Chair of Automatic Control

Technische Universität München

Krylov subspace model reduction for bilinear and MIMO quadratic-bilinear systems

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Workshop on Model Reduction Methods and Optimization

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Outline

I. Model Reduction for Bilinear Systems

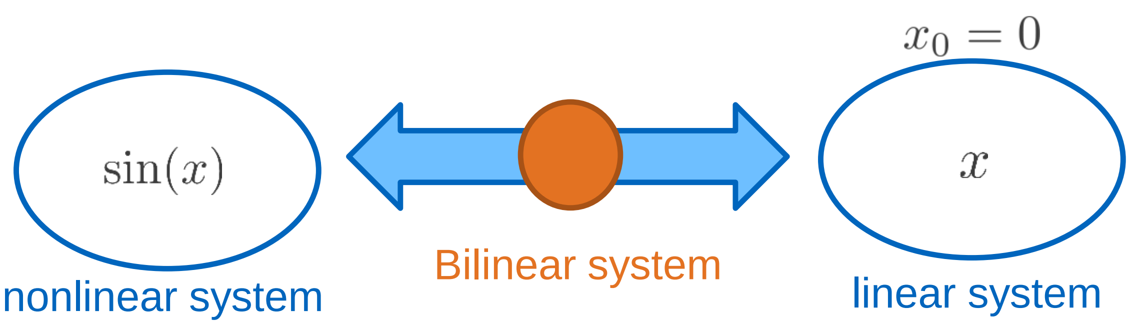
- Projective MOR of bilinear systems
- Bilinear systems theory
- Interpolation-based model reduction via Krylov subspaces
- $\mathcal{H}_2$ optimal model reduction of bilinear systems
- $\mathcal{H}_2$ pseudo-optimal reduction
- Numerical example
- Summary and Outlook

II. Model Reduction for Quadratic-Bilinear Systems

- Transfer function concepts and Krylov reduction for SISO systems
- Transfer function concepts and Krylov reduction for MIMO systems
- Numerical example
- Summary and Outlook

Motivation

- **Bilinear systems** are a special class of nonlinear systems (weakly nonlinear)
- Interface between fully nonlinear and linear systems



- The analogy between linear and bilinear systems allows us to **transfer** some of the **existing linear reduction techniques to the bilinear case**

Nonlinear state equation

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\boxed{\det(\mathbf{E}) \neq 0}$$

Carleman bilinearization
[Rugh '81]

Bilinear model

$$\mathbf{\Sigma}: \quad \begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{j=1}^{m} \mathbf{N}_j \mathbf{x}(t) u_j(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \;\; \mathbf{x}(0) = \mathbf{x}_0 \end{aligned}$$

$\mathbf{E}, \mathbf{A}, \mathbf{N}_j \in \mathbb{R}^{n \times n}$

$\mathbf{B} \in \mathbb{R}^{n \times m}$; $\mathbf{C} \in \mathbb{R}^{p \times n}$

Projective Reduction of Bilinear Systems

Bilinear model

$$\mathbf{\Sigma}: \quad \begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{j=1}^{m} \mathbf{N}_j \mathbf{x}(t) u_j(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \;\; \mathbf{x}(0) = \mathbf{x}_0 \end{aligned}$$

$\mathbf{E}, \mathbf{A}, \mathbf{N}_j \in \mathbb{R}^{n \times n}$

$\mathbf{B} \in \mathbb{R}^{n \times m}; \;\; \mathbf{C} \in \mathbb{R}^{p \times n}$

$n \gg r$

Projection

$$\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r} \quad \Rightarrow \quad \mathbf{E}_r = \mathbf{W}^T \mathbf{E} \mathbf{V}, \quad \mathbf{A}_r = \mathbf{W}^T \mathbf{A} \mathbf{V}, \quad \mathbf{N}_{r,j} = \mathbf{W}^T \mathbf{N}_j \mathbf{V},$$

$$\mathbf{B}_r = \mathbf{W}^T \mathbf{B}, \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}$$

Reduced bilinear model

$$\mathbf{\Sigma}_r: \quad \begin{aligned} \mathbf{E}_r\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\mathbf{x}_r(t) + \sum_{j=1}^{m} \mathbf{N}_{r,j} \mathbf{x}_r(t) u_j(t) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r(t) \end{aligned}$$

$\mathbf{E}_r, \mathbf{A}_r, \mathbf{N}_{r,j} \in \mathbb{R}^{r \times r}$

$\mathbf{B}_r \in \mathbb{R}^{r \times m}; \;\; \mathbf{C}_r \in \mathbb{R}^{p \times r}$

[Rugh '81]

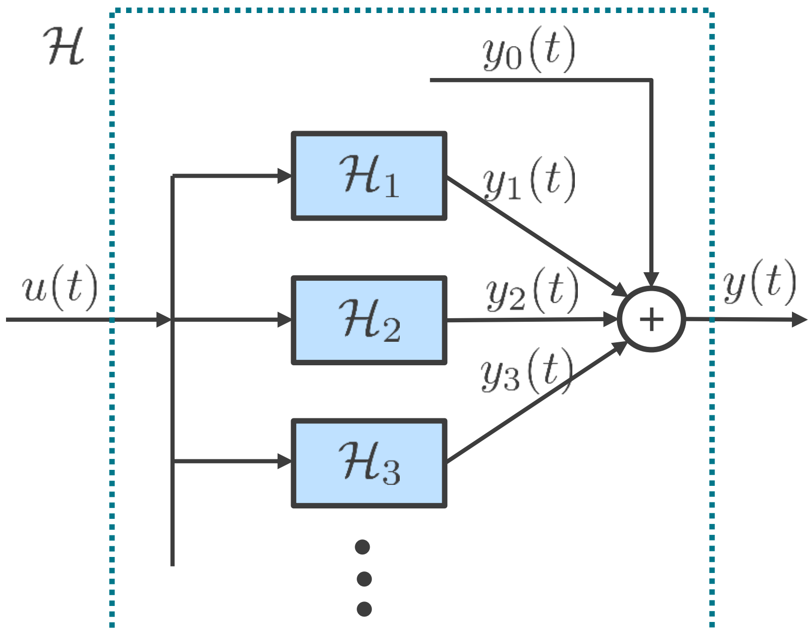
Output Response and Transfer Functions of Bilinear Systems

$$\boxed{y(t) = \sum_{k=0}^{\infty} y_k(t) = \sum_{k=0}^{\infty} \mathcal{H}_k\left[u(t)\right]}$$

$y_k(t)$: output of *k*-th homogenous subsystem

$\mathcal{H}_k$: *k*-th order Volterra operator

$y_0 = \mathcal{H}_0$: constant output



[Rugh '81]

- Within this framework, the input-output representation is given by

$$y(t) = \sum_{k=1}^{\infty} y_k(t_1, \ldots, t_k)$$
$$= \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} g_k(t_1, \ldots, t_k)\, u(t-t_1) \ldots u(t-t_k) \mathrm{d}t_k \ldots \mathrm{d}t_1$$

- Definition by convolution integrals

Input-output representation

$$y(t) = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underbrace{\mathbf{c}^T e^{\mathbf{E}^{-1}\mathbf{A}\tau_k} \mathbf{E}^{-1}\mathbf{N} \cdots \mathbf{E}^{-1}\mathbf{N} e^{\mathbf{E}^{-1}\mathbf{A}\tau_2} \mathbf{E}^{-1}\mathbf{N} e^{\mathbf{E}^{-1}\mathbf{A}\tau_1} \mathbf{E}^{-1}\mathbf{b}}_{g_k(\tau_1, \ldots, \tau_k)}$$
$$\times u(t-\tau_k) \cdots u(t-\tau_k - \ldots - \tau_1)\, \mathrm{d}\tau_k \cdots \mathrm{d}\tau_1$$

***k*-th order transfer function of a bilinear system**

$$G_k(s_1, \ldots, s_k) = \mathbf{c}^T (s_k\mathbf{E} - \mathbf{A})^{-1} \mathbf{N} \cdots \mathbf{N} (s_2\mathbf{E} - \mathbf{A})^{-1} \mathbf{N} (s_1\mathbf{E} - \mathbf{A})^{-1} \mathbf{b}$$

[Rugh ’81], [Flagg ’12]

- First three subsystems:

$$k = 1: \quad G_1(s_1) = \mathbf{c}^T(s_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$$

$$k = 2: \quad G_2(s_1,s_2) = \mathbf{c}^T\boxed{(s_2\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}}(s_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$$

$$k = 3: \quad G_3(s_1,s_2,s_3) = \mathbf{c}^T\boxed{(s_3\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}(s_2\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}}(s_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$$

$\mathcal{H}_2$ norm for bilinear systems

$$||\mathbf{\Sigma}||^2_{\mathcal{H}_2} := \mathrm{tr}\left(\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{(2\pi)^k}\mathbf{G}_k(j\omega_1,\ldots,j\omega_k)\mathbf{G}_k^T(j\omega_1,\ldots,j\omega_k)\mathrm{d}\omega_1\cdots\mathrm{d}\omega_k\right)$$

- **P** and **Q** satisfy the following bilinear Lyapunov equations:

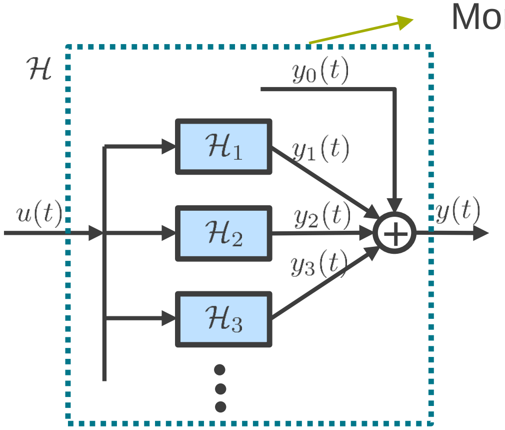
Bilinear Lyapunov equations

$$\mathbf{A}\mathbf{P}\mathbf{E}^T + \mathbf{E}\mathbf{P}\mathbf{A}^T + \boxed{\sum_{j=1}^{m}\mathbf{N}_j\mathbf{P}\mathbf{N}_j^T} + \mathbf{B}\mathbf{B}^T = \mathbf{0},$$

$$\mathbf{A}^T\mathbf{Q}\mathbf{E} + \mathbf{E}^T\mathbf{Q}\mathbf{A} + \boxed{\sum_{j=1}^{m}\mathbf{N}_j^T\mathbf{Q}\mathbf{N}_j} + \mathbf{C}^T\mathbf{C} = \mathbf{0}$$

Volterra series-based interpolation:

Enforcing multipoint interpolation of the underlying Volterra series



Volterra series interpolation [Flagg/Gugercin '15]

Set of interpolation points: $S = \{s_1, \ldots, s_r\}$

$$\sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}}^{r} \eta_{l_1,\ldots,l_{k-1},j} \mathbf{G}_k(s_{l_1}, \ldots, s_{l_{k-1}}, s_j) = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}}^{r} \eta_{l_1,\ldots,l_{k-1},j} \mathbf{G}_{k,r}(s_{l_1}, \ldots, s_{l_{k-1}}, s_j)$$

This approach interpolates the weighted series at the interpolation points $s_1, \ldots, s_r$

Weighting matrices: $\mathbf{U}_V = \{u_{i,j}\}, \mathbf{U}_W = \{\hat{u}_{i,j}\} \in \mathbb{R}^{r \times r}$

$$\eta_{l_1,\ldots,l_{k-1},j} = u_{j,l_{k-1}} u_{l_{k-1},l_{k-2}} \ldots u_{l_2,l_1} \text{ for } k \geq 2 \text{ and } \eta_{l_1} = 1 \text{ for } l_1 = 1, \ldots, r$$

Weights and shifts are defined by the user

Example: $\eta_{1,2,3} = u_{3,2} \cdot u_{2,1}$

Explicit computation of Volterra series-based interpolation [Flagg/Gugercin '15]

$$\mathbf{v}_j = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}}^{r} \eta_{l_1,\ldots,l_{k-1},j} (\mathbf{A} - s_j \mathbf{E})^{-1} \mathbf{N} (\mathbf{A} - s_{l_{k-1}} \mathbf{E})^{-1} \mathbf{N} \ldots \mathbf{N} (\mathbf{A} - s_1 \mathbf{E})^{-1} \mathbf{b}$$

$$\mathbf{w}_j = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}}^{r} \hat{\eta}_{l_1,\ldots,l_{k-1},j} (\mathbf{A}^T - \mu_j \mathbf{E}^T)^{-1} \mathbf{N}^T (\mathbf{A}^T - \mu_{l_{k-1}} \mathbf{E}^T)^{-1} \mathbf{N}^T \ldots \mathbf{N}^T (\mathbf{A}^T - \mu_1 \mathbf{E}^T)^{-1} \mathbf{c}$$

Link Krylov-Sylvester

$$\tilde{\mathbf{V}} = [\mathbf{v}_1, \ldots, \mathbf{v}_r] \in \mathbb{R}^{n \times r}$$

$$\tilde{\mathbf{W}} = [\mathbf{w}_1, \ldots, \mathbf{w}_r] \in \mathbb{R}^{n \times r}$$

Volterra Sylvester equations

$$\mathbf{A}\tilde{\mathbf{V}} - \mathbf{E}\tilde{\mathbf{V}}\mathbf{S}_V - \mathbf{N}\tilde{\mathbf{V}}\mathbf{U}_V^T = \mathbf{b}\mathbf{e}^T$$

$$\mathbf{A}^T\tilde{\mathbf{W}} - \mathbf{E}^T\tilde{\mathbf{W}}\mathbf{S}_W^T - \mathbf{N}^T\tilde{\mathbf{W}}\mathbf{U}_W^T = \mathbf{c}^T\mathbf{e}$$

$\mathcal{H}_2$ optimal model reduction of bilinear systems

Goal of $\mathcal{H}_2$ optimality

$$\|\mathbf{\Sigma} - \mathbf{\Sigma}_r\|_{\mathcal{H}_2} = \min_{\dim(\mathbf{H}_r)=r} \|\mathbf{\Sigma} - \mathbf{H}_r\|_{\mathcal{H}_2}$$

- minimizing the approximation error $\|\mathbf{\Sigma} - \mathbf{\Sigma}_r\|_{\mathcal{H}_2}$

Error system

$$\mathbf{\Sigma}_{err} := \mathbf{\Sigma} - \mathbf{\Sigma}_r$$

$\Rightarrow$

$\mathcal{H}_2$ norm of the error system

$$E^2 := \|\mathbf{\Sigma}_{err}^2\|_{\mathcal{H}_2}^2 := \|\mathbf{\Sigma} - \mathbf{\Sigma}_r\|_{\mathcal{H}_2}^2$$

Necessary conditions for $\mathcal{H}_2$ optimality [Benner/Breiten '12]

First order necessary conditions:

I) $\boxed{\dfrac{\partial E^2}{\partial \tilde{\mathbf{C}}_{ij}} = 0}$ $\Leftrightarrow$ $\mathbf{G}_r(-\overline{\lambda}_i)\tilde{\mathbf{B}}_i = \mathbf{G}(-\overline{\lambda}_i)\tilde{\mathbf{B}}_i$

II) $\boxed{\dfrac{\partial E^2}{\partial \tilde{\mathbf{B}}_{ij}} = 0}$ $\Leftrightarrow$ $\tilde{\mathbf{C}}_i^T\mathbf{G}(-\overline{\lambda}_i) = \tilde{\mathbf{C}}_i^T\mathbf{G}_r(-\overline{\lambda}_i)$

III) $\boxed{\dfrac{\partial E^2}{\partial \lambda_i} = 0}$ $\Leftrightarrow$ $\tilde{\mathbf{C}}_i^T\mathbf{G}'(-\overline{\lambda}_i)\tilde{\mathbf{B}}_i = \tilde{\mathbf{C}}_i^T\mathbf{G}_r'(-\overline{\lambda}_i)\tilde{\mathbf{B}}_i$

IV) $\boxed{\dfrac{\partial E^2}{\partial \tilde{\mathbf{N}}_{ij}} = 0}$

$\mathcal{H}_2$ optimal model reduction of bilinear systems

$\mathcal{H}_2$ optimality for bilinear systems

$$\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_r\|_{\mathcal{H}_2} = \min_{\dim(\mathbf{H}_r)=r} \|\boldsymbol{\Sigma} - \mathbf{H}_r\|_{\mathcal{H}_2}$$

$\boldsymbol{\Sigma}_r$ satisfies

$$\boxed{\frac{\partial E^2}{\partial \tilde{\mathbf{C}}_{ij}} = 0} \qquad \boxed{\frac{\partial E^2}{\partial \tilde{\mathbf{N}}_{ij}} = 0} \qquad \boxed{\frac{\partial E^2}{\partial \tilde{\mathbf{B}}_{ij}} = 0} \qquad \boxed{\frac{\partial E^2}{\partial \lambda_i} = 0}$$

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}}^{r} \eta_{l_1,\ldots,l_{k-1},j} G_k(s_{l_1},\ldots,s_{l_{k-1}},s_j)$$
$$= \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}}^{r} \eta_{l_1,\ldots,l_{k-1},j} G_{k,r}(s_{l_1},\ldots,s_{l_{k-1}},s_j)$$

$\phi_{l_1,\cdots,l_k}$: reduced order residues

λ_{l_i} : reduced order poles

[Flagg/Gugercin '15]

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} G_k(-\overline{\lambda}_{l_1},\cdots,-\overline{\lambda}_k) = \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} G_{k,r}(-\overline{\lambda}_{l_1},\cdots,-\overline{\lambda}_k),$$

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} \left(\sum_{j=1}^{k}\frac{\partial}{\partial s_j} G_k(-\overline{\lambda}_{l_1},\cdots,-\overline{\lambda}_k)\right) = \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} \left(\sum_{j=1}^{k}\frac{\partial}{\partial s_j} G_{k,r}(-\overline{\lambda}_{l_1},\cdots,-\overline{\lambda}_k)\right)$$

$\mathcal{H}_2$ optimality

$$\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_r\|_{\mathcal{H}_2} = \min_{\dim(\mathbf{H}_r)=r} \|\boldsymbol{\Sigma} - \mathbf{H}_r\|_{\mathcal{H}_2}$$

$\boldsymbol{\Sigma}_r$ satisfies

$$\boxed{\frac{\partial E^2}{\partial \tilde{\mathbf{C}}_{ij}} = 0} \quad \boxed{\frac{\partial E^2}{\partial \tilde{\mathbf{N}}_{ij}} = 0} \quad \boxed{\frac{\partial E^2}{\partial \tilde{\mathbf{B}}_{ij}} = 0} \quad \boxed{\frac{\partial E^2}{\partial \lambda_i} = 0} \Leftrightarrow \boxed{\begin{aligned} \boldsymbol{\Sigma}(-\overline{\lambda}_i) &= \boldsymbol{\Sigma}_r(-\overline{\lambda}_i) \\ \boldsymbol{\Sigma}'(-\overline{\lambda}_i) &= \boldsymbol{\Sigma}_r'(-\overline{\lambda}_i) \end{aligned}}$$

$\mathcal{H}_2$ pseudo-optimality

$\mathcal{L} = \{\lambda_1, \cdots, \lambda_n\}$: fixed reduced poles

$\mathcal{G}(\mathcal{L})$: Subset of reduced models

$\boldsymbol{\Sigma}_r$ satisfies

$$||\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_r||_{\mathcal{H}_2} = \min_{\mathbf{H}_r \in \mathcal{G}(\mathcal{L})} ||\boldsymbol{\Sigma} - \mathbf{H}_r||_{\mathcal{H}_2}$$

if

$$\boxed{\boldsymbol{\Sigma}(-\overline{\lambda}_i) = \boldsymbol{\Sigma}_r(-\overline{\lambda}_i)}$$

$$\boxed{\sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} G_k(-\overline{\lambda}_{l_1}, \cdots, -\overline{\lambda}_k) = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} G_{k,r}(-\overline{\lambda}_{l_1}, \cdots, -\overline{\lambda}_k),}$$

$$\sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} \left(\sum_{j=1}^{k} \frac{\partial}{\partial s_j} G_k(-\overline{\lambda}_{l_1}, \cdots, -\overline{\lambda}_k) \right) = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_k=1}^{r} \phi_{l_1,\cdots,l_k} \left(\sum_{j=1}^{k} \frac{\partial}{\partial s_j} G_{k,r}(-\overline{\lambda}_{l_1}, \cdots, -\overline{\lambda}_k) \right)$$

Notation

$$\mathbf{B}_\perp = \mathbf{B} - \mathbf{E}\mathbf{V}\mathbf{E}_r^{-1}\mathbf{B}_r$$

$$\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\mathbf{S}_V - \sum_{j=1}^{m}\mathbf{N}_j\mathbf{V}\mathbf{U}_V^T = \mathbf{B}\mathbf{R}_V$$

$$\mathbf{A}_r\mathbf{P}_r\mathbf{E}_r^T + \mathbf{E}_r\mathbf{P}_r\mathbf{A}_r^T + \sum_{j=1}^{m}\mathbf{N}_{r,j}\mathbf{P}_r\mathbf{N}_{r,j}^T + \mathbf{B}_r\mathbf{B}_r^T = \mathbf{0}$$

$$\mathbf{A}\mathbf{X}\mathbf{E}_r^T + \mathbf{E}\mathbf{X}\mathbf{A}_r^T + \sum_{j=1}^{m}\mathbf{N}_j\mathbf{X}\mathbf{N}_{r,j}^T + \mathbf{B}\mathbf{B}_r = \mathbf{0}$$

$\lambda_i(\mathbf{S}_V)$: Shifts
$\mathbf{U}_V$: Weights
$\mathbf{R}_V$: Right tangential directions

New conditions for pseudo-optimality for bilinear systems

i) $\mathbf{S}_V = -\mathbf{P}_r\mathbf{A}_r^T\mathbf{E}_r^{-T}\mathbf{P}_r^{-1}$

ii-1) $\mathbf{E}_r^{-1}\mathbf{B}_r + \mathbf{P}_r\mathbf{R}_V^T = \mathbf{0}$

ii-2) $\mathbf{P}_r^{-1}\mathbf{U}_V^T + \mathbf{N}_r^T\mathbf{E}_r^{-T}\mathbf{P}_r^{-1} = \mathbf{0}$

iii) $\mathbf{S}_V\mathbf{P}_r + \mathbf{P}_r\mathbf{S}_V^T - \mathbf{P}_r\mathbf{R}_V^T\mathbf{R}_V\mathbf{P}_r + \mathbf{P}_r\mathbf{U}_V\mathbf{N}_r^T\mathbf{E}_r^{-T} = \mathbf{0}$
$\Leftrightarrow \mathbf{P}_r^{-1}\mathbf{S}_V + \mathbf{S}_V^T\mathbf{P}_r^{-1} - \mathbf{U}_V\mathbf{P}_r^{-1}\mathbf{U}_V^T - \mathbf{R}_V^T\mathbf{R}_V = \mathbf{0}$

iv) $\mathbf{X} = \mathbf{V}\mathbf{P}_r$

v) $\mathbf{A}\hat{\mathbf{P}}\mathbf{E}^T + \mathbf{E}\hat{\mathbf{P}}\mathbf{A}^T + \sum_{j=1}^{m}\mathbf{N}_j\hat{\mathbf{P}}\mathbf{N}_j^T + \mathbf{B}\mathbf{B}^T = \mathbf{B}_\perp\mathbf{B}_\perp^T$

vi) $\mathbf{P}_r^{-1} = \mathbf{E}_r^T\mathbf{Q}_f\mathbf{E}_r$

$\mathcal{H}_2$ pseudo-optimal reduction of bilinear systems

BIPORK: Bilinear pseudo-optimal rational Krylov

Algorithm 1 Bilinear pseudo-optimal rational Krylov (BIPORK)

Input: $\mathbf{V}$, $\mathbf{S}_V$, $\mathbf{U}_V$, $\mathbf{R}_V$, $\mathbf{C}$, such that $\mathbf{A}\mathbf{V}-\mathbf{E}\mathbf{V}\mathbf{S}_V-\mathbf{N}\mathbf{V}\mathbf{U}_V^T=\mathbf{B}\mathbf{R}_V$ is satisfied

Output: $\mathcal{H}_2$ pseudo-optimal reduced model $\mathbf{\Sigma}_r$

1: $\mathbf{P}_r^{-1}$: solution of condition iii): $\mathbf{P}_r^{-1}\mathbf{S}_V+\mathbf{S}_V^T\mathbf{P}_r^{-1}-\mathbf{U}_V\mathbf{P}_r^{-1}\mathbf{U}_V^T-\mathbf{R}_V^T\mathbf{R}_V=\mathbf{0}$

2: $\mathbf{N}_r=-(\mathbf{P}_r^{-1})^{-1}\mathbf{U}_V\mathbf{P}_r^{-1}$ condition ii-2)

3: $\mathbf{B}_r=-\left(\mathbf{P}_r^{-1}\right)^{-1}\mathbf{R}_V^T$ condition ii-1)

4: $\mathbf{A}_r=\mathbf{S}_V+\mathbf{B}_r\mathbf{R}_V+\mathbf{N}_r\mathbf{U}_V^T$, $\mathbf{E}_r=\mathbf{I}_r$, $\mathbf{C}_r=\mathbf{C}\mathbf{V}$

Advantages and properties of BIPORK

- ROM is globally optimal within a subset: $||\mathbf{\Sigma}-\mathbf{\Sigma}_r||_{\mathcal{H}_2}=\min\limits_{\mathbf{H}_r\in\mathcal{G}(\mathcal{L})}||\mathbf{\Sigma}-\mathbf{H}_r||_{\mathcal{H}_2}$
- Eigenvalues of ROM: $\Lambda(\mathbf{S}_V)=\Lambda(-\mathbf{E}_r^{-1}\mathbf{A}_r)$

  → choice of the shifts is twice as important
- Stability preservation in the ROM can be ensured (choice of shifts & weights)
- Low numerical effort required: solution of a bilinear Lyapunov equation and two linear system of equations, both of reduced order

Heat Transfer Model: Bilinear boundary controlled heat transfer system

Heat equation
$x_t = \Delta x$ $\dfrac{\partial x}{\partial t} = \dfrac{\partial^2 x}{\partial z_1^2} + \dfrac{\partial^2 x}{\partial z_2^2} \quad \text{on unit square } \Omega = [0,1] \times [0,1]$

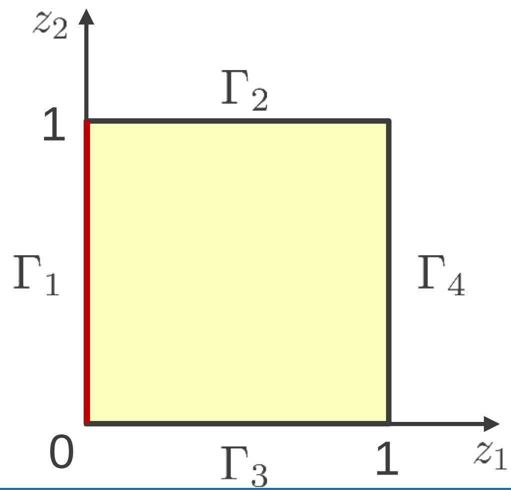
Boundary conditions
$n \cdot \left(\dfrac{\partial x}{\partial z_1} + \dfrac{\partial x}{\partial z_2} \right) = \boxed{(x-1)\,u} \quad \text{on } \Gamma_1$ $x = 0 \quad \text{on } \Gamma_2, \Gamma_3, \Gamma_4$

[Benner/Breiten ’12]

- Spatial discretization on an equidistant $k \times k$ grid together with the boundary conditions yields:

  $\Rightarrow \; \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\,\mathbf{x}\,u + \mathbf{b}u$ of dimension $n = k^2$

- Output: $y = \mathbf{c}^T\mathbf{x} = \dfrac{1}{k^2}\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}\mathbf{x}$



$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$: boundary domains

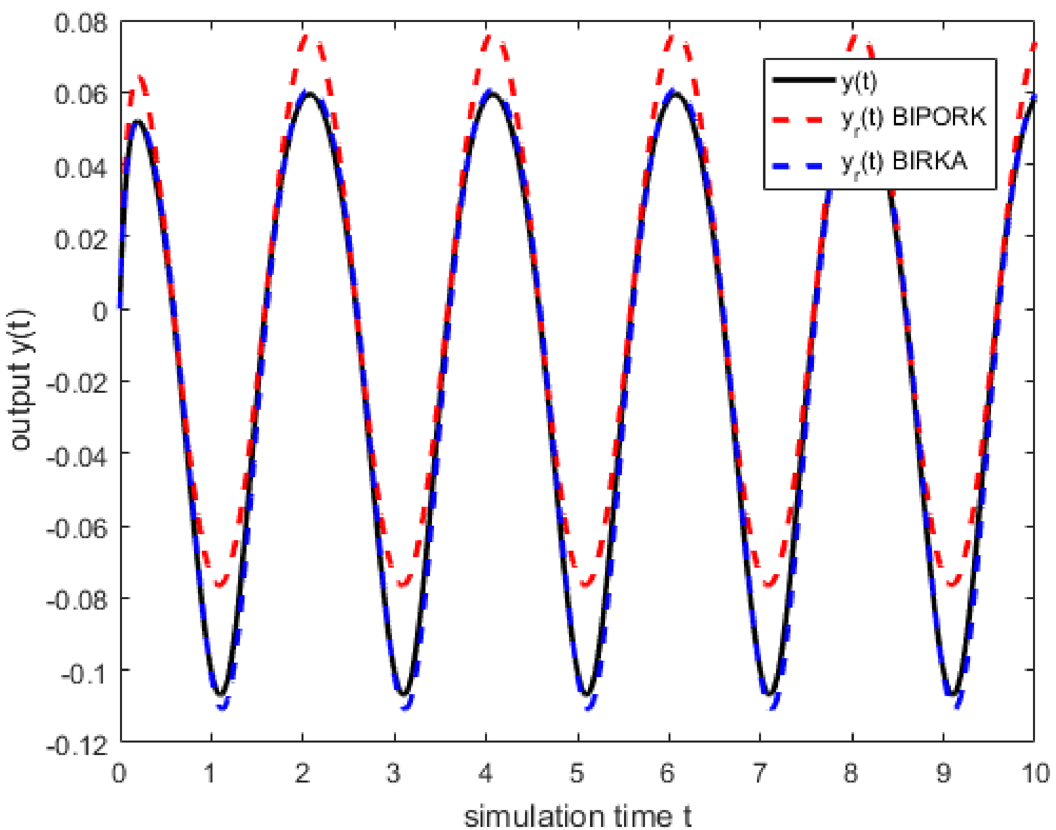
BIRKA: $\mathcal{H}_2$ optimality
No convergence after 50 iterations

BIPORK: $\mathcal{H}_2$ pseudo-optimality

$$s_0 = \begin{bmatrix} 10 & 20 & 30 & 40 & 50 & 60 \end{bmatrix}$$

$$\mathbf{U}_V = \mathrm{diag}([1e^{-10}\ 2e^{-10}\ 7e^{-10}\ 5e^{-10}\ 7e^{-10}\ 8e^{-10}])$$

Output response for $u(t) = \cos(\pi t)$:



$k = 50;\ \boxed{n = 2500 \quad r = 6}$

Summary and Outlook

Summary:

- **Goal:** Reduction of high dimensional nonlinear systems
- Approximation of a nonlinear system by a bilinear system using Volterra theory
- Systems theory and Krylov-based model reduction for bilinear systems
- $\mathcal{H}_2$ pseudo-optimal model reduction for bilinear systems
  - Derivation of new conditions for $\mathcal{H}_2$ pseudo-optimality for bilinear systems
  - Bilinear pseudo-optimal rational Krylov (BIPORK) – **conditions ii-1), ii-2), iii)**

Outlook:

- Solution of bilinear Lyapunov equations with BIPORK and the link with the alternating direction implicit (ADI) method: **conditions iv)-v)**

  BI-LR-ADI = RKSM + BIPORK

- Cumulative reduction (CuRe) for bilinear systems: **condition vi)**

References

[Benner/Breiten ’12] Interpolation-based H2-model reduction of bilinear control systems. SIAM Journal on Matrix Analysis and Applications

[Flagg ’12] Interpolation Methods for the Model Reduction of Bilinear Systems, PhD thesis

[Flagg/Gugercin ’15] Multipoint Volterra series interpolation and H2 optimal model reduction of bilinear systems, SIAM Journal on …

[Rugh ’81] Nonlinear system theory. The Volterra/Wiener Approach

[Wolf ’14] H2 Pseudo-Optimal Model Order Reduction, PhD thesis

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SISO Quadratic-bilinear systems

SISO Quadratic-bilinear model

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{H}\,\mathbf{x}\otimes\mathbf{x} + \mathbf{N}\,\mathbf{x}\,u + \mathbf{b}\,u$$

$$y = \mathbf{c}^T\,\mathbf{x}$$

$\mathbf{E}, \mathbf{A}, \mathbf{N} \in \mathbb{R}^{n\times n}$

$\mathbf{H} \in \mathbb{R}^{n\times n^2}$: Hessian tensor

$\mathbf{b}, \mathbf{c} \in \mathbb{R}^{n}$

- A large class of smooth nonlinear control-affine systems can be equivalently transformed into quadratic-bilinear systems
- The transformation is exact (no approximation), but not unique and the state-space dimension is slightly increased after the transformation
- Input-output behaviour can be characterized by generalized transfer functions
  → allows us to use Krylov-based reduction methods

SISO Quadratic-bilinear systems

Variational equation approach

- Assumption: Nonlinear system is assumed to be a series of homogeneous nonlinear subsystems
- Considering input of the form $\alpha u(t)$ and comparing coefficients of α^i yields:

$$
\begin{aligned}
\mathbf{E}\dot{\mathbf{x}}_1(t) &= \mathbf{A}\mathbf{x}_1(t) + \mathbf{b}u(t) \\
\mathbf{E}\dot{\mathbf{x}}_2(t) &= \mathbf{A}\mathbf{x}_2(t) + \mathbf{H}(\mathbf{x}_1(t) \otimes \mathbf{x}_1(t)) + \mathbf{N}\mathbf{x}_1(t)u(t) \\
\mathbf{E}\dot{\mathbf{x}}_3(t) &= \mathbf{A}\mathbf{x}_3(t) + \mathbf{H}(\mathbf{x}_1(t) \otimes \mathbf{x}_2(t) + \mathbf{x}_2(t) \otimes \mathbf{x}_1(t)) + \mathbf{N}\mathbf{x}_2(t)u(t) \\
&\vdots
\end{aligned}
$$

Generalized transfer functions via growing exponential approach

$$G_1(s_1) = -\mathbf{c}^T(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{b} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{b} \qquad \boxed{\mathbf{A}_s = \mathbf{A} - s\mathbf{E}}$$

$$G_2(s_1, s_2) = -\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{b}) - \mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$$

$\Downarrow$ $\boxed{\mathbf{H} \text{ is symmetric}}$

$$G_2(s_1, s_2) = -\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[2 \cdot \mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b}) - \mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$$

$\Downarrow$ $\boxed{s_1 = s_2 = \sigma}$

$$G_2(\sigma, \sigma) = -\mathbf{c}^T\mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{b} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{b}) - \frac{1}{2}\mathbf{N}\mathbf{A}_{\sigma}^{-1}\mathbf{b}\right]$$

Moments

$$G_1(s_1) = -\mathbf{c}^T(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{b} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{b}$$

$$\boxed{\mathbf{A}_s = \mathbf{A} - s\mathbf{E}}$$

$$\boxed{\frac{\partial}{\partial s}\mathbf{A}_s^{-1}(s) = -\mathbf{A}_s^{-1}\frac{\partial \mathbf{A}_s}{\partial s}\mathbf{A}_s^{-1} = \mathbf{A}_s^{-1}\mathbf{E}\mathbf{A}_s^{-1}}$$

$$\Downarrow$$

$$\frac{\partial G_1}{\partial s_1} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b}$$

$$G_2(s_1, s_2) = -\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{b}) - \mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$$

$$\Downarrow$$

$$\begin{aligned}
\frac{\partial G_2}{\partial s_1} = &- \mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{E}\mathbf{A}_{s_1+s_2}^{-1}\mathbf{H}[\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b}] \\
&- \mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{H}[\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b}] \\
&+ \frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{E}\mathbf{A}_{s_1+s_2}^{-1}\mathbf{N}[\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b}] \\
&+ \frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{N}[\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b}]
\end{aligned}$$

SISO Quadratic-bilinear systems

Theorem: Two-sided rational interpolation

Let $\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$ be nonsingular, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\mathbf{H}_r = \mathbf{W}^T\mathbf{H}(\mathbf{V}\otimes\mathbf{V})$, $\mathbf{N}_r = \mathbf{W}^T\mathbf{N}\mathbf{V}$, $\mathbf{b}_r = \mathbf{W}^T\mathbf{b}$, $\mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}$ with $\mathbf{V}$, $\mathbf{W}\in\mathbb{R}^{n\times r}$ having full rank such that

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\ldots,k}\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma_i}^{-1}[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}) - \mathbf{N}\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}]\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\ldots,k}\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c},\ \mathbf{A}_{\sigma_i}^{-T}[\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c}) - \frac{1}{2}\mathbf{N}^T\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c})]\}$$

with $\sigma_i \notin \{\Lambda(\mathbf{A},\mathbf{E}),\Lambda(\mathbf{A}_r,\mathbf{E}_r\}$.

Then:

$$G_1(\sigma_i) = G_{1,r}(\sigma_i)$$

$$G_1(2\sigma_i) = G_{1,r}(2\sigma_i)$$

$$G_2(\sigma_i,\sigma_i) = G_{2,r}(\sigma_i,\sigma_i)$$

$$\frac{\partial G_2}{\partial s_j}(\sigma_i,\sigma_i) = \frac{\partial G_{2,r}}{\partial s_j}(\sigma_i,\sigma_i)$$

Proof for the theorem: see [Benner/Breiten '15]

MIMO Quadratic-bilinear systems

MIMO Quadratic-bilinear model

$$\boxed{\mathbf{E}}\,\dot{\mathbf{x}} = \boxed{\mathbf{A}}\,\mathbf{x} + \boxed{\mathbf{H}}\,\mathbf{x}\otimes\mathbf{x} + \boxed{\sum_{j=1}^{m}\mathbf{N}_j}\,\mathbf{x}\,u_j + \boxed{\mathbf{B}}\,\mathbf{u}$$

$$\mathbf{y} = \boxed{\mathbf{C}}\,\mathbf{x}$$

$\mathbf{E}, \mathbf{A}, \mathbf{N}_j \in \mathbb{R}^{n\times n}$

$\mathbf{H} \in \mathbb{R}^{n\times n^2}$: Hessian tensor

$\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$

$\mathcal{N} \in \mathbb{R}^{n\times n\times m}$

$\mathcal{N}(:,:,j) = \mathbf{N}_j \in \mathbb{R}^{n\times n}$

$$\begin{aligned}\bar{\mathbf{N}} &= \begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2 & \dots & \mathbf{N}_m\end{bmatrix} \in \mathbb{R}^{n\times n\cdot m}\\ &= \mathbf{N}^{(1)}\end{aligned}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}\mathbf{E}\,\dot{\mathbf{x}} &= \mathbf{A}\,\mathbf{x} + \mathbf{H}\,(\mathbf{x}\otimes\mathbf{x}) + \bar{\mathbf{N}}\,(\mathbf{u}\otimes\mathbf{x}) + \mathbf{B}\,\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\,\mathbf{x}\end{aligned}}$$

Generalized transfer matrices via growing exponential approach

$$\mathbf{G}_1(s_1) = -\mathbf{C}(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{B} = -\mathbf{C}\mathbf{A}_{s_1}^{-1}\mathbf{B}$$

$$\mathbf{G}_2(s_1, s_2) = -\frac{1}{2}\mathbf{C}\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{B}) - \bar{\mathbf{N}}\left(\mathbf{I}_m \otimes (\mathbf{A}_{s_1}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B})\right)\right]$$

$$\Downarrow \quad \boxed{s_1 = s_2 = \sigma}$$

$$\mathbf{G}_2(\sigma, \sigma) = -\frac{1}{2}\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\left[2 \cdot \mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) - \bar{\mathbf{N}}\left(\mathbf{I}_m \otimes 2 \cdot \mathbf{A}_{\sigma}^{-1}\mathbf{B}\right)\right]$$

$$\Downarrow$$

$$\mathbf{G}_2(\sigma, \sigma) = -\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) - \bar{\mathbf{N}}\left(\mathbf{I}_m \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}\right)\right]$$

Moments

$$\begin{aligned}\frac{\partial \mathbf{G}_2}{\partial s_1}(\sigma, \sigma) = &-\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{E}\mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) \\ &-\frac{1}{2}\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{E}\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B} \quad + \quad \mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{E}\mathbf{A}_{\sigma}^{-1}\mathbf{B}) \\ &+\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{E}\mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) \\ &+\frac{1}{2}\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{E}\mathbf{A}_{\sigma}^{-1}\mathbf{B})\end{aligned}$$

New Theorem: Two-sided rational interpolation (Block-Krylov)

Let $\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$ be nonsingular, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\mathbf{H}_r = \mathbf{W}^T\mathbf{H}(\mathbf{V}\otimes\mathbf{V})$, $\bar{\mathbf{N}}_r = \mathbf{W}^T\bar{\mathbf{N}}(\mathbf{I}_m\otimes\mathbf{V})$, $\mathbf{B}_r = \mathbf{W}^T\mathbf{B}$, $\mathbf{C}_r = \mathbf{C}\mathbf{V}$ with $\mathbf{V}$, $\mathbf{W}\in\mathbb{R}^{n\times r}$ having full rank such that

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\ldots,k}\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}, \mathbf{A}_{2\sigma_i}^{-1}[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B})]\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\ldots,k}\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T, \mathbf{A}_{\sigma_i}^{-T}[(\mathbf{H}+\mathbf{J})^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T) - \bar{\mathbf{N}}^{(2)}(\mathbf{I}_m\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T)]\}$$

with $\sigma_i \notin \{\Lambda(\mathbf{A},\mathbf{E}), \Lambda(\mathbf{A}_r,\mathbf{E}_r\}$.

Then:

$$\mathbf{G}_1(\sigma_i) = \mathbf{G}_{1,r}(\sigma_i)$$

$$\mathbf{G}_1(2\sigma_i) = \mathbf{G}_{1,r}(2\sigma_i)$$

$$\mathbf{G}_2(\sigma_i,\sigma_i) = \mathbf{G}_{2,r}(\sigma_i,\sigma_i)$$

$$\frac{\partial\mathbf{G}_2}{\partial s_j}(\sigma_i,\sigma_i) = \frac{\partial\mathbf{G}_{2,r}}{\partial s_j}(\sigma_i,\sigma_i)$$

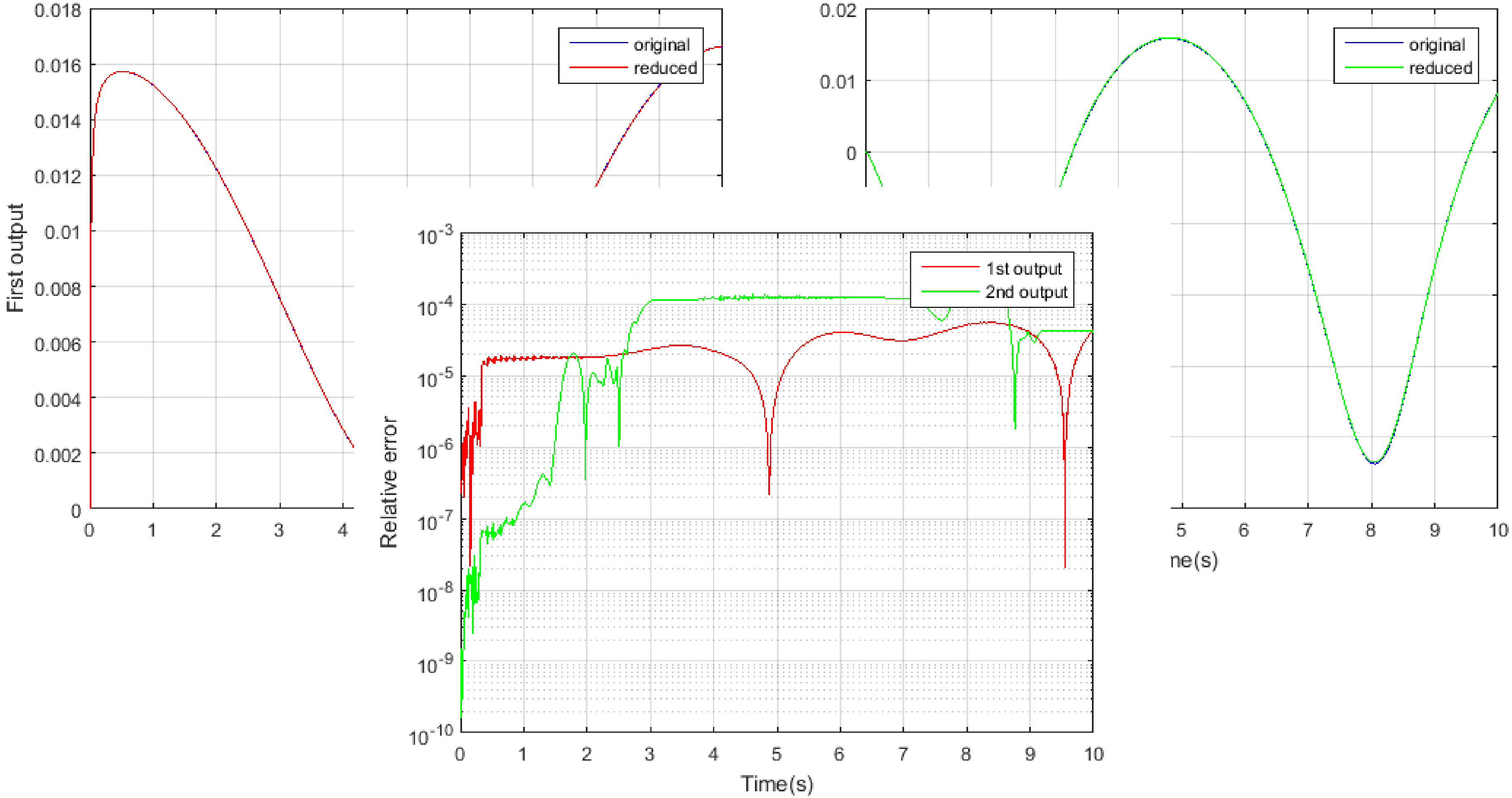
Proof for the theorem almost done. Still having some issues with the proof of the first moment of $\mathbf{G}_2$

Nonlinear RC circuit: Made MIMO by adding another input current source: $m = p = 2$

Block-Krylov reduction with 6 optimal expansion points from IRKA

$$n = 1000$$
$$r = 36$$

$$r = N_{s_0} \cdot (m + m^2)$$



Summary and Outlook

Summary:

- **Goal:** Reduction of high dimensional nonlinear systems
- Equivalent transformation of a nonlinear system into a quadratic-bilinear system
- Systems theory and Krylov-based reduction for SISO quadratic-bilinear systems
- Systems theory and Krylov-based reduction for MIMO quadratic-bilinear systems
  - New proposition for block and tangential Krylov approach
  - First numerical examples show promising results

Outlook:

- Quadratic-bilinear MOR
  - Stability-preserving two-sided rational Krylov for QBDAEs?
  - Choice of optimal expansion points? Comparison with T-QB-IRKA
  - Reduction for quadratic-bilinear DAEs

References

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Thank you for your attention!