Coding Schemes for Discrete Memoryless Broadcast Channels with Rate-Limited Feedback

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Objective
Investigate whether and how rate-limited feedback increases the
nonfeedback capacity of some discrete memoryless broadcast channels
(DMBC).

Definitions
Definition 1: A DMBC is called strictly less-noisy if
\( I(U; Y_1) > I(U; Y_2) \)
holds for all auxiliary \( U - X - (Y_1, Y_2) \) with \( I(U; Y_2) > 0 \).

Definition 2: A DMBC is called strictly essentially less-noisy \((Y_2 > Y_1)\) if
given any \( P_{X|Y_1} \), \( I(U; Y_1) > I(U; Y_2) \)
holds for all auxiliary \( U - X - (Y_1, Y_2) \) with \( I(U; Y_2) > 0 \), where \( P_{X|Y_1} \) is a
sufficient class. Given a BC, for any joint pmf \( P_{UX} \), there exists a joint pmf
\( P_{UX|Y_1} \) that satisfies
\[
\sum_{u,v,x} P_{UX|Y_1}(u,v,x) \leq P_{UX}(u,x)
\]
\[
l_{1}(U) \leq l_{1}(U; Y_1)
\]
\[
l_{2}(U) \leq l_{2}(U; Y_2)
\]
\[
l_{1}(V; X|Y_1) \leq l_{1}(V; X|Y_W)
\]
\[
l_{2}(V; X) \leq l_{2}(V; X|Y_1)
\]
\[
l_{1}(V; X|Y_1) = l_{1}(V; X|Y_W)
\]
\[
l_{2}(V; X) = l_{2}(V; X|Y_1)
\]

Remark 1: Superposition coding is optimal for strictly (essentially) less-noisy
DMBC for some \( P_{UX|Y_1} \).

Non Achievable Regions

Theorem 1:
For DMBCs with feedback, the capacity region \( C_{R_{\text{FB}}} \) includes the region \( R_{\text{FB}} \):
\[
R_{1} \leq I(U; Y_1|Q)
\]
\[
R_{2} \leq I(U; Y_2|Q - I(U; Y_1|U, Y_2, Q))
\]
\[
R_{2} \leq I(U; Y_2|U, Y_1|Q)
\]
for some pmf \( P_{Q|Y_1,Y_2,U}P_{Y_1|Y_2,U}P_{Y_2|U,Q} \) satisfying
\[
I(U; Y_1|U, Y_2, Q) \leq R_{\text{FB}}
\]
and
\[
I(U; Y_2|U, Y_1|Q) \leq R_{\text{FB}}
\]

Theorem 2:
For DMBCs with feedback, \( C_{R_{\text{FB}}} \) includes the region \( R_{\text{FB}2} \):
\[
R_{1} \leq I(U; Y_1|Q)
\]
\[
R_{2} \leq I(U; Y_2|Q - I(U; Y_1|U, Y_2, Q))
\]
\[
R_{2} \leq I(U; Y_2|U, Y_1|Q)
\]
for some pmf \( P_{Q|Y_1,Y_2,U}P_{Y_1|Y_2,U}P_{Y_2|U,Q} \) satisfying
\[
I(U; Y_1|U, Y_2, Q) \leq R_{\text{FB}}
\]
and
\[
I(U; Y_2|U, Y_1|Q) \leq R_{\text{FB}}
\]

Usefulness of Feedback

Theorem 3:
Assume \( R_{\text{FB}} > 0 \). For strictly essentially less-noisy DMBCs:
1. If \((R_1 > 0, R_2 > 0) \in (\text{int}(C_{R_{\text{FB}}})) \cap (\text{int}(C_{R_{\text{FB}2}})) \), then \((R_1, R_2) \in \text{int}(C_{R_{\text{FB}2}})) \).
2. If \( C_{R_{\text{FB}}} \neq C_{R_{\text{FB}2}} \), then \( C_{R_{\text{FB}}} \subset C_{R_{\text{FB}2}} \), i.e. feedback strictly increases the nonfeedback capacity region.

Examples

\( Y_1 = X \oplus Z \), with indep. \( Z \sim \text{Bern}(p) \) and \( 0 < p_1 \leq p_2 \leq 1/2 \)
\( X \rightarrow Y_1 \sim \text{BSC}(p) \) and \( X \rightarrow Y_2 \sim \text{BEC}(\epsilon) \)
\( 0 < \epsilon < H(p) \).

Summary

- Feedback can increase the capacity of strictly essentially less-noisy DMBC.
- Feedback can even increase the capacity for some more capable but not strictly essentially less-noisy DMBC.
- Recover all previously known capacity and degrees of freedom results for memoryless BCs with feedback.
- Results hold up under noisy feedback links if the receivers can code over them.

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Coding Schemes

Scheme achieving \( R_{\text{FB}} \):
- Codebook: In block \( b \in [1: B + 1] \), generate
\[
u_{3}^{2}(m_{a}, b_{a} - 1) = \sum_{u_{2} \in \mathcal{A}_{2}} P_{Y_2}(u_{2} | a_{2}, u_{1}) \text{ for } m_{a} \in [1: 2^{m_{a}}] \]
\[
u_{2}^{2}(m_{a}, b_{a} - 1) = \sum_{u_{2} \in \mathcal{A}_{2}} P_{Y_2}(u_{2} | a_{2}, u_{1}) \text{ for } m_{a} \in [1: 2^{m_{a}}] \]
for \( b_{a} \in [1: 2^{m_{a}}] \).
- Transmitter: In block \( b \in [1: B + 1] \), sends
\[
u_{3}^{2}(m_{a}, b_{a} - 1) \text{ to } b_{a} \text{ back to the transmitter}
- Receiver 2: looks for an index \( m_{a}^{*} \) s.t.
\[
Q_{2}^{*}(m_{a}^{*}, b_{a}^{*} - 1) \text{ s.t.}
\]

Note: \( R_{\text{FB}} \) is achieved by applying Marton’s coding, two-sided feedback and backward decoding to the scheme for \( R_{\text{FB}} \).