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A note on the Galambos copula and its associated Bernstein function

Abstract: There is an infinite exchangeable sequence of random variables $\{X_k\}_{k \in \mathbb{N}}$ such that each finite-dimensional distribution follows a min-stable multivariate exponential law with Galambos survival copula, named after [7]. A recent result of [15] implies the existence of a unique Bernstein function Ψ associated with $\{X_k\}_{k \in \mathbb{N}}$ via the relation $\Psi(d) =$ exponential rate of the minimum of d members of $\{X_k\}_{k \in \mathbb{N}}$. The present note provides the Lévy–Khinchin representation for this Bernstein function and explores some of its properties.

Keywords: Galambos copula, Bernstein function, min-stable multivariate exponential distribution, strong IDT process, infinite divisibility

MSC: 62H20, 62H05

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1 Introduction

A d -dimensional random vector (X_1, \dots, X_d) follows a min-stable multivariate exponential law (MSMVE) if $\min\{c_1 X_{i_1}, \dots, c_k X_{i_k}\}$ has a (univariate) exponential law for all $1 \leq i_1 < \dots < i_k \leq d$ and constants $c_1, \dots, c_k > 0$. This is the case if and only if its components have (univariate) exponential laws and its survival copula is of extreme-value kind, see [13, Theorem 6.2, p. 174]. A sequence of random variables $\{X_k\}_{k \in \mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be MSMVE if $\min\{c_1 X_{i_1}, \dots, c_k X_{i_k}\}$ has a (univariate) exponential distribution for arbitrary $c_1, \dots, c_k > 0$ and $i_1 < i_2 < \dots < i_k, k \in \mathbb{N}$. Provided such a sequence is in addition *exchangeable*, i.e. the law of $\{X_{\sigma(k)}\}_{k \in \mathbb{N}}$ is invariant under bijections $\sigma : \mathbb{N} \rightarrow \mathbb{N}$, the article [15] provides a canonical stochastic construction by means of non-decreasing stochastic processes which are *strong infinitely divisible with respect to time (strong IDT)*, a notion that was introduced and investigated in [6, 9, 16]. A stochastic process $\{H_t\}_{t \geq 0}$ is strong IDT if for each $n \in \mathbb{N}$ one has

$$\{H_t\}_{t \geq 0} \stackrel{d}{=} \left\{ \sum_{i=1}^n H_{t/n}^{(i)} \right\}_{t \geq 0},$$

where $\stackrel{d}{=}$ denotes equality in law and $\{H_t^{(i)}\}_{t \geq 0}$ are independent copies of $\{H_t\}_{t \geq 0}, i \in \mathbb{N}$. Indeed, there is a one-to-one relationship between exchangeable MSMVE sequences and right-continuous, non-decreasing strong IDT processes starting from $H_0 = 0$ and satisfying $\lim_{t \rightarrow \infty} H_t = \infty$, which is induced by the stochastic model

$$X_k := \inf\{t > 0 : H_t > \epsilon_k\}, \quad k \in \mathbb{N}, \quad (1)$$

where $\{\epsilon_k\}_{k \in \mathbb{N}}$ is an iid sequence of unit mean exponential random variables, independent of $\{H_t\}_{t \geq 0}$. One well-studied subfamily of non-decreasing strong IDT processes is the family of (killed) Lévy subordinators. However, there exist also proper increasing strong IDT processes that are not Lévy subordinators, and their rigorous study is an interesting topic for further research. In particular, the present article studies an exchangeable MSMVE sequence which is associated with a non-Lévy strong IDT process, whose probability law, however, is unknown.

The present note contributes the following two points to the study of the law of this unknown strong IDT process:

- Section 2 explains that for each $\theta > 0$ there is an exchangeable MSMVE sequence such that the exponential rate of the minimum $\min\{X_1, \dots, X_d\}$ is given by

$$\lambda_\theta(d) := - \sum_{k=1}^d \binom{d}{k} (-1)^k k^{-\theta}, \quad d \in \mathbb{N}. \tag{2}$$

These rates stem from an MSMVE distribution first introduced in [7].

- Given the first bullet point, it follows from [15] that there exists a unique Bernstein function Ψ_θ such that $\Psi_\theta(d) = \lambda_\theta(d)$ for all $d \in \mathbb{N}$. A Bernstein function is defined on $[0, \infty)$, non-negative, starting at zero, and is smooth on $(0, \infty)$ with completely monotone first derivative. For background on Bernstein functions we recommend [19]. Section 3 provides the Lévy–Khinchin representation of Ψ_θ and studies its properties.

2 The stochastic construction of the Galambos MSMVE sequence

For the sake of simplified notation, we introduce the following definition.

Definition 2.1 (Cumulative hazard process). *A $[0, \infty]$ -valued stochastic process $H = \{H_t\}_{t \geq 0}$ is called cumulative hazard process if it satisfies $H_0 = 0$, is right-continuous, non-decreasing, and $\lim_{t \rightarrow \infty} H_t = \infty$ almost surely.*

There is a one-to-one relationship between cumulative hazard processes and infinite exchangeable sequences of random variables with support in $[0, \infty)$, induced by the canonical construction (1), which is an immediate consequence of De Finetti’s Theorem. The original references are [4, 5], popular generalizations to more general state spaces are achieved in, e.g., [12, 17].

Let M_1, M_2, \dots be an iid sequence of positive random variables with Laplace transform φ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where we assume that

$$\theta := - \lim_{s \downarrow 0} s (\varphi^{-1})'(s) / \varphi^{-1}(s) \in (0, \infty)$$

exists. For instance, if $\varphi(x) = (1 + x)^{-1/\vartheta}$ for $\vartheta \in (0, \infty)$ (Laplace transform of a certain Gamma distribution), then $\theta = \vartheta$ exists. We apply the conventions $-\log(0) := \infty, \varphi^{-1}(0) := \infty$. For each $n \in \mathbb{N}$ we define the cumulative hazard process

$$H_t^{(n)} := \begin{cases} - \sum_{i=1}^n \log(1 - e^{-M_i \varphi^{-1}(t/n)}), & 0 \leq t < n \\ \infty, & t \geq n \end{cases}. \tag{3}$$

Applying construction (1) with the processes $H = H^{(n)}$ for each n , we obtain sequences of random variables $\{X_k^{(n)}\}_{k \in \mathbb{N}}$.

- For each $k, n \in \mathbb{N}$ the survival function of $X_k^{(n)}$ is given by

$$\mathbb{P}(X_k^{(n)} > t) = \mathbb{E} \left[e^{-H_t^{(n)}} \right] = \left(1 - \frac{t}{n} \right)^n, \quad 0 \leq t \leq n. \tag{4}$$

In particular, as $n \rightarrow \infty$, the law of $X_k^{(n)}$ converges weakly to a unit exponential law.

- If C_φ denotes the Archimedean copula (see, e.g., [3] for background on the latter) generated by φ , i.e.

$$C_\varphi(u_1, \dots, u_d) = \varphi(\varphi^{-1}(u_1) + \dots + \varphi^{-1}(u_d)), \quad u_1, \dots, u_d \in [0, 1], \quad d \geq 1,$$

we observe for $0 \leq t_1, \dots, t_d \leq n$ that

$$\begin{aligned} \mathbb{P}(X_1^{(n)} > t_1, \dots, X_d^{(n)} > t_d) &= \mathbb{E}\left[e^{-(H_{t_1}^{(n)} + \dots + H_{t_d}^{(n)})}\right] = \mathbb{E}\left[\prod_{k=1}^d \left(1 - e^{-M_1 \varphi^{-1}(t_k/n)}\right)\right]^n \\ &= \mathbb{E}\left[1 + \sum_{k=1}^d (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq d} \left(e^{-M_1 \varphi^{-1}(t_{i_1}/n)} + \dots + e^{-M_1 \varphi^{-1}(t_{i_k}/n)}\right)\right]^n \\ &= \left(1 + \sum_{k=1}^d (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq d} C_\varphi\left(\frac{t_{i_1}}{n}, \dots, \frac{t_{i_k}}{n}\right)\right)^n \\ &= \left(1 + \frac{(-n) \sum_{k=1}^d (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq d} C_\varphi\left(\frac{t_{i_1}}{n}, \dots, \frac{t_{i_k}}{n}\right)}{n}\right)^n. \end{aligned}$$

By [3, Theorem 3.1], the sequence

$$a_n := n \sum_{k=1}^d (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq d} C_\varphi\left(\frac{t_{i_1}}{n}, \dots, \frac{t_{i_k}}{n}\right), \quad n \in \mathbb{N},$$

converges to

$$\ell_\theta(t_1, \dots, t_d) := t_1 + \dots + t_d - \sum_{I:|I|\geq 2} (-1)^{|I|} \left(\sum_{k \in I} t_k^\theta\right)^{-1/\theta}. \tag{5}$$

The function ℓ_θ in (5) is a so-called *stable tail dependence function* associated with an MSMVE in the sense that $\exp(-\ell_\theta(t_1, \dots, t_d))$ is the survival function of an MSMVE distribution. Alternatively, $\exp(-\ell_\theta(-\log(u_1), \dots, -\log(u_d)))$ is an extreme-value copula. More precisely, it corresponds to the so-called Galambos copula, named after [7]. Since we have for arbitrary sequences $\{a_n\}_{n \in \mathbb{N}}$ that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a \Leftrightarrow \lim_{n \rightarrow \infty} a_n = a,$$

we conclude

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_1^{(n)} > t_1, \dots, X_d^{(n)} > t_d) = e^{-\ell_\theta(t_1, \dots, t_d)},$$

i.e. the distribution of $(X_1^{(n)}, \dots, X_d^{(n)})$ converges to the Galambos MSMVE as $n \rightarrow \infty$, for arbitrary $d \in \mathbb{N}$. Concluding, we have a sequence of infinite sequences of random variables, such that the finite-dimensional distributions of them converge weakly to Galambos MSMVEs.

- As a special case of the previous bullet point, we also get for $m_1, \dots, m_d \in \mathbb{N}$ and $t_1, \dots, t_d \geq 0$ arbitrary that

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[e^{-(m_1 H_{t_1}^{(n)} + \dots + m_d H_{t_d}^{(n)})}\right] = \exp\left(-\ell_\theta(\underbrace{t_1, \dots, t_1}_{m_1 \text{ times}}, \underbrace{t_2, \dots, t_2}_{m_2 \text{ times}}, \dots, \underbrace{t_d, \dots, t_d}_{m_d \text{ times}})\right).$$

Since the (multivariate) Laplace transforms of finite- (d -)dimensional distributions from $\{H_t^{(n)}\}$ are completely determined by their values on \mathbb{N}^d by the Theorem of Stone–Weierstrass, we can conclude that the following statement holds true.

Lemma 2.2 (Existence of the Galambos strong IDT process). *There is a strong IDT cumulative hazard process H such that*

$$\mathbb{E}\left[e^{-(H_{t_1} + \dots + H_{t_d})}\right] = e^{-\ell_\theta(t_1, \dots, t_d)}, \quad t_1, \dots, t_d \geq 0.$$

In particular, the exchangeable sequence $\{X_k\}_{k \in \mathbb{N}}$ defined via (1) from the process H has Galambos MSMVEs as finite-dimensional distributions.

Proof. The computations above imply that the cumulative hazard processes $H^{(n)}$ converge weakly to some cumulative hazard process H , whose finite-dimensional distributions are given as claimed in the statement. Now $H^{(n)}$ is of the structural form

$$H_t^{(n)} = \sum_{i=1}^n A_{t/n}^{(i)}, \quad A_t^{(i)} := -\log \left(1 - e^{-M_i \varphi^{-1}(t)} \right), \quad t \geq 0,$$

where $A^{(1)}, A^{(2)}, \dots$ are iid cumulative hazard processes. From this observation it is not difficult to observe that the limiting cumulative hazard process H must be strong IDT. □

Lemma 2.2 rigorously verifies a statement made in [15, Example 5.1] about the existence of H . Unfortunately, the stochastic nature of H is unknown. It is an interesting open problem to describe it conveniently and, in particular, find a simulation algorithm for paths of H , because this would imply an efficient sampling algorithm for the Galambos MSMVE (and associated Galambos copula) in arbitrary dimension along the stochastic construction (1). Notice in particular that H cannot be a jump process, because its jumps would induce positive probabilities for events such as $\{X_1 = X_2\}$, which is not the case for the Galambos MSMVE. As a first step into studying the stochastic behavior of H the next section studies the infinitely divisible law of H_t for fixed $t > 0$.

Figure 1 shows a bivariate scatter plot which should approximately resemble a scatter plot from the Galambos copula $(u_1, u_2) \rightarrow \exp(-\ell_\theta(-\log(u_1), -\log(u_2)))$ for $\theta = 1$. It is generated by simulating iid samples of the random vector $(X_1^{(n)}, X_2^{(n)})$ from the canonical construction (1) with the approximating cumulative hazard process (3), for $n = 100$, and then transforming to uniform marginals by applying the survival function (4) to both components, i.e. visualized are samples of

$$(U_1^{(n)}, U_2^{(n)}) := \left(\max \{ (1 - X_1^{(n)}/n)^n, 0 \}, \max \{ (1 - X_2^{(n)}/n)^n, 0 \} \right).$$

The involved Laplace transform has been specified as $\varphi(x) = (1 + x)^{-1/\theta}$ with $\theta = 1$, resulting in a unit exponential distribution for the involved random variables M_1, M_2, \dots . Furthermore, based on $N := 1000000$ iid samples of $(U_1^{(n)}, U_2^{(n)})$ for varying n , denoted $(U_1^{(n)}(k), U_2^{(n)}(k))$ for $k = 1, \dots, N$, we computed the empirical Spearman’s Rho

$$\hat{\rho}_N^{(n)} := \frac{\sum_{k=1}^N \left(U_1^{(n)}(k) - \frac{1}{N} \sum_{l=1}^N U_1^{(n)}(l) \right) \left(U_2^{(n)}(k) - \frac{1}{N} \sum_{l=1}^N U_2^{(n)}(l) \right)}{\sqrt{\sum_{k=1}^N \left(U_1^{(n)}(k) - \frac{1}{N} \sum_{l=1}^N U_1^{(n)}(l) \right)^2 \sum_{k=1}^N \left(U_2^{(n)}(k) - \frac{1}{N} \sum_{l=1}^N U_2^{(n)}(l) \right)^2}}$$

and compared it with the theoretical Spearman’s Rho of the limiting Galambos copula, which is given by

$$\rho = 12 \int_0^1 (2 - (x^{-\theta} + (1-x)^{-\theta})^{-1/\theta})^{-2} dx - 3 \stackrel{(\theta=1)}{\approx} 0.5874,$$

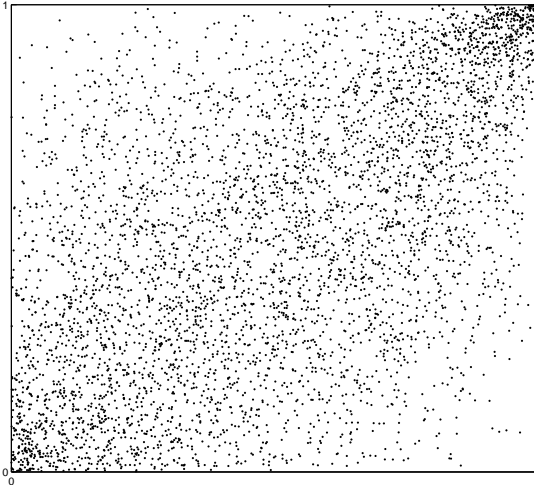
see, e.g., [8]. It is observed that the empirical value for Spearman’s Rho approximates the theoretical value with increasing n , as implied by the theory.

3 The Galambos Bernstein function

In the previous paragraph we have constructed an exchangeable MSMVE sequence $\{X_k\}_{k \in \mathbb{N}}$ and a non-decreasing, strong IDT process H such that

$$\mathbb{P}(X_1 > x_1, \dots, X_d > x_d) = \mathbb{E} \left[e^{-H_{x_1} - \dots - H_{x_d}} \right] = e^{-\ell_\theta(x_1, \dots, x_d)}, \quad x_1, \dots, x_d > 0.$$

It is known from [15] that there exists a Bernstein function Ψ_θ such that the law of H_t is infinitely divisible with Laplace transform $\exp(-t \Psi_\theta)$, each $t \geq 0$. However, according to [15] this Bernstein function is not known in



n	$\hat{\rho}_N^{(n)}$
5	0.5642
10	0.5757
25	0.5827
50	0.5863
75	0.5850
100	0.5873

Fig. 1. Left: scatter plot from the survival copula of $(X_1^{(n)}, X_2^{(n)})$ constructed with the cumulative hazard process (3), for $n = 100$ and $\varphi(x) = (1 + x)^{-1}$ (5000 samples are visualized). Right: simulations of the empirical Spearman’s Rho $\hat{\rho}_N^{(n)}$ based on $N = 1000000$ iid samples and varying n .

closed form except for the case $\theta = 1$. The only thing that is known is that the exponential rate of the minimum $\min\{X_1, \dots, X_d\}$ is given by

$$\Psi_\theta(d) = \ell_\theta(\underbrace{1, \dots, 1}_{d \text{ times}}) = - \sum_{k=1}^d \binom{d}{k} (-1)^k k^{-\theta}, \quad d \in \mathbb{N}. \tag{6}$$

The following proposition derives the Lévy–Khinchin representation of the Bernstein function Ψ_θ for arbitrary $\theta > 0$. To this end, recall that any Bernstein function Ψ has a Lévy–Khinchin representation

$$\Psi(x) = b x + \int_0^\infty (1 - e^{-xt}) \nu(dt) + \nu(\{\infty\}) 1_{\{x>0\}}, \quad x \geq 0,$$

with a constant $b \geq 0$ and a measure ν on $(0, \infty]$ satisfying $\int_{(0, \infty]} \min\{t, 1\} \nu(dt) < \infty$, called the *Lévy measure* associated with Ψ .

Proposition 3.1 (The Galambos Bernstein function). *The Bernstein function Ψ_θ associated with the Galambos MSMVE is given by the Lévy–Khinchin representation*

$$\Psi_\theta(x) = \int_0^\infty (1 - e^{-tx}) \frac{e^{-t}}{1 - e^{-t}} \left(\log \left(\frac{1}{1 - e^{-t}} \right) \right)^{\theta-1} \frac{dt}{\Gamma(\theta)}, \quad x \geq 0.$$

Proof. Fix θ and consider the sequence $a_k := \Psi_\theta(k + 1) - \Psi_\theta(k)$, $k \in \mathbb{N}_0$. From (6) we conclude that

$$a_k = \sum_{i=0}^k \binom{k}{i} (-1)^i (i + 1)^{-\theta}, \quad k \in \mathbb{N}_0.$$

Let τ be a Γ -distributed random variable with density $f_\tau(x) = x^{\theta-1} e^{-x} / \Gamma(\theta)$, $x > 0$. From the knowledge about the Laplace transform of the Γ -distribution it is observed that $\{(k + 1)^{-\theta}\}_{k \in \mathbb{N}_0}$ is the moment sequence of $\exp(-\tau)$, i.e. $(k + 1)^{-\theta} = \int_0^\infty \exp(-kx) f_\tau(x) dx$. Consequently,

$$a_k = \sum_{i=0}^k \binom{k}{i} (-1)^i (i + 1)^{-\theta} = \int_0^\infty f_\tau(x) \sum_{i=0}^k \binom{k}{i} (-1)^i \exp(-ix) dx = \int_0^\infty f_\tau(x) (1 - e^{-x})^k dx,$$

implying that $\{a_k\}_{k \in \mathbb{N}_0}$ equals the moment sequence of the random variable $X := 1 - \exp(-\tau)$. The density f_X of X is given by

$$f_X(x) = f_\tau(-\log(1-x))/(1-x) = \left(\log\left(\frac{1}{1-x}\right)\right)^{\theta-1} / \Gamma(\theta), \quad x \in (0, 1).$$

An application of [14, Lemma 5.3] implies that the Lévy measure of Ψ_θ is given by

$$\nu(dt) = \frac{e^{-t}}{1-e^{-t}} f_X(e^{-t}) dt = \frac{e^{-t}}{1-e^{-t}} \left(\log\left(\frac{1}{1-e^{-t}}\right)\right)^{\theta-1} / \Gamma(\theta) dt,$$

which implies the claim. \square

Interestingly, (the infinitely divisible law associated with) the Bernstein function of Proposition 3.1 seems to be completely unstudied in the academic literature. The following corollary collects some properties of the Bernstein function Ψ_θ and, hence, about the stochastic nature of H .

Corollary 3.2 (Properties of Ψ_θ). *The following properties are satisfied by Ψ_θ .*

- (a) *The Lévy measure is not finite.*
- (b) $\theta \in \mathbb{N} \Rightarrow \Psi_\theta$ *is complete, i.e. the associated infinitely divisible law is from the so-called Bondesson class.*
- (c) *The associated infinitely divisible law is self-decomposable if and only if $\theta \geq 1$.*
- (d) *For $x \geq 0$ arbitrary, $\lim_{\theta \downarrow 0} \Psi_\theta(x) = 1_{\{x>0\}}$ and $\lim_{\theta \rightarrow \infty} \Psi_\theta(x) = x$.*

Notice that statement (d) is actually not obvious from the Lévy–Khinchin representation alone. Furthermore, the Bondesson class is a large subclass of infinitely divisible laws on the half-line which was introduced in [2] under the name g.c.m.e.d. laws. These are characterized by those Bernstein functions whose Lévy measure has a completely monotone density with respect to Lebesgue measure, called complete Bernstein functions, cf. [19, Chapters 6 and 7]. Moreover, an infinitely divisible law on the half-line $(0, \infty)$ is called self-decomposable if its Laplace transform φ is such that $x \mapsto \varphi(x)/\varphi(cx)$ is completely monotone for all $c \in (0, 1)$, cf. [19, Definition 5.12, p. 41].

Proof. Observe that $(-\log(1 - \exp(-t)))' = -\exp(-t)/(1 - \exp(-t))$, which implies by substitution for arbitrary $\theta > 0$ that

$$\nu((0, \infty)) = \int_0^\infty \frac{e^{-t}}{1-e^{-t}} \left(\log\left(\frac{1}{1-e^{-t}}\right)\right)^{\theta-1} \frac{dt}{\Gamma(\theta)} = \int_0^\infty u^{\theta-1} \frac{du}{\Gamma(\theta)} = \infty.$$

- (b) Clearly, $t \mapsto \exp(-t)$ is completely monotone and $t \mapsto 1 - \exp(-t)$ is a Bernstein function (associated Lévy measure is Dirac measure at 1). Hence, the function $t \mapsto \exp(-t)/(1 - \exp(-t))$ is completely monotone, since (i) the reciprocal of a Bernstein function is completely monotone and (ii) the product of two completely monotone functions is again completely monotone. To see this, (ii) follows from [19, Corollary 1.6], and (i) is verified from the fact that $g(x) = 1/x$ is completely monotone (it is the Laplace transform of Lebesgue measure on $(0, \infty)$, see [19, Thm. 1.4]) and hence $g \circ f$ is c.m. for an arbitrary Bernstein function f by [19, Thm. 3.6 (ii)]. So for $\theta = 1$ we already observe that Ψ_1 is complete. For $\theta \in \{2, 3, 4, \dots\}$, we show that the function $t \mapsto (-\log(1 - \exp(-t)))^{\theta-1}$ is completely monotone as well, which then implies the claim, as products of completely monotone functions are completely monotone again. If φ is completely monotone and $\beta \in \mathbb{N}$, then φ^β is also completely monotone since the set of completely monotone functions is closed under multiplication, which was used before. Moreover, the function $t \mapsto -\log(1 - \exp(-t))$ is c.m., since its first derivative equals $t \mapsto -\exp(-t)/(1 - \exp(-t))$, and the latter function is the negative of a completely monotone function (as was just explained). This ultimately proves that $t \mapsto (-\log(1 - \exp(-t)))^{\theta-1}$ is completely monotone for all $\theta \in \{2, 3, 4, \dots\}$.
- (c) Recall that an infinitely divisible law is self-decomposable if and only if it has a Lévy measure of the form $k(t) dt$ with $t \mapsto tk(t)$ non-increasing, see [18, Chapter 3, Sections 16–17]. In the present situation we have

$k(t) = k_\theta(t) = \exp(-t)/(1 - \exp(-t))(-\log(1 - \exp(-t)))^{\theta-1}/\Gamma(\theta)$. Furthermore, we compute

$$\frac{d}{dt}(t k_\theta(t)) = \frac{(-\log(1 - \exp(-t)))^{\theta-2}}{(e^t - 1)^2 \Gamma(\theta)} \left(-\theta t + t + (e^t(t - 1) + 1) \log(1 - e^{-t}) \right),$$

which is strictly smaller than zero if and only if

$$\theta > 1 + \frac{e^t(t - 1) + 1}{t} \log(1 - e^{-t}) =: h(t).$$

Now $h(t)$ is a continuous function on $(0, \infty)$ with $\lim_{t \downarrow 0} h(t) = 1$, which can be observed by an application of L'Hospital's rule, and $h(t) < 1$, which follows from the fact that $e^t(t - 1) + 1 > 0$ and $\log(1 - e^{-t}) < 0$. Consequently, for $\theta \geq 1$ the function $t \mapsto t k_\theta(t)$ has strictly negative derivative for all $t > 0$ and the law is self-decomposable, as claimed. For $\theta < 1$, however, we find a non-empty interval on which $h(t) > \theta$. Consequently, the function $t \mapsto t k_\theta(t)$ is increasing on that interval and the law not self-decomposable.

(d) Defining the Bernstein functions $\Psi_0(x) := 1_{\{x>0\}}$ and $\Psi_\infty(x) := x, x \geq 0$, it follows from (6) that

$$\lim_{\theta \downarrow 0} \Psi_\theta(d) = \Psi_0(d), \quad \lim_{\theta \rightarrow \infty} \Psi_\theta(d) = \Psi_\infty(d), \quad d \in \mathbb{N}_0.$$

Now the claim follows from Lemma 4.1 in the Appendix. □

4 Conclusion

It was shown that there exists a strong IDT process H based on which random vectors with Galambos survival copula can be constructed via (1). The present note has further collected some stochastic properties of H and its associated Bernstein function.

Acknowledgement: Helpful comments by three anonymous referees on earlier versions of this manuscript are gratefully appreciated.

Appendix

Lemma 4.1 (Technical Lemma). *Let $\Psi_0, \Psi_1, \Psi_2, \dots$ be Bernstein functions, such that $\lim_{n \rightarrow \infty} \Psi_n(d) = \Psi_0(d)$ for all $d \in \mathbb{N}_0$. Then $\lim_{n \rightarrow \infty} \Psi_n(x) = \Psi_0(x)$ for all $x \geq 0$.*

Proof. For each $n \in \mathbb{N}_0$ there exists a unique probability law π_n on $[0, \infty]$ such that $\exp(-\Psi_n)$ equals the Laplace transform of π_n . Now fix $n \in \mathbb{N}_0$. By Hausdorff's moment problem, see [10, 11], we find a unique probability law μ_n on the unit interval $[0, 1]$ such that $\exp(-\Psi_n(d)) = \int_{[0,1]} y^d \mu_n(dy)$ for all $d \in \mathbb{N}_0$, since the sequence $\{\exp(-\Psi_n(d))\}_{d \in \mathbb{N}_0}$ is completely monotone starting at one. The function $f_n(x) := \int_{[0,1]} y^x \mu_n(dy)$ is completely monotone by Bernstein's Theorem, see [1] (if $X \sim \mu_n$, it equals the Laplace transform of the random variable $-\log(X)$ taking values in $[0, \infty]$ with the convention $-\log(0) := \infty$). Laplace transforms are completely determined by their values on \mathbb{N}_0 due to the uniqueness in Hausdorff's result, hence $f_n(x) = \exp(-\Psi_n(x))$ for all $x \geq 0$. By assumption, we observe for arbitrary $d \in \mathbb{N}_0$ that

$$\int_{[0,1]} y^d \mu_n(dy) = \exp(-\Psi_n(d)) \rightarrow \exp(-\Psi_0(d)) = \int_{[0,1]} y^d \mu_0(dy), \quad n \rightarrow \infty.$$

Since the polynomials are dense in the set of continuous (and bounded) functions on the compact interval $[0, 1]$, it follows that μ_n tends weakly to μ_0 . This in turn implies the claim, since for arbitrary $x \geq 0$ we observe

$$\Psi_n(x) = -\log \left(\int_{[0,1]} y^x \mu_n(dy) \right) \rightarrow -\log \left(\int_{[0,1]} y^x \mu_0(dy) \right) = \Psi_0(x), \quad n \rightarrow \infty.$$

□

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Received December 4, 2013; accepted February 19, 2014.