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Dispersion Analysis of Periodic Structures by Solving Corresponding Excitation Problems

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Abstract. Eigenproblems of periodic structures with complicated material distribution are solved efficiently by solving corresponding excitation problems. A unit cell with periodic boundary conditions is discretised and handled by a doubly periodic hybrid finite element boundary integral technique, which even considers complex propagation constants. Instead of solving algebraic eigenproblems, the analogy to resonance problems is exploited which gives improved physical insight into the problem. Moreover, open problem types can be handled. Numerical results for composite right/left-handed waveguides in one and three dimensions confirm the presented approach.

Keywords. Waveguides, periodic problems, eigenproblems, resonators.

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1 Introduction

Linear systems are completely characterised by their eigensolutions and eigenvalues. With regard to electromagnetic devices, the eigensolutions and eigenvalues reveal the performance of these devices immediately, which promotes an efficient design procedure. By knowing the eigensolutions and eigenvalues, the response of a system to any kind of excitation can be predicted by a mere projection of the eigensolutions onto the excitation. Hence, the determination of eigensolutions and eigenvalues is of main scientific interest. Any electromagnetic field problem can be considered as an eigenproblem by excluding any form of excitation. Presumably every numerical field solving procedure can be employed for solving eigenproblems if the underlying discretised operator, which describes the field problem with respect to the boundary conditions, is considered without exciting sources. A collection of solution methods for numerical eigenproblems is provided in [9,17]. The numerical eigenproblem solvers included in most of the commercial software packages handle eigenproblems in form of a linear equation system, which becomes large if many discretisation unknowns are present. Consequently, eigenproblems often result as expensive or even impossible to solve. In particular the convergence behaviour of numerical solvers applied to nonlinear eigenproblems is mostly not satisfying. Nonlinear eigenproblems arise from integral equation formulations or problems including a perfectly matched layer (PML). The physical cause for convergence failure is often not even obvious. Furthermore, unphysical solutions together with PML are often detected but which are desired to avoid. An attempt is started in [2], where a-priori knowledge of the field pattern shall help to distinguish wanted from unphysical modes already during the solution process. The purpose is to improve the efficiency of the iterative Jacobi–Davidson algorithm built-in in the software package of CST MWS [1]. The application area is yet still restricted to simple structures such as dielectric substrates with air holes without metal. Likewise, many eigenproblem computations are limited to relatively small eigenproblem sizes as for instance in the analysis of the dispersion behaviour of waveguides, mostly meaning to solve two-dimensional eigenproblems. However, periodic configurations in one, two or three dimensions have gained notable interest in recent time especially in context with artificial materials or metamaterials. Although the periodic nature allows a reduction of the solution process to an elementary periodic unit cell, the numerical description and the corresponding discretisation of the 3D unit cell can fast lead to a huge number of unknowns. The structures normally exhibit elaborate material and geometry variations within small dimensions.

Nonetheless, one possibility to obtain a solution for large eigenproblems is to reduce the number of unknowns. If the dispersion analysis of periodic configurations is demanded the fields in the periodic boundaries can be expanded by a modal series. The modes are typically the eigenmodes of the host uniform waveguide, which is periodically loaded. The unit cell is excited mode-wise and the scattering or S-parameters reliably result from the driven full-wave simulation. The S-matrix is related to the transfer or T-matrix and the final eigenproblem is constituted by the usually small number of T-parameters. The procedure works well if a small number of port modes is sufficient to accurately formulate the field problem. If systems are concerned which

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are in particular operated in the fundamental mode to ensure a stable and predictable performance, the method is very well suited. Yet for open problems, the discrete and finite set of modal solutions becomes continuous and extends to infinity. The series expansion needs to be replaced by an integral formulation. However, the expansion method can deliver approximate results [18, 20], where the waveguide is well separated from the open half space.

Another way to facilitate eigenproblems is a specific form of the so-called Trefftz method [10, 12] relying on the method of fundamental solutions [8, 11, 15]. The particular solutions are there generated by exciting the solution domain by sources from outside. The procedure is extended to generalised Sturm–Liouville eigenproblems [16] and to the use of internally placed sources [14, 15]. The advantage of these methods is that the algebraic solution of the eigenproblem can be circumvented by instead solving a corresponding excitation problem repeated times.

Similar procedures have been known in electrical engineering for a long time. The resonance behaviour of filters and resonators is investigated by measurements and equivalent circuits by applying an excitation and by observing the frequency-dependent response of the system to the excitation. From the course of the reaction to the excitation, the resonance frequency identical with the eigenvalue can be directly inferred. This analogy is exploited in the method presented in this article. An eigenproblem is solved as a resonator problem by finding the singular point ω_{res} while an excitation is defined [5, 6]. Internal excitations are applied, e.g. by impressed currents or in form of (discrete) ports, which are adequately coupled to the field problem. Thus, circuit related considerations are possible, which are also usable for problem regularisations as well as for energy and power inspections. A further benefit is that the excitation can be specifically placed and the refining search in the iterative solution process is localised. Problematic PML modes are not excited at all.

The approach explained above is implemented in a hybrid finite element boundary integral (FEBI) solver for doubly periodic structures, which is well suited for one- and two-dimensional artificial waveguide structures. In the FE part of the code, an edge-based periodic boundary condition (PBC) is defined while a Floquet-mode based periodic Green's function serves for the spectral domain integral equation. Open problems can perfectly be managed due to the BI part. Moreover, the solver is upgraded in the PBCs in that the solver becomes suitable for complex propagation constants and thus evanescent and leaky mode representations are supported. These modes are typically not promoted by standard numerical solvers. As alternative, the proposed technique can be combined with other electromagnetic field solvers, e.g. CST MWS [1].

First, a composite right/left-handed hollow waveguide is investigated, where the right-handed and left-handed dispersion behavior is clearly verified. A three-dimensional synthesized artificial material composition follows where the solution method easily presents the eigenvalues of the waveguide modes.

2 Periodic Eigenproblem Formulation

The analysis of periodic configurations reduces to the analysis of one unit cell subject to periodic boundary conditions according to the Bloch-Floquet theorem. Eigensolutions are characterised as functions which do not change except for a multiplicative constant, the eigenvalue, if the corresponding operator is applied to them. Concerning a unit cell as element of a periodic array as illustrated by Figure 1, the eigenvector $\boldsymbol{v} = (\ldots, a'_m, \ldots, b'_m, \ldots)$ contains the field quantities, e.g. the incident a and reflected b modes, on one side of the unit cell. It is mapped by a factor, the propagation constant or the transfer matrix T, onto the field quantities on the opposite side according to

$$\mathrm{e}^{\gamma p}\boldsymbol{v} = \boldsymbol{T}\boldsymbol{v} \quad \text{or}$$
$$(\boldsymbol{T} - \mathrm{e}^{\gamma p}\boldsymbol{I})\boldsymbol{v} = 0. \tag{1}$$

The length of the unit cell is p. Equation (1) can also be expressed by

$$L(\omega, \gamma)\boldsymbol{v} = 0, \tag{2}$$

which is an operator L depending on discrete pairs of frequency ω and propagation constant γ. Various pairs of ω, γ eventually render the dispersion behaviour. Thus, the corresponding eigensolutions result as null space of the operator. In discretised versions, the operator is a matrix, which must become singular. Considering the equations for shunt and series resonant circuits

$$Y(\omega)U = \left[\mathrm{j}\left(\omega C - \frac{1}{\omega L}\right) + G_l\right]U = 0, \tag{3}$$

$$Z(\omega)I = \left[\mathrm{j}\left(\omega L - \frac{1}{\omega C}\right) + R_l\right]I = 0, \tag{4}$$

respectively, they become singular, too and yield the eigenvalues as resonant frequencies ω_{res}. The circuits are pictured in Figure 2 showing a capacitor C, an inductance L and a loss conductance G_l or a loss resistance R_l, respectively. For lossy resonators, ω_{res} will become a complex

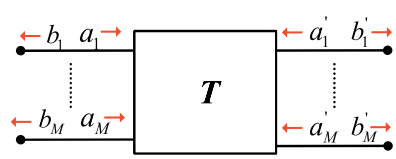

Figure 1. The $\boldsymbol{T}$-matrix acts as mapping factor for a unit cell.

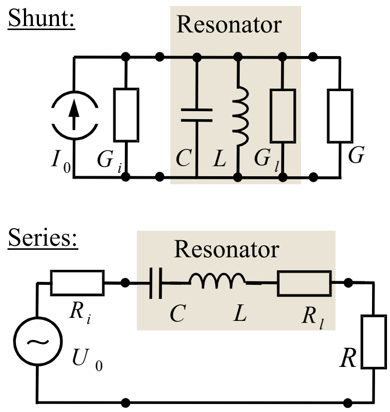

Figure 2. Equivalent circuit of shunt and series resonators with excitation and load.

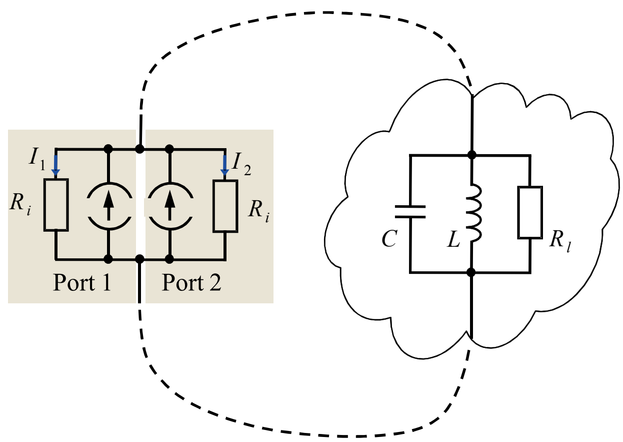

Figure 3. Excitation of general eigenproblem by ports according to circuit equivalence.

quantity, which in turn is real for vanishing G_l or R_l, respectively. In case of circuits, it is common practice to apply an excitation and to monitor the reaction (voltage or current) to the excitation. In addition, the generator or load resistance can serve as regularisation tool for the singular resonance behaviour without affecting the resonant frequency. Following this strategy, the boundary value problems are excited for various (ω, γ) combinations as illustrated in Figure 3 for an excitation by ports. In a numerical implementation, a pair of ports as shown in Figure 3 excite the problem and the S-parameters are to observe. However, the coupling of the ports may introduce parasitic reactive coupling elements. The coupling elements influence the curve of the S-parameters. As one measure, the parasitic elements can be removed by knowing the coupling model. As another measure, another observable can be chosen e.g. the electric field because the field in the solution domain will still reliably indicate the resonance meaning the eigenvalue.

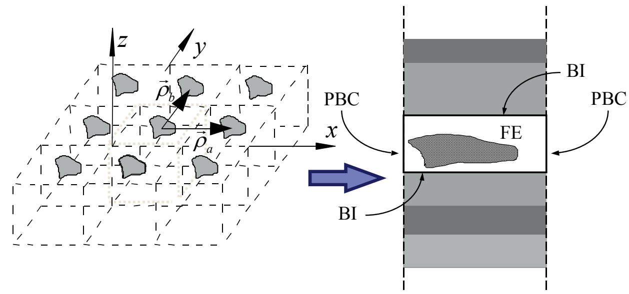

Figure 4. The combination of FE method and BI part in the hybrid FEBI technique.

3 Numerical Implementation

In this work, a periodic hybrid finite element boundary integral (FEBI) method according to [4] and [3] is employed and applied to doubly periodic array configurations. Due to the Bloch–Floquet theorem, the field solution of the structure is to factor in a periodic and a non-periodic part according to

$$\boldsymbol{E}(\boldsymbol{r} + m\boldsymbol{\rho}_a + n\boldsymbol{\rho}_b) = \boldsymbol{E}(\boldsymbol{r})\,\mathrm{e}^{-j\boldsymbol{k}_{t00}\cdot(m\boldsymbol{\rho}_a + n\boldsymbol{\rho}_b)}, \quad (5)$$

$$\boldsymbol{H}(\boldsymbol{r} + m\boldsymbol{\rho}_a + n\boldsymbol{\rho}_b) = \boldsymbol{H}(\boldsymbol{r})\,\mathrm{e}^{-j\boldsymbol{k}_{t00}\cdot(m\boldsymbol{\rho}_a + n\boldsymbol{\rho}_b)}. \quad (6)$$

$\boldsymbol{\rho}_a$ and $\boldsymbol{\rho}_b$ are the lattice vectors in x- and y-direction, respectively, as illustrated in Figure 4. $\boldsymbol{k}_{t00}$ corresponds to γ in Equation (2) and is considered as complex with respect to the PBCs. The subscript $t00$ specifies that the wavevector is in parallel to xy-plane and indicates the (00)th Floquet mode within the 2D Floquet mode series [3, 4]. According to the Floquet theorem, only one periodic unit cell is discretised and computed by the FE technique. The planar top and bottom surfaces of the unit cell mesh are incorporated by the BI part, where appropriate spectral domain 2D periodic Green's functions are implemented such that complex $\boldsymbol{k}_{t00}$ are possible. The Green's functions account for the influence of possible homogeneous material layers above and below the FE unit cell.

According to the above described solution procedure, the FE unit cell is to excite correspondingly which can be in form of impressed volume currents or impressed electrical field strengths. This is realisable by discrete ports in a full-wave driven simulation. The electric field strength as well as any other kind of integral measure can be observed at individual locations. To accelerate the search of appropriate pairs of $(\omega, \boldsymbol{k}_{t00})$, i.e. the eigenvalue, bisectional search algorithms in 1D together with start vector estimation from previous solutions within the iterative linear equation solver are utilised.

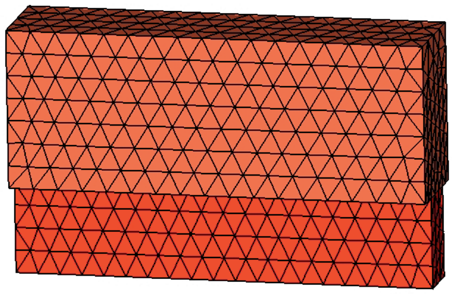

Figure 5. A unit cell of the metallic rectangular waveguide with dielectric-filled corrugations according to [7] meshed with tetrahedra.

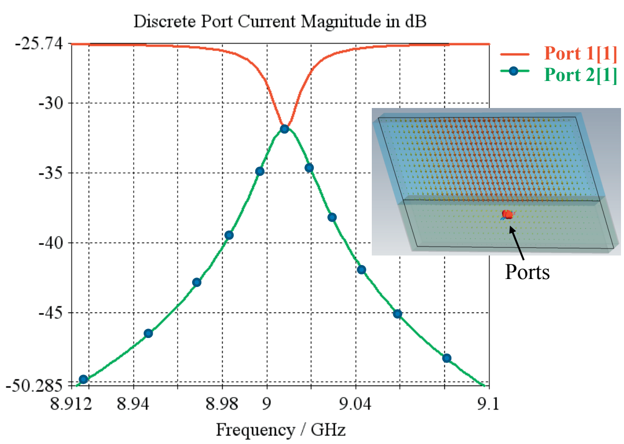

Figure 6. The currents for the discrete ports are positioned in the corrugation as shown in the inset.

4 Numerical Results

As first reference model to validate the proposed eigenvalue computation technique serves a dielectric-filled corrugated hollow waveguide, which has already been computed in [7, 19]. A unit cell of this periodical composition is discretised with tetrahedral mesh as visualised in Figure 5 to be analysed by the hybrid FEBI method according to [3, 4] as described in the previous section. Since the problem is closed only the FE portion is relevant for the solution. Currents can be impressed in some of the tetrahedra. Equivalently, it is possible to excite the unit cell model by discrete ports in CST MWS [1]. The discrete ports can exemplarily placed in the corrugation. Due to the high-impedance environment of the ports, the port resistors should exhibit a high value e.g. 5000 Ω not to disturb the authentic field distribution. At resonance, identical port currents without losses are registered, as shown in Figure 6, which also demonstrates the total power loss to be equally distributed over the two port resistors. The course of the discrete port currents additionally discloses the resonance behaviour. By changing the position of the discrete ports the resonance behaviour might be slightly shifted because of parasitic effects. However, by

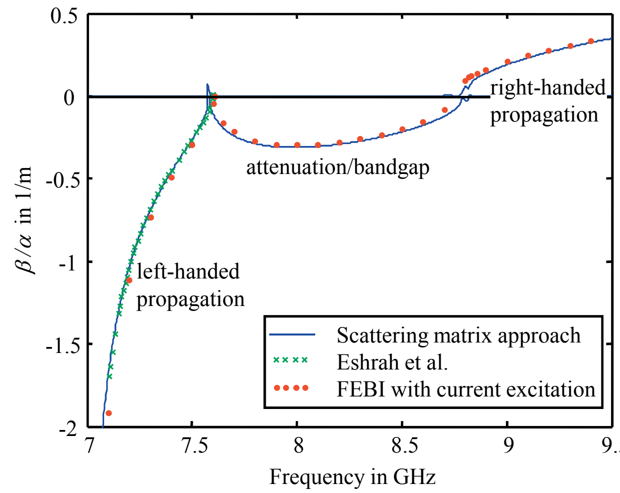

Figure 7. Dispersion diagram of hollow waveguide with dielectric-filled corrugations.

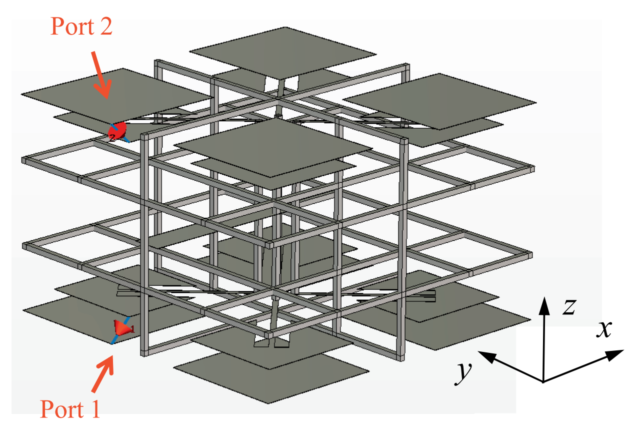

Figure 8. The metallic structure of the 3D unit cell according to the structure presented in [21] but modified in order to suppress the distorted plane wave mode.

choosing another observable, e.g. the electric field strength, the true resonance behaviour can be maintained. Whenever the electric field strength exhibits peaks the eigenvalues $(\omega, \boldsymbol{k}_{t00})$ are determined. Eventually, they can be plotted in the dispersion diagram as displayed in Figure 7. The results obtained by employing the FEBI method is compared to [19] and [7], where the latter technique only reveals a solution for the frequency region where left-handed wave propagation takes place. The FEBI method moreover provides the attenuation constant and a right-handed phase constant as well as the scattering matrix approach [13] does. The results of both methods agree well with each other.

The next problem is a 3D periodic unit cell as given in Figure 8. Two dielectric substrates (suppressed in the Figure) are contained in the metallic structure which is a metamaterial unit cell. It reveals left-handed dispersion behaviour at some frequencies as described in more detail in [21]. Because of the complexity of the model and the large

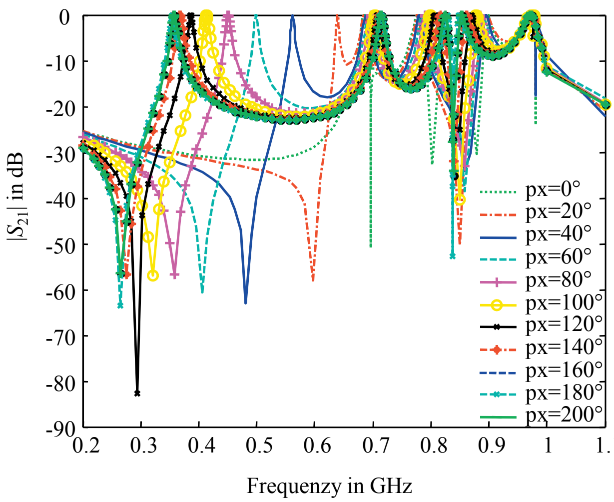

Figure 9. Maximum indicates eigensolution.

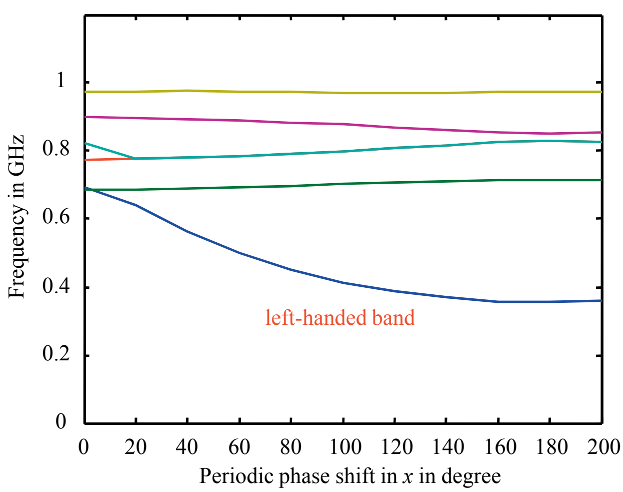

Figure 10. Dispersion diagram of 3D metamaterial unit cell shown in Figure 8.

number of unknowns resulting from the discretisation, the model would be a challenge for many eigenproblem solvers. Moreover, a series expansion of the fields in mode functions as required for the SMA [13, 19] becomes difficult because hardly any homogeneous waveguide regions can be found and rather an endless number of waveguide modes would be necessary for an accurate description of the field problem. Nevertheless, the proposed eigenproblem formulation provides reliable results. Observing the transmission parameter, which is displayed in Figure 9 for a varying periodic phase difference, an eigensolution is found whenever a maximum is reached. The results for a swept periodic phase difference are presented in the dispersion diagram in Figure 10 for the first modes propagating in the transverse x-direction.

5 Conclusions

Eigenproblems of periodically constituted configurations have been successfully solved by corresponding excitation problems subject to periodic boundary conditions. Moreover, the resulting resonance problems deliver a physical understanding of the dispersion behaviour of the periodic structures. The excitation can be placed selectively what allows a more efficient solution procedure. Even complex propagation constants can be solved for. Commercial field solvers can be used for a convenient solution of the excitation problems, which opens up the possibility for finding the attenuation constants of propagating modes by perturbation methods.

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