

To examine the cross ratio phases that can occur when dealing with Euclidean phirotopes, we introduce new notation:

$$\varkappa := \frac{0}{0} \quad \text{and}$$
$$\infty := \frac{c}{0} \quad \text{for all } c \in \mathbb{C} \setminus \{0\}.$$

The cross ratio phase of indices that correspond to real points in the realisation is either real number or, if the denominator contains a term that equals zero, ∞ or $\varkappa$:

$$\operatorname{cr}_\varphi (a, b \mid c, d)_e \in \mathbb{R} \cup \{\infty, \varkappa\} \quad \forall a, b, c, d, e \in \mathcal{E} \setminus \{i, j\}. \tag{5.1}$$

Furthermore, $\mathtt{I} = \overline{\mathtt{J}}$ implies

$$\operatorname{cr}_\varphi (a, b \mid c, d)_i = \overline{\operatorname{cr}_\varphi (a, b \mid c, d)_j} \quad \forall a, b, c, d \in \mathcal{E} \setminus \{i, j\}. \tag{5.2}$$

The Equations (5.1) and (5.2) provide us with a tool for checking whether or not a phirotope is Euclidean.

Theorem 5.2 (Characterisation of Euclidean phirotopes)
A realisable non-chirotopal rank-3 phirotope φ on the index set $\mathcal{E}$, $|\mathcal{E}| \geq 7$ is Euclidean, if there are two indices $i, j \in \mathcal{E}$ such that $\varphi_{\setminus\{i,j\}}$ is not a near-pencil and

$$\operatorname{cr}_\varphi (a, b \mid c, d)_e \in \mathbb{R} \cup \{\infty, \varkappa\} \quad \forall a, b, c, d, e \in \mathcal{E} \setminus \{i, j\} \quad \text{and} \tag{5.3}$$
$$\operatorname{cr}_\varphi (a, b \mid c, d)_i = \overline{\operatorname{cr}_\varphi (a, b \mid c, d)_j} \quad \forall a, b, c, d \in \mathcal{E} \setminus \{i, j\}. \tag{5.4}$$

The indices i and j are called the *special indices*, all other indices are called the *ordinary indices*.

Proof. Let φ be a realisable rank-3 phirotope on $\mathcal{E} = \{1, 2, \ldots, n+2\}$, $n \geq 5$, that yields at least one non-real cross ratio phase. The condition (5.3) implies that there are two elements $i, j \in \mathcal{E}$, such that the deletion $\varphi_{\setminus\{i,j\}}$ is chirotopal. It follows that there is a reorientation $\varrho \in (\mathcal{S}^1)^n$ of $\varphi_{\setminus\{i,j\}}$ such that $(\varphi_{\setminus\{i,j\}})^\varrho$ is a chirotope. W.l.o.g., set $i = n+1$ and $j = n+2$. We define a new reorientation vector $\tilde{\varrho} \in (\mathcal{S}^1)^{n+2}$ via

$$\tilde{\varrho}_k = \begin{cases} \varrho_k, & \text{if } k \in [n], \\ 1, & \text{else.} \end{cases}$$

Consider the reoriented phirotope $\varphi^{\tilde{\varrho}}$. As its deletion $(\varphi^{\tilde{\varrho}})_{\setminus\{i,j\}}$ is a chirotope, it holds true that

$$\varphi^{\tilde{\varrho}}(a, b, c) \in \mathbb{R} \quad \forall a, b, c \in \mathcal{E} \setminus \{i, j\}. \tag{5.5}$$

W.l.o.g., $\varphi|_{[4]}$ is uniform as $\varphi_{\setminus\{n+1,n+2\}}$ is not a near-pencil. We choose the first four vectors of a realisation $P_1, \ldots, P_4$ to be four distinct points in $\mathbb{RP}^2$ corresponding to the indices 1 to 4, respectively. All loops of the phirotope will be realised by the zero vector. For every other index $k \in [n]$ there exists at least one ordered triple that is mapped to a non-zero number and contains k and two indices from $[4]$. With the Equation (5.5) we conclude that the point P_k of the realisation has to have real coordinates as well. So we have seen that the reoriented phirotope $\varphi^{\tilde{\varrho}}$ can be realised as a vector configuration $V = (P_1, P_2, \ldots, P_n, P_i, P_j)$ with n real vectors $P_1, P_2, \ldots, P_n \in \mathbb{RP}^2$. The points P_i and P_j necessarily have complex coordinates, as φ is non-chirotopal by assumption.

As a next step, we will show that P_i and P_j have to be complex conjugates. This will enable us to find a transformation that leaves all real points real and maps P_i to $\mathtt{I}$ and P_j to $\mathtt{J}$. To show that $P_i = \overline{P_j}$, we consider condition (5.4), with which for all $P_a, P_b, P_c, P_d \in V \setminus (P_i, P_j)$ it holds true that

$$\omega\left(\operatorname{cr}(P_a, P_b | P_c, P_d)_{P_i}\right) = \overline{\omega\left(\operatorname{cr}(P_a, P_b | P_c, P_d)_{P_j}\right)}.$$

For fixed P_i, one solution to this is certainly given by $P_j \sim \overline{P_i}$. We have seen in the Lemma 4.20 that realisations of phirotopes are rigid. Thus, $P_j \sim \overline{P_i}$ is the unique solution.

It remains to be show that P_i and P_j can be mapped to $\mathtt{I}$ and $\mathtt{J}$, respectively, by a projective transformation that leaves the coordinates of all other points real. This can be done via a transformation that is given by a matrix

$$T = \begin{pmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

We require $T \cdot P_i = \mathtt{I}$. This yields six real equations in nine real variables: With the complex coordinates of $P_i = \begin{pmatrix} p_1, & p_2, & p_3 \end{pmatrix}^T$ split into real and imaginary parts $p_k = r_k + \mathrm{i} \cdot i_k$, the first column reads as follows:

$$t_1 r_1 + t_2 r_2 + t_3 r_3 = 0 \qquad \wedge \qquad t_1 i_1 + t_2 i_2 + t_3 i_3 = -1.$$

We can choose any of the solutions to these two real equations in the three real variables t_1, t_2, t_3. Similarly, the second and third column of T can be determined. With this, we have found a transformation that maps P_i to $\mathtt{I}$ but leaves all real points real.
The requirement $T \cdot P_j = \mathtt{J}$ is automatically met as

$$T \cdot P_j = T \cdot \overline{P_i} = \overline{T \cdot P_j} = \overline{\mathtt{I}} = \mathtt{J}.$$

All in all, we have seen that conditions (5.3) and (5.4) are enough to guarantee that a realisable non-chirotopal phirotope permits a realisation in which two points are $\mathtt{I}$ and $\mathtt{J}$ and all other points have real coordinates. □

Whenever a phirotope exhibits a structure similar to that described by Equations (5.1) and (5.2) we call the phirotope "Euclidean" – regardless of whether or not it is realisable. We use the characterisation given in the Theorem 5.2 as definition:

Definition 5.3 (Euclidean phirotope)
A non-chirotopal rank-3 phirotope φ on $\mathcal{E}$, $|\mathcal{E}| \geq 7$, is called *Euclidean*, if and only if there are two indices $i, j \in \mathcal{E}$ such that $\varphi_{\setminus\{i,j\}}$ is not a near-pencil and the Equations (5.3) and (5.4) are satisfied.

In the following sections, we will look at some Euclidean concepts and theorems that can be expressed using phirotopes. These theorems are then already valid at the abstract level of phirotopes and do not require a realisation.

In part 2 of the Lemma 3.2 we have seen that for rank-2 phirotopes the cross ratio of four points can be reconstructed from the cross ratio phase of the corresponding indices, if the latter is not a real number. As this reconstruction is needed for the Euclidean theorems on the level of phirotopes, we will see how the reconstruction works in rank 2 and then seek to extend it to rank 3.

Consider a realisable non-chirotopal phirotope φ on $\mathcal{E}$ and four indices $a, b, c, d \in \mathcal{E}$ such that $\mathrm{cr}_\varphi(a, b \,|\, c, d) \notin \mathbb{R}$. Our goal is to reconstruct the exact value of $\mathrm{cr}(A, B|C, D)$, where $A, B, C, D \in \mathbb{C}^2/\mathbb{R}^+$ are the points of a realisation of φ that correspond to a, b, c, d, respectively. To this end, consider the involved cross ratio phases, which are the phases of the cross ratios (see Equation 3.1). The cross ratio phases are all given by the values of the phirotope. Setting $\lambda := \mathrm{cr}(A, B|C, D)$, we obtain:

$$\operatorname{cr}_{\varphi}(a,b\,|\,c,d) = \frac{\lambda}{|\lambda|} \quad (5.6) \qquad \operatorname{cr}_{\varphi}(a,b\,|\,d,c) = \frac{|\lambda|}{\lambda} \quad (5.7)$$

$$\operatorname{cr}_{\varphi}(a,c\,|\,b,d) = \frac{1-\lambda}{|1-\lambda|} \quad (5.8) \qquad \operatorname{cr}_{\varphi}(a,c\,|\,d,b) = \frac{|1-\lambda|}{1-\lambda} \quad (5.9)$$

$$\operatorname{cr}_{\varphi}(a,d\,|\,b,c) = \frac{1-\frac{1}{\lambda}}{\left|1-\frac{1}{\lambda}\right|} \quad (5.10) \qquad \operatorname{cr}_{\varphi}(a,d\,|\,c,b) = \frac{\frac{\lambda}{\lambda-1}}{\left|\frac{\lambda}{\lambda-1}\right|} \quad (5.11)$$

Note that $\lambda \notin \mathbb{R}$ (as $\lambda \in \mathbb{R}$ would violate $\operatorname{cr}_{\varphi}(a,b\,|\,c,d) \notin \mathbb{R}$, as can be seen in the Equation (5.6)). Some pairs of these equations determine λ, while others do not. The Equations (5.6) and (5.7), for example, essentially contain the same information as they are just the reciprocal of one another. The same holds true for (5.8) and (5.9) as well as for (5.10) and (5.11). All other pairs determine λ. We take a closer look at the exemplary calculation for one pair, for example for (5.6) and (5.8):

$$(5.6): \quad \operatorname{cr}_{\varphi}(a,b\,|\,c,d) = \frac{\lambda}{|\lambda|} = \frac{\lambda}{\sqrt{(\lambda\cdot\overline{\lambda})}}$$
$$\Rightarrow \overline{\lambda} = \frac{\lambda}{\operatorname{cr}_{\varphi}(a,b\,|\,c,d)^2}$$

This will be inserted into equation (5.8) and then we solve for λ which yields:

$$\lambda = \frac{\left(1-\operatorname{cr}_{\varphi}(a,c\,|\,b,d)^2\right)\cdot \operatorname{cr}_{\varphi}(a,b\,|\,c,d)^2}{\operatorname{cr}_{\varphi}(a,b\,|\,c,d)^2 - \operatorname{cr}_{\varphi}(a,c\,|\,b,d)^2} \tag{5.12}$$

The second solution for λ of this equation is not feasible, as it is $\lambda = 1 \in \mathbb{R}$. The Table E.1 in the appendix lists the remaining depictions of λ that are obtained by similar calculations.

Although we have carried out the calculations in rank 2 to keep the formulae small, the same calculations allow us to construct $\operatorname{cr}(A,B|C,D)_E$ from $\operatorname{cr}_{\varphi}(a,b\,|\,c,d)_{\{e\}}$ for some rank-3 phirotope φ on an index set containing a, b, c, d, and e. So, we have seen that we can reconstruct the value of cross ratios whenever the corresponding cross ratio phase is non-real. Thus, we can express every Euclidean property that can be verified using complex cross ratios in terms of phirotopes. Before we list some examples, we state a lemma that helps to make the following calculations clearer.

Lemma 5.4
Let φ be a Euclidean rank-3 phirotope on $\mathcal{E}$ and let $i, j \in \mathcal{E}$ be the special indices. If A, B, C, D, E, A', B', C', D', and E' are the points of a realisation of φ and correspond to the indices a, b, c, d, e, a', b', c', d', and e', respectively, then

$$\exists r \in \mathbb{R} : \qquad \operatorname{cr}(A, B|C, D)_E = r \cdot \operatorname{cr}_\varphi(a, b \,|\, c, d)_{\{e\}} .$$

and

$$\begin{aligned} & \operatorname{cr}(A, B|C, D)_E = r \cdot \operatorname{cr}(A', B'|C', D')_{E'} \text{ for some } r \in \mathbb{R} \\ \iff \quad & \operatorname{cr}_\varphi(a, b \,|\, c, d)_{\{e\}} = \operatorname{cr}_\varphi(a', b' \,|\, c', d')_{\{e'\}} . \end{aligned}$$

Proof. The first part of the lemma follows from Equation (3.1) which says $\operatorname{cr}_\varphi(a, b \,|\, c, d)_{\{e\}} = \omega\left(\operatorname{cr}(A, B|C, D)_E\right)$.
Applying the phase function on both sides of the second part implies "⇒". The other direction follows with Equation (3.1) as well. □

5.2. Euclidean properties and theorems

Definition 5.5 (Comparison of angles)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. Furthermore, let $a, b, c, x, y, z \in \mathcal{E} \setminus \{i, j\}$. If and only if

$$\operatorname{cr}_\varphi(b, c \,|\, i, j)_{\{a\}} = \operatorname{cr}_\varphi(y, z \,|\, i, j)_{\{x\}} \notin \{0, \infty, \varkappa\},$$

we say that *the angles $\sphericalangle_a(b, c)$ and $\sphericalangle_x(y, z)$ are equal.*

As the phirotope at hand might not be realisable, the concept of angles (and later also that of lines) introduced here is completely abstract.

Lemma 5.6 (Comparison of angles)
If the phirotope in the Definition 5.5 is realisable and A, B, C, X, Y, Z, $\mathtt{I}$ and $\mathtt{J}$ are the points of the realisation that correspond to the indices mentioned in that definition, then

$$\sphericalangle_A(B, C) = \sphericalangle_X(Y, Z).$$

Proof. From projective geometry we know that

$$\sphericalangle_A(B,C) = \sphericalangle_X(Y,Z) \qquad \Leftrightarrow \qquad \mathrm{cr}(B,C|\mathtt{I},\mathtt{J})_A = \mathrm{cr}(Y,Z|\mathtt{I},\mathtt{J})_X\,.$$

With the Lemma 5.4 the claim follows. □

In fact, we cannot only compare angles but we can measure them.

Definition 5.7 (Measurement of angles)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i,j \in \mathcal{E}$. The *angle* $\sphericalangle_a(b,c)$ between the lines ab and ac (modulo π) is defined as

$$\sphericalangle_a(b,c) := \frac{1}{2\mathrm{i}} \ln \left(\mathrm{cr}_\varphi\left(b,c\,|\,i,j\right)_{\{a\}}\right),$$

whenever $\mathrm{cr}_\varphi\left(b,c\,|\,i,j\right)_{\{a\}} \notin \{0,\infty,\varkappa\}$.

Lemma 5.8 (Measurement of angles)
If the phirotope in the Definition 5.7 is realisable and A, B, C, $\mathtt{I}$, and $\mathtt{J}$ are the points of the realisation that correspond to the indices mentioned in that definition, then

$$\sphericalangle_A(B,C) = \sphericalangle_a(b,c).$$

Proof. From projective geometry we know that

$$\sphericalangle_A(B,D) \bmod \pi = \frac{1}{2\mathrm{i}} \ln\left(\mathrm{cr}(B,C|\mathtt{I},\mathtt{J})_A\right).$$

As $\mathrm{cr}_\varphi\left(b,c\,|\,i,j\right)_{\{a\}} \notin \{0,\infty,\varkappa\}$, we obtain:

$$\begin{aligned}
\mathrm{cr}_\varphi\left(b,c\,|\,i,j\right)_{\{a\}} &= \frac{\varphi(a,b,i)\cdot\varphi(a,c,j)}{\varphi(a,b,j)\cdot\varphi(a,c,i)} \\
&= \frac{[A,B,\mathtt{I}][A,C,\mathtt{J}]}{[A,B,\mathtt{J}][A,C,\mathtt{I}]} \cdot \left|\frac{[A,B,\mathtt{J}][A,C,\mathtt{I}]}{[A,B,\mathtt{I}][A,C,\mathtt{J}]}\right| \\
&= \mathrm{cr}(B,C|\mathtt{I},\mathtt{J})_A
\end{aligned}$$

The last line follows from $\mathtt{I} \sim \overline{\mathtt{J}}$ and $A,B,C \in \mathbb{R}^3$ because this implies that the absolute value of the cross ratio is 1. It follows that $\ln\left(\mathrm{cr}_\varphi\left(b,c\,|\,i,j\right)_{\{a\}}\right) = \ln\left(\mathrm{cr}(B,C|\mathtt{I},\mathtt{J})_A\right)$. □

Corollary 5.9 (Perpendicularity)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. Let $a, b, c \in \mathcal{E} \setminus \{i, j\}$. The angle $\sphericalangle_a(b, c)$ is a right angle, if and only if

$$\mathrm{cr}_\varphi\,(b, c \,|\, i, j)_{\{a\}} = -1.$$

We also say that the lines ab and ac are perpendicular.

Definition 5.10 (Co-circularity)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. Let $a, b, c, d \in \mathcal{E} \setminus \{i, j\}$. If and only if

$$\mathrm{cr}_\varphi\,(a, c \,|\, b, d)_{\{j\}} = \mathrm{cr}_\varphi\,(a, c \,|\, b, d)_{\{i\}} \notin \{0, \infty, \varkappa\},$$

we say that *a, b, c, and d are co-circular.*

Lemma 5.11 (Co-circularity)
If the phirotope in the Definition 5.10 is realisable and A, B, C, D, $\mathtt{I}$ and $\mathtt{J}$ are the points of the realisation that correspond to the indices mentioned in that definition, then A, B, C and D are co-circular.

Proof. The four points A, B, C, and D are co-circular, if and only if $\mathrm{cr}(A, C|B, D)_{\mathtt{J}} = \mathrm{cr}(A, C|B, D)_{\mathtt{I}}$. With the Lemma 5.4 the claim follows. □

Definition 5.12 (Comparison of lengths)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. Let $a, b, c \in \mathcal{E} \setminus \{i, j\}$ such that $\varphi(a, b, c) \neq 0$. If and only if

$$\mathrm{cr}_\varphi\,(a, c \,|\, i, j)_{\{b\}} = \mathrm{cr}_\varphi\,(b, a \,|\, i, j)_{\{c\}} \notin \{0, \infty, \varkappa\},$$

we say that *the lengths $|a, b|$ and $|a, c|$ are equal.*

Lemma 5.13 (Comparison of lengths)
If the phirotope in the Definition 5.12 is realisable and A, B, C, $\mathtt{I}$ and $\mathtt{J}$ are the points of the realisation that correspond to the indices mentioned in that definition, then

$$|A, B| = |A, C|\,.$$

Proof. It holds true that $|A, B| = |A, C|$, if and only if $\mathrm{cr}(A, C|\mathtt{I}, \mathtt{J})_B = \mathrm{cr}(B, A|\mathtt{I}, \mathtt{J})_C$. (This corresponds to the angles $\sphericalangle_B(A, C)$ and $\sphericalangle_C(B, A)$ being equal, cf. Lemma 5.6. Thus, A, B, and C form an isosceles triangle.) The Lemma 5.4 yields the claim. □

Definition 5.14 (Parallelism)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. Let $a, b, x, y \in \mathcal{E} \setminus \{i, j\}$. If and only if

$$\varphi(a, b, i) \cdot \varphi(x, y, j) = \varphi(a, b, j) \cdot \varphi(x, y, i) \neq 0,$$

we say that *the lines ab and xy are parallel.*

Lemma 5.15 (Parallelism)
If the phirotope in the Definition 5.14 is realisable and A, B, X, Y, $\mathtt{I}$ and $\mathtt{J}$ are the points of the realisation that correspond to the indices mentioned in that definition, then the lines AB and XY are parallel.

Proof. In $\mathbb{RP}^2$, the three lines spanned by A and B, X and Y, and $\mathtt{I}$ and $\mathtt{J}$, respectively, are concurrent, if and only if

$$[A, B, \mathtt{I}][X, Y, \mathtt{J}] = [A, B, \mathtt{J}][X, Y, \mathtt{I}].$$

Multiplying this with $\frac{[B,X,\mathtt{J}]}{[B,X,\mathtt{I}]}$ and rearranging terms, this is equivalent to

$$\mathrm{cr}(A, X|\mathtt{I}, \mathtt{J})_B = \mathrm{cr}(Y, B|\mathtt{I}, \mathtt{J})_X\,.$$

Applying the Lemma 5.4 yields the claim. □

With this list of properties, we can state and prove some theorems of Euclidean geometry. Below, Krummeck, and Richter-Gebert already proved Miquel's theorem in terms of phirotopes (cf. [BKR03]) – although they did not use Euclidean phirotopes for this. The theorem of Miquel holds true for all phirotopes, not only for Euclidean ones. They also showed that the circular Pappus' theorem does not hold true for phirotopes.

For theorems on Euclidean phirotopes, we start by proving a simple theorem:

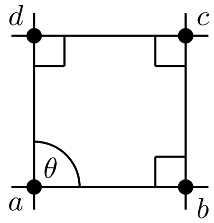


Figure 5.1.: Schematic representation of the Lemma 5.16. The angle θ is also a right angle, as the proof shows.

Lemma 5.16

Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. If for some indices $a, b, c, d \in \mathcal{E} \setminus \{i, j\}$ it holds true that the following pairs of lines are perpendicular:

$$ab \perp bc, \qquad bc \perp cd, \qquad cd \perp ad,$$

then also the two lines cd and ad are perpendicular.

Proof. As φ is Euclidean, we can express the perpendicularities as follows:

$$\begin{aligned}
ab \perp bc: &\qquad \varphi(b,a,i) \cdot \varphi(b,c,j) = -\varphi(b,a,j) \cdot \varphi(b,c,i),\\
bc \perp cd: &\qquad \varphi(c,b,i) \cdot \varphi(c,d,j) = -\varphi(c,b,j) \cdot \varphi(c,d,i),\\
cd \perp ad: &\qquad \varphi(d,c,i) \cdot \varphi(d,a,j) = -\varphi(d,c,j) \cdot \varphi(d,a,i).
\end{aligned}$$

Multiplying all equations and cancelling terms that appear on both sides yields

$$\begin{aligned}
&\varphi(b,a,i) \cdot \varphi(d,a,j) = -\varphi(b,a,j) \cdot \varphi(d,a,i)\\
\iff &\varphi(a,b,i) \cdot \varphi(a,d,j) = -\varphi(a,b,j) \cdot \varphi(a,d,i)\\
&\operatorname{cr}_{\varphi}(b,d \,|\, i,j)_{\{a\}} = -1.
\end{aligned}$$

Thus, the lines ab and ad are also perpendicular. $\square$

Similarly, we can transfer real binomial proofs as for example listed in [CRG95] to phirotopes.

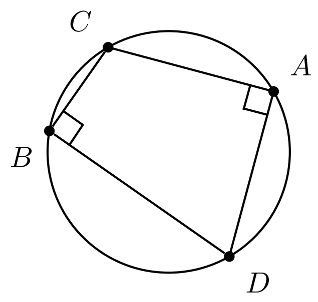


Figure 5.2.: Schematic representation of the Theorem of Thales, see Lemma 5.17

Lemma 5.17 (Theorem of Thales)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. If for some indices $a, b, c, d \in \mathcal{E} \setminus \{i, j\}$ it holds true that $ac \perp ad$ and $bc \perp bd$, then a, b, c, and d are co-circular.

A schematic representation of the Theorem of Thales can be found in the Figure 5.2.

Proof. From the perpendicularity requirements we obtain:

$$\begin{aligned}
ac \perp ad : \qquad & \varphi(a,c,i) \cdot \varphi(a,d,j) = -\,\varphi(a,c,j) \cdot \varphi(a,d,i), \\
bd \perp bc : \qquad & \varphi(b,c,j) \cdot \varphi(b,d,i) = -\,\varphi(b,c,i) \cdot \varphi(b,d,j).
\end{aligned}$$

Multiplying both equations and rearranging the terms yields:

$$\mathrm{cr}_\varphi\,(a, b \,|\, c, d)_{\{i\}} = \mathrm{cr}_\varphi\,(a, b \,|\, c, d)_{\{j\}}\,.$$

Thus, a, b, c, and d are co-circular. $\square$

The fact that we can measure angles allows us to prove theorems about angles much like we are used to.

Lemma 5.18 (Sum of angles of a triangle)
Let φ be a Euclidean phirotope on $\mathcal{E}$ with the special indices $i, j \in \mathcal{E}$. For every triple of indices $a, b, c \in \mathcal{E} \setminus \{i, j\}$, it holds true that

$$\sphericalangle_a(b,c) + \sphericalangle_b(c,a) + \sphericalangle_c(a,b) \equiv 0 \mod \pi.$$

Proof.

$$\begin{aligned}
&(\sphericalangle_a(b,c) + \sphericalangle_b(c,a) + \sphericalangle_c(a,b)) \bmod \pi \\
&= \frac{1}{2\mathrm{i}}\left(\ln\left(\mathrm{cr}_\varphi\,(b, c \,|\, i, j)_{\{a\}}\right) + \ln\left(\mathrm{cr}_\varphi\,(c, a \,|\, i, j)_{\{b\}}\right) + \ln\left(\mathrm{cr}_\varphi\,(a, b \,|\, i, j)_{\{c\}}\right)\right) \bmod \pi \\
&= \frac{1}{2\mathrm{i}} \ln(1) \bmod \pi \\
&= 0 \bmod \pi.
\end{aligned}$$

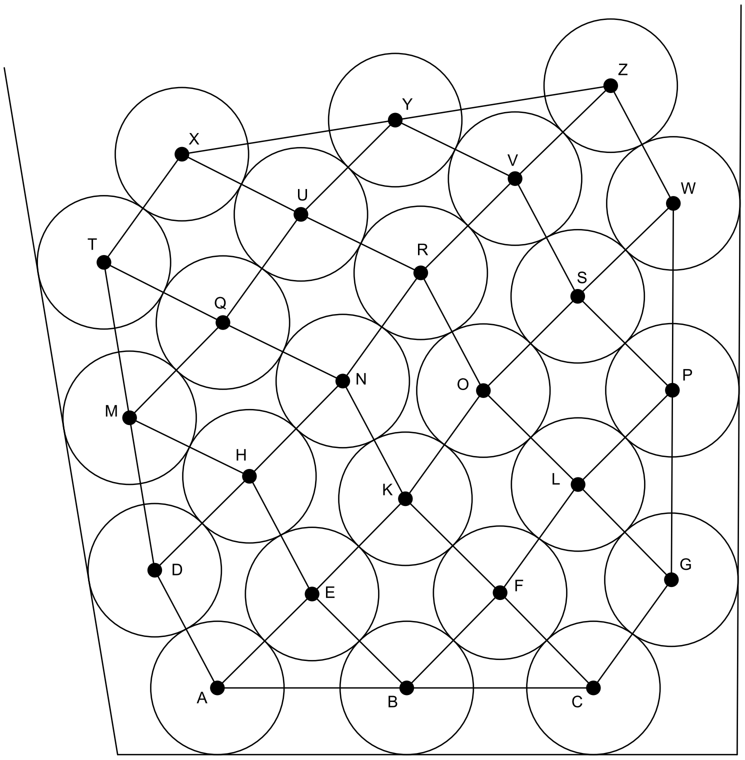


Figure 5.3.: The theorem of 24 wine bottles in a box.

□

The next theorem that we want to prove using Euclidean phirotopes is best understood by considering the construction given in the Figure 5.3. We stack wine bottles of equal radii into a box such that in the lowermost row there are three bottles, on top of that four, then again three and so on. After having inserted 24 bottles into the box, we notice that onto the top three bottles we can put a plank that rests on all three bottles. Mathematically, we formulate this as follows:

Theorem 5.19 (Bottles theorem)
Consider the configuration given in the Figure 5.3. It consists of 13 rhombi that all have the same side length ς and 8 isosceles triangles whose legs also have length ς. If the points A, B, C and D, M, T and G, P, W are collinear respectively, then so are the points X, Y, Z.

Proof. We will prove this theorem using phirotopes. Consider the Euclidean phirotope φ on $\mathcal{E} = \{a, b, \ldots, z, i, j\}$ with special indices i, and j. Each index corresponds to the vertex with the same but upper case label.

The assumptions of the theorem are the following: There are 8 isosceles triangles. This implies that certain angles are the same. Exemplarily, consider the isosceles triangle A, B, E

with base A, B. It holds true that $\sphericalangle_A(B,E) = \sphericalangle_B(E,A)$. For the phirotope, this means that $\mathrm{cr}_\varphi\,(b,e\,|\,i,j)_{\{a\}} = \mathrm{cr}_\varphi\,(e,a\,|\,i,j)_{\{b\}}$. The isosceles triangles are

triangle	$\triangle ABE$	$\triangle BCF$	$\triangle GPL$	$\triangle PWS$	$\triangle ZYV$	$\triangle YXU$	$\triangle TMQ$	$\triangle MDH$
base	A,B	B,C	G,P	P,W	Z,Y	Y,X	T,M	M,D

Furthermore, there are 13 rhombi. They also imply that certain angles are equal. For example the rhombus $\diamondsuit AEHD$ implies that $\sphericalangle_A(ED) = \sphericalangle_H(DE)$, $\sphericalangle_D(AH) = \sphericalangle_E(HA)$, and that $\sphericalangle_A(DE) + \sphericalangle_D(HA) = 180°$. Thus,

$$\begin{aligned}
&\mathrm{cr}_\varphi\,(e,d\,|\,i,j)_{\{a\}} = \mathrm{cr}_\varphi\,(d,e\,|\,i,j)_{\{h\}}\\
&\mathrm{cr}_\varphi\,(a,h\,|\,i,j)_{\{d\}} = \mathrm{cr}_\varphi\,(h,a\,|\,i,j)_{\{e\}}\\
&\mathrm{cr}_\varphi\,(e,d\,|\,i,j)_{\{a\}} \cdot \mathrm{cr}_\varphi\,(a,d\,|\,i,j)_{\{d\}} = 1
\end{aligned}$$

As can also read off the Figure 5.3, the rhombi of the configuration are

$$\begin{array}{ccccc}
\diamondsuit QUXT & & \diamondsuit RVYU & & \diamondsuit SWZV\\
& \diamondsuit NRUQ & & \diamondsuit OSVR & \\
\diamondsuit HMNQ & & \diamondsuit KORN & & \diamondsuit LPSO\\
& \diamondsuit EKNH & & \diamondsuit FLOK & \\
\diamondsuit AEHD & & \diamondsuit BFKE & & \diamondsuit CGLF
\end{array}$$

And finally, there are three collinearities: (A,B,C), (D,M,T), and (G,P,W), which gives $\varphi(a,b,c) = 0$, $\varphi(d,m,t) = 0$, and $\varphi(g,p,w) = 0$. Our goal is to deduce that also $\varphi(x,y,z) = 0$.

To simplify the calculations, we assign names to some cross ratios:

$$\begin{aligned}
\alpha &:= \mathrm{cr}_\varphi\,(e,d\,|\,i,j)_{\{a\}}\,, & \beta &:= \mathrm{cr}_\varphi\,(b,e\,|\,i,j)_{\{a\}}\,, & \gamma &:= \mathrm{cr}_\varphi\,(g,f\,|\,i,j)_{\{c\}}\,,\\
\delta &:= \mathrm{cr}_\varphi\,(f,e\,|\,i,j)_{\{b\}}\,, & \epsilon &:= \mathrm{cr}_\varphi\,(h,m\,|\,i,j)_{\{d\}}\,, & \eta &:= \mathrm{cr}_\varphi\,(p,l\,|\,i,j)_{\{g\}}\,.
\end{aligned}$$

The sum of angles in a triangle is 180°. Thus,

$$\begin{array}{lccccc}
\triangle ABE & \Rightarrow & \mathrm{cr}_\varphi\,(e,a\,|\,i,j)_{\{b\}} = \beta & \text{and} & \mathrm{cr}_\varphi\,(a,b\,|\,i,j)_{\{e\}} = \beta^{-2},\\
\triangle MDH & \Rightarrow & \mathrm{cr}_\varphi\,(d,h\,|\,i,j)_{\{m\}} = \epsilon & \text{and} & \mathrm{cr}_\varphi\,(m,d\,|\,i,j)_{\{h\}} = \epsilon^{-2},\\
\triangle GPL & \Rightarrow & \mathrm{cr}_\varphi\,(l,g\,|\,i,j)_{\{p\}} = \eta & \text{and} & \mathrm{cr}_\varphi\,(g,p\,|\,i,j)_{\{l\}} = \eta^{-2}.
\end{array}$$

From $\varphi(a,b,c) = 0$ and with the Grassmann-Plücker relation $(b,a\,|\,b,c,i,j)$ it follows that

$\mathrm{cr}_\varphi\,(a, c \,|\, i, j)_b = 1$. Thus,

$$\mathrm{cr}_\varphi\,(e, a \,|\, i, j)_b \cdot \mathrm{cr}_\varphi\,(f, e \,|\, i, j)_b \cdot \mathrm{cr}_\varphi\,(c, f \,|\, i, j)_b = 1,$$
$$\Rightarrow \mathrm{cr}_\varphi\,(c, f \,|\, i, j)_{\{b\}} = \beta^{-1}\delta^{-1}.$$

In the isosceles triangle $\triangle BCF$ we obtain:

$$\triangle BCF \qquad \Rightarrow \qquad \mathrm{cr}_\varphi\,(f, b \,|\, i, j)_{\{c\}} = \beta^{-1}\delta^{-1} \qquad \text{and} \qquad \mathrm{cr}_\varphi\,(b, c \,|\, i, j)_{\{f\}} = \beta^2\delta^2.$$

The relationship of angles in a rhombus gives:

$$\Diamond AEHD:$$
$$\Rightarrow \quad \mathrm{cr}_\varphi\,(d, e \,|\, i, j)_{\{h\}} = \alpha, \quad \mathrm{cr}_\varphi\,(a, h \,|\, i, j)_{\{d\}} = \alpha^{-1}, \quad \text{and} \quad \mathrm{cr}_\varphi\,(h, a \,|\, i, j)_{\{e\}} = \alpha^{-1}.$$
$$\Diamond BFKE$$
$$\Rightarrow \quad \mathrm{cr}_\varphi\,(e, f \,|\, i, j)_{\{k\}} = \delta, \quad \mathrm{cr}_\varphi\,(b, k \,|\, i, j)_{\{e\}} = \delta^{-1}, \quad \text{and} \quad \mathrm{cr}_\varphi\,(k, b \,|\, i, j)_{\{f\}} = \delta^{-1}.$$
$$\Diamond CGLF$$
$$\Rightarrow \quad \mathrm{cr}_\varphi\,(f, g \,|\, i, j)_{\{l\}} = \gamma, \quad \mathrm{cr}_\varphi\,(c, l \,|\, i, j)_{\{f\}} = \gamma^{-1}, \quad \text{and} \quad \mathrm{cr}_\varphi\,(l, c \,|\, i, j)_{\{g\}} = \gamma^{-1}.$$

Around one vertex, the angles add up to 360° which can be seen in terms of phirotopes by the fact that all cross ratios cancel. As an example, we show this for vertex E:

$$\mathrm{cr}_\varphi\,(a, b \,|\, i, j)_{\{e\}} \cdot \mathrm{cr}_\varphi\,(b, k \,|\, i, j)_{\{e\}} \cdot \mathrm{cr}_\varphi\,(k, h \,|\, i, j)_{\{e\}} \cdot \mathrm{cr}_\varphi\,(h, a \,|\, i, j)_{\{e\}} = 1$$

Thus,

$$\mathrm{cr}_\varphi\,(k, h \,|\, i, j)_{\{e\}} = \alpha\beta^2\delta.$$

The same argument applied to F gives $\mathrm{cr}_\varphi\,(l, k \,|\, i, j)_{\{f\}} = \beta^{-2}\gamma\delta^{-1}$.

By applying the same arguments to every rhombus, triangle or angles around one vertex, we propagate the information through the configuration. Finally, we arrive at

$$\mathrm{cr}_\varphi\,(x, u \,|\, i, j)_{\{y\}} = \beta^{-1}\epsilon^{-1}\eta^{-1},$$
$$\mathrm{cr}_\varphi\,(u, v \,|\, i, j)_{\{y\}} = \delta^{-1}\epsilon^{2}\eta^{2},$$
$$\mathrm{cr}_\varphi\,(v, z \,|\, i, j)_{\{y\}} = \beta\delta\epsilon^{-1}\eta^{-1}.$$

By multiplying all these, we obtain

$$\mathrm{cr}_\varphi\,(x, z \,|\, i, j)_{\{y\}} = \mathrm{cr}_\varphi\,(x, u \,|\, i, j)_{\{y\}} \cdot \mathrm{cr}_\varphi\,(u, v \,|\, i, j)_{\{y\}} \cdot \mathrm{cr}_\varphi\,(v, z \,|\, i, j)_{\{y\}} = 1.$$

Thus, in the Grassmann-Plücker relation $(y, x \mid y, z, i, j)$, we obtain

$$\varphi(x, y, z) \cdot \varphi(y, i, j) \in \{-1, 0, +1\}. \tag{5.13}$$

As φ is a Euclidean phirotope, it can be reoriented such that $\varphi_{\setminus\{i,j\}}$ is a chirotope and $\varphi(x, y, z)$ is real. Cross ratios – and thus all our calculations so far – are invariant under reorientation. Thus, (5.13) has to hold true for all reorientations, and as $\varphi(y, i, j) \neq 0$ it follows that $\varphi(x, y, z) = 0$. $\square$

5.3. Open problem: Extending chirotopes to Euclidean phirotopes

The following question is still open: Which rank-3 chirotopes can be extended to Euclidean phirotopes? The question is trivial, of course, for realisable chirotopes. By adding the points `I` and `J` to the realisation and then determining the phirotope of the extended configuration, the chirotope has been extended to a (realisable) Euclidean phirotope.

If all Euclidean phirotopes were realisable, then only realisable chirotopes were extendible to Euclidean phirotopes. There are, however, non-realisable Euclidean phirotopes. The following Example 5.20 constructs such a phirotope.

Example 5.20 To construct a non-realisable Euclidean phirotope, we start by considering the rank-3 vector configuration $V = (A, B, C, D, E, \mathtt{I}, \mathtt{J}) \subset \mathbb{RP}^2$ with

$$A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}.$$

The phirotope φ_V of this configuration is Euclidean. Its image consists of $\binom{7}{3} = 35$ values. Its domain is $\{a, b, c, d, e, i, j\}^3$, where we assume that the index a corresponds to the point A and so on. We obtain a non-realisable phirotope φ' by perturbing φ_V as follows:

$$\varphi'(\lambda) = \begin{cases} \omega(\det(A, E, \mathtt{I}) + 0.1), & \text{if } \lambda = (a, e, i), \\ \omega(\det(A, E, \mathtt{J}) + 0.1), & \text{if } \lambda = (a, e, j), \\ \varphi_V(\lambda), & \text{else.} \end{cases}$$

We check the non-violation of the $5 \cdot \binom{7}{5} = 105$ Grassmann-Plücker relations with the help of the Cinderella program depicted in the Appendix D. The perturbation yields a Euclidean phirotope, as

$$\overline{\varphi(a,e,j)} = \overline{\omega(\det(A,E,\mathtt{J})+0.1)} = \omega(\det(A,E,\mathtt{I})+0.1) = \varphi(a,e,i).$$

Furthermore, φ' is not realisable, as every five-point condition that contains exactly one of the terms $\varphi(a,e,i)$ or $\varphi(a,e,j)$ does not yield zero: If in the five-point condition all but one terms are fixed, then also the last one is fixed. Thus, the perturbation of $\varphi(a,e,i)$ and $\varphi(a,e,j)$ results in the five-point condition not being satisfied any more.

The perturbation in the above example is chosen relatively small in order not to violate the phirotope axiom (φ2), that is, in order not to obviously violate the Grassmann-Plücker relations. A bigger perturbation – although not violating the non-realisability and the Euclideaness, if applied symmetrically to $\varphi(a,e,i)$ and $\varphi(a,e,j)$ – might yield a structure that is no longer a phirotope.

So far we have only encountered chirotopes that can be extended to Euclidean phirotopes. The question whether all chirotopes can be extended to Euclidean phirotopes has yet to be answered.

6. Why Phirotopes are Boring

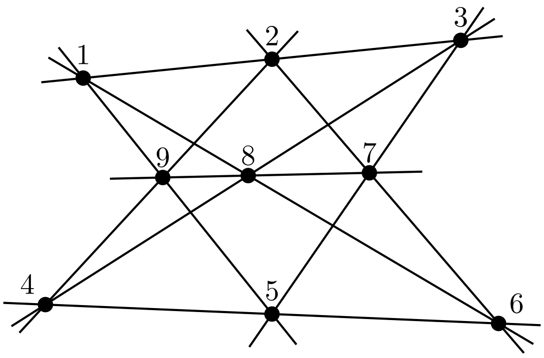


Figure 6.1.: The theorem of Pappus.

The theorem of Pappus (cf. Figure 6.1) gives rise to the smallest non-realisable uniform oriented matroid in rank 3. The non-realisable arrangement is obtained by perturbing all collinearities. It was found by Ringel (cf. [Rin56]), and Grünbaum conjectured in [Grü72] that it is indeed the smallest instance. Later, Goodmann and Pollack proved this conjecture (cf. [GP80]).

The theorem of Pappus is a good starting point for producing non-realisable oriented matroids. Perturbing one collinearity is already enough to obtain a (not uniform) non-realisable oriented matroid.

When dealing with non-chirotopal complex oriented matroids, the realisability can be decided using the five-point condition. Thus, exactly one of the two following cases has to be true: Either, the theorem of Pappus can be proven using the five-point condition, as the theorem of Pappus holds true for all realisable phirotopes, or, the theorem of Pappus holds true for all non-chirotopal phirotopes irrespective of their realisability. This would mean that already the phirotope axioms and the non-chirotopality force the theorem of Pappus to hold true. In fact, the second case is true: the theorem of Pappus holds true for *all* non-chirotopal phirotopes and it is impossible to prove the theorem of Pappus using the five-point condition.

In the course of this chapter we will see that neither the theorem of Pappus nor that of Desargues can be used to construct non-realisable non-chirotopal phirotopes. Both of them always hold true for non-chirotopal phirotopes – regardless of realisability. The absence of configurations that stem from a perturbed theorem of Pappus is the reason for this chapter's name, which is to be taken with a pinch of salt.

For the proofs in this chapter, we will use bi-quadratic final polynomials. Final polynomials were introduced in [BG87], [BS87], and [BRS90]. Dress, Sturmfels, and Whiteley independently pointed out that every non-realisable (real) oriented matroid has a final polynomial that proves its non-realisability (cf. [BS89] and [Whi91]). Much effort was made to construct final polynomials algorithmically and in that course, Bokowski and Richter-Gebert in [BR90] introduced bi-quadratic final polynomials, a class of final polynomials that can be computed very efficiently (cf. [RG92b], [RG93], [FMNRG09]).

We start the chapter by extending the Lemma 4.5 to non-uniform phirotopes:

Lemma 6.1
Let φ be a rank-3 phirotope on $\mathcal{E}$ with $a, b, c, d, e, f \in \mathcal{E}$. If $\mathrm{cr}_\varphi\,(a, b \,|\, c, d)_{\{f\}} = \pm 1$ and $\mathrm{cr}_\varphi\,(a, b \,|\, c, e)_{\{f\}} = \pm 1$, then all cross ratio phases on the indices a, b, c, d, and e seen from $\{f\}$ are real, 0, or ∞.

Proof. Let $\mathrm{cr}_\varphi\,(a, b \,|\, c, d)_{\{f\}} = \pm 1$. Thus, it holds true that

$$\varphi(f, a, c)\varphi(f, b, d) = \pm\varphi(f, a, d)\varphi(f, b, c). \tag{6.1}$$

Consider the Grassmann-Plücker relation $(f, a \,|\, f, b, c, d)$ for which there are r_1, r_2, $r_3 \in \mathbb{R}^+$ such that

$$r_1 \cdot \varphi(f, a, b)\varphi(f, c, d) - r_2 \cdot \varphi(f, a, c)\varphi(f, b, d) + r_3 \cdot \varphi(f, a, d)\varphi(f, b, c) = 0.$$

The Equation 6.1 yields that for this to hold true, the first term $\varphi(f, a, b)\varphi(f, c, d)$ has to be zero, or equal to $\pm\varphi(f, a, c)\varphi(f, b, d)$ (and $\pm\varphi(f, a, d)\varphi(f, b, c)$). Thus, it holds true that

$$\mathrm{cr}_\varphi\,(\pi(a), \pi(b) \,|\, \pi(c), \pi(d))_{\{f\}} \in \{-1, 0, +1, \infty\} \quad \forall \pi \in S_4.$$

The same holds true for all $\mathrm{cr}_\varphi\,(\pi(a), \pi(b) \,|\, \pi(c), \pi(e))_{\{f\}}$ which is also equal to ± 1 according to the assumption of the lemma. By dividing

$$\pm 1 = \mathrm{cr}_\varphi\,(a, b \,|\, c, d)_{\{f\}} : \mathrm{cr}_\varphi\,(a, b \,|\, c, e)_{\{f\}} = \mathrm{cr}_\varphi\,(a, b \,|\, e, d)_{\{f\}}$$

we furthermore obtain that

$$\mathrm{cr}_\varphi\,(\pi(a), \pi(b) \,|\, \pi(d), \pi(e))_{\{f\}} \in \{-1, 0, +1, \infty\} \quad \forall \pi \in S_4.$$

□

With this, we are ready to state the main theorem of this chapter.

Theorem 6.2
There is no non-chirotopal non-Pappus phirotope.

The proof of this theorem has the following structure: We will prove that if exactly eight of the nine collinearities of the theorem of Pappus are present in a phirotope, then it is chirotopal. (If all nine collinearities are present, then the phirotope can be non-chirotopal. For an example of concrete coordinates see Appendix F.)

To prove this, we will use bi-quadratic final polynomials to deduce from the given eight collinearities equations that contain the term that in a realisable setting would be the ninth collinearity.

Then we will see how this already forces many cross ratio phases to be real. In the last step, we will show that indeed all cross ratio phases are real (or contain a term which is zero) and, thus, the phirotope is chirotopal. This is done by an exhaustive enumeration of all cross ratio phases in Mathematica.

Proof of Theorem 6.2. Consider the non-chirotopal phirotope φ on $\mathcal{E} = [9]$ that has the same combinatorics as the point configuration given in Figure 6.1, but assume that it does not have the non-base $\{7, 8, 9\}$: We collect all non-bases in the set $\mathcal{K}$,

$$\mathcal{K} = \Big\{(1,2,3),(1,5,9),(1,6,8),(2,4,9),(2,6,7),(3,4,8),(3,5,7),(4,5,6)\Big\},$$

and assume

$$\varphi(\lambda) = 0 \;\Leftrightarrow\; \lambda \in \mathcal{K}.$$

The first part of the proof will be to show that all cross ratio phases that contain the three indices 7, 8, and 9 and are viewed from one of the indices 7, 8, or 9 are real valued or contain a zero term. The eight given collinearities cause Grassmann-Plücker relations to contract. For example, as $\varphi(1, 2, 3) = 0$, according to the Lemma 4.14 it holds true that

$$\varphi(1,2,4)\varphi(1,3,7) = \varphi(1,2,7)\varphi(1,3,4). \tag{6.2}$$

This means that the cross ratio phase $\mathrm{cr}_\varphi\,(2,3\,|\,4,7)_{\{1\}}$ is real valued and non-zero. In fact, with the same argument we obtain that

$$\mathrm{cr}_\varphi\,(b,c\,|\,x,y)_{\{a\}} \in \mathbb{R} \setminus \{0\} \tag{6.3}$$

for all pairwise different $a, b, c, x, y \in [9]$ such that $\varphi(a, b, c) = 0$. Note that interchanging two indices that form a pair (for example b and c) will yield the reciprocal of the cross ratio phase.

This is real if and only if the cross ratio phase we started with was real. Up to such interchanges there are

$$8 \cdot 3 \cdot \binom{6}{2} - 3 \cdot 6 - 3 = 339$$

different cross ratio phases: 8 is the number of collinearities (a, b, c) that we have postulated, 3 is the number of possibilities for choosing the index that appears in every index triple, $\binom{6}{2}$ is the number of pairs of the remaining 6 indices that can be chosen for x and y. However, we double counted some cross ratio phases: in the Grassmann-Plücker relation the first term might consist of two collinearities. For example, in the Grassmann-Plücker relation

$$\varphi(1,2,3)\varphi(1,6,8) - \varphi(1,2,6)\varphi(1,3,8) + \varphi(1,2,8)\varphi(1,3,6)$$

both $\varphi(1,2,3)$ and $\varphi(1,6,8)$ equal zero. Each of the indices in [6] is involved in 3 collinearities (thus we have to subtract $6 \cdot 3$), each of the indices in $\{7,8,9\}$ is involved in two collinearities (thus we further subtract 3).

We now use bi-quadratic final polynomials to derive cross ratio phases containing $\varphi(7,8,9)$. Note that all non-bases yield equalities, similar to the Equation 6.2:

$$\begin{aligned}
\varphi(1,2,3) &= 0 &&\Rightarrow \varphi(1,2,7)\varphi(1,3,4) = \varphi(1,2,4)\varphi(1,3,7),\\
\varphi(1,5,9) &= 0 &&\Rightarrow \varphi(1,5,7)\varphi(1,9,4) = \varphi(1,4,5)\varphi(1,7,9),\\
\varphi(1,6,8) &= 0 &&\Rightarrow \varphi(1,4,6)\varphi(1,7,8) = \varphi(1,6,7)\varphi(1,8,4),\\
\varphi(2,4,9) &= 0 &&\Rightarrow \varphi(1,2,4)\varphi(4,7,9) = \varphi(1,9,4)\varphi(2,4,7),\\
\varphi(2,6,7) &= 0 &&\Rightarrow \varphi(1,6,7)\varphi(2,4,7) = \varphi(1,2,7)\varphi(6,4,7),\\
\varphi(3,4,8) &= 0 &&\Rightarrow \varphi(1,4,8)\varphi(3,4,7) = \varphi(1,3,4)\varphi(8,4,7),\\
\varphi(3,5,7) &= 0 &&\Rightarrow \varphi(1,3,7)\varphi(5,4,7) = \varphi(1,5,7)\varphi(3,4,7),\\
\varphi(4,5,6) &= 0 &&\Rightarrow \varphi(1,4,5)\varphi(6,4,7) = \varphi(1,4,6)\varphi(5,4,7).
\end{aligned}$$

These Grassmann-Plücker relations were generated by the indices 1, 4, and 7 and that of the respective collinearity. Multiplying all left sides and all right sides and cancelling out terms that appear on both sides yields:

$$\varphi(1,7,8)\varphi(4,7,9) = \varphi(1,7,9)\varphi(4,8,7) \tag{6.4}$$

Thus, also the cross ratio phase $\mathrm{cr}_\varphi\,(1,4\,|\,8,9)_{\{7\}}$ is real valued.

With the same method we can in principle deduce all cross ratio phases of the forms $\mathrm{cr}_\varphi\,(x,y\,|\,8,9)_{\{7\}}$, $\mathrm{cr}_\varphi\,(x,y\,|\,9,7)_{\{8\}}$, and $\mathrm{cr}_\varphi\,(x,y\,|\,7,8)_{\{9\}}$. The total number of cross ratio phases of the these forms is $3 \cdot 13 = 39$, as for each index 7, 8, and 9, all but two pairs of indices from [6]

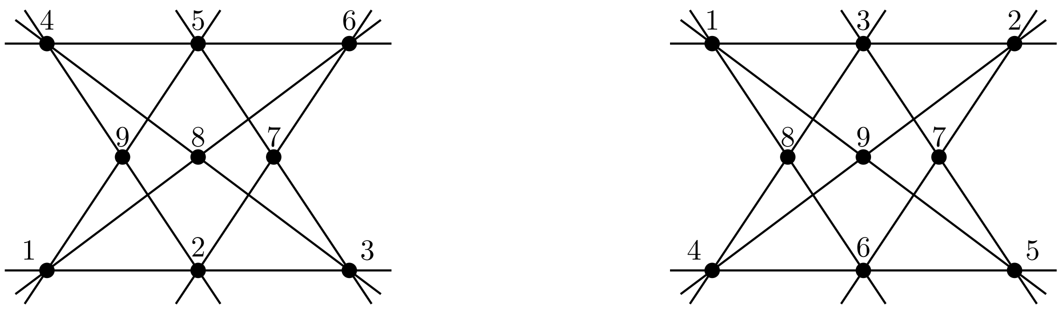


Figure 6.2.: The generators of the automorphism group of the examined configuration that leave the index 7 invariant.

can be chosen as x and y (two pairs each form a collinearity with the respective index 7, 8, or 9). To shorten the proof, we exploit the symmetry of the theorem of Pappus: The theorem of Pappus is completely symmetrical in the indices 7, 8, and 9. Thus, it suffices to find bi-quadratic final polynomials viewed from 7. There are automorphisms of the configuration that leave the position of the index 7 invariant, see Figure 6.2. Note that the second generator of the automorphism group of the theorem of Pappus that leaves the index 7 invariant interchanges the indices 8 and 9. Interchanging indices within a pair in the cross ratio phase can be neglected as it yields the reciprocal but does not have any influence on the imaginary part of the value of the cross ratio phase. Then, the following pairs of indices have bi-quadratic final polynomials that are equivalent with respect to the symmetry:

$$
\begin{aligned}
&(1,2) \sim (1,3) \sim (4,5) \sim (4,6), && (1,5) \sim (1,6) \sim (2,4) \sim (3,4),\\
&(2,3) \sim (5,6), && (2,5) \sim (3,6),\\
&(1,4), && (2,6) \sim (3,5).
\end{aligned}
$$

Thus, it suffices to find bi-quadratic final polynomials for only one of each of these six equivalence classes to guarantee that all bi-quadratic final polynomials can be found. We have listed six such bi-quadratic final polynomials in the Appendix G.

In the second step of the proof, we use the cross ratio phases that were proven to be real with bi-quadratic final polynomials to argue that further cross ratio phases have to be real:
If we consider the Grassmann-Plücker relation $(1,7\,|\,4,7,8,9)$, we know that there are r_1, r_2, $r_3 \in \mathbb{R}^+$ such that

$$r_1 \cdot \varphi(1,7,4,)\varphi(7,8,9) + r_2 \cdot \varphi(1,7,8)\varphi(4,7,9) - r_3 \cdot \varphi(1,7,9)\varphi(4,7,8) = 0.$$

Applying the Equation 6.4 we know that the last two terms have opposing phases. We have assumed that we do not have more than eight zeros in the image of the phirotope. Therefore, the phase of the term $\varphi(1,7,4,)\varphi(7,8,9)$ has to be one of the two phases in order not to obviously violate the Grassmann-Plücker relation. Hence, it holds true that

$$\varphi(7,8,9) = \pm \frac{\varphi(1,7,8)\varphi(4,7,9)}{\varphi(1,4,7)} \tag{6.5}$$

and, thus, the cross ratio phase $\mathrm{cr}_\varphi\,(1,9\,|\,8,4)_{\{7\}}$ is real valued and non-zero. Note that while multiplying degenerate Grassmann-Plücker relations in order to obtain new ones is a linear operation, the deduction that gave the Equation 6.5 is not.

The third step was done by an exhaustive search in Mathematica (cf. Appendix H or `https://www-m10.ma.tum.de/users/schaar/No_Non-Pappus_Phirotope.nb`): From the cross ratio phases that are now known to be real, we can deduce all remaining cross ratio phases that do not contain a zero with a bi-quadratic final polynomial. This is a contradiction to the assumption that φ is chirotopal and, thus, the set $\{7,8,9\}$ cannot be a basis. $\square$

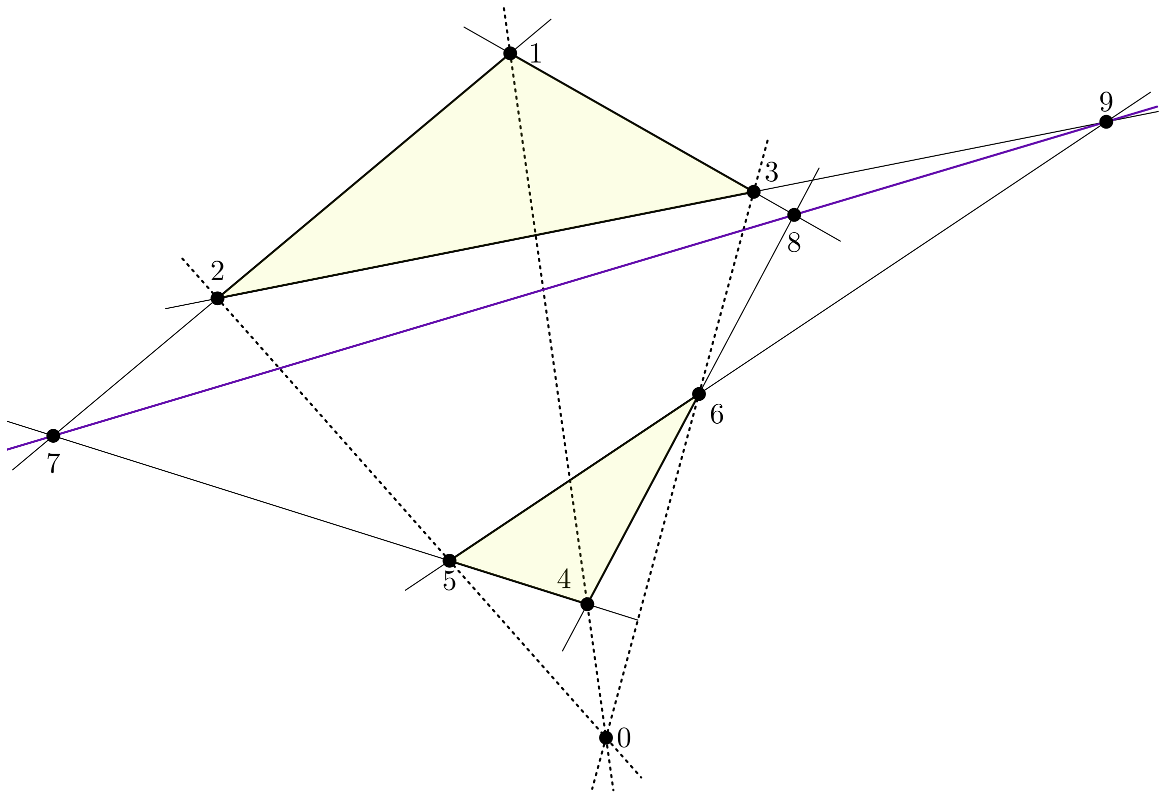


Figure 6.3.: The theorem of Desargues.

Theorem 6.3
There is no non-chirotopal non-Desargues phirotope.

Proof. A perturbation of the theorem of Desargues (cf. Figure 6.3) is only possible if the phirotope is chirotopal. The proof works similarly to the proof of the Theorem 6.2:

Firstly, we collect all cross ratios that are known to be real as they stem from Grassmann-Plücker relations that consist of only two terms (the last one contains a collinearity).
Secondly, we generate further cross ratios that are known to be real by linearly combining cross ratios of the first step.
Thirdly, we examine those cross ratios from the second step whose numerators and denominators are two terms of a Grassmann-Plücker relation that does not contain a zero. Up to a minus, the last term of this Grassmann-Plücker relation then has to have the same phase as the numerator or denominator and, thus, we obtain two new cross ratios both of which are real.
Fourthly, we linearly combine all cross ratios that are now known to be real and obtain a larger set of real cross ratios.

While the proof for the theorem of Pappus was already complete at this step as all cross ratios have been proven to be real, the theorem of Desargues needs another iteration of the process: An exhaustive search in Mathematica shows that not all cross ratios can be generated by a linear combination yet. So we again perform the third step and obtain new cross ratios that are known to be real due to the interaction of a cross ratio that is already known to be real and a Grassmann-Plücker relation that contains the numerator and denominator of this cross ratio but no zero.
Finally, another exhaustive search in Mathematica proves that now all cross ratios are real as the remaining ones can be generated by a linear combination of real cross ratios.

The Mathematica code corresponding to this proof can be found in the Appendix H or downloaded here: `https://www-m10.ma.tum.de/users/schaar/No_Non-Pappus_Phirotope.nb`. □

Note that while the Theorems 6.2 and 6.3 make a statement about non-bases that have to be present in a phirotope, it might still be the case that a phirotope exhibits all non-bases of the theorem of Pappus but is not realisable. This happens, if in a uniform minor the five-point condition is violated. The structure that is only concerned with the bases and non-bases of a phirotope is the "underlying matroid".

Definition 6.4 (Underlying matroid)
Let φ be a rank-d phirotope on $\mathcal{E}$. The function $\mathcal{M}_\varphi : \mathcal{E}^d \to \{0,1\}$

$$\mathcal{M}_\varphi(\lambda) = \begin{cases} 0, & \text{if } \varphi(\lambda) = 0, \\ 1, & \text{else.} \end{cases}$$

defines a matroid on $\mathcal{E}$ which is called the *underlying matroid of* φ.

The Theorems 6.2 and 6.3 showed that we are facing major obstacles when trying to perturb incidence theorems. With the help of bi-quadratic final polynomial, we have proven that the phirotopes of at least two incidence theorems cannot be perturbed in such a way that the result is still a phirotope. However, incidence theorems are the main source of non-realisable configurations. This leads us to the following conjecture.

Conjecture
If a (non-realisable) matroid admits a bi-quadratic final polynomial, then this matroid cannot be the underlying matroid of a non-chirotopal phirotope.

The next step would be to examine the matroid Ω_{14}^{-} from [RG96], a non-realisable matroid that does not admit a bi-quadratic final polynomial. We speculate that one would nevertheless be able to find a non-chirotopal phirotope that has Ω_{14}^{-} as underlying matroid. Thus, our stronger conjecture is the following.

Conjecture
The underlying matroids of non-chirotopal phirotopes are realisable.

7. Open Problems and Conjectures

7.1. Pseudolines

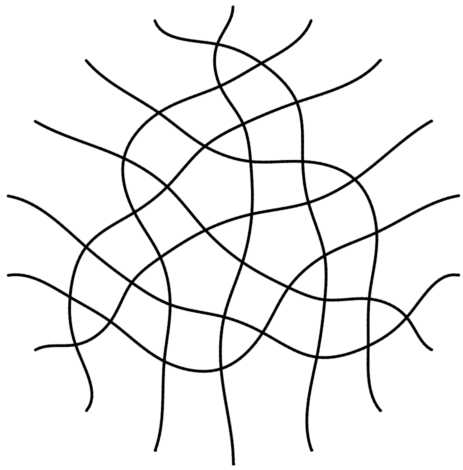


Figure 7.1.: The pseudoline arrangement used in the logo of the Chair of Geometry and Visualisation at the Technical University of Munich.

In (real) oriented matroid theory, pseudolines play an important role. With the help of pseudolines, it is possible to visualise even non-realisable oriented matroids of rank 3. For example, the logo of the chair at which this thesis is created is a pseudoline arrangement, see Figure 7.1. It depicts the smallest non-realisable rank-3 oriented matroid, namely the perturbed configuration of the theorem of Pappus.

Up until now, we have lacked a similar way of visualising (non-realisable) phirotopes. The main reason for that is – yet again – the non-discrete range of phirotopes. The range of chirotopes, $\{-1, 0, +1\}$, can (roughly speaking) be transferred to "left" and "right" relations in pseudoline arrangements. For phirotopes, a similar arrangement would furthermore need to carry information about the phases. Furthermore, a pseudoline arrangement is stretchable, if and only if the corresponding chirotope is realisable. It would be desirable if the equivalent of pseudoline arrangements for phirotopes also exhibited this property.

7.2. Geometric interpretation of the five-point condition

The five-point condition is a sum over products of phirotope values. In the realisable case, the phirotope values are the phases of determinants of points with complex coordinates. In rank 2, consider the phase of the determinant of two arbitrary points $A, B \in \mathbb{CP}^1$. W.l.o.g., let $A = \begin{pmatrix} a, & 1 \end{pmatrix}^T$ and $B = \begin{pmatrix} b, & 1 \end{pmatrix}^T$ with $a, b \in \mathbb{C}$. Then, it holds true that

$$\omega(\det(A, B)) = \frac{a - b}{|a - b|},$$

which can be thought of as the direction from the complex point a to the complex point b. We understand that multiplying two complex numbers corresponds to adding their phases. So the products of phirotope values might be interpreted as the sum of all angles of the directions of the vectors pointing from one complex point to the next. Furthermore, we visualise adding or subtracting two complex numbers as concatenating the vectors that represent these numbers.

Up until now, however, we have not been able to fit all these interpretations into one general picture. Thus, we are still looking for a geometric interpretation of the five-point condition.

7.3. Singularities in the space of phirotopes

It is an open problem to further investigate the relationship between chirotopal phirotopes and non-chirotopal phirotopes. We have seen that chirotopes are phirotopes. In fact, chirotopes can be considered singularities in the space of phirotopes. It is a common effect that in singular situations structures become ambiguous. The intersection of two lines, for example, is exactly one point – except for the case in which the two lines coincide. Likewise, the realisations of non-chirotopal phirotopes are rigid while the realisations of chirotopal phirotopes are not.

Often, the relation of the singularity and the ambient space is worth a close examination. An example of this is given by quadrilateral sets. Six points on a common line form a quadrilateral set, if they are the projection of the six intersections of four lines, see Figure 7.2. A limit process

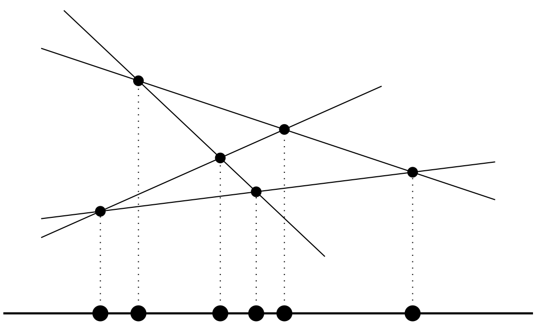


Figure 7.2.: A quadrilateral set obtained by parallelly projecting the six points of intersection of four lines onto another line.

in which we let the four lines converge will yield one line on which there are six points. A

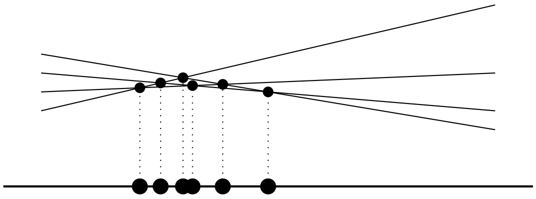


Figure 7.3.: If the four generating lines of a quadrilateral set are almost the same, we are close to a singular situation.

natural question to ask is the following: Consider six points on a line. Is there a limit process as described above such that these six points are the limit? This question is old and for details the reader is referred to [RG11]. The answer can also be found in said reference: Six points $A, B, C, D, E, F \in \mathbb{RP}^1$ form a quadrilateral set, if their coordinates satisfy

$$[A, E][B, F][C, D] = [A, F][B, D][C, E].$$

The questions that concern us in the context of phirotopes are the following:

- Is there a limit process that transforms phirotopes into chirotopes?
- Is there a characterisation of chirotopes that are the limit of this process?
- Which properties are invariant under taking the limit in this sense? Can we use phirotopes and limit processes to further investigate chirotopes?

7.4. More on incidence theorems on phirotopes and on the five-point condition

Since the findings in the Chapter 6 were intriguing, we would like to acquire a deeper understanding of the subject matter. Rather than having an exhaustive enumeration programmed in Mathematica, we would prefer a readable proof that would help us gain further insights. The most important questions here are:

- *Why* is it not possible to bend the lines that are subject of the theorems of Pappus and Desargues?
- Are there incidence theorems that do *not* always hold true on the level of phirotopes? Do configurations exist that contain all collinearities that are the requirements of an incidence theorem but lack the one that corresponds to the conclusion? Or are all incidence theorems on phirotopes always true?

- The five-point condition guarantees realisability of non-chirotopal phirotopes. With the results from Chapter 6 it seems that the phirotope axioms themselves already exclude a lot of configurations that might have been non-realisable. Loosely speaking, after violating the Grassmann-Plücker relations, violating an incidence theorem is the next naive thing one could try to construct a non-realisable phirotope. This calls for further investigations of the five-point condition. Why is it able to filter out the non-realisable phirotopes of the "not naively non-realisable" (that is to say: all) phirotopes?
- With the help of an answer to the last point: can we (further) reduce and simplify the five-point condition?

Of course, we would also be very interested in proofs or counterexamples for the two conjectures that were already presented in the Chapter 6:

Conjecture
If a (non-realisable) matroid admits a bi-quadratic final polynomial, then this matroid cannot be the underlying matroid of a non-chirotopal phirotope.

Conjecture
The underlying matroids of non-chirotopal phirotopes are realisable.

Appendices

A. Example of a Non-Realisable Uniform Rank-2 Phirotope

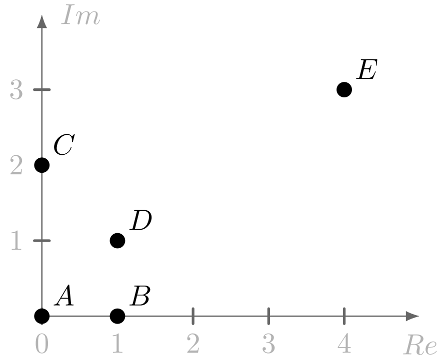


Figure A.4.: The configuration that is used as a starting point to construct a non-realisable phirotope.

As a starting point, we use the configuration given in the Figure A.4. This yields the following phirotope:

$$
\begin{aligned}
&\varphi(a,b) = -1, && \\
&\varphi(a,c) = -\mathrm{i}, && \varphi(b,c) = \frac{1}{\sqrt{5}}(1-2\mathrm{i}), \\
&\varphi(a,d) = \frac{1}{\sqrt{2}}(-1-\mathrm{i}), && \varphi(b,d) = -\mathrm{i}, \\
&\varphi(a,e) = \frac{1}{5}(-4-3\mathrm{i}), && \varphi(b,e) = \frac{1}{\sqrt{2}}(-1-\mathrm{i}), \\
&&& \\
&\varphi(c,d) = \frac{1}{\sqrt{2}}(-1+\mathrm{i}), && \\
&\varphi(c,e) = \frac{1}{\sqrt{17}}(-4-\mathrm{i}), && \varphi(d,e) = \frac{1}{\sqrt{13}}(-3-2\mathrm{i}).
\end{aligned}
$$

We perturb this sightly to obtain a new phirotope φ' on $\{a,b,c,d,e\}$ which is

$$
\varphi'(x,y) = \begin{cases} \frac{1}{5}(-4-3\mathrm{i}), & \text{if } (x,y) = (d,e) \\ \varphi(x,y), & \text{else.} \end{cases}
$$

To see that this is still a phirotope, consider all Grassmann-Plücker relations, in which $\varphi'(d,e)$ occurs:

$(a \mid b, d, e)$:

$$15 \cdot (\varphi(a,b) \cdot \varphi'(d,e)) - 25 \cdot (\varphi(a,d) \cdot \varphi(b,e)) + 20 \cdot (\varphi(a,e) \cdot \varphi(b,d)) = 0,$$

$(a \mid c, d, e)$:

$$\begin{aligned} 19\sqrt{2} \cdot (\varphi(a,c) \cdot \varphi'(d,e)) - 5\sqrt{17} \cdot (\varphi(a,d) \cdot \varphi(c,e)) \\ +27 \cdot (\varphi(a,e) \cdot \varphi(c,d)) = 0, \end{aligned}$$

$(b \mid c, d, e)$:

$$4\sqrt{5} \cdot (\varphi(b,c) \cdot \varphi'(d,e)) - \sqrt{17} \cdot (\varphi(b,d) \cdot \varphi(c,e)) + 7 \cdot (\varphi(b,e) \cdot \varphi(c,d)) = 0.$$

As all factors are positive, the phirotope axiom (φ2) is satisfied. The five-point condition, however, will not yield zero (but approximately 0.11 − 0.04i).

B. Verifying Dependencies in the Five-Point Condition with Python (Equation 4.12)

In [1]:
```
from sympy import *
```

In [2]:
```
ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf = symbols('ab ac ad ae af
bc bd be bf cd ce cf')
```

In [3]:
```
t= Matrix([ac*bc,ab*bc,ab*ac])
s= Matrix([cf,bf,af])
f= Matrix([ce,be,ae])
v= Matrix([cd,bd,ad])
```

In [4]:
```
a= t.row_join(f).row_join(v)
c= t.row_join(s).row_join(v)
e= t.row_join(s).row_join(f)
A= a.det()
C= c.det()
E= e.det()
```

In [5]:
```
b= Matrix([cd*ce*ab,bd*be*ac,ad*ae*bc]).row_join(f).row_join(v)
B= b.det()
d= Matrix([cd*cf*ab,bd*bf*ac,ad*af*bc]).row_join(s).row_join(v)
D= d.det()
f= Matrix([ce*cf*ab,be*bf*ac,ae*af*bc]).row_join(s).row_join(f)
F= f.det()
```

In [6]:
```
B = -B
D = -D
F = -F
```

In [7]:
```
U= ad*be*bf*af-ae*bd*bf*af
V= ae*be*bd*af-ad*ae*be*bf
W= bd*ad*ae*bf-ad*bd*be*af
```

In [8]:
```
X= ab*ae*bf-ab*be*af
Y= ab*bd*af-ab*ad*bf
Z= ad*ab*be-bd*ab*ae
```

In [9]:
```
factor(-A*C*F*W-A*D*E*V- A*D*F*Z-B*D*E*X-B*C*E*U-B*C*F*Y)
```

Out[9]:
```
0
```

C. Verifying Dependencies in the Five-Point Condition with Python (Equation 4.13)

In [1]:
```
from sympy import *
```

In [2]:
```
ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf = symbols('ab ac ad ae af
bc bd be bf cd ce cf')
```

In [3]:
```
t= Matrix([ac*bc,ab*bc,ab*ac])
s= Matrix([cf,bf,af])
f= Matrix([ce,be,ae])
v= Matrix([cd,bd,ad])
```

In [4]:
```
a= t.row_join(f).row_join(v)
c= t.row_join(s).row_join(v)
e= t.row_join(s).row_join(f)
A= a.det()
C= c.det()
E= e.det()
```

In [5]:
```
b= Matrix([cd*ce*ab,bd*be*ac,ad*ae*bc]).row_join(f).row_join(v)
B= b.det()
d= Matrix([cd*cf*ab,bd*bf*ac,ad*af*bc]).row_join(s).row_join(v)
D= d.det()
f= Matrix([ce*cf*ab,be*bf*ac,ae*af*bc]).row_join(s).row_join(f)
F= f.det()
```

In [6]:
```
B = -B
D = -D
F = -F
```

In [7]:
```
de=B/A
df=D/C
ef=F/E
```

In [8]:
```
U= ad*be*bf*af-ae*bd*bf*af
V= ae*be*bd*af-ad*ae*be*bf
W= bd*ad*ae*bf-ad*bd*be*af
```

In [9]:
```
X= ab*ae*bf-ab*be*af
Y= ab*bd*af-ab*ad*bf
Z= ad*ab*be-bd*ab*ae
```

In [10]:
```
factor(U*de+V*df+W*ef+X*de*df+Y*de*ef+Z*df*ef)
```

Out[10]:
```
0
```

D. Example of a Non-Realisable Euclidean Phirotope

The following CindyScript code generates a non-realisable Euclidean phirotope and checks that all requirements are fulfilled. The program was written in Cinderella.2 (version 2.8). Syntax highlighting was done with CindyScriptPygments by von Gagern (cf. [vG16]).

Slot "Initialization":

```
//generation of all quintuples used in GPRs
n=7;
list=[];
apply(1..n,aa,
  rest=(1..n--[aa]);
  apply(1..(n-4),bb,
    apply((bb+1)..(n-3),cc,
      apply((cc+1)..(n-1),dd,
        apply((dd+1)..(n-1),ee,
           list=list++[[aa,rest_bb,rest_cc,rest_dd,rest_ee]];
        );
      );
    );
  );
);
Permut=[[1,2,3,4],[1,2,4,3],[1,3,2,4],[1,3,4,2],[1,4,2,3], [1,4,3,2],
  [2,1,3,4],[2,1,4,3],[2,3,1,4],[3,1,2,4], [3,1,4,2],[3,2,1,4]];
```

Slot "Draw":

```
//functions used to determine the signum of a permtation
sgn(s):=if(s~=0,0,if(s<0,-1,1));
mom=apply(1..n,(1,#,#^2));
sig(a,b,c):=sgn(det(mom_a,mom_b,mom_c));

//function to draw the terms of GPRs
ph(v):=if(v~=0,0,v/|v|)*.3;

//start configuration, the points k and l are I and J,
//respectively.
a=(0,0,1);
```

```
b=(1,0,1);
c=(0,2,1);
d=(1,1,1);
e=(4,3,1);
k=(-i,1,0);
l=(i,1,0);
pts=(a,b,c,d,e,k,l);

//perturbation of the phirotope values [1,5,6] and [1,5,7]
ind=triples(1..n);
pert=[];
apply(ind,pert:#=0);
pert:[1,5,6]=0.1;
pert:[1,5,7]=0.1;

//function that evaluates all phirotope values
phval(i,j,k):=ph(pert:(sort([i,j,k]))+det(pts_(sort([i,j,k]))))*sig(i,j,k);

//the routine good determines whether all GPRs are not violated
good(a,b,c):=(
s=sort([a,b,c],|#|);
erg=false;
a=s_1/s_3;
b=s_2/s_3;
c=1;
if(a~=0,
  if(b/|b|~=-c/|c|,erg=true),
    if((im(a)~!=0) % (im(b)~!=0),
    det=det((gauss(a)++[1],gauss(b)++[1],gauss(c)++[1]));
    dd1=det((gauss(a),gauss(b)));
    dd2=det((gauss(b),gauss(c)));
    dd3=det((gauss(c),gauss(a)));
    if(dd1*det~>0 & dd2*det~>0 & dd3*det~>0 ,erg=true),
    if((re(a)<0)%(re(b)<0),erg=true)
  );
);
if(erg==false,println("no phirotope"));
```

```
erg
);

//functions that draws the terms of the GPRs on the canvas
drawgp(a,b,c,d,e):=(
  l1=phval(a,b,c)*phval(a,d,e);
  l2=-phval(a,b,d)*phval(a,c,e);
  l3=phval(a,b,e)*phval(a,c,d);
  col=if(good(l1,l2,l3),(0,1,0),(1,0,0));
  draw((0,0),gauss(ph(l1)));
  draw((0,0),gauss(ph(l2)));
  draw((0,0),gauss(ph(l3)));
  draw(gauss(ph(l1)),size->2,color->col);
  draw(gauss(ph(l2)),size->2,color->col);
  draw(gauss(ph(l3)),size->2,color->col);
    drawtext((-.4,-.7),a+":"+""+b+""+c+""+d+""+e,size->7);
);
drawgp(l):=drawgp(l_1,l_2,l_3,l_4,l_5);

break=sqrt(length(list))*1.4;
scale(1/break*20);
x=0;
y=0;
apply(list,
   translate((x,y));
   drawgp(#);
   translate((-x,-y));
   x=x+1;
   if(x>break,
     x=0;y=y+1.2
   );
);

//generation of the terms of the five-point formula
xi(a,b,c,d,e):=(
phval(a,b,c)^2* phval(b,c,d)^2* phval(c,d,e)^2* phval(d,e,a)^2* phval(e,a,b)^2;
);
```

```
Quint=[];
forall(pairs(1..7),p,
        Quint=append(Quint,1..7--p);
);

real=0;
pf=0;
E=[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,1,0],[0,0,0,0,1]];

//check if the five-point formula yields zero
forall(Quint,L,
  repeat(length(Permut),i,
    per=Permut_i;
        s=det([E_(per_1),E_(per_2),E_(per_3),E_(per_4),E_5]);
        pf=pf+ s* xi(L_(per_1),L_(per_2),L_(per_3),L_(per_4),L_5);
        );
        if(pf~!=0,real=real+1);
);
if(real==0,println("realisable"),println("non-realisable"));

//check if the phirotope is still chirotopal
chiro=0;
forall((2,3,4),p,
  forall((2,3,4),q,
    d1=phval(1,5,6)*phval(p,q,6)/(phval(1,q,6)*phval(p,5,6));
    d2=phval(1,5,7)*phval(p,q,7)/(phval(1,q,7)*phval(p,5,7));
    if(d1!=conjugate(d2),chiro=chiro+1);
  );
);
if(chiro==0,println("chirotopal"),println("non-chirotopal"));
```

E. Reconstruction of the Cross Ratio from Cross Ratio Phases

Table E.1.: Different ways to construct the cross ratio values out of the phirotope values

Used formulae	Result for $\mathrm{cr}(A, B \vert C, D)$
(5.6) & (5.8)	$\dfrac{\left(1 - \mathrm{cr}_\varphi(a,c\,\vert\,b,d)^2\right)\cdot \mathrm{cr}_\varphi(a,b\,\vert\,c,d)^2}{\mathrm{cr}_\varphi(a,b\,\vert\,c,d)^2 - \mathrm{cr}_\varphi(a,c\,\vert\,b,d)^2}$
(5.6) & (5.9)	$\dfrac{\mathrm{cr}_\varphi(a,b\,\vert\,c,d)^2\cdot\left(\mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2 - 1\right)}{\mathrm{cr}_\varphi(a,b\,\vert\,c,d)^2\cdot \mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2 - 1}$
(5.6) & (5.10)	$\dfrac{\mathrm{cr}_\varphi(a,b\,\vert\,c,d)^2\cdot \mathrm{cr}_\varphi(a,d\,\vert\,b,c)^2 - 1}{\mathrm{cr}_\varphi(a,d\,\vert\,b,c)^2 - 1}$
(5.6) & (5.11)	$\dfrac{\mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2 - \mathrm{cr}_\varphi(a,b\,\vert\,c,d)^2}{\mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2 - 1}$
(5.7) & (5.8)	$\dfrac{\mathrm{cr}_\varphi(a,c\,\vert\,b,d)^2 - 1}{\mathrm{cr}_\varphi(a,b\,\vert\,d,c)^2\cdot \mathrm{cr}_\varphi(a,c\,\vert\,b,d)^2 - 1}$
(5.7) & (5.9)	$\dfrac{-\,\mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2 + 1}{\mathrm{cr}_\varphi(a,b\,\vert\,d,c)^2 - \mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2}$
(5.7) & (5.10)	$\dfrac{\mathrm{cr}_\varphi(a,d\,\vert\,b,c)^2 - \mathrm{cr}_\varphi(a,b\,\vert\,d,c)^2}{\mathrm{cr}_\varphi(a,b\,\vert\,d,c)^2\cdot\left(\mathrm{cr}_\varphi(a,d\,\vert\,b,c)^2 - 1\right)}$
(5.7) & (5.11)	$\dfrac{\mathrm{cr}_\varphi(a,b\,\vert\,d,c)^2\cdot \mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2 - 1}{\mathrm{cr}_\varphi(a,b\,\vert\,d,c)^2\cdot\left(\mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2 - 1\right)}$
(5.8) & (5.10)	$\dfrac{\mathrm{cr}_\varphi(a,c\,\vert\,b,d)^2 - 1}{\mathrm{cr}_\varphi(a,d\,\vert\,b,c)^2 - 1}$
(5.8) & (5.11)	$\dfrac{\mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2\cdot\left(-\mathrm{cr}_\varphi(a,c\,\vert\,b,d)^2 + 1\right)}{\mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2 - 1}$
(5.9) & (5.10)	$\dfrac{-\,\mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2 + 1}{\mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2\cdot\left(\mathrm{cr}_\varphi(a,d\,\vert\,b,c)^2 - 1\right)}$
(5.9) & (5.11)	$\dfrac{\mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2\cdot\left(\mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2 - 1\right)}{\mathrm{cr}_\varphi(a,c\,\vert\,d,b)^2\cdot\left(\mathrm{cr}_\varphi(a,d\,\vert\,c,b)^2 - 1\right)}$

F. Example of a Non-Chirotopal Pappus' Configuration

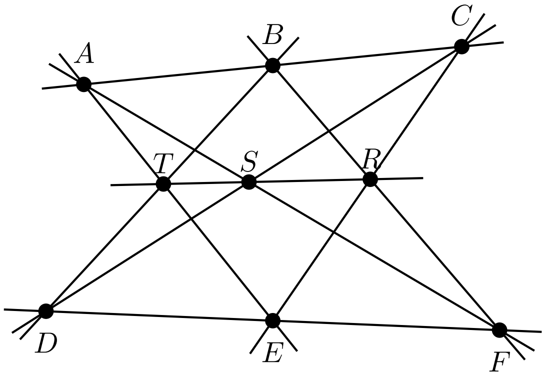


Figure F.5.: The theorem of Pappus.

A non-chirotopal Pappos' configuration is, for example, given by the following coordinates:

$$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \mathrm{i} \\ 2 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad E = \begin{pmatrix} \mathrm{i} \\ \mathrm{i} \\ 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$R = \begin{pmatrix} i \\ 2 \\ 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1+\mathrm{i} \\ -2\mathrm{i} \\ -2 \end{pmatrix}.$$

The combinatorics of this configuration can also be read off Figure F.5. The configuration is clearly non-chirotopal. For example, the phase of the cross ratio

$$\omega\left(\mathrm{cr}(C, S|E, F)_A\right) = -\mathrm{i}$$

is not real.

G. Bi-Quadratic Final Polynomials for the Theorem of Pappus

Bi-quadratic final polynomial for $\mathrm{cr}_\varphi\,(8,9\,|\,1,2)_{\{7\}}$

$$
\begin{array}{lcr}
\varphi(1,2,3)=0 & \Rightarrow & \varphi(1,2,5)\varphi(1,3,4)=\varphi(1,2,4)\varphi(1,3,5)\\
\varphi(2,4,9)=0 & \Rightarrow & -\varphi(9,2,7)\varphi(4,1,2)=\varphi(9,1,2)\varphi(4,7,2)\\
\varphi(1,5,9)=0 & \Rightarrow & -\varphi(1,2,9)\varphi(1,5,7)=\varphi(1,2,5)\varphi(1,7,9)\\
\varphi(3,4,8)=0 & \Rightarrow & -\varphi(4,6,3)\varphi(4,8,1)=\varphi(4,1,3)\varphi(4,6,8)\\
\varphi(3,5,7)=0 & \Rightarrow & -\varphi(5,1,3)\varphi(5,6,7)=\varphi(5,6,3)\varphi(5,7,1)\\
\varphi(1,6,8)=0 & \Rightarrow & -\varphi(8,1,7)\varphi(8,4,6)=\varphi(8,1,4)\varphi(8,6,7)\\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(7,2,4)\varphi(7,8,6)=\varphi(7,4,6)\varphi(7,8,2)\\
\varphi(4,5,6)=0 & \Rightarrow & \varphi(6,3,5)\varphi(6,7,4)=\varphi(6,3,4)\varphi(6,7,5)\\
\hline
 & \Rightarrow & \varphi(7,8,1)\varphi(7,9,2)=\varphi(7,8,2)\varphi(7,9,1)
\end{array}
$$

Bi-quadratic final polynomial for $\mathrm{cr}_\varphi\,(8,9\,|\,2,3)_{\{7\}}$

$$
\begin{array}{lcr}
\varphi(1,2,3)=0 & \Rightarrow & \varphi(1,2,5)\varphi(1,3,8)=\varphi(1,2,8)\varphi(1,3,5)\\
\varphi(1,5,9)=0 & \Rightarrow & \varphi(9,2,5)\varphi(1,5,7)=\varphi(1,2,5)\varphi(9,5,7)\\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(1,2,8)\varphi(1,6,7)=\varphi(1,2,6)\varphi(1,7,8)\\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(6,1,2)\varphi(6,7,4)=\varphi(6,2,4)\varphi(6,7,1)\\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(7,2,4)\varphi(7,8,6)=\varphi(7,4,6)\varphi(7,8,2)\\
\varphi(3,5,7)=0 & \Rightarrow & -\varphi(5,1,3)\varphi(5,7,4)=\varphi(5,3,4)\varphi(5,7,1)\\
\varphi(3,4,8)=0 & \Rightarrow & \varphi(8,1,4)\varphi(8,3,7)=\varphi(8,1,3)\varphi(8,4,7)\\
\varphi(1,6,8)=0 & \Rightarrow & -\varphi(8,1,7)\varphi(8,4,6)=\varphi(8,1,4)\varphi(8,6,7)\\
\varphi(4,5,6)=0 & \Rightarrow & \varphi(5,3,4)\varphi(6,2,4)=\varphi(5,2,4)\varphi(6,3,4)\\
\varphi(2,4,9)=0 & \Rightarrow & \varphi(9,2,7)\varphi(4,5,2)=\varphi(9,2,5)\varphi(4,7,2)\\
\varphi(3,5,7)=0 & \Rightarrow & -\varphi(7,3,4)\varphi(7,9,5)=\varphi(7,4,5)\varphi(7,9,3)\\
\varphi(3,4,8)=0 & \Rightarrow & \varphi(4,6,3)\varphi(4,7,8)=\varphi(4,6,8)\varphi(4,7,3)\\
\hline
 & \Rightarrow & \varphi(7,8,2)\varphi(7,9,3)=\varphi(7,8,3)\varphi(7,9,2)
\end{array}
$$

Bi-quadratic final polynomial for $\mathrm{cr}_{\varphi}\,(8,9\,|\,1,4)_{\{7\}}$

$$
\begin{array}{lcr}
\varphi(1,2,3)=0 & \Rightarrow & \varphi(1,2,7)\varphi(1,3,4)=\varphi(1,2,4)\varphi(1,3,7)\\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(7,1,6)\varphi(7,2,4)=\varphi(7,1,2)\varphi(7,4,6)\\
\varphi(2,4,9)=0 & \Rightarrow & -\varphi(4,1,2)\varphi(4,7,9)=\varphi(4,7,2)\varphi(4,9,1)\\
\varphi(3,5,7)=0 & \Rightarrow & -\varphi(7,1,3)\varphi(7,4,5)=\varphi(7,1,5)\varphi(7,3,4)\\
\varphi(3,4,8)=0 & \Rightarrow & -\varphi(4,7,3)\varphi(4,8,1)=\varphi(4,1,3)\varphi(4,7,8)\\
\varphi(4,5,6)=0 & \Rightarrow & \varphi(6,1,5)\varphi(6,7,4)=\varphi(6,1,4)\varphi(6,7,5)\\
\varphi(1,5,9)=0 & \Rightarrow & -\varphi(1,4,9)\varphi(1,5,7)=\varphi(1,4,5)\varphi(1,7,9)\\
\varphi(1,6,8)=0 & \Rightarrow & -\varphi(1,4,6)\varphi(1,7,8)=\varphi(1,4,8)\varphi(1,6,7)\\
\varphi(4,5,6)=0 & \Rightarrow & \varphi(5,6,7)\varphi(5,1,4)=\varphi(5,7,4)\varphi(5,6,1)\\
\hline
 & \Rightarrow & \varphi(7,8,1)\varphi(7,9,4)=\varphi(7,8,4)\varphi(7,9,1)
\end{array}
$$

Bi-quadratic final polynomial for $\mathrm{cr}_{\varphi}\,(8,9\,|\,1,5)_{\{7\}}$:

$$
\begin{array}{lcr}
\varphi(1,2,3)=0 & \Rightarrow & \varphi(1,2,8)\varphi(1,3,4)=\varphi(1,2,4)\varphi(1,3,8)\\
\varphi(2,4,9)=0 & \Rightarrow & -\varphi(9,2,5)\varphi(4,1,2)=\varphi(9,1,2)\varphi(4,5,2)\\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(6,2,5)\varphi(6,7,1)=\varphi(6,1,2)\varphi(6,7,5)\\
\varphi(1,5,9)=0 & \Rightarrow & -\varphi(9,1,2)\varphi(9,5,7)=\varphi(9,1,7)\varphi(9,2,5)\\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(1,2,6)\varphi(1,7,8)=\varphi(1,2,8)\varphi(1,6,7)\\
\varphi(3,4,8)=0 & \Rightarrow & \varphi(8,1,3)\varphi(8,4,7)=\varphi(8,1,4)\varphi(8,3,7)\\
\varphi(3,4,8)=0 & \Rightarrow & -\varphi(4,7,3)\varphi(4,8,1)=\varphi(4,1,3)\varphi(4,7,8)\\
\varphi(4,5,6)=0 & \Rightarrow & \varphi(5,2,4)\varphi(5,6,7)=\varphi(5,6,2)\varphi(5,7,4)\\
\varphi(3,5,7)=0 & \Rightarrow & -\varphi(7,4,5)\varphi(7,8,3)=\varphi(7,3,4)\varphi(7,8,5)\\
\hline
 & \Rightarrow & \varphi(7,8,1)\varphi(7,9,5)=\varphi(7,8,5)\varphi(7,9,1)
\end{array}
$$

Bi-quadratic final polynomial for $\mathrm{cr}_\varphi\,(8,9\,|\,2,5)_{\{7\}}$:

$$\begin{array}{lcr}
\varphi(1,2,3)=0 & \Rightarrow & \varphi(1,2,8)\varphi(1,3,5)=\varphi(1,2,5)\varphi(1,3,8) \\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(6,2,4)\varphi(6,7,1)=\varphi(6,1,2)\varphi(6,7,4) \\
\varphi(1,6,8)=0 & \Rightarrow & -\varphi(1,2,6)\varphi(1,7,8)=\varphi(1,2,8)\varphi(1,6,7) \\
\varphi(1,6,8)=0 & \Rightarrow & -\varphi(8,1,4)\varphi(8,6,7)=\varphi(8,1,7)\varphi(8,4,6) \\
\varphi(1,5,9)=0 & \Rightarrow & \varphi(1,2,5)\varphi(9,5,7)=\varphi(1,5,7)\varphi(9,2,5) \\
\varphi(3,5,7)=0 & \Rightarrow & -\varphi(5,3,4)\varphi(5,7,1)=\varphi(5,1,3)\varphi(5,7,4) \\
\varphi(3,4,8)=0 & \Rightarrow & \varphi(8,1,3)\varphi(8,4,7)=\varphi(8,1,4)\varphi(8,3,7) \\
\varphi(3,4,8)=0 & \Rightarrow & \varphi(4,6,8)\varphi(4,7,3)=\varphi(4,6,3)\varphi(4,7,8) \\
\varphi(4,5,6)=0 & \Rightarrow & \varphi(5,2,4)\varphi(6,3,4)=\varphi(5,3,4)\varphi(6,2,4) \\
\varphi(2,4,9)=0 & \Rightarrow & \varphi(9,2,5)\varphi(9,4,7)=\varphi(9,2,7)\varphi(9,4,5) \\
\varphi(2,4,9)=0 & \Rightarrow & \varphi(4,5,9)\varphi(4,7,2)=\varphi(4,5,2)\varphi(4,7,9) \\
\varphi(2,6,7)=0 & \Rightarrow & -\varphi(7,4,6)\varphi(7,8,2)=\varphi(7,2,4)\varphi(7,8,6) \\
\varphi(3,5,7)=0 & \Rightarrow & -\varphi(7,4,5)\varphi(7,8,3)=\varphi(7,3,4)\varphi(7,8,5) \\
\hline
 & \Rightarrow & \varphi(7,8,2)\varphi(7,9,5)=\varphi(7,8,5)\varphi(7,9,2)
\end{array}$$

Bi-quadratic final polynomial for $\mathrm{cr}_\varphi\,(8,9\,|\,2,6)_{\{7\}}$:

$$\begin{array}{lcr}
\varphi(2,6,7)=0 & \Rightarrow & \varphi(7,1,6)\varphi(7,8,2)=\varphi(7,1,2)\varphi(7,8,6) \\
\varphi(2,6,7)=0 & \Rightarrow & \varphi(7,1,2)\varphi(7,9,6)=\varphi(7,1,6)\varphi(7,9,2) \\
\hline
 & \Rightarrow & \varphi(7,8,2)\varphi(7,9,6)=\varphi(7,8,6)\varphi(7,9,2)
\end{array}$$

H. Proofs of the Theorems of Pappus and Desargues

This Mathematica program can also be downloaded here: `https://www-m10.ma.tum.de/users/schaar/No_Non-Pappus_Phirotope.nb`.

```
(*This program proves that the theorems of Pappus and Desargues are always
true for phirotopes.
This is done by showing that if some cross ratios are real, one can deduce
that the rest of the cross ratios are also real.
The cross ratios that are known to be real at the beginning are the ones
that can be deduced from a Grassmann-Plücker relation (GPR) which contains
a zero term.

Sets of five elements are stored to keep track of the cross ratios
that are already known to be real. Each set corresponds to one cross
ratio (and one GPR) which is generated from the set by the function CR
(or pl, for the GPR).*)

(*Input Cell*)
Clear[Coll, n, Conclusion];
n = 10; (*number of points of incidence theorem*)

(*insert collinearities of prerequisite of incidence theorem into Coll.
The points must be numbered 1,...,n*)

(*for Pappus use n=9 and
Coll = Map[Sort,{{1,2,3}, {1,8,6}, {1,9,5}, {2,9,4}, {2,7,6}, {3,8,4},
   {3,7,5}, {4,5,6}}];*)

(* for Desargues use n=10 and*)
Coll = Map[Sort, {{1,2,7}, {1,3,8}, {1,4,10}, {2,9,3}, {2,5,10},
   {3,6,10}, {4,7,5}, {4,8,6}, {5,6,9}}];

(*insert collinearity of conclusion here*)
Conclusion = Sort[{7, 8, 9}];


(*gerenation of the set of all bases*)
```

```
Clear[Bases, pl, inQ, gpSets, setsForRows];
Bases = Complement[Subsets[Range[n], {3}], Coll];

(*the function pl[{x_,a_,b_,c_,d_}] generates the sets that are part
of the GPR (x,a|x,b,c,d) *)
pl[{x_ , a_ , b_ , c_ , d_ }] := {{x, a, b}, {x, c, d}, {x, a, c}, {x, b, d},
   {x, a, d}, {x, b, c}};

(*gerenation of the list setsForRows that contains all sets of five
points that have a GPR (according to pl), which contains at least one
zero term. Those correspond to real cross ratios. *)
inQ[l_] := (Length[Intersection[Map[Sort, pl[l]], Coll]] >= 1);
gpSets = Join @@ (Map[Table[RotateRight[#, k], {k, 0, 4}] &,
     Subsets[Range[n], {5}]]);
setsForRows = Cases[gpSets, _?inQ];


Clear[collPos, fuse, signList, entrySign];
(*position of collinearity in g-p-realtion*)
collPos[p_] := FirstPosition[Map[Sort, p], Intersection[Coll,
   Map[Sort, p]][[1]]][[1]];
fuse[{a_ , b_ , c_ , d_ , e_ , 0}] := {1, 1, -1, -1, 0, 0};
fuse[{a_ , b_ , c_ , d_ , 0, f_ }] := {1, 1, -1, -1, 0, 0};
fuse[{a_ , b_ , c_ , 0, e_ , f_ }] := {-1, -1, 0, 0, 1, 1};
fuse[{a_ , b_ , 0, d_ , e_ , f_ }] := {-1, -1, 0, 0, 1, 1};
fuse[{a_ , 0, c_ , d_ , e_ , f_ }] := {0, 0, 1, 1, -1, -1};
fuse[{0, b_ , c_ , d_ , e_ , f_ }] := {0, 0, 1, 1, -1, -1};

(*signList generates a list that contains the signs of the logarithm
of the terms in the GPR*)
signList[p_] := fuse[ReplacePart[Map[Signature, p], collPos[p] -> 0]];

(*entrySign generates a vector of length Length[Bases] that assigns
to each basis its sign the logarithm of the GPR p_*)
entrySign[p_ , b_] :=If[FirstPosition[Map[Sort, #], b][[1]] == "NotFound",
   0, signList[#][[FirstPosition[Map[Sort, #], b][[1]]]]] &[pl[p]];
```

```
(*the matrix M contains the logarithmic representation of all GPRs
with one zero term*)
Clear[M];
M = Table[entrySign[setsForRows[[k]], Bases[[l]]],
   {k, 1, Length[setsForRows]}, {l, 1, Length[Bases]}];


Clear[collPos2, signList2, entrySign2];
(*collPos2 position of the collinearity including 'Conclusion' in the GPR*)
collPos2[p_] := FirstPosition[Map[Sort, p],
   Intersection[Join[Coll, {Conclusion}], Map[Sort, p]][[1]]][[1]];
signList2[p_] := fuse[ReplacePart[Map[Signature, p], collPos2[p] -> 0]];

(*entrySign2 generates vectors of the signs of logarithms for GPRs
that also contain the sequence 'Conclusion' *)
entrySign2[p_, b_] := If[FirstPosition[Map[Sort, #], b][[1]] == "NotFound",
   0, signList2[#][[FirstPosition[Map[Sort, #], b][[1]]]]] &[pl[p]];

(*tests is the set of all GPRs on the set Join[Conlusion,{x,y}] for all
   x,y in the indexset.*)
Clear[tests];
tests = Join @@ Table[Map[Join[RotateRight[Conclusion, k], #] &,
     Subsets[Complement[Range[n], Conclusion], {2}]], {k, 1, 3}];


(*in NewGPR all GPRs that can be obtained by a linear combination of GPRs
   in 'setsForRows' are stored. If Length[NewGPR][Equal]Length[tests],
   then all are can be obtained by a linear combination and are thus real.
   Then the proof is complete already at this step*)
(*If the theorem of Pappus is examined, this will complete the proof.
   The theorem of Desargues needs another iteration of linearly combining
   cross ratios and applying non-linear arugmentation steps*)
Clear[isinSpan, NewGPR];
isinSpan[p_, M_] := (MatrixRank[M] == MatrixRank[Append[M,
   Table[entrySign2[p, Bases[[k]]], {k, 1, Length[Bases]}]]]);
NewGPR = Cases[tests, _?(isinSpan[#, M] &)];
```

```
Length[NewGPR] == Length[tests]

Out[]= False

(*To all GPRs in NewGPR, we apply the non-linear argumentation step (last
   term has to point in one of the directions of the previous two).
We build a new, bigger matrix Nonl that contains the signs of the
   logarithms of the cross ratios that are now known to be real. The
   function CR is designed in a way such that it contains exactly the
   cross ratio obtained by the non-linear argument *)
Clear[CR, fuseCr, entrySignCr, Nonl];
CR[p_] := {{p[[1]], p[[2]], p[[3]]}, {p[[1]], p[[4]], p[[5]]},
    {p[[1]], p[[2]], p[[5]]}, {p[[1]], p[[3]], p[[4]]}};
fuseCr[p_] := {+1, +1, -1, -1};
entrySignCr[p_, b_] := If[FirstPosition[Map[Sort, #], b][[1]] == "NotFound",
   0, fuseCr[#][[FirstPosition[Map[Sort, #], b][[1]]]]] &[CR[p]];
Nonl = Join[M, Table[entrySignCr[k, Bases[[l]]], {k, NewGPR},
   {l, 1, Length[Bases]}]];

(*now we generate ALL possible cross ratios, that do not contain a 0, and
   their corresponding logarithmic rows*)
Clear[noColl, AllCR, testCr];
(*test whether some term of the cross ratio of the input is zero*)
noColl[l_] := (Length[Intersection[Map[Sort, CR[l]], Coll]] == 0);

(*AllCR generates the list of the sets whose cross ratio (CR[.]) contain
   the same elements as the input but for which we have to check
   separately if they are real. If the cross ratios of all three sets are
   real, then all cross ratios on these elements are real. *)
AllCR[{a_, b_, c_, d_, e_}] := {{a, b, c, d, e}, {a, b, c, e, d},
   {a, b, d, c, e}};

(*testCr contains cross ratios of which one has to test whether or not
   they are real/in the span of 'Nonl'.*)
testCr = Cases[Join @@ Table[AllCR[k], {k, gpSets}], _?noColl];

(*search all the cross ratios that are still not in the Span of Nonl.
```

```
   If WhatIsLeft is empty [Rule] all are in the Span*)
(*As this iterates over all cross ratios that might still be real or
   non-real, the evaluation of this cell might take a while*)
Clear[WhatIsLeft];
isinSpan[p_ , Nonl] := (MatrixRank[Nonl] == MatrixRank[Append[Nonl,
   Table[entrySignCr[p, Bases[[k]]], {k, 1, Length[Bases]}]]]);
WhatIsLeft = DeleteCases[testCr, _?(isinSpan[#, Nonl] &)];

(*Test whether there is one non-linear argumentation step such that the
   cross ratios in WhatIsLeft can be deduced*)
Clear[NonlinearStepReach, Reachable, nonReachable];
NonlinearStepReach[x_] := (isinSpan[AllCR[x][[2]], Nonl] ||
   isinSpan[AllCR[x][[3]], Nonl]);
Reachable = Cases[WhatIsLeft, _?NonlinearStepReach];
nonReachable = Complement[WhatIsLeft, Reachable];

(*now append the logarithms of the signs of all the cross ratios that
are reachable by one nonlinear argumentation step (and which are thus
real) to the matrix*)
Clear[Nonl2];
Nonl2 = Join[Nonl, Table[entrySignCr[k, Bases[[l]]], {k, Reachable},
   {l, 1, Length[Bases]}]];

(*Test if now all still remaining cross ratios can also be deduced
and are thus real. If only 'True' is obtained [Rule] they do.*)
DeleteDuplicates[Table[MatrixRank[Nonl2] == MatrixRank[ Append[Nonl2,
   Table[entrySignCr[l, Bases[[k]]], {k, 1, Length[Bases]}]]],
   {l, nonReachable}]]

Out[]= {True}
```

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