A Universal Constant for Exponential Riesz Sequences

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Abstract. The aim of this paper is to study certain correlations between lower and upper bounds of exponential Riesz sequences, in particular between sharp lower and upper bounds, where we show that the product of the sharp bounds of an exponential Riesz sequence is bounded from above by a universal constant. The result is applied to the norms of coefficient and frame operators and their inverses.

Keywords: Sharp bounds, exponential Riesz sequences, exponential Riesz bases, exponential frames

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1. Introduction

Let $H$ be a separable Hilbert space over $\mathbb{C}$. A sequence $\Phi = (\varphi_n)_{n \in \mathbb{Z}}$ of elements in $H$ is called a Riesz-Fischer sequence or a Bessel sequence for $H$, if there is a constant $A > 0$ or $B > 0$ such that for all numbers $n \in \mathbb{N}$ and $c_{-n}, \ldots, c_n \in \mathbb{C}$

$$A \sum_{j=-n}^{n} |c_j|^2 \leq \left\| \sum_{j=-n}^{n} c_j \varphi_j \right\|_H^2$$

(1)

or

$$\left\| \sum_{j=-n}^{n} c_j \varphi_j \right\|_H^2 \leq B \sum_{j=-n}^{n} |c_j|^2,$$

(2)

respectively. If $\Phi$ is both a Bessel and a Riesz-Fisher sequence, it is called a Riesz sequence for $H$. The constants $A$ and $B$ are called lower and upper bounds, respectively. It is an easy matter to check that the supremum of all lower bounds and the infimum of all upper bounds of a Riesz sequence is again a lower bound and an upper bound, which we denote by $A_{\text{opt}}(\Phi)$ and $B_{\text{opt}}(\Phi)$, respectively. The constants $A_{\text{opt}}(\Phi)$ and $B_{\text{opt}}(\Phi)$ are called the sharp lower and the sharp upper bounds for $\Phi$, respectively.

In this paper, we shall be concerned with exponential Riesz sequences for $L^2(-\sigma, \sigma)$, i.e. with Riesz sequences for $L^2(-\sigma, \sigma)$ of the form $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$, where $\sigma > 0$ and $(\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of complex numbers. From (1) and (2) it follows readily that if $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$
is a Riesz sequence for $L^2(-\sigma, \sigma)$, then the sequence $(\Im \lambda_n)_{n \in \mathbb{Z}}$ of imaginary parts must be uniformly bounded.

Young (see [9: Proposition 1/Corollary 1] and [10: Chapter 4/Propositions 2 and 3, Theorem 3 and Chapter 2/Remark following Theorem 17]) has shown that for a sequence $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ of exponentials to be a Riesz sequence for $L^2(-\sigma, \sigma)$ it is sufficient that it be a Riesz-Fischer sequence for $L^2(-\sigma, \sigma)$, provided $(\Im \lambda_n)_{n \in \mathbb{Z}}$ is uniformly bounded. We shall have a closer look at this result and the occurring bounds. In particular, if $\sigma, \tau > 0$ and $(\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of complex numbers such that

$$\sup_{n \in \mathbb{Z}} |\Im \lambda_n| \leq \tau$$

and $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ is a Riesz-Fischer sequence for $L^2(-\sigma, \sigma)$ with lower bound $A$, we shall construct an upper bound $B$ for $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$, depending only on $A$, $\sigma$ and $\tau$ (Proposition 1). From this we shall obtain a universal constant for the product of the sharp bounds of exponential Riesz sequences (Theorem 1).

The results of this paper are part of the author’s doctoral thesis [5: Chapter 4].

2. Results

We need the following

**Definition 1.** A sequence $(\lambda_n)_{n \in \mathbb{Z}}$ of complex numbers is called **separated** by $\delta > 0$, if $\inf_{n \neq m} |\lambda_n - \lambda_m| \geq \delta$. The sequence is called **separated**, if there is some $\delta > 0$ such that $(\lambda_n)_{n \in \mathbb{Z}}$ is separated by $\delta$.

**Proposition 1.** Let $\tau \geq 0$, $\sigma > 0$, $A > 0$ and $(\lambda_n)_{n \in \mathbb{Z}}$ be a sequence of complex numbers satisfying (3), such that $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ is a Riesz-Fischer sequence for $L^2(-\sigma, \sigma)$ with lower bound $A$. Then there holds:

1. $(\lambda_n)_{n \in \mathbb{Z}}$ is separated by $\delta = \delta(A, \sigma, \tau)$, where

$$\delta(A, \sigma, \tau) = \frac{1}{\sigma} \log \left( 1 + e^{-\sigma \tau} \sqrt{\frac{A}{\sigma}} \right).$$

2. $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ is a Riesz sequence for $L^2(-\sigma, \sigma)$ with upper bound $B = B(A, \sigma, \tau)$, where

$$B(A, \sigma, \tau) = \frac{2}{\sigma} \left(e^{2\sigma(\tau+1)} - 1\right) \left(1 + \frac{2}{\delta}\right)^2$$

$$= \frac{2}{\sigma} \left(e^{2\sigma(\tau+1)} - 1\right) \left(1 + \frac{2\sigma}{\log \left(1 + e^{-\sigma \tau} \sqrt{\frac{A}{\sigma}} \right)} \right)^2. \quad (4)$$

**Remark 1.** Proposition 1 states more explicitly a result of Young (see [9: Proposition 1, Corollary 1] and [10: Chapter 4/Propositions 3 + 2, Theorem 3 and Chapter 2/Remark following Theorem 17]) who proved that, under the assumptions of Proposition 1, the sequence $(\lambda_n)_{n \in \mathbb{Z}}$ must be separated and that $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ is a Riesz sequence.

From Proposition 1 we shall obtain
Theorem 1. For every $\sigma > 0$ and every $\tau \geq 0$, there exists a constant $C(\sigma, \tau) > 0$ such that the following holds:

If $(\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of complex numbers satisfying (3) and such that $\Phi = (e^{i\lambda_n \bullet})_{n \in \mathbb{Z}}$ is a Riesz sequence for $L^2(-\sigma, \sigma)$, then the product of the sharp bounds $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$ is bounded from above by $C(\sigma, \tau)$:

$$A_{opt}(\Phi) B_{opt}(\Phi) \leq C(\sigma, \tau).$$

The constant $C(\sigma, \tau)$ can be chosen as

$$C(\sigma, \tau) = 256 e^{4 \sigma \tau + 2 \sigma} (\sigma + \frac{1}{8})^2.$$

Remark 2. There is no universal constant $D(\sigma, \tau) > 0$ such that $A_{opt}(\Phi) B_{opt}(\Phi) \geq D(\sigma, \tau)$ for all exponential Riesz sequences $\Phi = (e^{i\lambda_n \bullet})_{n \in \mathbb{Z}}$ satisfying (3).

Counterexample. For $0 < \varepsilon < 1$ we define

$$\lambda_n^\varepsilon = \begin{cases} n & \text{for } n \in \mathbb{Z} \setminus \{0\} \\ 1 - \varepsilon & \text{for } n = 0 \end{cases} \quad \text{and} \quad \Phi^\varepsilon = (e^{i\lambda_n^\varepsilon \bullet})_{n \in \mathbb{Z}}.$$

From the orthonormality of $\left(\frac{1}{\sqrt{2\pi}} e^{in\bullet}\right)_{n \in \mathbb{Z}}$ in $L^2(-\pi, \pi)$ it follows that, for $0 < \varepsilon < 1$, $\Phi^\varepsilon$ is a Riesz sequence for $L^2(-\pi, \pi)$ and $B_{opt}(\Phi^\varepsilon) \leq 4\pi$. However, from $\|e^{i\lambda_n^\varepsilon \bullet} - e^{i\lambda_0^\varepsilon \bullet}\|_{L^2(-\pi, \pi)}^2 \to 0$ for $\varepsilon \to 0$ we conclude $A_{opt}(\Phi^\varepsilon) \to 0$ for $\varepsilon \to 0$. This shows $A_{opt}(\Phi^\varepsilon) B_{opt}(\Phi^\varepsilon) \to 0$ for $\varepsilon \to 0$.

Definition 2. A sequence $\Phi = (\varphi_n)_{n \in \mathbb{Z}}$ in a separable Hilbert space $H$ over $\mathbb{C}$ is called a frame for $H$ (cf. Duffin and Schaeffer [3: Section 3]), if there exist constants $A > 0$ and $B > 0$ such that for all $f \in H$

$$A \|f\|_H^2 \leq \sum_{n \in \mathbb{Z}} |(f, \varphi_n)_H|^2 \leq B \|f\|_H^2.$$

The constants $A$ and $B$ are called lower and upper frame bounds, respectively. The operators

$$T_\Phi : H \to l^2(\mathbb{Z}), \quad f \mapsto ((f, \varphi_n)_H)_{n \in \mathbb{Z}}$$

and

$$S_\Phi : H \to H, \quad f \mapsto \sum_{n \in \mathbb{Z}} (f, \varphi_n)_H \varphi_n$$

are called the coefficient operator and the frame operator, respectively, corresponding to the frame $\Phi$.

Remark 3. From Definition 2 it follows easily that the coefficient operator corresponding to a frame is an injective, bounded linear operator, with bounded inverse on its range. Furthermore, it can be shown that $S_\Phi$ is a well-defined, bijective, bounded linear map (cf. Duffin and Schaeffer [3: Section 3]; the sum converges in the norm of $H$).

Definition 3. A frame in a separable Hilbert space over $\mathbb{C}$ is called an exact frame (or a Riesz basis), if it is no longer a frame after any of its elements is removed.

Remark 4. Every exact frame is a Riesz sequence with the same bounds (cf. Duffin and Schaeffer [3: Lemma X] and Közlów [4: Section II.1/Corollary 1 to Theorem 8]).
Corollary 1. For the constant $C(\sigma, \tau)$ of Theorem 1 there holds:

If $(\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of complex numbers satisfying (3) and such that $\Phi = (e^{i\lambda_n \bullet})_{n \in \mathbb{Z}}$ is an exact frame for $L^2(-\sigma, \sigma)$, then we have the following inequalities for the norms of the coefficient and frame operator and their inverses:

\[
\|T_\Phi\| \leq \sqrt{C(\sigma, \tau)} \|T_\Phi^{-1}\|
\]
\[
\|S_\Phi\| \leq C(\sigma, \tau) \|S_\Phi^{-1}\|
\]
\[
\|T_\Phi\|^2 \leq C(\sigma, \tau) \|S_\Phi^{-1}\|.
\]

Remark 5. A statement analog to Corollary 1 does not hold for arbitrary (non-exact) exponential frames.

Counterexample. We define the sequence $\Phi_m = (e^{i\frac{x}{m} \bullet})_{n \in \mathbb{Z}}$ where $m \in \mathbb{N}$. It can be shown that $\Phi_m$ is a frame for $L^2(-\pi, \pi)$, but $\|T_{\Phi_m}\|^2 = \|T_{\Phi_m}^{-1}\|^{-2} = \|S_{\Phi_m}\| = \|S_{\Phi_m}^{-1}\|^{-1} = 2\pi m$.

Definition 4. Denote by $PW^2_{\sigma}$ the Paley-Wiener space, consisting of all entire functions of exponential type at most $\sigma$, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. The norm on $PW^2_{\sigma}$ is the usual $L^2(\mathbb{R})$-norm.

For a sequence $\Lambda = (\lambda_n)_{n \in \mathbb{Z}}$ of complex numbers, the operator $I_\Lambda$ is defined by

\[
I_\Lambda : PW^2_{\sigma} \rightarrow \mathbb{C}^\mathbb{Z}, \quad F \mapsto (F(\lambda_n))_{n \in \mathbb{Z}}.
\]

Remark 6. If, for $\Lambda = (\lambda_n)_{n \in \mathbb{Z}}, (e^{i\lambda_n \bullet})_{n \in \mathbb{Z}}$ is an exact frame for $L^2(-\sigma, \sigma)$, then $I_\Lambda$ defines a bijective, bounded linear operator from $PW^2_{\sigma}$ onto $l^2(\mathbb{Z})$ (i.e. $\Lambda$ is a complete interpolating sequence, cf. Young [10: Chapter 4/Theorem 9]).

From Corollary 1 we obtain

**Corollary 2.** For the constant $C(\sigma, \tau)$ of Theorem 1 there holds:

If $\Lambda = (\lambda_n)_{n \in \mathbb{Z}}$ is a sequence of complex numbers satisfying (3) and such that $(e^{i\lambda_n \bullet})_{n \in \mathbb{Z}}$ is an exact frame for $L^2(-\sigma, \sigma)$, and if $I_\Lambda : PW^2_{\sigma} \rightarrow l^2(\mathbb{Z})$ is the bijection considered above, then

\[
\|I_\Lambda\| \leq \frac{\sqrt{C(\sigma, \tau)}}{2\pi} \|I_\Lambda^{-1}\|.
\]

3. Proofs

We shall need the following

**Lemma 1** (see [6: Lemma 1]). Let $\delta, \sigma > 0$, $\tau \geq 0$, and $(\lambda_n)_{n \in \mathbb{Z}}$ be a sequence of complex numbers, separated by $\delta$ and satisfying (3). Then for all functions $F$ of the Paley-Wiener space $PW^2_{\sigma}$ the inequality

\[
\sum_{n \in \mathbb{Z}} |F(\lambda_n)|^2 \leq \frac{e^{2\sigma(\tau+1)} - 1}{\pi\sigma} \left(1 + \frac{2}{\delta}\right)^2 \int_{\mathbb{R}} |F(x)|^2 dx
\]

(7)

holds.

It should be noted that for separated real sequences the first inequality of type (7) (with a different constant) was given by Plancherel and Pólya [8: p. 126].
Proof of Proposition 1.

1. For the following proof that $(\lambda_n)_{n \in \mathbb{Z}}$ is separated we use similar arguments as Pavlov did in [7: Theorem 1]. Let $k, n \in \mathbb{Z}$, where $k \neq n$. Then we have from (1)

$$2A = A(|1|^2 + |-1|^2) \leq \int_{-\sigma}^{\sigma} |e^{i\lambda_k x} - e^{i\lambda_n x}|^2 dx = \int_{-\sigma}^{\sigma} |e^{i\lambda_k x}|^2 |1 - e^{i(\lambda_n - \lambda_k)x}|^2 dx \leq e^{2\sigma \tau} \int_{-\sigma}^{\sigma} |1 - e^{i(\lambda_n - \lambda_k)x}|^2 dx.$$ 

(8)

For $x \in (-\sigma, \sigma)$ we have

$$|1 - e^{i(\lambda_n - \lambda_k)x}| = \left| \sum_{m=1}^{\infty} \frac{(i(\lambda_n - \lambda_k)x)^m}{m!} \right| \leq \sum_{m=1}^{\infty} \frac{(|\lambda_n - \lambda_k||\sigma|)^m}{m!} = e^{(|\lambda_n - \lambda_k||\sigma| - 1)}.$$

Hence we conclude from (8)

$$2A \leq e^{2\sigma \tau} 2\sigma (e^{|\lambda_n - \lambda_k||\sigma| - 1})^2.$$

From this it follows easily that

$$|\lambda_n - \lambda_k| \geq \frac{1}{\sigma} \log \left( 1 + e^{-\sigma \tau} \sqrt{\frac{A}{\sigma}} \right) = \delta(A, \sigma, \tau),$$

i.e. $(\lambda_n)_{n \in \mathbb{Z}}$ is separated by $\delta = \delta(A, \sigma, \tau)$.

2. From Lemma 1 and assertion 1 we derive (7) with $\delta = \delta(A, \sigma, \tau)$. By the Paley-Wiener theorem, this is equivalent to

$$\sum_{n \in \mathbb{Z}} |(f, e^{i\lambda_n \cdot})|^2 \leq \frac{2}{\sigma} (e^{2\sigma(\tau+1)} - 1) \left( 1 + \frac{2}{\delta} \right)^2 \|f\|^2$$

for all $f \in L^2(-\sigma, \sigma)$. By a theorem of Boas [2: Theorem 1] (cf. Young [10: Chapter 4/Theorem 3]) the latter inequality is equivalent to $(e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ being a Bessel sequence with upper bound $B = B(A, \sigma, \tau)$, as defined by (4).

Proof of Theorem 1. Let $\sigma > 0, \tau \geq 0$, and $(\lambda_n)_{n \in \mathbb{Z}}$ be a sequence of complex numbers satisfying (3) and such that $\Phi = (e^{i\lambda_n \cdot})_{n \in \mathbb{Z}}$ is a Riesz sequence with sharp bounds $A = A_{opt}(\Phi)$ and $B_{opt}(\Phi)$. From Proposition 1 we conclude

$$B_{opt}(\Phi) \leq \frac{2(e^{2\sigma(\tau+1)} - 1)}{\sigma} \left( 1 + \frac{2\sigma}{\log \left( 1 + e^{-\sigma \tau} \sqrt{\frac{A}{\sigma}} \right)} \right)^2. \quad (9)$$
From
\[ A \leq \|e^{i\lambda_n \cdot \bullet}\|_{L^2(-\sigma, \sigma)}^2 \leq 2\sigma e^{2\sigma\tau} \quad (n \in \mathbb{Z}) \]
we derive
\[ \frac{1}{\sqrt{8}} e^{-\sigma\tau} \sqrt{\frac{A}{\sigma}} \leq \frac{1}{2}. \]  
(10)

Using \( \log(1 + x) \geq x \) for \( x \in [0, \frac{1}{2}] \), we thus obtain
\[ \log \left(1 + \frac{e^{-\sigma\tau}}{\sqrt{\frac{A}{\sigma}}}\right) \geq \log \left(1 + \frac{1}{\sqrt{8}} e^{-\sigma\tau} \sqrt{\frac{A}{\sigma}}\right) \geq \frac{1}{2} \frac{1}{\sqrt{8}} e^{-\sigma\tau} \sqrt{\frac{A}{\sigma}}. \]

Using (9) and (10), we conclude
\[ B_{opt}(\Phi) \leq \frac{2}{\sigma}(e^{2\sigma(\tau+1)} - 1) \left(1 + 4\sqrt{8} \sigma e^{\sigma\tau} \sqrt{\frac{\sigma}{A}}\right)^2 \]
\[ \leq \frac{2}{\sigma}(e^{2\sigma(\tau+1)} - 1) \left(1 + \sigma\right)^2 \left(4\sqrt{8} \sigma e^{\sigma\tau} \sqrt{\frac{\sigma}{A}}\right)^2 \]
\[ \leq 256 e^{4\sigma\tau + 2\sigma} \left(\frac{1}{8} + \sigma\right)^2 \frac{1}{\sigma}. \]

This shows \( A_{opt}(\Phi) B_{opt}(\Phi) \leq C(\sigma, \tau) \) for \( C(\sigma, \tau) \) defined by (6) \( \blacksquare \)

Proof of Corollary 1. Let \( \Phi = (e^{i\lambda_n \cdot \bullet})_{n \in \mathbb{Z}} \) be an exact frame for \( L^2(-\sigma, \sigma) \). Since the frame bounds of the exact frame \( \Phi \) coincide with the bounds of \( \Phi \) as a Riesz sequence (Remark 4), we conclude that for all \( f \in L^2(-\sigma, \sigma) \)
\[ A_{opt}(\Phi) \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |(f, e^{i\lambda_n \cdot \bullet})|^2 \leq B_{opt}(\Phi) \|f\|^2 \]  
(11)

and that \( A_{opt}(\Phi) \) and \( B_{opt}(\Phi) \) are the best possible constants in this inequality. From this we derive
\[ \|T_\Phi\| = \sqrt{B_{opt}(\Phi)} \quad \text{and} \quad \|T^{-1}_\Phi\| = \frac{1}{\sqrt{A_{opt}(\Phi)}}. \]  
(12)

The norms of \( S_\Phi \) and \( S^{-1}_\Phi \) also can be expressed by the best constants occuring in (11). It holds
\[ \|S_\Phi\| = B_{opt}(\Phi) \quad \text{and} \quad \|S^{-1}_\Phi\| = \frac{1}{A_{opt}(\Phi)} \]  
(13)

(cf. Benedetto and Walnut [1: Theorem 3.2/a]). Thus the result follows from inequalities (5), (12) and (13) \( \blacksquare \)

Proof of Corollary 2. It follows from Corollary 1 and the Paley-Wiener theorem (cf. Young [10: Chapter 2/Theorem 18]) \( \blacksquare \)
Open questions.

1. For what classes of exact frames (not of exponential type) do analogues to Theorem 1 and Corollary 1 exist? It can be thought, e.g., of the class of all exact Gabor frames with discretisation parameters \( t_0 \) and \( \omega_0 \) such that \( t_0 \omega_0 = 1 \).

2. What is the best constant \( C_{opt}(\sigma, \tau) \) fulfilling inequality (5)? We conjecture that \( C_{opt}(\pi, 0) = 4\pi^2 \).

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References


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