Simulating Lévy-frailty copulas built from α-stable Lévy subordinators

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Abstract

The efficiency of three methodologies to simulate Lévy-frailty copulas (a subclass of Marshall–Olkin copulas) built from α-stable Lévy subordinators is investigated. We compare a method based on the recursive formula for general exchangeable Marshall–Olkin copulas, the simulation of the involved α-stable subordinator on a fine grid, and the simulation of the approximation of the α-stable subordinator by a compound Poisson process. We measure efficiency in terms of computational speed, considering different values for the dimension of the copula and the parameter α of the subordinator.

Keywords: Lévy-frailty copula; stable Lévy subordinator; Marshall–Olkin distribution.

1 Introduction

In this work we focus on Lévy-frailty copulas [18, 17, 19, 20]. These constitute the extendible subfamily of Marshall–Olkin copulas. The Marshall-Olkin distribution is based on the intuitive idea of a system influenced by exogenous shocks [21, 3]. It is constructed via fatal shocks, causing extinction times of elements in a system. Due to their stochastic properties, such as exponential marginals, as well as an intuitive construction, Marshall–Olkin distributions became of interest in different applications. Besides, they are not absolutely continuous. When exchangeable, they assign positive probability to the event that several components take the same value. In addition, having extreme-value survival copulas, they are useful in applications in extreme-value theory (see [13]).

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Lévy-frailty copulas are based on a stochastic model with conditionally independent and identically distributed components (c.f. [17, Chapter 4]). This has massive advantages in simulation compared to more general Marshall–Olkin copulas. They impose dependence between initially independent exponential random variables through first-passage times of Lévy subordinators and their analytical form is parameterized in terms of the Laplace exponent of the involved Lévy subordinator. Through this research we exemplary consider Lévy-frailty copulas built from an α-stable Lévy subordinator. The convenient Laplace exponent of this family makes the resulting copula analytically tractable. α-stable distributions follow a heavy tailed distribution with non-existing first moment. More information and further references can be found in [1, 4, 23].

We compare three simulation techniques to simulate Lévy-frailty copulas. We present our results in terms of simulation efficiency, measured by computational time, and compare the algorithms taking into consideration different dimensions of the copula and different values for the index of stability α. In the first technique we use a recursive algorithm to simulate exchangeable Marshall–Olkin copulas. In the second method we simulate the α-stable subordinator on a fine grid and mimic the canonical construction of the Lévy-frailty model. In the last ansatz, we approximate the α-stable subordinator by a compound Poisson process. Besides the obvious use in Monte Carlo studies involving Lévy-frailty copulas, another application is the efficient simulation of the exponential functional of an α-stable Lévy subordinator. Distributional properties of random numbers of such type have been studied in a number of papers (see, e.g., [8]).

The paper is organized as follows: Section 2 recalls the mathematical background and describes the simulation algorithms used. In Section 3 we present the results that show the computational time of each algorithm to simulate Lévy-frailty copulas built from an α-stable Lévy-subordinator as well as an overview on the general comparison of the techniques. Section 4 concludes.

2 Mathematical background and simulation strategies

Throughout this investigation the variables $(X_1, \ldots, X_d)$ are interpreted as lifetimes, hence we describe the dependence between them using survival functions. The link between a survival function and a (survival) copula is given by the analogous version of Sklar's Theorem [25], stating that a multivariate survival function $\hat{F}(x_1, \ldots, x_d) := \mathbb{P}(X_1 > x_1, \ldots, X_d > x_d)$ can be split into its survival marginal functions, $\hat{F}_1, \ldots, \hat{F}_d$, and a copula $\hat{C} : [0,1]^d \to [0,1]$:

$$\hat{F}(x_1, \ldots, x_d) = \hat{C}(\hat{F}_1(x_1), \ldots, \hat{F}_d(x_d)), \quad (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

The aim of this paper is to analyze different numerical techniques to simulate Lévy-frailty copulas. Lévy subordinators, $\{\Lambda_t\}_{t \geq 0}$, are a.s. non-decreasing Lévy processes. Examples are the Poisson process, the IG-process, and the Gamma process, among others. Lévy subordinators can be characterized through their Laplace transform. The Laplace exponent of a subordinator Λ is denoted $\Psi : [0, \infty) \to [0, \infty)$ and the Laplace transform of Λ satisfies $\mathbb{E}[e^{-x\,\Lambda_t}] = e^{-t\Psi(x)}$, $x \geq 0$, $t \geq 0$. The Lévy–Khintchine formula (see [16, 14, 15]) provides a closed-form expression for the Laplace exponent of a Lévy subordinator:

$$\Psi(x) = \mu x + \int_{(0,\infty]} \left(1 - e^{-tx}\right) \nu(dt), \quad x \geq 0,$$

where $\mu \geq 0$ is a non-negative drift and ν is the Lévy measure of the Lévy subordinator, satisfying

$$\nu(B) := \mathbb{E}\left[|\{s \in (0,1] \,:\, \Delta\Lambda_t \neq 0,\, \Delta\Lambda_t \in B\}|\right], \quad B \in \mathcal{B}((0,\infty]), \tag{1}$$

such that $\int_{(0,1]} t\nu(dt) < \infty$ and $\nu\left((0,\epsilon]\right) < \infty$, $\forall \epsilon > 0$. $\mathcal{B}(\cdot)$ is the Borel σ-algebra and $\{\Delta\Lambda_t := \Lambda_t - \Lambda_{t^-}\}_{t\geq 0}$ the jump process of $\{\Lambda_t\}_{t\geq 0}$. A Lévy subordinator Λ is said to be α-stable, $\alpha \in (0,1)$, if it has zero drift $\mu = 0$ and Lévy measure $\nu(dt) = \frac{\alpha}{\Gamma(1-\alpha)} t^{-(1+\alpha)} \mathbb{1}_{\{t\geq 0\}} dt$. The Lévy measure is absolutely continuous with respect to the Lebesgue measure and its Laplace exponent is given by

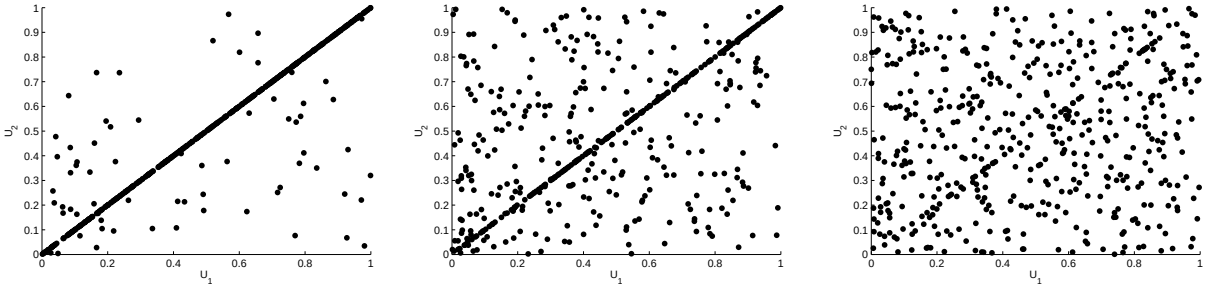
$$\Psi(x) = x^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-xt}\right) t^{-(1+\alpha)} dt, \quad x \geq 0.$$

For a more detailed background on Lévy subordinators, we refer the reader to [4, 5, 23, 6, 10, 22].

Lévy-frailty copulas were introduced in [18]. They are characterized by the Laplace exponent of a Lévy subordinator. More precisely, let Ψ be the Laplace exponent of a Lévy-subordinator Λ satisfying $\Psi(1) = 1$. Then, the associated Lévy-frailty copula is defined as

$$C_\Psi(u_1, \ldots, u_d) := \prod_{i=1}^{d} u_{(i)}^{\Psi(i)-\Psi(i-1)}, \tag{2}$$

where $u_{(1)} < \ldots < u_{(d)}$ is the ordered sequence of $u_1, \ldots, u_d \in [0,1]$.



(a) Lévy-frailty copula built from 0.1-stable Lévy subordinator

(b) Lévy-frailty copula built from 0.5-stable Lévy subordinator

(c) Lévy-frailty copula built from 0.9-stable Lévy subordinator

Figure 1 Scatterplots of 500 samples of a two dimensional Lévy-frailty copula built from an α-stable Lévy subordinator with indexes $\alpha = 0.1$, $\alpha = 0.5$, and $\alpha = 0.9$. We observe that the dependence decreases in α.

Simulation Methodologies

The first simulation technique presented in this work is based on the simulation of exchangeable Marshall–Olkin copulas. Although there are different ways to characterize such copulas (see, e.g., [19, 20]), within this work we define them using d-monotone sequences. A sequence $\{a_0, a_1, \ldots, a_{d-1}\}$ satisfying the condition $(-1)^j \Delta^j a_k \geq 0$, $k = 0, 1, \ldots, d-1$, $j = 0, 1, \ldots, d-k-1$, where Δ is the difference operator $\Delta a_k := a_{k+1} - a_k$, $k \in \mathbb{N}_0$, is called d-monotone. Let $\{a_0, \ldots, a_{d-1}\}$ be a

d-monotone sequence with $a_0 = 1$. Then, the family of exchangeable Marshall–Olkin (eMO) copulas is defined as $eMO := \{\prod_{i=1}^{d} u_{(i)}^{a_{i-1}}\}$, where $u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(d)}$ is the ordered list of $u_1, \ldots, u_d \in [0,1]$. A completely monotone sequences, $\{a_k\}_{k\in\mathbb{N}_0}$, satisfies the stronger condition $(-1)^j \Delta^j a_k \geq 0$, $k, j \in \mathbb{N}_0$. If a d-monotone sequence can be extended to a completely monotone sequence, then one can derive the extendible subfamily of the Marshall–Olkin distribution. This subfamily can be constructed via first-passage times of Lévy subordinators and constitutes the stochastic model behind Lévy-frailty copulas

$$X_k := \inf\{t > 0 : \Lambda_t \geq E_k\}, \quad k \in \{1, \ldots, d\}, \quad d \in \mathbb{N}, \tag{3}$$

where $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of i.i.d. unit exponential random variables and $\{\Lambda_t\}_{t\geq 0}$ is an independent Lévy subordinator with $\Psi(1) = 1$.

The construction in Equation (3) is the main tool of the second simulation technique presented in this paper. [20, Theorem 3.2] proves that the survival copula behind $(X_1, \ldots, X_d)$, $d \geq 2$, built in (3) is the exchangeable Marshall–Olkin copula from (2). For each completely monotone sequence $\{a_k\}_{k\in\mathbb{N}_0}$ there exists a unique Lévy subordinator satisfying $a_k = \Psi(k+1) - \Psi(k)$, see [20, Theorem 3.4].

Algorithm 1 (Simulation of eMO copulas)
Due to the link $a_k = \Psi(k+1) - \Psi(k)$ between Lévy-frailty copulas and (its superclass of) exchangeable Marshall–Olkin copulas, we can use any algorithm for eMOs to simulate a Lévy-frailty copula with Laplace exponent Ψ. Exchangeable Marshall–Olkin distributions can be simulated recursively; afterwards, the margins are transformed to uniform distributions on $[0,1]$. Intuitively, consider a system with d functional components. This system is affected by external shocks that destroy the components. First of all, it is simulated how long it takes until the first shock occurs. By min-stability, this is an exponentially distributed waiting time whose intensity is the sum of all shocks' intensities. Then, it is simulated how many components are destroyed by this shock. The number H of components destroyed at the first event follows a discrete probability distribution given by

$$\mathbb{P}(H = k) = \frac{\binom{d}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} a_{d-k+j}}{\sum_{j=0}^{d-1} a_j}, \quad 1 \leq k \leq d \in \mathbb{N}.$$

By exchangeability, it is not needed to know which components are destroyed, so we can assume w.l.o.g. that the last H components are destroyed, and we randomly permute the components in the very last step. By min stability, we can proceed identically with the subsystem having $d - H$ functional components. This procedure is called recursively until all components are destroyed. A pseudo code for this algorithm and its formal justification can be found in [20, pp. 135–136].

Algorithm 2 (Simulating the α-stable Lévy subordinator on a grid)
This algorithm is based on Equation (3). We simulate the α-stable Lévy subordinator on a fine grid, with dt denoting the size of each step in time. The simulation of paths of an α-stable subordinator is achieved via the cumulative sum of independent $(dt)^{\frac{1}{\alpha}} \mathcal{S}(\alpha)$-distributed random variables, i.e.

$$\Lambda_{t+dt} = \Lambda_t + (dt)^{\frac{1}{\alpha}} \mathcal{S}(\alpha), \quad \alpha \in (0,1),\ t \geq 0, \tag{4}$$

where $\mathcal{S}(\alpha)$ is an α-stable random variable independent of the previously drawn ones. The first passage-times are computed via

$$X_k = \inf\{t > 0 : \Lambda_t \geq E_k\},$$

where $\{E_k\}_{k=1,\dots,d}$ are unit exponential i.i.d. random variables. Clearly, there exists a discretization bias, so the finer one chooses the grid, the more accurate is the simulation. However, considering a finer grid has an immediate influence on the computational effort.

There is a practical consideration that can be taken into account. Instead of checking whether the subordinator Λ_t reaches the barriers E_k, $k = 1, \dots, d$, at every node of the discretized temporal path, one can first sample the vector $(E_1, \dots, E_d)$ and sort it afterwards to $E_{(1)} < \dots < E_{(d)}$. This way, the condition in Equation (2) does not have to be checked more than once through the whole temporal path, i.e. as soon as $\Lambda_t > E_{(k)}$, we obtain $X_{(k)}$, $k = 1, \dots, d$, and we continue analyzing when $\Lambda_{t^*} > E_{(k+1)}$, $t^* > t$ until we obtain $(X_{(1)}, \dots, X_{(d)})$. Applying the order statistics of $(E_1, \dots, E_d)$ on $(X_{(1)}, \dots, X_{(d)})$, $X_{(k)} = X_{|E_{(k)}=E_{\tilde{k}}} = X_{\tilde{k}}$, $\tilde{k} = 1, \dots, d$, we get $(X_1, \dots, X_d)$. The advantage of this pre-ordering is visible for big values of the dimension d (see Figure 2).

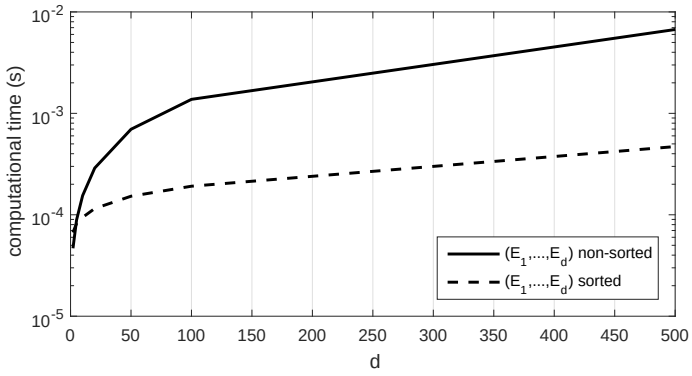


Figure 2 Computational effort (logarithmic) of Algorithm 2 depending on whether the vector $(E_1, \dots, E_d)$ has been sorted or not. The advantage is visible for big values of the dimension d.

The last technique considered is based on the simulation of an approximation of the α-stable subordinator; a suitable compound Poisson process. Considering a sequence of i.i.d. random variables $\{Y_k\}_{k\geq 1}$ and a standard Poisson process $\{N_t\}_{t\geq 0}$, independent of $\{Y_k\}_{k\geq 1}$, the compound Poisson process, $\{X_t\}_{t\geq 0}$, can be represented as $X_t = \sum_{k=1}^{N_t} Y_k$. For details on compound Poisson processes we refer the reader to [23, 24, 1]. From the Lévy–Itô decomposition we know that Lévy subordinators can be written as

$$\Lambda_t = \mu\, t + \int_0^t \int_{x>0} x\, m^{\lambda}(ds, dx),$$

where $x = \Delta\Lambda_t \in [0,\infty)$ represents the size of jumps of the jump process $\{\Delta\Lambda_t := \Lambda_t - \Lambda_{t^-}\}$ and

$$m^{\Lambda}(\omega;[0,t]\times A) := |(s,\Delta\Lambda_s(\omega)\in[0,t]\times A)|, \quad A\subset(0,\infty],\ \forall\omega\in\Omega,$$

is a random jump measure, see [23, Theorem 19.3]. Therefore, Lévy subordinators can be expressed as a combination of a deterministic drift process $\{\mu\, t\}_{t\geq 0}$ and jumps (so called *big jumps* with absolute jump size bigger than $\varepsilon \geq 0$ and *small jumps* with absolute jump size smaller than $\varepsilon \geq 0$). Since subordinators are finite variation processes, i.e. $\int_{(0,1]} t\, v(dt) < \infty$, and in case of α-stable Lévy subordinators $\mu = 0$, α-stable Lévy subordinators can be approximated in the following way:

$$\begin{aligned}\Lambda_t^{\varepsilon} &\approx \sum_{s<t} \Delta\Lambda_s\, \mathbb{1}_{\{\Delta\Lambda_s\geq\varepsilon\}} + \mathbb{E}\left[\sum_{s<t}\Delta\Lambda_s\, \mathbb{1}_{\{0<\Delta\Lambda_s<\varepsilon\}}\right] \\ &\overset{3}{=} \sum_{s<t}\Delta\Lambda_s\, \mathbb{1}_{\{\Delta\Lambda_s\geq\varepsilon\}} + \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\, t. \end{aligned} \tag{5}$$

Note that the term $\frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\, t$ now serves as drift $\mu\, t$. Next, we need the Lévy measure and the Laplace transform of the approximating process Λ_t^{ε}. The Lévy measure contains the information of the average number of jumps in unit time and the magnitude of jumps. The intensity is given by:

$$\int_{\varepsilon}^{\infty} \frac{\alpha}{\Gamma(1-\alpha)} t^{-(1+\alpha)} dt = \frac{1}{\Gamma(1-\alpha)}\left(\varepsilon^{-\alpha} - \lim_{t\to\infty} t^{-\alpha}\right) = \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)}.$$

Lemma 1 (Characteristics of the approximating compound Poisson process)
Let $\{\Lambda_t\}_{t\geq 0}$ be an α-stable Lévy subordinator and let $\{\Lambda_t^{\varepsilon}\}_{t\geq 0}$ be the approximation described in Equation (5). Then, the Lévy measure of Λ_t^{ε} is given by

$$\nu_{\varepsilon}(dt) := \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)} \cdot \frac{\varepsilon^{\alpha}\,\alpha}{t^{1+\alpha}}\, \mathbb{1}_{\{t\geq\varepsilon\}}\, dt, \quad 0 \leq \varepsilon \ll 1,\ \alpha\in(0,1). \tag{6}$$

Proof

$$\begin{aligned}\nu(dt) &= \frac{\alpha}{\Gamma(1-\alpha)\, t^{1+\alpha}}\, \mathbb{1}_{\{t\geq 0\}}\, dt \approx \frac{\alpha\cdot\varepsilon^{-\alpha}\cdot\varepsilon^{\alpha}}{\Gamma(1-\alpha)\, t^{1+\alpha}}\, \mathbb{1}_{\{t\geq\varepsilon\}}\, dt \\ &= \underbrace{\frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)}}_{\text{intensity of jumps}} \cdot \frac{\varepsilon^{\alpha}\,\alpha}{t^{1+\alpha}}\, \mathbb{1}_{\{t\geq\varepsilon\}}\, dt = \nu_{\varepsilon}(dt),\end{aligned}$$

[3] Note that

$$\begin{aligned}\mathbb{E}\left[\sum_{s<t}\Delta\Lambda_s\, \mathbb{1}_{\{0<\Delta\Lambda_s<\varepsilon\}}\right] &= \int_0^t \int_{[0,\infty)} x\, \mathbb{1}_{\{0<\Delta\Lambda_s<\varepsilon\}}\, \nu(dx)ds \\ &= \int_0^t \int_{[0,\infty)} x\, \frac{\alpha}{\Gamma(1-\alpha)}\, x^{-(1+\alpha)}\, dxds \\ &= \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\cdot t.\end{aligned}$$

where $\frac{\varepsilon^\alpha \alpha}{t^{1+\alpha}} \mathbb{1}_{\{t \geq \varepsilon\}}$ *is the density of the Pareto distribution (see, e.g., [2]).* □

We conclude that the compound Poisson approximation of an α-stable subordinator is based on a sum of *big jumps* following a Pareto distribution and the expected value of the *small jumps* as drift. Let us now compute the Laplace exponent of the approximated process $\{\Lambda_t^\varepsilon\}_{t \geq 0}$.

Lemma 2 (Laplace exponent of the approximating compound Poisson process) *Let* $\{\Lambda_t^\varepsilon\}_{t \geq 0}$ *be the approximated subordinator of the* α*-stable subordinator described in Equation* (5)*. Then, the Laplace exponent of* Λ_t^ε *is given by*

$$\Psi_\varepsilon(x) = \frac{(1-\alpha)\,\Gamma(1-\alpha, \varepsilon\, x)\, x^\alpha + \alpha\, \varepsilon^{1-\alpha}\, x + (1-\alpha)\, \varepsilon^{-\alpha}\, (1 - e^{-\varepsilon\, x})}{\alpha\, \varepsilon^{1-\alpha} + (1-\alpha)\, \varepsilon^{-\alpha}\, (1 - e^{-\varepsilon}) + (1-\alpha)\, \Gamma(1-\alpha, \varepsilon)}, \tag{7}$$

such that $\Gamma(\cdot)$ *is the Gamma function and* $\Gamma(\cdot, \cdot)$ *the upper incomplete Gamma function*[4]*.*

Proof

$$\Lambda_t^\varepsilon = \sum_{s<t} \Delta\Lambda_s\, \mathbb{1}_{\{\Delta\Lambda_s \geq \varepsilon\}} + \frac{\alpha\, \varepsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \cdot t$$

where big jumps, $\sum_{s \leq t} \Delta\Lambda_s \mathbb{1}_{\{\Delta\Lambda_s \geq \varepsilon\}}$*, follow the Pareto distribution.*

From the Lévy–Khintchine formula,

$$\begin{aligned}
\hat{\Psi}_\varepsilon(x) &= \mu\, x + \int_{(0,\infty]} (1 - e^{-tx}) \nu_\varepsilon(dt) \\
&= \frac{\alpha\, \varepsilon^{1-\alpha}}{(1-\alpha)\, \Gamma(1-\alpha)}\, x + \frac{\alpha}{\Gamma(1-\alpha)} \int_{(0,\infty]} (1 - e^{-t\, x}) \frac{1}{t^{1+\alpha}}\, \mathbb{1}_{\{t > \varepsilon\}} dt =
\end{aligned}$$

note that $1 - e^{-t\, x} = \int_0^t x e^{-x\, y} dy$*,*

$$= \frac{\alpha\, \varepsilon^{1-\alpha}}{(1-\alpha)\, \Gamma(1-\alpha)}\, x + \frac{\alpha}{\Gamma(1-\alpha)} \int_{(\varepsilon,\infty]} \left[\int_0^t x e^{-x\, y} dy \right] \frac{1}{t^{1+\alpha}} dt,$$

[4] The upper incomplete Gamma function is defined as

$$\Gamma(b, s) := \int_s^\infty u^{b-1}\, e^{-u}\, du, \quad \mathcal{R}(b) > 0,\ s \in \mathbb{Z},$$

that satisfies $\Gamma(b) = \Gamma(b, 0)$.

now we change the order of the integrals,

$$
\begin{aligned}
&= \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\varepsilon}\left[\int_{\varepsilon}^{\infty}\frac{1}{t^{1+\alpha}}dt\right]xe^{-x\,y}dy \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)}\int_{\varepsilon}^{\infty}\left[\int_{y}^{\infty}\frac{1}{t^{1+\alpha}}dt\right]xe^{-x\,y}dy \\
&= \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)}\int_0^{\infty}xe^{-x\,y}dy + \frac{\alpha}{\Gamma(1-\alpha)}\,\frac{x}{\alpha}\int_{(\varepsilon,\infty]}y^{-\alpha}e^{-x\,y}dy,
\end{aligned}
$$

choosing $x\,y = u$,

$$
\begin{aligned}
&= \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)}(1-e^{-\varepsilon\,x}) + \frac{x}{\Gamma(1-\alpha)}\int_{\varepsilon\,x}^{\infty}\frac{u^{-\alpha}}{x^{1-\alpha}}\,e^{-u}du \\
&= \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}\,x + \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)}(1-e^{-\varepsilon\,x}) + \frac{x^{\alpha}}{\Gamma(1-\alpha)}\Gamma(1-\alpha,\varepsilon\,x).
\end{aligned}
$$

Note that $\Psi_{\varepsilon}(1) = 1$, *and*

$$
\hat{\Psi}_{\varepsilon}(1) = \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)} + \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)}(1-e^{-\varepsilon}) + \frac{\Gamma(1-\alpha,\varepsilon)}{\Gamma(1-\alpha)},
$$

therefore,

$$
\begin{aligned}
\Psi_{\varepsilon}(x) &= \frac{\hat{\Psi}_{\varepsilon}(x)}{\hat{\Psi}_{\varepsilon}(1)} \\
&= \frac{(1-\alpha)\,\Gamma(1-\alpha,\varepsilon\,x)\,x^{\alpha} + \alpha\,\varepsilon^{1-\alpha}\,x + (1-\alpha)\,\varepsilon^{-\alpha}\,(1-e^{-\varepsilon x})}{\alpha\,\varepsilon^{1-\alpha} + (1-\alpha)\,\varepsilon^{-\alpha}\,(1-e^{-\varepsilon}) + (1-\alpha)\,\Gamma(1-\alpha,\varepsilon)}.
\end{aligned}
$$

□

Algorithm 3 (Approximation by a compound Poisson subordinator)
In this approach we approximate the α*-stable Lévy subordinator by a compound Poisson process. We simulate its jumps via a Pareto distribution and replace small jumps by their expected value; a term considered as drift. We simulate the process in* (5) *and we compute the first-passage times by* (3). *The pseudo code for this algorithm is given in [20, page 151]. The drift and intensity are given by*

$$
\mu = \frac{\alpha\,\varepsilon^{1-\alpha}}{(1-\alpha)\,\Gamma(1-\alpha)}, \quad \beta = \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)},
$$

and the parameters of the jump distribution are $\boldsymbol{\theta} = (\kappa, \alpha)$, *where* $\kappa > 0$ *is the scale parameter and* $\alpha \in (0,1)$ *the shape parameter of the Pareto distribution.*

Remark 1
It is possible to generate Pareto distributed (pd) random numbers from generalized Pareto distribution (gpd) available in most simulation softwares. The relation between both distributions is given by

$$gpd(x;\mu,\sigma,\xi) = gpd\left(x;\kappa,\frac{\kappa}{\alpha},\frac{1}{\alpha}\right) = pd(x;\kappa,\alpha),$$

$\mu \in (-\infty,\infty)$ is the location parameter, $\sigma \in (0,\infty)$ the scale parameter, and $\xi \in (-\infty,\infty)$ the shape parameter of the generalized Pareto distribution. Standard references for the Pareto distribution and the generalized Pareto distribution are [2, 11].

Since in Algorithm 3 the Lévy-frailty copula is simulated using an approach that is based on an approximation, we analyze the quality of the approximation in the following lemma.

Lemma 3 (Difference of the copula and its approximation)
Let C_{Ψ_ε} be the Lévy-frailty copula parameterized in terms of Ψ_ε. Then the quality of the approximation between C_{Ψ_ε} and C_Ψ is given by

$$\varepsilon \leq \left(\frac{\delta(1-\alpha)\Gamma(1-\alpha,1)}{4(d-1)}\right)^{\frac{1}{1-\alpha}} \Rightarrow \|C_{\Psi_\varepsilon} - C_\Psi\|_\infty \leq \delta, \quad \delta \geq 0,\, \varepsilon \geq 0. \tag{8}$$

Proof
We need to find $\varepsilon = \varepsilon(\delta,\alpha,d) > 0$ that satisfies

$$\|C_{\Psi_\varepsilon}(u_1,\ldots,u_d) - C_\Psi(u_1,\ldots,u_d)\|_\infty = \sup_{u_1,\ldots,u_d\in[0,1]} |C_{\Psi_\varepsilon}(u_1,\ldots,u_d) - C_\Psi(u_1,\ldots,u_d)| \leq \delta. \tag{9}$$

Recall that $\{a_k = \Psi(k+1) - \Psi(k) = (k+1)^\alpha - k^\alpha\}_{k\in\mathbb{N}_0}$ is a completely monotone sequence, i.e., $(-1)^j\Delta^j a_k \geq 0$, $k = 0,1,\ldots,d$, $j = 0,1,\ldots,d-k-1$, so it is a decreasing sequence, i.e.

$$1 = a_0 \geq a_1 \geq \ldots \geq a_k \geq \ldots$$

Let us consider the bivariate case in (9):

$$\sup_{u_1,u_2\in[0,1]} \left|u_{(1)}u_{(2)}^{a_{1,\varepsilon}} - u_{(1)}u_{(2)}^{a_1}\right| = \sup_{u_1,u_2\in[0,1]} u_{(1)}\left|u_{(2)}^{a_{1,\varepsilon}} - u_{(2)}^{a_1}\right| \leq \left|u_{(2)}^{a_{1,\varepsilon}} - u_{(2)}^{a_1}\right| \leq \delta,$$

and note that,

$$\begin{aligned}
&\sup_{u_1,\ldots,u_d\in[0,1]} |C_{\Psi_\varepsilon}(u_1,\ldots,u_d) - C_\Psi(u_1,\ldots,u_d)| \\
&= \sup_{u_1,\ldots,u_d\in[0,1]} \left|u_{(1)}u_{(2)}^{a_{1,\varepsilon}}\cdot\ldots\cdot u_{(d)}^{a_{d-1,\varepsilon}} - u_{(1)}u_{(2)}^{a_1}\cdot\ldots\cdot u_{(d)}^{a_{d-1}}\right| \\
&\leq \left|u_{(d)}^{a_{d-1,\varepsilon}} - u_{(d)}^{a_{d-1}}\right| + \left|u_{(d-1)}^{a_{d-2,\varepsilon}} - u_{(d-1)}^{a_{d-2}}\right| + \ldots + \left|u_{(2)}^{a_{1,\varepsilon}} - u_{(2)}^{a_1}\right|.
\end{aligned}$$

Let us now work with one of the terms on the sum above: $\left|u_{(k+1)}^{a_{k,\varepsilon}} - u_{(k+1)}^{a_k}\right|$. *We apply the mean value theorem, considering* $f(x,t) = x^t$. *Then,*

$$|f(x,y_1) - f(x,y_2)| \leq \left| \sup_{t\in[y_1,y_2]} \frac{\partial}{\partial t} f(x,t)(y_1 - y_2) \right| = \left| x^t \log(x)(y_1 - y_2) \right|, \quad y_1 > y_2 \in \mathbb{R}.$$

Since $x \in [0,1]$,

$$|f(x,y_1) - f(x,y_2)| \leq \left| x^t \log(x) \right| |y_1 - y_2| \leq |y_1 - y_2|.$$

Now choosing $y_1 = a_{k,\varepsilon}$ *and* $y_2 = a_k$:

$$\begin{aligned}
&\left|u_{(k+1)}^{a_{k,\varepsilon}} - u_{(k+1)}^{a_k}\right| \\
&\quad \leq |a_{k,\varepsilon} - a_k| \\
&\quad = \left| \frac{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}\left(e^{-\varepsilon\cdot k}(1-e^{-\varepsilon})\right)}{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)} \right. \\
&\qquad \left. + \frac{(1-\alpha)\left[\Gamma(1-\alpha,\varepsilon(k+1))\,(k+1)^\alpha - \Gamma(1-\alpha,\varepsilon k)\,k^\alpha\right]}{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)} - \left[(k+1)^\alpha - k^\alpha\right] \right|,
\end{aligned}$$

note that $\Gamma(1-\alpha,\varepsilon(k+1)) \leq \Gamma(1-\alpha,\varepsilon k)$,

$$\begin{aligned}
&\leq \left| \frac{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}\left(e^{-\varepsilon\cdot k}(1-e^{-\varepsilon})\right) + (1-\alpha)\Gamma(1-\alpha,\varepsilon k)\left[(k+1)^\alpha - k^\alpha\right]}{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)} \right. \\
&\quad \left. - \left((k+1)^\alpha - k^\alpha\right) \right|,
\end{aligned}$$

since $\Gamma(1-\alpha,\varepsilon k) \leq \Gamma(1-\alpha,\varepsilon)$ *and* $e^{-\varepsilon k} \leq 1$,

$$\begin{aligned}
&\leq \left| \frac{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)\left[(k+1)^\alpha - k^\alpha\right]}{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)} - \left((k+1)^\alpha - k^\alpha\right) \right| \\
&= \left| \frac{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + \left[(k+1)^\alpha - k^\alpha\right]\left[-\alpha\varepsilon^{1-\alpha} - (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon})\right]}{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)} \right|,
\end{aligned}$$

we apply now the triangular inequality,

$$\leq \frac{\left|\alpha\varepsilon^{1-\alpha}\right| + \left|(1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon})\right| + \left|\left[(k+1)^\alpha - k^\alpha\right]\left[-\alpha\varepsilon^{1-\alpha} - (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon})\right]\right|}{\left|\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)\right|},$$

recall that $(k+1)^\alpha - k^\alpha \leq 1$,

$$\leq \frac{2\alpha\varepsilon^{1-\alpha} + 2(1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon})}{\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon)} \overset{(*)}{\leq},$$

due to $\alpha\varepsilon^{1-\alpha} \to 0$ *and* $(1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) \to 0$ *when* $\varepsilon \searrow 0$*, the denominator becomes*

$$\alpha\varepsilon^{1-\alpha} + (1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon}) + (1-\alpha)\Gamma(1-\alpha,\varepsilon) \to (1-\alpha)\Gamma(1-\alpha,\varepsilon),$$

so,

$$\overset{(*)}{\leq} \frac{2\alpha\varepsilon^{1-\alpha} + 2(1-\alpha)\varepsilon^{-\alpha}(1-e^{-\varepsilon})}{(1-\alpha)\Gamma(1-\alpha,\varepsilon)},$$

since $\alpha \in (0,1)$ *and* $\Gamma(1-\alpha,\varepsilon) \geq \Gamma(1-\alpha,1)$*,*

$$\leq \frac{2\varepsilon^{1-\alpha} + 2\varepsilon^{-\alpha}(1-e^{-\varepsilon})}{(1-\alpha)\Gamma(1-\alpha,1)},$$

w.l.o.g. we can consider $\varepsilon \in [0,1]$*, so* $(1-e^{-\varepsilon}) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\varepsilon^n}{n!} \leq \varepsilon$*,*

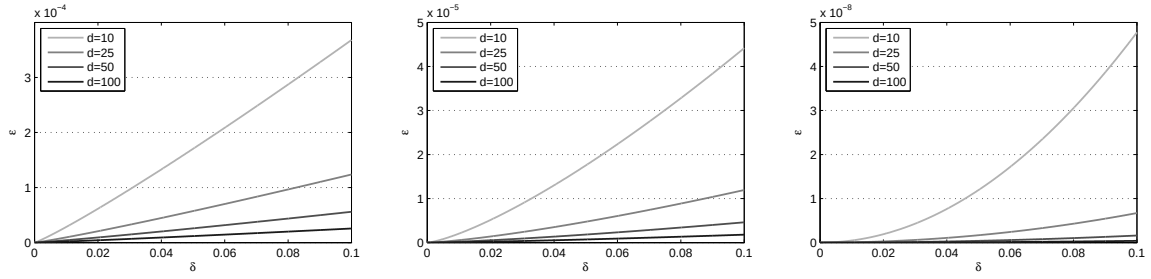
$$\leq \frac{4\varepsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha,1)}.$$

So, if we get the value of ε *such that* $\frac{4\varepsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha,1)} \leq \frac{\delta}{d-1}$*,* $\delta \in \mathbb{R}$*, then*

$$\begin{aligned}
&\sup_{u_1,\ldots,u_d\in[0,1]} |C_{\Psi_\varepsilon}(u_1,\ldots,u_d) - C_{\Psi}(u_1,\ldots,u_d)| \\
&\leq \left|u_{(d)}^{\theta_{d-1,\varepsilon}} - u_{(d)}^{\theta_{d-1}}\right| + \left|u_{(d-1)}^{\theta_{d-2,\varepsilon}} - u_{(d-1)}^{\theta_{d-2}}\right| + \ldots + \left|u_{(2)}^{\theta_{1,\varepsilon}} - u_{(2)}^{\theta_1}\right| \\
&\leq (d-1)\frac{4\varepsilon^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha,1)} \leq \delta.
\end{aligned}$$

□

Figure 3 describes how the parameter ϵ performs depending on the values of δ, the index α of the stable subordinator, and dimension d of the copula.



(a) $\alpha = 0.1$. (b) $\alpha = 0.25$. (c) $\alpha = 0.5$.

Figure 3 Behavior of the parameter ε in terms of δ in Equation (8) for different values of the parameter α and dimension of a copula, d.

3 Results of our simulation study

We now analyze the efficiency of the algorithms described in the previous section. First, we study the computational times for each algorithm for different dimensions d and parameters α. Afterwards, we compare the efficiency of the three algorithms. We analyze the average time of a sample of $k = 100\,000$ scenarios, i.e. we simulate k times the involved copula and compute the average time[5]. We choose $\alpha = 0.1$, $\alpha = 0.25$, and $\alpha = 0.5$ to include different dependence levels.

Algorithm 1

	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
d	CpTime (s)		
2	0.0003	0.0003	0.0004
5	0.0010	0.0012	0.0021
10	0.0033	0.0045	0.0104
20	0.0129	0.0193	0.0641

Table 1 Computational time for Algorithm 1 for different d and α. For higher dimensions and bigger values of α, the algorithm becomes computationally more expensive.

Obviously, the effort is increasing in d, just as expected. One can also observe that for bigger values of α, i.e. for weaker dependence between the elements in the system, the time required to simulate the copula is bigger. This can be explained by the fact that in cases with strong dependence the probability to destroy more elements at once is bigger, so that the algorithm requires less recursions.

Algorithm 2

This algorithm simulates the α-stable Lévy subordinator on a discrete grid and afterwards computes the first-passage times via $X_k := \inf\{t > 0 : \Lambda_t \geq E_k\}$ (see Table 2).

Algorithm 3

We approximate the α-stable Lévy subordinator by a compound Poisson process (see Table 3).

[5]Computational times were computed using Matlab R2009a on a 2.4 GHz PC.

	CpTime (s) ($dt = 10^{-3}$)			CpTime (s) ($dt = 10^{-4}$)		
d	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
2	0.0017	0.0016	0.0012	0.0135	0.0134	0.0105
5	0.0017	0.0016	0.0013	0.0136	0.0137	0.0106
10	0.0017	0.0016	0.0013	0.0136	0.0141	0.0106
20	0.0017	0.0016	0.0013	0.0136	0.0141	0.0107

Table 2 Computational time for Algorithm 2 for different d and α. The paths of the α-stable Lévy subordinator are generated on a grid with step size dt. There are no significant differences regarding the dimension and the parameter α of the copula. However, the computational time increases when we consider a finer grid.

	$\alpha = 0.1$			$\alpha = 0.25$			$\alpha = 0.5$		
d	CpTime (s)	ε	δ	CpTime (s)	ε	δ	CpTime (s)	ε	δ
2	0.0002	10^{-3}	0.0218	0.0002	10^{-4}	0.0175	0.0003	10^{-6}	0.0132
5	0.0002	10^{-4}	0.0110	0.0002	10^{-5}	0.0080	0.0008	10^{-7}	0.0167
10	0.0002	10^{-4}	0.0247	0.0002	10^{-5}	0.0181	0.0011	10^{-7}	0.0375
20	0.0002	10^{-4}	0.0522	0.0002	10^{-5}	0.0381	0.0044	10^{-8}	0.0251

Table 3 Computational time for Algorithm 3 for different d and α. We consider different values of ε in order to get accurate results regarding δ. Regarding the dimension, when α is small, the differences are not significant. However, when the dependence becomes weaker, i.e. α increases, the algorithm becomes more expensive in d. Concerning α, for stronger dependence the algorithm is faster.

3.1 Overall comparison

We can observe that, besides very small dimensions, Algorithm 1 is the slowest, independent of α. It is, however, the only algorithm without discretization bias. If we compare Algorithms 2 and 3, we can conclude that the third algorithm is more efficient than the second one, regardless of the size of the time step in Algorithm 2. The reason resides in how each algorithm checks the canonical condition given in Equation (3). Note that the algorithm based on simulating the compound Poisson process checks this canonical condition only when the process jumps, while the technique based on simulating the α-stable subordinator checks at each discrete point of the grid until the condition is satisfied. So the computational effort in the last method is lower than in the second one. Besides the cheaper computational cost of Algorithm 3, it has another advantage compared to Algorithm 2. While in Algorithm 2 one can not determine the accuracy of the results, in the case of the last simulation technique the accuracy is given in terms of δ, i.e. one can fix the aimed value for δ and depending on this value, compute the value of ε. Due to the low computational effort of Algorithm 3, we additionally simulate high-dimensional copulas and study the resulting computational times.

	$\alpha = 0.1$			$\alpha = 0.25$			$\alpha = 0.5$		
d	CpTime (s)	ε	δ	CpTime (s)	ε	δ	CpTime (s)	ε	δ
50	0.0002	10^{-5}	0.0169	0.0003	10^{-6}	0.0175	0.0354	10^{-9}	0.0204
100	0.0003	10^{-5}	0.0349	0.0003	10^{-6}	0.0353	0.0528	10^{-9}	0.0413

Table 4 Computational times for the simulation of Lévy-frailty copulas in high dimensions using Algorithm 3. The computational cost is higher for bigger copulas when the index α takes values close to 0.5. The differences are not significant for small values of α. Regarding the dependence level, the algorithm is slower when α gets close to 0.5.

Application: The exponential functional of a Lévy subordinator

We consider an example, where the simulation of Lévy-frailty copulas built from α-stable subordinators (in large dimensions) is applied. In [12] it is proven that an exponential functional of a Lévy subordinator, in its terminal value, converges in distribution to the arithmetic mean of a sum of dependent variables that follow the Marshall–Olkin distribution:

$$\lim_{d \nearrow \infty} \frac{X_1 + \ldots + X_d}{d} \overset{\mathcal{L}}{=} I_\infty,$$

where the random variable $I_\infty = \int_0^\infty e^{-\Lambda_t} dt$ is the exponential functional of a Lévy subordinator Λ (standard references on exponential functional of Lévy subordinators can be found in, e.g., [9, 7]). Therefore, the presented algorithms can be used to simulate the underlying Lévy-fraity copulas in high dimensional cases and thus analyze the behavior of the exponential functional of Lévy subordinators.

4 Conclusion

We studied the efficiency of three algorithms to simulate Lévy-frailty copulas built from α-stable subordinators. We measured the efficiency in terms of computational time. We compared the algorithms in different dimensions d and for different parameters α. With respect to the dimension, one can observe that the first method (an unbiased approach based on exchangeable Marshall–Olkin distributions) is faster for small copulas. Studying the results for different values of α, this algorithm is also more efficient when α gets smaller, i.e. for copulas with strong dependence. The second algorithm simulates the α-stable Lévy subordinator in a fine grid and mimics the canonical construction given in Equation (3). We simulated copulas considering different dimensions, dependence levels, and time discretizations. The last technique approximates the α-stable subordinator by a compound Poisson process, having Pareto distributed jumps and a suitable drift. In this ansatz, it is possible to control the discretization error, as shown analytically. The results obtained do not show significant differences regarding simulation time for $\alpha = 0.1$ and $\alpha = 0.25$, but when $\alpha = 0.5$, the algorithm needs more time to sample the copula. This difference becomes even more pronounced when considering higher dimensions, e.g. $d = 50$ and $d = 100$.

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