Math. Proc. Camb. Phil. Soc. (1991), **110**, 137
Printed in Great Britain

On a question of R. Godement about the spectrum of positive, positive definite functions

BY MOHAMMED B. BEKKA
Mathematisches Institut, Technische Universität München, Arcisstraße 21, *W*-8000 *München* 2, *Germany*

(*Received* 18 *October* 1990)

Introduction

Let G be a locally compact group, and let $P(G)$ be the convex set of all continuous, positive definite functions ϕ on G normalized by $\phi(e) = 1$, where e denotes the group unit of G. For $\phi \in P(G)$ the *spectrum* $\operatorname{sp}\phi$ of ϕ is defined as the set of all indecomposable $\psi \in P(G)$ which are limits, for the topology of uniform convergence on compact subsets of G, of functions of the form

$$x \mapsto \sum_{i,j=1}^{n} c_i \bar{c}_j \phi(x_j^{-1} x x_i), \quad \text{where } c_1, \ldots, c_n \in \mathbb{C}, x_1, \ldots, x_n \in G$$

(see [5], p. 43). Denoting by π_ϕ the cyclic unitary representation of G associated with ϕ, it is clear that $\operatorname{sp}\phi$ consists of all $\psi \in P(G)$ for which π_ψ is irreducible and weakly contained in π_ϕ (see [3], chapter 18).

Let 1_G be the constant positive definite function 1 on G. R. Godement asks in [5], p. 77, whether G always has the following property which we denote by (P) throughout this paper:

(P) If $\phi \in P(G)$ and if ϕ is positive in the usual sense, i.e. $\phi(x) \geqslant 0$ for all $x \in G$, then $1_G \in \operatorname{sp}\phi$.

Examples of positive, positive definite functions are given by functions of the form $f * f^\sim$, where f is a positive function on G with compact support, $f^\sim$ is defined by $f^\sim(x) = \overline{f(x^{-1})}$ and $*$ denotes convolution (for further examples, see Section 1 of this paper). Actually, Godement was mainly interested in the question whether 1_G is the limit, uniformly on compact subsets, of such positive definite functions $f * f^\sim$ ([5], problème 5, p. 76). It is today well known that this is equivalent to the amenability of G (see [7] or [8]). Thus, a necessary condition in order that (P) holds is that G is amenable. As observed in [5], p. 77, it is easy to see that compact or abelian groups satisfy (P). The main purpose of this paper is to show that (P) is satisfied by all exponentially bounded groups (for example, all compact extensions of nilpotent groups) and by all solvable discrete groups.

We do not know whether (P) is equivalent to amenability. However, for connected groups G we show that amenability of G is equivalent to the following (weaker) version (P*) of (P):

(P*) If $\phi \in P(G)$ and if $\phi \geqslant 0$, then $1_G \in \operatorname{sp}_d \phi$,

where $\operatorname{sp}_d \phi$ denotes the spectrum of ϕ when ϕ is considered as a positive definite function on G_d, the group G viewed as a discrete group.

1. *Some remarks*

Let π be a continuous unitary representation of the locally compact group G in the Hilbert space $(H_\pi, \langle \cdot , \cdot \rangle)$. A unit vector $\xi \in H_\pi$ will be called a *positive vector* for π, if $\operatorname{Re}\langle \pi(x)\xi, \xi\rangle \geqslant 0$ for all $x \in G$. Since $\operatorname{Re}\langle \pi(\cdot)\xi, \xi\rangle \in P(G)$ such vectors yield examples of positive, positive definite functions.

Examples 1·1. (i) The (left) regular representation λ_G of a locally compact group G has positive vectors. Recall that λ_G acts on $L^2(G)$ by left translation

$$(\lambda_G(x)f)(y) = {}_xf(y) = f(x^{-1}y) \quad (x, y \in G, f \in L^2(G)).$$

More generally, each induced representation $\operatorname{ind}_H^G 1_H$ (where H is a closed subgroup of G and 1_H the trivial one-dimensional representation of H) has positive vectors.

(ii) The conjugation representation γ_G of G, which acts on $L^2(G)$ by the unitary operators

$$(\gamma_G(x)f)(y) = \Delta(x)^{\frac{1}{2}}f(x^{-1}yx) \quad (x, y \in G, f \in L^2(G))$$

(where Δ denotes the modular function of G), has positive vectors.

(iii) If $\phi \in P(G)$, then $\phi\bar{\phi} = |\phi|^2$ is a positive, positive definite function on G. Hence the (inner) tensor product $\pi \otimes \bar{\pi}$ has positive vectors for each unitary representation π of G, $\bar{\pi}$ denoting the conjugate representation of π.

(iv) Let ψ be a real negative definite function on G. Then $\phi = \exp\psi$ is a positive, positive definite function on G (see [1], 8·4). An example of such a positive definite function on $\mathbb{R}^n$ is the function

$$x \mapsto \exp(-\|x\|^2), \quad \|x\| \text{ denoting the Euclidean norm.}$$

It is easy to translate (P) into a property of unitary representations with positive vectors. Indeed, consider the following property (P′) of G which is formally stronger than (P):

(P′) If π is a unitary representation of G with a positive vector, then π contains weakly 1_G.

PROPOSITION 1·2. (P) *and* (P′) *are equivalent for every locally compact group* G.

Proof. Let π be a unitary representation of G with positive vector $\xi \in H_\pi$. Let $\phi(x) = \operatorname{Re}\langle \pi(x)\xi, \xi\rangle$, $x \in G$. If (P) holds, then 1_G is weakly contained in π_ϕ which is a subrepresentation of $\pi \oplus \bar{\pi}$. Thus 1_G is weakly contained in $\pi \oplus \bar{\pi}$, and this implies that 1_G is weakly contained in π. ∎

Suppose G is abelian. Then each $\phi \in P(G)$ is the Fourier transform $\hat{\mu}$ of a probability measure μ on the dual group $\hat{G}$. It is easy to see (cf. [1], 4·14) that $\phi \geqslant 0$ if and only if μ is a positive definite measure on $\hat{G}$. On the other hand, $\operatorname{sp}\phi = \operatorname{supp}\mu$. So, (P) for abelian groups is a consequence of the following result, valid for arbitrary locally compact groups:

PROPOSITION 1·3. *Let μ be a positive definite* (*not necessarily bounded*) *measure on the locally compact group G. If $\mu \neq 0$, then $e \in \operatorname{supp}\mu$.*

Proof. Suppose $e \notin \operatorname{supp}\mu$. Then there is some neighbourhood U of e such that $\mu(f) = 0$ for all $f \in C_{00}(G)$ with $\operatorname{supp} f \subseteq U$, where $C_{00}(G)$ denotes the space of all compactly supported functions on G. Choose a net $(f_\alpha)_\alpha$ in $C_{00}(G)$ with $f_\alpha \geqslant 0$, $\|f_\alpha\|_1 = 1$ and $\operatorname{supp} f_\alpha \to \{e\}$. Then $\lim_\alpha \mu(f_\alpha^* * f_\alpha) = 0$, and hence by the Cauchy–Schwarz inequality

$$|\mu(f)|^2 = \lim_\alpha |\mu(f * f_\alpha)|^2 \leqslant \lim_\alpha \mu(f^* * f)\,\mu(f_\alpha^* * f_\alpha) = 0$$

for all $f \in C_{00}(G)$, a contradiction. ∎

The following (probably well known) fact should be mentioned in relation with property (P′).

PROPOSITION 1·4. *Let π be a unitary representation of the locally compact group G in the Hilbert space H_π. If there exist $\xi \in H_\pi$ and $c > 0$ such that* $\operatorname{Re}\langle \pi(x)\xi, \xi\rangle \geqslant c$ *for all $x \in G$, then π contains 1_G as subrepresentation (that is, there is $\eta \in H_\pi$, $\eta \neq 0$ with $\pi(x)\eta = \eta$ for all $x \in G$).*

Proof. Let X be the closed convex hull of $\{\pi(x)\xi; x \in G\}$ in H_π. Let η be the unique element of X with minimal norm. Then $\operatorname{Re}\langle \eta, \xi\rangle \geqslant c$, hence $\eta \neq 0$. Moreover, $\pi(x)\eta = \eta$ for all $x \in G$. ∎

2. *Exponentially bounded groups*

Recall that a locally compact group is called exponentially bounded if $\lim_n |K^n|^{1/n} = 1$ for each compact neighbourhood K of e, where $|\cdot|$ denotes Haar measure and $K^n = \{k_1 \ldots k_n; k_i \in K\}$. Clearly, groups of polynomial growth are exponentially bounded. The converse is true for connected groups, but not in general. Exponentially bounded groups are unimodular and amenable (cf. [**6**]).

THEOREM 2·1. *Exponentially bounded groups have property (P).*

Proof. Let G be such a group, and let $\phi \in P(G)$ with $\phi \geqslant 0$. Let K be a compact neighbourhood of e with $K = K^{-1}$, and let $\epsilon > 0$. Then there is $n \in \mathbb{N}$ such that

$$\int_{K^{n+1}\times K^{n+1}} \phi(z^{-1}y)\,dy\,dz \leqslant (1+\epsilon)\int_{K^n \times K^n} \phi(z^{-1}y)\,dy\,dz \qquad (*)$$

where dy (respectively dz) denotes the Haar measure on G. Indeed, otherwise

$$|K^{n+1}|^2 \geqslant \int_{(K^{n+1})^2} \phi(z^{-1}y)\,dy\,dz > (1+\epsilon)^n \int_{K\times K} \phi(z^{-1}y)\,dy\,dz$$

for all $n \in \mathbb{N}$. As $\int_{K\times K} \phi(z^{-1}y)\,dy\,dz > 0$, this would be a contradiction to $\lim |K^n|^{1/n} = 1$. Choose $n \in \mathbb{N}$ such that $(*)$ holds. Let $f = \chi_{K^n}$ be the characteristic function of K^n. Let π be the unitary representation of G associated to ϕ with Hilbert space H_π. Let $\xi \in H_\pi$ be such that $\phi(x) = \langle \pi(x)\xi, \xi\rangle$, $x \in G$. Then

$$\|\pi(f)\xi\|^2 = \int_G f^* * f(x)\,\phi(x)\,dx > 0,$$

since $f^* * f(e)\,\phi(e) > 0$ and $f^* * f(x)\,\phi(x) \geqslant 0$ for all $x \in G$. Now let

$$\psi(x) = \frac{1}{\|\pi(f)\xi\|^2}\langle \pi(x)\pi(f)\xi, \pi(f)\xi\rangle, x \in G.$$

Then ψ is associated to π. Moreover, for each $x \in G$

$$|\psi(x)-1|^2 = \frac{1}{\|\pi(f)\,\xi\|^4}|\langle \pi({}_xf-f)\,\xi, \pi(f)\,\xi\rangle|^2 \leqslant \frac{\|\pi({}_xf-f)\,\xi\|^2}{\|\pi(f)\,\xi\|^2}$$

$$= \int_{G\times G} ({}_xf-f)(y)\,({}_xf-f)(z)\,\phi(z^{-1}y)\,dy\,dz \Big/ \int_{G\times G} f(y)f(z)\,\phi(z^{-1}y)\,dy\,dz$$

$$= \int_{(xK^n\Delta K^n)^2} \phi(z^{-1}y)\,dy\,dz \Big/ \int_{(K^n)^2} \phi(z^{-1}y)\,dy\,dz,$$

where Δ denotes the symmetric difference. Now, (*) implies that for $x \in K$

$$\int_{(xK^n\Delta K^n)^2} \phi(z^{-1}y)\,dy\,dz \leqslant \int_{(K^{n+1})^2\setminus(K^n)^2} \phi(z^{-1}y)\,dy\,dz + \int_{(K^n\setminus xK^n)^2} \phi(z^{-1}y)\,dy\,dz$$

$$\leqslant \epsilon \int_{(K^n)^2} \phi(z^{-1}y)\,dy\,dz + \int_{(x^{-1}K^n\setminus K^n)^2} \phi(z^{-1}y)\,dy\,dz$$

$$\leqslant 2\epsilon \int_{(K^n)^2} \phi(z^{-1}y)\,dy\,dz,$$

since $x^{-1} \in K$. Hence $|\psi(x)-1|^2 \leqslant 2\epsilon$ for all $x \in K$. ∎

Remark 2·2. The proof of the above theorem yields a more precise result which will be used in the sequel: G being exponentially bounded, if $\phi \in P(G)$ and $\phi \geqslant 0$, then 1_G is uniform limit on compact subsets of G of functions of the form

$$x \mapsto \sum_{i,j=1}^{n} c_i c_j \phi(x_j^{-1} x x_i),$$

where all $c_i \geqslant 0$ and $x_i \in G$. Indeed, the vector $\pi(f)\,\xi$ in the above proof belongs to the closure of the cone generated by $\{\pi(x)\,\xi; x \in G\}$.

Remark 2·3. The above theorem applies, in particular, to nilpotent locally compact groups. The simplest solvable group not covered by Theorem 2.1 is the $(ax+b)$-group $G = \{(a,b); a \in \mathbb{R}^*, b \in \mathbb{R}\}$ with multiplication law $(a,b)(a',b') = (aa', ab'+b)$. But it is easy to show that (P) holds for G. Indeed, let N be the normal subgroup $\{(1,b); b \in \mathbb{R}\}$. Up to unitary equivalence, the unitary irreducible representations of G are the characters of the abelian quotient G/N and two infinite-dimensional representations π^+ and π^- which contain weakly 1_G (cf. [**4**], theorem 5·1). Let $\phi \in P(G)$ with $\phi \geqslant 0$. If π^+ or π^- is weakly contained in π_ϕ, then 1_G is weakly contained in π_ϕ. In the other case, π_ϕ factors to a representation of G/N, and the result follows since (P) holds for G/N.

Remark 2·4. Let G be nilpotent. Then, except 1_G, no irreducible representation of G has a positive vector. Let π be an irreducible representation of G with a positive vector, and let Z be the centre of G. Then $\pi|Z$ has a positive vector. Hence π is trivial on Z. Considering the ascending central series of G and repeating this argument, we see that $\pi = 1_G$.

In contrast with this, let G be a connected semisimple non-compact Lie group, and let P be a minimal parabolic subgroup. Then the principal series representation $\pi = \mathrm{ind}_P^G 1_P$ is irreducible (cf., e.g., [**9**]) and has positive vectors. Observe that 1_G is not weakly contained in π.

THEOREM 2·5. *Discrete solvable groups have property* (P).

Proof. Let G be such a group, and let $\phi \in P(G)$ with $\phi \geqslant 0$. Let

$$G = G_n \supseteq G_{n-1} \supseteq \ldots \supseteq G_0 = \{e\}$$

be a composition series with abelian factors G_i/G_{i-1}, $1 \leqslant i \leqslant n$. First we show by induction on i the following: for each $0 \leqslant i \leqslant n$ there is a net $(\psi_\alpha)_\alpha$ in $P(G)$ with $\psi_\alpha \geqslant 0$ such that $\lim_\alpha \psi_\alpha(x) = 1$ for all $x \in G_i$ and such that π_{ψ_α} is weakly contained in π_ϕ for all α.

The case $i = 0$ being trivial (take $\psi_\alpha = \phi$), suppose there is such a net $(\psi_\alpha)_{\alpha \in A}$ for some i. Let ψ be a limit point of $(\psi_\alpha)_{\alpha \in A}$ in the weak $*$-topology $\sigma(l^\infty(G), l^1(G))$. Then $\psi \in P(G)$ and $\psi \geqslant 0$. Moreover

$$\psi(x) = \lim \psi_\alpha(x) = 1 \quad \text{for all } x \in G_i.$$

Hence $\psi \mid G_{i+1}$ factors to a positive, positive definite function of G_{i+1}/G_i. Thus, by Theorem 2·1 (see Remark 2·2), there is a net $(\psi'_\beta)_\beta$ in $P(G_{i+1}/G_i)$ of the form

$$\psi'_\beta(x) = \sum c_k c_l \psi(x_l^{-1} x x_k), x \in G_{i+1},$$

where all $c_k \geqslant 0$ and $x_k \in G_{i+1}$, such that $\lim \psi'_\beta(x) = 1$ for all $x \in G_{i+1}$. Observe that in fact $\psi'_\beta \in P(G)$ and $\psi'_\beta \geqslant 0$. Moreover $\pi_{\psi'_\beta} = \pi_\psi$. Hence each $\pi_{\psi'_\beta}$ is weakly contained in $\{\pi_{\psi_\alpha}; \alpha \in A\}$ which is weakly contained in π_ϕ. Our claim is thus proved, and we get a net $(\psi_\alpha)_\alpha \in P(G)$ such that $\lim \psi_\alpha(x) = 1$ for all $x \in G_n = G$ and such that each π_{ψ_α} is weakly contained in π_ϕ. Hence 1_G is weakly contained in π_ϕ. ∎

We now turn to property (P*) from the Introduction. Observe that (P*) can be formulated as follows: if π is a unitary representation of G with positive vectors, then 1_G is weakly contained in π, when π and 1_G are viewed as representations of G_d.

THEOREM 2·6. *For a connected locally compact group G the following are equivalent*:

(i) *G is amenable*;

(ii) *G has property* (P*).

Proof. If G has property (P*), then 1_G is weakly contained in the regular representation λ_G, when both representations are considered as representations of G_d. As is well known, this is equivalent to the amenability of G (cf. [**7**], proposition 8·8). Suppose G is amenable. Let N be the closure of the commutator subgroup of G. By [**2**], proposition 3, N has polynomial growth. Let $\phi \in P(G)$, $\phi \geqslant 0$. By Theorem 2·1 (see Remark 2·2), there is a net $(\psi_\alpha)_\alpha$ in $P(G)$ with $\psi_\alpha \geqslant 0$ such that $\lim_\alpha \psi_\alpha(x) = 1$ for all $x \in N$, and such that π_{ψ_α} is weakly contained in π_ϕ for all α. Viewing now G as discrete group, the method of proof of Theorem 2·5 yields some $\psi \in P(G_d)$, $\psi \geqslant 0$ with $\psi \mid N = 1$ and such that π_ψ is weakly contained in π_ϕ (as representations of G_d). Since G/N is abelian, 1_G is weakly contained in π_ψ, and the result follows. ∎

Remark 2·7. It is well known that 1_G is limit, for the compact convergence, of positive definite functions of the form $f * f^\sim$, $f \in C_{00}(G)$, $f \geqslant 0$, if (and only if) G is amenable ([**7**, **8**]). Actually, using Reiter's property (P_2) (cf. [**8**], chapter 8, §3), it is easy to prove the following more precise result. Let G be a locally compact unimodular amenable group. Let $\phi \in P(G)$ with $\phi \geqslant 0$, and let π be the associated representation. If ϕ is integrable or if π is square-integrable (i.e. all matrix coefficients of π are square-integrable), then 1_G is weakly contained in π.

It is a pleasure to thank Professor G. Schlichting for stimulating discussions.

REFERENCES

[1] C. BERG and G. FORST. *Potential Theory on Locally Compact Abelian Groups* (Springer-Verlag, 1975).

[2] J. BOIDOL. Group algebras with a unique C^*-norm. *J. Funct. Anal.* **56** (1984), 220–232.

[3] J. DIXMIER. *C^*-algebras* (North-Holland, 1977).

[4] J. M. G. FELL. Weak containment and induced representations of groups. *Canad. J. Math.* **14** (1962) 237–268.

[5] R. GODEMENT. Les fonctions de type positif et la théorie des groupes. *Trans. Amer. Math. Soc.* **63** (1948), 1–84.

[6] Y. GUIVARC'H. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France* **101** (1973), 333–379.

[7] J. P. PIER. *Amenable Locally Compact Groups* (Wiley, 1984).

[8] H. REITER. *Classical Harmonic Analysis and Locally Compact Groups* (Oxford University Press, 1968).

[9] N. WALLACH. Cyclic vectors and irreducibility for principal series representations. *Trans. Amer. Math. Soc.* **158** (1971), 107–113.