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Two-component viscous flows with density-dependent transition layer

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We present a model describing flows which undergo a phase change caused by chemical reactions in the material flow. In mathematical terms, we consider a free boundary problem for the one-dimensional, compressible Navier–Stokes system coupled with an Allen–Cahn equation. One free boundary arises at the boundary of the flow to a vacuum state with a jump in the density. Another approximate free boundary arises due to the Allen–Cahn equation modeling the phase transition of the flow. We consider a density-dependent viscosity $\mu = \rho^{\theta}$ and a density-dependent transition layer of thickness $\delta = \tilde{\delta}\rho^{-1}$. We establish the existence of a unique weak global solution, provided that $\theta < 1/3$.

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1. Introduction

We investigate the kinetics of gaseous or liquid flows which undergo a phase transition. We study the dynamics of fluid structures as well as the dynamics of phase field structures which describes the motion of the transition layer. In mathematical terms, the governing equations that include transport phenomena, such as viscosity and diffusion, are coupled with a phase field equation with a non-conserved phase parameter (see [6]).

Many authors have studied problems of two-component viscous fluids with free boundaries separating the components (see [11], [2], [12]). Problems including the influence of environmental factors, for example the temperature, were considered by [8], [16], [10] or [11]. Besides the temperature, other effects have an important influence, for example chemical reactions. In our investigations we focus on this aspect. The flow is assumed to be chemically in non-equilibrium. The entire chemical process consists of single reactions with distinct reactants. We model this as a whole and describe it as one phase change in the material flow. In this way, the single reactions are incorporated in the ansatz of the Helmholtz free energy (see (8) below).

The isothermal material flow in n dimensions is governed by the compressible Navier–Stokes system which can be written, in Eulerian coordinates, as

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{1}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + \tau) = -\rho g, \tag{2}$$

where $x \in \Omega(t) \subset \mathbb{R}^n$ and $t > 0$. The unknown functions ρ and v denote the density and velocity, respectively.

The tensor τ has the form

$$\tau = P - S, \qquad S = \mu_1(\rho)\,\mathrm{div}\, v\mathbb{I} - \mu_2(\rho)\left(\frac{1}{2}\big(\nabla v + (\nabla v)^T\big) - \frac{1}{n}\,\mathrm{div}\, v\mathbb{I}\right), \tag{3}$$

where P is the pressure tensor and S is the stress tensor. The Lamé coefficients μ_1 and μ_2 are usually positive constants, but we assume that they depend on ρ. Varying viscosity is motivated by numerous applications, for example for shallow water models (see [5]) and gaseous star models in astrophysics (see [15]). Later in Section 7, we prove the uniqueness for the problem (1), (2), (9) with density dependent Lamé coefficients (see (3)) and, of course, with boundary conditions which will be explained below. The uniqueness is based on the estimates of ρ, that is, $\rho \geqslant c$ and $\rho \leqslant C$. It is excluded that in the fluid flow an additional cavity occurs. In this connection we mention [9], where a constant viscosity is considered and a non-uniqueness result is shown.

In equation (2), the vector g is a function of x of class $L^2(\mathbb{R}^n)$. The term ρg stands for an external force. For applications in astrophysics it could contain for example the gravitation.

The boundary $\partial\Omega(t)$ consists of a fixed part Γ_D and a free boundary $\Gamma_f(t)$, hence $\partial\Omega(t) = \Gamma_D \cup \Gamma_f(t)$. We presume Dirichlet boundary conditions for the fixed part Γ_D, that is,

$$v = 0 \quad \text{on } \Gamma_D. \tag{4}$$

The free boundary $\Gamma_f(t)$ represents the interface between the liquid or gaseous medium and the vacuum. We presume that it travels at the same velocity as the fluid particles. Thus, the well-known jump condition

$$[\tau]_{\Gamma_f(t)}\vec{n}_{\Gamma_f(t)} = 0 \quad \text{on } \Gamma_f(t) \tag{5}$$

must hold (see [13]). Here it is assumed that the surface tension is equal to zero.

The Navier–Stokes equations are coupled with a phase field equation of Allen–Cahn type describing the development of a phase change in the material. The evolution of this phase change is mainly forced by a Helmholtz free energy density f. Let Φ be the phase field parameter. We assume that f only depends on ρ, Φ and $\nabla\Phi$.

Thus, we enforce the ansatz for the non-inert system (see [1], [17], [22])

$$\rho(\partial_t\Phi + v\cdot\nabla\Phi) = -\lambda_{\tilde{\delta}}\frac{\delta f}{\delta\Phi}. \tag{6}$$

Here $\delta f/\delta\Phi$ denotes the variational gradient of f with respect to Φ, that is,

$$\frac{\delta f}{\delta\Phi} = f_{|\Phi} - \mathrm{div}(f_{|\nabla\Phi}),$$

where $f_{|\Phi}$ and $f_{|\nabla\Phi}$ denote the derivatives with respect to Φ and $\nabla\Phi$.

The coefficient $\lambda_{\tilde{\delta}} > 0$ could be a positive function depending on ρ and Φ. For simplicity, here we assume that $\lambda_{\tilde{\delta}}$ is a constant of order $O(1/\tilde{\delta})$, where $\tilde{\delta} > 0$ is a positive constant.

At the boundary $\partial\Omega(t)$ we presume zero Neumann boundary conditions, that is,

$$\nabla\Phi\cdot\vec{n}_{\partial\Omega(t)} = 0 \quad \text{on } \partial\Omega(t). \tag{7}$$

In this paper f has the form

$$f(\rho,\Phi,\nabla\Phi) := \rho\left(W(\Phi) + \Psi(\rho,\Phi) + \delta(\rho)^2\frac{|\nabla\Phi|^2}{2}\right), \tag{8}$$

where $\delta(\rho) > 0$ is an arbitrary function of $\rho > 0$. Here W stands for the double-well potential with two minima at the same height (for example see (14)), and Ψ models the deviation from the thermodynamical equilibrium (for example see (15)). The term $\delta(\rho)^2|\nabla\Phi|^2/2$ acts as 'surface energy'.[1]

From (6) we infer

$$\lambda_{\tilde{\delta}}^{-1}\rho\,(\partial_t\Phi + v\cdot\nabla\Phi) = -\,\rho W_{|\Phi} - \rho\Psi_{|\Phi} + \operatorname{div}(\rho\delta(\rho)^2\nabla\Phi), \tag{9}$$

where $t > 0$, $x \in \Omega(t)$.

It is well-known that if we apply phase-field methods to multiphase Navier–Stokes equations, then an additional 'surface tension'[2] arises in the pressure tensor P. Thus, the system (1), (2), (9) is energetically and thermodynamically consistent (see [1], [12], [20]). Further, from general thermodynamics, we define the effective pressure p_f by $-p_f = f - \rho f_{|\rho}$. Then the pressure tensor in the Navier–Stokes equations has to be $P = p_f\mathbb{I} + \nabla\Phi \otimes f_{|\nabla\Phi}$, so that the entropy principle is fulfilled (see, for example, [1, §4]). Hence, we obtain

$$\begin{aligned} P &\equiv P(\rho,\Phi,\nabla\Phi) = (-f + \rho f_{|\rho})\mathbb{I} + \nabla\Phi\otimes f_{|\nabla\Phi} \\ &= (-\rho(W(\Phi)+\Psi(\rho,\Phi)) + \rho(W(\Phi)+\Psi(\rho,\Phi)+\rho\Psi_{|\rho}(\rho,\Phi)))\mathbb{I} \\ &\quad + \left[-\rho\frac{\delta(\rho)^2}{2}|\nabla\Phi|^2 + \rho\left(\frac{\delta(\rho)^2}{2} + \rho\delta(\rho)\delta'(\rho)\right)|\nabla\Phi|^2\right]\mathbb{I} + \rho\delta(\rho)^2\nabla\Phi\otimes\nabla\Phi \\ &= \underbrace{\rho^2\Psi_{|\rho}(\rho,\Phi)}_{=:p_1(\rho,\Phi)}\mathbb{I} + \underbrace{\rho^2\delta(\rho)\delta'(\rho)|\nabla\Phi|^2\mathbb{I} + \rho\delta(\rho)^2\nabla\Phi\otimes\nabla\Phi}_{=:P_2(\rho,\nabla\Phi)}. \end{aligned} \tag{10}$$

Here P_2 denotes the part of P which depends on $\nabla\Phi$. It stands for the 'surface tension' between the different phases and is a symmetric tensor. We want to presume that P_2 is trace free. With this restriction, we ensure that $P_2(\rho,\nabla\Phi)$ just represents a shear force. We mention that the volume force in τ, arising in the general case, is governed by $\operatorname{div} v$ only.

In order to get the tensor $P_2(\rho,\nabla\Phi)$ trace free, we compute

$$\begin{aligned} 0 &= \operatorname{trace} P_2(\rho,\nabla\Phi) \\ &= \operatorname{trace}\begin{pmatrix} \rho^2\delta(\rho)\delta'(\rho)|\nabla\Phi|^2 + \rho\delta(\rho)^2|\partial_{x_1}\Phi|^2 & & \rho\delta(\rho)^2\partial_{x_1}\Phi\partial_{x_n}\Phi \\ \vdots & \ddots & \vdots \\ \rho\delta(\rho)^2\partial_{x_n}\Phi\partial_{x_1}\Phi & & \rho^2\delta(\rho)\delta'(\rho)|\nabla\Phi|^2 + \rho\delta(\rho)^2|\partial_{x_n}\Phi|^2 \end{pmatrix} \\ &= \sum_{k=1}^{n}[\rho^2\delta(\rho)\delta'(\rho)|\nabla\Phi|^2 + \rho\delta(\rho)^2|\partial_{x_k}\Phi|^2] \\ &= \rho\delta(\rho)(n\rho\delta'(\rho) + \delta(\rho))|\nabla\Phi|^2. \end{aligned}$$

Then $\delta(\rho)$ must satisfy, since $\delta(\rho) > 0$ and $\rho > 0$,

$$n\rho\delta'(\rho) + \delta(\rho) = 0.$$

[1,2] These notions are motivated by the free boundary problem arising from the Allen–Cahn equation as $\delta(\rho) \to 0$.

For this, we have the general solution

$$\delta(\rho) = \tilde{\delta}\frac{1}{\rho^\alpha} \quad \text{with} \quad \alpha := \frac{1}{n}. \tag{11}$$

In one dimension the coefficient of $|\nabla\Phi|^2/2$ in (8) is $\rho\delta(\rho)^2 = \tilde{\delta}^2\rho^{-1}$. In two dimensions the coefficient is $\rho\delta(\rho)^2 = \tilde{\delta}^2$ and therefore constant. In three dimensions we have $\rho\delta(\rho)^2 = \tilde{\delta}^2\rho^{1/3}$ and so on.

So far we have introduced system (1), (2), (9) with boundary conditions (4), (5), (7) in higher dimensions. An open problem is to show the global existence of a weak solution and its uniqueness. But for a feasible mathematical treatment, in a first step, we want to analyze the above model system (1), (2), (9) with boundary conditions (4), (5), (7) in one dimension.

2. Model in one dimension

The one-dimensional isothermal equations can be written as

$$\partial_t\rho + \partial_x(\rho v) = 0, \tag{12}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + P(\rho, \Phi, \partial_x\Phi)) = \partial_x(\mu(\rho)\partial_x v) - \rho g, \tag{13}$$

where $0 < x < \xi(t)$, $t > 0$, $x \in \mathbb{R}$. Here $x = 0$ stands for the fixed boundary and $x = \xi(t)$ for the free boundary. We introduce as viscosity coefficient

$$\mu(\rho) := \rho^\theta, \qquad \theta > 0.$$

This means μ is sublinear in ρ and is expected to vanish when ρ becomes zero. For compressible Navier–Stokes systems with density-dependent viscosity, the continuous dependence of the solutions on the initial data can be shown (see [7]).

In identity (8) we define

$$W(\Phi) := c_1(\Phi\ln\Phi + (1-\Phi)\ln(1-\Phi)) + c_2\Phi(1-\Phi), \tag{14}$$

where $0 < \Phi < 1$ and c_1, c_2 are positive constants with $c_2 > c_1$. The first summand stands for the well-known logarithmic part and the last for the phase interaction. For the logarithmic part see [3], where the usage of the logarithmic potential is discussed, and physics literature is quoted (see moreover [14], [4]). The constants $c_2 > c_1$ are chosen so that W represents a double-well potential with two minima at $\Phi \approx 1$ and $\Phi \approx 0$. The exact values depend on the choice of c_1 and c_2.

As deviation from the equilibrium we set

$$\Psi(\rho, \Phi) := \Phi\frac{f_1(\rho)}{\rho} + (1-\Phi)\frac{f_2(\rho)}{\rho}. \tag{15}$$

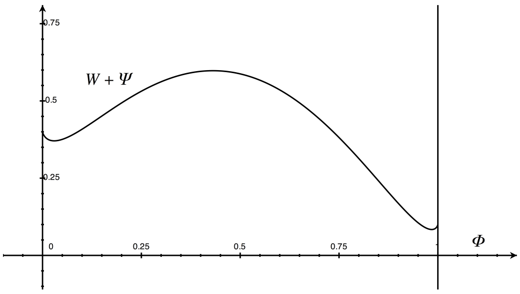
Here $f_1(\rho)$ and $f_2(\rho)$ are the free energies of the two phases. In this paper we set for simplicity

$$f_1(\rho) := \rho^2 b_1 \quad \text{and} \quad f_2(\rho) := \rho^2 b_2,$$

where b_1 and b_2 are real constants. Thus Ψ is an affine function.

In our model, Ψ introduces the chemical induction. The coefficients b_1 and b_2 stand for the chemical potentialities of the phases: b_1 measures the possibility of phase 1 to transform into phase 2, and b_2 measures the possibility of phase 2 to transform into phase 1.

All the constants c_1, c_2, and b_1 and b_2 are to be set so that $W + \Psi$ remains a double-well potential, though its minima need not be situated at the same height. For example, in the figure we see the plot of $W + \Psi$ for the values $c_1 = 1.1$, $c_2 = 4.4$, $b_1 = 0.1$, $b_2 = 0.4$, and $\rho = 1$:



We mention that the derivatives of $W + \Psi$ at $\Phi = 0$ and $\Phi = 1$ go to infinity.

From equation (11), we know that in one dimension

$$\delta(\rho) = \tilde{\delta}\rho^{-1} \tag{16}$$

where $\tilde{\delta} > 0$. The function δ represents the gradient penalty coefficient and is expected to regulate the thickness of the transition layer. For small ρ the gradient penalty coefficient is increased and a thicker interface will minimize the total free energy of the system. This means that regions residing nearly in vacuum state force a large transition layer.

Altogether, the Helmholtz free energy density f (see (8)), becomes

$$f(\rho, \Phi, \partial_x\Phi) = \rho[c_1(\Phi \ln \Phi + (1-\Phi)\ln(1-\Phi)) + c_2\Phi(1-\Phi) + \rho(\Phi b_1 + (1-\Phi)b_2)] + \frac{\tilde{\delta}^2}{2}\frac{1}{\rho}|\partial_x\Phi|^2.$$

In this way, f is a convex function with respect to the density ρ, and non-convex in Φ. Further, equation (9) reads

$$\lambda_{\tilde{\delta}}^{-1}\rho(\partial_t\Phi + v\cdot\partial_x\Phi) = -c_1\rho\ln\left(\frac{\Phi}{1-\Phi}\right) - c_2\rho(1-2\Phi) - b_d\rho^2 + \partial_x(\delta(\rho)^2\rho\partial_x\Phi), \tag{17}$$

where $t > 0$, $0 < x < \xi(t)$ and

$$b_d := b_1 - b_2.$$

Since $n = 1$, the fact that P_2 is trace free (see (10)) implies $P_2(\rho, \partial_x\Phi) = 0$. Thus, the pressure P is independent of $\partial_x\Phi$. Then $P = p_1(\rho, \Phi)$ ($\mathbb{I} = 1$ in one space dimension) obeys the following law:

$$P = p_1(\rho, \Phi) = \rho^2\Psi_{|\rho}(\rho, \Phi) = \rho^2(b_d\Phi + b_2).$$

This calculation, where the derivatives of Φ can be completely eliminated, is only possible in one space dimension.

Equations (12), (13) and (17) fulfill the entropy principle (see [20]).

We consider system (12), (13), (17) with the initial data

$$\rho(0, x) = \rho_0(x), \qquad v(0, x) = v_0(x), \qquad \Phi(0, x) = \Phi_0(x). \tag{18}$$

Further, we consider the boundary conditions. The boundary $x = 0$ is the fixed boundary, where the velocity v is set equal to zero (see condition (4)). For the phase parameter Φ we presume zero Neumann boundary conditions (see condition (7)). Hence, we write

$$v(t, 0) = 0, \tag{19}$$

$$\partial_x \Phi(t, 0) = 0. \tag{20}$$

The boundary $x = \xi(t)$ constitutes one of the free boundaries (besides the approximate free boundary). As we mentioned in Section 1 (see (5)), it represents the interface between the liquid or gaseous medium and the vacuum. We presume that it travels at the same velocity as the fluid particles

$$\frac{\mathrm{d}\xi}{\mathrm{d}t}(t) = v(t, \xi(t)), \tag{21}$$

and the conditions (5) and (7) read respectively

$$(p - \mu\partial_x v)(t, \xi(t)) = 0, \tag{22}$$

$$\partial_x \Phi(t, \xi(t)) = 0. \tag{23}$$

The aim of this article is to show the global existence of a weak solution and its uniqueness. In the subsequent considerations the transformation to Lagrangian coordinates is essential. The free boundary problem (12), (13), (17), (18)–(23) is transformed into a phase field problem with fixed boundary. Then we can show that regions with zero mass density in the interior of the domain are impossible. In Theorem 18 we prove that the mass density is bounded from below by a positive constant depending on the data. This feature justifies the transformation to Lagrangian coordinates in order to prove existence and is essential to guarantee uniqueness. Further, we also show that regions with arbitrarily high mass density cannot occur (see Theorem 13).

After the transformation to Lagrangian coordinates, we discretize the space variable and use the line method.

3. Weak formulation of the problem and basic assumptions

We rewrite the equations (12), (13), (17) in Lagrangian coordinates

$$z = \int_0^x \rho(t, x')\,\mathrm{d}x'.$$

We normalize the total mass to 1, that is,

$$\int_0^{\xi(t)} \rho(t, x')\,\mathrm{d}x' = 1.$$

The above problem (12), (13), (17), (18)–(23) is thus transformed and the functions (ρ, v, Φ), $0 < \Phi < 1$, satisfy the following fixed boundary problem:

$$\partial_t \rho + \rho^2 \partial_z v = 0, \tag{24}$$

$$\partial_t v + \partial_z p(\rho, \Phi) = \partial_z(\mu(\rho)\rho\partial_z v) - g, \tag{25}$$

$$\lambda_{\tilde{\delta}}^{-1} \partial_t \Phi = -c_1 \ln\left(\frac{\Phi}{1-\Phi}\right) - c_2(1-2\Phi) - b_d \rho + \partial_z(\tilde{\delta}^2 \partial_z \Phi) \tag{26}$$

in $t > 0$ and $0 < z < 1$, where

$$p(\rho, \Phi) = \rho^2(b_d \Phi + b_2). \tag{27}$$

The boundary conditions become

$$v(t, 0) = 0, \tag{28}$$

$$(p - \mu\rho\partial_z v)(t, 1) = 0, \tag{29}$$

$$\partial_z \Phi(t, 0) = \partial_z \Phi(t, 1) = 0, \tag{30}$$

and the initial condition

$$(\rho, v, \Phi)(0, z) = (\rho_0, v_0, \Phi_0)(z), \qquad 0 \leqslant z \leqslant 1. \tag{31}$$

The transformed system (24)–(31) is equivalent to the old one, that is, (12), (13), (17), (18)–(23) written in Eulerian coordinates, if the density is bounded from below by a positive constant (see Theorem 18).

In this paper we consider the following assumptions:

(A.1) $\rho_0 \in C^{0,1}([0, 1])$ and $\rho_0(z) \geqslant \underline{\rho}_0$, where $\underline{\rho}_0 > 0$ is a constant;
(A.2) $v_0 \in C^1([0, 1])$ and $\frac{\mathrm{d}}{\mathrm{d}x} v_0 \in C^{0,1}([0, 1])$;
(A.3) $\Phi_0 \in C^1([0, 1])$ and $0 \leqslant \Phi_0 \leqslant 1$;
(A.4) $g \in L^2(\mathbb{R})$;
(A.5) $0 < \theta < 1/3$.

REMARK. The constant $\theta < 1/3$ is needed for the estimate of ρ from below (see Theorem 18).

DEFINITION 1 A triple (ρ, v, Φ) is called a *global weak solution* for the system (24)–(31) if

$$\rho \in L^\infty([0, T] \times [0, 1]) \cap C^1([0, T]; L^2(0, 1)), \tag{32}$$

$$v \in L^\infty([0, T] \times [0, 1]) \cap C^1([0, T]; L^2(0, 1)), \tag{33}$$

$$\Phi \in L^\infty([0, T] \times [0, 1]) \cap C^1([0, T]; L^2(0, 1)), \tag{34}$$

$$\rho^{\theta+1} \partial_z v \in L^\infty([0, T] \times [0, 1]) \cap C^0([0, T]; L^2(0, 1)), \tag{35}$$

$$\partial_z \Phi \in L^\infty([0, T] \times [0, 1]) \cap C^0([0, T]; L^2(0, 1)), \tag{36}$$

for all T, and if the following equations hold:

$$\partial_t \rho + \rho^2 \partial_z v = 0 \tag{37}$$

for almost all $z \in (0, 1)$ and for any $t \geqslant 0$, and

$$\int_0^1 [\varphi \partial_t v - \partial_z \varphi (p - \mu \rho \partial_z v) + \varphi g] \, \mathrm{d}z = 0, \tag{38}$$

$$\int_0^1 \left[\varphi \lambda_{\tilde{\delta}}^{-1} \partial_t \Phi + \varphi \left\{ c_1 \ln\left(\frac{\Phi}{1 - \Phi} \right) + c_2 (1 - 2\Phi) + b_d \rho \right\} + \partial_z \varphi \tilde{\delta}^2 \partial_z \Phi \right] \mathrm{d}z = 0 \tag{39}$$

for any test function $\varphi \in C_0^\infty((0, 1])$ and for almost all $t \in [0, T]$.

Under assumptions (A.1)–(A.5), we will prove the existence of a global weak solution to the initial boundary value problem (24)–(31) in the sense of Definition 1.

Our main result is the following

THEOREM 2 Suppose (ρ_0, v_0, Φ_0) satisfy (A.1)–(A.4) and θ satisfies (A.5). The constants b_1 and b_2 are nonnegative. Then the initial boundary value problem (24)–(31) admits a global weak solution (ρ, v, Φ) satisfying (32)–(36) and

$$1/C \leqslant \rho(t, z) \leqslant C \quad \text{for almost all } (t, z) \in [0, T] \times [0, 1],$$
$$0 < \Phi < 1 \quad \text{for almost all } (t, z) \in [0, T] \times [0, 1].$$

The idea for the estimate for ρ from below (see the proof of Theorem 18) is taken from [19]. Our result is a generalization of that work. We deal with the existence and uniqueness of solution for the one-dimensional Navier–Stokes equations in connection with a phase field equation where an approximate free boundary arises.

Above and in what follows, we use C for a positive generic constant, which depends only on the initial data, on the time T, and on the choice of the constants b_1, b_2, c_1, and c_2.

In order to make the argument below clearer (Section 4), we introduce a new dependent variable

$$w := 1 - 2\Phi. \tag{40}$$

Then the ln-term has the form

$$-\ln\left(\frac{\Phi}{1 - \Phi} \right) = \ln\left(\frac{1 + w}{1 - w} \right) \quad \text{if } -1 < w < +1.$$

We rewrite the system (24)–(31) in the new variable w. The functions (ρ, v, w), $-1 < w < +1$, satisfy the following initial boundary problem:

$$\partial_t \rho + \rho^2 \partial_z v = 0, \tag{41}$$

$$\partial_t v + \partial_z \tilde{p}(\rho, w) = \partial_z (\mu(\rho) \rho \partial_z v) - g, \tag{42}$$

$$\lambda_{\tilde{\delta}}^{-1} \partial_t w = -2c_1 \ln\left(\frac{1 + w}{1 - w} \right) + 2c_2 w + 2b_d \rho + \partial_z (\tilde{\delta}^2 \partial_z w) \tag{43}$$

in $t > 0$ and $0 < z < 1$, where

$$\tilde{p}(\rho, w) := p(\rho, (1 - w)/2) = \rho^2 \left(b_d \frac{1}{2} (1 - w) + b_2 \right). \tag{44}$$

The boundary conditions become

$$v(t,0)=0, \tag{45}$$

$$(\tilde{p}-\mu\rho\partial_z v)(t,1)=0, \tag{46}$$

$$\partial_z w(t,0)=\partial_z w(t,1)=0, \tag{47}$$

and the initial condition

$$(\rho, v, w)(0,z)=(\rho_0, v_0, 1-2\Phi_0)(z), \qquad 0\leqslant z\leqslant 1. \tag{48}$$

4. Difference scheme and estimates

One difficulty in solving equations (41)–(48) lies in the logarithmic non-linearity. The method to handle this is to introduce a regularized system, in which $\ln[(1+w)/(1-w)]$ is replaced by polynomials. The ln-term has the form

$$\ln\left(\frac{1+w}{1-w}\right)=2\sum_{k=0}^{\infty}\frac{1}{2k+1}w^{2k+1} \quad \text{if } -1<w<+1.$$

Now let us define, for each $M\in\mathbb{N}$, the approximation

$$a_M(w):=2\sum_{k=0}^{M}\frac{1}{2k+1}w^{2k+1} \quad \text{for } w\in\mathbb{R}. \tag{49}$$

In this way, the ln-term is approximated by a function which is extended on the whole $\mathbb{R}^1$. As M tends to infinity, the corresponding series diverges outside $(-1,1)$. We have

$$a_M(w)\xrightarrow{M\to\infty}\ln\left(\frac{1+w}{1-w}\right) \quad \text{locally uniformly for } -1<w<+1.$$

Further, we define a non-negative, smooth cut-off function $\eta\in C_0^\infty(\mathbb{R})$ by

$$\eta(w)=\begin{cases} b_1\frac{1}{2}(1-w)+b_2\frac{1}{2}(1+w) & \text{for all } w\in(-1-\kappa_1, 1+\kappa_1),\\ 0 & \text{for all } w\in(-\infty,-1-\kappa_2)\cup(1+\kappa_2,+\infty),\end{cases}$$

where κ_1, κ_2 with $\kappa_2>\kappa_1$ are positive constants. The constant κ_1 is chosen so that

$$b_1\frac{1}{2}(1-w)+b_2\frac{1}{2}(1+w)>0 \quad \text{for } w\in(-1-\kappa_1, 1+\kappa_1). \tag{50}$$

Both functions, $a_M(w)$ and $\eta(w)$, will be used in the Approximate (N, M)-Problem below.

We discretize the derivatives of equations (41)–(43) with respect to z using the line method. For the Navier–Stokes equations in one dimension, (41), (42), this discretization is well-known (see [19] and [21]). It gives the equations (51) and (52) below. A similar technique applied to the phase field equation (43) results in (53). On the way, we replace the ln-term by its approximation a_M (see (49)). Additionally, we replace $2\rho b_d$ in (43) by $-4\rho\eta'(w)$ and $b_d\frac{1}{2}(1-w)+b_2$ in (44) by $\eta(w)$. These modifications are necessary to establish the existence of a solution of the spatial discrete approximate problem (see the Approximate (N, M)-Problem below). Later we will see that the solution w of the approximate problem remains bounded for $-1-\varepsilon < w < 1+\varepsilon$ (see Lemma 11), so that we can neglect the cut-off function η.

Finally, this leads to the following scheme:

APPROXIMATE (N, M)-PROBLEM The approximate problem has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{n-1} + \rho_{n-1}^2 \frac{v_n - v_{n-1}}{h} = 0, \tag{51}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v_n + \frac{\tilde{p}_n - \tilde{p}_{n-1}}{h} = \frac{1}{h}\left[\mu_n\rho_n\frac{v_{n+1}-v_n}{h} - \mu_{n-1}\rho_{n-1}\frac{v_n - v_{n-1}}{h}\right] - g_n, \tag{52}$$

$$\begin{aligned}\lambda_{\tilde{\delta}}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}w_n = &-2c_1 a_M(w_n) + 2c_2 w_n - 4\rho_{n-1}\eta'(w_n)\\ &+\tilde{\delta}^2\frac{1}{h}\left[\frac{w_{n+1}-w_n}{h} - \frac{w_n - w_{n-1}}{h}\right],\end{aligned} \tag{53}$$

for $n = 1, \ldots, N$, where $h = 1/N$, $N \in \{2^i : i \in \mathbb{N}\}$ being a large natural number which determines the division of the interval $[0, 1]$ into N intervals of length h.

Further, we discretize equation (44) and get

$$\tilde{p}_{n-1} = \rho_{n-1}^2\eta(w_n).$$

The boundary conditions (45)–(47) are discretized as

$$v_0(t) = 0 \quad \text{and} \quad \left(\tilde{p}_N - \mu_N\rho_N\frac{v_{N+1}-v_N}{h}\right)(t) = 0, \tag{54}$$

$$\frac{w_1 - w_0}{h}(t) = 0 \quad \text{and} \quad \frac{w_{N+1}-w_N}{h}(t) = 0, \tag{55}$$

and the initial conditions (48) are discretized as

$$\rho_{n-1}(0) = \rho_0((n-1)h) > 0, \tag{56}$$

$$v_n(0) = v_0(nh), \tag{57}$$

$$w_n(0) = 1 - 2\Phi_0(nh). \tag{58}$$

□

By the theory of ordinary differential equations, the Cauchy problem (51)–(58) admits for all (N, M) a unique local in time solution, that is, there exist a $t^* > 0$ and a function

$$t \mapsto (\rho_{n-1}^{NM}(t), v_n^{NM}(t), w_n^{NM}(t))_{n=1,\ldots,N} \in \mathbb{R}^{3N}$$

in the class $C^0([0, t^*)) \cap C^1((0, t^*))$. Here $[0, t^*)$ can be chosen as the maximal existence interval of this solution. By (51) and (A.1), we see that $\rho_{n-1}^{NM}(t) > 0$ for $0 < t < t^*$. We denote by $(\rho^{NM}, v^{NM}, w^{NM})$ the function $t \mapsto (\rho_{n-1}^{NM}(t), v_n^{NM}(t), w_n^{NM}(t))_{n=1,\ldots,N}$, where $(\rho_{n-1}^{NM}, v_n^{NM}, w_n^{NM})$ is the local in time solution of (51)–(58) for each $n = 1, \ldots, N$.

Now, we want to show that this local solution of the Approximate (N, M)-Problem exists for all t, that is, for $0 \leqslant t < t^* = T$ (the time T can be chosen arbitrarily). For that we will show that the local in time solution of the Approximate (N, M)-Problem satisfies a priori estimates, first of all the energy estimate (see Theorem 7). The next three propositions are needed for the energy estimate.

PROPOSITION 3 Let $(\rho^{NM}, v^{NM}, w^{NM})$ be a local in time solution of problem (51)–(58). Then in the maximal existence interval we have

$$\sum_{n=1}^{N} h\frac{1}{4}\left\{\lambda_{\tilde{\delta}}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left[4c_1\sum_{k=0}^{M}\frac{(w_n^{NM})^{2k+2}}{(2k+2)(2k+1)}\right] + c_2\frac{\mathrm{d}}{\mathrm{d}t}[(1-w_n^{NM})(1+w_n^{NM})] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\tilde{\delta}^2}{2}\left(\frac{w_n^{NM}-w_{n-1}^{NM}}{h}\right)^2\right]\right\} + \sum_{n=1}^{N} h\rho_{n-1}^{NM}\frac{\mathrm{d}}{\mathrm{d}t}\eta(w_n^{NM}) = 0. \quad (59)$$

Proof. After multiplying equation (53) with $h\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM}$, we sum it up from $n = 1$ to $n = N$. This yields

$$\sum_{n=1}^{N} h\left\{\lambda_{\tilde{\delta}}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM}\right)^2 + 2c_1a_M(w_n^{NM})\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM} - c_2\frac{\mathrm{d}}{\mathrm{d}t}\left((w_n^{NM})^2\right) - \tilde{\delta}^2\frac{1}{h}\left(\frac{w_{n+1}^{NM}-w_n^{NM}}{h} - \frac{w_n^{NM}-w_{n-1}^{NM}}{h}\right)\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM}\right\} = -4\sum_{n=1}^{N} h\rho_{n-1}^{NM}\frac{\mathrm{d}}{\mathrm{d}t}\eta(w_n^{NM}).$$

We calculate further the fourth summand on the left-hand side:

$$-\sum_{n=1}^{N}\tilde{\delta}^2\left(\frac{w_{n+1}^{NM}-w_n^{NM}}{h} - \frac{w_n^{NM}-w_{n-1}^{NM}}{h}\right)\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM}$$
$$= +\sum_{n=1}^{N}\tilde{\delta}^2\left(\frac{w_n^{NM}-w_{n-1}^{NM}}{h}\right)\frac{\mathrm{d}}{\mathrm{d}t}(w_n^{NM}-w_{n-1}^{NM}) - \tilde{\delta}^2\frac{w_{N+1}^{NM}-w_N^{NM}}{h}\frac{\mathrm{d}}{\mathrm{d}t}w_N^{NM} + \tilde{\delta}^2\frac{w_1^{NM}-w_0^{NM}}{h}\frac{\mathrm{d}}{\mathrm{d}t}w_0^{NM}$$

and after exploiting the boundary conditions (55),

$$= +\sum_{n=1}^{N} h\tilde{\delta}^2\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n^{NM}-w_{n-1}^{NM}}{h}\right)^2.$$

Further, by (49),

$$2c_1a_M(w_n^{NM})\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM} = \frac{\mathrm{d}}{\mathrm{d}t}\left[4c_1\sum_{k=0}^{M}\frac{(w_n^{NM})^{2k+2}}{(2k+2)(2k+1)}\right].$$

By inserting this in the above equation, we derive that

$$\sum_{n=1}^{N} h\left\{\lambda_{\tilde{\delta}}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left[4c_1\sum_{k=0}^{M}\frac{(w_n^{NM})^{2k+2}}{(2k+2)(2k+1)}\right] + c_2\frac{\mathrm{d}}{\mathrm{d}t}[(1-w_n^{NM})(1+w_n^{NM})] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\tilde{\delta}^2}{2}\left(\frac{w_n^{NM}-w_{n-1}^{NM}}{h}\right)^2\right]\right\} = -4\sum_{n=1}^{N} h\rho_{n-1}^{NM}\frac{\mathrm{d}}{\mathrm{d}t}\eta(w_n^{NM}). \qquad \square$$

PROPOSITION 4 Let $(\rho^{NM}, v^{NM}, w^{NM})$ be a local in time solution of problem (51)–(58). Then in the maximal existence interval we have

$$\sum_{n=1}^{N} h\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((v_n^{NM})^2) - \sum_{n=1}^{N} h\tilde{p}_{n-1}^{NM}\frac{v_n^{NM}-v_{n-1}^{NM}}{h} + \sum_{n=1}^{N} h\mu_{n-1}^{NM}\rho_{n-1}^{NM}\left(\frac{v_n^{NM}-v_{n-1}^{NM}}{h}\right)^2 + \sum_{n=1}^{N} hg_n v_n^{NM} = 0. \tag{60}$$

Proof. We multiply equation (52) with hv_n^{NM} and sum up. By using the boundary conditions (54), the desired equation follows immediately. □

PROPOSITION 5 Let $(\rho^{NM}, v^{NM}, w^{NM})$ be a local in time solution of problem (51)–(58). Then in the maximal existence interval we have

$$\sum_{n=1}^{N} h\frac{\mathrm{d}}{\mathrm{d}t}\rho_{n-1}^{NM}\eta(w_n^{NM}) + \sum_{n=1}^{N} h(\rho_{n-1}^{NM})^2\eta(w_n^{NM})\frac{v_n^{NM}-v_{n-1}^{NM}}{h} = 0. \tag{61}$$

Proof. Multiplying (51) with $h\eta(w_n^{NM})$ we obtain the assertion by summation. □

Now, we define the energy function $t \mapsto \mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})(t)$ for the Approximate (N, M)-Problem. It represents the energy of the discrete scheme.

DEFINITION 6 We define

$$\begin{aligned}
&\mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})(t)\\
&\quad := \sum_{n=1}^{N} h\frac{1}{2}(v_n^{NM})^2(t) + \sum_{n=1}^{N} h\frac{1}{4}\left[4c_1\sum_{k=0}^{M}\frac{(w_n^{NM})^{2k+2}}{(2k+2)(2k+1)}\right](t)\\
&\qquad + c_2\sum_{n=1}^{N} h\frac{1}{4}[(1-w_n^{NM})(1+w_n^{NM})](t) + \sum_{n=1}^{N} h\frac{1}{4}\left[\frac{\tilde{\delta}^2}{2}\left(\frac{w_n^{NM}-w_{n-1}^{NM}}{h}\right)^2\right](t)\\
&\qquad + \sum_{n=1}^{N} h[\rho_{n-1}^{NM}\eta(w_n^{NM})](t)\\
&\qquad + \int_0^t \sum_{n=1}^{N} h\frac{1}{4}\lambda_{\tilde{\delta}}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}w_n^{NM}\right)^2 \mathrm{d}\tau + \int_0^t \sum_{n=1}^{N} h(\rho_{n-1}^{NM})^{\theta+1}\left(\frac{v_n^{NM}-v_{n-1}^{NM}}{h}\right)^2 \mathrm{d}\tau.
\end{aligned}$$

In the next theorem we show that the energy $\mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})$ of the discrete system is uniformly bounded on any bounded time interval $[0, T]$.

THEOREM 7 Let b_1 and b_2 be non-negative constants. There is an M^*, independent of T, such that for all $M \geqslant M^*$,

$$\mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})(t) \leqslant C^*(\rho_0, v_0, w_0, g, T) \tag{62}$$

and

$$\mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})(t) \geqslant -C,$$

for all $t \in [0, T]$.

Proof. We add all equations (59), (60) and (61). Next we integrate over $[0, t]$ and take assumptions (A.1)–(A.4) on the initial functions and g into account. Then we get

$$\begin{aligned}
&\mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})(t) \\
&\quad\leqslant \sum_{n=1}^{N} h\frac{1}{2}(v_n^{NM})^2(0) + \sum_{n=1}^{N} h\frac{1}{4}\left[4c_1 \sum_{k=0}^{M} \frac{(w_n^{NM})^{2k+2}}{(2k+2)(2k+1)}\right](0) \\
&\qquad + c_2 \sum_{n=1}^{N} h\frac{1}{4}[(1 - w_n^{NM})(1 + w_n^{NM})](0) + \sum_{n=1}^{N} h\frac{1}{4}\left[\frac{\tilde{\delta}^2}{2}\left(\frac{w_n^{NM} - w_{n-1}^{NM}}{h}\right)^2\right](0) \\
&\qquad + \sum_{n=1}^{N} h(\rho_{n-1}^{NM}\eta(w_n^{NM}))(0) + \int_0^t \sum_{n=1}^{N} hg_n^2 \,\mathrm{d}\tau + \int_0^t \sum_{n=1}^{N} h(v_n^{NM})^2 \,\mathrm{d}\tau \\
&\quad\leqslant C^*(\rho_0, v_0, w_0, g) + \int_0^t \sum_{n=1}^{N} h(v_n^{NM})^2 \,\mathrm{d}\tau.
\end{aligned}$$

From the Gronwall lemma we get the desired inequality $\mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})(t) \leqslant C^*(\rho_0, v_0, w_0, g, T)$. Further, because $\rho_{n-1}^{NM}\eta(w_n^{NM}) > 0$ and the ln-term approximation is positive (see (49)),

$$\begin{aligned}
&\mathcal{E}_{NM}(\rho^{NM}, v^{NM}, w^{NM})(t) \\
&\quad\geqslant \sum_{n=1}^{N} h\left[c_1 \sum_{k=0}^{M} \frac{(w_n^{NM})^{2k+2}}{(2k+2)(2k+1)}(t) + c_2\frac{1}{4}[(1 - w_n^{NM})(1 + w_n^{NM})](t)\right] \\
&\quad\geqslant \sum_{n=1}^{N} h\left[c_1 \sum_{k=2}^{M} \frac{(w_n^{NM})^{2k+2}}{(2k+2)(2k+1)}(t) + c_2\frac{1}{4} - \frac{3}{16}\frac{c_2^2}{c_1}\right] \\
&\quad\geqslant c_2\frac{1}{4} - \frac{3}{16}\frac{c_2^2}{c_1} \geqslant -C(c_1, c_2),
\end{aligned} \tag{63}$$

where $C(c_1, c_2) := \left|c_2\frac{1}{4} - \frac{3}{16}\frac{c_2^2}{c_1}\right|$. That is, $\mathcal{E}_{NM}$ is bounded from below. □

Further, we have

LEMMA 8 Let $N \in \mathbb{N}$. For all n, $1 \leqslant n \leqslant N$,

$$\liminf_{M\to\infty} \inf_{t\in[0,T]} (1 - w_n^{NM}(t)) \geqslant 0 \quad \text{and} \quad \liminf_{M\to\infty} \inf_{t\in[0,T]} (1 + w_n^{NM}(t)) \geqslant 0.$$

Proof. We prove the assertion by contradiction. We assume that there exist $\tilde{n} \in \{1, \ldots, N\}$ and $\tilde{\varepsilon} > 0$, and for all M, there are $\tilde{M}$ with $\tilde{M} \geqslant M$ and $\tilde{t} \in [0, T]$, such that $w_{\tilde{n}}(\tilde{t}) > 1 + \tilde{\varepsilon}$. Then from

Theorem 7 (see estimates (62) and (63)), we know

$$\begin{aligned}C^*(\rho_0, v_0, w_0, g, T) &\geqslant \mathcal{E}_{N\tilde{M}}(\rho^{N\tilde{M}}, v^{N\tilde{M}}, w^{N\tilde{M}})(\tilde{t})\\ &\geqslant \sum_{\substack{n=1\\ n\neq\tilde{n}}}^{N} h\Bigg[c_1 \sum_{k=0}^{\tilde{M}} \frac{(w_n^{N\tilde{M}})^{2k+2}}{(2k+2)(2k+1)}(\tilde{t}) + c_2\frac{1}{4}[(1-w_n^{N\tilde{M}})(1+w_n^{N\tilde{M}})](\tilde{t})\Bigg]\\ &\quad + h\Bigg[c_1 \sum_{k=2}^{\tilde{M}} \frac{(w_{\tilde{n}}^{N\tilde{M}})^{2k+2}}{(2k+2)(2k+1)}(\tilde{t}) + c_2\frac{1}{4} - \frac{3}{16}\frac{c_2^2}{c_1}\Bigg]\\ &> -\frac{N-1}{N}C + \frac{1}{N}\Bigg[c_1 \sum_{k=2}^{M} \frac{(1+\tilde{\varepsilon})^{2k+2}}{(2k+2)(2k+1)} + c_2\frac{1}{4} - \frac{3}{16}\frac{c_2^2}{c_1}\Bigg].\end{aligned}$$

The left-hand side depends only on (ρ_0, v_0, w_0), g and T, that is, $C^*(\rho_0, v_0, w_0, g, T)$ is fixed. But it is possible to choose M so that the sum on the right-hand side exceeds $C^*(\rho_0, v_0, w_0, g, T)$. Therefore $w_{\tilde{n}}^{N\tilde{M}}(t) \leqslant 1 + \tilde{\varepsilon}$.

In the same way we can prove that $w_{\tilde{n}}^{N\tilde{M}}(t) \geqslant -1 - \tilde{\varepsilon}$. □

REMARK 9 Lemma 8 equivalently reads: For all $N \in \mathbb{N}$ and all $\varepsilon > 0$ there exists an $M^*(N, \varepsilon)$ such that for all $M \geqslant M^*(N, \varepsilon)$,

$$1 - w_n^{NM}(t) \geqslant -\varepsilon \quad \text{and} \quad 1 + w_n^{NM}(t) \geqslant -\varepsilon$$

for all $t \in [0, T]$, $n \in \{1, \ldots, N\}$.

From now on we fix $\varepsilon < \kappa_1$. For each $N \in \mathbb{N}$ we choose $M = M(N)$ large enough so that

$$1 - w_n^{NM(N)}(t) \geqslant -\varepsilon \quad \text{and} \quad 1 + w_n^{NM(N)}(t) \geqslant -\varepsilon$$

for all $t \in [0, T]$ and $n \in \{1, \ldots, N\}$.

NOTATION 10 From now on (up to the end of Section 6) we skip N and M in the notation of the solution of the Approximate (N, M)-Problem and write

$$(\rho_{n-1}, v_n, w_n) = (\rho_{n-1}^{NM(N)}, v_n^{NM(N)}, w_n^{NM(N)}). \qquad \square$$

So we have $1 - w_n \geqslant -\varepsilon$ and $1 + w_n \geqslant -\varepsilon$. As a consequence of Theorem 7 and Lemma 8, we obtain

LEMMA 11 For any given $N \in \mathbb{N}$, system (51)–(58) has a unique global solution (ρ_{n-1}, v_n, w_n) for $t \geqslant 0$. Moreover, $\rho_{n-1} > 0$ and $-1 - \varepsilon \leqslant w_n \leqslant 1 + \varepsilon$ for $t \geqslant 0$.

REMARK 12 From Lemma 11 and property (50) we know that

$$\eta(w_n) = b_1\frac{1}{2}(1 - w_n) + b_2\frac{1}{2}(1 + w_n) = b_d\frac{1}{2}(1 - w_n) + b_2 > 0.$$

An important problem in the theory of compressible Navier–Stokes equations is to guarantee that the mass density stays finite.

THEOREM 13 There exists C independent of $N \in \mathbb{N}$ such that

$$\rho_{n-1}(t) \leqslant C \quad \text{for } t \in [0, T].$$

Proof. We know from (51) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_{n-1}^{\theta}) = \theta \rho_{n-1}^{\theta-1} \frac{\mathrm{d}}{\mathrm{d}t}\rho_{n-1} = -\theta \rho_{n-1}^{\theta+1} \frac{v_n - v_{n-1}}{h}. \tag{64}$$

Now, sum up (52) from $n = k$ to $n = N$, and multiply it with h. Taking (64) into account, we get the following important equation:

$$\sum_{n=k}^{N} h \frac{\mathrm{d}}{\mathrm{d}t} v_n - \rho_{k-1}^2 \left(b_d \frac{1}{2}(1 - w_k) + b_2 \right) + \rho_N^2 \left(b_d \frac{1}{2}(1 - w_{N+1}) + b_2 \right) + \sum_{n=k}^{N} h g_n$$
$$= \sum_{n=k}^{N} \left(-\frac{1}{\theta} \left[\frac{\mathrm{d}}{\mathrm{d}t}\rho_n^{\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\rho_{n-1}^{\theta} \right] \right) = \rho_N^{\theta+1} \frac{v_{N+1} - v_N}{h} + \frac{1}{\theta} \frac{\mathrm{d}}{\mathrm{d}t}\rho_{k-1}^{\theta}. \tag{65}$$

From the boundary condition (54), integration from 0 to t yields

$$\frac{1}{\theta}\rho_{k-1}^{\theta}(t) + \int_0^t \rho_{k-1}^2 \left(b_d \frac{1}{2}(1 - w_k) + b_2 \right) \mathrm{d}\tau$$
$$= \frac{1}{\theta}\rho_{k-1}^{\theta}(0) + \sum_{n=k}^{N} h v_n(t) - \sum_{n=k}^{N} h v_n(0) + \int_0^t \sum_{n=k}^{N} h g_n \, \mathrm{d}\tau$$
$$\leqslant \frac{1}{\theta}\rho_0^{\theta}((k-1)h) + \left(\sum_{n=1}^{N} h v_n^2(t) \right)^{1/2} + \left(\sum_{n=1}^{N} h v_0^2(nh) \right)^{1/2} + C \leqslant C,$$

because of (A.1), (A.4), Theorem 7 and estimate (63) in the proof of Theorem 7. Since $-1 - \varepsilon \leqslant w_k \leqslant 1 + \varepsilon$ by Lemma 8, $\rho_{k-1} > 0$, and $\theta > 0$ by (A.5), the left-hand side is greater than 0. Therefore, we have $\rho_{n-1}^{\theta}(t) \leqslant C$. □

PROPOSITION 14 Under assumptions (A.1)–(A.4), we have

$$\int_0^t \sum_{n=1}^{N} h \left| \frac{\mathrm{d}}{\mathrm{d}t}\rho_{n-1} \right|^2 \mathrm{d}\tau \leqslant C.$$

Proof. We multiply equation (51) with $h \frac{\mathrm{d}}{\mathrm{d}t}\rho_{n-1}$, sum it up from $n = 1$ to $n = N$ and integrate over $(0, t)$. By using Theorem 13, Young's inequality and the fact that

$$\int_0^t \sum_{n=1}^{N} h \rho_{n-1}^{\theta+1} \left(\frac{v_n - v_{n-1}}{h} \right)^2 \mathrm{d}\tau \leqslant C,$$

which is a consequence of Theorem 7, the assertion follows directly. □

LEMMA 15 Under assumptions (A.1)–(A.4), we have

$$\sum_{n=1}^{N} h \left| \frac{\mathrm{d}w_n}{\mathrm{d}t} \right|^2 (t) \leqslant C \quad \text{for } 0 \leqslant t \leqslant T, \tag{66}$$

$$\int_0^t \sum_{n=1}^N h\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n - w_{n-1}}{h}\right)\right|^2 \mathrm{d}\tau \leqslant C \qquad \text{for } 0 \leqslant t \leqslant T, \tag{67}$$

$$\sum_{n=1}^N h|a_{M(N)}(w_n)|^2(t) \leqslant C \qquad \text{for } 0 \leqslant t \leqslant T. \tag{68}$$

Proof. Differentiating equation (53) with respect to t gives

$$\lambda_{\tilde{\delta}}^{-1}\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_n = -2c_1 a'_{M(N)}(w_n)\frac{\mathrm{d}}{\mathrm{d}t}w_n + 2c_2\frac{\mathrm{d}}{\mathrm{d}t}w_n + 2b_d\frac{\mathrm{d}}{\mathrm{d}t}\rho_{n-1} + \tilde{\delta}^2\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_{n+1}-w_n}{h} - \frac{w_n - w_{n-1}}{h}\right).$$

We multiply the equation with $h\frac{\mathrm{d}w_n}{\mathrm{d}t}$ and sum it up over n from 1 to N. The last term on the right-hand side is integrated by parts over space in the discrete sense. The first term on the right-hand side can be neglected, since $a'_{M(N)} \geqslant 0$. Then we get

$$\frac{1}{2}\lambda_{\tilde{\delta}}^{-1}\sum_{n=1}^N h\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\frac{\mathrm{d}w_n}{\mathrm{d}t}\right|^2\right) + \tilde{\delta}^2\sum_{n=1}^N h\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n - w_{n-1}}{h}\right)\right|^2 \leqslant (2c_2 + b_d^2)\sum_{n=1}^N h\left|\frac{\mathrm{d}w_n}{\mathrm{d}t}\right|^2 + \sum_{n=1}^N h\left|\frac{\mathrm{d}\rho_{n-1}}{\mathrm{d}t}\right|^2.$$

We integrate over $[0, t]$ and infer

$$\frac{1}{2}\lambda_{\tilde{\delta}}^{-1}\sum_{n=1}^N h\left|\frac{\mathrm{d}w_n}{\mathrm{d}t}\right|^2(t) + \int_0^t \tilde{\delta}^2\sum_{n=1}^N h\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n - w_{n-1}}{h}\right)\right|^2 \mathrm{d}\tau \leqslant \frac{1}{2}\lambda_{\tilde{\delta}}^{-1}\sum_{n=1}^N h\left|\frac{\mathrm{d}w_n}{\mathrm{d}t}\right|^2(0) + C\int_0^t\sum_{n=1}^N h\left|\frac{\mathrm{d}w_n}{\mathrm{d}t}\right|^2\mathrm{d}\tau + \int_0^t\sum_{n=1}^N h\left|\frac{\mathrm{d}\rho_{n-1}}{\mathrm{d}t}\right|^2\mathrm{d}\tau.$$

Theorem 7, Proposition 14 and the Gronwall lemma imply that the right-hand side is bounded. Thus, the estimates (66) and (67) hold pointwise in $t \in [0, T]$.

Further, we conclude by multiplying (53) with $ha_{M(N)}(w_n)$ that

$$2c_1\sum_{n=1}^N h|a_{M(N)}(w_n)|^2(t) + \tilde{\delta}^2\sum_{n=1}^N h\left|\frac{w_n - w_{n-1}}{h}\right|^2 \underbrace{a'_{M(N)}(w_{n-1} + \beta_n(w_n - w_{n-1}))(t)}_{\geqslant 0} \leqslant \frac{1}{2\varepsilon}\sum_{n=1}^N h\lambda_{\tilde{\delta}}^{-2}\left|\frac{\mathrm{d}w_n}{\mathrm{d}t}\right|^2(t) + \frac{1}{2\varepsilon}\sum_{n=1}^N h(4c_2^2|w_n|^2(t) + 4b_d^2\rho_{n-1}^2(t)) + \frac{3\varepsilon}{2}\sum_{n=1}^N h|a_{M(N)}(w_n)|^2(t).$$

Choosing ε so small that $2c_1 - \frac{3}{2}\varepsilon > 0$, we conclude, by Lemma 8, Theorem 13 and inequality (66), that estimate (68) holds pointwise in $t \in [0, T]$. □

5. Positivity of mass density

With equations (12), (13), (17) we model natural phenomena in which the medium core has a free boundary, separating medium and vacuum. We are interested in whether inner vacuum regions can develop within the solid core. This is known as the *vacuum problem* (see [9]). Inner vacuum regions are qualitatively not the same as the initially given outer vacuum region. These inner cavities emerge as a result of vanishing material, still wetted with the material flow (see [9]), and therefore they have a completely different boundary behavior than outer vacuum regions.

In this section, we are going to establish the basic estimates in order to exclude such additional islands without mass in the inner core. Exactly, we want to show, in the discrete scheme, that $\rho_{n-1}(t) \geqslant C > 0$ for $t \in [0, T]$ (see Theorem 18).

LEMMA 16 Under the assumptions made above,

$$\sum_{n=1}^{N} h\rho_{n-1}^{\theta-1}(t) \leqslant C.$$

Proof. Multiplying equation (51) with $h\rho_{n-1}^{\theta-2}$ we get

$$\frac{1}{\theta-1} h \frac{\mathrm{d}}{\mathrm{d}t}(\rho_{n-1}^{\theta-1}) = -h\rho^{\theta} \frac{v_n - v_{n-1}}{h} \qquad \text{for all } n = 1, \dots, N.$$

Integrating from 0 to t and summing up over n from 1 to N we derive, by assumption (A.1) and Theorem 7,

$$\begin{aligned}\frac{1}{1-\theta}\sum_{n=1}^{N} h\rho_{n-1}^{\theta-1}(t) &\leqslant \frac{1}{1-\theta}\sum_{n=1}^{N} h\rho_{0}^{\theta-1}((n-1)h) + \frac{1}{2}\int_0^t \sum_{n=1}^{N} h\rho_{n-1}^{\theta-1}\,\mathrm{d}\tau \\ &\quad + \frac{1}{2}\int_0^t \sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left(\frac{v_n - v_{n-1}}{h}\right)^2 \mathrm{d}\tau \\ &\leqslant C(1/\underline{\rho}_0) + \frac{1}{2}\int_0^t \sum_{n=1}^{N} h\rho_{n-1}^{\theta-1}\,\mathrm{d}\tau + C.\end{aligned}$$

The assertion follows from the Gronwall lemma. □

The next lemma constitutes the central technical point in our calculation. If we write its assertion without our discretizing scheme in a continuous manner, it means that $\mathrm{d}\rho^{\theta}/\mathrm{d}z$ can be controlled in the L^2-norm.

LEMMA 17 Assuming (A.1)–(A.5) it follows that

$$\sum_{n=1}^{N} h\left(\frac{\rho_n^{\theta} - \rho_{n-1}^{\theta}}{h}\right)^2(t) \leqslant C.$$

Proof. From equation (52) and identity (64), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_n^{\theta} - \rho_{n-1}^{\theta}}{h}\right)(t) = -\theta\left[\frac{\mathrm{d}}{\mathrm{d}t}v_n + \frac{-\rho_n^2 w_{n+1} + \rho_{n-1}^2 w_n}{h}\tfrac{1}{2}b_d + \frac{\rho_n^2 - \rho_{n-1}^2}{h}\left(\tfrac{1}{2}b_d + b_2\right) + g_n\right](t).$$

Multiplying with $\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)^2 = -\theta\frac{\mathrm{d}v_n}{\mathrm{d}t}\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h} + \theta\left(\frac{\rho_n^2 w_{n+1}-\rho_{n-1}^2 w_n}{h}\tfrac{1}{2}b_d - \frac{\rho_n^2-\rho_{n-1}^2}{h}\left(\tfrac{1}{2}b_d+b_2\right)\right)\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h} - \theta g_n\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}.$$

Summing up from $n=1$ to $n=N$, multiplying with h, and integrating from 0 to t, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^N h\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)^2(t)\\
&\quad= \frac{1}{2}\sum_{n=1}^N h\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)^2(0) - \int_0^t\sum_{n=1}^N h\theta\frac{\mathrm{d}v_n}{\mathrm{d}t}\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\,\mathrm{d}\tau\\
&\qquad+\int_0^t\sum_{n=1}^N h\theta\left(\frac{\rho_n^2 w_{n+1}-\rho_{n-1}^2 w_n}{h}\tfrac{1}{2}b_d - \frac{\rho_n^2-\rho_{n-1}^2}{h}\left(\tfrac{1}{2}b_d+b_2\right)\right)\cdot\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\,\mathrm{d}\tau\\
&\qquad-\int_0^t\sum_{n=1}^N h\theta g_n\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\,\mathrm{d}\tau.
\end{aligned}$$

Using assumption (A.1) and partial integration we calculate

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^N h\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)^2(t) \leqslant C &- \sum_{n=1}^N h\left(\theta v_n\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)(t) + \sum_{n=1}^N h\left(\theta v_n\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)(0)\\
&+\int_0^t\sum_{n=1}^N h\theta v_n\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)\mathrm{d}\tau\\
&+\int_0^t\sum_{n=1}^N h\theta\left(\frac{\rho_n^2-\rho_{n-1}^2}{h}\tfrac{1}{2}b_d w_{n+1} + \frac{w_{n+1}-w_n}{h}\rho_{n-1}^2\tfrac{1}{2}b_d\right.\\
&\left.-\frac{\rho_n^2-\rho_{n-1}^2}{h}\left(\tfrac{1}{2}b_d+b_2\right)\right)\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)\mathrm{d}\tau\\
&-\int_0^t\sum_{n=1}^N h\theta g_n\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\,\mathrm{d}\tau.
\end{aligned}\tag{69}$$

Now, we deal with the second last summand on the right-hand side of (69). We split it into two terms. The first term is, using the mean value theorem,

$$\begin{aligned}
&\int_0^t\sum_{n=1}^N h\theta\frac{\rho_n^2-\rho_{n-1}^2}{h}\left(\tfrac{1}{2}b_d w_{n+1}-\left(\tfrac{1}{2}b_d+b_2\right)\right)\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)\mathrm{d}\tau\\
&\quad= +\int_0^t\sum_{n=1}^N h\,\underbrace{(b_d(w_{n+1}-1)-2b_2)}_{\leqslant 0\text{ (Remark 12)}}(\rho_{n-1}+\beta_n(\rho_n-\rho_{n-1}))^{2-\theta}\left(\frac{\rho_n^\theta-\rho_{n-1}^\theta}{h}\right)^2\mathrm{d}\tau,
\end{aligned}$$

where $0 < \beta_n < 1$. Further, by Young's inequality and Theorem 13, the second term becomes

$$\int_0^t \sum_{n=1}^N h\theta \tfrac{1}{2} b_d \frac{w_{n+1} - w_n}{h} \rho_{n-1}^2 \left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h} \right) \mathrm{d}\tau$$
$$\leqslant C \int_0^t \sum_{n=1}^N h \left(\frac{w_{n+1} - w_n}{h} \right)^2 \mathrm{d}\tau + C \int_0^t \sum_{n=1}^N h \left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h} \right)^2 \mathrm{d}\tau.$$

From Theorem 7 and the boundary conditions, this is

$$\leqslant C + C \int_0^t \sum_{n=1}^N h \left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h} \right)^2 \mathrm{d}\tau.$$

The last integral on the right-hand side of (69) is handled by using Young's inequality and assumption (A.4).

Then, under assumptions (A.1), (A.2) on the initial functions, inequality (69) becomes

$$\begin{aligned}
&\frac{1}{2} \sum_{n=1}^N h \left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h} \right)^2 (t) + \int_0^t \sum_{n=1}^N h\theta^2 v_n \frac{\mathrm{d}v_n}{\mathrm{d}t} \\
&\quad + \int_0^t \sum_{n=1}^N h(b_d(1 - w_{n+1}) + 2b_2)(\rho_{n-1} + \beta_n(\rho_n - \rho_{n-1}))^{2-\theta} \left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h} \right)^2 \mathrm{d}\tau \\
&\leqslant C + \frac{\theta^2}{2\varepsilon} \sum_{n=1}^N h v_n^2(t) + \frac{\varepsilon}{2} \sum_{n=1}^N h \left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h} \right)^2 (t) + C \\
&\quad + \int_0^t \sum_{n=1}^N h\theta^2 v_n \left(\frac{\rho_n^2 w_{n+1} - \rho_{n-1}^2 w_n}{h} \tfrac{1}{2} b_d - \frac{\rho_n^2 - \rho_{n-1}^2}{h} \left(\tfrac{1}{2} b_d + b_2 \right) \right) \mathrm{d}\tau \\
&\quad + C + C \int_0^t \sum_{n=1}^N h \left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h} \right)^2 \mathrm{d}\tau.
\end{aligned} \tag{70}$$

On the left-hand side of (70) the second summand equals

$$\theta^2 \sum_{n=1}^N h \frac{1}{2} v_n^2(t) - \theta^2 \sum_{n=1}^N h \frac{1}{2} v_0^2(nh).$$

Now, we evaluate the fifth summand on the right-hand side. We split it into

$$\int_0^t \sum_{n=1}^N h\theta^2 v_n \frac{\rho_{n-1}^2 (w_{n+1} - w_n)}{h} \tfrac{1}{2} b_d \,\mathrm{d}\tau + \int_0^t \sum_{n=1}^N h\theta^2 v_n \left(w_{n+1} \tfrac{1}{2} b_d - (\tfrac{1}{2} b_d + b_2) \right) \frac{\rho_n^2 - \rho_{n-1}^2}{h} \,\mathrm{d}\tau.$$

The first part satisfies

$$\int_0^t \sum_{n=1}^N h\theta^2 v_n \frac{\rho_{n-1}^2 (w_{n+1} - w_n)}{h} \tfrac{1}{2} b_d \,\mathrm{d}\tau \leqslant C \int_0^t \sum_{n=1}^N h \left(\frac{w_{n+1} - w_n}{h} \right)^2 \mathrm{d}\tau + C \int_0^t \sum_{n=1}^N h\rho_{n-1}^4 v_n^2 \,\mathrm{d}\tau.$$

The remaining part becomes

$$\begin{aligned}
&\int_0^t \sum_{n=1}^N h\theta^2 v_n\big(w_{n+1}\tfrac{1}{2}b_d - (\tfrac{1}{2}b_d + b_2)\big)\frac{\rho_n^2 - \rho_{n-1}^2}{h}\,\mathrm{d}\tau\\
&\quad = \int_0^t \sum_{n=1}^N h\theta(b_d(w_{n+1} - 1) - 2b_2)(\rho_{n-1} + \beta_n(\rho_n - \rho_{n-1}))^{2-\theta}\cdot\left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h}\right)v_n\,\mathrm{d}\tau\\
&\quad \leqslant C\int_0^t \sum_{n=1}^N h\left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h}\right)^2\mathrm{d}\tau + \int_0^t \sum_{n=1}^N h(\rho_{n-1} + \beta_n(\rho_n - \rho_{n-1}))^{4-2\theta}v_n^2\,\mathrm{d}\tau.
\end{aligned}$$

We choose ε very small, recall that $4 - 2\theta > 0$ and apply Theorems 7 and 13. Then inequality (70) becomes

$$\sum_{n=1}^N h\left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h}\right)^2(t) \leqslant C + C\int_0^t \sum_{n=1}^N h\left(\frac{\rho_n^\theta - \rho_{n-1}^\theta}{h}\right)^2\mathrm{d}\tau.$$

The Gronwall lemma yields the assertion. □

THEOREM 18 Let the assumptions above, that is, (A.1)–(A.5), be satisfied. Then

$$\rho_{n-1}(t) \geqslant C > 0,$$

where C is independent of h.

Proof. It is the same argument as in [19, Proposition 6]. We give the proof for completeness. There is an $\tilde{n}$ $(1 \leqslant \tilde{n} \leqslant N)$ with $\rho_{\tilde{n}-1} = \max_{1\leqslant n\leqslant N}\rho_{n-1}$. Due to Lemma 16 and $\theta - 1 < 0$, we get $\rho_{\tilde{n}-1}^{\theta-1} \leqslant \sum_{n=1}^N h\rho_{n-1}^{\theta-1} \leqslant C$. Applying Lemma 17 and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\rho_{n-1}^{\theta-1} &= \rho_{\tilde{n}-1}^{\theta-1} + \sum_{k=\tilde{n}}^{n-1}(\rho_k^{\theta-1} - \rho_{k-1}^{\theta-1})\\
&= \rho_{\tilde{n}-1}^{\theta-1} + \sum_{k=\tilde{n}}^{n-1} h\frac{\theta-1}{\theta}(\rho_{k-1} + \beta_k(\rho_k - \rho_{k-1}))^{-1}\frac{\rho_k^\theta - \rho_{k-1}^\theta}{h}\\
&\leqslant C + C\Big(\sum_{n=1}^N h(\rho_{n-1} + \beta_n(\rho_n - \rho_{n-1}))^{-2}\Big)^{1/2}\bigg(\sum_{n=1}^N h\left(\frac{\rho_k^\theta - \rho_{k-1}^\theta}{h}\right)^2\bigg)^{1/2}\\
&\leqslant C + C\Big(\sum_{n=1}^N h(\rho_{n-1} + \beta_n(\rho_n - \rho_{n-1}))^{-2}\Big)^{1/2}. \qquad (71)
\end{aligned}$$

By Lemma 16, we have

$$\sum_{n=1}^N h(\rho_{n-1} + \beta_n(\rho_n - \rho_{n-1}))^{-2} \leqslant \max_{1\leqslant n\leqslant N}\rho_{n-1}^{-\theta-1}\sum_{n=1}^N h\rho_{n-1}^{\theta-1} \leqslant C\max_{1\leqslant n\leqslant N}\rho_{n-1}^{-\theta-1}.$$

Inserting this in (71) yields

$$\rho_{n-1}^{\theta-1} \leqslant C + C \max_{1\leqslant n\leqslant N} \rho_{n-1}^{-(\theta+1)/2}.$$

By assumption (A.5), $0 < \theta < 1/3$, hence $1-\theta > (\theta+1)/2$, we derive that there is a positive constant C, depending on T, for which $\rho_{n-1} \geqslant C$. □

We have settled the vacuum problem, and so it is guaranteed that no inner cavities can arise within the medium core. Now, we want to prove an additional feature of the movement from the free boundary which separates the medium core and the vacuum. A consequence of Lemma 19 below will be that the free boundary travels with finite velocity.

LEMMA 19 Under assumptions (A.1)–(A.5), we have

$$\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2(t) + \int_0^t \sum_{n=1}^{N} h\left[\rho_{n-1}^{\theta+1}\left(\frac{\mathrm{d}v_n/\mathrm{d}t - \mathrm{d}v_{n-1}/\mathrm{d}t}{h}\right)^2\right](\tau)\,\mathrm{d}\tau \leqslant C, \tag{72}$$

$$\left|\left(\rho_{n-1}^{\theta+1}\frac{v_n - v_{n-1}}{h}\right)(t)\right| \leqslant C. \tag{73}$$

Proof. Differentiate equation (52) with respect to t and multiply it by $h\mathrm{d}v_n/\mathrm{d}t$. Then summing over $n=1$ to $n=N$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2\right] - \sum_{n=1}^{N} h\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}b_d\frac{\rho_n^2 w_{n+1} - \rho_{n-1}^2 w_n}{h} - \left(\tfrac{1}{2}b_d + b_2\right)\frac{\rho_n^2 - \rho_{n-1}^2}{h}\right)\frac{\mathrm{d}v_n}{\mathrm{d}t}\\
&= \sum_{n=1}^{N} h\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho_n^{\theta+1}\frac{v_{n+1}-v_n}{h} - \rho_{n-1}^{\theta+1}\frac{v_n - v_{n-1}}{h}\right]\frac{1}{h}\frac{\mathrm{d}v_n}{\mathrm{d}t} - \sum_{n=1}^{N} hg_n\frac{\mathrm{d}v_n}{\mathrm{d}t}\\
&= -\sum_{n=1}^{N} h\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{n-1}^{\theta+1}\frac{v_n - v_{n-1}}{h}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_N^{\theta+1}\frac{v_{N+1}-v_N}{h}\right)\frac{\mathrm{d}v_N}{\mathrm{d}t} - \sum_{n=1}^{N} hg_n\frac{\mathrm{d}v_n}{\mathrm{d}t}\\
&= -\sum_{n=1}^{N} h(\theta+1)\rho_{n-1}^{\theta}\frac{\mathrm{d}\rho_{n-1}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\\
&\quad - \sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\right|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_N^{\theta+1}\frac{v_{N+1}-v_N}{h}\right)\frac{1}{h}\frac{\mathrm{d}v_N}{\mathrm{d}t} - \sum_{n=1}^{N} hg_n\frac{\mathrm{d}v_n}{\mathrm{d}t}.
\end{aligned}$$

Further, we have

$$\begin{aligned}
&-\sum_{n=1}^{N} h\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}b_d\frac{\rho_n^2 w_{n+1} - \rho_{n-1}^2 w_n}{h} - \left(\tfrac{1}{2}b_d + b_2\right)\frac{\rho_n^2 - \rho_{n-1}^2}{h}\right)\frac{\mathrm{d}v_n}{\mathrm{d}t}\\
&\qquad = +\sum_{n=1}^{N} h\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}b_d\rho_{n-1}^2 w_n - \left(\tfrac{1}{2}b_d + b_2\right)\rho_{n-1}^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\\
&\qquad\quad - \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}b_d\rho_N^2 w_{N+1} - \left(\tfrac{1}{2}b_d + b_2\right)\rho_N^2\right)\frac{\mathrm{d}v_N}{\mathrm{d}t}.
\end{aligned}$$

Altogether, by using the boundary condition (54) we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2\right] + \sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\right|^2 \\
&\quad = -\sum_{n=1}^{N} h\left(\tfrac{1}{2}b_d 2\rho_{n-1}\frac{\mathrm{d}\rho_{n-1}}{\mathrm{d}t}w_n + \tfrac{1}{2}b_d\rho_{n-1}^2\frac{\mathrm{d}w_n}{\mathrm{d}t} - \left(\tfrac{1}{2}b_d + b_2\right)2\rho_{n-1}\frac{\mathrm{d}\rho_{n-1}}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right) \\
&\qquad + \sum_{n=1}^{N} h(\theta+1)\rho_{n-1}^{\theta+2}\left(\frac{v_n - v_{n-1}}{h}\right)^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right) - \sum_{n=1}^{N} hg_n\frac{\mathrm{d}v_n}{\mathrm{d}t}.
\end{aligned}
$$

By Young's inequality,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2\right] + \sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\right|^2 \\
&\leqslant \frac{\varepsilon}{2}\sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\right|^2 + \frac{1}{2\varepsilon}\sum_{n=1}^{N} h\rho_{n-1}^{5-\theta}\underbrace{(b_d(w_n - 1) - 2b_2)^2}_{\leqslant C \text{ (Remark 12)}}\left|\frac{v_n - v_{n-1}}{h}\right|^2 \\
&\quad + \frac{\varepsilon}{2}\sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\right|^2 + \frac{1}{2\varepsilon}\sum_{n=1}^{N} h\tfrac{1}{4}b_d^2\rho_{n-1}^{3-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}t}w_n\right|^2 \\
&\quad + \frac{\varepsilon}{2}\sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\right|^2 + \frac{1}{2\varepsilon}\sum_{n=1}^{N} h(\theta+1)^2\rho_{n-1}^{3+\theta}\left|\frac{v_n - v_{n-1}}{h}\right|^4 \\
&\quad + \frac{1}{2}\sum_{n=1}^{N} hg_n^2 + \frac{1}{2}\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2. \qquad (74)
\end{aligned}
$$

From (64), (65), and (54) we can conclude

$$
\rho_{n-1}^{2+2\theta}\left|\frac{v_n - v_{n-1}}{h}\right|^2 \leqslant C\left(\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2 + C\right),
$$

and $\rho_{n-1}^{5-\theta} \leqslant C\rho_{n-1}^{1+\theta}$ by Theorem 13. Further, by Remark 12, Theorem 13, and Theorem 18, we have $1/C \leqslant \rho_{n-1} \leqslant C$ and $-1-\varepsilon \leqslant w_n \leqslant 1+\varepsilon$.

Inserting this in (74), by choosing ε small, we apply Lemma 15 to derive

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2\right] + C\sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_n - v_{n-1}}{h}\right)\right|^2 \\
&\qquad \leqslant C\left(\sum_{n=1}^{N} h\rho_{n-1}^{\theta+1}\left|\frac{v_n - v_{n-1}}{h}\right|^2\right)\left(C\sum_{n=1}^{N} h\left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2 + C\right).
\end{aligned}
$$

Due to Theorem 7, the first factor on the right-hand side is in $L^1(0, T)$. Then the Gronwall lemma yields the assertion. □

LEMMA 20 Under assumptions (A.1)–(A.5), we have

$$\left|\frac{w_n - w_{n-1}}{h}(t)\right| \leqslant C. \tag{75}$$

Proof. For $k = 1, \ldots, N$, equation (53) has the form

$$\lambda_{\tilde{\delta}}^{-1} \frac{\mathrm{d}}{\mathrm{d}t} w_k = -2c_1 a_{M(N)}(w_k) + 2c_2 w_k + 2\rho_{k-1} b_d + \tilde{\delta}^2 \frac{1}{h}\left[\frac{w_{k+1} - w_k}{h} - \frac{w_k - w_{k-1}}{h}\right]. \tag{76}$$

We multiply (76) with h and sum it up from $k = 1$ to $k = n$ to obtain

$$\lambda_{\tilde{\delta}}^{-1} \sum_{k=1}^{n} h \frac{\mathrm{d}}{\mathrm{d}t} w_k = -2c_1 \sum_{k=1}^{n} h a_{M(N)}(w_k) + 2c_2 \sum_{k=1}^{n} h w_k + 2b_d \sum_{k=1}^{n} h\rho_{k-1} + \tilde{\delta}^2 \frac{w_{n+1} - w_n}{h}.$$

By using Lemma 15, inequalities (66) and (68), Lemma 8, and Theorem 13, we get the assertion. □

PROPOSITION 21 We have

(i) $$\sum_{n=1}^{N} h \left|\frac{\rho_n(t) - \rho_{n-1}(t)}{h}\right|^2 \leqslant C,$$

(ii) $$\sum_{n=1}^{N} h \left|\frac{v_n(t) - v_{n-1}(t)}{h}\right|^2 \leqslant C,$$

(iii) $$\sum_{n=1}^{N} h \left|\frac{w_n(t) - w_{n-1}(t)}{h}\right|^2 \leqslant C,$$

(iv) $$\sum_{n=1}^{N} h \left|\frac{1}{h}\left(\rho_n^{\theta+1} \frac{v_{n+1} - v_n}{h}(t) - \rho_{n-1}^{\theta+1} \frac{v_n - v_{n-1}}{h}(t)\right)\right|^2 \leqslant C,$$

(v) $$\sum_{n=1}^{N} h \left|\frac{1}{h}\left(\frac{w_{n+1} - w_n}{h}(t) - \frac{w_n - w_{n-1}}{h}(t)\right)\right|^2 \leqslant C.$$

Proof. (i) is valid because of Lemma 17, Theorem 13 and Theorem 18. (ii) is due to (73) and Theorem 18. We have shown (iii) in Theorem 7. Further, we prove (iv): From (52), Lemma 19, assumption (A.4), and items (i) and (iii), we have

$$\begin{aligned}
&\sum_{n=1}^{N} h \left|\frac{1}{h}\left(\rho_n^{\theta+1} \frac{v_{n+1} - v_n}{h}(t) - \rho_{n-1}^{\theta+1} \frac{v_n - v_{n-1}}{h}(t)\right)\right|^2 \\
&\quad \leqslant 2 \sum_{n=1}^{N} h \left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2 \\
&\qquad + 2 \sum_{n=1}^{N} h \left|\frac{\rho_n^2 (b_d \frac{1}{2}(1 - w_{n+1}) + b_2) - \rho_{n-1}^2 (b_d \frac{1}{2}(1 - w_n) + b_2)}{h}\right|^2 + 2 \sum_{n=1}^{N} h |g_n|^2 \\
&\quad \leqslant 2 \sum_{n=1}^{N} h \left|\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2 + C \sum_{n=1}^{N} h \left|\frac{\rho_n - \rho_{n-1}}{h}\right|^2 + C \sum_{n=1}^{N} h \left|\frac{w_{n+1} - w_n}{h}\right|^2 + 2 \sum_{n=1}^{N} h |g_n|^2 \\
&\quad \leqslant C.
\end{aligned}$$

We show (v):

$$\sum_{n=1}^{N} h\left|\frac{1}{h}\left(\frac{w_{n+1}-w_n}{h}(t)-\frac{w_n-w_{n-1}}{h}(t)\right)\right|^2$$
$$\leqslant C\sum_{n=1}^{N} h\left|\frac{\mathrm{d}w_n}{\mathrm{d}t}\right|^2+C\sum_{n=1}^{N} h|a_{M(N)}(w_n)|^2+C\sum_{n=1}^{N} h|w_n|^2+C\sum_{n=1}^{N} h|\rho_{n-1}|^2$$
$$\leqslant C,$$

by Lemma 15. □

PROPOSITION 22 We have

(i) $$\sum_{n=1}^{N} h|\rho_{n-1}(t)-\rho_{n-1}(s)|^2\leqslant C|t-s|^2,$$

(ii) $$\sum_{n=1}^{N} h|v_n(t)-v_n(s)|^2\leqslant C|t-s|^2,$$

(iii) $$\sum_{n=1}^{N} h\left|\rho_{n-1}^{\theta+1}\frac{v_n-v_{n-1}}{h}(t)-\rho_{n-1}^{\theta+1}\frac{v_n-v_{n-1}}{h}(s)\right|^2\leqslant C|t-s|,$$

(iv) $$\sum_{n=1}^{N} h|w_n(t)-w_n(s)|^2\leqslant C|t-s|^2,$$

(v) $$\sum_{n=1}^{N} h\left|\frac{w_n-w_{n-1}}{h}(t)-\frac{w_n-w_{n-1}}{h}(s)\right|^2\leqslant C|t-s|.$$

Proof. From Lemma 15 it follows that $\sum_{n=1}^{N} h\left|\frac{\mathrm{d}}{\mathrm{d}t}w_n\right|^2\leqslant C$, which implies (iv). We show (v):

$$\sum_{n=1}^{N} h\left|\frac{w_n-w_{n-1}}{h}(t)-\frac{w_n-w_{n-1}}{h}(s)\right|^2\leqslant\int_0^T\sum_{n=1}^{N} h\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n-w_{n-1}}{h}\right)\right|^2\mathrm{d}\tau\,|t-s|^2\leqslant C|t-s|^2$$

by Lemma 15, (67). □

6. Convergence of approximate solutions to a global weak solution

We interpolate ρ_{n-1}, v_n, and w_n in the following way. For any $t\in[0,T]$ define $\rho_h(t,z)$, $v_h(t,z)$, and $w_h(t,z)$ for $(n-1)h\leqslant z<nh$ by

$$\left.\begin{aligned}
&\rho_h(t,z)=\rho_{n-1}(t),\\
&v_h(t,z)=\frac{1}{h}[(z-(n-1)h)v_n(t)+(nh-z)v_{n-1}(t)],\\
&w_h(t,z)=\frac{1}{h}[(z-(n-1)h)w_n(t)+(nh-z)w_{n-1}(t)],
\end{aligned}\right\}\tag{77}$$

so that for $(n-1)h \leqslant z < nh$,

$$\partial_z v_h(t,z) = \frac{1}{h}(v_n(t) - v_{n-1}(t)),$$
$$\partial_z w_h(t,z) = \frac{1}{h}(w_n(t) - w_{n-1}(t)).$$

We define

$$v_h^R(t,z) = v_n(t), \quad v_h^L(t,z) = v_{n-1}(t) \quad \text{for } (n-1)h \leqslant z < nh,$$
$$w_h^R(t,z) = w_n(t), \quad w_h^L(t,z) = w_{n-1}(t) \quad \text{for } (n-1)h \leqslant z < nh,$$
$$g_h^R(t,z) = g_n(t), \quad g_h^L(t,z) = g_{n-1}(t) \quad \text{for } (n-1)h \leqslant z < nh,$$

and, further,

$$a_h(x) := a_{M([1/h])}(x).$$

From (73) and (75) we have

$$|v_h - v_h^R| = \mathcal{O}(h), \qquad |v_h - v_h^L| = \mathcal{O}(h), \tag{78}$$
$$|w_h - w_h^R| = \mathcal{O}(h), \qquad |w_h - w_h^L| = \mathcal{O}(h). \tag{79}$$

We define, with $\mu_h(t,z) = (\rho_h(t,z))^\theta$,

$$u_h(t,z) := \mu_h(t,z)\rho_h(t,z)\partial_z v_h(t,z). \tag{80}$$

By (77) for $(n-1)h \leqslant z < nh$ we obtain

$$u_h(t,z) = \rho_{n-1}^{\theta+1}(t)\frac{v_n(t) - v_{n-1}(t)}{h}.$$

This yields

LEMMA 23 The approximate solution (ρ_h, v_h, w_h) constructed above satisfies for all (t,z):

(i) $1/C \leqslant \rho_h(t,z) \leqslant C$,

(ii) $|v_h(t,z)| \leqslant C$,

(iii) $-1 \leqslant w_h(t,z) \leqslant 1$,

(iv) $|\partial_z v_h(t,z)| \leqslant C$,

(v) $|u_h(t,z)| \leqslant C$,

(vi) $|\partial_z w_h(t,z)| \leqslant C$. □

From Propositions 21 and 22 we directly derive Lemmas 24 and 25 below in the continuous case.

LEMMA 24 Let $r \in \mathbb{R}_+$. We have

(i) $\|\rho_h(t,\cdot+r) - \rho_h(t,\cdot)\|^2_{L^2(0,1-r)} \leqslant Cr$,

(ii) $\|v_h^R(t,\cdot+r) - v_h^R(t,\cdot)\|^2_{L^2(0,1-r)} \leqslant Cr$,

(iii) $\|w_h^R(t,\cdot+r) - w_h^R(t,\cdot)\|^2_{L^2(0,1-r)} \leqslant Cr$,

(iv) $\|u_h(t,\cdot+r) - u_h(t,\cdot)\|^2_{L^2(0,1-r)} \leqslant Cr$,

(v) $\|\partial_z w_h(t,\cdot+r) - \partial_z w_h(t,\cdot)\|^2_{L^2(0,1-r)} \leqslant Cr$.

Proof. (i) With the help of Proposition 21(i), we calculate

$$\begin{aligned}\int_0^{1-r} |\rho_h(t,z+r)-\rho_h(t,z)|^2\,\mathrm{d}z &\leqslant \sum_{n=1}^{N-[rN]} h|\rho_{n+[rN]}(t)-\rho_{n-1}(t)|^2\\ &\leqslant \sum_{n=1}^{N-[rN]} h\sum_{k=0}^{[rN]} h\left|\frac{\rho_{n+k}(t)-\rho_{n+k-1}(t)}{h}\right|^2\\ &= \sum_{k=0}^{[rN]} h\sum_{n=1}^{N-[rN]} h\left|\frac{\rho_{n+k}(t)-\rho_{n+k-1}(t)}{h}\right|^2 \leqslant rC.\end{aligned}$$

(ii)–(v) Use Proposition 21(ii)–(v), analogously to the proof of (i). □

LEMMA 25 We have

(i) $$\int_0^1 |\rho_h(t,z)-\rho_h(s,z)|^2\,\mathrm{d}z \leqslant C|t-s|^2,$$

(ii) $$\int_0^1 |v_h^R(t,z)-v_h^R(s,z)|^2\,\mathrm{d}z \leqslant C|t-s|^2,$$

(iii) $$\int_0^1 \left|u_h(t,z)-u_h(s,z)\right|^2\mathrm{d}z \leqslant C|t-s|,$$

(iv) $$\int_0^1 |w_h^R(t,z)-w_h^R(s,z)|^2\,\mathrm{d}z \leqslant C|t-s|^2,$$

(v) $$\int_0^1 \left|\partial_z w_h(t,z)-\partial_z w_h(s,z)\right|^2\mathrm{d}z \leqslant C|t-s|.$$

Proof. (i) We estimate, with the use of Proposition 22(i),

$$\int_0^1 |\rho_h(t,z)-\rho_h(s,z)|^2\,\mathrm{d}z \leqslant \sum_{n=1}^{N} h|\rho_{n-1}(t)-\rho_{n-1}(s)|^2 \leqslant C|t-s|^2.$$

To prove (ii)–(v) use Proposition 22(ii)–(v). □

THEOREM 26 There is a subsequence $h\to 0$ such that on $\{(t,z) : 0<z<1, t>0\}$:

(i) $\rho_h \to \rho$ almost everywhere;
(ii) $v_h^R \to v$ almost everywhere;
(iii) $v_h \to v$ almost everywhere;
(iv) $w_h^R \to w$ almost everywhere;
(v) $w_h \to w$ almost everywhere;
(vi) $u_h \to u$ almost everywhere;
(vii) $\partial_z w_h \to \partial_z w$ almost everywhere.

Proof. (i) is a direct consequence of the following statement:

$$\rho_h \to \rho \quad \text{in } L^2(0,T;L^2(0,1)) \tag{81}$$

for a subsequence $h\to 0$.

First, for an arbitrary sequence $h \to 0$, we will show that $\{\rho_h\}_h$ is a compact set in the Banach space $L^2(0, T; L^2(0, 1))$. By the Riesz theorem, classifying compact sets in L^2, we must show two properties. The first property,

$$\sup_h \int_0^T \int_0^1 |\rho_h(t, z)|^2 \, \mathrm{d}z \, \mathrm{d}t < \infty,$$

follows directly from Lemma 23(i). For the second property,

$$\sup_h \int_0^{T-s_1} \int_0^{1-s_2} |\rho_h(t + s_1, z + s_2) - \rho_h(t, z)|^2 \, \mathrm{d}z \, \mathrm{d}t \to 0$$

for $(s_1, s_2) \in \mathbb{R}^2$ with $|(s_1, s_2)| \to 0$, we calculate

$$\begin{aligned} \int_0^{T-s_1} \int_0^{1-s_2} & |\rho_h(t + s_1, z + s_2) - \rho_h(t, z)|^2 \, \mathrm{d}z \, \mathrm{d}t \\ & \leqslant \int_0^{T-s_1} \int_0^{1-s_2} |\rho_h(t + s_1, z + s_2) - \rho_h(t, z + s_2)|^2 \, \mathrm{d}z \, \mathrm{d}t \\ & \quad + \int_0^{T-s_1} \int_0^{1-s_2} |\rho_h(t, z + s_2) - \rho_h(t, z)|^2 \, \mathrm{d}z \, \mathrm{d}t. \end{aligned} \tag{82}$$

For the first summand on the right-hand side of (82), we use Lemma 25:

$$\begin{aligned} \int_0^{T-s_1} \int_0^{1-s_2} & |\rho_h(t + s_1, z + s_2) - \rho_h(t, z + s_2)|^2 \, \mathrm{d}z \, \mathrm{d}t \\ & \leqslant C|s_1|^2 (T - s_1) \to 0 \quad \text{as } |(s_1, s_2)| \to 0. \end{aligned}$$

For the second summand, we apply Lemma 24 to get

$$\int_0^{T-s_1} \int_0^{1-s_2} |\rho_h(t, z + s_2) - \rho_h(t, z)|^2 \, \mathrm{d}z \, \mathrm{d}t \leqslant C(T - s_1)|s_2| \to 0 \quad \text{as } |(s_1, s_2)| \to 0.$$

Thus, the set $\{\rho_h\}_h$ is compact in $L^2(0, T; L^2(0, 1))$. Consequently, there is a subsequence $h_\ell \to 0$ such that ρ_{h_ℓ} converges to a limit function ρ in $L^2(0, T; L^2(0, 1))$. It follows directly that there is a further subsequence h_{ℓ_j} such that $\rho_{h_{\ell_j}} \to \rho$ almost everywhere on $\{(t, z) : 0 < z < 1, t > 0\}$.

(ii)–(vii) Because of Lemmas 23, 24, and 25, the functions v_h, w_h, u_h and $\partial_z w_h$ have the same features as the function ρ_h. Namely, by Lemma 23, they are all bounded, and by Lemmas 24 and 25, each of these functions is uniformly continuous in t and uniformly continuous in z, in the L^2-norm. Therefore, the theorem follows by the same argument as above. To save space, we avoid here a repetition. □

Now, we show that the limit functions are weak solutions. From Theorem 26 it follows directly that

$$\partial_t \rho + \rho^2 \partial_z v = 0 \quad \text{for almost all } z \in (0, 1) \text{ and for any } t \geqslant 0.$$

Let us show that the weak formulation (38) of (25) holds. Let $\varphi \in C_0^\infty((0, 1])$. We define $\varphi_n := \varphi(nh)$ and $\varphi_h(z) := \frac{1}{h}[(z - (n - 1)h)\varphi_n + (nh - z)\varphi_{n-1}]$ for $(n - 1)h \leqslant z < nh$. Further,

$\varphi_h^R(z) := \varphi_n$ and $\varphi_h^L(z) := \varphi_{n-1}$ for $(n-1)h \leqslant z < nh$. Multiplying (52) by $h\varphi_n = h\varphi(nh)$ and summing from $n = 1$ to $n = N$, we get

$$\begin{aligned} 0 = &\sum_{n=1}^{N} h\varphi_n \frac{\mathrm{d}}{\mathrm{d}t} v_n \\ &+ \sum_{n=1}^{N} h\varphi_n \frac{1}{h}\left[(\tilde{p}_n - \tilde{p}_{n-1}) - \left(\mu_n\rho_n \frac{v_{n+1} - v_n}{h} - \mu_{n-1}\rho_{n-1}\frac{v_n - v_{n-1}}{h}\right)\right] + \sum_{n=1}^{N} h\varphi_n g_n. \end{aligned}$$

We write the sums as integrals over step functions where the last term of the sum is uniformly estimated using Lemma 23. Since φ is smooth and vanishes at $z = 0$, we have

$$\sum_{n=1}^{N} h\varphi_n \frac{\mathrm{d}}{\mathrm{d}t} v_n(t) = \int_0^1 \varphi_h^R(z)\partial_t v_h^R(t,z)\,\mathrm{d}z + \mathcal{O}(h),$$

$$\begin{aligned} \sum_{n=1}^{N} h\varphi_n \frac{1}{h}&\left[(\tilde{p}_n - \tilde{p}_{n-1}) - \left(\mu_n\rho_n \frac{v_{n+1} - v_n}{h} - \mu_{n-1}\rho_{n-1}\frac{v_n - v_{n-1}}{h}\right)\right](t) \\ &= -\int_0^1 \partial_z\varphi_h(z)(\tilde{p}_h - u_h)(t,z)\,\mathrm{d}z + \mathcal{O}(h), \end{aligned}$$

where we have used estimates (78), (79), and definition (80),

$$\sum_{n=1}^{N} h\varphi_n g_n = \int_0^1 \varphi_h^R(z) g_h^R(z)\,\mathrm{d}z + \mathcal{O}(h).$$

Because of Lemma 23 and Theorem 26, as $h \to 0$, we see that equation (38) holds for the limit function.

Now, we deal with the third equation (26) (see (39) in the weak formulation). From estimate (68) (see Lemma 15) it follows that for every $t \in (0, T)$,

$$\int_0^1 |a_h(w_h^R(t,z))|^2\,\mathrm{d}z = \sum_{n=1}^{N} h|a_{M(N)}(w_n)|^2(t) \leqslant C. \tag{83}$$

For an arbitrary small $\varepsilon \in (0, 1)$ and for every $t \in (0, T)$, we let

$$\mathcal{N}_h^\varepsilon(t) := \{z \in (0,1) : |w_h^R(t,z)| > 1 - \varepsilon\}, \quad \mathcal{N}^\varepsilon(t) := \{z \in (0,1) : |w(t,z)| > 1 - \varepsilon\}.$$

From (83), we have

$$\begin{aligned} C \geqslant \left[\int_{\mathcal{N}_h^\varepsilon(t)} |a_h(w_h^R(t,z))|^2\,\mathrm{d}z\right]^{1/2} &\geqslant |\mathcal{N}_h^\varepsilon(t)|^{1/2}\left[\inf_{z\in\mathcal{N}_h^\varepsilon(t)}\left(2\sum_{k=0}^{M(N)} \frac{(w_h^R(t,z))^{2k+1}}{2k+1}\right)^2\right]^{1/2} \\ &\geqslant |\mathcal{N}_h^\varepsilon(t)|^{1/2}\, 2\sum_{k=0}^{M(N)} \frac{(1-\varepsilon)^{2k+1}}{2k+1}, \end{aligned}$$

so that

$$|\mathcal{N}_h^\varepsilon(t)|^{1/2} \leqslant \frac{C}{2\sum_{k=0}^{M(N)} \frac{(1-\varepsilon)^{2k+1}}{2k+1}}.$$

Then the Fatou Lemma yields

$$0 \leqslant |\mathcal{N}^\varepsilon(t)|^{1/2} = \int_0^1 \chi_{\mathcal{N}^\varepsilon(t)}(z)\,\mathrm{d}z = \int_0^1 \liminf_{h\to 0} \chi_{\mathcal{N}_h^\varepsilon(t)}(z)\,\mathrm{d}z$$
$$\leqslant \liminf_{h\to 0} |\mathcal{N}_h^\varepsilon(t)| \leqslant \frac{C}{\ln\left(\frac{2-\varepsilon}{\varepsilon}\right)} \xrightarrow{\varepsilon\to 0} 0,$$

and therefore

$$\mathcal{N}(t) := |\{z \in (0,1) : |w(t,z)| \geqslant 1\}| = 0.$$

Now, we have to prove that $a_h(w_h^R) \to \ln\left(\frac{1+w}{1-w}\right)$ almost everywhere in $(0,T)\times(0,1)$, namely, for $z \notin \mathcal{N}(t)$. Then $|w_h^R(t,z)| \leqslant 1-\varepsilon$ for some $\varepsilon > 0$. We compute

$$\left|a_h(w_h^R)(t,z) - \ln\left(\frac{1+w_h^R}{1-w_h^R}\right)(t,z)\right| \leqslant 2 \sum_{k=M(N)+1}^{\infty} \frac{|w_h^R(t,z)|^{2k+1}}{2k+1}$$
$$\leqslant 2\frac{1}{2M(N)+3} \sum_{k=M(N)+1}^{\infty} (1-\varepsilon)^{2k+1}$$
$$\leqslant 2\frac{1}{2M(N)+3}\frac{1}{\varepsilon} \xrightarrow{N\to\infty} 0,$$

since $M(N) \to \infty$ as $N \to \infty$. Therefore

$$\left|a_h(w_h^R) - \ln\left(\frac{1+w}{1-w}\right)\right| \leqslant \left|a_h(w_h^R) - \ln\left(\frac{1+w_h^R}{1-w_h^R}\right)\right| + \left|\ln\left(\frac{1+w_h^R}{1-w_h^R}\right) - \ln\left(\frac{1+w}{1-w}\right)\right|,$$

where the right-hand side converges to 0 as $h \to 0$.

We conclude from (83) that the almost everywhere limit $\ln(\frac{1+w}{1-w})$ is in $L^\infty(0,T;L^2(0,1))$. Moreover, by (83), the weak convergence

$$\int_0^1 \varphi(z) a_h(w_h^R(t,z))\,\mathrm{d}z \xrightarrow{h\to 0} \int_0^1 \varphi(z)\ln\left(\frac{1+w(t,z)}{1-w(t,z)}\right)\mathrm{d}z$$

holds for a subsequence $h \to 0$. Also

$$\sum_{n=1}^N h\varphi_n \lambda_{\tilde\delta}^{-1} \frac{\mathrm{d}}{\mathrm{d}t} w_n(t) = \int_0^1 \varphi_h^R(z)\lambda_{\tilde\delta}^{-1}\partial_t w_h(t,z)\,\mathrm{d}z + \mathcal{O}(h),$$
$$\sum_{n=1}^N h\varphi_n \tilde\delta^2 \frac{1}{h}\left[\frac{w_{n+1}-w_n}{h} - \frac{w_n - w_{n-1}}{h}\right](t) = -\int_0^1 \partial_z\varphi_h(z)\tilde\delta^2\partial_z w_h(t,z)\,\mathrm{d}z + \mathcal{O}(h),$$
$$\sum_{n=1}^N h\varphi_n a_{M(N)}(w_n)(t) = \int_0^1 \varphi_h^R(z) a_h(w_h^R(t,z))\,\mathrm{d}z + \mathcal{O}(h).$$

From the discrete equation (53), choosing $M = M(N)$, it follows that, as $h \to 0$,

$$\int_0^1 \left[\varphi\lambda_{\tilde\delta}^{-1}\partial_t w + \varphi\left\{2c_1\ln\left(\frac{1+w}{1-w}\right) - 2c_2 w - 2b_d\rho\right)\right\} + \partial_z\varphi\tilde\delta^2\partial_z w\right]\mathrm{d}z = 0.$$

Since $\Phi = \frac{1}{2}(1-w)$, this gives equation (39).

Altogether, Theorem 2 is proved.

7. Uniqueness of the weak solution

THEOREM 27 Assume (A.1)–(A.5). Let (ρ_1, v_1, Φ_1) and (ρ_2, v_2, Φ_2) be weak solutions of (24)–(31) satisfying (32)–(39). Then $\rho_1 = \rho_2$, $v_1 = v_2$, and $\Phi_1 = \Phi_2$.

Proof. Taking the difference of (26) yields

$$\int_0^1 \varphi \partial_t (\Phi_2 - \Phi_1)\,\mathrm{d}z + \int_0^1 \varphi c_1 \left[\ln\left(\frac{\Phi_2}{1-\Phi_2}\right) - \ln\left(\frac{\Phi_1}{1-\Phi_1}\right)\right]\mathrm{d}z$$
$$= \int_0^1 \varphi 2c_2(\Phi_2 - \Phi_1)\,\mathrm{d}z - \int_0^1 \varphi b_d(\rho_2 - \rho_1)\,\mathrm{d}z - \int_0^1 \partial_z\varphi \tilde{\delta}^2 \partial_z(\Phi_2 - \Phi_1)\,\mathrm{d}z$$

for all $\varphi \in C_0^\infty((0,1])$. Thus, there exists a sequence of test functions φ in $C_0^\infty((0,1])$ converging to $(\Phi_2 - \Phi_1)(t,\cdot)$ in $C([0,1])$ and $\partial_z\varphi \to \partial_z(\Phi_2 - \Phi_1)(t,\cdot)$ in $L^1((0,1))$. Therefore, going to the limit, we get

$$\frac{1}{2}\int_0^1 |\Phi_2 - \Phi_1|^2(t,\cdot)\,\mathrm{d}z + c_1 \int_0^t \int_0^1 \left[\ln\left(\frac{\Phi_2}{1-\Phi_2}\right) - \ln\left(\frac{\Phi_1}{1-\Phi_1}\right)\right](\Phi_2 - \Phi_1)\,\mathrm{d}z\,\mathrm{d}\tau$$
$$+ \tilde{\delta}^2 \int_0^t \int_0^1 |\partial_z(\Phi_2 - \Phi_1)|^2\,\mathrm{d}z\,\mathrm{d}\tau$$
$$\leqslant C \int_0^t \int_0^1 (\Phi_2 - \Phi_1)^2\,\mathrm{d}z\,\mathrm{d}\tau + \frac{b_d^2}{2}\int_0^t \int_0^1 (\rho_2 - \rho_1)^2\,\mathrm{d}z\,\mathrm{d}\tau.$$

We have

$$c_1 \int_0^t \int_0^1 \left[\ln\left(\frac{\Phi_2}{1-\Phi_2}\right) - \ln\left(\frac{\Phi_1}{1-\Phi_1}\right)\right](\Phi_2 - \Phi_1)\,\mathrm{d}z\,\mathrm{d}\tau \geqslant 0,$$

since $x \mapsto \ln(x/(1-x))$ is a monotone function. Applying the Gronwall lemma to this estimate, we get

$$\int_0^1 |\Phi_2 - \Phi_1|^2(t,\cdot)\,\mathrm{d}z \leqslant C \int_0^t \int_0^1 |\rho_2 - \rho_1|^2\,\mathrm{d}z\,\mathrm{d}\tau. \tag{84}$$

Now, we do the same procedure with (25) as we have just done with (26). Then

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}|v_2 - v_1|^2\,\mathrm{d}z + \int_0^1 \partial_z(v_2 - v_1)(\mu(\rho_2)\rho_2\partial_z v_2 - \mu(\rho_1)\rho_1\partial_z v_1)\,\mathrm{d}z$$
$$= \int_0^1 \partial_z(v_2 - v_1)(\rho_2^2(b_d\Phi_2 + b_2) - \rho_1^2(b_d\Phi_1 + b_2))\,\mathrm{d}z.$$

Since $\mu(\rho) = \rho^\theta$ and

$$\mu(\rho_2)\rho_2\partial_z v_2 - \mu(\rho_1)\rho_1\partial_z v_1 = \rho_1^{1+\theta}\partial_z(v_2 - v_1) + (\rho_2^{1+\theta} - \rho_1^{1+\theta})\partial_z v_2,$$

from Young's inequality we derive

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}|v_2-v_1|^2\,\mathrm{d}z + \int_0^1 \rho_1^{1+\theta}|\partial_z(v_2-v_1)|^2\,\mathrm{d}z\\
&\quad\leqslant \frac{1}{2\varepsilon}\int_0^1 [(\rho_2^2-\rho_1^2)b_d\Phi_2 + (\Phi_2-\Phi_1)b_d\rho_1^2 + (\rho_2^2-\rho_1^2)b_2]^2\,\mathrm{d}z\\
&\qquad + \frac{\varepsilon}{2}\int_0^1 |\partial_z(v_2-v_1)|^2\,\mathrm{d}z + \frac{1}{2\varepsilon}\int_0^1 (\rho_2^{1+\theta}-\rho_1^{1+\theta})^2|\partial_z v_2|^2\,\mathrm{d}z + \frac{\varepsilon}{2}\int_0^1 |\partial_z(v_2-v_1)|^2\,\mathrm{d}z.
\end{aligned}$$

As $1/C \leqslant \rho_1$, we choose ε so small that $(1/C)^{1+\theta} - 2\varepsilon \geqslant 0$ and obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 |v_2-v_1|^2\,\mathrm{d}z + \frac{1}{2}\int_0^t\int_0^1 \rho_1^{1+\theta}|\partial_z(v_2-v_1)|^2\,\mathrm{d}z\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t\int_0^1 b_d^2\Phi_2^2(\rho_2-\rho_1)^2\,\mathrm{d}z\,\mathrm{d}\tau + C\int_0^t\int_0^1 b_d^2\rho_1^4(\Phi_2-\Phi_1)^2\,\mathrm{d}z\,\mathrm{d}\tau\\
&\qquad + C\int_0^t\int_0^1 b_2^2(\rho_2-\rho_1)^2\,\mathrm{d}z\,\mathrm{d}\tau + C\int_0^t\int_0^1 (\rho_2-\rho_1)^2|\partial_z v_2|\,\mathrm{d}z\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t\int_0^1 (\rho_2-\rho_1)^2\,\mathrm{d}z\,\mathrm{d}\tau
\end{aligned}\tag{85}$$

due to inequality (84) and $|\partial_z v_2| \leqslant C$ by Lemma 23 and Theorem 26.

Now, we rewrite equation (24) $\frac{\partial}{\partial t}\frac{1}{\rho} = \frac{\partial v}{\partial z}$, take the difference and multiply it by $1/\rho_2 - 1/\rho_1$. Then we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2 = \partial_z(v_2-v_1)\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right).$$

We integrate over $[0, t] \times [0, 1]$ to get

$$\begin{aligned}
\frac{1}{2}\int_0^1\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2\mathrm{d}z &= \int_0^t\int_0^1 \partial_z(v_2-v_1)\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)\mathrm{d}z\,\mathrm{d}\tau\\
&\leqslant \int_0^t\int_0^1 \rho_1^{1+\theta}|\partial_z(v_2-v_1)|^2\,\mathrm{d}z\,\mathrm{d}\tau + \frac{1}{2}\int_0^t\int_0^1 \frac{1}{\rho^{1+\theta}}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2\mathrm{d}z\,\mathrm{d}\tau.
\end{aligned}$$

Since $\frac{1}{\rho_2} - \frac{1}{\rho_1} = \frac{\rho_1-\rho_2}{\rho_1\rho_2}$, from (85) we get

$$\int_0^1 (\rho_2-\rho_1)^2\,\mathrm{d}z \leqslant C\int_0^t\int_0^1 (\rho_2-\rho_1)^2\,\mathrm{d}z\,\mathrm{d}\tau.$$

So $\rho_2 = \rho_1$ a.e. Now (84) and (85) imply that $v_2 = v_1$ and $\Phi_2 = \Phi_1$ a.e. □

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