

# ON THE DECAY - AND THE SMOOTHNESS BEHAVIOR OF THE FOURIER TRANSFORM, AND THE CONSTRUCTION OF SIGNALS HAVING STRONG DIVERGENT SHANNON SAMPLING SERIES

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## ABSTRACT

In this work, we show by means of the technique inspired by the Banach-Steinhaus Thm., that typically the Fourier transform of an integrable signal decays arbitrarily slowly toward the infinity, and has an arbitrary weak worst continuity/smoothness behaviour. However, the corresponding characterization can only be given weakly by means of the limit superior. Those statements gives therefore a tightening of the famous Riemann-Lebesgue's Lemma. Furthermore, we give a construction of functions, whose Fourier transform decays slowly than an arbitrary given decay rate. Inspired by that, we are also able to give an alternative proof of the strong divergence of the Shannon sampling series [1] for signals in the Paley-Wiener space  $\mathcal{PW}_{\omega_g}^1$ , band-limited to an arbitrary  $\omega_g \in \mathbb{R}^+$ . The corresponding construction of signals is stronger than the existent one given by Boche and Farell, and gives a new insight into the divergence phenomenon of the Shannon sampling series

**Index Terms**— Fourier transform, Riemann-Lebesgue Lemma, Decay behaviour, Smoothness/continuity behaviour, Divergence of Sampling Series

## 1. INTRODUCTION

The Fourier series and Fourier transform (FT) is without doubt an important tool not only in the field of modern signal - and information processing, but also in other engineering sciences. For instance, they contribute greatly to the sampling theory [2], time-frequency analysis [3], and wavelet theory [4], which constitute the foundations of today's digital world [5, 6, 7]. One of the important results concerning to the Fourier series and FT is the so called Riemann-Lebesgue's Lem. (RLL). Its version for the FT asserts that the FT of an integrable signal  $f \in L^1(\mathbb{R})$  on the real line has a regular behaviour, in the sense that it is continuous, and vanishes at infinity, i.e. it converges to 0 as  $|\omega| \rightarrow \infty$ . Analogously, for the Fourier series, it says that the Fourier coefficients of a w.l.o.g.  $2\pi$ -periodic functions, which are each integrable on w.l.o.g.  $[-\pi, \pi]$ , decay with increasing index toward 0.

Thus, the RLL for FT might tempt someone to think optimistically, that time-limited signals, which is clearly integrable, are approximately band-limited, since all of the frequencies sufficiently far away from the origin might be approaching zero fastly. Analogously, the RLL for Fourier series might tempt someone to think, that any given signal can be approximated very well by a few dominant Fourier coefficients of low indexes, because most of the Fourier

coefficients are near zero. We aim to show in this work that such thoughtless conclusions are false. Specifically, we aim to show, that there exists not only an  $f \in L^1(\mathbb{R})$ , whose FT decays arbitrarily slowly toward 0, rather such  $f$  can be found, s.t. its time occupation is essentially contained in w.l.o.g.  $[-\pi, \pi]$  (this result can be generalized to any other  $[-t_g, t_g]$ ,  $t_g > 0$ , by simple rescaling). We shall even see, that such property is a "typical" property for the considered signal spaces ( $L^1(\mathbb{R})$ ,  $L^1([-\pi, \pi])$ ). Furthermore, since FT and IFT are almost identical, those statements can easily modified, as to give the statements concerning to the decay behaviour, among others, of band-limited signals. Notice also later that this observation can also be transferred into the context of Fourier series. An application of the spaces  $L^1([-\pi, \pi])$  and  $L^1(\mathbb{R})$  in engineering is given for instance in optimal control [8].

As we shall see, the statements mentioned previously is in some sense weak, since it can only be given by means of the limit superior. So, it is natural to ask, whether at least one of those can be strengthened. Specifically, we ask ourselves: Does there exists for a function  $M$  specifying a certain decay rate, a function  $f \in L^1(\mathbb{R})$ , for which  $\{M(|\omega|)|\hat{f}(\omega)|\}_{\omega \in \mathbb{R}}$  diverges strongly, i.e.  $\lim_{|\omega| \rightarrow \infty} M(|\omega|)|\hat{f}(\omega)| = \infty$ ? This question will be answered in this work by means of a specific construction.

The so-called Shannon's sampling series (SSS) [5] is probably one of the prominent example of an reconstruction process in the signal processing. Its significance is founded by the fact, that it ensures perfect reconstruction of band-limited square-integrable signals from its samples, taken by the best possible sufficiently high rate - the Nyquist rate. Since this initial result, many sampling theorems of different directions have been developed, aiming to broaden the signal classes, for which the SSS holds. Furthermore, the modes of convergence now constitutes an entire area of research. Some excellent overviews concerning to those aspects can be found in [9, 10, 11, 12, 13]. However, the convergence behaviour of the SSS for the broad signal class  $\mathcal{PW}_{\omega_g}^1$ , i.e. the space of signals, which are band-limited to  $\omega_g > 0$  (see Sec. 2), is rather interesting. In [14], it was shown that the SSS for  $\mathcal{PW}_{\omega_g}^1$  converges locally, while in [15], it was shown that the SSS for  $\mathcal{PW}_{\omega_g}^1$  diverges globally, in the sense that there exists a signal  $f \in \mathcal{PW}_{\omega_g}^1$  (this behaviour holds even for "typical" signals in  $\mathcal{PW}_{\omega_g}^1$ , i.e. for residual sets (see Sec. 2) in  $\mathcal{PW}_{\omega_g}^1$ ), s.t.:

$$\limsup_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{\omega_g} \sum_{k=-N}^N f\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\omega_g(t - \frac{k\pi}{\omega_g}))}{t - \frac{k\pi}{\omega_g}} \right| = \infty. \quad (1)$$

Notice that the latter is given only weakly by means of the limit su-

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perior. A stronger result concerning to the divergence phenomenon was given in [1]. Specifically, there it was shown, by means of a specific construction, that there exists  $f \in \mathcal{PW}_{\omega_g}^1$ , s.t.:

$$\lim_{N \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{\omega_g} \sum_{k=-N}^N f\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\omega_g(t - \frac{k\pi}{\omega_g}))}{t - \frac{k\pi}{\omega_g}} \right| = \infty, \quad (2)$$

i.e. that the SSS diverges strongly (globally). Excellent overviews concerning to the divergence phenomenon of the SSS and its importance for the field of signal processing can be found in [16]. Motivated by the result mentioned in the previous paragraph, we shall also give an alternative proof of the strong divergence of the SSS. Besides, it shall be obvious, that the corresponding construction found in this work is stronger than in [1].

For sake of completeness, we also aim to specify the second statement of the RLL concerning to the continuity behaviour of the FT of an integrable signal. Particularly, it shall be shown, that typically the FT of an integrable signal possesses an arbitrary weak continuity/smoothness behaviour.

## 2. NOTATIONS AND BASIC NOTIONS

An operator denotes simply a linear mapping between vector spaces. Let  $\mathcal{X}_1$  and  $\mathcal{X}_2$  be normed spaces, and  $T : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ . The norm of the operator  $T$  is given by:  $\|T\| := \sup_{\|x\|_{\mathcal{X}_1} \leq 1} \|Tx\|_{\mathcal{X}_2} = \sup_{\|x\|_{\mathcal{X}_1} = 1} \|Tx\|_{\mathcal{X}_2}$ . An operator is said to be *bounded*, if its norm is finite. For operator is boundedness equivalent with continuity. The space of all bounded operators between  $\mathcal{X}_1$  and  $\mathcal{X}_2$  is denoted by  $\mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$ . We call an operator mapping from a vector space  $\mathcal{X}$  to  $\mathbb{C}$ , as *functional on  $\mathcal{X}$* . The space of functionals on  $\mathcal{X}$  is denoted by  $\mathcal{X}^*$ .

Let  $p \in [1, \infty]$ , and  $\mathcal{X} \subset \mathbb{R}$ . We denote the *Lebesgue space of  $p$ -integrable functions* on  $\mathcal{X}$  by  $L^p(\mathcal{X})$ . For  $f \in L^1(\mathbb{R})$ , we define the *Fourier transform (FT)* of  $f$  by  $\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t} dt$  and the *inverse Fourier transform (IFT)* by  $\check{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega)e^{i\omega t} d\omega$ . For  $p \in [1, \infty]$  and  $\omega_g \in \mathbb{R}^+$ , we define the *Paley-Wiener space  $\mathcal{PW}_{\omega_g}^p$*  as the space of signals  $f : \mathbb{R} \rightarrow \mathbb{C}$ , which is representable as the IFT of a function  $\hat{f} \in L^p([-\omega_g, \omega_g])$ .  $\mathcal{PW}_{\omega_g}^p$  can be seen as the space of signals band-limit  $\omega_g$ . In most cases, we treat signals band-limited to  $\pi$ , since the results can easily be extended to any other band-limit by some simple rescaling. A detailed treatment of those basic functional analytic notions and Lebesgue spaces can be found in [17, 18].

Let  $\mathcal{B}$  be a Banach space. A set  $\mathcal{M} \subseteq \mathcal{B}$  is said to be *nowhere dense*, if the inner of the closure of  $\mathcal{M}$  is empty. One may visualize a nowhere dense set as a set "being perforated with holes" [19]. A set  $\mathcal{M} \subseteq \mathcal{B}$  is said to be of *1. category*, if it can be represented as a countable union of nowhere dense sets. The complement of a set of 1. category is defined as a *residual set*. Topologically, sets of 1. category can be seen as a small set, and visualized as a set "approximable" by sets "being perforated with holes". Accordingly, residual sets, each as a complementary set of a set of 1. category, can be seen as a large set. The Baire category Thm. ensures that this categorization of sets of a Banach spaces is non-trivial, by showing that the whole Banach space  $\mathcal{B}$  is not "small" in this sense, or can even not be "approximated" by such sets, i.e. it can not be written as the union of sets of 1. category, and that the residual sets are dense in  $\mathcal{B}$ , and closed under countable intersection. A property that holds for a residual subset of  $\mathcal{B}$  is called a *generic property*. A generic property might not holds for all elements of  $\mathcal{B}$ , but for

"typical" elements of  $\mathcal{B}$ . The so-called Banach-Steinhaus Thm. [20], which is one of the central results in functional analysis, constitutes a consequence of the Baire category Thm. . One of its version can be expressed as follows:

**Theorem 1 (The Principle of Condensation of Singularity):** Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be Banach spaces. Given a family  $\Phi$  in  $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2)$ . If it holds  $\sup_{T \in \Phi} \|T\|_{\mathcal{B}_1 \rightarrow \mathcal{B}_2} = \infty$ , then there exists an  $x_* \in \mathcal{B}_1$  for which  $\sup_{T \in \Phi} \|Tx_*\|_{\mathcal{B}_2} = \infty$ . Furthermore, the set of such  $x_*$  is a residual set in  $\mathcal{B}_1$ .

For more detailed treatment of the Baire category Thm., and the Banach-Steinhaus Thm., we refer to standard textbooks such as [19, 17, 21, 18]

## 3. THE DECAY BEHAVIOUR OF THE FT

Without doubt, the following Thm. is of fundamental significance in the signal processing:

**Theorem 2 (Riemann-Lebesgue Lemma (RLL)):** Let  $f \in L^1(\mathbb{R})$ . The FT  $\hat{f}$  of  $f$  is continuous. Furthermore, it vanishes at infinity, in the sense that  $\lim_{|\omega| \rightarrow \infty} \hat{f}(\omega) = 0$ .

The RLL asserts that the FT of an integrable function possesses in some sense a regular behaviour. It is natural to ask, how "regular" the FT of such a function might be. Firstly, we ask ourselves how fast might be the decay of the FT of such functions. In the following Thm., we give the corresponding answer, even for the case where the signal is (almost) concentrated to the intervall w.l.o.g.  $[-\pi, \pi]$ :

**Theorem 3:** Let  $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$  be an arbitrary monotonically non-decreasing function, with  $\lim_{\omega \rightarrow \infty} M(|\omega|) = +\infty$ . Then there exists a function  $f_* \in L^1(\mathbb{R})$ , with  $f_*(t) = 0$ , for a.e.  $|t| > \pi$ , such that it holds:

$$\limsup_{|\omega| \rightarrow \infty} M(|\omega|) |\hat{f}_*(\omega)| = \infty. \quad (3)$$

Furthermore, the set of such function is a residual set in  $L^1([-\pi, \pi])$ .

**Sketch of proof:** Let be  $\omega \in \mathbb{R}$ . Define  $\Psi_\omega \in L^1([-\pi, \pi])$  by  $\Psi_\omega f := \int_{-\pi}^{\pi} f(t)e^{-i\omega t} dt$ . It is not hard to see that:

$$\|\Psi_\omega\| = \sup_{\|f\|_{L^1([-\pi, \pi])} \leq 1} |\Psi_\omega f| = 1. \quad (4)$$

By means of the function sequence of functions  $f_n$ ,  $n \in \mathbb{N}$ , given by  $f_n(t) = n$ , for  $|t| \leq 1/2n$ , and  $f_n(t) = 0$  else, one can show the non-trivial part of above statement, i.e.  $\|\Psi_\omega\| \geq 1$ . Now, the statement (4) asserts that the norm of  $\Psi_{\omega, M} \in L^1([-\pi, \pi])^*$  given by  $\Psi_{\omega, M} f := M(|\omega|)\Psi_\omega(f) = M(|\omega|)\hat{f}(\omega)$ , is  $\|\Psi_{\omega, M}\| = M(\omega)$ . Thus, it yields  $\sup_{\omega \in \mathbb{R}_0^+} \|\Psi_{\omega, M}\| = \infty$ , since by assumption  $\sup_{\omega \in \mathbb{R}_0^+} M(\omega) = \lim_{\omega \rightarrow \infty} M(\omega) = \infty$ . Thm. 1 asserts, that there exists a function  $f_* \in L^1([-\pi, \pi])$  s.t.  $\limsup_{|\omega| \rightarrow \infty} |\Psi_{\omega, M} f_*| = \infty$ , and correspondingly the set of such functions is residual in  $L^1([-\pi, \pi])$ . ■

**Remark 1:** Since the FT and the IFT behave almost equally, Thm. 3 can be applied up to a minor modification to describe the decay behaviour of band-limited signals. also other discussions found in this section can appropriately be modified.

Notice that by simply changing the domain of  $\Psi_\omega$  and  $\Psi_{\omega, M}$ , from  $L^1([-\pi, \pi])$  to  $L^1(\mathbb{R})$ , in the proof of Thm. 3, the statement in Thm. 3 can be modified as follows: The set of functions  $f_* \in L^1(\mathbb{R})$ , for which (3) holds, is a residual set in  $L^1(\mathbb{R})$ . Furthermore, Thm.

3 might by some efforts also be modified, as to give the following more general statement: Let  $p \in [1, \infty]$ , and  $M$  is a function fulfilling the requirements given in Thm. 3. Then there exists a signal  $f_* \in L^p(\mathbb{R})$ , with  $f_*(t) = 0$ , for a.e.  $|t| > \pi$ , such that  $\lim_{|\omega| \rightarrow \infty} M(|\omega|) |\widehat{f}_*(\omega)| = \infty$ . Furthermore, the set of such functions is a residual set in  $L^p([-\pi, \pi])$ . Thm. 3 and the latter discussions assert that generically time-limited (or equivalently regarding to Rem. 1: band-limited -) signals are even not approximatively band-limited (resp.: band-limited), where  $f \in L^p([-\pi, \pi])$  is approximatively band-limited means, that  $\forall \epsilon > 0$ , there exists  $\omega_\epsilon > 0$ , s.t.  $|f(\omega)| \leq \epsilon$ ,  $\forall |\omega| > \omega_\epsilon$ .

Besides, the attentive readers would recognize that Thm. 3 can be modified as to give a characterization of the decay behaviour of the Fourier coefficients of a periodic function  $f$ , integrable on the interval  $[-\pi, \pi]$ , i.e.  $f \in L^1(\mathbb{T})$ , in the following sense: Let  $M : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$  be an arbitrary monotonically increasing function (or equivalently: a monotonically increasing sequence), for which  $\lim_{n \rightarrow \infty} M(n) = \infty$ . Then there exists a function  $f_* \in L^1(\mathbb{T})$ , for which it holds  $\limsup_{|n| \rightarrow \infty} M(|n|) |\widehat{f}_*(n)| = \infty$ , where in this case  $\widehat{f}_*(n)$  denotes the fourier coeff. of  $f_*$ . Furthermore, the set of such functions  $f_*$  is a residual set in  $L^1(\mathbb{T})$ .

#### 4. CONSTRUCTION OF A FUNCTION STRONGLY DECAYING SLOWER THAN A FIXED RATE

Next, we shall show, that even for some functions  $f$  generally in  $L^1(\mathbb{R})$  strong divergence may occur for  $M(|\omega|) |\widehat{f}(\omega)|$ , where  $M$  is a fixed decay rate. To do this, we first need the following Thm.:

**Theorem 4:** Let  $G : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$  be an arbitrary a monotonically decreasing continuous function, for which  $\lim_{\omega \rightarrow \infty} G(\omega) = 0$  holds. Then there exists a function  $f \in L^1(\mathbb{R})$  such that  $|\widehat{f}(\omega)| \geq G(|\omega|)$ , for all  $\omega \in \mathbb{R}$ . Furthermore, the function  $f$  can be found s.t.  $\widehat{f}$  is real and non-negative.

**Sketch of proof:** For  $n \in \mathbb{N}$ , consider the Fejèr kernel  $g_n$  (see textbooks, e.g. [22, 23]), whose FT is given by  $\widehat{g}_n(\omega)$ , for  $|\omega| \leq 2^n$ , and 0 else. It is well-known that  $\int_{-\infty}^{\infty} g_n(t) dt = 1$ , and therefore:  $\|g_n\|_{L^1(\mathbb{R})} = 1$ . Now, for each  $n \in \mathbb{N}$ , define the function  $g_n^*$  by  $g_n^* := 2g_{n+1} - g_n$ . For  $n \in \mathbb{N}$ , one can give the FT of  $g_n^*$  explicitly by  $\widehat{g}_n^*(\omega) = 1$ , for  $|\omega| \leq 2^n$ ,  $\widehat{g}_n^*(\omega) = 1 - \frac{|\omega - 2^{n+1}|}{2^{n+1} - 2^n}$ , for  $2^n < |\omega| \leq 2^{n+1}$ , and 0 else. Furthermore, by the triangle inequality, and by the fact that  $\|g_n\|_{L^1(\mathbb{R})} = 1$ , for each  $n \in \mathbb{N}$ , we can conclude that  $\|g_n^*\|_{L^1(\mathbb{R})} \leq 3$ ,  $\forall n \in \mathbb{N}$ . Next, take an arbitrary function  $G$ , which fulfills the requirement given in the Thm. For each steps  $k \in \mathbb{N}$ , choose  $n_k \in \mathbb{N}$  sufficiently large enough, s.t.  $n_k > n_{k-1}$  and  $G(2^{n_k}) \leq \frac{1}{2} G(2^{n_{k-1}})$ , where  $n_0 = 0$ . One can easily show by induction involving the fact that  $G$  vanishes at infinity, that the choice is possible. Further, define the function  $f_k := f_{k-1} + G(2^{n_{k-1}}) g_{n_{k-1}}^*$ , where  $f_0 = 0$ . Observe by induction that by this procedure, we have a pointwise monotonically increasing sequence of real and non-negative functions  $\{\widehat{f}_k\}$ , which fulfills:

$$\forall k \in \mathbb{N} : \widehat{f}_k(\omega) \geq G(|\omega|), \quad \forall |\omega| \leq 2^{n_k} \quad (5)$$

Now, notice that,  $\forall k \in \mathbb{N}$ ,  $f_k$  can be written as  $f_k = \sum_{l=1}^k G(2^{n_l}) g_{n_l}^*$ . Correspondingly, one may easily obtain for  $k, k' \in \mathbb{N}$ ,  $k \geq k'$ :

$$\|f_k - f_{k'}\|_{L^1(\mathbb{R})} \leq \sum_{l=k'+1}^k G(2^{n_l}) \|g_{n_l}^*\|_{L^1(\mathbb{R})} \leq 3G(0) \sum_{l=k'+1}^k \frac{1}{2^l},$$

Thus from above computations and from the fact that the series  $\sum_{k=1}^{\infty} 1/2^k$  converges, one can easily conclude, that  $\{f_k\}$  is a Cauchy sequence in  $L^1(\mathbb{R})$ , and accordingly, completeness of  $L^1(\mathbb{R})$  asserts the existence of  $f \in L^1(\mathbb{R})$ , for which  $\lim_{n \rightarrow \infty} f_n = f$  w.r.t. the norm of  $L^1(\mathbb{R})$ . By the continuity of the FT seen as an element of operator between  $L^1(\mathbb{R})$  and  $C_0$ , it follows that  $\widehat{f}_k$  converges uniformly to  $\widehat{f}$ , and clearly also pointwise. Now, let  $\omega \in \mathbb{R}$  be arbitrary but fixed. There exists of course an  $k_0 \in \mathbb{N}$ , s.t.  $|\omega| \leq 2^{k_0}$ . From the fact that  $\{\widehat{f}_n(\omega)\}_n$  is monotonically increasing,  $\widehat{f}_n(\omega)$  converges to  $\widehat{f}(\omega)$ , and (5), we have as desired  $\widehat{f}(\omega) \geq \widehat{f}_{2^{n_{k_0}}}(\omega) \geq G(|\omega|)$ . ■

**Theorem 5:** Let  $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$  be an arbitrary monotonically increasing continuous function, with  $M(0) > 0$  and  $\lim_{|\omega| \rightarrow \infty} M(\omega) = +\infty$ . Then there exists a function  $f_* \in L^1(\mathbb{R})$ , for which it holds:

$$\lim_{|\omega| \rightarrow \infty} |\widehat{f}_*(\omega)| M(|\omega|) = +\infty.$$

Further,  $f_*$  can be chosen, s.t.  $\widehat{f}_*$  is real and non-negative.

**Proof:** Define the function  $G := 1/\sqrt{M}$ . Notice that  $G$  fulfills the requirements given in Thm. 4. Thence, there exists a function  $f_*$ , for which  $|\widehat{f}_*(\omega)| M(|\omega|) \geq \sqrt{M(|\omega|)}$ , for all  $\omega \in \mathbb{R}$ , which immediately gives the desired statement. ■

**Remark 2:** Similar to Rem. 1, Thm. 4 and Thm. 5 can appropriately be modified to handle functions on the frequency domain.

#### 5. A NEW PROOF OF THE STRONG DIVERGENCE OF SHANNON'S SAMPLING SERIES (SSS)

In this section, we aim to give a construction of functions in  $\mathcal{PW}_{\omega_g}^1$ , where  $\omega_g \in \mathbb{R}^+$ , whose corresponding SSS diverges strongly. To be more specific, we want construct a universal function  $f \in L^1(\mathbb{R})$ , such that for each  $\omega_g \in \mathbb{R}^+$ , we can in turn construct another function  $f_*$  by means of  $f$ , such that the SSS of the band-limited interpolation  $\check{f}_{*,\omega_g}$  of the IFT of  $f_*$ , strongly diverges, i.e. (2) holds for  $\check{f}_{*,\omega_g}$ . Notice that, as the adjective "strong" asserts, this sort of divergence, in comparison to the lim sup/weak divergence of SSS (see (1)) abdacts the existence of a subsequence  $\{N_l\}_l$ , for which:

$$\limsup_{k \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \frac{1}{\omega_g} \sum_{k=-N_l}^{N_l} \check{f}_{*,\omega_g} \left( \frac{k\pi}{\omega_g} \right) \frac{\sin(\omega_g(t - \frac{k\pi}{\omega_g}))}{t - \frac{k\pi}{\omega_g}} \right| < \infty, \quad (6)$$

and hence the convergence of the series given in (6) to  $\check{f}_{*,\omega_g}$ . For signal processing aspect, (2) abnegates the existence of an adaptive method, based on clever choice of such subsequence  $\{N_l\}_l$ , to reconstruct signals in  $\mathcal{PW}_{\pi}^1$  by means of SSS. This construction is clearly stronger than the existent one given in [1], because it is there given for each  $\omega_g \in \mathbb{R}^+$  by an explicit construction.

To fulfill this task, first suppose that we have a function  $G : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$  fulfilling the conditions given in Thm. 4 (we shall soon discuss about the choice of such function), for which the following holds:

$$\lim_{N \rightarrow \infty} \sum_{k=0}^N G \left( \frac{k\pi}{\omega_g} \right) \frac{1}{N + \frac{1}{2} - k} = \infty, \quad \forall \omega_g \in \mathbb{R}^+. \quad (7)$$

Thm. 4 and Rem. 2 asserts that we can find a function  $f \in L^1(\mathbb{R})$ , whose IFT is real and non-negative, and fulfills  $\check{f}(t) \geq G(|t|)$ , for

all  $t \in \mathbb{R}$ . In particular, we have for every  $\omega_g \in \mathbb{R}^+$ :

$$\check{f}\left(\frac{k\pi}{\omega_g}\right) \geq G\left(\frac{k\pi}{\omega_g}\right), \quad \forall k \in \mathbb{Z}. \quad (8)$$

Now, let  $\omega_g \in \mathbb{R}^+$  be fixed. Define another function  $f_*$  by  $f_*(\omega) := f(\omega + \omega_g)$ ,  $t \in \mathbb{R}$ . It is not hard to see that  $f_* \in L^1(\mathbb{R})$ , and that the following holds:

$$\check{f}_*\left(\frac{k\pi}{\omega_g}\right) = (-1)^k \check{f}\left(\frac{k\pi}{\omega_g}\right), \quad k \in \mathbb{Z}. \quad (9)$$

Of course we can give the band-limited interpolation  $\check{f}_{*,\omega_g} \in \mathcal{PW}_{\omega_g}^1$  of  $\check{f}_*$ , for which it holds:

$$\check{f}_{*,\omega_g}\left(\frac{k\pi}{\omega_g}\right) = \check{f}_*\left(\frac{k\pi}{\omega_g}\right), \quad (10)$$

by setting:  $f_{*,\omega_g}(\omega) := \sum_{k=-\infty}^{+\infty} f_*(\omega + 2\omega_g k)$ ,  $\forall |\omega| \leq \omega_g$ , and 0 else (see e.g. [24]).

Notice that for  $N \in \mathbb{N}$ , the finite SSS of  $\check{f}_{*,\omega_g}$  (2) at the time instance  $\tilde{t}_N := t_N(\pi/\omega_g)$ , where  $t_N := (N + (1/2))$  can be written as  $\sum_{k=-N}^N \check{f}_{*,\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\pi(t_N - k))}{\pi(t_N - k)}$ . Now, by the addition Thm. for trigonometric functions and the relations (10) and (9), one may obtain:

$$\sum_{k=-N}^N \check{f}_{*,\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\pi(t_N - k))}{\pi(t_N - k)} = \frac{\sin(\pi t_N)}{\pi} \sum_{k=-N}^N \frac{\check{f}\left(\frac{k\pi}{\omega_g}\right)}{N + \frac{1}{2} - k}.$$

By giving explicitly the value of  $\sin(\pi t_N)$  in above expression, noticing that  $\check{f}$  is non-negative, and by (8), we have:

$$\left| \sum_{k=-N}^N \check{f}_{*,\omega_g}\left(\frac{k\pi}{\omega_g}\right) \frac{\sin(\pi(t_N - k))}{\pi(t_N - k)} \right| \geq \frac{1}{\pi} \sum_{k=0}^N \frac{G\left(\frac{k\pi}{\omega_g}\right)}{N + \frac{1}{2} - k},$$

Collecting all the previous observations, and by assumption (7), it is not hard to see that (2) holds for  $\check{f}_{*,\omega_g}$ , as desired. Of course, by (10), (2) holds also for  $f_*$ .

Now, it remains to construct the function  $G$ , for which (7) holds. Notice that it is sufficient to require that  $\forall \omega_g \in \mathbb{R}^+$ ,  $\lim_{N \rightarrow \infty} G((N\pi)/\omega_g) \log(N + 2) = \infty$ . For instance, the function  $G$  given by  $G(t) = 1$ , for  $t \leq 10$ , and for  $t > 10$ ,  $G(t) = \log(\log(10))/\log(\log(t))$ , fulfills above condition and hence (7).

## 6. ON THE SMOOTHNESS OF THE FT

The RLL (Thm. 2) asserts besides that the FT of an integrable function is continuous. The following Thm. gives the corresponding specification of this statement:

**Theorem 6:** Let  $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  an arbitrary monotonically increasing continuous function, with  $\mu(0) := \lim_{h \rightarrow 0^+} \mu(h) = 0$ . Given an arbitrary point  $\omega_* \in \mathbb{R}$ . Then the set of all  $f \in L^1(\mathbb{R})$ , for which:

$$\limsup_{h \rightarrow 0} \frac{|\hat{f}(\omega_* + h) - \hat{f}(\omega_*)|}{\mu(h)} = +\infty,$$

holds, is a residual set.

**Sketch of proof:** For fixed  $\omega \in \mathbb{R}$ , we have  $\hat{f}(\omega_* + h) - \hat{f}(\omega_*) := \int_{-\infty}^{+\infty} f(t) e^{-i\omega_* t} (e^{iht} - 1) dt$ , where  $h > 0$ . We aim to analyze for  $\omega_* \in \mathbb{R}$  and  $h > 0$ , the behaviour of  $\Psi_{\omega_*,h} \in L^1(\mathbb{R})^*$ , given by:  $\Psi_{\omega_*,h} f := \int_{-\infty}^{+\infty} f(t) e^{-i\omega_* t} (e^{iht} - 1) dt$ . Now, for  $c \in \mathbb{R}^+$ ,

define the function  $f_c$ , by  $f_c(t) := ce^{i\omega_* t}$ , for  $|t| \leq 1/2c$ , and  $f_c(t) := 0$  else. By simple computations, one obtains:  $\Psi_{\omega_*,h} f_c = [\text{sinc}(\frac{h}{2c}) - 1]$ . For a fixed choice of  $h > 0$ , set  $c_* = h/2\pi$ , which yields the estimation  $|\Psi_{\omega_*,h} f_{c_*}| = 1$ , implying  $\|\Psi_{\omega_*,h}\| \geq 1, \forall h \in \mathbb{R}^+$ . Now let  $\mu$  be an arbitrary function fulfilling the requirements given in this Thm. Define  $\Psi_{\omega_*,h,\mu} \in L^1(\mathbb{R})^*$  by  $\Psi_{\omega_*,h,\mu} f := (\Psi_{\omega_*,h} f)/\mu(h)$ . From the latter estimation of  $\|\Psi_{\omega_*,h}\|$ , we have  $\|\Psi_{\omega_*,h,\mu}\| \geq 1/\mu(h)$ , and correspondingly  $\lim_{h \rightarrow 0} \|\Psi_{\omega_*,h,\mu}\| \geq \lim_{h \rightarrow 0} 1/\mu(h) = +\infty$ . Thus  $\sup_{h > 0} \|\Psi_{\omega_*,h,\mu}\| = +\infty$ , and correspondingly by Thm. 3, we obtain the desired result. ■

**Remark 3:** Let be  $\omega \in \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{C}$  be a continuous function, and let  $\gamma_{g,\omega}$  denotes the (local) modulus of continuity (MOC) of  $g$  at  $\omega$ , i.e. a continuous monotonically increasing function  $\gamma_{g,\omega} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  vanishing at 0, and for which it holds:  $\forall h > 0 : |g(\omega + h) - g(\omega)| \leq \gamma_{g,\omega}(|h|)$ . In some sense, MOC specifies the continuity behaviour of the function  $g$  at  $\omega$ , and gives a measure on the smoothness of  $g$  at the point  $\omega$ . So we may alternatively formulate Thm. 6 as follows: Given a frequency  $\omega_* \in \mathbb{R}$ , and a function  $\mu$  satisfying the conditions in that Thm.. Then typically functions in  $L^1(\mathbb{R})$  have a FT, which does not admit  $\mu$  as the MOC at  $\omega^*$ .

**Remark 4:** We are also capable to give a stronger result than that, given in Thm. 6, in the following sense:

"Let  $\mu$  be a function fulfilling the requirements given in Thm. 6. The set  $\mathcal{D}_\mu$  of all  $f \in L^1(\mathbb{R})$ , such that the set:

$$\mathcal{P}_{\text{Div}}^{(\mu)}(f) := \left\{ \omega \in \mathbb{R} : \limsup_{h \rightarrow 0} \frac{|\hat{f}(\omega + h) - \hat{f}(\omega)|}{\mu(h)} = +\infty \right\}$$

is a residual set in  $\mathbb{R}$ , forms a residual set in  $L^1(\mathbb{R})$ ."

Roughly, this means that typically the FT of an integrable function is arbitrarily weak continuous on typical points on the real line. The corresponding proof shall be given in the subsequence work.

## 7. DISCUSSION - OUTLOOK - RELATIONS TO PRIOR WORKS

In this work, we are able to show, by means of the technique inspired by the Banach-Steinhaus Thm., some tightening of the famous RLL, which constitutes one of the important foundation of signal processing: the FT of an integrable signal does typically not behave regularly, in the sense, that generically the FT of an integrable signal decays arbitrarily slowly toward 0 and has arbitrarily weak continuity behaviour, although the results are given weakly by means of limit superior. Up until now, it was uncertain, whether the RLL can be tightened, without giving further constraints to the considered signals (e.g. differentiability). Furthermore, the behaviour concerning to the decay of the FT given in this work holds also even for the more restrictive class  $L^p([- \omega_g, \omega_g])$ ,  $\omega_g > 0$ , and  $p \in [1, \infty]$  of concentrated signals. All the results found in this work holds of course also for the IFT of integrable signals, as to give insight to the decay behaviour and continuity behaviour of i.a. band-limited signals.

Also, we are able to give a construction of a signal, which strongly has an arbitrarily slow decay behaviour. However, as [25] might assert, the set of such signals is topologically small in  $L^1(\mathbb{R})$ . The corresponding technique for proving that result gives also new sight to the strong divergence phenomenon [1] of the SSS and a stronger proof for the divergence of SSS for band-limited signals  $f \in \mathcal{PW}_{\omega_g}^1$ , where  $\omega_g \in \mathbb{R}^+$ .

## 8. REFERENCES

- [1] H. Boche and B. Farrell, “Strong divergence of reconstruction procedures for the paley-wiener space  $\mathcal{PW}_\pi^1$  and the hardy space  $\mathcal{H}^1$ ,” *Journal of Approximation Theory*, vol. 183, pp. 98–117, 2014.
- [2] J. Higgins, *Sampling theory in Fourier and signal analysis: foundations*. Oxford University Press, 1996.
- [3] K. Gröchenig, *Foundations of Time-Frequency Analysis*. Birkhäuser, 2001.
- [4] I. Daubechies, *Ten lectures on wavelets*. SIAM, 1992, vol. 61.
- [5] C. Shannon, “Communication in the presence of noise,” *Proc. IRE*, vol. 37, no. 1, pp. 10–21, 1949.
- [6] D. Gabor, “Theory of communication. part 1: The analysis of information,” *Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering*, vol. 93, no. 26, pp. 429–441, 1946.
- [7] S. Mallat, “A theory for multiresolution signal decomposition: the wavelet representation,” *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, vol. 11, no. 7, pp. 674–693, 1989.
- [8] M. Nagahara, D. E. Quevedo, and D. Nesić, “Maximum hands-off control: a paradigm of control effort minimization,” *IEEE Transactions on Automatic Control*, vol. 61, no. 4, 2016.
- [9] A. J. Jerri, “The shannon sampling theorem - its various extensions and applications: A tutorial review,” *Proceedings of the IEEE*, vol. 65, no. 11, pp. 1565 – 1596, Nov. 1977.
- [10] J. R. Higgins, “Five short stories about the cardinal series,” *Bull. Amer. Math. Soc.*, vol. 12, no. 1, pp. 45 – 89, 1985.
- [11] P. L. Butzer, W. Splettster, and R. L. Stens, “The sampling theorem and linear prediction in signal analysis,” *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 90, no. 1, pp. 1 – 70, Jan. 1988.
- [12] F. Marvasti, *Nonuniform Sampling: Theory and Practice*. Kluwer Academic / Plenum Publishers, 2001.
- [13] H. Boche and U. J. Mönich, *New perspectives on Approximation and Sampling Theory: Applied and Numerical Harmonic Analysis*. Springer International Publishing, 2014, ch. Signal and System Approximation from General Measurements, pp. 115 – 148.
- [14] J. L. Brown, “On the error in reconstructing a nonbandlimited function by means of the bandpass sampling theorem,” *Journal of Mathematical Analysis and Applications*, vol. 18, pp. 75–84, 1967.
- [15] H. Boche and U. J. Mönich, “There exists no globally uniformly convergent reconstruction for the paley-wiener space  $\mathcal{PW}_\pi^1$  of bandlimited functions sampled at nyquist rate,” *IEEE Trans. Signal Process.*, vol. 56, no. 7, p. 31703179, July 2008.
- [16] H. Boche and V. Pohl, *Sampling Theory - A Renaissance*, ser. Applied and Numerical Harmonic Analysis. Birkhäuser Mathematics, 2015, ch. System Approximations and Generalized Measurements in Modern Sampling Theory.
- [17] L. V. Kantorovich and G. P. Akilov, *Functional analysis in normed spaces*. New York: Pergamon Press, 1964.
- [18] W. Rudin, *Functional Analysis*, 2nd ed. McGraw-Hill Book Company, 1991, int. Ed. 1991.
- [19] J. Oxtoby, *Measure and Category : A Survey of the Analogies between Topological and Measure Spaces*, ser. Graduate texts in mathematics. New York: Springer-Verlag, 1980.
- [20] S. Banach and H. Steinhaus, “Sur le principe de la condensation de singularités,” *Fund. Math.*, vol. 9, pp. 50–61, 1927.
- [21] W. Rudin, *Real and Complex Analysis*, 3rd ed. McGraw-Hill Book Company, 1921, int. ed. 1987.
- [22] V. Pohl and H. Boche, *Advanced Topics in System and Signal Theory: A Mathematical Approach*, ser. Foundations in Signal Processing, Communications and Networking. Springer-Verlag, 2009, vol. 4.
- [23] A. Zygmund, *Trigonometric Series*, 3rd ed. Cambridge University Press, 1935, vol. 1.
- [24] H. Boche and E. Tampubolon, “On the existence of the band-limited interpolation of a non-band-limited signals,” *In Preparation*, 2015.
- [25] H. Boche and U. J. Mönich, “A general approach for convergence analysis of adaptive sampling-based signal processing,” in *Sampling Theory and Applications (SampTA), 2015 International Conference on*, May 2015, pp. 211–215.