Interpolation-based model reduction of nonlinear systems

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Motivation for Nonlinear Model Order Reduction

Given a large-scale nonlinear control system of the form

\[
\Sigma : \begin{cases}
    E\dot{x}(t) = f(x(t)) + B u(t), \\
    y(t) = C x(t), \quad x(0) = x_0
\end{cases} \quad x(t) \in \mathbb{R}^n
\]

with \( E \in \mathbb{R}^{n \times n}, f(x(t)) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{q \times n} \)

Simulation, design, control and optimization cannot be done efficiently!

\[ \downarrow \text{MOR} \]

Reduced order model

\[
\Sigma_r : \begin{cases}
    E_r\dot{x}_r(t) = f_r(x_r(t)) + B_r u(t), \\
    y_r(t) = C_r x_r(t), \quad x_r(0) = x_{r,0}
\end{cases} \quad x_r(t) \in \mathbb{R}^r, \ r \ll n
\]

with \( E_r \in \mathbb{R}^{r \times r}, f_r(x_r(t)) : \mathbb{R}^r \rightarrow \mathbb{R}^r \) and \( B_r \in \mathbb{R}^{r \times m}, C_r \in \mathbb{R}^{q \times r} \)

Goal:
\[ y_r(t) \approx y(t) \]
Projective nonlinear MOR

Approximation in the subspace $\mathcal{V} = \text{span}(EV)$

$$x = V x_r + e, \quad V \in \mathbb{R}^{n \times r}$$

Procedure:
1. Replace $x$ by its approximation
2. Reduce the number of equations (via projection with $\Pi = EV(W^T EV)^{-1}W^T$)
3. Petrov-Galerkin condition

$$\begin{align*}
E_r & \quad E_r x_r = W^T f(V x_r) + W^T B u \\
\dot{x}_r & \quad y \approx y_r = C_r x_r
\end{align*}$$

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Model Order Reduction (MOR)

Large-scale nonlinear model

\[
\begin{align*}
\Sigma : \quad & \begin{cases}
E \dot{x}(t) = f(x(t)) + Bu(t), \\
y(t) = Cx(t), \quad x(0) = x_0
\end{cases} \\
E \in \mathbb{R}^{n \times n}, & \quad f(x(t)) : \mathbb{R}^n \rightarrow \mathbb{R}^n \\
B \in \mathbb{R}^{n \times m}, & \quad C \in \mathbb{R}^{q \times n}
\end{align*}
\]

Reduced order model (ROM)

\[
\begin{align*}
\Sigma_r : \quad & \begin{cases}
E_r \dot{x}_r(t) = f_r(x_r(t)) + B_r u(t), \\
y_r(t) = C_r x_r(t), \quad x_r(0) = x_{r,0}
\end{cases} \\
E_r \in \mathbb{R}^{r \times r}, & \quad f_r(x_r(t)) : \mathbb{R}^r \rightarrow \mathbb{R}^r \\
B_r \in \mathbb{R}^{r \times m}, & \quad C_r \in \mathbb{R}^{q \times r}
\end{align*}
\]
Challenges of Nonlinear Model Reduction

• Nonlinear systems can exhibit complex behaviours
  – Multiple equilibria
  – Stable, unstable or semi-stable limit cycles
  – Chaotic behaviours

• Input-output behaviour of nonlinear systems cannot be described with the help of transfer functions, the state-transition matrix or the convolution (only possible for special cases)

• Choice of the reduced order basis
  – Projection bases should comprise the most dominant directions of the state-space
  – Existing approaches:
    → Simulation-based methods
    → Volterra-based approaches
    → Quadratic-bilinear-based techniques

• Expensive evaluation of the full-order vector of nonlinearities $f(Vx_r(t))$
  – Approximation by so-called hyper-reduction techniques: EIM, DEIM, Gappy-POD, GNAT, ECSW, …
Overview of existing nonlinear model reduction methods

• Classification in
  1. Simulation- or trajectory-based methods
  2. Volterra-based approaches (bilinear)
  3. Polynomialization- and variational analysis-based techniques (quadratic-bilinear)

or

  a) Time domain approaches (Simulation- or trajectory-based approaches)
  b) Frequency domain approaches (Interpolation-based methods: bilinear & QBMOR)

or

  i. Strong nonlinear approaches (POD, NL-BT, Empirical Gramians, TPWL, QBMOR)
  ii. Weakly nonlinear approaches (Bilinear models)

• Methods:
  1. POD, Nonlinear Balanced Truncation (NL-BT), Empirical Gramians, TPWL
  2. Bilinear systems (BT, bilinear RK, BIRKA, Loewner Framework,...)
  3. Quadratic-bilinear (BT, RK)
Overview

\[ \dot{x}(t) = f(x(t)) + Bu(t) \]
\[ y(t) = Cx(t) \]

Simulation-based methods

- Proper Orthogonal Decomposition (POD)
- Nonlinear Balanced Truncation
- Empirical Gramians
- Trajectory piecewise linear approximation (TPWL)

Volterra-based methods

- Bilinear Balanced Truncation
- Bilinear Rational Krylov
- Bilinear IRKA
- Bilinear Loewner Framework

Quadratic-bilinear methods

- Balanced Truncation for QBDAEs
- Two-sided Rational Krylov for QBDAEs
Proper Orthogonal Decomposition (POD)

Starting point: \[ \dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + B\mathbf{u}(t) \]
\[ y(t) = C\mathbf{x}(t) \]

1. Choose suitable training input signals \( \mathbf{u}_1(t), \mathbf{u}_2(t), \ldots, \mathbf{u}_t(t) \)

2. Take snapshots from simulated full-order state trajectories

\[ \mathbf{X}_{(n,t\cdot N)} = [\mathbf{x}^{u_1}(t_1), \mathbf{x}^{u_1}(t_2), \ldots, \mathbf{x}^{u_1}(t_N), \mathbf{x}^{u_2}(t_1), \mathbf{x}^{u_2}(t_2), \ldots] \]

3. Perform singular value decomposition (SVD) of the snapshot matrix \( \mathbf{X} \)

\[ \mathbf{X} = \mathbf{M}_{(n,n)} \mathbf{\Sigma}_{(n,n)} \mathbf{N}^T_{(n,t\cdot N)} \approx \mathbf{M}_r \mathbf{\Sigma}_r \mathbf{N}^T_r \]

4. Reduced order basis: \( \mathbf{V} = \mathbf{M}_r \in \mathbb{R}^{n \times r} \)

Advantages
- Straightforward data-driven method
- Error bound for approximation error
- Optimal in least squares sense:
  \[ \min_{\text{rank}(\mathbf{X}_r) = r} \| \mathbf{X} - \mathbf{X}_r \|_2 \]

Drawbacks
- Simulation of full-order model for different input signals required
- SVD of large snapshot matrix \( \mathbf{X} \)
- Training input dependency
Trajectory Piecewise-Linear Approximation (TPWL)

**Starting point:**
\[ \dot{E}x(t) = f(x(t)) + Bu(t) \]
\[ y(t) = Cx(t) \]

1. Linearize original nonlinear model along simulated state trajectory

\[ \dot{E}x(t) = \sum_{i=1}^{s} \omega_i(x) (f(x_i) + A_i(x - x_i)) + Bu(t) \]
\[ y(t) = Cx(t) \]

2. Reduce linearized models with well-known linear model reduction techniques (e.g. POD, Balanced Truncation, Rational Krylov, ...)

3. Construct reduced order model as weighted sum of linearized reduced models:

\[ W^T E V \dot{x}_r(t) = \sum_{i=1}^{s} \omega_i(x_r(t)) W^T (f(x_i) - A_i x_i) + \sum_{i=1}^{s} \omega_i(x_r(t)) (W^T A_i V x_r) + W^T Bu(t) \]
\[ y_r(t) = CVx_r(t) \]

Weighting functions
\[ \sum_{i=1}^{s} \omega_i(x_r(t)) = 1, \quad \omega_i(x_r(t)) \geq 0 \]
Trajectory Piecewise-Linear Approximation (TPWL)

**Offline stage**
1. Simulation of full-order model for several appropriate training input signals
2. Selection of linearization points (number $s$ and distance $\delta$) and linearization at selected points
3. Reduction of all linearized models
4. Choice of weighting function (e.g. Gaussian, $\text{sinc}$ squared, trapezoidal, triangular, ...)

**Online stage**
1. Calculation of the weights according to the current state
2. Computation of reduced model as convex combination of linearized reduced models

**Advantages**
- Strong nonlinear approach
- Linear model reduction techniques can be used
- No hyper-reduction step necessary

**Drawbacks**
- Simulation, linearization and reduction of full-order models
- Many degrees of freedom ($s, \delta, \omega_i$)
- Training input dependency
Trajectory Piecewise-Linear Approximation (TPWL)

Variations and extensions of the TPWL approach

- Fast approximate simulation
  - select the linearization points using the linearized or the reduced trajectory

- Reduction of the linearized models
  - Using **global projection matrices**:
    \[ V = \begin{bmatrix} V_1^{(1)} & V_1^{(2)} & V_2^{(1)} & V_2^{(2)} & \ldots & V_s^{(1)} & V_s^{(2)} \end{bmatrix} \]
    \[ \text{span}\{V_i^{(1)}\} = \mathcal{K}_r ((A_i - s_0E)^{-1}E, (A_i - s_0E)^{-1}B) \]
    \[ \text{span}\{V_i^{(2)}\} = \mathcal{K}_r ((A_i - s_0E)^{-1}E, (A_i - s_0E)^{-1}(f(x_i) - A_ix_i)) \]
  - Using **local projection matrices**:
    \[ V_1 = \begin{bmatrix} V_1^{(1)} & V_1^{(2)} \end{bmatrix}, \ldots, V_s = \begin{bmatrix} V_s^{(1)} & V_s^{(2)} \end{bmatrix} \]
    \[ \rightarrow \text{Computation of state transformations to common subspace are necessary} \]

- Generation of **stable TPWL reduced models**

- Reduction of **nonlinear, parametric models** using TPWL + pMOR by Matrix Interpolation

- Reduction of **nonlinear DAE models** (e.g. electrostatic beam, IMTEK) using TPWL
Overview

Simulation-based methods

\[
\begin{align*}
    \dot{\mathbf{x}}(t) &= f(\mathbf{x}(t)) + \mathbf{B}u(t) \\
    y(t) &= \mathbf{C}\mathbf{x}(t)
\end{align*}
\]

- Proper Orthogonal Decomposition (POD)
- Nonlinear Balanced Truncation
- Empirical Gramians
- Trajectory piecewise linear approximation (TPWL)

Volterra-based methods

\[
\begin{align*}
    \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + N\mathbf{x}u + \mathbf{B}u(t) \\
    y(t) &= \mathbf{C}\mathbf{x}(t)
\end{align*}
\]

- Bilinear Balanced Truncation
- Bilinear Rational Krylov
- Bilinear IRKA
- Bilinear Loewner Framework

Quadratic-bilinear methods

\[
\begin{align*}
    \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + H(\mathbf{x} \otimes \mathbf{x}) + N\mathbf{x}u + \mathbf{B}u(t) \\
    y(t) &= \mathbf{C}\mathbf{x}(t)
\end{align*}
\]

- Balanced Truncation for QBDAEs
- Two-sided Rational Krylov for QBDAEs
Carleman linearization

Starting point: \( \dot{E}x(t) = f(x(t)) + Bu(t) \)
\[ y(t) = Cx(t) \]

Goal: Approximation of (weakly) nonlinear systems by Carleman linearization

- Taylor series representation:
\[
f(x) = f(x_0) + \left[ \frac{f^{(1)}(x_0)}{1!} (x - x_0) \right]_{A_1} + \left[ \frac{f^{(2)}(x_0)}{2!} (x - x_0)^2 \right]_{A_2} + \left[ \frac{f^{(3)}(x_0)}{3!} (x - x_0)^3 \right]_{A_3} + \cdots
\]

Assumptions:
- \( x_0 = 0 \)
- \( f(x_0) = 0 \)

\[
f(x) = A_1 x + A_2 (x \otimes x) + A_3 (x \otimes x \otimes x) + \cdots = \sum_{k=1}^{\infty} A_k x^{(k)} \approx \sum_{k=1}^{N} A_k x^{(k)}
\]

- State-space model:
\[
\dot{E}x(t) = \sum_{k=1}^{N} A_k x^{(k)} + Bu(t)
\]
\[ y(t) = Cx(t) \]
Carleman bilinearization

Starting point:

\[ \mathbf{E} \dot{x}(t) = \sum_{k=1}^{N} \mathbf{A}_k \mathbf{x}^{(k)} + \mathbf{B}u(t) \]

\[ y(t) = \mathbf{C}x(t) \]

Goal: Bilinear model

- Consider differential equations for \( \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \ldots, \mathbf{x}^{(N)} \)

\[
\mathbf{E}^{(2)} \frac{d}{dt} \mathbf{x}^{(2)} = \frac{d}{dt} (\mathbf{E}x \otimes \mathbf{E}x) = \mathbf{E} \dot{x} \otimes \mathbf{E}x + \mathbf{E}x \otimes \mathbf{E} \dot{x} \\
= \left( \sum_{k=1}^{N} \mathbf{A}_k \mathbf{x}^{(k)} + \mathbf{B}u \right) \otimes \mathbf{E}x + \mathbf{E}x \otimes \left( \sum_{k=1}^{N} \mathbf{A}_k \mathbf{x}^{(k)} + \mathbf{B}u \right) \\
= \sum_{k=1}^{N-1} [\mathbf{A}_k \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{A}_k] \mathbf{x}^{(k+1)} + [\mathbf{B} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{B}] \mathbf{x} \otimes u
\]

\[
\mathbf{E}^{(3)} \frac{d}{dt} \mathbf{x}^{(3)} = \frac{d}{dt} (\mathbf{E}x \otimes \mathbf{E}x \otimes \mathbf{E}x) \\
\vdots
\]

- Bilinear model:

\[ \mathbf{E}^{\otimes} \dot{x}^{\otimes} = \mathbf{A}^{\otimes} x^{\otimes} + \mathbf{N}^{\otimes} x^{\otimes} u + \mathbf{B}^{\otimes} u \]

with

\[ x^{\otimes} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \vdots \\ \mathbf{x}^{(N)} \end{bmatrix} \]

\[ y = \mathbf{C}^{\otimes} x^{\otimes} \]
Carleman bilinearization: example

Starting point:
\[ \dot{E}x(t) = f(x(t)) + Bu(t) \]
\[ y(t) = Cx(t) \]

Carleman linearization
\[ N = 2 \]
\[ \dot{E}x = A_1x + A_2(x \otimes x) + Bu \]
\[ y = Cx \]

Carleman bilinearization
\[ x^\otimes = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \]
\[ E^{(2)} \frac{d}{dt} x^{(2)} = \frac{d}{dt}(Ex \otimes Ex) = E\dot{x} \otimes Ex + Ex \otimes E\dot{x} \]
\[ = (A_1x + A_2(x \otimes x) + Bu) \otimes Ex \]
\[ + Ex \otimes (A_1x + A_2(x \otimes x) + Bu) \]
\[ = [A_1 \otimes E + E \otimes A_1] x \otimes x + [B \otimes E + E \otimes B] x \otimes u \]

Bilinear model:
\[ \begin{bmatrix} E & 0 \\ 0 & E \otimes E \end{bmatrix} \dot{x}^\otimes = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \otimes E + E \otimes A_1 \end{bmatrix} x^\otimes + \begin{bmatrix} 0 & 0 \\ B \otimes E + E \otimes B & 0 \end{bmatrix} x^\otimes u + \begin{bmatrix} B \\ 0 \end{bmatrix} u \]
\[ y = \begin{bmatrix} C & 0 \end{bmatrix} x^\otimes \]
State-Space Representation of Bilinear Systems

Consider bilinear SISO systems of the form

\[
\begin{align*}
E \dot{x} &= A x + N x u + b u \\
y &= c^T x
\end{align*}
\]

with \(E, A, N \in \mathbb{R}^{n \times n}\) and \(b, c \in \mathbb{R}^n\).

- Many (weakly) nonlinear systems can be approximated by bilinear systems through Carleman bilinearization
  
  **Drawback:** Dimension of the bilinear model is significantly higher than the original state dimension \(\rightarrow\) only applicable for medium-sized (weakly) nonlinear systems

- Linear in input and linear in state, but not jointly linear in state and input

- **Advantage:** Close relation to linear systems, a lot of well-known concepts can be extended, e.g. transfer functions, Gramians, Sylvester and Lyapunov equations.
Output response and Transfer Functions of Bilinear Systems

Some background on Volterra theory

- **Output response** expressed by Volterra series: \( y(t) = \sum_{k=1}^{\infty} y_k(t) \)

\[
y_k(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_k} \left[ c^T e^{E^{-1}A t_k} E^{-1} N \cdots E^{-1} N e^{E^{-1}A t_k} E^{-1} b \right] u(t - t_1 - \cdots - t_k) \cdots u(t - t_k) \, dt_k \cdots dt_1
\]

**Impulse response / kernel of \( k \)th degree**

\[
y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_k} g_k(t_1, \ldots, t_k) u(t - t_1 - \cdots - t_k) \cdots u(t - t_k) \, dt_k \cdots dt_1
\]

- **Multivariable Laplace-transform:**

\[
G_1(s_1) = c^T (s_1 E - A)^{-1} b
\]

\[
G_2(s_1, s_2) = c^T (s_2 E - A)^{-1} N (s_1 E - A)^{-1} b
\]

\[
G_3(s_1, s_2, s_3) = c^T (s_3 E - A)^{-1} N (s_2 E - A)^{-1} N (s_1 E - A)^{-1} b
\]

\[\vdots\]

\[
G_k(s_1, \ldots, s_k) = c^T (s_k E - A)^{-1} N \cdots N (s_2 E - A)^{-1} N (s_1 E - A)^{-1} b
\]
Model Reduction of Bilinear Systems

Volterra-based methods

\[ \dot{x} = Ax + Nxu + Bu \]
\[ y = Cx \]

- [Al-Baiyat ’93], [Benner/Damm ’11]
- Solution of two bilinear Lyapunov equations
- [Phillips ’00], [Bai/Skoogh ’06], [Breiten/Damm ’10]
- Multimoment-Matching for bilinear systems
- [Zhang/Lam ’02], [Benner/Breiten ’12], [Flagg ’12]
- H2-optimal model reduction for bilinear systems
- [Flagg ’12], [Antoulas ’14]
- Data-driven interpolation-based approach
MOR for Bilinear Systems: Multimoment-Matching

Multimoments for bilinear systems: [Bai/Skoogh ’06], [Breiten/Damm ’10]

• Transfer function:

\[ G_k(s_1, \ldots, s_k) = c^T (s_k E - A)^{-1} N \cdots N (s_2 E - A)^{-1} N (s_1 E - A)^{-1} b \]

• Make use of Neumann expansion

• Expansion in a multivariable Maclaurin series

\[ G_k(s_1, \ldots, s_k) = \sum_{l_k=1}^{\infty} \cdots \sum_{l_1=1}^{\infty} m(l_1, \ldots, l_k) \cdot (s_1 - \sigma_1)^{l_1-1} \cdots (s_k - \sigma_k)^{l_k-1} \]

• Multimoments:

\[ m(l_1, \ldots, l_k) = (-1)^k c^T (A - \sigma_k E)^{-l_k} N \cdots N (A - \sigma_2 E)^{-l_2} N (A - \sigma_1 E)^{-l_1} b \]

• Markov parameters:

\[ m^\infty(l_1, \ldots, l_k) = c^T A^{l_k-1} N \cdots N A^{l_2-1} N A^{l_1-1} b \]

with \[ G_k(s_1, \ldots, s_k) = \sum_{l_k=1}^{\infty} \cdots \sum_{l_1=1}^{\infty} m^\infty(l_1, \ldots, l_k) \cdot s_1^{-l_1} \cdots s_k^{-l_k} \]
MOR for Bilinear Systems: Multimoment-Matching

Multimoment-Matching: [Bai/Skoogh ’06], [Feng/Benner ’07], [Breiten/Damm ’10]

1. Calculation of the Krylov subspaces:

\[
\begin{align*}
\text{span}\{V^{(1)}\} &= \mathcal{K}_{r_1} \left( (A - \sigma_1 E)^{-1}E, (A - \sigma_1 E)^{-1}b \right) \\
\text{span}\{V^{(2)}\} &= \mathcal{K}_{r_2} \left( (A - \sigma_2 E)^{-1}E, (A - \sigma_2 E)^{-1}NV^{(1)}U^T \right) \\
& \vdots \\
\text{span}\{V^{(j)}\} &= \mathcal{K}_{r_j} \left( (A - \sigma_j E)^{-1}E, (A - \sigma_j E)^{-1}NV^{(j-1)}U^T \right), \ j = 2, \ldots, J
\end{align*}
\]

\[
\text{span}\{V\} = \bigcup_{j=1}^{J} \text{colspan}\{V^{(j)}\}
\]

2. Computation of the reduced order model:

\[
\begin{align*}
E_r &= W^TEV, \quad A_r = W^TAV, \quad N_r = W^TNV, \quad b_r = W^Tb, \quad c_r^T = c^TV
\end{align*}
\]

Example:

- 1st subsystem: \( r_1 = 4, \sigma_1 \)

\[
\begin{align*}
V^{(1)} &= \left[ (A - \sigma_1 E)^{-1}b, (A - \sigma_1 E)^{-1}E(A - \sigma_1 E)^{-1}b, \ldots \right] \\
W^{(1)} &= \left[ (A - \sigma_1 E)^{-T}c, (A - \sigma_1 E)^{-T}E^T(A - \sigma_1 E)^{-T}c, \ldots \right]
\end{align*}
\]

- 2nd subsystem: \( r_2 = 2, \sigma_2 \)

\[
\begin{align*}
V^{(2)} &= \left[ (A - \sigma_2 E)^{-1}NV^{(1)}U^T, (A - \sigma_2 E)^{-1}E(A - \sigma_2 E)^{-1}NV^{(1)}U^T \right] \\
W^{(2)} &= \left[ (A - \sigma_2 E)^{-T}N^TW^{(1)}U^T, (A - \sigma_2 E)^{-T}E^T(A - \sigma_2 E)^{-T}N^TW^{(1)}U^T \right]
\end{align*}
\]

\[
\begin{align*}
m(l_1) &= m_r(l_1) \\
& \text{for } l_1 = 1, \ldots, r_1
\end{align*}
\]

\[
\begin{align*}
m(l_1, l_2) &= m_r(l_1, l_2) \\
& \text{for } l_1 = 1, \ldots, r_1, \quad l_2 = 1, \ldots, r_2
\end{align*}
\]
MOR for Bilinear Systems: Multimoment-Matching

**Multimoment-Matching:** [Bai/Skoogh ’06], [Feng/Benner ’07], [Breiten/Damm ’10]

1. Calculation of the Krylov subspaces:

\[
\begin{align*}
\text{span}\{V^{(1)}\} &= \mathcal{K}_{r_1} \left( (A - \sigma_1 E)^{-1} E, (A - \sigma_1 E)^{-1} b \right) \\
\text{span}\{V^{(2)}\} &= \mathcal{K}_{r_2} \left( (A - \sigma_2 E)^{-1} E, (A - \sigma_2 E)^{-1} NV^{(1)} U^T \right) \\
\vdots & \\
\text{span}\{V^{(j)}\} &= \mathcal{K}_{r_j} \left( (A - \sigma_j E)^{-1} E, (A - \sigma_j E)^{-1} NV^{(j-1)} U^T \right), \quad j = 2, \ldots, J
\end{align*}
\]

\[
\text{span}\{V\} = \bigcup_{j=1}^{J} \text{colspan}\{V^{(j)}\}
\]

2. Computation of the reduced order model:

\[
E_r = W^T EV, \quad A_r = W^T AV, \quad N_r = W^T NV, \quad b_r = W^T b, \quad c_r^T = c^T V
\]

**Open questions/problems:**

- How to choose the expansion points?
  → Optimal expansion points via $\mathcal{H}_2$-optimal model reduction (bilinear IRKA)
- How many moments should be matched per subsystem?
- How many subsystems are necessary for a good approximation?
- Error bounds?
MOR for Bilinear Systems: $\mathcal{H}_2$-optimal model reduction

- $\mathcal{H}_2$-norm of a MIMO bilinear system:

$$\|\Sigma\|^2_{\mathcal{H}_2} := \text{tr} \left( \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} G_k(j\omega_1, \ldots, j\omega_k) G_k^*(j\omega_1, \ldots, j\omega_k) \, d\omega_k \cdots d\omega_1 \right)$$

Alternative calculation via

$$\|\Sigma\|^2_{\mathcal{H}_2} = \text{tr} \left( CPC^T \right) = \text{tr} \left( B^T QB \right)$$

where $P$ and $Q$ are the solutions of the following bilinear Lyapunov equations:

$$APE + EPA^T + \sum_{k=1}^{m} N_k P N_k^T + BB^T = 0$$

$$A^T QE + E^T QA + \sum_{k=1}^{m} N_k^T Q N_k + C^T C = 0$$

$$\quad \Rightarrow \quad \|\Sigma\|^2_{\mathcal{H}_2} = (\text{vec}(I_q))^T (C \otimes C) \left( -A \otimes E - E \otimes A - \sum_{k=1}^{m} N_k \otimes N_k \right)^{-1} (B \otimes B) \text{vec}(I_m)$$

- Error system: $\Sigma_e = \Sigma - \Sigma_r$

$$E_e = \begin{bmatrix} E & 0 \\ 0 & E_r \end{bmatrix}, \quad A_e = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \quad N_{k,e} = \begin{bmatrix} N_k & 0 \\ 0 & N_{k,r} \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ B_r \end{bmatrix}, \quad C_e = \begin{bmatrix} C & -C_r \end{bmatrix}$$
MOR for Bilinear Systems: $\mathcal{H}_2$-optimal model reduction

- $\mathcal{H}_2$-norm of error system:

$$E^2 = \|\Sigma_e\|^2_{\mathcal{H}_2} = \|\Sigma - \Sigma_r\|^2_{\mathcal{H}_2} = \text{tr} \left( C_e P_e C_e^T \right) = \text{tr} \left( B_e^T Q_e B_e \right)$$

where $P_e$ and $Q_e$ are the solutions of the following bilinear Lyapunov equations:

$$A_e P_e E_e + E_e P_e A_e^T + \sum_{k=1}^{m} N_{k,e} P_e N_{k,e}^T + B_e B_e^T = 0$$

$$A_e^T Q_e E_e + E_e^T Q_e A_e + \sum_{k=1}^{m} N_{k,e}^T Q_e N_{k,e} + C_e^T C_e = 0$$

Assume the reduced model $\Sigma_r$ is given by its eigenvalue decomposition:

$$E_r^{-1} A_r = R \Lambda R^{-1}, \quad \tilde{N}_k = R^{-1} E_r^{-1} N_{k,r} R, \quad \tilde{B} = R^{-1} E_r^{-1} B_r, \quad \tilde{C} = C_r R$$

$$\Rightarrow E^2 = f(A, \Lambda, N_k, \tilde{N}_k, B, \tilde{B}, C, \tilde{C}) \rightarrow \min$$

- Necessary conditions for $\mathcal{H}_2$-optimality:

1. $\frac{\partial E^2}{\partial C_{ij}} \equiv 0 \iff G(-\bar{\lambda}_{r,i}) \tilde{B}_i^T = G_r(-\bar{\lambda}_{r,i}) \tilde{B}_i^T$

2. $\frac{\partial E^2}{\partial \tilde{B}_{ij}} \equiv 0 \iff \tilde{C}_i^T G(-\bar{\lambda}_{r,i}) = \tilde{C}_i^T G_r(-\bar{\lambda}_{r,i})$

3. $\frac{\partial E^2}{\partial \lambda_{r,i}} \equiv 0 \iff \tilde{C}_i^T G'(-\bar{\lambda}_{r,i}) \tilde{B}_i^T = \tilde{C}_i^T G'_r(-\bar{\lambda}_{r,i}) \tilde{B}_i^T$

4. $\frac{\partial E^2}{\partial \tilde{N}_{k,ij}} \equiv 0$
Bilinear IRKA approach

**Algorithm 1 Bilinear Iterative Rational Krylov Algorithm (BIRKA)**

**Input:** $E, A, N_k, B, C, E_r, A_r, N_{k,r}, B_r, C_r$

**Output:** $E_r^{\text{opt}}, A_r^{\text{opt}}, N_{k,r}^{\text{opt}}, B_r^{\text{opt}}, C_r^{\text{opt}}$

1. while (change in $\Lambda > \epsilon$) do
2. \[ E_r^{-1} A_r = RA R^{-1}, \tilde{N}_k = R^{-1} E_r^{-1} N_{k,r} R, \tilde{B} = R^{-1} E_r^{-1} B_r, \tilde{C} = C_r R \]
3. \[ \text{vec}(V) = \left( -\Lambda \otimes E - E \otimes A - \sum_{k=1}^{m} \tilde{N}_k \otimes N_k \right)^{-1} (\tilde{B} \otimes B) \text{vec}(I_m) \]
4. \[ \text{vec}(W) = \left( -\Lambda \otimes E - E \otimes A^T - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k^T \right)^{-1} (\tilde{C}^T \otimes C)^T \text{vec}(I_q) \]
5. \[ V = \text{orth}(V), W = \text{orth}(W) \]
6. \[ E_r = W^T E V, A_r = W^T A V, N_r = W^T N V, B_r = W^T B, C_r = C V \]
7. end while
8. \[ E_r^{\text{opt}} = E_r, A_r^{\text{opt}} = A_r, N_{k,r}^{\text{opt}} = N_{k,r}, B_r^{\text{opt}} = B_r, C_r^{\text{opt}} = C_r \]
MOR for Linear Systems: $H_2$ pseudo-optimal reduction

- **Duality:** Krylov subspaces with Sylvester equations
  \[
  \text{span}\{V\} = \mathcal{K}_r \left((A - s_0E)^{-1}E, (A - s_0E)^{-1}B\right) \\
  \text{span}\{W\} = \mathcal{K}_r \left((A - s_0E)^{-T}E^T, (A - s_0E)^{-T}C^T\right)
  \]

- **$H_2$-optimality vs. $H_2$ pseudo-optimality**

<table>
<thead>
<tr>
<th>$H_2$-optimality</th>
<th>$H_2$ pseudo-optimality</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem:</strong></td>
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</tr>
<tr>
<td>$|G - G_r|<em>{H_2} = \min</em>{\dim(\tilde{G}_r) = r} |G - \tilde{G}<em>r|</em>{H_2}$</td>
<td>$\Lambda = {\lambda_1, \ldots, \lambda_r}, \lambda_i \in \mathbb{C}^-$</td>
</tr>
<tr>
<td><strong>Necessary conditions for local $H_2$-optimality (SISO):</strong> (Meier-Luenberger)</td>
<td>$|G - G_r|<em>{H_2} = \min</em>{\tilde{G}_r \in \mathcal{G}(\Lambda)} |G - \tilde{G}<em>r|</em>{H_2}$</td>
</tr>
<tr>
<td>$G(-\bar{\lambda}<em>{r,i}) = G_r(-\bar{\lambda}</em>{r,i})$</td>
<td><strong>Necessary and sufficient condition for global $H_2$ pseudo-optimality:</strong></td>
</tr>
<tr>
<td>$G'(-\bar{\lambda}_{r,i}) = G'<em>r(-\bar{\lambda}</em>{r,i})$</td>
<td>$G(-\bar{\lambda}<em>{r,i}) = G_r(-\bar{\lambda}</em>{r,i})$</td>
</tr>
<tr>
<td>$G_r$ minimizes the $H_2$ error locally within the set of all ROMs of order $r$</td>
<td><strong>Pseudo-optimal means optimal in a certain subset</strong></td>
</tr>
<tr>
<td></td>
<td>$G_r$ minimizes the $H_2$ error globally within the subset of all ROMs of order $r$ with poles $\Lambda$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
AV - EVS &= BL \\
A^T W - E^T WS^T &= C^T L \\
\lambda_i(S) &= s_0 : \text{shifts} \\
L : \text{tangential directions}
\end{align*}
\]
MOR for Linear Systems: $H_2$ pseudo-optimal reduction

Notation:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Equation</th>
<th>Known/Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gramian</td>
<td>$A_r P_r E_r^T + E_r P_r A_r^T + B_r B_r^T = 0$</td>
<td>(known)</td>
</tr>
<tr>
<td>Scalar product</td>
<td>$A X E_r^T + E X A_r^T + B B_r^T = 0$</td>
<td>(unknown)</td>
</tr>
<tr>
<td>Krylov</td>
<td>$A V - E V S = B_L$</td>
<td>(known)</td>
</tr>
<tr>
<td>Projection</td>
<td>$B_{\perp} = B - E V E_r^{-1} B_r$</td>
<td>(known)</td>
</tr>
</tbody>
</table>

New conditions for pseudo-optimality [Wolf ‘14]:

Let $V$ be a basis of a Krylov subspace. Let $G_r(s)$ be the reduced model obtained by projection with $W$. Then, the following conditions are equivalent:

i) $S = -P_r A_r^T E_r^{-T} P_r^{-1}$

ii) $E_r^{-1} B_r + P_r L^T = 0$

iii) $S P_r + P_r S^T - P_r L^T L P_r = 0$

iv) $X = V P_r$

v) $A \hat{P} E^T + E \hat{P} A^T + B B^T = B_{\perp} B_{\perp}^T$

vi) $P_r^{-1} = E_r^* \hat{Q}_r E_r$
MOR for Linear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

**PORK: Pseudo-optimal rational Krylov**

<table>
<thead>
<tr>
<th>Algorithm 1 Pseudo-optimal rational Krylov (PORK)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> V, S, L, C, such that $AV - EVS = BL$ is satisfied</td>
</tr>
<tr>
<td><strong>Output:</strong> $\mathcal{H}_2$ pseudo-optimal reduced model $G_r(s) = C_r(sE_r - A_r)^{-1}B_r$</td>
</tr>
<tr>
<td>1. $P_r^{-1} = \text{lyap}(-S^T, L^T)$</td>
</tr>
<tr>
<td>2. $B_r = -(P_r^{-1})^{-1}L^T$</td>
</tr>
<tr>
<td>3. $A_r = S + B_rL$, $E_r = I$, $C_r = CV$</td>
</tr>
</tbody>
</table>

**Advantages and properties of PORK:**

- ROM is globally optimal within a subset: $\|G - G_r\|_{\mathcal{H}_2} = \min_{\tilde{G}_r \in G(\Lambda)} \|G - \tilde{G}_r\|_{\mathcal{H}_2}$
- Eigenvalues of ROM: $\Lambda(S) = \Lambda(-E_r^{-1}A_r)$, choice of the shifts is twice as important
- Stability preservation in the ROM can be ensured
- Low numerical effort required: solution of a Lyapunov equation and a linear system of equations, both of reduced order.
MOR for Bilinear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

- **Duality:** Bilinear Krylov subspaces with bilinear Sylvester equations [Flagg ’12]

  \[
  \text{span}\{V^{(1)}\} = \mathcal{K}_{r_1} \left( (A - \sigma_1 E)^{-1} E, (A - \sigma_1 E)^{-1} B \right) \\
  \text{span}\{V^{(j)}\} = \mathcal{K}_{r_j} \left( (A - \sigma_j E)^{-1} E, (A - \sigma_j E)^{-1} N V^{(j-1)} U^T \right), \quad j = 2, \ldots, J \\
  \text{span}\{W^{(1)}\} = \mathcal{K}_{r_1} \left( (A - \sigma_1 E)^{-T} E^T, (A - \sigma_1 E)^{-T} C^T \right) \\
  \text{span}\{W^{(j)}\} = \mathcal{K}_{r_j} \left( (A - \sigma_j E)^{-T} E^T, (A - \sigma_j E)^{-T} N^T W^{(j-1)} U^T \right), \quad j = 2, \ldots, J
  \]

\[
\begin{align*}
  AV - EVS - NVU^T &= BL \\
  A^T W - E^T WS^T - N^T WU^T &= C^T L
\end{align*}
\]

\[\lambda_i(S) = s_0 : \text{shifts} \quad L : \text{tangential directions} \quad U^T : \text{weights}\]

Can we derive new conditions for pseudo-optimality for bilinear systems?
MOR for Bilinear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

Notation:

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<th>$A_r P_r E_r^T + E_r P_r A_r^T + N_r P_r N_r^T + B_r B_r^T = 0$ (known)</th>
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New conditions for pseudo-optimality for bilinear systems:

Let $V$ be a basis of a Krylov subspace. Let $\Sigma_r$ be the reduced model obtained by projection with $W$. Then, the following conditions are equivalent:

1) $S = -P_r A_r^T E_r^{-T} P_r^{-1}$
2) $E_r^{-1}B_r + P_r L_r^T = 0$
3) $E_r^{-1}N_r P_r + P_r U_r = 0$
4) $SP_r + P_r S_r^T - P_r L_r^T L_r P_r + P_r U_r N_r^T E_r^{-T} = 0$
5) $X = VP_r$
6) Work In Progress (WIP)
BIPORK: Bilinear pseudo-optimal rational Krylov

Algorithm 1 Bilinear pseudo-optimal rational Krylov (BIPORK)

Input: $V, S, U, L, C$, such that $AV - EVS - NVU^T = BL$ is satisfied

Output: $\mathcal{H}_2$ pseudo-optimal reduced model $\Sigma_r$

1. $P_r^{-1}$: solution of bil. Lyap. equation: $S^TP_r^{-1} + P_r^{-1}S - UP_r^{-1}U^T - L^TL = 0$
2. $N_r = -(P_r^{-1})^{-1}UP_r^{-1}$
3. $B_r = -(P_r^{-1})^{-1}L^T$
4. $A_r = S + B_r L + N_r U^T$, $E_r = I$, $C_r = CV$
Summary and Outlook

**Summary:**

- **Goal:** Reduction of high dimensional nonlinear systems
- Simulation-based, Volterra-based and quadratic-bilinear-based approaches
- Model reduction for bilinear systems (BT, Krylov, BIRKA, Loewner)
- $\mathcal{H}_2$ pseudo-optimal model reduction for bilinear systems
  - Derivation of new conditions for $\mathcal{H}_2$ pseudo-optimality for bilinear systems
  - Bilinear pseudo-optimal Rational Krylov (BIPORK)

**Outlook:**

- Solution of bilinear Lyapunov equations with BIPORK:
  \[ \text{BI-LR-ADI} = \text{RKSM} + \text{BIPORK} \]
- Cumulative reduction of bilinear systems
- Quadratic-bilinear MOR
  - Stability-preserving two-sided rational Krylov for QBDAEs?
  - IRKA for QBDAEs? Algorithm for choosing optimal expansion points?
Discussion and open problems

Feedback and hints relating the following topics are welcome:

• Numerical solvers (direct and/or indirect) for nonlinear matrix equations
  a) Direct solvers
    ▪ Direct solvers for bilinear Sylvester and Lyapunov equations
  b) Indirect solvers
    ▪ Bilinear low-rank ADI method
    ▪ Bilinear Extended Krylov Subspace Method (EKSM)
    ▪ Other Krylov-based iterative solvers, e.g. CG, PCG, BiCG, BiCGstab

• Error bounds for bilinear systems
  – Existing approaches or literature?

• Nonlinear, parametric benchmarks
  – Parametric Nonlinear RC-Ladder?
  – Parametric Nonlinear Heat Transfer (IMTEK)?
  – …
Thank you for your attention!