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Characterizations of Orthogonal Polynomials and Harmonic Analysis on Polynomial Hypergroups

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"God exists since mathematics is consistent, and the Devil exists since we cannot prove it."

(André Weil)[1]

[1] [GCG98, p. 251]

Abstract

We characterize specific classes of orthogonal polynomials in terms of properties which come from (or are related to) harmonic and functional analysis such as cohomology. We consider polynomial hypergroups, which are commutative and come along with a sophisticated harmonic analysis, and, in the first part of the thesis, give a sufficient criterion and a necessary criterion for their ℓ^1-algebras to be weakly amenable. These criteria will be based on growth and smoothness conditions, asymptotics, shift operators and various further ingredients such as the Plancherel isomorphism and the fundamental lemma of the calculus of variations. Moreover, we extensively study point amenability (i.e., the nonexistence of nonzero bounded point derivations w.r.t. symmetric characters) w.r.t. such ℓ^1-algebras. Both of these amenability notions have been known to correspond to certain problems concerning the derivatives of the underlying orthogonal polynomials—and to be surprisingly rarely satisfied; the latter contrasts with L^1-algebras of locally compact groups.

Considering suitable ultraspherical polynomials, we show that there exist polynomial hypergroups whose ℓ^1-algebra is weakly amenable but fails to be (right character) amenable, which solves a problem that has been open for some years. In the second part of the thesis, we completely characterize point and weak amenability for the classes of Jacobi, symmetric Pollaczek and associated ultraspherical polynomials (and for two further classes) by identifying the corresponding parameter regions. Besides our general criteria, each of these classes requires specific analytical techniques: the characterization concerning the Jacobi polynomials makes use of their asymptotics, the Fourier expansions of their derivatives, suitable approximations and inheritance via homomorphisms. The result for symmetric Pollaczek polynomials relies on a transformation into a system whose derivatives are more accessible concerning asymptotic behavior. Our solution for the associated ultraspherical polynomials will benefit from a fruitful interplay between hypergeometric and absolutely continuous Fourier series.

The third part of the thesis deals with symmetric, suitably normalized orthogonal polynomial sequences $(P_n(x))_{n\in\mathbb{N}_0}$ within which we characterize the class of ultraspherical polynomials in terms of certain constancy properties of the Fourier coefficients belonging to $(P'_{2n-1}(x))_{n\in\mathbb{N}}$. Such characterizations may be motivated by amenability considerations, and our result improves previous work of Lasser–Obermaier in terms of the whole sequence $(P'_n(x))_{n\in\mathbb{N}}$; in fact, we shall uncover some redundancy. We obtain similar characterizations for the discrete and—more involved—continuous q-ultraspherical polynomials via $(D_{q^{-1}}P_{2n-1}(x))_{n\in\mathbb{N}}$ and $(\mathcal{D}_q P_{2n-1}(x))_{n\in\mathbb{N}}$, respectively, where D_q denotes the q-difference operator and $\mathcal{D}_q$ denotes the Askey–Wilson operator; these characterizations sharpen earlier results of Ismail–Obermaier. Finally, we characterize a large subclass of the continuous q-ultraspherical polynomials via the averaging operator $\mathcal{A}_q$ and explicitly show that this characterization does not extend to the whole class.

Besides these main results, the thesis contains several elaborated motivating examples and additional discussions, recalls important basics, provides background information, and, finally, briefly explains two possible projects w.r.t. postdoctoral research (Outlook).

Zusammenfassung

Wir charakterisieren spezifische Klassen orthogonaler Polynome über Eigenschaften, welche aus der harmonischen Analysis und Funktionalanalysis wie der Kohomologie kommen (oder Verwandtschaft dazu aufweisen). Wir betrachten polynomiale Hypergruppen – diese sind kommutativ und gehen mit einer eleganten harmonischen Analysis einher – und präsentieren im ersten Teil der Arbeit ein hinreichendes Kriterium sowie ein notwendiges Kriterium dafür, dass ihre ℓ^1-Algebren schwach mittelbar sind. Diese Kriterien gründen auf Wachstums- und Glattheitsbedingungen, asymptotischem Verhalten, Shift-Operatoren und diversen weiteren Bestandteilen wie dem Plancherel-Isomorphismus und dem Fundamentallemma der Variationsrechnung. Des Weiteren untersuchen wir ausgiebig Punktmittelbarkeit (d.h. die Nichtexistenz nicht-verschwindender beschränkter Punktderivationen bzgl. symmetrischer Charaktere) bzgl. solcher ℓ^1-Algebren. Von beiden dieser Mittelbarkeitsbegriffe war bereits bekannt, dass sie gewissen Problemstellungen entsprechen, die die Ableitungen der zugrunde liegenden orthogonalen Polynome betreffen, sowie dass sie überraschend selten erfüllt sind; letzteres steht im Kontrast zu L^1-Algebren lokalkompakter Gruppen.

Indem wir geeignete ultrasphärische Polynome betrachten, zeigen wir, dass es polynomiale Hypergruppen gibt, deren ℓ^1-Algebra schwach mittelbar, jedoch nicht (rechts-Charakter-) mittelbar ist – was ein Problem löst, dass für einige Jahre offen gewesen ist. Im zweiten Teil der Arbeit charakterisieren wir Punktmittelbarkeit und schwache Mittelbarkeit vollständig für die Klassen der Jacobi-, symmetrischen Pollaczek- und assoziierten ultrasphärischen Polynome (sowie für zwei weitere Klassen), indem wir die zugehörigen Parameterregionen identifizieren. Neben unseren allgemeinen Kriterien benötigt jede dieser Klassen spezifische analytische Techniken: Die die Jacobi-Polynome betreffende Charakterisierung benutzt deren asymptotisches Verhalten, die Fourier-Entwicklungen ihrer Ableitungen, geeignete Approximationen sowie Vererbung über Homomorphismen. Das Resultat für die symmetrischen Pollaczek-Polynome stützt sich auf eine Transformation in ein System, dessen Ableitungen zugänglicher bzgl. asymptotischen Verhaltens sind. Unsere Lösung für die assoziierten ultrasphärischen Polynome profitiert von einem fruchtbaren Zusammenspiel zwischen hypergeometrischen Reihen und absolut stetigen Fourierreihen.

Der dritte Teil der Arbeit befasst sich mit symmetrischen, geeignet normalisierten Folgen orthogonaler Polynome $(P_n(x))_{n\in\mathbb{N}_0}$, innerhalb derer wir die Klasse der ultrasphärischen Polynome über gewisse Konstantheitseigenschaften der Fourierkoeffizienten, die zu $(P_{2n-1}'(x))_{n\in\mathbb{N}}$ gehören, charakterisieren. Solche Charakterisierungen können über Mittelbarkeitsbetrachtungen motiviert werden. Unser Resultat verbessert ein früheres von Lasser–Obermaier, welches die gesamte Folge $(P_n'(x))_{n\in\mathbb{N}}$ in Betracht zog und bzgl. dessen wir eine Redundanz erkennen werden. Wir erhalten ähnliche Charakterisierungen für die diskreten und – was komplizierter ist – kontinuierlichen q-ultrasphärischen Polynome über $(D_{q^{-1}}P_{2n-1}(x))_{n\in\mathbb{N}}$ bzw. $(\mathcal{D}_q P_{2n-1}(x))_{n\in\mathbb{N}}$, wobei D_q den q-Differenzenoperator und $\mathcal{D}_q$ den Askey–Wilson-Operator bezeichnet; diese Charakterisierungen verschärfen frühere Resultate von Ismail–Obermaier. Schlussendlich charakterisieren wir eine große Teilklasse der kontinuierlichen q-ultrasphärischen Polynome über den Averaging-Operator $\mathcal{A}_q$ und zeigen explizit, dass diese Charakterisierung nicht für die gesamte Klasse gilt.

Neben diesen Hauptresultaten enthält die Arbeit mehrere ausgearbeitete motivierende Beispiele und zusätzliche Diskussionen, wiederholt wichtige Grundlagen, stellt Hintergrundinformationen zur Verfügung und erklärt abschließend kurz zwei mögliche Projekte mit Blick auf Postdoktorandenforschung (Outlook).

List of publications

This thesis is a publication-based dissertation, based on our papers [Kah15] and [Kah16].

- Our paper "Orthogonal polynomials and point and weak amenability of ℓ^1-algebras of polynomial hypergroups" [Kah15] was first published in Constructive Approximation in

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  published by Springer Science+Business Media New York. The author respects the "Copyright Transfer Statement" which can be found in the appendix of this thesis. Springer holds the copyright of [Kah15]. However, the "Copyright Transfer Statement" explicitly permits the author to include the final published journal article in other publications (such as his dissertation).

- Our paper "Characterizations of ultraspherical polynomials and their q-analogues" [Kah16] was first published in Proceedings of the American Mathematical Society in

  Proc. Amer. Math. Soc. 144 (2016), no. 1, 87–101,
  electronically published on September 4, 2015, DOI: http://dx.doi.org/10.1090/proc/12640 (to appear in print),

  published by American Mathematical Society. The author electronically signed the "Consent to Publish" which can be found in the appendix of this thesis. The American Mathematical Society holds the copyright of [Kah16]. However, the "Consent to Publish" explicitly permits the author to use (part or all of) the work in his own future publications.

Both papers can be found in the appendix.

The author gratefully acknowledges the possibility provided by Springer Science+Business Media New York to include [Kah15] in his dissertation.

The author gratefully acknowledges the possibility provided by the American Mathematical Society, Providence, RI to include [Kah16] in his dissertation.

Due to the publication-based character, parts of this thesis are very similar to our papers [Kah15, Kah16].

Preface and acknowledgments

This work is located at a crossing point between the branches of functional, harmonic and Fourier analysis on the one hand, and the theory of orthogonal polynomials and special functions on the other hand. The focus of our research can be divided into three parts, contained in three corresponding main sections:

- *Part I:* general results on weak amenability—and on the nonexistence of nonzero bounded point derivations w.r.t. symmetric characters—concerning ℓ^1-algebras of polynomial hypergroups (on $\mathbb{N}_0$); weak amenability for ℓ^1-algebras corresponding to ultraspherical polynomials, and the solution to the (previously open) problem whether it is possible that a weakly amenable ℓ^1-algebra of a polynomial hypergroup fails to be amenable. See Section 2 and [Kah15].
- *Part II:* weak amenability—and the nonexistence of nonzero bounded point derivations—for ℓ^1-algebras corresponding to important one-parameter generalizations of the ultraspherical polynomials (Jacobi, symmetric Pollaczek, associated ultraspherical). See Section 3 and [Kah15].
- *Part III:* new characterizations of ultraspherical, discrete q-ultraspherical and continuous q-ultraspherical polynomials in terms of the derivative, the q-difference operator D_q and the Askey–Wilson operator $\mathcal{D}_q$, respectively; new characterization of certain continuous q-ultraspherical polynomials in terms of the averaging operator $\mathcal{A}_q$—and the proof that the latter characterization does not hold for the whole class of continuous q-ultraspherical polynomials. See Section 4 and [Kah16].

The main (and most of the further) results of this thesis can be found in the papers [Kah15, Kah16]; however, the thesis contains also a few results which are not contained in the papers. For instance, the notes at the end of Section 4.4 are a sharpening of the corresponding parts in [Kah16]. Furthermore, in Section 3 we now also consider the notions of α-amenability (or, closely related, φ-amenability) and right character amenability (without exception, the corresponding situations w.r.t. amenability itself are easier to see and have been known before). Concerning an important step in the proof of Theorem 3.2 (which is also [Kah15, Theorem 4.1]), we present a faster yet less elementary variant, see Remark 3.1.

The thesis contains several detailed "motivating examples": Section 2.1 reconsiders weak amenability for the "minimal" (but nevertheless interesting) example of the Chebyshev polynomials of the first kind and discusses arising problems, which motivates our sufficiency criterion Theorem 2.1 [Kah15, Theorem 2.3]. Via an estimation using Dougall's formula, the asymptotic behavior of the gamma function, the power mean inequality and the Riemann zeta function, Section 2.5 applies this sufficiency criterion to the ultraspherical polynomials corresponding to the parameter region properly between the Chebyshev polynomials of the first kind and the Legendre polynomials. On the one hand, this provides first examples of polynomial hypergroups whose ℓ^1-algebra is weakly amenable but fails to be amenable. In fact, we will even obtain simultaneous failure of right character amenability. On the other hand, Section 2.5 motivates the considerably more involved approach for Theorem 3.1 [Kah15, Theorem 3.1] on general Jacobi polynomials. Section 3.1 applies our necessary criterion for weak amenability, Theorem 2.2 [Kah15, Theorem 2.2], to the "non-ultraspherical" subclass of the symmetric Pollaczek polynomials in a rather elementary way (via Euler's infinite product formula for the complex gamma function, elementary approximation and Stirling's formula). This result is then improved in Theorem 3.2 towards establishing even the existence of a nonzero bounded point derivation. Furthermore, it motivates the similar yet no longer elementary study of the associated ultraspherical polynomials leading to Theorem 3.3 [Kah15, Theorem 5.1].

With few exceptions, what cannot be found in the main sections of this thesis are the detailed proofs. Concerning the latter, we explicitly refer to our papers—in which the details are given. Our motivation for writing Section 2, Section 3 and Section 4 was a twofold one: on the one hand, the purpose is to provide a thorough overview, including elaborated motivations for our results (and for our research at all), some background and also some additional information—all of this in a more extensive way than this would have been possible in the paper versions. On the other hand, the purpose is to achieve a concise presentation which is limited to outlines or sketches of those proofs that can be found in the papers [Kah15] and [Kah16] (which are included in this publication-based dissertation).

The introductory Section 1 starts with the Banach–Tarski paradox and Tarski's theorem as historical motivation for amenability considerations and recalls some of the most important amenability notions for groups, hypergroups and—most important for our purposes—Banach algebras: we particularly recall the notions of amenability (Johnson), weak amenability (Bade–Curtis–Dales and Johnson), right character amenability (Kaniuth–Lau–Pym and Monfared) and point amenability (nonexistence of nonzero bounded point derivations w.r.t. symmetric characters, regarded as a global property). We also recall the most important facts about (polynomial) hypergroups and their basic harmonic analysis. Very roughly speaking, the main difference between a group and the more general notion of a hypergroup is that the convolution of two Dirac measures need no longer be a Dirac measure again but still a (more general) probability measure in the latter case; the algebraic group operations are generalized to the hypergroup convolution and involution, which are required to satisfy certain compatibility and non-degeneracy properties. Polynomial hypergroups on $\mathbb{N}_0$ were introduced by Lasser in the 1980s, have a sophisticated harmonic analysis and provide a very rich example class. For the L^1-algebra of a locally compact group G, amenability and right character amenability are known to correspond to the amenability of G (where Johnson's characterization "G amenable $\Leftrightarrow$ $L^1(G)$ amenable" can be regarded as basic motivation to consider amenability notions for Banach algebras), whereas weak amenability and point amenability are always satisfied. For the ℓ^1-algebra of a polynomial hypergroup, however, the situation is very different: although each polynomial hypergroup is known to be amenable in the hypergroup sense, even point amenability of the ℓ^1-algebra, which is the weakest of the four abovementioned properties for Banach algebras, is often not satisfied. Furthermore, the individual behavior strongly depends on the inducing sequence $(P_n(x))_{n\in\mathbb{N}_0}$ of orthogonal polynomials (or on the orthogonalization measure μ) and can quickly lead to major challenges in the theory of orthogonal polynomials and special functions. A very convenient criterion of Lasser (2007) states that the ℓ^1-algebra of a polynomial hypergroup necessarily fails to be amenable whenever the Haar weights tend to infinity. This allows to rule out amenability for most of the naturally occurring examples. Also for right character and point amenability convenient criteria have been available; concerning point amenability, there is a criterion of Lasser (2009) which relates the existence of nonzero bounded point derivations to the derivatives of the underlying orthogonal polynomials.

For weak amenability, however the situation has been much worse: even for ultraspherical polynomials $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$, which for $\alpha \geq -\frac{1}{2}$ form the surely most prominent example class for polynomial hypergroups, the situation has been open for the parameter region properly between the Chebyshev polynomials of the first kind and the Legendre polynomials (i.e., for $\alpha \in \left(-\frac{1}{2}, 0\right)$; as already mentioned above, we will give the solution in Section 2.5). Moreover, there has not been a *convenient* criterion for weak amenability. A characterization of Lasser (2007) in terms of the Fourier coefficients[2] of the (inducing polynomials') derivatives, which will be precisely recalled in Section 1.4, has suffered from several general barriers (such as the requirement of deep—and frequently not explicitly available—knowledge about the inducing

[2] 'Fourier coefficients' means coefficients in expansions w.r.t. the basis $\left\{\frac{1}{\int_{\mathbb{R}} P_n^2(x)\,\mathrm{d}\mu(x)} P_n(x) : n \in \mathbb{N}_0\right\}$.

orthogonal polynomials, and the requirement to deal with the whole space ℓ^∞, despite the fact that many of the powerful tools of harmonic analysis are restricted to proper subspaces). It has also suffered from a lack of applicability: to our knowledge, its sufficient direction has been successfully applied only to examples where also the stronger notion of amenability holds (e.g., to the Chebyshev polynomials of the first kind), and its necessary direction has been successfully applied only to examples where the structure of the underlying orthogonal polynomials is very specific and allows for explicit computations (e.g., to the ultraspherical polynomials for $\alpha \geq 0$) or where one might also argue via point derivations instead (e.g., ultraspherical polynomials for $\alpha \geq \frac{1}{2}$). The two main results of Section 2 are our sufficiency criterion and our necessary criterion mentioned above, which are both based on Lasser's characterization but overcome the described problems in a satisfying way: the sufficient criterion Theorem 2.1 relies on limiting behavior of orthogonal polynomials, growth conditions, the Plancherel isomorphism and several further ingredients such as the fundamental lemma of the calculus of variations, and its combination with inheritance via homomorphisms will turn out to be strong enough to establish weak amenability whenever this property occurs in the whole classes of Jacobi, symmetric Pollaczek and associated ultraspherical polynomials. The necessary criterion Theorem 2.2 relies on shift operators and shows that the ℓ^1-algebra cannot be weakly amenable if the orthogonalization measure does not behave sufficiently "badly". It will essentially contribute to ruling out weak amenability when this property fails in the three just mentioned classes. Section 2 provides also helpful results concerning point amenability of ℓ^1-algebras of polynomial hypergroups.

Theorem 3.1, Theorem 3.2 and Theorem 3.3 are the main results of Section 3. They provide the following characterizations, which are full descriptions of point and weak amenability for the classes of Jacobi, symmetric Pollaczek and associated ultraspherical polynomials:

- The ℓ^1-algebra induced by the Jacobi polynomials $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ is point amenable if and only if $\alpha < \frac{1}{2}$, and weakly amenable if and only if $\alpha < 0$.
- The ℓ^1-algebra induced by the symmetric Pollaczek polynomials $(Q_n^{(\alpha,\lambda)}(x))_{n\in\mathbb{N}_0}$ is point amenable if and only if $\alpha < \frac{1}{2}$ and $\lambda = 0$, and weakly amenable if and only if $\alpha < 0$ and $\lambda = 0$.
- The ℓ^1-algebra induced by the associated ultraspherical polynomials $(A_n^{(\alpha,\nu)}(x))_{n\in\mathbb{N}_0}$ is point amenable if and only if $\alpha < \frac{1}{2}$, and weakly amenable if and only if $\alpha < 0$ and $\nu = 0$.

Concerning the general regions from which the parameters are taken such that a polynomial hypergroup is induced, and concerning earlier partial results, we refer to Section 3. Besides applications of general results of Section 2, establishing these characterizations requires specific analytical techniques for each of the three classes. The Jacobi polynomials are tackled via the Chu–Vandermonde and the Pfaff–Saalschütz identity, the Stolz–Cesàro theorem, a technical induction argument and inheritance via homomorphisms, and via their asymptotics. In contrast to the purely ultraspherical case considered in Section 2.5, explicit computations seem to be rather impracticable at some stages—and it will be one of our tasks to avoid them as far as possible. Our proof concerning point amenability for the symmetric Pollaczek polynomials is essentially based on a transformation into a system with easier asymptotic behavior, and on a subsequent estimation of the derivatives at 0. While, for instance, the Jacobi polynomials can be represented as a terminating ${}_2F_1$ hypergeometric series in a rather convenient way, such estimations are much more involved for the symmetric Pollaczek polynomials (for which expedient explicit computations seem to be out of reach—for instance, in the hypergeometric representation of the symmetric Pollaczek polynomials, x occurs both in the argument and in a parameter). Our result on the associated ultraspherical polynomials is based on Euler's transformation for hypergeometric functions, on the location of the zeros of hypergeometric functions, on Pringsheim's theorem and on an own result concerning absolute continuity of Fourier series (which might be of interest of its own). Another important generalization of the ultraspherical polynomials, namely the class of continuous q-ultraspherical (or Rogers) polynomials, has already been known to contain no example such that $\ell^1(h)$ is at least point amenable. Finally, in Section 3.5 we study two further classes (random walk polynomials and cosh-polynomials). We note at this stage that two auxiliary results, namely Lemma 2.4 (i) and Lemma 3.1, can also be found in the author's Master's thesis [Kah12] (the latter with a different proof, however).

Theorem 3.1, Theorem 3.2 and Theorem 3.3 can also be seen as characterizations of certain (subclasses of classes of) orthogonal polynomials. Characterizing specific orthogonal polynomials has a long history. Section 4 is devoted to another type of such characterizations: as already mentioned above, the Fourier coefficients of the derivatives of the underlying orthogonal polynomials play a crucial role with regard to weak amenability of ℓ^1-algebras of polynomial hypergroups. The simplification of the proof of Theorem 3.1 when restricting oneself to the purely ultraspherical subcase (as presented in the motivating Section 2.5) is partially reasoned in the very simple form these Fourier coefficients take for the ultraspherical polynomials; a more detailed discussion is given in Section 4.1. In fact, this simple form, a striking constancy property, *characterizes* the ultraspherical polynomials: in 2008, Lasser and Obermaier found that for a sequence $(P_n(x))_{n\in\mathbb{N}_0}$ of symmetric random walk polynomials[3] with normalization point $A>0$ (i.e., $P_n(A)=1$ $(n\in\mathbb{N}_0)$) and orthogonalization measure μ the following are equivalent: (i) $P_n(x)=P_n^{(\alpha)}(x)$ $(n\in\mathbb{N}_0)$ for some $\alpha>-1$, (ii) $A=1$ and $P_n'(x)=P_n'(1)P_{n-1}^*(x)$ $(n\in\mathbb{N})$. Here, $(P_n^*(x))_{n\in\mathbb{N}_0}$ is the random walk polynomial sequence w.r.t. the normalization point A and the measure $\mathrm{d}\mu^*(x):=(A^2-x^2)\,\mathrm{d}\mu(x)$ (which is well-defined because $\mathsf{supp}\,\mu\subseteq[-A,A]$ for symmetric random walk polynomial sequences). The condition $P_n'(x)=P_n'(1)P_{n-1}^*(x)$ $(n\in\mathbb{N})$ is a concise reformulation of the aforementioned constancy property of the Fourier coefficients.

This result by Lasser and Obermaier has experienced improvements in two directions: on the one hand, in 2011 Ismail and Obermaier found analogues for the classes of discrete and continuous q-ultraspherical polynomials (later, Ismail and Simeonov obtained also extensions to symmetric Al-Salam–Chihara, symmetric Askey–Wilson and symmetric Meixner–Pollaczek polynomials, and the author's Master's thesis [Kah12] contains suitable extensions to the classes of Jacobi and generalized Chebyshev polynomials); the characterization of the discrete q-ultraspherical polynomials $(P_n(x;\alpha:q))_{n\in\mathbb{N}_0}$ uses the q-difference operator D_q (more precisely, $D_{q^{-1}}$), and the characterization of the continuous q-ultraspherical polynomials $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$ is in terms of another q-generalization of the classical derivative, namely in terms of the Askey–Wilson operator $\mathcal{D}_q$ ($q\in(0,1)$; $\alpha,\beta\in\left(0,\frac{1}{\sqrt{q}}\right)$). On the other hand, we have shown in our Master's thesis [Kah12] that the Lasser–Obermaier characterization remains valid if (ii) is replaced by the apparently weaker condition "$A=1$ and $P_{2n-1}'(x)=P_{2n-1}'(1)P_{2n-2}^*(x)$ $(n\in\mathbb{N})$". In fact, it suffices to require a constancy property as described above only for odd indices and only at some carefully chosen points. In contrast to the original Lasser–Obermaier result, this sharpening does no longer follow from older, more "classical" results of Hahn and Al-Salam–Chihara.

Theorem 4.4 [Kah16, Theorem 2.1] is a modification of the just mentioned result of our Master's thesis [Kah12]: Theorem 4.4 is no longer restricted to random walk polynomials but considers more general (symmetric) orthogonal polynomial sequences $(P_n(x))_{n\in\mathbb{N}_0}$. Since this more general setting does no longer contain a condition which a priori enforces that $\mathsf{supp}\,\mu\subseteq[-A,A]$ (or the boundedness of $\mathsf{supp}\,\mu$, or at least the uniqueness of μ, at all), it also requires a more general interpretation of the sequence $(P_n^*(x))_{n\in\mathbb{N}_0}$. Since this improvement does not affect the actual proof, however, we shall mainly focus on the classes of discrete and continuous q-ultraspherical polynomials in Section 4: the main purpose is to sharpen the Ismail–Obermaier results on the

[3] In this thesis, the expression "random walk polynomials" will be used with two different meanings: the current meaning, which corresponds to that in Section 4, differs from the meaning in Section 3.

discrete and continuous q-ultraspherical polynomials in the same way as we have been able to sharpen the Lasser–Obermaier result on purely ultraspherical polynomials in our Master's thesis [Kah12] and in Theorem 4.4; such a project (as part of the author's dissertation) was announced in the outlook of [Kah12]. In this context, the two main results are Theorem 4.5 [Kah16, Theorem 2.2] and Theorem 4.6 [Kah16, Theorem 2.3].

While Theorem 4.5 (discrete q-ultraspherical polynomials) can be established via an induction argument that resembles the proof of Theorem 4.4 (ultraspherical polynomials), Theorem 4.6 (i.e., the analogous characterization of the continuous q-ultraspherical polynomials) requires additional and more sophisticated strategies. The actual reason for the more technical and involved argument is that while the product formula for the q-difference operator basically resembles that of the classical derivative, the product formula for the Askey–Wilson operator essentially relies on an additional operator $\mathcal{A}_q$, an averaging operator. This leads to the problem that one has to simultaneously tackle (determinacy problems concerning) the additional Fourier coefficients w.r.t. this averaging operator. An important idea to overcome this problem is to consider the functions $n \mapsto \mathcal{A}_q[xP_n(x)]$ and determinacy of corresponding Fourier coefficients. Moreover, our proof will rely on a detailed study and some kind of "simultaneous involvement" of the continuous q-ultraspherical polynomials themselves; the latter yields a conclusion which avoids further tedious calculations and considerably shortens the argument.

The fourth main result of Section 4, Theorem 4.7 [Kah16, Theorem 2.4], is a characterization of the continuous q-ultraspherical polynomials with $\beta \leq 1$ in terms of the averaging operator $\mathcal{A}_q$. Provided $q \in (0,1)$, $\beta \in (0,1]$, $P_n\left(\frac{\sqrt{\beta}}{2} + \frac{1}{2\sqrt{\beta}}\right) = 1$ $(n \in \mathbb{N}_0)$ and $P_2(0) = -\beta\frac{1-q}{1-\beta^2 q}$, it yields the equivalence of the following two properties:

(i) $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$,

(ii) the quotient $\frac{\int_{\mathbb{R}} \mathcal{A}_q P_{n+1}(x) P_{n-1}(x)\,\mathrm{d}\mu(x)}{\int_{\mathbb{R}} \mathcal{D}_q P_{n+1}(x) P_n(x)\,\mathrm{d}\mu(x)}$ is independent of $n \in \mathbb{N}$.

Again, this will be shown via an induction argument. The second part of Theorem 4.7 states that this characterization does not hold for the whole class of continuous q-ultraspherical polynomials; more precisely, the characterization becomes wrong if the condition "$\beta \in (0,1]$" is replaced by the weaker condition "$\beta \in \left(0, \frac{1}{\sqrt{q}}\right)$" (which worked in Theorem 4.6). We will establish an explicit counterexample based on the sequence $\left(P_n\left(x;\frac{5}{4}\,\middle|\,\frac{1}{2}\right)\right)_{n\in\mathbb{N}_0}$.

The thesis concludes with a short outlook on possible future research, with a collection of important symbols, with the appendix—which, besides the papers [Kah15, Kah16] with the detailed proofs, the Springer "Copyright Transfer Statement" and the AMS "Consent to Publish", includes summaries of the papers—and, finally, with the references. The BibTeX code of most of the references was taken from the AMS website (MathSciNet), often with slight adjustments.

Apart from pointing out relations to own works, we deliberately refrain from giving references in a preface; however, the precise references will be given in the later parts of this thesis.

In the course of our research, we have frequently used mathematical software (Maple)—to get ideas and conjectures, observe possible simplifications, identify possible asymptotic and limiting or growth behavior, get an idea of numerical values, "check" calculations, plot graphs and so on. Since we think that it is reasonable and standard in modern mathematics to use such valuable auxiliary tools for the described purposes, we will refrain from pointing out the single usages in this thesis. However, we want to emphasize that such mathematical software is not needed to understand the final proofs. The thesis, including our papers [Kah15, Kah16], can be read without any computer usage; in particular, long calculations have been simplified so far that they do not require a computer algebra system and can be made by hand (in reasonable time).

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1. From the Banach–Tarski paradox to amenability notions for hypergroups and orthogonal polynomials

Parts of Section 1 are very similar to our publication [Kah15].

1.1. Historical motivation

In 1924, Banach and Tarski found a result which has become famous far beyond the mathematical community: a ball in $\mathbb{R}^3$ can be split into a finite number of pieces in such a way that these pieces can be reassembled into two balls of the original size; more generally, given any two bounded sets $A, B \subseteq \mathbb{R}^3$ with nonempty interior, it is possible to split A into finitely many pieces and rearrange them to a partition of B, using nothing but rigid motions. In his foreword to [Wag93], Mycielski calls this 'Banach–Tarski paradox' the "most surprising result of theoretical mathematics" [Wag93, p. xi]. Without any doubt, it is indeed counterintuitive from a physical point of view, more precisely: it strikingly shows that a mathematical result, in particular a highly nonconstructive one (in 1964, Solovay showed that the Banach–Tarski paradox is not contained in ZF or at least in ZF + DC, i.e., ZF + the 'axiom of dependent choice', cf. [Wag93, Chapter 13]), may be far away from being compatible with "everyday experience", and only of narrow significance concerning physical theories. From a purely mathematical point of view, the (very subjective) question whether one finds the Banach–Tarski paradox surprising or not may be related to the personal attitude towards the axiom of choice (which, due to Gödel and Cohen, is logically independent of ZF): the constructivist might argue that a counterintuitive, or even "unnatural", axiom should not be expected to produce intuitive theorems, whereas the formalist, or the "pragmatic" analyst who benefits from the Hahn–Banach theorem, from Alaoglu's theorem and from many other results and concepts which would not be available in ZF, has to (and will) approve that his kind of mathematics intrinsically enforces phenomena like the Banach–Tarski paradox (and, in particular, sets which are not Lebesgue-measurable). Across all factions, however, it should be quite unexpected that—despite the generalizations to n dimensions, $n \geq 4$, are valid [Wag93, Chapter 5]—the Banach–Tarski paradox has no analogue in $\mathbb{R}$ or $\mathbb{R}^2$; this is a consequence of Banach's theorem (1923) which shows that the Lebesgue-measures on $\mathbb{R}$ and $\mathbb{R}^2$ extend to isometry-invariant finitely additive measures on $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}(\mathbb{R}^2)$, respectively [Wag93, Corollary 10.9].

The proof of the Banach–Tarski paradox is easier than one might expect; a very readable version, which does not necessarily require to deal with the extensive background theory, can be found in the first three chapters of [Wag93]. A very helpful tool is the concept of 'paradoxical sets' and 'paradoxical groups':

Definition. Let X be a set, and let G be a group that acts on X. Then a subset E of X is called 'G-paradoxical' if there exist a partition $\{A_1, \ldots, A_m, B_1, \ldots, B_n\}$, $m, n \in \mathbb{N}$, of E and $a_1, \ldots, a_m, b_1, \ldots, b_n \in G$ such that as well $\{a_1(A_1), \ldots, a_m(A_m)\}$ as $\{b_1(B_1), \ldots, b_n(B_n)\}$ is a partition of E. G itself is called 'paradoxical' if G is G-paradoxical w.r.t. the left translations.

With this definition, the Banach–Tarski paradox—from now on, we refer to the three-dimensional "standard form" (two balls out of one), and not to the "strong form" (arbitrary bounded sets with nonempty interior)—reads as follows: every ball in $\mathbb{R}^3$ is G_3-paradoxical, where G_3 denotes the isometry group of $\mathbb{R}^3$. The central part of the proof given in [Wag93] is the following ingredient [Wag93, Proposition 1.10]:

Proposition 1.1. *Let X be a set, and let G be a paradoxical group that acts freely (i.e., without nontrivial fixed points) on X. Then X is G-paradoxical.*

In the proof of Proposition 1.1, the axiom of choice is needed to obtain a partition $\{g(N) : g \in G\}$ of X, where $N \subseteq X$ and the G-orbits in X intersect at single points; such a—highly nonconstructive—“coordinate system”, consisting of the orbits on the one hand and the images of N under the group action on the other hand, allows to transfer a “paradoxical decomposition” of G to a paradoxical decomposition of X. Based on Proposition 1.1, in [Wag93] the remaining proof of the Banach–Tarski paradox is done via a specific paradoxical subgroup of $SO(3)$ (a free group of rank 2; partially going back to von Neumann), via the resulting ‘Hausdorff paradox’ for spheres, via an argument which goes back to Sierpiński and makes it possible to handle group actions with at last countably many fixed points and, finally, via the transition from spheres to balls (and the Banach–Schröder–Bernstein theorem if one is interested in the strong form). Roughly speaking, the overall approach is to obtain a rather explicit decomposition w.r.t. highly non-explicit “coordinates” provided by the axiom of choice.

Being older than 90 years now, the Banach–Tarski paradox has experienced many (deep) improvements. In 1991, Pawlikowski proved that the Banach–Tarski paradox is already implied by the Hahn–Banach theorem [Paw91], which is well-known to be weaker than the axiom of choice (and which is also weaker than Alaoglu’s theorem [Lux62, Lux69, Pin72]—Alaoglu’s theorem is known to be still weaker than the axiom of choice [HL71, Lux69]; cf. also [Edw75]). Much earlier results concern minimizing the number of necessary pieces in the decomposition (in 1947, Robinson showed that five pieces are enough yet also necessary, cf. [Wag93, Chapter 4]) and infinite versions (even getting a continuum of spheres from one, Sierpiński 1945, cf. [Wag93, Chapter 6]). In 1994, Dougherty and Foreman gave a decomposition using pieces which have the property of Baire (solution to Marczewski’s problem) [DF94]. In 2005, Wilson established a continuous movement version (solution to de Groot’s problem) [Wil05]. Moreover, the lack of analogues in one or two dimensions suggested another eye-catching question which became known as Tarski’s circle-squaring problem and was answered positively by Laczkovich in 1990: it is possible to cut a disc into finitely many pieces and rearrange them to a square (of necessarily the same area) [Lac90]. Laczkovich’s proof uses the axiom of choice and about 10^{50} pieces, and it shows that the rearrangement can even be done by using only translations; again there is a recent continuous movement version [Wil05]. Related results are the—much older—von Neumann paradoxes for the plane and line, see [Wag93, Chapter 7].

More important for our purposes, however, is Tarski’s theorem: In the light of the Banach–Tarski paradox and Banach’s theorem, it is an interesting question how the (non)existence of paradoxical decompositions is related to the existence of finitely additive measures. The answer was given by Tarski in 1938 [Wag93, Corollary 9.2]:

Theorem 1.1 (Tarski’s theorem)**.** *Let X be a set, let G be a group that acts on X, and let E be a subset of X. The following are equivalent:*

(i) *E is not G-paradoxical,*

(ii) *there exists a G-invariant finitely additive measure $\nu : \mathcal{P}(X) \to [0, \infty]$ with $\nu(E) = 1$.*

Of course, the nontrivial direction is “(i) ⇒ (ii)”. Applying Theorem 1.1 to G itself (and the action via left translations), one obtains [Run02, Corollary 0.2.11]:

Corollary 1.1. *Let G be a group. The following are equivalent:*

(i) *G is not paradoxical,*

(ii) *there exists a left-invariant finitely additive measures $\nu : \mathcal{P}(G) \to [0, 1]$ with $\nu(G) = 1$,*

(iii) *there exists $m \in \ell^\infty(G)^*$ such that $\langle \delta_g * \phi, m\rangle = \langle \phi, m\rangle$ $(g \in G, \phi \in \ell^\infty(G))$ and $\langle 1, m\rangle = \|m\| = 1$; here, $*$ means the convolution of the Dirac measure δ_g with the function ϕ (w.r.t. the discrete topology).*

The Banach–Tarski paradox and Tarski’s theorem influenced various areas of analysis and can be regarded as origin of a large and fruitful branch of harmonic analysis—the branch which deals with *amenability* and related concepts.

1.2. Amenability notions for groups and Banach algebras

Corollary 1.1 motivates the definition of an amenable locally compact group [Run02, Definition 1.1.1; Definition 1.1.3; Definition 1.1.4]:

Definition 1.1. A locally compact group G is called 'amenable' if there exists a left-invariant 'mean' on $L^\infty(G)$, i.e., $m \in L^\infty(G)^*$ such that $\langle \delta_g * \phi, m\rangle = \langle \phi, m\rangle$ $(g \in G, \phi \in L^\infty(G))$ and $\langle 1, m\rangle = \|m\| = 1$; $*$ means the convolution of the Dirac measure δ_g with the function ϕ.

In particular, a locally compact group G is not paradoxical if and only if G is amenable w.r.t. the discrete topology; if so, then G is also amenable w.r.t. the original topology (making it a locally compact group) [Run02, Corollary 1.1.10]. The expression "amenable" was introduced by Day, maybe as a pun (cf. [Run02, Chapter 1.5] for further information).

A left-invariant mean on $L^\infty(G)$ (G being a locally compact group) can be characterized as follows [Run02, Proposition 1.1.2]: a linear functional $m : L^\infty(G) \to \mathbb{C}$ is a left-invariant mean on $L^\infty(G)$ if and only if $\langle \delta_g * \phi, m\rangle = \langle \phi, m\rangle$ $(g \in G, \phi \in L^\infty(G))$, $\langle 1, m\rangle = 1$ and $\langle \phi, m\rangle \geq 0$ ($\phi \in L^\infty(G)$ with $\phi \geq 0$).

There are several sufficient conditions for a locally compact group to be amenable: for instance, finite groups, compact groups, solvable (in particular, Abelian) groups, locally finite groups (i.e., every finite set of elements generates a finite subgroup), and closed subgroups of amenable groups are amenable; every elementary group is amenable, see [Run02, Chapter 1] and [Wag93, Chapter 10].

If a locally compact group contains a closed subgroup which is a free group of rank 2, then it is not amenable [Run02, Corollary 1.2.8]; however, the von Neumann conjecture, i.e., that every non-amenable group contains a subgroup which is a free group of rank 2, was disproved by Ol'shanskii in 1980 [Ol′80].

There are several characterizations of amenable groups. For the moment, we restrict ourselves to the following characterizations in terms of Reiter's conditions P_1 and P_p [Rei68, 3.2 and 6 in Chapter 8] [Pie84, Chapter 2.6] and the Følner condition [Run02, p. 35] [Pie84, Chapter 2.7]:

Proposition 1.2. *Let G be a locally compact group. The following are equivalent:*

(i) *G is amenable,*

(ii) *G satisfies 'Reiter's condition P_1', i.e., for every $\epsilon > 0$ and for every compact subset C of G, there exists a positive function $\phi \in L^1(G)$ with $\|\phi\|_1 = 1$ such that $\|\delta_g * \phi - \phi\|_1 < \epsilon$ $(g \in C)$,*

(iii) *for at least one $p \in [1, \infty)$, G satisfies 'Reiter's condition P_p', i.e., for every $\epsilon > 0$ and for every compact subset C of G, there exists a positive function $\phi \in L^p(G)$ with $\|\phi\|_p = 1$ such that $\|\delta_g * \phi - \phi\|_p < \epsilon$ $(g \in C)$,*

(iv) *for every $p \in [1, \infty)$, G satisfies Reiter's condition P_p,*

(v) *G satisfies the 'Følner condition', i.e., for every $\epsilon > 0$ and for every compact subset C of G, there exists a Borel subset E of G with $0 < \nu_G(E) < \infty$ such that $\frac{\nu_G(gE\Delta E)}{\nu_G(E)} < \epsilon$ $(g \in C)$.*

In Proposition 1.2 (v), ν_G denotes the (more precisely, a fixed) left Haar measure of G, and Δ denotes the symmetric difference. Another good reference is the monograph [Pat88]. In the following, we recall how amenable groups can be characterized in terms of their L^1-algebras and properties which come from cohomology [Dal00].

Recall that, given a Banach algebra A, a Banach space X is called a 'Banach A-bimodule' if there exist continuous bilinear mappings $A \times X \to X$, $(a, x) \mapsto a \bullet x$ and $(a, x) \mapsto x \circ a$, such that $a \bullet (b \bullet x) = ab \bullet x$, $(x \circ a) \circ b = x \circ ab$ and $a \bullet (x \circ b) = (a \bullet x) \circ b$ for all $a, b \in A$ and $x \in X$.

Of course, A itself is a Banach A-bimodule (via the algebra multiplication). More interesting, if X is a Banach A-bimodule, then its dual X^* becomes a Banach A-bimodule—the 'dual module'—via $A \times X^* \to X^*$, $a \bullet f(x) := f(x \circ a)$ and $f \circ a(x) := f(a \bullet x)$ $(x \in X)$. Furthermore, if $\varphi \in \Delta(A)$, where $\Delta(A)$ denotes the 'character space' of A (i.e., the set of 'characters'—i.e., nonzero homomorphisms from A into $\mathbb{C}$),[4] then $\mathbb{C}$ becomes a Banach A-bimodule via $a \bullet x := x \circ a := \varphi(a)x$ $(a \in A, x \in \mathbb{C})$. Following the reference, we denote this Banach A-bimodule by $\mathbb{C}_\varphi$.

If A is commutative, then a Banach A-bimodule X is called a 'Banach A-module' if $a \bullet x = x \circ a$ $(a \in A, x \in X)$. Trivially, if A is commutative, then A itself is a Banach A-module, and if X is a Banach A-module, then so is X^*; furthermore, if $\varphi \in \Delta(A)$, then $\mathbb{C}_\varphi$ is a Banach A-module.

Recall that a linear mapping D from a Banach algebra A into a Banach A-bimodule X is called a 'derivation' if it satisfies the "product rule"

$$D(ab) = a \bullet D(b) + D(a) \circ b \ (a, b \in A),$$

and an 'inner derivation' if

$$D(a) = a \bullet x - x \circ a \ (a \in A)$$

for some $x \in X$. Obviously, each inner derivation is a bounded derivation.

These concepts, which belong to cohomology, enabled Johnson to find the following characterization of amenable groups [Joh72]:

Theorem 1.2 (Johnson's characterization)**.** *Let G be a locally compact group. The following are equivalent:*

(i) *G is amenable,*

(ii) *for every Banach $L^1(G)$-bimodule X, every bounded derivation from $L^1(G)$ into the dual module X^* is an inner derivation.*

This remarkable result was Johnson's motivation to "extend" the definition of amenability to arbitrary Banach algebras, in the following way [Joh72]:

Definition. A Banach algebra A is called 'amenable' if for every Banach A-bimodule X every bounded derivation from A into the dual module X^* is an inner derivation.

There are several ways of characterizing amenable Banach algebras. One of these ways is via approximate diagonals: a Banach algebra A is amenable if and only if there exists a 'bounded approximate diagonal' for A, i.e., a bounded net $(m_\alpha)_{\alpha \in I} \subseteq A \widehat{\otimes} A$ such that $\lim_\alpha (a \cdot m_\alpha - m_\alpha \cdot a) = 0$ and $\lim_\alpha (\pi(m_\alpha)a) = a$ $(a \in A)$; another characterization is in terms of virtual diagonals: A is amenable if and only if there exists a 'virtual diagonal' for A, i.e., $M \in (A \widehat{\otimes} A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi^{**}(M) \cdot a = a$ $(a \in A)$ [Run02, Theorem 2.2.4].

The definition of an amenable Banach algebra and its characterizations suggest an abundance of generalizations (or sharpenings). For instance, a Banach algebra A is called 'essentially

[4] If the character space $\Delta(A)$ of a (commutative) Banach algebra A is endowed with the Gelfand topology, then $\Delta(A)$ is called the 'structure space' of A.

amenable' if the defining condition holds at least for 'neo-unital' Banach A-bimodules X (i.e., $A \bullet X \circ A = X$) [GL04]; if a Banach algebra is unital (or at least has a bounded approximate identity), then the notions of amenability and essential amenability coincide [GL04, p. 231].

A Banach algebra A is called 'approximately amenable' if for every Banach A-bimodule X every bounded derivation from A into the dual module X^* is the strong limit of a net of inner derivations [GL04], and A is called 'pseudo-amenable' if there exists an approximate diagonal for A (which need not necessarily be bounded) [GZ07]; if A has a bounded approximate identity, the notions of approximate amenability and pseudo-amenability coincide [GZ07, Proposition 3.2].

A Banach algebra A is called 'pointwise amenable' if for every Banach A-bimodule X, for every bounded derivation from A into the dual module X^*, and for every $a \in A$, there exists $x \in X^*$ such that $D(a) = a \bullet x - x \circ a$ [DL10].

A is called 'contractible' or 'super-amenable' if for every Banach A-bimodule X every bounded derivation from A into X is an inner derivation [Dal00, Run02].

In this thesis, we shall extensively study the following property [Joh88]:

Definition. A Banach algebra A is called 'weakly amenable' if every bounded derivation from A into A^* is an inner derivation.

Concerning weak amenability, there exist several characterizations and generalizations [Dal00, Run02] again—for instance, considering iterated duals, which yields the notion of permanent weak amenability: a Banach algebra A is called 'permanently weakly amenable' if, for every $n \in \mathbb{N}$, every bounded derivation from A into the nth dual $A^{(n)}$ is an inner derivation; for commutative Banach algebras, permanent weak amenability reduces to weak amenability [DGG98]. Trivially, a commutative Banach algebra A is weakly amenable if and only if there exists no nonzero bounded derivation from A into A^* [BCD87]. Furthermore, a commutative Banach algebra A is weakly amenable if and only if for every Banach A-module X every bounded derivation from A into X is zero [BCD87].

The preceding amenability notions are only an excerpt of the various possibilities, and it is not our aim to give a complete survey. We note that some of the notions suggest further variations (e.g., 'sequential/uniform/weak-$*$ approximate amenability') or reasonable combinations (e.g., 'approximate weak amenability'); of course, there is an abundance of resulting implications, surprising coincidences—and open problems. The corresponding literature is extensive.

If A is a commutative Banach algebra, one has the following implications:[5] "A contractible" $\Rightarrow$ (trivial) "A amenable" $\Rightarrow$ (trivial) "A pointwise amenable" $\Rightarrow$ [DL10, Theorem 1.5.4] "A approximately amenable" $\Rightarrow$ [GZ07, Corollary 3.4] "A pseudo-amenable" $\Rightarrow$ [GZ07, Corollary 3.7] "A weakly amenable". Moreover, if A is also semisimple, then there exists no nonzero continuous derivation from A into A itself (this is a consequence of the Singer–Werner theorem [Dal00, Corollary 2.7.20]).[6] Furthermore: if a Banach algebra A is commutative and contractible, then A is semisimple and finite-dimensional [Dal00, Corollary 2.8.49].

Johnson's analogue to Theorem 1.2 w.r.t. weak amenability is the following [Joh91]:

Theorem 1.3. *For any locally compact group G, $L^1(G)$ is weakly amenable.*

[5] In fact, not all of these implications require commutativity.

[6] In fact, the following remarkable result holds (as a consequence of a theorem of Thomas [Dal00, Theorem 5.2.48]): if A is a semisimple commutative Banach algebra, then there exists neither a continuous nor a discontinuous nonzero derivation from A into A.

A remarkable sharpening of Theorem 1.3 is that $L^1(G)$ is permanently weakly amenable for every locally compact group G [CGZ09, p. 3179]. Moreover, given any locally compact group G, then essential amenability, approximate amenability and pseudo-amenability of $L^1(G)$ are all equivalent to G being amenable: since every locally compact group has a bounded approximate identity [Kan09, Chapter 1.3], the assertion reduces to '$L^1(G)$ is approximately amenable if and only if G is amenable', which is shown in [GL04, Theorem 3.2]. The question when $L^1(G)$ is pointwise amenable seems to be open [DL10, p. 13]. $L^1(G)$ is contractible if and only if G is finite [Run02, Exercise 4.1.7].

We now recall two local concepts of amenability. Let A be a Banach algebra, and let $\varphi \in \Delta(A)$.

- A is called 'φ-amenable' if for every Banach A-bimodule X such that $a \bullet x = \varphi(a)x$ $(a \in A, x \in X)$ every bounded derivation from A into the dual module X^* is an inner derivation [KLP08b].
- A linear functional $D : A \to \mathbb{C}$ is called a 'point derivation on A at φ' if

$$D(ab) = \varphi(a)D(b) + \varphi(b)D(a) \ (a, b \in A)$$

[Dal00].

Observe that a point derivation on A at φ is a derivation from A into $\mathbb{C}_\varphi$; it is inner if and only if it is zero (which, of course, holds without any commutativity assumptions on A).

A being φ-amenable is equivalent to the existence of $m \in A^{**}$ such that $\langle f \cdot a, m\rangle = \varphi(a)\langle f, m\rangle$ $(a \in A, f \in A^*)$ and $\langle \varphi, m\rangle = 1$ [KLP08b, p. 85; Theorem 1.1];[7] such an m is called a 'φ-mean' [KLP08b]. Of course, this reminds to means in the context of amenable groups. In fact, for any locally compact group G, $L^1(G)$ is amenable w.r.t. the trivial character if and only if G is amenable [KLP08a, p. 942]. The local concept of φ-amenability yields the definition of the following global property: A is called 'right character amenable' if A is φ-amenable for every $\varphi \in \Delta(A)$ and A has a bounded right approximate identity [KLP08a, Mon08];[8] so $L^1(G)$ is right character amenable if and only if G is amenable.

It is not difficult to see that if there exists a nonzero bounded point derivation at $\varphi \in \Delta(A)$, then A is not φ-amenable [KLP08b, Remark 2.4]. Moreover, A fails to be weakly amenable if there exists a nonzero bounded point derivation for some $\varphi \in \Delta(A)$ [Dal00, Theorem 2.8.63].

Concerning ℓ^1-algebras of polynomial hypergroups, which shall be considered in this thesis, the theory of φ-amenability and point derivations has turned out to be especially rich when one restricts oneself to characters $\varphi \in \Delta_s(A)$ ('Hermitian character/structure space'), i.e., $\varphi \in \Delta(A)$ such that $\varphi(a^*) = \overline{\varphi(a)}$ $(a \in A)$ (provided A is a Banach $*$-algebra), cf. the approaches presented in [Las09a, Las09b, Las09c]. Therefore, we make the following definition (which, despite the similar name, must not be confused with the notion of pointwise amenability recalled above):

Definition. We call a Banach $*$-algebra A 'point amenable' if for all $\varphi \in \Delta_s(A)$ there exists no nonzero bounded point derivation on A at φ.

With regard to the implications recalled above, we note that a Banach $*$-algebra which is not point amenable can neither be right character amenable nor weakly amenable.

[7] $f \cdot a \in A^*$ is defined by $f \cdot a(b) = f(ab)$ $(b \in A)$.

[8] Since every amenable Banach algebra has a bounded approximate identity [Dal00, Theorem 2.9.57], every amenable Banach algebra is also right character amenable.

1.3. Hypergroups and orthogonal polynomials: some basic harmonic analysis

Motivated by the Banach–Tarski paradox, in Subsection 1.2 we recalled some amenability properties of locally compact groups G and the corresponding L^1-algebras. It is a natural question to ask whether, and how, results for the group case transfer to generalizations of groups—one might in particular think of semigroups, and indeed the literature on their harmonic analysis, including amenability notions, is extensive. In this thesis, we consider a (generally) very different generalization of locally compact groups: hypergroups. We briefly recall their general definition and some basics, following the presentation in [BH95]—which is a refinement of Jewett's concept [Jew75]; the concepts of Dunkl (1973) and Spector (1975) are similar. Our central results deal with hypergroups which are discrete; in this case, the hypergroup axioms simplify and can be found in [Las05], for instance (cf. below).

Let K be a locally compact Hausdorff space, and let $C(K)$, $C_b(K)$ and $C_c(K)$ denote the sets of (complex-valued) continuous functions on K, bounded continuous functions on K and continuous functions on K with compact support, respectively. We assume that $C_b(K)$ is endowed with the $\|.\|_\infty$-norm. Furthermore, $C_c(K)$ shall carry the topology which is obtained as inductive limit of the spaces $C_E(K) := \{f \in C_c(K) : \mathsf{supp}\, f \subseteq E\}$, $E \subseteq K$ compact, where each of the spaces $C_E(K)$ shall be endowed with the $\|.\|_\infty$-norm.

Let $M(K)$ denote the set of (complex) Radon measures on K, i.e., the set of continuous linear functionals on $C_c(K)$, and let $\|\mu\| := \sup\{|\mu(f)| : f \in C_c(K) \text{ with } \|f\|_\infty \leq 1\}$ for every $\mu \in M(K)$. Let $M_+(K)$ denote the subset of positive measures, let $M^b(K) := \{\mu \in M(K) : \|\mu\| < \infty\}$ denote the set of bounded Radon measures, and, finally, let $M^1(K) := \{\mu \in M(K) : \mu \geq 0 \text{ and } \|\mu\| = 1\}$ denote the subset of probability measures on K. Via integration theory, there are identifications with functions on the Borel σ-algebra on K. We think that the functional analytic approach to hypergroups is a rather elegant and natural one. The spaces $M^b(K)$ and $C_b(K)$ are a dual pair; in the following, $M^b(K)$ shall be endowed with the $\sigma(M^b(K), C_b(K))$ topology (which is called the 'Bernoulli topology').

We assume that the set $\mathcal{C}(K) := \{C \subseteq K : C \text{ compact and } C \neq \emptyset\}$ is endowed with the 'Michael topology', the topology on $\mathcal{C}(K)$ which is given by the subbasis of all $\mathcal{C}_U(V) := \{C \in \mathcal{C}(K) : C \cap U \neq \emptyset \text{ and } C \subseteq V\}$, $U, V \subseteq K$ open.

Definition. A nonvoid locally compact Hausdorff space K, together with a (second) binary operation $\omega : M^b(K) \times M^b(K) \to M^b(K)$ ('convolution') and a mapping $\widetilde{.} : K \to K$ ('involution'), is called a 'hypergroup' if the following conditions hold:

- $\omega(\delta_x, \delta_y) \in M^1(K)$ and $\mathsf{supp}\ \omega(\delta_x, \delta_y) \in \mathcal{C}(K)$ for every $x, y \in K$, and the mappings $K \times K \to M^1(K)$, $(x, y) \mapsto \omega(\delta_x, \delta_y)$ and $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto \mathsf{supp}\ \omega(\delta_x, \delta_y)$ are continuous,
- together with ω, the $\mathbb{C}$-linear space $M^b(K)$ is an algebra,
- there exists a (necessarily unique) 'unit element' $e \in K$ such that $\omega(\delta_e, \delta_x) = \delta_x = \omega(\delta_x, \delta_e)$ $(x \in K)$,
- $\widetilde{.}$ is a homeomorphism and, for all $x, y \in K$, one has $\widetilde{\widetilde{x}} = x$, $\omega(\delta_x, \delta_y)(\widetilde{f}) = \omega(\delta_{\widetilde{y}}, \delta_{\widetilde{x}})(f)$ $(f \in C_c(K)$, where $\widetilde{f} \in C_c(K)$ is defined via $\widetilde{f}(x) := f(\widetilde{x}))$, and $e \in \mathsf{supp}\ \omega(\delta_x, \delta_y) \Leftrightarrow x = \widetilde{y}$.

If ω makes the $\mathbb{C}$-linear space $M^b(K)$ even a commutative algebra, then K is called a 'commutative' hypergroup. If K is endowed with the discrete topology, then K is called a 'discrete' hypergroup.

As already mentioned above, the definition becomes considerably simpler in the discrete case, cf. [BH95] and [Las05, p. 56; Definition 2.1]: a nonvoid set K, together with a mapping ω

(convolution) which maps $K \times K$ into conv $\{\delta_k : k \in K\}$, the convex hull[9] of Dirac *functions* on K, and a bijective mapping $\widetilde{\cdot} : K \to K$ (involution), is a discrete hypergroup if

- ω is associative, i.e.,

$$\sum_{k \in K} \omega(y,z)(k)\omega(x,k) = \sum_{k \in K} \omega(x,y)(k)\omega(k,z)$$

  for all $x, y, z \in K$ (note that the sums are finite),

- there exists a (necessarily unique) unit element $e \in K$ such that $\omega(e,x) = \delta_x = \omega(x,e)$ $(x \in K)$,

- for all $x, y \in K$, one has $\widetilde{\widetilde{x}} = x$, $\omega(x,y)(\widetilde{k}) = \omega(\widetilde{y},\widetilde{x})(k)$ $(k \in K)$, and

$$e \in \mathsf{supp}\, \omega(x,y) \Leftrightarrow x = \widetilde{y}. \tag{1.1}$$

Observe that, in contrast to the original definition, we have defined the convolution ω (corresponding to a discrete hypergroup) on $K \times K$ rather than on $M^b(K) \times M^b(K)$—and mapping to functions on K rather than to measures in $M^b(K)$. On the one hand, we have decided to do so to keep consistent with the cited literature; on the other hand, it is easy to see that the two approaches can be identified which each other (identifying x with δ_x and using bilinear extensions). We refer to [BH95] and [Las05] for more details.

A (discrete) group G may be considered as a discrete hypergroup via $\omega(x,y) := \delta_{xy}$ and $\widetilde{x} := x^{-1}$ $(x, y \in G)$ [Las05, Remark on p. 57]; in an analogous way, each locally compact group may be considered as a hypergroup via its usual convolution structure [Jew75, p. 17 Proposition 2]. In contrast to the group case, a hypergroup need not have an “algebraic structure” which is independent from its entire, in particular also “topological” structure: to each locally compact group corresponds a discrete group (which has the same algebraic properties)—however, if K is a hypergroup such that for all $x, y \in K$ there is an element “xy” in K with $\omega(\delta_x, \delta_y) = \delta_{xy}$, then K is already a locally compact group (via $(x,y) \mapsto xy$) [Jew75, p. 17 Proposition 1]. Cf. also the notes at the beginning of [Jew75, Section 7.1].

Hypergroups are interesting for several reasons. One reason is that they cover many examples which can be rather different from the group or semigroup setting (for instance, double cosets). Another—surely not less important—reason is that hypergroups have a rich harmonic analysis. For any $x \in K$ and $f \in C(K)$, one can define the ‘(left) translation’ $T_x f : K \to \mathbb{C}$ of f by x via

$$T_x f(y) := \int_K f \,\mathrm{d}\omega(\delta_x, \delta_y).^{10} \tag{1.2}$$

Note that the right hand side of (1.2) is well-defined for every $f \in C(K)$ because $\mathsf{supp}\, \omega(\delta_x, \delta_y)$ is compact. If $f \in C_c(K)$, then $T_x f \in C_c(K)$ $(x \in K)$. A measure $\nu_K \in M_+(K)$, $\nu_K \neq 0$, such that $\nu_K(f) = \nu_K(T_x f)$ $(f \in C_c(K), x \in K)$ (‘left-invariance’), is called a ‘Haar measure’. Up to a positive real factor, a Haar measure is unique. For many specific types of hypergroups (for instance, for commutative, compact or discrete hypergroups), the existence of such a Haar measure is known; particularly for discrete hypergroups, the existence of a Haar measure is easily seen, and the Haar measure takes a very simple form [Las05, Theorem 2.1]. There is an article on the arXiv which states that every hypergroup (however, in the sense of Spector) bears a Haar measure [Cha12] (we shall not make use of this). The translation can be defined for more general functions; we shall need the following: for any $p \in [1, \infty]$, $f \in L^p(K)$ (w.r.t. a

[9] finite convex combinations

[10] To avoid any confusion, we note that our reference [BH95] writes T^x instead of T_x, whereas T_x in [BH95] can mean something different.

fixed Haar measure) and $x \in K$, one has $T_x f \in L^p(K)$ (and $\|T_x f\|_p \leq \|f\|_p$). We refer to the monograph [BH95] for more details.

We shall not recall the further basic concepts of harmonic analysis on hypergroups in full generality (anyway, many of these concepts are limited to the commutative case, of course), but restrict ourselves to those types this thesis mainly deals with, namely to polynomial hypergroups.

Polynomial hypergroups were introduced by Lasser in the 1980s [Las83]. They provide an abundance of examples for hypergroups which, on the one hand, are very different from groups, and, on the other hand, nevertheless show a great diversity among themselves: the individual behavior strongly depends on $(P_n(x))_{n\in\mathbb{N}_0}$, the inducing orthogonal polynomial sequence. All polynomial hypergroups have in common that many concepts of harmonic analysis and Gelfand theory take a rather concrete form; hence, one may regard polynomial hypergroups as an elegant way to study orthogonal polynomials via methods from functional and harmonic analysis. Of course, one may also think of them as a valuable possibility to obtain many examples in functional and harmonic analysis—in particular, in the theory of Banach algebras—which come from the theory of orthogonal polynomials and special functions. In fact, the topic is located at a fruitful crossing point between the areas. In the following, we refer to [Las83] and [Las05] if not stated otherwise.

Let $a_0 > 0$, $b_0 < 1$, $c_0 := 0$, $(a_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \subseteq (0,1)$ and $(b_n)_{n\in\mathbb{N}} \subseteq [0,1)$ satisfy $a_n + b_n + c_n = 1$ $(n \in \mathbb{N}_0)$, and let $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ be a sequence of polynomials that is given by the three-term recurrence relation $P_0(x) := 1$, $P_1(x) := \frac{1}{a_0}(x - b_0)$,

$$P_1(x)P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) \; (n \in \mathbb{N}). \tag{1.3}$$

Trivially, $P_n(1) = 1$ $(n \in \mathbb{N}_0)$. As a crucial condition for obtaining a hypergroup structure, additionally assume that 'property (P)' holds, i.e., that the linearization coefficients $g(m,n;k)$ defined by the expansions

$$P_m(x)P_n(x) = \sum_{k=0}^{m+n} \underbrace{g(m,n;k)}_{\overset{!}{\geq} 0 \text{ (P)}} P_k(x) \; (m,n \in \mathbb{N}_0) \tag{1.4}$$

are all nonnegative. As a consequence of the theory of orthogonal polynomials, in particular Favard's theorem (cf. [Chi78, I-Theorem 4.4, II-Theorem 3.1]), $(P_n(x))_{n\in\mathbb{N}_0}$ is orthogonal w.r.t. a unique probability (Borel) measure μ on $\mathbb{R}$ with $|\mathsf{supp}\,\mu| = \infty$; the support of μ is contained in the set

$$\widehat{\mathbb{N}_0} := \left\{x \in \mathbb{R} : \sup_{n\in\mathbb{N}_0} |P_n(x)| < \infty\right\} = \left\{x \in \mathbb{R} : \max_{n\in\mathbb{N}_0} |P_n(x)| = 1\right\},$$

which contains 1 (obvious) and is a compact subset of $[1 - 2a_0, 1]$.[11] Orthogonality yields

$$g(m,n;|m-n|), g(m,n;m+n) \neq 0 \; (m,n \in \mathbb{N}_0) \tag{1.5}$$

and

$$g(m,n;k) = 0 \; (m,n \in \mathbb{N}_0, k < |m-n|), \tag{1.6}$$

i.e., (1.4) reduces to

$$P_m(x)P_n(x) = \sum_{k=|m-n|}^{m+n} \underbrace{g(m,n;k)}_{\overset{!}{\geq} 0 \text{ (P)}} P_k(x) \; (m,n \in \mathbb{N}_0)$$

[11] The uniqueness of μ is a consequence of the compact support: if there was a different orthogonalization measure ν, then neither μ nor ν could have compact support, cf. [Chi78, II-Theorem 3.2; II-Theorem 5.6]. Alternatively, the uniqueness of μ can be obtained directly from the three-term recurrence relation and the conditions on $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}$; apply [Chi78, II-Theorem 5.6; IV-Theorem 2.2].

(which, of course, can be seen as an extension of the three-term recurrence relation (1.3)). Therefore, one has

$$\sum_{k=|m-n|}^{m+n} g(m,n;k) = 1 \ (m,n \in \mathbb{N}_0). \tag{1.7}$$

Defining $\omega : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}^{\mathbb{N}_0}$ and $\widetilde{\cdot} : \mathbb{N}_0 \to \mathbb{N}_0$ by $\omega(m,n) := \sum_{k=|m-n|}^{m+n} g(m,n;k)\delta_k$ and $\widetilde{n} := n$, property (P) and (1.7) imply that $\mathbb{N}_0$ becomes a commutative discrete hypergroup with unit element 0, a 'polynomial hypergroup'; ω maps into conv $\{\delta_n : n \in \mathbb{N}_0\}$. If $m,n \in \mathbb{N}_0$ with $mn \neq 0$, then, due to (1.5), $\mathsf{supp}\ \omega(m,n)$ has at least two elements (in sharp contrast to the group case, cf. above). Note that (1.6) (and hence orthogonality) is very important for the "non-degeneracy property" (1.1) to be satisfied.

In a series of papers starting with [Szw92b], Szwarc gave several sufficient conditions for the crucial property (P). [Szw03] provides an abstract characterization of property (P). To our knowledge, there is no simple and convenient explicit characterization which is just in terms of $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}$.

Sometimes it is more convenient to consider different normalizations of the polynomials: let $(\rho_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ and $(p_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ denote the sequences of monic and *orthonormal*[12] polynomials corresponding to $(P_n(x))_{n\in\mathbb{N}_0}$, respectively. One has $\rho_0(x) = p_0(x) = 1$ and

$$x\rho_n(x) = \rho_{n+1}(x) + \beta_n\rho_n(x) + \alpha_n^2\rho_{n-1}(x) \ (n \in \mathbb{N}_0),$$
$$xp_n(x) = \alpha_{n+1}p_{n+1}(x) + \beta_n p_n(x) + \alpha_n p_{n-1}(x) \ (n \in \mathbb{N}_0),$$

where $\alpha_0 := 0$, $\alpha_1 := a_0\sqrt{c_1}$, $\alpha_n := a_0\sqrt{c_n a_{n-1}}$ $(n \in \mathbb{N}\backslash\{1\})$, $\beta_0 := b_0$ and $\beta_n := a_0 b_n + b_0$ $(n \in \mathbb{N})$.[13] Some of the cited results from [Chi78] use the monic normalization.

For any $n \in \mathbb{N}_0$, the 'translation operator' (or 'shift operator') T_n maps a function $f : \mathbb{N}_0 \to \mathbb{C}$ to the translation $T_n f : \mathbb{N}_0 \to \mathbb{C}$ of f by n, which reads

$$T_n f(m) = \sum_{k=|m-n|}^{m+n} g(m,n;k)f(k) = T_m f(n) \ (m \in \mathbb{N}_0).$$

The Haar measure, normalized such that $\{0\}$ is mapped to 1, is just the counting measure on $\mathbb{N}_0$ weighted by the 'Haar weights', i.e., the values of the 'Haar function' $h : \mathbb{N}_0 \to [1,\infty)$ defined by

$$h(n) = \frac{1}{\int_{\mathbb{R}} P_n^2(x)\,\mathrm{d}\mu(x)} = \frac{1}{g(n,n;0)} = p_n^2(1);$$

obviously, each of the following two conditions is equivalent to the preceding definition of h:

- $h(0) = 1$, $h(1) = \frac{1}{c_1}$ and
$$h(n+1) = \frac{a_n}{c_{n+1}}h(n) \ (n \in \mathbb{N}),$$
- $h(0) = 1$ and
$$g(m,n;k)h(n) = g(m,k;n)h(k) \ (m,n \in \mathbb{N}_0, k \in \{|m-n|,\dots,m+n\}).$$

[12] with positive leading coefficients

[13] To obtain well-definedness in the preceding recurrence relations, we make the—widely common—convention that $(\alpha_0^2 = \alpha_0 =)0$ times something undefined shall be 0.

For $p \in [1,\infty)$, let the $\|.\|_p$-norms and the corresponding spaces be defined w.r.t. the Haar measure:

$$\ell^p(h) := \{f : \mathbb{N}_0 \to \mathbb{C} : \|f\|_p < \infty\}$$

with

$$\|f\|_p := \left(\sum_{k=0}^{\infty} |f(k)|^p h(k)\right)^{\frac{1}{p}};$$

moreover, let $\ell^\infty(h) := \ell^\infty$. For any $n \in \mathbb{N}_0$, T_n is a nonexpansive operator in $B(\ell^p(h))$ ($p \in [1,\infty]$). Trivially, if $f \in c_0$, then also $T_n f \in c_0$, and if $f \in c_{00}$, then $T_n f \in c_{00}$.[14] For any $p \in [1,\infty)$ and $q := \frac{p}{p-1} \in (1,\infty]$, one has the duality $(\ell^p(h))^* \cong \ell^q(h)$ via

$$\langle f, g\rangle := \sum_{k=0}^{\infty} f(k)g(k)h(k) \ (f \in \ell^p(h), g \in \ell^q(h)).$$

In the same manner, the duality $(c_0)^* \cong \ell^1(h)$ holds. Note that $\langle f, g\rangle$ is also well-defined if $g \in \ell^1(h)$ and $f \in \ell^\infty$ arbitrary. Concerning inclusions, the following holds: if $p, q \in [1,\infty]$ with $p \leq q$, then $\ell^p(h) \subseteq \ell^q(h)$ (as the Haar weights are bounded from below by 1).

Given $f \in \ell^p(h)$ and $g \in \ell^q(h)$, where $p \in [1,\infty]$, $q := \frac{p}{p-1} \in [1,\infty]$, the 'convolution' $f * g : \mathbb{N}_0 \to \mathbb{C}$ of f and g is defined by

$$f * g(n) := \langle T_n f, g\rangle .$$

The following hold:

$$\begin{aligned} & f * g \in \ell^\infty, \\ & f * g = g * f, \\ & f \in \ell^1(h) \Rightarrow f * g \in \ell^q(h) \text{ and } \|f * g\|_q \leq \|f\|_1 \|g\|_q . \end{aligned} \tag{1.8}$$

We note that if $1 < p, q < \infty$, then (1.8) (i.e., the commutativity of the convolution) is shown in [Las05]. If $f \in \ell^1(h)$ and $g \in \ell^\infty$, one can establish (1.8) in the following way: first, it can be seen elementarily from the definitions that the convolution commutes if $f \in c_{00}$. Then, one uses an approximation argument and the fact that, for any $f \in \ell^1(h)$, $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$ and $\|g * f\|_\infty \leq \|f\|_1 \|g\|_\infty$.

Defining $(\epsilon_n)_{n\in\mathbb{N}_0} \subseteq c_{00}$ via the expansions

$$P_n(x) = \sum_{k=0}^{n} \epsilon_n(k) P_k(x) h(k),\ \epsilon_n(n+1) := \epsilon_n(n+2) := \ldots := 0 \ (n \in \mathbb{N}_0, x \in \mathbb{R}),$$

or, equivalently, via

$$\epsilon_n = \frac{1}{h(n)} \delta_n \ (n \in \mathbb{N}_0),$$

it is obvious that $\|\epsilon_n\|_1 = 1$ and

$$\epsilon_m * \epsilon_n = \sum_{k=|m-n|}^{m+n} g(m,n;k)\epsilon_k,\ \epsilon_n * g = T_n g$$

for all $m, n \in \mathbb{N}_0$, $g \in \ell^\infty$; therefore, the convolution $*$ can be seen as an extension of the hypergroup convolution ω.

[14] As widely common, we denote by c_0 and c_{00} the subspaces of ℓ^∞ which consist of the null sequences and the sequences with finite support, respectively.

Endowing $\ell^2(h)$ with the inner product $\ell^2(h) \times \ell^2(h) \to \mathbb{C}$, $(f,g) \mapsto \langle f, \overline{g} \rangle$, $\ell^2(h)$ becomes a Hilbert space.

Endowing $\ell^1(h)$ with the convolution $*$ and complex conjugation (as involution), $\ell^1(h)$ becomes a semisimple commutative Banach $*$-algebra with unit $\epsilon_0 = \delta_0$, the 'ℓ^1-algebra' of the polynomial hypergroup. $\ell^1(h)$ acts on the (unital) Banach $\ell^1(h)$-module $\ell^p(h)$ by convolution for each $p \in [1, \infty]$, and ℓ^∞ is the dual module of $\ell^1(h)$, see [Las07] or [Las09c].

The structure space $\Delta(\ell^1(h))$ of $\ell^1(h)$ can be identified with

$$\mathcal{X}^b(\mathbb{N}_0) := \left\{ z \in \mathbb{C} : \sup_{n \in \mathbb{N}_0} |P_n(z)| < \infty \right\} = \left\{ z \in \mathbb{C} : \max_{n \in \mathbb{N}_0} |P_n(z)| = 1 \right\}$$

via the homeomorphism $\mathcal{X}^b(\mathbb{N}_0) \to \Delta(\ell^1(h))$, $z \mapsto \varphi_z$,

$$\varphi_z(f) := \sum_{k=0}^{\infty} f(k) \overline{P_k(z)} h(k) \; (f \in \ell^1(h)).$$

In the same way, $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) \cap \mathbb{R}$ can be identified with the Hermitian structure space $\Delta_s(\ell^1(h))$. Given any $f \in \ell^1(h)$, under this identification the Gelfand transform of f, restricted to $\Delta_s(\ell^1(h))$, becomes the 'Fourier transform' $\widehat{f} : \widehat{\mathbb{N}_0} \to \mathbb{C}$ of f,

$$\widehat{f}(x) := \sum_{k=0}^{\infty} f(k) P_k(x) h(k);$$

one has $\widehat{f} \in C(\widehat{\mathbb{N}_0})$, $\left\| \widehat{f} \right\|_\infty \leq \|f\|_1$ and

$$\widehat{f * g} = \widehat{f} \, \widehat{g} \; (f, g \in \ell^1(h)). \tag{1.9}$$

The mapping $\widehat{\cdot} : \ell^1(h) \to C(\widehat{\mathbb{N}_0})$, $f \mapsto \widehat{f}$ is called the 'Fourier transformation' on $\ell^1(h)$. The Fourier transformation is injective: if $f \in \ell^1(h)$ and $\widehat{f}|_{\mathsf{supp}\, \mu} = 0$, then $f = 0$.[15]

We need to recall another identification: given any $x \in \widehat{\mathbb{N}_0}$, the ('symmetric') 'character'[16] ${}_x\alpha \in \ell^\infty \backslash \{0\}$ belonging to x is defined by

$${}_x\alpha(n) := P_n(x) \; (n \in \mathbb{N}_0).$$

One has

$$T_m \, {}_x\alpha(n) = {}_x\alpha(m) \, {}_x\alpha(n) \; (m, n \in \mathbb{N}_0) \tag{1.10}$$

and, which is obvious,

$$|{}_x\alpha(n)| \leq 1 \; (n \in \mathbb{N}_0).$$

The map $\vartheta : \widehat{\mathbb{N}_0} \to \{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$, $\vartheta(x) := {}_x\alpha$ is a bijection, and endowing $\{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$ with the topology inherited from $\widehat{\mathbb{N}_0}$ via ϑ (this is the topology of pointwise convergence), the Plancherel–Levitan theorem yields a unique regular positive bounded Borel measure π on $\{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$ with

$$\|f\|_2^2 = \int |\widehat{f} \circ \vartheta^{-1}|^2 \, \mathrm{d}\pi \; (f \in \ell^1(h)).$$

[15] Identifying $\Delta(\ell^1(h))$ with $\mathcal{X}^b(\mathbb{N}_0)$, the Gelfand transform of a function $f \in \ell^1(h)$ becomes $\mathcal{F}f : \mathcal{X}^b(\mathbb{N}_0) \to \mathbb{C}$, $\mathcal{F}f(z) := \sum_{k=0}^{\infty} f(k) \overline{P_k(z)} h(k)$; sometimes, also $\mathcal{F}f$ is called 'Fourier transform' of f (and $\mathcal{F}$ is called 'Fourier transformation', too). For our purposes, we only need $\widehat{f} = \mathcal{F}f|_{\widehat{\mathbb{N}_0}}$, however.

[16] The expression "symmetric character" (or "Hermitian character") is used both for the elements of the Hermitian structure space $\Delta_s(\ell^1(h))$, i.e., for the symmetric characters in the "Banach algebraic sense", and for the symmetric characters in the "hypergroup sense", whose definition is recalled now. Of course, these notions can be identified with each other (since both of them can be identified with elements of $\widehat{\mathbb{N}_0} \subseteq \mathbb{R}$, see below).

π is called Plancherel measure, and there exists exactly one isometric isomorphism from $\ell^2(h)$ to $L^2(\{_x\alpha : x \in \widehat{\mathbb{N}_0}\}, \pi)$ such that the image of $f \in \ell^1(h)$ and $\widehat{f} \circ \vartheta^{-1}$ coincide as elements of $L^2(\{_x\alpha : x \in \widehat{\mathbb{N}_0}\}, \pi)$; this isometric isomorphism is called the Plancherel isomorphism. Under the identification ϑ, the Plancherel measure π reduces to the orthogonalization measure μ, and the Plancherel isomorphism to the (uniquely determined) isometric isomorphism $\mathcal{P} : \ell^2(h) \to L^2(\mathbb{R}, \mu)$ which satisfies

$$\widehat{f} = \mathcal{P}(f) \ (f \in \ell^1(h))$$

in $L^2(\mathbb{R}, \mu)$. Therefore, it is justified to use the names 'Plancherel measure' and 'Plancherel isomorphism' also for μ and $\mathcal{P}$, respectively, which shall be done throughout this thesis. The inverse Plancherel isomorphism $\mathcal{P}^{-1}$ satisfies

$$\mathcal{P}^{-1}(F)(k) = \int_{\mathbb{R}} F(x) P_k(x) \, \mathrm{d}\mu(x) \ (F \in L^2(\mathbb{R}, \mu), k \in \mathbb{N}_0).$$

One has

$$\mathcal{P}(f * g) = \mathcal{P}(f)\mathcal{P}(g) \ (f \in \ell^1(h), g \in \ell^2(h)),$$

which follows from (1.9) by approximation.

Some of the many known explicit examples such that $\widehat{\mathbb{N}_0} \subsetneqq \mathcal{X}^b(\mathbb{N}_0)$ and $1 \notin \operatorname{supp} \mu$ (hence, in particular, also $\operatorname{supp} \mu \subsetneqq \widehat{\mathbb{N}_0}$) can be found in Section 3.3 and Section 3.5. Recall that such properties are very different from Abelian locally compact groups.

1.4. Hypergroups and orthogonal polynomials: various types of amenability

Concerning (generalized notions of) amenability for polynomial (or other) hypergroups, there exist several approaches—which, on the one hand, very naturally arise from their group analogues because many of the concepts we recalled in Subsection 1.2 can be transferred to hypergroups in a reasonable and frequently straight forward way, but, on the other hand, need (and in most cases do) no longer bear the equivalences or rather general results which are valid in the group case. We divide these different approaches into two parts (which, of course, are not independent from each other): transference of means (and Reiter's conditions, Følner condition and so on)—and considerations of the L^1-algebras (which can be of very different type compared to L^1-algebras of locally compact groups).

In the following, let K be a hypergroup which has a (fixed) Haar measure.

Means w.r.t. hypergroups are extensively studied in [Ska92]. Eight years earlier, some results and basic definitions were already given in S. Wolfenstetter's dissertation [Wol84]. The following definition is taken from [Ska92] and the straight forward analogue to Definition 1.1:

Definition 1.2. *K is called 'amenable' if there exists a left-invariant 'mean' on $L^\infty(K)$, i.e., $m \in L^\infty(K)^*$ such that $\langle T_x\phi, m\rangle = \langle \phi, m\rangle$ $(x \in K, \phi \in L^\infty(K))$ and $\langle 1, m\rangle = \|m\| = 1$.*

Each commutative hypergroup is amenable (this follows from the Markov–Kakutani fixed point theorem) [Ska92, Example 3.3 (a)], and, easier to see, each compact hypergroup is amenable [Ska92, Example 3.3 (b)]. Moreover, closed subgroups of amenable hypergroups are amenable [Ska92, Proposition 3.5].

In view of Proposition 1.2, we note that K is amenable if and only if K satisfies Reiter's condition P_1 [Ska92, Theorem 4.1].[17] However, this is not equivalent to the corresponding condition P_2 being satisfied; Reiter's condition P_2 w.r.t. hypergroups, which is stronger than

[17]'Reiter's condition P_p', $p \in [1, \infty)$, means that for every $\epsilon > 0$ and for every compact subset C of K there exists a positive function $\phi \in L^p(K)$ with $\|\phi\|_p = 1$ such that $\|T_x\phi - \phi\|_p < \epsilon$ $(x \in C)$ [Ska92, LS11].

P_1 [Ska92, Theorem 4.3; Example 4.6], is also studied in [FL00] (in a modified way). Further noteworthy references concerning (modified) Reiter's condition(s) in the context of hypergroups are [FLS04, FLS05, Geb92, LS11]. [HHL10] deals with a Følner type condition and strongly translation-invariant means on polynomial hypergroups, cf. also [Hof12]. Strongly invariant means on commutative hypergroups are studied in [LO12].

If the L^1-algebra ([BH95]; we omit recalling the precise general definition, since in this thesis we shall concentrate on ℓ^1-algebras of polynomial hypergroups[18]) of a hypergroup is amenable, then the hypergroup is amenable [Ska92, Proposition 4.9]. The converse is not true: in fact, an amenable hypergroup need not even have a point amenable L^1-algebra, which can easily be seen from our results presented in Section 3 (but has been known before, see various examples in [Las09b], for instance).

From now on, we restrict ourselves to polynomial hypergroups again, induced by sequences $(P_n(x))_{n\in\mathbb{N}_0}$ as in Subsection 1.3. Since these are commutative, they are all amenable and hence satisfy Reiter's condition P_1. Moreover, [Ska92, Lemma 4.5] clarifies the situation w.r.t. Reiter's condition P_2: P_2 is satisfied if and only if $1 \in \mathsf{supp}\ \mu$ (further characterizations of this property are given in [LOW13, LS11]). Since these results on amenability in the hypergroup sense and on Reiter's conditions are already very satisfying and since there is a considerable amount of literature on these topics (cf. above), we shall from now on focus on amenability properties which concern *the Banach algebra* $\ell^1(h)$.

Since $\ell^1(h)$ is a commutative, unital and semisimple (and, obviously, infinite-dimensional) Banach $*$-algebra, one has the following (cf. Subsection 1.2):

- $\ell^1(h)$ is never contractible,
- the notions of amenability and essential amenability coincide for $\ell^1(h)$,
- the notions of weak amenability and permanent weak amenability coincide for $\ell^1(h)$,
- if $\ell^1(h)$ is weakly amenable or right character amenable, then $\ell^1(h)$ is also point amenable,
- there exists no nonzero derivation from $\ell^1(h)$ into $\ell^1(h)$.

There are interesting necessary criteria for amenability of $\ell^1(h)$:

Theorem 1.4. *If $\ell^1(h)$ is amenable, then both of the following hold:*

(i) *$h(n) \not\to \infty$ $(n \to \infty)$ [Las07, Theorem 3],*

(ii) *there exists some $\epsilon > 0$ such that for all $x, y \in \widehat{\mathbb{N}_0}$ with $x \neq y$ there is some $f \in \ell^1(h)$ with $\|f\|_1 = 1$ and $|\widehat{f}(x) - \widehat{f}(y)| \geq \epsilon$ [Las09c, Proposition 1].*

(i) is very convenient. (ii), which is a consequence of a result of Gourdeau [Gou89], seems to be less convenient—however, it is interesting because it provides a kind of separation result.

Most of the naturally occurring examples satisfy $h(n) \to \infty$ $(n \to \infty)$ and hence do not have amenable ℓ^1-algebras. A property which has extensively been studied is the notion of α-amenability [Las09a], coming from the notion of φ-amenability:

Definition. Let $\alpha \in \{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$. $\ell^1(h)$ is called 'α-amenable' if one (and hence every) of the following equivalent conditions holds:

[18] As common in the group case, we write "ℓ^1" when explicitly referring to the discrete topology, and "L^1" otherwise.

(i) $\ell^1(h)$ is φ-amenable with $\varphi \in \Delta_s(\ell^1(h))$ given by

$$\varphi(f) := \sum_{k=0}^{\infty} \alpha(k)f(k)h(k) \ (f \in \ell^1(h)),$$

(ii) there exists $m \in (\ell^\infty)^*$ such that $\langle T_n f, m\rangle = \alpha(n)\langle f, m\rangle$ $(n \in \mathbb{N}_0, f \in \ell^\infty)$ and $\langle \alpha, m\rangle = 1$ (such an m is called an 'α-mean'),

(iii) there is some $M \geq 1$ such that 'Reiter's condition $P_1(\alpha, M)$' holds, i.e., for every $\epsilon > 0$ and for every finite subset C of $\mathbb{N}_0$, there exists $f \in \ell^1(h)$ with $\|f\|_1 \leq M$ and $\sum_{k=0}^{\infty} \alpha(k)f(k)h(k) = 1$ such that $\|T_n f - \alpha(n)f\|_1 < \epsilon$ $(n \in C)$.[19]

Consequently, if $\mathcal{X}^b(\mathbb{N}_0)\backslash\widehat{\mathbb{N}_0} = \emptyset$, then $\ell^1(h)$ is right character amenable if and only if $\ell^1(h)$ is α-amenable for every $\alpha \in \{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$. In contrast to amenable groups, right character amenability of $\ell^1(h)$ need not be satisfied although every polynomial hypergroup is amenable in the hypergroup sense (see [Las09c, p. 792], for instance; we will obtain further examples in Section 3). However, $\ell^1(h)$ is always 1-amenable, i.e., α-amenable w.r.t. the trivial character $\alpha = {}_1\alpha$; as in the group case, this obviously just corresponds to the amenability (of each polynomial hypergroup—amenability in the hypergroup sense) [Las09a]. A characterization of α-amenability in terms of the 'Glicksberg–Reiter property' is given in [Las09a]. Several further investigations have been made; we just mention [FLS04, FLS05, LP10] and that there are some works by Azimifard, including [Azi10] and [Azi09] (arXiv). The following [FLS04, Theorem 4.4] is of particular interest:

Theorem 1.5. *If $x \in \widehat{\mathbb{N}_0}$ with $\mu(x) = 0$ and $\ell^1(h)$ is ${}_x\alpha$-amenable, then $P_n(x) \not\to 0$ $(n \to \infty)$.*

[FLS04, Theorem 4.10], which should be compared to Theorem 1.4 (i), tells:

Proposition 1.3. *Assume that $(P_n(x))_{n\in\mathbb{N}_0}$ is of 'Nevai class $M(0,1)$', i.e., $\alpha_n \to \frac{1}{2}$ $(n \to \infty)$ and $\beta_n \to 0$ $(n \to \infty)$ (cf. the recurrence relation for the orthonormal sequence $(p_n(x))_{n\in\mathbb{N}_0}$),[20] and assume also that $(P_n(x))_{n\in\mathbb{N}_0}$ is of 'bounded variation type', i.e., $\sum_{n=1}^{\infty}(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|) < \infty$. Then $(-1,1) \subseteq \widehat{\mathbb{N}_0}$, and: if $x \in (-1,1)$ and $\ell^1(h)$ is ${}_x\alpha$-amenable, then $h(n) \not\to \infty$ $(n \to \infty)$.*

Moreover, from [FLS04, Proposition 4.2] one easily obtains the following useful result concerning ${}_{-1}\alpha$-amenability (cf. [FLS04, Example 4.6; Remark 4.8]):

Proposition 1.4. *If $b_n \equiv 0$ and $\frac{h(n)}{\sum_{k=0}^{n} h(k)} \to 0$ $(n \to \infty)$, then $-1 \in \widehat{\mathbb{N}_0}$ and $\ell^1(h)$ is ${}_{-1}\alpha$-amenable.*

Identifying $\Delta_s(\ell^1(h))$ with $\widehat{\mathbb{N}_0}$, a point derivation w.r.t. a Hermitian character becomes a linear functional $D_x : \ell^1(h) \to \mathbb{C}$ satisfying

$$D_x(f * g) = \widehat{f}(x)D_x(g) + \widehat{g}(x)D_x(f) \ (f, g \in \ell^1(h)),$$

where $x \in \widehat{\mathbb{N}_0}$ is fixed [Las09b]. Therefore, it is justified to call such D_x 'point derivation' (on $\ell^1(h)$ at x) again, which shall be done throughout the thesis. Using the notation of [Las09b], we write "$D_x \neq 0$ exists" if there is a nonzero *bounded* point derivation at x, and "$D_x = 0$" otherwise. Hence, $\ell^1(h)$ is point amenable if and only if $D_x = 0$ for all $x \in \widehat{\mathbb{N}_0}$.

[19] Concerning (iii), there exists a simpler characterization in terms of T_1 (instead of T_n), see [FLS04, Theorem 4.1].

[20] Nevai class $M(0,1)$ has some interesting consequences for polynomial hypergroups—for instance, concerning connection coefficients [LR93].

The identifications yield that if $x \in \widehat{\mathbb{N}_0}$ and $D_x \neq 0$ exists, then $\ell^1(h)$ is neither weakly amenable nor ${}_x\alpha$-amenable. Concerning the latter, [Las09b, Proposition 5] provides a slight improvement.

Given $x \in \widehat{\mathbb{N}_0}$, it can be rather difficult—and require deep knowledge about the underlying orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}_0}$—to see whether $D_x \neq 0$ exists or $D_x = 0$. Nevertheless, there are some trivial cases: one always has $D_1 = 0$, and if $b_n \equiv 0$, then $-1 \in \widehat{\mathbb{N}_0}$ and $D_{-1} = 0$ [Las09b, Proposition 3]. The following characterization [Las09b, Theorem 1] relates the existence of nonzero bounded point derivations to the sequence $(P_n'(x))_{n\in\mathbb{N}_0}$ of derivatives, which makes it extremely useful:

Theorem 1.6. *If $x \in \widehat{\mathbb{N}_0}$, then $D_x \neq 0$ exists if and only if $\{P_n'(x) : n \in \mathbb{N}_0\}$ is bounded. Hence, $\ell^1(h)$ is point amenable if and only if $\{P_n'(x) : n \in \mathbb{N}_0\}$ is unbounded for all $x \in \widehat{\mathbb{N}_0}$.*

In [Las09b], several more criteria are presented, involving spectral sets, homomorphisms, growth conditions, Nevai class $M(0,1)$ and bounded variation type (cf. Proposition 1.3). More important for our purposes, however, is Lasser's analogue to Theorem 1.6 concerning weak amenability (Theorem 1.7 below), which involves the Fourier coefficients associated with $P_n'(x)$:

let $(\kappa_n)_{n\in\mathbb{N}_0} \subseteq c_{00}$ be defined via the expansions

$$\kappa_0 := 0,\ P_n'(x) = \sum_{k=0}^{n-1} \kappa_n(k)P_k(x)h(k),\ \kappa_n(n) := \kappa_n(n+1) := \ldots := 0\ (n \in \mathbb{N}, x \in \mathbb{R}),$$

or, equivalently, via

$$\kappa_n = \mathcal{P}^{-1}(P_n')\ (n \in \mathbb{N}_0).$$

Remark 1.1. Compared to [Las07] (and also [Las09c]), our sequence $(\kappa_n)_{n\in\mathbb{N}_0}$ coincides with Lasser's original definition up to the constant factor a_0.

The following is [Las07, Theorem 2] (or [Las09c, Theorem 2]):

Theorem 1.7. *$\ell^1(h)$ is weakly amenable if and only if $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is unbounded for all $\varphi \in \ell^\infty\backslash\{0\}$.*

There are two special situations in which the *failure* of weak amenability may be seen rather quickly from Theorem 1.7:

- $(\kappa_n)_{n\in\mathbb{N}_0}$ is explicitly known and easily seen to be uniformly bounded (i.e., $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\} = \{\|\kappa_n * \epsilon_0\|_\infty : n \in \mathbb{N}_0\}$ is bounded). For instance, this is the case for the ultraspherical polynomials $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$ with $\alpha \geq 0$ (see [Las07, Corollary 1]). Recall that, given some $\alpha > -1$, $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$ is given by its orthogonalization measure

$$\mathrm{d}\mu(x) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1)^2}(1-x^2)^\alpha \chi_{(-1,1)}(x)\,\mathrm{d}x \tag{1.11}$$

  and the normalization $P_n^{(\alpha)}(1) = 1$ $(n \in \mathbb{N}_0)$,[21] or, equivalently, in terms of

$$b_n \equiv 0,\ a_n \equiv 1 - b_n - c_n,\ c_n := \frac{n}{2n+2\alpha+1}\ (n \in \mathbb{N});$$

  property (P) holds (and hence a polynomial hypergroup is induced) if and only if $\alpha \geq -\frac{1}{2}$, and the Haar weights are given by $h(0) = 1$ and

$$h(n) = \frac{(2n+2\alpha+1)(2\alpha+2)_{n-1}}{n!}\ (n \in \mathbb{N}); \tag{1.12}$$

[21] We use the following (widely common) notation: for any subset $A \subseteq \mathbb{R}$ let $\chi_A : \mathbb{R} \to \{0,1\}$ be defined by $\chi_A(x) := \begin{cases} 1, & x \in A, \\ 0, & \text{else}. \end{cases}$

moreover, $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) = \mathsf{supp}\,\mu = [-1,1]$ [Las05, Section 6].[22] One has

$$\|\kappa_n\|_\infty = \frac{2n+2\alpha+1}{h(n)}\ (n \in \mathbb{N}) \tag{1.13}$$

[Las07, (10); (11)]. It is obvious that

$$h(n) = \Theta(n^{2\alpha+1})\ (n \to \infty),\ \text{i.e.,}\ h(n) = \mathcal{O}(n^{2\alpha+1})\ (n \to \infty)\ \text{and}\ \liminf_{n\to\infty} \frac{|h(n)|}{n^{2\alpha+1}} > 0 \tag{1.14}$$

(cf. also (2.25) in Section 2.5). Hence, $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ is bounded if and only if $\alpha \geq 0$.

For $\alpha = -\frac{1}{2}$, one obtains the Chebyshev polynomials of the first kind $(T_n(x))_{n\in\mathbb{N}_0}$, i.e.,

$$P_n^{(-\frac{1}{2})}(x) = T_n(x)\ (n \in \mathbb{N}_0);$$

this is the only example among the ultraspherical polynomials such that $\ell^1(h)$ is even amenable ([Las07, Corollary 3], Theorem 1.4 (i)).[23] The problem of weak amenability for the parameter region $\alpha \in \left(-\frac{1}{2}, 0\right)$, which was suggested in [Las09c, Section 3] and has been open for some years, shall be solved in this thesis, see Section 2.5; note that, by (1.13), $\|\kappa_n\|_\infty \to \infty$ $(n \to \infty)$ in this case. For $\alpha = 0$, one gets the Legendre polynomials; for $\alpha = \frac{1}{2}$, one has the Chebyshev polynomials of the second kind.

- $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is bounded for some $\varphi \in \{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$; since, due to (1.10), ${}_x\alpha * \kappa_n(m) = \sum_{k=0}^n T_m\, {}_x\alpha(k)\kappa_n(k)h(k) = P_m(x)\sum_{k=0}^n \kappa_n(k)P_k(x)h(k) = P_m(x)P_n'(x)$ for each $x \in \widehat{\mathbb{N}_0}$ and $m,n \in \mathbb{N}_0$, and since consequently $\|\kappa_n *\, {}_x\alpha\|_\infty = |P_n'(x)|$, the boundedness of $\{\|\kappa_n *\, {}_x\alpha\|_\infty : n \in \mathbb{N}_0\}$ corresponds to the existence of a nonzero bounded point derivation at x (due to Theorem 1.6) [Kah15, Section 1]. In other words: the general implication "weakly amenable ⇒ point amenable" (cf. Subsection 1.2) reduces to the fact that point amenability is equivalent to the unboundedness of $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ for all $\varphi \in \{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$, which is a proper subset of $\ell^\infty\backslash\{0\}$,[24] whereas weak amenability is equivalent to the unboundedness of $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ for *all* $\varphi \in \ell^\infty\backslash\{0\}$, regardless of whether φ is a character or not.

In general, a direct application of Theorem 1.7 may be very difficult and require deep knowledge about the concrete sequence $(P_n(x))_{n\in\mathbb{N}_0}$, however; several problems occur: on the one hand, in many interesting examples explicit formulas for $(\kappa_n)_{n\in\mathbb{N}_0}$ and the convolution (which relies on the linearization coefficients $g(m,n;k)$) are not available or at least cumbersome—therefore, explicit computations are tedious or, in many cases, even impossible. On the other hand, the space ℓ^∞ is "too large" to be "manageable" via the basic concepts of harmonic analysis—many of these are limited to $\ell^1(h)$ (or at least $\ell^2(h)$) or to $\{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$.

Apart from this thesis (and our corresponding paper [Kah15]), we are aware of only one example to which Theorem 1.7 has successfully been applied in order to *establish* weak amenability: this is the sequence $(T_n(x))_{n\in\mathbb{N}_0}$ of Chebyshev polynomials of the first kind [Las07, Corollary 2], for which both $(\kappa_n)_{n\in\mathbb{N}_0}$ and $*$ take a very convenient form—but, as already recalled above, $\ell^1(h)$

[22] The reference contains a typo concerning the Haar weights—the correct formula (1.12) can be obtained from (3.2) by setting $\alpha = \beta$.

[23] Further examples of polynomial hypergroups such that $\ell^1(h)$ is amenable are provided by the Bernstein–Szegő polynomials [LP10].

[24] That $\{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$ is indeed a *proper* subset of $\ell^\infty\backslash\{0\}$ can be seen as follows: for instance, $\epsilon_0 \notin \{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\}$—for otherwise one would have $0 = P_1(x) = P_2(x) = \ldots$ for some $x \in \widehat{\mathbb{N}_0}$, which contradicts orthogonality [Chi78, I-Theorem 5.3]. (Of course, an even easier argument to see that the subset is proper would be to take into account that $\{{}_x\alpha : x \in \widehat{\mathbb{N}_0}\} \subseteq \{\varphi \in \ell^\infty : \varphi(0) = 1\}$.)

is even known to be amenable (with completely different proofs using approximate diagonals). Despite the latter fact, the proof of [Las07, Corollary 2] is interesting concerning generalizations, cf. [Las07, remark after the proof of Corollary 3]. We refer to Section 2.1 below for a modified version which will indeed serve as a kind of motivation for the research presented in Section 2.2, Section 2.5 and Section 3.2.

It is one of the purposes of this thesis to deduce a necessary criterion which can be as "easy" to check as the convenient necessary criteria for amenability and α-amenability (and hence right character amenability) recalled above, and to deduce a sufficient criterion which, in particular, enables us to solve a problem which has been open for some years, namely whether "$\ell^1(h)$ weakly amenable" already implies "$\ell^1(h)$ amenable" (or at least "$\ell^1(h)$ right character amenable"). Concerning this problem, our sufficient criterion will enable us to give explicit examples such that these implications do *not* hold (for instance, the abovementioned ultraspherical polynomials for the parameter region $\alpha \in \left(-\frac{1}{2}, 0\right)$ will turn out to be suitable examples, see Section 2.5). These might also be the first examples of commutative, non-compact hypergroups in general which have such properties.

The related question whether there exists a polynomial hypergroup such that $\ell^1(h)$ is point amenable but fails to be weakly amenable and right character amenable has already been answered positively by [Las07, Las09b, Las09c], also in terms of ultraspherical polynomials: on the one hand, the ultraspherical polynomials yield point amenable $\ell^1(h)$ if and only if $\alpha < \frac{1}{2}$ (where $D_x \neq 0$ exists for each $x \in (-1, 1)$ if $\alpha \geq \frac{1}{2}$) [Las09b, Corollary 1]. On the other hand, the ultraspherical polynomials yield right character amenable $\ell^1(h)$ if and only if $\alpha = -\frac{1}{2}$ [Las09c, p. 792]—then, $\ell^1(h)$ is also amenable (Chebyshev polynomials of the first kind). Finally, recall that $\ell^1(h)$ fails to be weakly amenable if $\alpha \geq 0$ (cf. above).

The little q-Legendre polynomials [Las05, Section 6] provide an example of a polynomial hypergroup such that $\ell^1(h)$ is right character amenable yet non-amenable, see [Las09c, p. 792].

Remark. We note at this stage that in the references [FLS04, Las07, Las09b, Las09c], which we cite frequently in this thesis, the additional assumption $b_0 \geq 0$ was made. However, none of the results we cite becomes false if this additional assumption is dropped (easy to see; the assumption $b_0 \geq 0$ has no meaning). The class of Jacobi polynomials, see Section 3, contains examples where $b_0 < 0$.

2. Point and weak amenability of ℓ^1-algebras of polynomial hypergroups: general results

Parts of Section 2 are very similar to our publication [Kah15].

If not stated otherwise, let $(P_n(x))_{n\in\mathbb{N}_0}$ always be as in Section 1.3 (without further specification).

2.1. Motivating example: Chebyshev polynomials of the first kind reconsidered

We define $(f_n)_{n\in\mathbb{N}_0} \subseteq c_{00}$ by

$$p_n^2(x) = \sum_{k=0}^{2n} f_n(k)P_k(x)h(k),\ f_n(2n+1) := f_n(2n+2) := \ldots := 0\ (n \in \mathbb{N}_0, x \in \mathbb{R}),$$

or, equivalently, by

$$f_n = \mathcal{P}^{-1}(p_n^2)\ (n \in \mathbb{N}_0), \tag{2.1}$$

and we define $(F_n)_{n\in\mathbb{N}_0} \subseteq c_{00}$ by

$$F_n := \frac{1}{n+1}\sum_{k=0}^{n} f_k\ (n \in \mathbb{N}_0). \tag{2.2}$$

It is easy to see that f_n and F_n are nonnegative, and that f_n is explicitly given by

$$f_n(k) = \begin{cases} g(n,k;n), & k \leq 2n, \\ 0, & \text{else.} \end{cases} \tag{2.3}$$

Moreover, for the sake of brevity, we define $\sigma : \mathbb{N} \to \mathbb{R}\backslash\{0\}$ by

$$\sigma(n) := \kappa_n(n-1).$$

In the following, we consider the simplest example of a polynomial hypergroup on $\mathbb{N}_0$; this is provided by the Chebyshev polynomials of the first kind, i.e., $(T_n(x))_{n\in\mathbb{N}_0} = \left(P_n^{(-\frac{1}{2})}(x)\right)_{n\in\mathbb{N}_0}$. The Chebyshev polynomials of the first kind are given via the simplest possible recurrence coefficients: $b_n \equiv 0$ and $a_n = c_n = \frac{1}{2}$ $(n \in \mathbb{N})$. The orthogonalization measure reduces to $\mathrm{d}\mu(x) = \omega_T(x)\,\mathrm{d}x$, where $\omega_T : \mathbb{R} \to [0,\infty)$,

$$\omega_T(x) := \frac{1}{\pi}(1-x^2)^{-\frac{1}{2}}\chi_{(-1,1)}(x). \tag{2.4}$$

One has

$$\cos(nx) = T_n(\cos x)\ (n \in \mathbb{N}_0, x \in \mathbb{C}) \tag{2.5}$$

[AS64, 22.3.15],

$$g(m,n;k) = \frac{1}{2}[\delta_{|m-n|}(k) + \delta_{m+n}(k)]\ (m,n \in \mathbb{N}_0, k \in \{|m-n|,\ldots,m+n\}), \tag{2.6}$$

and the Haar function reduces to

$$h(n) = 2 - \delta_0(n)\ (n \in \mathbb{N}_0). \tag{2.7}$$

All of these basics are well-known. $(f_n)_{n\in\mathbb{N}_0}$ (via (2.3) and (2.6)), $(F_n)_{n\in\mathbb{N}_0}$ and σ are explicitly given by

$$f_n = \begin{cases} \epsilon_0, & n = 0, \\ \epsilon_0 + \epsilon_{2n}, & \text{else,} \end{cases} \tag{2.8}$$

$$F_n = \frac{n}{n+1}\epsilon_0 + \frac{1}{n+1}\sum_{k=0}^{n}\epsilon_{2k},$$

$$\sigma(n) = n.$$

Moreover, one has

$$\kappa_n = \sigma(n) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} h(n-1-2k)\epsilon_{n-1-2k} \ (n \in \mathbb{N}) \tag{2.9}$$

[Las07, p. 186]. We now give a—more generalizable—variant to [Las07, Corollary 2] concerning the proof of weak amenability of the Banach algebra $\ell^1(h)$ that corresponds to the Chebyshev polynomials of the first kind. This variant can be regarded as motivation for our sufficient criterion presented in the next subsection. As in [Las07], we use Theorem 1.7. Hence, let $\varphi \in \ell^\infty$ such that $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is bounded, and let $C := \sup_{n\in\mathbb{N}_0} \|\kappa_n * \varphi\|_\infty$. We divide our proof into four steps:

Step 1: it is shown in [Las07, proof of Corollary 2] that (in our notation)

$$\kappa_{n+1} * \epsilon_n - \kappa_n * \epsilon_{n+1} = \sigma(n+1)F_n \ (n \in \mathbb{N}_0). \tag{2.10}$$

Consequently, $\|F_n * \varphi\|_\infty \leq \frac{2C}{n+1}$ $(n \in \mathbb{N}_0)$, and therefore

$$\|F_n * \varphi\|_\infty \to 0 \ (n \to \infty). \tag{2.11}$$

Step 2: let $T := \sup_{n\in\mathbb{N}} \frac{\sigma(n+2)}{\sigma(n)} (= 3)$. Using (2.9), for every $n \in \mathbb{N}$ the following calculation holds:

$$\frac{\kappa_{n+2} * \varphi(0)}{\sigma(n+2)} - \frac{\kappa_n * \varphi(0)}{\sigma(n)} = h(n+1)T_{n+1}\varphi(0) = h(n+1)\varphi(n+1). \tag{2.12}$$

Consequently,

$$(n+2)|\varphi(n+1)| = \frac{1}{2}\left|\kappa_{n+2} * \varphi(0) - \frac{\sigma(n+2)}{\sigma(n)}\kappa_n * \varphi(0)\right| \leq \frac{1}{2}(C+TC) = 2C, \tag{2.13}$$

which implies that

$$\varphi \in \mathcal{O}(n^{-1}) \subseteq c_0. \tag{2.14}$$

Step 3: let

$$F := \epsilon_0. \tag{2.15}$$

Obviously, $f_n \to F$ $(n \to \infty)$ pointwise and

$$\sup_{n\in\mathbb{N}_0} \|f_n\|_1 < \infty. \tag{2.16}$$

Hence, $(F_n)_{n\in\mathbb{N}_0}$ converges to F pointwise (even uniformly), and

$$\sup_{n\in\mathbb{N}_0} \|F_n\|_1 < \infty. \tag{2.17}$$

Now given any $m \in \mathbb{N}_0$, we can use that T_m is nonexpansive w.r.t. the $\|.\|_p$-norms to conclude that $T_mF_n \overset{*}{\rightharpoonup} T_mF$ w.r.t. $\sigma(\ell^1(h), c_0)$ [CP03, p. 405; p. 413], which, in view of Step 2, yields $F_n * \varphi(m) = \langle T_mF_n, \varphi\rangle \to \langle T_mF, \varphi\rangle = F * \varphi(m)$ $(n \to \infty)$, i.e., $F_n * \varphi$ converges to $F * \varphi$ pointwise. With Step 1, we obtain from this that

$$\|F_n * \varphi - F * \varphi\|_\infty \to 0 \ (n \to \infty), \tag{2.18}$$

and,

Step 4, that $F * \varphi = 0$, i.e., $\varphi = 0$ (since $F = \epsilon_0$). Using Theorem 1.7, we see that $\ell^1(h)$ is indeed weakly amenable. Note that, blowing up the conclusion at the beginning of the present Step 4, it reads

$$(\forall m \in \mathbb{N}_0 : \langle T_mF, \varphi\rangle = 0) \Rightarrow \varphi = 0. \tag{2.19}$$

We note again that $\ell^1(h)$ is even amenable [Las07, Corollary 3] (or also [Wol84, Proposition 5.4.4])—so the preceding proof of weak amenability is interesting only in view of possible generalizations to other, more complicated polynomial hypergroups. We chose our presentation with a view to such generalizations (and, to this end, inserted (2.18) and (2.19), which the reader might have found clumsy and needless at first sight). Searching for generalizations, it is remarkable that Step 1 (both the result (2.11) and the central argument (2.10)) holds in large generality whenever $\sigma(n) \to \infty$ $(n \to \infty)$, see the next subsection. Besides this observation, however, again several problems occur:

- *Problem 1:* Step 2 essentially relied on the (very) special structure of the Chebyshev polynomials of the first kind: the important intermediate result (2.14) was due to the simple explicit formula for the sequence $(\kappa_n)_{n\in\mathbb{N}}$ (2.9) (and also due to the finiteness of T).

- *Problem 2:* Step 3 made use of the existence of a simple weak-$*$ limit F (2.15) of the sequence $(F_n)_{n\in\mathbb{N}_0}$ w.r.t. $\sigma(\ell^1(h), c_0)$. In general, such a limit need not exist at all: the Banach–Steinhaus theorem yields that (2.17) is necessary for the existence of a weak-$*$ limit, and in Subsection 2.5 we shall see examples such that (2.17) is violated (ultraspherical polynomials for $\alpha > -\frac{1}{2}$). Although a pointwise—and even uniform—limit (which, without suitable further information, is of little worth, of course) exists under rather general conditions, see the notes at the end of Subsection 2.2, it need not be of "simple" structure (or at least explicitly available) in general. All in all, a reasonable generalization of (2.18) will be one of the main tasks.

- *Problem 3:* even if one comes up to a situation which can be compared to the beginning of Step 4, the implication in (2.19) might be false or at least very nontrivial, depending on F: it might happen that the set $\{T_m F : m \in \mathbb{N}_0\}$ is simply "not large enough" to allow the conclusion. Therefore, also Step 4 essentially relied on the special structure of the Chebyshev polynomials of the first kind.

In the next subsection, we shall present the announced sufficient criterion for weak amenability of $\ell^1(h)$. The Chebyshev weight ω_T (2.4) will play a crucial role because it naturally arises when considering the limiting behavior of orthogonal polynomials.

2.2. Establishing weak amenability: growth conditions and asymptotics

We give a sufficient criterion whose combination with inheritance via homomorphisms shall allow us to establish weak amenability of $\ell^1(h)$ for all of those Jacobi, symmetric Pollaczek and associated ultraspherical polynomial sequences such that weak amenability of $\ell^1(h)$ holds. Before, the only known example in this class was the sequence $(T_n(x))_{n\in\mathbb{N}_0}$ of Chebyshev polynomials of the first kind. As already outlined in the previous subsection, Step 1 of our motivating example (which considered just these Chebyshev polynomials of the first kind) generalizes without noteworthy difficulties: the proof of the following Lemma 2.1, which is [Kah15, Lemma 2.2] in our corresponding paper, is based on ideas of [Las07, Section 2, in part. Corollary 2], puts them into a more general framework, and uses, as already done in [Las07], the well-known 'Christoffel–Darboux formula' which states that

$$\frac{1}{a_0 c_n h(n)} \sum_{k=0}^{n-1} h(k) P_k^2(x) = P_n'(x) P_{n-1}(x) - P_{n-1}'(x) P_n(x) \tag{2.20}$$

for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$ (cf. [Las07, (12)]).

Lemma 2.1. *Let $\varphi \in \ell^\infty$ such that $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is bounded. If $\sigma(n) \to \infty$ $(n \to \infty)$, then $\|F_n * \varphi\|_\infty \to 0$ $(n \to \infty)$.*

Roughly speaking, Problem 1, Problem 2 and Problem 3 described at the end of the previous subsection can be overcome with the following approach:
(1) do not overcome Problem 1 in generality, but make (2.14)—more precisely, a stronger form of (2.14) which imposes also properly sublinear growth of h and therefore yields both $\mathcal{O}(n^{-1}) \subseteq \ell^2(h)$ and, as needed to apply Lemma 2.1, $\sigma(n) \to \infty$ $(n \to \infty)$ (see Lemma 2.4 (i) below)—a condition.
(2) Overcome Problem 2 by considering a weak limit F of the sequence $(F_n)_{n\in\mathbb{N}_0} = \left(\frac{1}{n+1}\sum_{k=0}^{n} f_k\right)_{n\in\mathbb{N}_0}$ w.r.t. $\sigma(\ell^2(h), \ell^2(h))$ instead of a weak-$*$ limit w.r.t. $\sigma(\ell^1(h), c_0)$. This exists under considerably—and suitably—more general growth conditions than (2.16), and our further strategy will not require to know it in an explicit form. Our central idea is to consider asymptotics and to use a result which goes back to Nevai and yields that—under suitable conditions—the polynomials $p_n^2(x)$ (which coincide with the functions $\widehat{f_n}(x)$ on $\widehat{\mathbb{N}_0}$) increasingly rapidly "oscillate" around a certain weak limit as $n \in \mathbb{N}_0$ increases. This limit involves the Chebyshev weight ω_T (2.4).
(3) Overcome Problem 3 via Hilbert space methods and the fundamental lemma of the calculus of variations [Dac04, Theorem 1.24].

The following auxiliary results correspond to [Kah15, Lemma 2.3] (Lemma 2.2) and [Kah15, Lemma 2.4] (Lemma 2.3).

Lemma 2.2. *If μ is absolutely continuous*[25] *and* supp $\mu = [-1,1]$*, and if furthermore $\mu' > 0$ a.e. in $[-1,1]$ and $\sup_{n\in\mathbb{N}_0}\int_{\mathbb{R}} p_n^4(x)\,\mathrm{d}\mu(x) < \infty$, then $\frac{\omega_T}{\mu'} \in L^2(\mathbb{R},\mu)$ and $\left\|F_n * \varphi - \mathcal{P}^{-1}\left(\frac{\omega_T}{\mu'}\right) * \varphi\right\|_\infty \to 0$ $(n\to\infty)$ for all $\varphi \in \ell^2(h)$.*[26]

Lemma 2.3. *Under the conditions of Lemma 2.2, the set $\left\{T_m\mathcal{P}^{-1}\left(\frac{\omega_T}{\mu'}\right) : m \in \mathbb{N}_0\right\}$ is total in the Hilbert space $\ell^2(h)$.*

Establishing theses lemmas is the actual work on the way to our sufficient criterion. The details are given in our paper [Kah15]. Besides the ingredients outlined above, our proof makes massive use of another result on asymptotic behavior (taken from Nevai's work [Nev79]), of a suitable weak convergence argument [BC09, Corollary A.8.8], of the Plancherel isomorphism, of appropriate approximation arguments, and, which is a consequence of the hypergroup structure (cf. Section 1.3), of the uniform boundedness of $\{P_n(x) : n \in \mathbb{N}_0\}$ on supp μ. Concerning Nevai's asymptotic result recalled above, our proof uses a version which is presented in [MNT87, Theorem 12.1] and provides a suitable strong convergence for the arithmetic means. Our strategy makes essential use of the additional regularity conditions on μ.

Based on Lemma 2.1, Lemma 2.2 and Lemma 2.3, we obtain the following sufficiency criterion, which is [Kah15, Theorem 2.3]:

Theorem 2.1. *If each of the conditions*

(i) $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ *is unbounded for all* $\varphi \in \ell^\infty \backslash \mathcal{O}(n^{-1})$,

(ii) μ *is absolutely continuous,* supp $\mu = [-1,1]$, $\mu' > 0$ *a.e. in* $[-1,1]$,

(iii) $h(n) = \mathcal{O}(n^\alpha)$ *(as* $n \to \infty$*) for some* $\alpha \in [0,1)$,

[25] If not stated otherwise, absolute continuity of the orthogonalization measure μ, the Radon–Nikodym derivative μ', as well as singular parts and so on, are always meant w.r.t. the Lebesgue–Borel measure on $\mathbb{R}$.

[26] Of course, the expression "$\frac{\omega_T}{\mu'} \in L^2(\mathbb{R},\mu)$" has to be interpreted in the sense that there exists a (unique) element of $L^2(\mathbb{R},\mu)$ that contains every Borel measurable continuation of the μ-a.e. defined quotient $\frac{\omega_T}{\mu'}$ to $\mathbb{R}$, and this element of $L^2(\mathbb{R},\mu)$ shall then be denoted by $\frac{\omega_T}{\mu'}$ again. In the following, we shall use this notation without further explanation.

(iv) $\sup_{n\in\mathbb{N}_0}\int_{\mathbb{R}} p_n^4(x)\,\mathrm{d}\mu(x)<\infty$

holds, then $\ell^1(h)$ *is weakly amenable.*

When applying Theorem 2.1 to examples later in this thesis, we will check condition (i) by verifying even the unboundedness of $\{|\kappa_n*\varphi(0)|:n\in\mathbb{N}_0\}$ for every $\varphi\in\ell^\infty\backslash\mathcal{O}(n^{-1})$; cf. also the motivating Subsection 2.1.

Obviously, condition (iv) of Theorem 2.1 is equivalent to the boundedness of $\{\|f_n\|_2:n\in\mathbb{N}_0\}$.

Using the lemmas, the proof of Theorem 2.1 can be done by transferring the conclusion of our motivating example in Subsection 2.1.

For the sake of completeness, we note that condition (ii) of Theorem 2.1 (i.e., that μ is absolutely continuous and in the 'Erdős class') has some interesting consequences by itself. On the one hand, it yields $\widehat{\mathbb{N}_0}=[-1,1]$: [Nev86, Theorem 4.5.6] tells that $\frac{h(n)}{\sum_{k=0}^{n-1}h(k)}\to 0\ (n\to\infty)$; therefore, h is of subexponential growth, which then implies $\widehat{\mathbb{N}_0}=\mathcal{X}^b(\mathbb{N}_0)=\mathsf{supp}\ \mu=[-1,1]$ as a consequence of [Las05, Corollary 6.1].[27] On the other hand, it yields the existence of a pointwise limit $F\in\ell^\infty$ of the sequence $(f_n)_{n\in\mathbb{N}_0}$, which then is also the uniform limit of $(F_n)_{n\in\mathbb{N}_0}$: this is a consequence of [MNT87, Theorem 11.1; Theorem 12.1]. Another consequence of Theorem 2.1 (ii) is that $(P_n(x))_{n\in\mathbb{N}_0}$ is of Nevai class $M(0,1)$ [Nev86, Theorem 4.5.7].

A different sufficient criterion for weak amenability of $\ell^1(h)$ was given via [Per11a, Proposition 2.19]; it involves the inverses of the functions F_n (in our notation) in $\ell^1(h)$ and the norms $\left\|F_n^{-1}\right\|_1$ ($n\in\mathbb{N}_0$; cf. [Per11a, Definition 2.11]). Even if the existence of these inverses is guaranteed,[28] it may be very difficult to estimate the norms $\left\|F_n^{-1}\right\|_1$. This problem, and the resulting lack of satisfying applicability, has already been pointed out in [Per11a]; to our knowledge, applications on concrete examples have not been given.

2.3. Ruling out weak amenability: shift operators and smoothness conditions

As recalled in Section 1.4, the Banach algebra $\ell^1(h)$ that corresponds to the sequence of ultraspherical polynomials $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$ with $\alpha\geq 0$ can be seen to be *not* weakly amenable by realizing that $(\kappa_n)_{n\in\mathbb{N}_0}$, which is explicitly known and of exactly the same structure as (2.9) (cf. Subsection 2.5 below), is uniformly bounded [Las07, Corollary 1]. In view of (2.9) (or, more precisely, its analogue (2.26) below), this just means that σ is bounded. Observe that the "critical value" $\alpha=0$ (which corresponds to the Legendre polynomials) has an interesting meaning concerning the orthogonalization measure (1.11): if $\alpha\geq 0$, then the Radon–Nikodym derivative μ' is continuous on $[-1,1]$ and continuously differentiable on $(-1,1)$, whereas if $\alpha<0$, then μ' behaves "worse". Theses observations yield the following questions concerning a general polynomial hypergroup:

- Is the boundedness of σ equivalent to the boundedness of $\{\|\kappa_n\|_\infty:n\in\mathbb{N}_0\}$? (Trivially, the boundedness of σ is necessary for the boundedness of $\{\|\kappa_n\|_\infty:n\in\mathbb{N}_0\}$.)
- Is $\{\|\kappa_n\|_\infty:n\in\mathbb{N}_0\}$ automatically bounded—and hence $\ell^1(h)$ not weakly amenable—whenever μ' is sufficiently "smooth"?

[27] An interesting characterization of $\widehat{\mathbb{N}_0}=\mathcal{X}^b(\mathbb{N}_0)=\mathsf{supp}\ \mu$ is given in [Per11b].

[28] For instance, this is satisfied if h is of subexponential growth because then $\widehat{\mathbb{N}_0}=\mathcal{X}^b(\mathbb{N}_0)=\mathsf{supp}\ \mu$ [Las05, Corollary 6.1] and $\widehat{F_n}>0$ on $\widehat{\mathbb{N}_0}$, so the existence follows from well-known general Gelfand theory. In particular, the existence of the inverses F_n^{-1} ($n\in\mathbb{N}_0$) would be guaranteed under the conditions of our sufficiency criterion Theorem 2.1. We note that there exists an interesting generalization of the cited ingredient [Las05, Corollary 6.1], see [BH95, Theorem 2.5.12] and [BH95, Corollary 2.5.13] (which correspond to results of Vogel [Vog87] and Voit [Voi88]), and cf. [Las05, Lemma 6.2].

If the answer to the first question was positive, this would be very convenient because the computation of σ is much easier than the computation of the whole sequence $(\kappa_n)_{n\in\mathbb{N}_0}$, see Lemma 2.4 (i) below. However, the answer is negative, see the notes at the end of Section 3.2. Nevertheless, the first part of Theorem 2.2 below provides a closely related characterization which involves the values $\kappa_n(0)$ (instead of $\sigma(n) = \kappa_n(n-1)$); for brevity, let $\tau : \mathbb{N}_0 \to \mathbb{R}$ be defined by

$$\tau(n) := \kappa_n(0).$$

The computation of τ is more involved than that of σ, but it can still be considerably easier than the computation of the whole sequence $(\kappa_n)_{n\in\mathbb{N}}$. In fact, Lemma 2.4 (ii) and (iii) will give explicit recurrence relations for τ; in Section 3.5, we shall apply these to the class of cosh-polynomials.

The answer to the second question ("μ' sufficiently 'smooth' $\Rightarrow \{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ bounded?") will turn out to be positive, which yields a necessary criterion for weak amenability which is as short as the necessary criteria for amenability, α-amenability and right character amenability recalled in Section 1.4. In the following, we state the announced results, which are [Kah15, Lemma 2.1] and [Kah15, Theorem 2.2].

Lemma 2.4. *The following hold:*

(i) *σ is explicitly given by*

$$\sigma(n) = \frac{n}{a_0 c_n h(n)} \ (n \in \mathbb{N}).$$

(ii) *τ satisfies the following coupled system of recursions: $\tau(0) = 0$, $\tau(1) = \frac{1}{a_0}$,*

$$\tau(2n) = -\frac{1}{g(n,n;2n)} \sum_{k=1}^{2n-1} g(n,n;k)\tau(k) \ (n \in \mathbb{N}),$$
$$\tau(2n+1) = \frac{1}{g(n,n+1;2n+1)} \left[\frac{n+1}{a_0 a_n h(n)} - \sum_{k=1}^{2n} g(n,n+1;k)\tau(k)\right] \ (n \in \mathbb{N}).$$

(iii) *If $b_n \equiv 0$ (symmetric case), then τ satisfies $\tau(2n) = 0$ ($n \in \mathbb{N}_0$) and the recurrence relation $\tau(1) = 1$,*

$$\tau(2n+1) = \frac{1}{g(n,n+1;2n+1)} \left[\frac{n+1}{a_n h(n)} - \sum_{k=1}^{n} g(n,n+1;2k-1)\tau(2k-1)\right] \ (n \in \mathbb{N}).$$

(iv) *For each $n \in \mathbb{N}_0$ the representation*

$$\kappa_n = \chi_{\{0,\ldots,n\}} T_n \tau$$

holds.

(v) *If $b_n \equiv 0$, then $\kappa_{2n-1}(2k+1) = \kappa_{2n}(2k) = 0$ ($n \in \mathbb{N}, k \in \mathbb{N}_0$).*

Theorem 2.2. *$\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ is bounded if and only if τ is bounded. If $\ell^1(h)$ is weakly amenable, then*

(i) *τ is unbounded;*

moreover, if $\ell^1(h)$ is weakly amenable (or, more generally, if (i) holds), then

(ii) *μ has a singular part or μ' is not absolutely continuous (as a function) on $[\min \mathsf{supp}\ \mu, \max \mathsf{supp}\ \mu]$.*

The detailed proofs are given in our paper [Kah15].

We note that Lemma 2.4 (i), which is also contained in the author's Master's thesis [Kah12], is immediate from integrating the Christoffel–Darboux formula (2.20) w.r.t. μ; under the additional assumption of symmetry (i.e., $b_n \equiv 0$, which we have not supposed), this result is already contained in [LO08]—with a completely different proof, however, which was generalized to a proof of the full assertion of Lemma 2.4 (i) by E. Perreiter then (who found Lemma 2.4 (i) independently of the author).

Lemma 2.4 (iv), which relates $(\kappa_n)_{n\in\mathbb{N}_0}$ to τ via the shift operators T_n, is the central tool for establishing Lemma 2.4 (ii) and (iii). Moreover, from Lemma 2.4 (iv) the necessity of the boundedness of $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ for the boundedness of τ (the sufficiency is trivial) can be seen as follows: if τ is bounded, then

$$\|\tau\|_\infty \geq \sup_{n\in\mathbb{N}_0} \|T_n\tau\|_\infty \geq \sup_{n\in\mathbb{N}_0} \|\kappa_n\|_\infty$$

because, as a consequence of the hypergroup structure, the shift operators T_n are nonexpansive on ℓ^∞ (see Section 1.3). The proof of Theorem 2.2 (ii) relies on Theorem 2.2 (i), an integration by parts argument and the uniform boundedness of $\{P_n(x) : n \in \mathbb{N}_0\}$ on $\operatorname{supp} \mu$. With regard to Theorem 2.2 (ii), it is interesting to observe that there are polynomial hypergroups whose orthogonalization measure μ is even purely discrete—an explicit example is provided by the little q-Legendre polynomials [Las05, Section 6]. As we already have recalled, the little q-Legendre polynomials yield right character amenable yet non-amenable $\ell^1(h)$.

Considering the ultraspherical polynomials again, (1.11) and (1.14) show that if μ' is absolutely continuous on $[-1, 1]$, then h is of at least linear growth. Via Theorem 2.2, this generalizes to the following observation for *arbitrary* polynomial hypergroups: if μ is absolutely continuous and μ' is absolutely continuous on the convex hull of $\operatorname{supp} \mu$, then τ—and hence σ—is bounded due to Theorem 2.2; in view of Lemma 2.4, this is only possible if h has at least linear growth. Compare this to the conditions of the sufficiency criterion Theorem 2.1.

2.4. Inheritance via homomorphisms and results on point amenability

As already mentioned, [Las09b] contains many results on point amenability of $\ell^1(h)$. In this subsection, we give some additional criteria. The following proposition, which corresponds to [Kah15, Proposition 2.1], gives two necessary criteria in the symmetric case (i.e., $b_n \equiv 0$):

Proposition 2.1. *Let* $b_n \equiv 0$.

(i) *If* $c_n a_{n-1} \leq \frac{1}{4}$ $(n \in \mathbb{N})$, *then* $\widehat{\mathbb{N}_0} = [-1, 1]$ *and* $D_x \neq 0$ *exists for all* $x \in (-1, 1)$. *Hence, if* $\ell^1(h)$ *is point amenable, then* $c_n a_{n-1} > \frac{1}{4}$ *for some* $n \in \mathbb{N}$.

(ii) *If* $\limsup_{n\to\infty} c_n < \frac{1}{2}$, *then* $0 \in \widehat{\mathbb{N}_0}$ *and* $D_0 \neq 0$ *exists. Hence, if* $\ell^1(h)$ *is point amenable, then* $\limsup_{n\to\infty} c_n \geq \frac{1}{2}$.

Proposition 2.1 (i) "isolates" an argument which is in principle already contained in [Las09b, Example 7] and which is based on nonnegativity of the connection coefficients to the Chebyshev polynomials of the second kind $\left(P_n^{(\frac{1}{2})}(x)\right)_{n\in\mathbb{N}_0}$ (cf. [Ask71, Theorem 1] or [Szw92a, Theorem 2]). The argument essentially uses that—as we already have recalled—the set $\left\{\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\frac{1}{2})}(x) : n \in \mathbb{N}_0\right\}$ is bounded for every $x \in (-1, 1)$ (and that one has $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) = \operatorname{supp} \mu = [-1, 1]$ for the Chebyshev polynomials of the second kind). The proof of Proposition 2.1 (ii) shows that the condition $\limsup_{n\to\infty} c_n < \frac{1}{2}$ enforces both $\{P_n(0) : n \in \mathbb{N}_0\}$ and $\{P_n'(0) : n \in \mathbb{N}_0\}$ to be bounded; hence, one can apply Theorem 1.6.

The following "inheritance via homomorphisms" result due to Lasser and Perreiter concerning weak amenability is from [LP10, in part. Theorem 2.2 and Proposition 3.1].

Theorem 2.3. *Let $(\widetilde{P_n(x)})_{n\in\mathbb{N}_0}$ induce another polynomial hypergroup on $\mathbb{N}_0$, and let $\widetilde{h}$ denote the corresponding Haar function. Let for each $n\in\mathbb{N}_0$ the connection coefficients $C_n(0),\ldots,C_n(n)$ be defined by $P_n(x)=\sum_{k=0}^n C_n(k)\widetilde{P_k(x)}$, and assume that $\sum_{k=0}^n |C_n(k)|\leq C$ $(n\in\mathbb{N}_0)$ for some fixed $C>0$. Then there exists a continuous homomorphism of Banach algebras with dense range from $\ell^1(h)$ into $\ell^1(\widetilde{h})$, and if $\ell^1(h)$ is weakly amenable, then so is $\ell^1(\widetilde{h})$.*

We have the following analogue to Theorem 2.3 for point amenability [Kah15, Proposition 2.2]:

Proposition 2.2. *Under the conditions of Theorem 2.3, let $\widetilde{\widehat{\mathbb{N}_0}}$ refer to $(\widetilde{P_n(x)})_{n\in\mathbb{N}_0}$. Then the following holds: if $x\in\widetilde{\widehat{\mathbb{N}_0}}$ and $D_x\neq 0$ exists on $\ell^1(\widetilde{h})$, then $x\in\widehat{\mathbb{N}_0}$ and $D_x\neq 0$ exists on $\ell^1(h)$. In particular: if $\ell^1(h)$ is point amenable, then so is $\ell^1(\widetilde{h})$.*

Proposition 2.2 is an extension of [Las09b, Proposition 6] (but can be established in the same way).

The detailed proofs are given in our paper [Kah15].

2.5. A first application of the sufficient criterion: ultraspherical polynomials and the solution to the problem "amenability vs. weak amenability"

We consider the ultraspherical polynomials $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$ for $\alpha\in\left(-\frac{1}{2},0\right)$ and, as a first application of Theorem 2.1, prove that $\ell^1(h)$ is weakly amenable in this case. This is much simpler than the generalization to Jacobi polynomials presented in Section 3.2. We preliminarily note that $(f_n)_{n\in\mathbb{N}_0}$ (2.1) is explicitly given by $f_n|_{\mathbb{N}_0\backslash\{0,2,\ldots,2n\}}=0$ and

$$f_n(2k)=\frac{(2k)!\left(\alpha+\frac{1}{2}\right)_k^2}{(2\alpha+1)_{2k}(k!)^2}\frac{\left(n+\alpha+\frac{1}{2}\right)\left(\alpha+\frac{1}{2}\right)_{n-k}n!(2\alpha+1)_{n+k}}{\left(n+k+\alpha+\frac{1}{2}\right)(n-k)!(2\alpha+1)_n\left(\alpha+\frac{1}{2}\right)_{n+k}}\ (n\in\mathbb{N}_0,k\in\{0,\ldots,n\}). \tag{2.21}$$

This is a consequence of (2.3) and the explicit formula for the linearization coefficients $g(m,n;k)$ for the ultraspherical polynomials (Dougall's formula), which reads, for any $m,n\in\mathbb{N}_0$,

$$\begin{aligned}g(m,n;k)=&j!\left(\alpha+\frac{1}{2}\right)_j\binom{m}{j}\binom{n}{j}\\&\times\frac{\left(m+n+\alpha+\frac{1}{2}-2j\right)\left(\alpha+\frac{1}{2}\right)_{m-j}\left(\alpha+\frac{1}{2}\right)_{n-j}(2\alpha+1)_{m+n-j}}{\left(m+n+\alpha+\frac{1}{2}-j\right)\left(\alpha+\frac{1}{2}\right)_{m+n-j}(2\alpha+1)_m(2\alpha+1)_n}\end{aligned} \tag{2.22}$$

if $k=m+n-2j$ with $j\in\{0,\ldots,\min\{m,n\}\}$, and $g(m,n;k)=0$, else, cf. [Las83, Section 3 (a)] or [Las05, p. 97], or also [AAR99, Theorem 6.8.2] or [Ask75, (5.7)] (the latter references contain some interesting historical notes concerning Dougall's formula). With

$$(x)_p=\frac{\Gamma(x+p)}{\Gamma(x)}\ (p\in\mathbb{R},x>\max\{0,-p\}), \tag{2.23}$$

after some rearrangements and cancellations (2.21) may be rewritten as

$$\begin{aligned}f_n(2k)=&\frac{\Gamma(2\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)^2}\left(n+\alpha+\frac{1}{2}\right)(n+2\alpha+1)_{-2\alpha}(2k+2\alpha+1)_{-2\alpha}(k+1)_{\alpha-\frac{1}{2}}^2\\&\times(n-k+1)_{\alpha-\frac{1}{2}}\left(n+k+\alpha+\frac{3}{2}\right)_{\alpha-\frac{1}{2}}\quad(n\in\mathbb{N}_0,k\in\{0,\ldots,n\}).\end{aligned} \tag{2.24}$$

In the same way, we can rewrite (1.12) as

$$h(n) = \frac{2n+2\alpha+1}{\Gamma(2\alpha+2)}(n+1)_{2\alpha} \; (n \in \mathbb{N}_0). \tag{2.25}$$

Having in mind the motivating example given in Subsection 2.1, i.e., the Chebyshev polynomials of the first kind, we note that Step 2 easily transfers to the new situation: as analogue to (2.9), one has

$$\kappa_n = \sigma(n) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} h(n-1-2k)\epsilon_{n-1-2k} \; (n \in \mathbb{N}) \tag{2.26}$$

with

$$\sigma(n) = \frac{n!}{(2\alpha+2)_{n-1}} \; (n \in \mathbb{N})$$

[Las07, (10); (11)], and for any $\varphi \in \ell^\infty$ such that $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is bounded, we can argue as follows: let $C := \sup_{n\in\mathbb{N}_0} \|\kappa_n * \varphi\|_\infty$ and $T := \sup_{n\in\mathbb{N}} \frac{\sigma(n+2)}{\sigma(n)} = \sup_{n\in\mathbb{N}} \frac{(n+2)(n+1)}{(n+2\alpha+2)(n+2\alpha+1)} (< 3)$. Via (2.26), we obtain an equation which reads exactly as (2.12). Using Lemma 2.4 (i), we get

$$(n+2)|\varphi(n+1)| = a_{n+1} \left| \kappa_{n+2} * \varphi(0) - \frac{\sigma(n+2)}{\sigma(n)} \kappa_n * \varphi(0) \right| \leq C + TC \leq 4C$$

for each $n \in \mathbb{N}$. Hence, $\varphi \in \mathcal{O}(n^{-1})$.

However, it is not possible to transfer Step 3: since the asymptotic behavior of the gamma function yields the well-known limit

$$\frac{(x)_p}{x^p} \to 1 \; (x \to \infty)$$

(cf. (2.23)), (2.24) and (2.25) imply that $\liminf_{n\to\infty} \frac{f_n(2n)h(2n)}{n^{\alpha+\frac{1}{2}}} > 0$. Consequently, $\|f_n\|_1 \to \infty$ $(n \to \infty)$ and also $\|F_n\|_1 \geq \frac{1}{n+1}\sum_{k=0}^{n} f_k(2k)h(2k) \to \infty$ $(n \to \infty)$. The latter shows that $(F_n)_{n\in\mathbb{N}_0}$ cannot have a weak-$*$ limit w.r.t. $\sigma(\ell^1(h), c_0)$, however (cf. Problem 2 in Subsection 2.1).

Nevertheless, we can prove that a satisfactory replacement—namely condition (iv) of Theorem 2.1—holds (condition (i) of Theorem 2.1 has already been established above, and condition (ii) and condition (iii) are obviously satisfied from (1.11) and (1.12)): using (2.24) and the asymptotic behavior of the gamma function, there exists $C_1 > 0$ such that, for all $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$, the estimation

$$\begin{aligned}
f_n(2k) \leq & C_1 \left(n+\alpha+\frac{1}{2}\right)(n+2\alpha+1)^{-2\alpha}(2k+2\alpha+1)^{-2\alpha}(k+1)^{2\alpha-1} \\
& \times (n-k+1)^{\alpha-\frac{1}{2}} \left(n+k+\alpha+\frac{3}{2}\right)^{\alpha-\frac{1}{2}} \leq \\
\leq & 2C_1(n+2)(n+2)^{-2\alpha}(k+1)^{-2\alpha}(k+1)^{2\alpha-1} \\
& \times [(n+2)-(k+1)]^{\alpha-\frac{1}{2}}(n+k+1)^{\alpha-\frac{1}{2}} = \\
= & \frac{2C_1}{k+1}(n+2)^{1-2\alpha}[(n+2)-(k+1)]^{\alpha-\frac{1}{2}}(n+k+1)^{\alpha-\frac{1}{2}} \leq \\
\leq & \frac{6C_1}{k+1}\left(\frac{(n+2)^2}{(n+2)^2-(k+1)^2}\right)^{\frac{1}{2}-\alpha}
\end{aligned}$$

holds, where the last inequality relies on $\frac{1}{n+k+1} \leq \frac{3}{(n+2)+(k+1)}$. Thus, taking into account the growth of h (1.14), we see that there exists $C_2 > 0$ such that, for all $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$,

$$\begin{aligned} f_n(2k)^2 h(2k) &\leq C_2 \left(\frac{(n+2)^2}{(k+1)[(n+2)^2 - (k+1)^2]} \right)^{1-2\alpha} = \\ &= C_2 \left[\frac{1}{k+1} + \frac{k+1}{(n+2)^2 - (k+1)^2} \right]^{1-2\alpha} \leq \\ &\leq C_2 \left[\frac{1}{k+1} + \frac{1}{n+1-k} \right]^{1-2\alpha} \leq \\ &\leq 2C_2 \left[\frac{1}{(k+1)^{1-2\alpha}} + \frac{1}{(n+1-k)^{1-2\alpha}} \right], \end{aligned}$$

where the last estimation uses the power mean inequality. Hence, we can conclude that

$$\int_{\mathbb{R}} p_n^4(x)\,\mathrm{d}\mu(x) = \|f_n\|_2^2 \leq 2C_2 \sum_{k=0}^{n} \left[\frac{1}{(k+1)^{1-2\alpha}} + \frac{1}{(n+1-k)^{1-2\alpha}} \right] < 4C_2 \zeta(1-2\alpha)$$

for every $n \in \mathbb{N}_0$, and obtain from our sufficient criterion Theorem 2.1 that $\ell^1(h)$ is not weakly amenable.

Since $h(n) \to \infty$ $(n \to \infty)$, $\ell^1(h)$ is not amenable due to Theorem 1.4 (i). For arbitrary $\alpha > -\frac{1}{2}$, $\ell^1(h)$ is ${}_x\alpha$-amenable if and only if $x = \pm 1$ [Las09c, p. 792]. However, recall that $\ell^1(h)$ is point amenable if and only if $\alpha < \frac{1}{2}$.

Remark 2.1. Of course, (2.24) and (2.25) hold true for any $\alpha > -\frac{1}{2}$, and $(f_n)_{n \in \mathbb{N}_0}$ converges pointwise to $F \in \ell^\infty$ given by $F|_{\mathbb{N}_0 \setminus \{0,2,\dots,2n\}} = 0$ and

$$F(2k) = \frac{\Gamma(2\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)^2} (2k+2\alpha+1)_{-2\alpha} (k+1)^2_{\alpha-\frac{1}{2}} \ (k \in \{0, \dots, n\}).$$

There is some $C_\alpha > 0$ such that, for each $N \in \mathbb{N}$,

$$\begin{aligned} &\sum_{k=1}^{N} F(2k)^2 h(2k) = \\ &= \frac{\Gamma(2\alpha+1)^2}{\Gamma\left(\alpha+\frac{1}{2}\right)^4 \Gamma(2\alpha+2)} \sum_{k=1}^{N} (2k+2\alpha+1)^2_{-2\alpha} (k+1)^4_{\alpha-\frac{1}{2}} (4k+2\alpha+1)(2k+1)_{2\alpha} \geq \\ &\geq C_\alpha \sum_{k=1}^{N} k^{2\alpha-1}. \end{aligned}$$

Therefore, if $\alpha \geq 0$, then $\|F\|_2^2 = \infty$, which yields $\sup_{n \in \mathbb{N}_0} \|f_n\|_2 = \infty$ (due to the Eberlein–Smulian theorem, or due to the sequential version of Alaoglu's theorem) or, in other words, that condition (iv) of Theorem 2.1 is violated. Obviously, also condition (iii) is violated if $\alpha \geq 0$; however neither condition (ii) (obvious) nor condition (i) is violated: concerning the latter, one can transfer the proof for $\alpha < 0$ given in this subsection.

Remark 2.2. As should be expected, the motivating example studied in Subsection 2.1 (Chebyshev polynomials of the first kind) can easily be reobtained from our sufficient criterion Theorem 2.1: w.r.t. condition (i) we refer to the short calculation given in Subsection 2.1 (Step 2); condition (ii) and condition (iii) are obviously satisfied from (2.4) and (2.7). That condition (iv) is satisfied, too, follows trivially from (2.8).

3. Complete characterizations of point and weak amenability for specific classes

Parts of Section 3 are very similar to our publication [Kah15].

In Section 2.5, we solved the problem of weak amenability for the class of ultraspherical polynomials. In the present section, we extend this result to important one-parameter generalizations of the ultraspherical polynomials, namely to the classes of Jacobi, symmetric Pollaczek and associated ultraspherical polynomials. We give complete characterizations of both point and weak amenability of the ℓ^1-algebras by specifying the corresponding parameter regions. In the following, we recall some basics and refer to [Las83, Sections 3 (a), (b)] and [Las94, Sections 3, 4].

- Given $\alpha,\beta>-1$, the sequence $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ of Jacobi polynomials which corresponds to α and β is determined by its orthogonalization measure

$$\mathrm{d}\mu(x)=\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}\chi_{(-1,1)}(x)\,\mathrm{d}x$$

and the normalization $R_n^{(\alpha,\beta)}(1)=1$ $(n\in\mathbb{N}_0)$, or, equivalently, via

$$\begin{aligned}a_0=\frac{2\alpha+2}{\alpha+\beta+2},\ a_n&=\frac{(\alpha+\beta+2)(n+\alpha+1)(n+\alpha+\beta+1)}{(\alpha+1)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\ (n\in\mathbb{N}),\\ c_n&=\frac{(\alpha+\beta+2)n(n+\beta)}{(\alpha+1)(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\ (n\in\mathbb{N}),\\ b_n&\equiv 1-a_n-c_n.\end{aligned}\tag{3.1}$$

The corresponding parameter region $D_P\subseteq(-1,\infty)^2$ such that property (P) holds if and only if $(\alpha,\beta)\in D_P$ is

$$D_P=\left\{(\alpha,\beta)\in\left[-\frac{1}{2},\infty\right)\times(-1,\infty):\alpha\geq\beta,a(a+5)(a+3)^2\geq(a^2-7a-24)b^2\right\},$$

where $a=\alpha+\beta+1$ and $b=\alpha-\beta$ [Gas70, Theorem 1]; one has $h(0)=1$ and

$$h(n)=\frac{(2n+\alpha+\beta+1)(\alpha+\beta+2)_{n-1}(\alpha+1)_n}{n!(\beta+1)_n}\ (n\in\mathbb{N}).\tag{3.2}$$

The hypergeometric representation of the Jacobi polynomials reads

$$R_n^{(\alpha,\beta)}(x)={}_2F_1\left(\left.\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix}\right|\frac{1-x}{2}\right)\ (n\in\mathbb{N}_0)$$

[KLS10, (9.8.1)].

- Given $\alpha>-\frac{1}{2}$, $\lambda\geq 0$, the sequence $(Q_n^{(\alpha,\lambda)}(x))_{n\in\mathbb{N}_0}$ of symmetric Pollaczek polynomials which corresponds to α and λ is determined by its orthogonalization measure

$$\mathrm{d}\mu(x)=C_{\alpha,\lambda}(1-x^2)^{\alpha}e^{\frac{\lambda x(2\arccos x-\pi)}{\sqrt{1-x^2}}}\left|\Gamma\left(\alpha+\frac{1}{2}+\frac{i\lambda x}{\sqrt{1-x^2}}\right)\right|^2\chi_{(-1,1)}(x)\,\mathrm{d}x,\tag{3.3}$$

where $C_{\alpha,\lambda}>0$ is a constant such that μ has total mass 1, and the normalization $Q_n^{(\alpha,\lambda)}(1)=1$ $(n\in\mathbb{N}_0)$. Equivalently, $(Q_n^{(\alpha,\lambda)}(x))_{n\in\mathbb{N}_0}$ is given via

$$b_n\equiv 0,\ a_n\equiv 1-b_n-c_n,\ c_n:=\frac{n}{2n+2\alpha+2\lambda+1}\frac{\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(2\lambda)^k}{(2\alpha+1)_k}}{\sum_{k=0}^{n}\binom{n}{k}\frac{(2\lambda)^k}{(2\alpha+1)_k}}\ (n\in\mathbb{N}).$$

The parameter region

$$D_Q := \left\{(\alpha,\lambda) \in \left(-\frac{1}{2},\infty\right) \times [0,\infty) : \text{(P) holds for } (Q_n^{(\alpha,\lambda)}(x))_{n\in\mathbb{N}_0}\right\}$$

satisfies $\left\{(\alpha,\lambda) \in \left(-\frac{1}{2},\infty\right) \times [0,\infty) : \alpha \geq 0 \text{ or } \lambda < \alpha + \frac{1}{2}\right\} \subseteq D_Q$, and one has

$$h(n) = \frac{(2n+2\alpha+2\lambda+1)(2\alpha+1)_n}{(2\alpha+2\lambda+1)n!}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{(2\lambda)^k}{(2\alpha+1)_k}\right]^2 \quad (n \in \mathbb{N}_0) \tag{3.4}$$

and

$$\alpha_n = \sqrt{\frac{n(n+2\alpha)}{(2n+2\alpha+2\lambda+1)(2n+2\alpha+2\lambda-1)}}\ (n \in \mathbb{N}_0),\ \beta_n \equiv 0. \tag{3.5}$$

The hypergeometric representation reads

$$\begin{aligned} Q_n^{(\alpha,\lambda)}(x) =& \frac{(x+i\sqrt{1-x^2})^n}{\sum_{k=0}^{n}\binom{n}{k}\frac{(2\lambda)^k}{(2\alpha+1)_k}} \\ &\times\ {}_2F_1\left(\begin{matrix}-n, \alpha+\frac{1}{2}+\frac{i\lambda x}{\sqrt{1-x^2}} \\ 2\alpha+1\end{matrix}\middle| 2-2x^2+2ix\sqrt{1-x^2}\right)\ (n \in \mathbb{N}_0, x \in (-1,1)) \end{aligned} \tag{3.6}$$

[Chi78, Chapter VI §5].

- Given $\alpha > -\frac{1}{2}$, $\nu \geq 0$, the sequence $(A_n^{(\alpha,\nu)}(x))_{n\in\mathbb{N}_0}$ of associated ultraspherical polynomials which corresponds to α and ν is determined by its orthogonalization measure

$$\begin{aligned} \mathrm{d}\mu(x) &= C_{\alpha,\nu}\frac{(1-x^2)^\alpha}{\left|{}_2F_1\left(\begin{matrix}\frac{1}{2}-\alpha, \nu \\ \alpha+\nu+\frac{1}{2}\end{matrix}\middle| 2x^2-1+2ix\sqrt{1-x^2}\right)\right|^2}\chi_{(-1,1)}(x)\,\mathrm{d}x = \\ &= C_{\alpha,\nu}\frac{(1-x^2)^\alpha}{\left|{}_2F_1\left(\begin{matrix}\frac{1}{2}-\alpha, \nu \\ \alpha+\nu+\frac{1}{2}\end{matrix}\middle| e^{2i\arccos x}\right)\right|^2}\chi_{(-1,1)}(x)\,\mathrm{d}x, \end{aligned} \tag{3.7}$$

where $C_{\alpha,\nu} > 0$ is a constant such that μ has total mass 1, and the normalization $A_n^{(\alpha,\nu)}(1) = 1$ $(n \in \mathbb{N}_0)$. Equivalently, $(A_n^{(\alpha,\nu)}(x))_{n\in\mathbb{N}_0}$ is determined via

$$\begin{aligned} &b_n \equiv 0,\ a_n \equiv 1-b_n-c_n, \\ &\forall n \in \mathbb{N} : c_n := \begin{cases} \frac{(n+\nu)(2\alpha+\nu)_{n+1}-(n+2\alpha+\nu)(\nu)_{n+1}}{(2n+2\alpha+2\nu+1)[(2\alpha+\nu)_{n+1}-(\nu)_{n+1}]}, & \alpha \neq 0, \\ \frac{n+\nu}{2n+2\nu+1}\frac{\sum_{k=0}^{n-1}\frac{1}{k+\nu}}{\sum_{k=0}^{n}\frac{1}{k+\nu}}, & \alpha = 0, \nu > 0, \\ \frac{n}{2n+1}, & \alpha = \nu = 0. \end{cases} \end{aligned} \tag{3.8}$$

Property (P) is satisfied without any restrictions on the parameter region, and one has

$$h(n) = \begin{cases} \frac{2n+2\alpha+2\nu+1}{4\alpha^2(2\alpha+2\nu+1)(\nu+1)_n(2\alpha+\nu+1)_n}[(2\alpha+\nu)_{n+1}-(\nu)_{n+1}]^2, & \alpha \neq 0, \\ \frac{2n+2\nu+1}{2\nu+1}\left(\nu\sum_{k=0}^{n}\frac{1}{k+\nu}\right)^2, & \alpha = 0, \nu > 0, \\ 2n+1, & \alpha = \nu = 0 \end{cases} \tag{3.9}$$

for each $n \in \mathbb{N}_0$. Moreover,

$$\alpha_n = \sqrt{\frac{(n+\nu)(n+2\alpha+\nu)}{(2n+2\alpha+2\nu+1)(2n+2\alpha+2\nu-1)}}\ (n \in \mathbb{N}_0),\ \beta_n \equiv 0. \tag{3.10}$$

Each of these classes satisfies $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) = \mathsf{supp}\,\mu = [-1,1]$. Hence, Reiter's condition P_2 is satisfied. Obviously, $(R_n^{(\alpha,\alpha)}(x))_{n\in\mathbb{N}_0} = (Q_n^{(\alpha,0)}(x))_{n\in\mathbb{N}_0} = (A_n^{(\alpha,0)}(x))_{n\in\mathbb{N}_0}$ for every $\alpha > -\frac{1}{2}$.

Another important generalization of the ultraspherical polynomials which leads to polynomial hypergroups is (a suitable subclass of) the class of continuous q-ultraspherical (or Rogers) polynomials [Las83, Section 3 (c)], cf. also Section 4 (up to normalization). It contains the ultraspherical polynomials as limiting cases but does not yield examples such that $\ell^1(h)$ is at least point amenable—in fact, $D_x = 0$ for $x = \pm 1$ and $D_x \neq 0$ exists for each $x \in (-1,1) = \widehat{\mathbb{N}_0}\backslash\{-1,1\}$ (cf. [Las09b, Section 4] or, more precisely, Theorem 1.6, [Las09b, Theorem 2] and [Vog87, Section 4.5]).

In [FLS04, Example 4.6], it is shown that if $P_n(x) = R_n^{(\alpha,\beta)}(x)$ $(n \in \mathbb{N}_0)$ with $(\alpha,\beta) \in D_P$ and $\alpha > -\frac{1}{2}$, then $\ell^1(h)$ is not ${}_x\alpha$-amenable for every $x \in (-1,1)$—and $\ell^1(h)$ is ${}_{-1}\alpha$-amenable if and only if $\alpha = \beta$ (ultraspherical polynomials, cf. Section 2.5); in particular, $\ell^1(h)$ is not right character amenable, and consequently not amenable. The latter is also obvious from Theorem 1.4 (i) and (3.2). The case $\alpha = \beta = -\frac{1}{2}$ corresponds to the Chebyshev polynomials of the first kind again, for which $\ell^1(h)$ is amenable.

If $P_n(x) = Q_n^{(\alpha,\lambda)}(x)$ $(n \in \mathbb{N}_0)$ with $(\alpha,\lambda) \in D_Q$, or if $P_n(x) = A_n^{(\alpha,\nu)}(x)$ $(n \in \mathbb{N}_0)$ with $\alpha > -\frac{1}{2}$, $\nu \geq 0$, then Theorem 1.4 (i), (3.4) and (3.9) show that $\ell^1(h)$ fails to be amenable. We improve this result and clarify the situation concerning α-amenability. Due to (3.5) and (3.10), in both cases $(P_n(x))_{n\in\mathbb{N}_0}$ is of Nevai class $M(0,1)$ and of bounded variation type (concerning the associated ultraspherical polynomials see also [Las09b, Example 4] for this). Moreover, $b_n \equiv 0$ and $\frac{h(n)}{\sum_{k=0}^n h(k)} \to 0$ $(n \to \infty)$ (cf. [Las94, Theorem 4.1] and [Las94, Theorem 3.1]). Hence, due to Proposition 1.3 and Proposition 1.4, $\ell^1(h)$ is ${}_x\alpha$-amenable if and only if $x = \pm 1$. Of course, this immediately yields analogous results concerning φ-amenability. In particular, $\ell^1(h)$ also fails to be right character amenable.[29]

3.1. A first application of the necessary criterion: weak amenability and smoothness for symmetric Pollaczek polynomials

As another kind of motivating example, we give a precise description of the parameter region $\subseteq D_Q$ for which $(Q_n^{(\alpha,\lambda)}(x))_{n\in\mathbb{N}_0}$ bears a weakly amenable ℓ^1-algebra; since the special case of purely ultraspherical polynomials has been clarified in Section 2.5, we may assume that $(\alpha,\lambda) \in D_Q$ with $\lambda > 0$ from now on. In the following, we show that $\mu' \in C^1(\mathbb{R})$ then (which can be regarded as an interesting elementary calculus exercise). Hence, $\ell^1(h)$ fails to be weakly amenable as a consequence of our necessary criterion Theorem 2.2. We decompose $\mu'|_{(-1,1)} = C_{\alpha,\lambda} v f \circ g$ with $v : (-1,1) \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : (-1,1) \to \mathbb{R}$,

$$v(x) := (1-x^2)^\alpha e^{\frac{\lambda x(2\arccos x - \pi)}{\sqrt{1-x^2}}},\ f(x) := \left|\Gamma\left(\alpha + \frac{1}{2} + i\lambda x\right)\right|^2,\ g(x) := \frac{x}{\sqrt{1-x^2}}$$

(3.3). Since

$$v'(x) = \frac{e^{\frac{\lambda x(2\arccos x - \pi)}{\sqrt{1-x^2}}}}{(1-x^2)^2}\left[(-2\alpha x\sqrt{1-x^2} - 2\lambda x\sqrt{1-x^2} - \lambda\pi + 2\lambda\arccos x)(1-x^2)^{\alpha+\frac{1}{2}}\right]$$

[29] These results concerning α-amenability for the Pollaczek and associated ultraspherical polynomials correct and extend earlier results of Azimifard on the Pollaczek and associated Legendre polynomials [Azi10, (III); (IV)]. In [Azi10, (III); (IV)], the special case $x = -1$ is overlooked; moreover, [Azi10, (III)] cites a wrong formula for the Haar weights of the associated Legendre polynomials.

for $x \in (-1,1)$, we see that

$$|v'(x)| \le \frac{e^{\frac{\lambda x(2\arccos x - \pi)}{\sqrt{1-x^2}}}}{(1-x^2)^2} \le \frac{e^{-\frac{\lambda}{\sqrt{1-x^2}}}}{(1-x^2)^2}$$

if $1-|x|$ is sufficiently small. Consequently, $v \in C^1(-1,1)$ and $\lim_{x\to\pm 1} v'(x) = 0 = \lim_{x\to\pm 1} v(x)$.
We now study the function f. Euler's infinite product formula for the complex gamma function states that

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}} \ (z \in \mathbb{C}, -z \notin \mathbb{N}_0)$$

[GK97, Proposition 15.1.6], and it is a consequence of this that

$$|\Gamma(x+iy)|^2 = \Gamma(x)^2 \prod_{n=0}^{\infty} \frac{(x+n)^2}{(x+n)^2+y^2} \ (x,y \in \mathbb{R}, -(x+iy) \notin \mathbb{N}_0) \tag{3.11}$$

[AS64, 6.1.25]. From (3.11) we obtain that

$$\log f(x) = 2\log\Gamma\left(\alpha+\frac{1}{2}\right) + \sum_{n=0}^{\infty} \underbrace{\log \frac{\left(\alpha+\frac{1}{2}+n\right)^2}{\left(\alpha+\frac{1}{2}+n\right)^2+\lambda^2 x^2}}_{=:\gamma_n(x)} \ (x \in \mathbb{R}).$$

Since $\gamma_n : \mathbb{R} \to \mathbb{R}$, $x \mapsto \gamma_n(x)$, is continuously differentiable for each $n \in \mathbb{N}_0$ with

$$\gamma_n'(x) = \frac{-2\lambda^2 x}{\left(\alpha+\frac{1}{2}+n\right)^2+\lambda^2 x^2} \ (x \in \mathbb{R}),$$

and since the series $\sum_{n=0}^{\infty} \gamma_n'(x)$ converges uniformly on compact intervals, we see that $\log f \in C^1(\mathbb{R})$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\log f(x) = -2\lambda^2 x \sum_{n=0}^{\infty} \frac{1}{\left(\alpha+\frac{1}{2}+n\right)^2+\lambda^2 x^2} \ (x \in \mathbb{R}).$$

Consequently, $f \in C^1(\mathbb{R})$ with

$$f'(x) = -2\lambda^2 x f(x) \sum_{n=0}^{\infty} \frac{1}{\left(\alpha+\frac{1}{2}+n\right)^2+\lambda^2 x^2} \ (x \in \mathbb{R}).$$

In particular, we have established the existence of a constant $C_1 > 0$ such that $|f'(x)| \le C_1|x|f(x)$ $(x \in \mathbb{R})$ and hence obtain

$$|(f\circ g)'(x)| \le C_1|g(x)|f\circ g(x)|g'(x)| = C_1\frac{|x|}{(1-x^2)^2} f\circ g(x) \ (x \in (-1,1)). \tag{3.12}$$

As a consequence of (a well-known variant of) Stirling's formula [AAR99, Corollary 1.4.4], there is a constant $C_2 > 0$ such that

$$|f\circ g(x)| \le C_2|g(x)|^{2\alpha}e^{-\lambda\pi|g(x)|} \ (x \in (-1,1));$$

in view of (3.12), this yields a constant $C_3 > 0$ such that

$$|(f\circ g)'(x)| \le C_3|g(x)|^{2\alpha+4}e^{-\lambda\pi|g(x)|}$$

if $x \in (-1,1)$ is such that $1-|x|$ is sufficiently small. So we can conclude that $f\circ g \in C^1(-1,1)$ and $\lim_{x\to\pm 1}(f\circ g)'(x) = 0 = \lim_{x\to\pm 1} f\circ g(x)$; putting all together, we see that indeed $\mu' \in C^1(\mathbb{R})$, and that, due to Theorem 2.2, $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ is bounded and $\ell^1(h)$ fails to be weakly amenable.

In Subsection 3.3, we shall see that $\ell^1(h)$ even fails to be point amenable whenever $\lambda > 0$—with a considerably more involved argument, however.

3.2. Jacobi polynomials: transition from the purely ultraspherical case, and how to avoid dealing with a ${}_9F_8$

Our main result on Jacobi polynomials is the following theorem, which corresponds to [Kah15, Theorem 3.1] and gives complete characterizations concerning point and weak amenability.

Theorem 3.1. *Let $(\alpha,\beta)\in D_P$, and let $P_n(x)=R_n^{(\alpha,\beta)}(x)$ $(n\in\mathbb{N}_0)$. Then $\ell^1(h)$ is*

(i) *point amenable if and only if $\alpha<\frac{1}{2}$ (where $D_x\neq 0$ exists for each $x\in(-1,1)$ if $\alpha\geq\frac{1}{2}$ and $D_{-1}\neq 0$ exists if and only if $\alpha\geq\beta+2$),*

(ii) *weakly amenable if and only if $\alpha<0$.*

Our proof of Theorem 3.1 relies on four lemmas, and we use the following notation:

$$\Delta:=\{(\alpha,\beta)\in D_P:\alpha+\beta+1\geq 0\}$$

and

$$\phi_n:=R_n^{(\alpha,\beta)}(-1)=(-1)^n\frac{(\beta+1)_n}{(\alpha+1)_n}\ (n\in\mathbb{N}_0).$$

Of course, ϕ_n depends on $\alpha,\beta>-1$. The explicit formula for ϕ_n follows from the second equation in [Ism09, (4.1.6)] (note that our normalization does not coincide with that used in [Ism09], cf. the first equation in [Ism09, (4.1.6)]). The asymptotics of the gamma function yield

$$\phi_n=\Theta(n^{\beta-\alpha})\ (n\to\infty). \tag{3.13}$$

The first lemma [Kah15, Lemma 3.1] gives an explicit formula for the sequence $(\kappa_n)_{n\in\mathbb{N}}$ and generalizes (2.9) and (2.26).

Lemma 3.1. *If $(\alpha,\beta)\in D_P$ and $P_n(x)=R_n^{(\alpha,\beta)}(x)$ $(n\in\mathbb{N}_0)$, then*

$$\kappa_n(k)=\sigma(n)\frac{\phi_n-\phi_k}{\phi_n-\phi_{n-1}}\ (n\in\mathbb{N},k\in\{0,\ldots,n\}). \tag{3.14}$$

Concerning asymptotic behavior, we obtain from (3.2), Lemma 2.4 (i), Lemma 3.1 and (3.13):

$$h(n)=\Theta(n^{2\alpha+1}),\ \sigma(n)=\Theta(n^{-2\alpha}),\ \tau(n)=\begin{cases}\Theta(n^{-\alpha-\beta}), & \alpha\neq\beta,\\ \mathcal{O}(n^{-2\alpha}), & \alpha=\beta\end{cases} \tag{3.15}$$

as $n\to\infty$.

In Section 2.5, we saw that Step 2 of Section 2.1 easily transfers from the Chebyshev polynomials of the first kind to arbitrary ultraspherical polynomials. It is considerably more involved (and the most technical part of Theorem 3.1) to obtain an analogue for non-symmetric Jacobi polynomials—i.e., to verify condition (i) of Theorem 2.1 (as far as a verification is necessary for our proof of Theorem 3.1; in fact, we shall see that it is not necessary to deal with the whole set $\{(\alpha,\beta)\in D_P:\alpha<0\}$ but it suffices to consider a proper subset and, after a successful application of Theorem 2.1 to this subset, to use inheritance via homomorphisms). Our second lemma [Kah15, Lemma 3.2] provides an auxiliary function which will become a crucial tool concerning the just outlined strategy.

Lemma 3.2. *If $(\alpha,\beta)\in D_P$, $\alpha\neq\beta$, and $P_n(x)=R_n^{(\alpha,\beta)}(x)$ $(n\in\mathbb{N}_0)$, then the function $\eta:\mathbb{N}\to[0,\infty)$,*

$$\eta(n):=\frac{n+2}{h(n+1)}\sum_{k=0}^{n}\frac{h(k)}{k+1}\left|\frac{\kappa_{n+2}(k)}{\sigma(n+2)}-\frac{\kappa_n(k)\tau(n+2)}{\tau(n)\sigma(n+2)}\right|,$$

is well-defined and satisfies $\lim_{n\to\infty}\eta(n)=\frac{\alpha-\beta}{2\alpha+1}$.

Concerning condition (iv) of Theorem 2.1, we have the following [Kah15, Lemma 3.3]:

Lemma 3.3. *If* $(\alpha,\beta) \in D_P$*,* $\alpha < 0$*, and* $P_n(x) = R_n^{(\alpha,\beta)}(x)$ $(n \in \mathbb{N}_0)$*, then* $\sup_{n\in\mathbb{N}_0} \int_{\mathbb{R}} p_n^4(x)\,\mathrm{d}\mu(x) < \infty$.

Finally, the fourth lemma [Kah15, Lemma 3.4] is helpful w.r.t. inheritance via homomorphisms—more precisely, w.r.t. applications of Theorem 2.3 and Proposition 2.2.

Lemma 3.4. *Let* $(\alpha,\beta),(\gamma,\delta) \in D_P$*, and let for each* $n \in \mathbb{N}_0$ *the connection coefficients* $C_n^P(\alpha,\beta;\gamma,\delta)(0),\ldots,C_n^P(\alpha,\beta;\gamma,\delta)(n)$ *be defined by* $R_n^{(\alpha,\beta)}(x) = \sum_{k=0}^n C_n^P(\alpha,\beta;\gamma,\delta)(k) R_k^{(\gamma,\delta)}(x)$*. Then each of the following conditions is sufficient for the existence of some* $C > 0$ *such that* $\sum_{k=0}^n |C_n^P(\alpha,\beta;\gamma,\delta)(k)| \leq C$ $(n \in \mathbb{N}_0)$*:* (A) $\gamma \leq \alpha$ and $\delta = \beta$, (B) $\gamma = \alpha$ and $\delta \geq \beta$, (C) $\gamma = \alpha - \xi$ and $\delta = \beta - \xi$ for some $\xi \geq 0$.

Lemma 3.1 can be established very quickly via the Jacobi forward shift

$$\frac{\mathrm{d}}{\mathrm{d}x} R_n^{(\alpha,\beta)}(x) = \frac{n(n+\alpha+\beta+1)}{2\alpha+2} R_{n-1}^{(\alpha+1,\beta+1)}(x)\ (n \in \mathbb{N}) \tag{3.16}$$

[Ism09, (4.1.6); (4.2.2)],[30] the connection coefficients w.r.t. the original sequence [Ism09, Section 9.1] (cf. also [Doh02]) and several ingredients from the theory of hypergeometric series (certain contiguous relations, Chu–Vandermonde identity [AAR99, Corollary 2.2.3], Pfaff–Saalschütz identity [AAR99, Theorem 2.2.6]). This way is presented in our paper [Kah15]. A very similar argument is used in [IS12]. Another direct proof is contained in the author's Master's thesis and based on kernel polynomials [Kah12, Section 3.4.2]. Alternatively, one could also use induction and a general recurrence relation for the sequence $(\kappa_n)_{n\in\mathbb{N}_0}$, cf. [Las07, (5)] and Remark 1.1, and cf. also (4.12) and (4.21) in Section 4 (which studies a different setting, however). This was done by E. Perreiter (independently of the author). Lemma 3.2 is a consequence of Lemma 3.1 and the Stolz–Cesàro theorem. Lemma 3.3 relies on ideas from [Sze75, Chapter 7.34] and suitable asymptotics of the Jacobi polynomials (which follow from [Sze75, Chapter 7.32] and are consequences of a formula of Mehler–Heine type and a formula of Darboux, or also of a formula of Hilb's type). If $\beta \geq -\frac{1}{2}$, the polynomials $p_n^4(x)$ $(n \in \mathbb{N}_0)$ even have a μ-integrable majorant on $(-1,1)$—namely the function $\mathbb{R} \to [0,\infty)$, $x \mapsto \frac{C}{(1-x)^{2\alpha+1}(1+x)^{2\beta+1}}\chi_{(-1,1)}(x)$, with $C \in \mathbb{R}$ independent of x. This is a consequence of [NEM94, Theorem 1] (which, however, is for non-symmetric Jacobi polynomials a rather deep result that has led to the Erdélyi–Magnus–Nevai conjecture which deals with the occurring constants C). Lemma 3.4 follows from results of [Bav71]. (B) and its proof via [Bav71] are already contained in E. Perreiter's dissertation [Per11a, Corollary 3.12].

Trying to prove Lemma 3.3 in a way which directly imitates the strategy presented in Section 2.5 for the purely ultraspherical case, i.e., transforming

$$\int_{\mathbb{R}} p_n^4(x)\,\mathrm{d}\mu(x) = \sum_{k=0}^{2n} [g(n,k;n)]^2 h(k)\ (n \in \mathbb{N}_0)$$

via (2.3) and estimating the right hand sides from above by using explicit formulas for the linearization coefficients $g(m,n;k)$, would be a "rather bad idea": for non-symmetric (and "non-special") Jacobi polynomials, these formulas are much more complicated than Dougall's formula (2.22)—Rahman's formula, for instance, which works (and explicitly shows property

[30] In contrast to (3.16), the Jacobi *backward* shift reads

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} R_n^{(\alpha+1,\beta+1)}(x) = (2\alpha+2)\left[R_1^{(\alpha,\beta)}(x) R_n^{(\alpha+1,\beta+1)}(x) - R_{n+1}^{(\alpha,\beta)}(x)\right]\ (n \in \mathbb{N}_0), \tag{3.17}$$

cf. [Ism09, (4.1.6); (4.2.5)] and (3.1). We shall refer to this later.

(P)) for the pairs $(\alpha, \beta) \in \Delta$, involves a ${}_9F_8$ hypergeometric series [Rah81].

In view of earlier results (cf. [Las09b, Section 4]), concerning Theorem 3.1 (i) it is just left to us to establish point amenability for $(\alpha, \beta) \in D_P \backslash \Delta$. Of course, this will follow from Theorem 3.1 (ii). More directly, it can be obtained from Proposition 2.2 and Lemma 3.4: if $(\alpha, \beta) \in D_P \backslash \Delta$, then $(\alpha + \xi, \beta) \in \Delta$ for some $\xi \in \left(0, -\alpha + \frac{1}{2}\right)$.

The failure of weak amenability for $\alpha \geq 0$ was shown in E. Perreiter's dissertation [Per11a, Corollary 3.12]; the proof corresponds to an application of Theorem 2.3 (inheritance via homomorphisms), Lemma 3.4 (B) and the failure of weak amenability for the ultraspherical polynomials corresponding to α.

The difficult part concerning the proof of Theorem 3.1 (ii) is to establish weak amenability for $\alpha < 0$. In view of Theorem 2.3 and Lemma 3.4, we may additionally assume that (α, β) is located in the interior of Δ. In this situation, the conditions of our sufficiency criterion Theorem 2.1 can be seen to be satisfied: (ii) and (iii) are clear, (iv) is a consequence of Lemma 3.3, and our proof of the remaining condition (i) can be outlined as follows:

let $\varphi \in \ell^\infty$ such that $\{\|\kappa_n * \varphi\|_\infty : n \in \mathbb{N}_0\}$ is bounded, let $C := \sup_{n \in \mathbb{N}_0} \|\kappa_n * \varphi\|_\infty$, let η be as in Lemma 3.2, let $S := \lim_{n \to \infty} \eta(n) < 1$ (the latter is a consequence of Lemma 3.2), let $T := \sup_{n \in \mathbb{N}} \left| \frac{\tau(n+2)}{\tau(n)} \right| < \infty$ (this follows from (3.15); with regard to well-definedness of quotients containing $\tau(n)$, note that we have $\tau(n) \neq 0$ $(n \in \mathbb{N})$ because $\alpha \neq \beta$), let $N \in \mathbb{N}$ such that $\eta(n) \leq \frac{S+1}{2}$ for all $n \in \mathbb{N}$ with $n \geq N$, and, finally, let $M := \max\left((N+1)\|\varphi\|_\infty, \frac{4+4T}{1-S} C\right)$. Then use induction on n to show that

$$|\varphi(n)| \leq \frac{M}{n+1}$$

for each $n \in \mathbb{N}_0$. This is the tedious non-symmetric analogue to Step 2 in Section 2.1 (and to the easier—still symmetric—generalization contained in Section 2.5). As in Section 2.1 and in Section 2.5, the details rely on a consideration of $(\kappa_n * \varphi(0))_{n \in \mathbb{N}_0}$.

The detailed proofs of Theorem 3.1 and the four lemmas can be found in our paper [Kah15].

In contrast to Abelian locally compact groups, for arbitrary polynomial hypergroups there is no Pontryagin duality (despite their commutativity): in fact, the only polynomial hypergroups for which $\widehat{\mathbb{N}_0}$ is a hypergroup w.r.t. pointwise multiplication and complex conjugation (and which are also 'Pontryagin' then [BH95, Proposition 2.4.18])—cf. [BH95, Chapter 2.4] for the precise definitions and some properties—correspond to Jacobi polynomials $(R_n^{(\alpha,\beta)}(x))_{n \in \mathbb{N}_0}$ with $(\alpha, \beta) \in D_P$ such that either $\beta \geq -\frac{1}{2}$ or $\alpha + \beta \geq 0$ [CS90] [BH95, Section 3.6]. Combining this with Theorem 3.1, we see that each of the cases "$\ell^1(h)$ amenable", "$\ell^1(h)$ weakly amenable but not right character amenable", "$\ell^1(h)$ point amenable but neither right character amenable nor weakly amenable" and "$\ell^1(h)$ not point amenable" can occur even in the class of Pontryagin polynomial hypergroups.

Theorem 3.1 has also some interesting consequences w.r.t. our general results presented in Section 2:

- Theorem 2.2 tells that the boundedness of the set $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ is equivalent to the boundedness of the single function τ. However, as outlined in Section 2.3, this is in general not equivalent to the boundedness of σ (whose computation would be considerably easier, see Lemma 2.4 (i)): if, for example, $P_n(x) = R_n^{(\alpha,\beta)}(x)$ $(n \in \mathbb{N}_0)$ with $\alpha > 0$, $\beta > -1$ and $\alpha + \beta < 0$, then $\tau(n) \to \infty$ $(n \to \infty)$ but $\sigma(n) \to 0$ $(n \to \infty)$. This follows from (3.15).

- (i) of our necessity criterion Theorem 2.2, i.e., the unboundedness of τ, is not sufficient for weak amenability of $\ell^1(h)$. We can take the examples of the previous item to see this. In particular, (ii) of Theorem 2.2, i.e., that μ is not absolutely continuous (as a measure) or μ' is not absolutely continuous on the convex hull of $\mathsf{supp}\ \mu$ (as a function), is not sufficient for weak amenability. Recall that the conditions of our sufficiency criterion Theorem 2.1 can be regarded as a suitable sharpening of (ii) of Theorem 2.2 (and also of (i) of Theorem 2.2); cf. the note at the end of Section 2.3.

- If α and β are chosen from the smaller region $\alpha \geq \frac{1}{2}$, $\beta > -1$, $\alpha + \beta < 0$, and if still $P_n(x) = R_n^{(\alpha,\beta)}(x)$ $(n \in \mathbb{N}_0)$, then $\ell^1(h)$ is not even point amenable (in fact, there exist nonzero bounded point derivations at every $x \in (-1, 1)$). Thus condition (i) (and hence condition (ii)) of Theorem 2.2 is also not sufficient for point amenability of $\ell^1(h)$.

- Finally, (ii) (and therefore (i)) of Theorem 2.2 is not necessary for point amenability of $\ell^1(h)$: if $0 \leq \alpha < \frac{1}{2}$ and $P_n(x) = P_n^{(\alpha)}(x)$ $(n \in \mathbb{N}_0)$, then $\ell^1(h)$ is point amenable (already due to [Las09b, Corollary 1]), but (ii) of Theorem 2.2 is not satisfied. The same examples show that neither condition (iii) nor condition (iv) of Theorem 2.1 is necessary for point amenability of $\ell^1(h)$ (cf. the notes at the end of Section 2.5, in part. Remark 2.1). The little q-Legendre polynomials provide examples which show that condition (ii) of Theorem 2.1 is not necessary for point amenability of $\ell^1(h)$ (cf. the notes at the end of Section 2.3).

- With regard to the comment at the end of Section 2.3, we note that if (ii) of Theorem 2.2 is satisfied, then h can nevertheless be of superlinear growth (see the examples above).

3.3. Symmetric Pollaczek polynomials: transformations and estimations of the derivatives

In the motivating Subsection 3.1, we used Theorem 2.2 to rule out weak amenability for all Pollaczek polynomials $(Q_n^{(\alpha,\lambda)}(x))_{n\in\mathbb{N}_0}$ with $(\alpha, \lambda) \in D_Q$, $\lambda > 0$. As announced in Subsection 3.1, we now sharpen this result and show that even point amenability fails in this situation. The following is [Kah15, Theorem 4.1].

Theorem 3.2. *Let $(\alpha, \lambda) \in D_Q$ and $P_n(x) = Q_n^{(\alpha,\lambda)}(x)$ $(n \in \mathbb{N}_0)$. Then $\ell^1(h)$ is*

(i) *point amenable if and only if $\alpha < \frac{1}{2}$ and $\lambda = 0$,*

(ii) *weakly amenable if and only if $\alpha < 0$ and $\lambda = 0$.*

(ii) is completely contained in the earlier parts of this thesis. Concerning the proof of (i), we distinguish three cases: the case $\alpha < \frac{1}{2}$ and $\lambda = 0$ corresponds just to point amenability of ultraspherical polynomials [Las09b, Corollary 1]. The case $\alpha + \lambda \geq \frac{\sqrt{8\alpha+5}}{2} - 1$ can be solved by extending an argument which has been given in [Las09b, Example 7] for (the subcase) $\alpha + \lambda \geq 1$. This argument corresponds to our Proposition 2.1 (i).

The remaining case $0 < \lambda < -\alpha + \frac{\sqrt{8\alpha+5}}{2} - 1$ is much more interesting and the actual difficulty in the proof of Theorem 3.2 (i): neither (i) nor (ii) of Proposition 2.1 applies;[31] our strategy will be quite different and can be outlined as the following idea: first transform the Pollaczek polynomials into a system of orthogonal polynomials which is closely related yet more convenient (particularly with regard to asymptotic behavior)—then, based on this transformation, show that the derivatives of the Pollaczek polynomials are bounded at 0, which yields a nonzero bounded point derivation due to Theorem 1.6.

[31] A very fast yet not trivial way to see that $\lim_{n\to\infty} c_n = \frac{1}{2}$ (in the whole class of symmetric Pollaczek polynomials) would be to apply a relationship to the Laguerre polynomials [Las94, Section 4] and Perron's formula in the complex plane [Sze75, Theorem 8.22.3].

The more convenient class we have in mind is the class of random walk polynomials[32], cf. [Las83, Section 3 (d)] [Las94, Sections 6, 7] in the following: given $a \geq 1$ and $b \geq 0$, the sequence $(S_n^{(a,b)}(x))_{n\in\mathbb{N}_0}$ of random walk polynomials which corresponds to a and b is determined by its orthogonalization measure

$$\mathrm{d}\mu(x) = C_{a,b}[4a-(a+1)^2x^2]^{\frac{b}{2a}-\frac{1}{2}} \exp\left\{\frac{(a-1)bx\left[2\arccos\left(\frac{a+1}{2\sqrt{a}}x\right)-\pi\right]}{2a\sqrt{4a-(a+1)^2x^2}}\right\}$$
$$\times\left|\Gamma\left(\frac{b}{2a}+\frac{i(a-1)bx}{2a\sqrt{4a-(a+1)^2x^2}}\right)\right|^2\chi_{\left(-\frac{2\sqrt{a}}{a+1},\frac{2\sqrt{a}}{a+1}\right)}(x)\,\mathrm{d}x$$

if $b > 0$ and

$$\mathrm{d}\mu(x) = C_{a,0}\frac{\sqrt{4a-(a+1)^2x^2}}{1-x^2}\chi_{\left(-\frac{2\sqrt{a}}{a+1},\frac{2\sqrt{a}}{a+1}\right)}(x)\,\mathrm{d}x$$

if $b = 0$, with suitable $C_{a,b}, C_{a,0} > 0$,[33] and the normalization $S_n^{(a,b)}(1) = 1$ $(n \in \mathbb{N}_0)$—or, equivalently, but in a much simpler form, via

$$b_n \equiv 0,\ a_n \equiv 1-b_n-c_n,\ c_n := \frac{n}{(a+1)n+b}\ (n\in\mathbb{N}). \tag{3.18}$$

Obviously,

$$S_n^{(1,b)}(x) = P_n^{\left(\frac{b-1}{2}\right)}(x)\ (b\geq 0, n\in\mathbb{N}_0). \tag{3.19}$$

Property (P) holds for all $a \geq 1$ and $b \geq 0$, and one has $\widehat{\mathbb{N}_0} = [-1,1]$ and

$$\mathcal{X}^b(\mathbb{N}_0) = \left\{z\in\mathbb{C} : \left|z-\frac{2\sqrt{a}}{a+1}\right| + \left|z+\frac{2\sqrt{a}}{a+1}\right| \leq 2\right\}.$$

The Haar weights satisfy $h(0) = 1$ and

$$h(n) = \begin{cases}\frac{a^n\left(\frac{b}{a}\right)_n[(a+1)n+b]}{bn!}, & b>0,\\ (a+1)a^{n-1}, & b=0\end{cases} \tag{3.20}$$

for $n \in \mathbb{N}$. The relationship to the Pollaczek polynomials reads as follows:

$$S_n^{(a,b)}(x) = \frac{Q_n^{\left(\frac{b}{2a}-\frac{1}{2},b\left(\frac{1}{a+1}-\frac{1}{2a}\right)\right)}\left(\frac{a+1}{2\sqrt{a}}x\right)}{Q_n^{\left(\frac{b}{2a}-\frac{1}{2},b\left(\frac{1}{a+1}-\frac{1}{2a}\right)\right)}\left(\frac{a+1}{2\sqrt{a}}\right)}\ (n\in\mathbb{N}_0) \tag{3.21}$$

and

$$Q_n^{(\alpha,\lambda)}(x) = \frac{S_n^{\left(\frac{2\alpha+2\lambda+1}{2\alpha-2\lambda+1},(2\alpha+1)\frac{2\alpha+2\lambda+1}{2\alpha-2\lambda+1}\right)}\left(\sqrt{1-\left(\frac{2\lambda}{2\alpha+1}\right)^2}x\right)}{S_n^{\left(\frac{2\alpha+2\lambda+1}{2\alpha-2\lambda+1},(2\alpha+1)\frac{2\alpha+2\lambda+1}{2\alpha-2\lambda+1}\right)}\left(\sqrt{1-\left(\frac{2\lambda}{2\alpha+1}\right)^2}\right)}\ (n\in\mathbb{N}_0), \tag{3.22}$$

provided $a \geq 1$, $b > 0$ in (3.21), and $\alpha > -\frac{1}{2}$, $0 \leq \lambda < \alpha + \frac{1}{2}$ in (3.22) (the denominators are nonzero).

The following lemma [Kah15, Lemma 4.1] is our central auxiliary tool:

[32] Concerning the relation to random walks, we refer to [AI84, Section 6]. To avoid confusions, we note that in Section 4 we will use the expression "random walk polynomials" in a rather different, much more general manner.

[33] We note that the case $b > 0$ is stated with a typo in [Las94, Section 7]. That our formula is the correct one can be seen from the proof of [Las94, Theorem (7.1)]; cf. also [AI84, Section 6].

Lemma 3.5. *Let $\alpha > -\frac{1}{2}$, $0 \leq \lambda < \alpha + \frac{1}{2}$. Let*

$$\rho := \sqrt{1 - \left(\frac{2\lambda}{2\alpha+1}\right)^2},\ \gamma := \sqrt{\frac{2\alpha - 2\lambda + 1}{2\alpha + 2\lambda + 1}},$$

and let

$$s_n := \frac{(2\alpha+2\lambda+1)(n+2\alpha+1)}{(2\alpha+1)(2n+2\alpha+2\lambda+1)},\ t_n := 1 - s_n = \frac{(2\alpha-2\lambda+1)n}{(2\alpha+1)(2n+2\alpha+2\lambda+1)}\ (n \in \mathbb{N}).$$

Then the recurrence relation $\psi_1 := \rho$,

$$\psi_{n+1} := \frac{\rho\psi_n - t_n}{s_n\psi_n}\ (n \in \mathbb{N}),$$

defines a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq [\gamma, \infty)$ which satisfies

$$\psi_n \geq \frac{2\lambda n + 2\alpha + 1}{2\lambda n + 2\alpha - 2\lambda + 1}\gamma$$

for each $n \in \mathbb{N}$.

Its elementary proof can be found in our paper [Kah15]. Based on Lemma 3.5, the proof of the remaining case $0 < \lambda < -\alpha + \frac{\sqrt{8\alpha+5}}{2} - 1$ (Theorem 3.2 (i)) can be sketched as follows: since $\lambda < \alpha + \frac{1}{2}$, we can set $S_n(x) := S_n^{(a,b)}(x)$ with $a := \frac{2\alpha+2\lambda+1}{2\alpha-2\lambda+1} > 1$ and $b := (2\alpha+1)\frac{2\alpha+2\lambda+1}{2\alpha-2\lambda+1} > 0$. In the following, we use the notation of Lemma 3.5. (3.22) yields

$$P_n'(0) = \rho\frac{S_n'(0)}{S_n(\rho)}\ (n \in \mathbb{N}_0),$$

and one has $S_0(x) = 1$, $S_1(x) = x$,

$$xS_n(x) = s_nS_{n+1}(x) + t_nS_{n-1}(x)\ (n \in \mathbb{N}).$$

It can be seen from the latter recurrence relation that

$$S_n'(0) = \mathcal{O}(n\gamma^n)\ (n \to \infty). \tag{3.23}$$

Furthermore, this recurrence relation implies that $S_n(\rho) = \prod_{k=1}^n \psi_k$ $(n \in \mathbb{N})$—hence, as a consequence of Lemma 3.5, we obtain

$$\frac{1}{S_n(\rho)} = \mathcal{O}(n^{-1}\gamma^{-n})\ (n \to \infty). \tag{3.24}$$

Putting all together, we can conclude that $\{P_n'(0) : n \in \mathbb{N}_0\}$ is bounded and apply Theorem 1.6 as desired. The details are given in our paper [Kah15].

Remark 3.1. Concerning the case $0 < \lambda < -\alpha + \frac{\sqrt{8\alpha+5}}{2} - 1$ (Theorem 3.2 (i)), we observe:

(i) Comparing the random walk polynomials to the symmetric Pollaczek polynomials, we could benefit from the random walk polynomials in a twofold way. On the one hand, their recurrence coefficients are of considerably easier structure. On the other hand—and more important, while $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n$ for the whole class of symmetric Pollaczek polynomials, the two limits are obviously different for the non-ultraspherical random walk polynomials: it was the latter property which enabled us to estimate $(S_n'(0))_{n\in\mathbb{N}_0}$ (cf. (3.23)).

(ii) With regard to the proof of (3.24), there is a variant which does not require to use Lemma 3.5 (nevertheless, we still use the notation of the lemma): applying [AI84, Section 6, in part. (6.30)], we obtain

$$\frac{1}{S_n(\rho)} = \Theta(n^{\alpha+\frac{1}{4}}\gamma^{-n}e^{-\sqrt{8\lambda n}})\ (n \to \infty), \tag{3.25}$$

which obviously yields (3.24). However, the proof of (3.25) via the cited ingredient is much less elementary than the proof of our Lemma 3.5; [AI84, (6.30)] relies on a relationship to the Laguerre polynomials and Perron's formula in the complex plane [Sze75, Theorem 8.22.3].

The strategy presented in Subsection 3.1 to rule out weak amenability whenever $\lambda > 0$ and the strategy presented in the current subsection to even rule out point amenability for those λ show that $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$ and $\{P_n'(0) : n \in \mathbb{N}_0\}$ are bounded—without the need of explicit computations of κ_n ($n \in \mathbb{N}_0$), or $\|\kappa_n\|_\infty$ ($n \in \mathbb{N}_0$), and $P_n'(0)$ ($n \in \mathbb{N}_0$). Such explicit computations seem to be out of reach for this example class (which, concerning κ_n ($n \in \mathbb{N}_0$), contrasts with the Jacobi polynomials studied in Subsection 3.2): this shows the worth of the (general) necessity result Theorem 2.2 and the worth of the (concrete) transformation into the random walk polynomials combined with the asymptotic approach. In particular, we think that trying to estimate $P_n'(0)$ ($n \in \mathbb{N}_0$) via (3.6) (or via similar hypergeometric representations for the random walk polynomials, cf. [AI84, Section 6]) would not be a better idea.

As a consequence of Theorem 3.2, we see that even the combination of $b_n \equiv 0$, the existence of an $n \in \mathbb{N}$ with $c_n a_{n-1} > \frac{1}{4}$ and the condition $\limsup_{n\to\infty} c_n \geq \frac{1}{2}$ (cf. Proposition 2.1) is not sufficient for point amenability.

Point and weak amenability for the random walk polynomials themselves will be characterized in Subsection 3.5.

3.4. Associated ultraspherical polynomials: interplay between hypergeometric and absolutely continuous Fourier series

Concerning the class of associated ultraspherical polynomials, point and weak amenability can be characterized as follows [Kah15, Theorem 5.1]:

Theorem 3.3. *Let $\alpha > -\frac{1}{2}$, $\nu \geq 0$, and $P_n(x) = A_n^{(\alpha,\nu)}(x)$ ($n \in \mathbb{N}_0$). Then $\ell^1(h)$ is*

(i) *point amenable if and only if $\alpha < \frac{1}{2}$ (where $D_x \neq 0$ exists for each $x \in (-1,1)$ if $\alpha \geq \frac{1}{2}$ but $D_{-1} = 0$ always),*

(ii) *weakly amenable if and only if $\alpha < 0$ and $\nu = 0$.*

The "if" part of (i) is shown in [Las09b, Example 4].[34] The "only if" part, as well as the stronger assertion stated in the brackets, can be seen via Proposition 2.1 (i); this is motivated by [Ask71, Section 3].[35] Taking into account (i) and the earlier results of this thesis, it remains to rule out weak amenability for the parameter region $\alpha < \frac{1}{2}$, $\nu > 0$. The subcase $\alpha \geq 0$ is not very difficult and can be tackled by an homomorphism inheritance argument—more precisely, via the nonnegativity of the connection coefficients to the ultraspherical polynomials $\{P_n^{(\alpha)}(x) : n \in \mathbb{N}_0\}$

[34] We note that the reference contains a small mistake: in general, we do not have $h(n) = \mathcal{O}(n^{2\alpha+1})$ ($n \to \infty$) as stated in [Las09b, Example 4], but $h(n) = \mathcal{O}(n^{2|\alpha|+1})$ if $\alpha \neq 0$ and $h(n) = \mathcal{O}(n^\gamma)$ for every $\gamma > 1$ if $\alpha = 0$, cf. (3.9). One does not have $h(n) = \mathcal{O}(n)$ in the latter case (except if $\nu = 0$). This mistake does not affect the conclusion, however.

[35] To avoid possible confusions, we note that the parametrization in [Ask71, Section 3] is slightly different from ours: the parameter μ in [Ask71, Section 3] corresponds to our parameter ν, whereas ν in the reference corresponds to $\alpha + \frac{1}{2}$ in our notation.

(see [Ask71, Section 3]) and Theorem 2.3; the special case $\alpha = 0$ has already been solved in [LP10, Corollary 3.4] by using this method.

The subcase $\alpha < 0$ is the actual difficulty in the proof of Theorem 3.3 (ii)—here, the connection coefficients to $\{P_n^{(\alpha)}(x) : n \in \mathbb{N}_0\}$ are still nonnegative [Ask71, Section 3], but the ℓ^1-algebra which corresponds to these ultraspherical polynomials is, as we already have seen, weakly amenable, and therefore the formerly successful homomorphism inheritance argument breaks down. Instead, our strategy for $\alpha < 0$ (which shall be assumed from now on) is based on our necessity criterion Theorem 2.2. The strategy uses a similar decomposition of the Radon–Nikodym derivative (which shall turn out to be absolutely continuous on $[-1, 1]$) as in the motivating Subsection 3.1; however, this is considerably more involved than ruling out weak amenability for the Pollaczek "counterparts" which were considered in Subsection 3.1. In the following, we give a brief sketch.

Since the numerators in (3.7) have singularities at the boundary of $\mathsf{supp}\ \mu$, we first use Euler's transformation formula for hypergeometric functions [AAR99, Theorem 2.2.5] to rewrite (3.7) as

$$\mathrm{d}\mu(x) = 16^{-\alpha} C_{\alpha,\nu} \frac{(1-x^2)^{-\alpha}}{\left|{}_2F_1\left(\begin{matrix} 2\alpha+\nu, \alpha+\frac{1}{2} \\ \alpha+\nu+\frac{1}{2} \end{matrix}\middle| e^{2i\arccos x}\right)\right|^2} \chi_{(-1,1)}(x)\,\mathrm{d}x. \tag{3.26}$$

Now, at least the numerator is absolutely continuous on $[-1, 1]$. Concerning the denominator of (3.26) and well-definedness, we note that ${}_2F_1\left(\begin{matrix} 2\alpha+\nu, \alpha+\frac{1}{2} \\ \alpha+\nu+\frac{1}{2} \end{matrix}\middle| \cdot\right)$ is absolutely convergent [AAR99, Theorem 2.1.2] and nonzero [Run71] on the unit circle. Observe that the denominator has also a meaning with respect to Fourier series. We shall crucially benefit from a fruitful interplay between hypergeometric and Fourier series, and we need the following auxiliary result on absolute continuity [Kah15, Lemma 5.1]:

Lemma 3.6. *Let $(\gamma_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}$ such that $\sum_{n=0}^{\infty} |\gamma_n| < \infty$, and assume that there exists an $N \in \mathbb{N}_0$ such that $0 \le \gamma_{n+1} \le \frac{n}{n+1}\gamma_n$ whenever $n \ge N$. Then*

(i) *both functions $F_c, F_s : [0, 2\pi] \to \mathbb{R}$, $F_c(x) := \sum_{n=0}^{\infty} \gamma_n \cos(nx)$, $F_s(x) := \sum_{n=1}^{\infty} \gamma_n \sin(nx)$, are absolutely continuous,*

(ii) *the function $G : [0, 2\pi] \to \mathbb{R}$, $G(x) := \left|\sum_{n=0}^{\infty} \gamma_n e^{-inx}\right|^2$, is absolutely continuous.*

Our proof of Lemma 3.6 makes use of an idea presented in [Tri68], various ingredients from the theory of trigonometric and Fourier series (in particular [Bar64, Chapters I§11, I§23; X§2-Theorem 1]) and Pringsheim's theorem. Having in mind the many already existing, frequently more elementary results on smoothness properties of Fourier series vs. decays of the corresponding coefficients, Lemma 3.6 might have some value of its own.

Turning back to the proof of Theorem 3.3 (ii) (failure of weak amenability for the case $\alpha < 0$, $\nu > 0$) and now decomposing $\mu'|_{[-1,1]} = 16^{-\alpha} C_{\alpha,\nu} \frac{v}{f \circ g}$ with $v : [-1, 1] \to \mathbb{R}$, $f : [0, 2\pi] \to \mathbb{R}$ and $g : [-1, 1] \to [0, 2\pi]$,

$$v(x) := (1-x^2)^{-\alpha},\ f(x) := \left|{}_2F_1\left(\begin{matrix} 2\alpha+\nu, \alpha+\frac{1}{2} \\ \alpha+\nu+\frac{1}{2} \end{matrix}\middle| e^{-ix}\right)\right|^2,\ g(x) := 2\pi - 2\arccos x,$$

first recall that v is absolutely continuous. Defining

$$\gamma_n := \frac{(2\alpha+\nu)_n \left(\alpha+\frac{1}{2}\right)_n}{\left(\alpha+\nu+\frac{1}{2}\right)_n n!}\ (n \in \mathbb{N}_0),$$

we see that $\gamma_n = \mathcal{O}(n^{2\alpha-1})$ (as $n \to \infty$) and that at least one of the sequences $(\gamma_n)_{n\in\mathbb{N}_0}$ or $(-\gamma_n)_{n\in\mathbb{N}_0}$ satisfies the conditions of Lemma 3.6. Hence, f is also absolutely continuous (and nonzero). Since g is absolutely continuous and increasing, we finally obtain that $\mu'|_{[-1,1]}$ is absolutely continuous. Hence, we can establish the boundedness of $\{\|\kappa_n\|_\infty : n \in \mathbb{N}_0\}$—and consequently rule out weak amenability of $\ell^1(h)$—via Theorem 2.2, as desired. As in the previous subsection, an explicit computation of κ_n $(n \in \mathbb{N}_0)$ or $\|\kappa_n\|_\infty$ $(n \in \mathbb{N}_0)$ seems to be out of reach, which shows the advantage of the approach provided by Theorem 2.2 via the (at this stage explicitly known) orthogonalization measure.

Again, the detailed proofs can be found in our paper [Kah15].

Albeit the previous strategy works for every pair $(\alpha,\nu) \in \left(-\frac{1}{2},0\right) \times (0,\infty)$, a simpler argument is available for the subregion fulfilling $\nu \geq -2\alpha$: as an immediate consequence of (3.7) and (3.26), or also from (3.8), one has

$$A_n^{(\alpha,\nu)}(x) = A_n^{(-\alpha,2\alpha+\nu)}(x) \ (n \in \mathbb{N}_0)$$

then; so if $\nu \geq -2\alpha$, the failure of weak amenability of $\ell^1(h)$ follows already from the easier parts of Theorem 3.3.

Comparing the classes of Jacobi, symmetric Pollaczek and associated ultraspherical polynomials to their common subclass, i.e., the ultraspherical polynomials, the three main results of this section—which are Theorem 3.1, Theorem 3.2 and Theorem 3.3—yield the following observations:

- both point and weak amenability are independent of the additional parameter (‘β’) of the Jacobi polynomials; the main task concerning the proof of Theorem 3.1 was establishing weak amenability for all $(\alpha,\beta) \in D_P$ with $\alpha < 0$;
- both point and weak amenability depend on the additional parameter (‘λ’) of the symmetric Pollaczek polynomials; the main task concerning the proof of Theorem 3.2 was ruling out point amenability for all $(\alpha,\lambda) \in D_Q$ with $0 < \lambda < -\alpha + \frac{\sqrt{8\alpha+5}}{2} - 1$;
- weak amenability depends on the additional parameter (‘ν’) of the associated ultraspherical polynomials, but point amenability is independent of this additional parameter; the most interesting part concerning the proof of Theorem 3.3 was ruling out weak amenability for all $(\alpha,\nu) \in \left(-\frac{1}{2},0\right) \times (0,\infty)$ with $\nu < -2\alpha$.

The behavior with regard to amenability, right character amenability and ${}_x\alpha$-amenability for $x \in (-1,1]$ does not depend on the additional parameters, and the behavior w.r.t. ${}_{-1}\alpha$-amenability depends on β but is independent of λ and μ, see the notes before Subsection 3.1.

3.5. Further classes: random walk polynomials and cosh-polynomials

Concerning the random walk polynomials, which were considered in Subsection 3.3 as an auxiliary tool, point and weak amenability can be characterized as follows [Kah15, Proposition 4.1]:

Proposition 3.1. *Let $a \geq 1$, $b \geq 0$, and $P_n(x) = S_n^{(a,b)}(x)$ $(n \in \mathbb{N}_0)$. Then $\ell^1(h)$ is*

(i) *point amenable if and only if $a = 1$ and $b < 2$,*

(ii) *weakly amenable if and only if $a = 1$ and $b < 1$.*

Based on the earlier results of this thesis, the proof is not difficult. It benefits from the very simple recurrence coefficients (3.18) and our Proposition 2.1 (ii). The details are given in our paper [Kah15].

We additionally note that a short calculation yields that (only) in the special case $a+b \geq 3$ also Proposition 2.1 (i) applies, which yields the stronger result that $D_x \neq 0$ exists for all $x \in (-1,1)$ then.

The failure of point amenability for the special cases $P_n(x) = S_n^{(a,0)}(x)$ $(n \in \mathbb{N}_0)$ with $a > 1$ (these are the Cartier–Dunau polynomials, which are connected with homogeneous trees [Las83, Section 3 (d)] [Las94, Sections 6, 7]) was already obtained in [Las09b, Example 5] by explicitly computing the derivatives at $x = 0$—hence, Proposition 3.1 extends that result; in contrast to [Las09b, Example 5], we have avoided such explicit computations. The failure of weak amenability can also be seen as in Subsection 3.1, i.e., via Theorem 2.2. However, in sharp contrast to the closely related symmetric Pollaczek polynomials, for the pairs (a,b) with $a > 1$ and $b > 0$ it is indeed the easier variant to rule out weak amenability via ruling out point amenability. In view of (3.20) and Theorem 1.4 (i), $\ell^1(h)$ is amenable if and only if $a = 1$ and $b = 0$ (Chebyshev polynomials of the first kind). As a consequence of Proposition 3.1, (3.19) and [Las09c, p. 792], this improves to the observation that the case $a = 1$, $b = 0$ is also the only one such that $\ell^1(h)$ is right character amenable. Reiter's condition P_2 is satisfied if and only if and only if $a = 1$ (because the latter is equivalent to $1 \in \operatorname{supp} \mu$), i.e., for the ultraspherical polynomials.

Another interesting example is the class of cosh-polynomials; the following basics can be found in [Las05, Section 6]: given $a \geq 0$, the sequence $(\Psi_n^{(a)}(x))_{n\in\mathbb{N}_0}$ of cosh-polynomials that corresponds to a is determined by its orthogonalization measure

$$\mathrm{d}\mu(x) = C_a \frac{1}{\sqrt{1-x^2\cosh^2 a}} \chi_{\left(-\frac{1}{\cosh a}, \frac{1}{\cosh a}\right)}(x)\,\mathrm{d}x,$$

$C_a > 0$ suitable, and the normalization $\Psi_n^{(a)}(1) = 1$ $(n \in \mathbb{N}_0)$. Equivalently, $(\Psi_n^{(a)}(x))_{n\in\mathbb{N}_0}$ is given by

$$b_n \equiv 0,\ a_n \equiv 1 - b_n - c_n,\ c_n := \frac{\cosh(a(n-1))}{2\cosh(an)\cosh a}\ (n \in \mathbb{N}).$$

Independently of a, property (P) is fulfilled, and the linearization coefficients take a very easy form:

$$g(m,n;k) = \frac{\cosh(ak)}{2\cosh(am)\cosh(an)}[\delta_{|m-n|}(k) + \delta_{m+n}(k)]\ (m,n \in \mathbb{N}_0, k \in \{|m-n|,\ldots,m+n\}).$$

The Haar weights satisfy

$$h(n) = [2 - \delta_0(n)]\cosh^2(an)\ (n \in \mathbb{N}_0).$$

Moreover, $\widehat{\mathbb{N}_0} = [-1,1]$ and

$$\mathcal{X}^b(\mathbb{N}_0) = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{\cosh a} \right| + \left| z + \frac{1}{\cosh a} \right| \leq 2 \right\}.$$

We have the following, which is [Kah15, Proposition 6.1], concerning amenability notions:

Proposition 3.2. *Let $a \geq 0$ and $P_n(x) = \Psi_n^{(a)}(x)$ $(n \in \mathbb{N}_0)$. The following are equivalent:* (i) *$a = 0$,* (ii) *$\ell^1(h)$ is amenable,* (iii) *$\ell^1(h)$ is weakly amenable,* (iv) *$\ell^1(h)$ is point amenable.*

Further equivalent conditions (which, in contrast to (i) – (iv), have not been contained in our paper [Kah15], however) are "(v) $\ell^1(h)$ is right character amenable" and "(vi) Reiter's condition P_2 is satisfied"; recall that the latter is equivalent to $1 \in \operatorname{supp} \mu$. The case $a = 0$ yields the Chebyshev polynomials of the first kind again. Since the most implications are trivial, it just remains to show that (iv) implies (i), which is immediate from our Proposition 2.1 (ii) (note that

$c_n \to \frac{1}{1+e^{2a}}$ as $n \to \infty$). In the special case $\cosh a \geq \sqrt{2}$, also Proposition 2.1 (i) applies (note that $c_1 a_0 = \frac{1}{2\cosh^2 a}$ and $c_n a_{n-1} = \frac{1}{4\cosh^2 a}$ ($n \geq 2$)). Another variant follows from Theorem 1.6 and the explicit values

$$|P_n'(0)| = \begin{cases} 0, & n \in \mathbb{N}_0 \text{ even}, \\ \frac{n\cosh a}{\cosh(an)}, & n \in \mathbb{N}_0 \text{ odd} \end{cases} \tag{3.27}$$

(easy to see). Concerning the failure of weak amenability for $a > 0$, we can also explicitly compute

$$\tau(n) = \begin{cases} 0, & n \in \mathbb{N}_0 \text{ even}, \\ \frac{n\cosh a}{\cosh(an)}, & n \in \mathbb{N}_0 \text{ odd} \end{cases} \tag{3.28}$$

via Lemma 2.4 (iii) and then apply Theorem 2.2 (i). Comparing (3.28) to (3.27), we get $\int_{\mathbb{R}} P_n'(x)\,\mathrm{d}\mu(x) = \tau(n) = |P_n'(0)|$ ($n \in \mathbb{N}_0$); this is rather striking because there is no general implication between the boundedness of τ and the boundedness of $\{P_n'(0) : n \in \mathbb{N}_0\}$ (cf. the classes considered at the end of Subsection 3.2).

4. Characterizations of ultraspherical and q-ultraspherical polynomials

Parts of Section 4 are very similar to our publication [Kah16].

4.1. Weak amenability reconsidered and a characterization of ultraspherical polynomials

Recall that deducing weak amenability of $\ell^1(h)$ for $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$, $-\frac{1}{2} < \alpha < 0$, from Theorem 2.1—see Section 2.5—was considerably easier than the full Jacobi analogue given in Theorem 3.1, primarily for the following two reasons: on the one hand, condition (iv) of Theorem 2.1 could be checked in a rather elementary way, based on the Plancherel isomorphism and a convenient explicit formula for the linearization coefficients $g(n, 2k; n)$ (Dougall's formula), and without the need to consider asymptotics. On the other hand, condition (i) of Theorem 2.1 could be verified as fast as for the Chebyshev polynomials of the first kind—due to the very simple form of the sequence $(\kappa_n)_{n\in\mathbb{N}_0}$, see (2.26), which can be reformulated as the observation that all of the functions $\kappa_{2n-1}|_{\{0,2,\ldots,2n-2\}}$, $\kappa_{2n-1}|_{\{1,3,\ldots,2n-1\}}(=0)$, $\kappa_{2n}|_{\{0,2,\ldots,2n-2\}}(=0)$ and $\kappa_{2n}|_{\{1,3,\ldots,2n-1\}}$ are *constant* for each $n \in \mathbb{N}$.

The equations $\kappa_{2n-1}|_{\{1,3,\ldots,2n-1\}} = 0$ and $\kappa_{2n}|_{\{0,2,\ldots,2n-2\}} = 0$ $(n \in \mathbb{N})$ are consequences of the symmetry ($b_n \equiv 0$, μ symmetric). However, the constancy of the functions $\kappa_{2n-1}|_{\{0,2,\ldots,2n-2\}}$ and $\kappa_{2n}|_{\{1,3,\ldots,2n-1\}}$ $(n \in \mathbb{N})$ is a striking property of the ultraspherical polynomials: in fact, the ultraspherical polynomials can be *characterized* via such properties. The first such characterization is due to Lasser and Obermaier [LO08]. It has been generalized to the classes of discrete and continuous q-ultraspherical polynomials by Ismail and Obermaier [IO11]; moreover, a refinement of the original Lasser–Obermaier result has been given in the author's Master's thesis [Kah12, Theorem 4.1 (i); Corollary 4.1 (i)]—Theorem 4.4 below, which is [Kah16, Theorem 2.1], is a modification of the result of the Master's thesis [Kah12, Theorem 4.1 (i); Corollary 4.1 (i)]. In the following, we shall precisely recall these results. We first recall some basic theory and introduce some basic notation.

Throughout the section, we consider sequences $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ of polynomials given by a recurrence relation of the form $P_0(x) = 1$,

$$xP_n(x) = a_n P_{n+1}(x) + c_n P_{n-1}(x) \; (n \in \mathbb{N}_0), \tag{4.1}$$

where $A > 0$, $c_0 := 0$, $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, $a_n := A - c_n$ $(n \in \mathbb{N}_0)$ and $c_n a_{n-1} > 0$ $(n \in \mathbb{N})$.[36] As a consequence of Favard's theorem and further standard results from the theory of orthogonal polynomials (cf. [Chi78, I-Theorem 4.1, I-Theorem 4.4, II-Exercise 1.2, II-Theorem 3.1]), this is equivalent to orthogonality w.r.t. a symmetric probability (Borel) measure μ on $\mathbb{R}$ with $|\text{supp}\,\mu| = \infty$ and the normalization $P_n(A) = 1$ $(n \in \mathbb{N}_0)$.[37]

If additionally

$$(c_n)_{n\in\mathbb{N}} \subseteq (0, A) \tag{RW}$$

holds, then $(P_n(x))_{n\in\mathbb{N}_0}$ is called a (symmetric) random walk polynomial sequence ('RWPS').[38]

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[36] Again, we make the widely common convention that ($c_0 =$)0 times something undefined shall be 0.

[37] Concerning the existence of *symmetric* μ, cf. [Chi78, I-Theorem 4.3] and [Chi82, p. 332]; note that if $(P_n(x))_{n\in\mathbb{N}_0}$ satisfies (4.1) and is orthogonal w.r.t. some (not necessarily symmetric) μ, then $(P_n(x))_{n\in\mathbb{N}_0}$ is also orthogonal w.r.t. the measure ν, $\nu(A) := \mu(-A)$ (A Borel subset of $\mathbb{R}$), so $(P_n(x))_{n\in\mathbb{N}_0}$ is orthogonal w.r.t. the symmetric measure $\frac{1}{2}[\mu + \nu]$.

[38] Concerning the relation to random walks, we refer to [CSvD98, vDS93]. In contrast to some authors, we do not generally require that $A = 1$ when using the expression RWPS. Moreover, to avoid any confusion, we mention again that the expression "random walk polynomials" (or "random walk polynomial sequence") as used in the present section is much more general than the corresponding expression used in Section 3.

This is equivalent to $\mathsf{supp}\ \mu \subseteq [-A, A]$.[39] In this case, μ is necessarily unique (not only among the symmetric orthogonalization measures—if there were any two different orthogonalization measures, then none of them could have compact support, cf. [Chi78, II-Theorem 3.2; II-Theorem 5.6] again). Consequently, $\mathrm{d}\mu^*(x) := (A^2 - x^2)\,\mathrm{d}\mu(x)$ uniquely defines a symmetric finite (Borel) measure μ^* on $\mathbb{R}$ with $|\mathsf{supp}\ \mu^*| = \infty$ and $\mathsf{supp}\ \mu^* \subseteq [-A, A]$, and we get a corresponding symmetric RWPS $(P_n^*(x))_{n\in\mathbb{N}_0}$ with $P_n^*(A) = 1$ $(n \in \mathbb{N}_0)$.

Of course, the ultraspherical polynomials $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$ are an example of a symmetric RWPS (for any $\alpha > -1$).

As usual, we denote by $(U_n(x))_{n\in\mathbb{N}_0}$ the Chebyshev polynomials of the second kind (normalized such that $U_n(1) = n + 1$), i.e., $U_n(x) = (n + 1)P_n^{(\frac{1}{2})}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(2 - \delta_{n-2k,0})T_{n-2k}(x)$ $(n \in \mathbb{N}_0)$ [AS64, 22.2.5; 22.12.2; 22.12.3]; moreover, let $U_{-1}(x) := 0$. It is well-known that

$$\sin((n + 1)x) = U_n(\cos x)\sin x \ (n \in \mathbb{N}_0, x \in \mathbb{C}) \tag{4.2}$$

[AS64, 22.3.16].

Given $q \in (0, 1)$ and $\alpha, \beta \in \left(0, \frac{1}{\sqrt{q}}\right)$, the corresponding—suitably normalized—sequences of discrete q-ultraspherical (or symmetric big q-Jacobi) polynomials $(P_n(x; \alpha : q))_{n\in\mathbb{N}_0}$ and continuous q-ultraspherical polynomials (or Rogers polynomials; in essence, symmetric continuous q-Jacobi polynomials) $(P_n(x; \beta|q))_{n\in\mathbb{N}_0}$ are defined via $A = \alpha\sqrt{q}$,

$$c_n = \alpha\sqrt{q}\frac{1 - q^n}{1 - \alpha^2 q^{2n}} \tag{4.3}$$

$(n \in \mathbb{N})$ and $A = \frac{\sqrt{\beta}}{2} + \frac{1}{2\sqrt{\beta}}$,

$$c_n = \frac{\sqrt{\beta}}{2}\frac{1 - q^n}{1 - \beta q^n} \tag{4.4}$$

$(n \in \mathbb{N})$, respectively [IO11]. Obviously, both the discrete q-ultraspherical polynomials $(P_n(x; \alpha : q))_{n\in\mathbb{N}_0}$ and the continuous q-ultraspherical polynomials $(P_n(x; \beta|q))_{n\in\mathbb{N}_0}$ are also examples of symmetric RWPS. The ultraspherical polynomials are limiting cases because

$$\lim_{q\to 1} P_n(x; q^{\alpha+\frac{1}{2}} : q) = P_n^{(\alpha)}(x)$$

and

$$\lim_{q\to 1} P_n(x; q^{\alpha+\frac{1}{2}}|q) = P_n^{(\alpha)}(x) \tag{4.5}$$

for every $\alpha > -1$, $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$; moreover, $T_n(x) = P_n(x; 1|q)$ and $U_n(x) = U_n\left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right) P_n(x; q|q)$ $(n \in \mathbb{N}_0)$ [KLS10]. Since we will not need them, we omit the explicit formulas for the orthogonalization measures. The orthogonalization measure μ of $(P_n(x; \alpha : q))_{n\in\mathbb{N}_0}$ is purely discrete and satisfies $\max \mathsf{supp}\ \mu = A$ [IO11]. If $\beta \leq 1$, the orthogonalization measure μ of $(P_n(x; \beta|q))_{n\in\mathbb{N}_0}$ is absolutely continuous (w.r.t. the Lebesgue–Borel measure on $\mathbb{R}$) and $\mathsf{supp}\ \mu = [-1, 1]$, whereas if $\beta > 1$, then point measures appear at $\pm A$ and $\max \mathsf{supp}\ \mu = A > 1$ [IO11]. Further information about these classes can be

[39] If $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ is a symmetric RWPS, then, as $a_n[P_{n+1}(x) - P_n(x)] = (x - A)P_n(x) + c_n[P_n(x) - P_{n-1}(x)]$ $(n \in \mathbb{N}_0)$ by (4.1), we have $P_n(x) \geq P_{n-1}(x) \geq 1$ for all $x \geq A$ and $n \in \mathbb{N}$ (which follows via (RW) and induction). This, however, implies that all zeros of the polynomials $P_n(x)$ $(n \in \mathbb{N})$ are located in $(-A, A)$—due to the latter, $\mathsf{supp}\ \mu \subseteq [-A, A]$ [Chi78, II-Theorem 3.2; II-Theorem 5.6]. This argument is motivated by a similar one in the author's Master's thesis [Kah12]. The reverse direction is contained in [LO08, p. 2493, p. 2494]—the slightly different setting w.r.t. the normalization point A does not affect the validity of the argument given in [LO08]. Cf. also [CSvD98, vDS93].

found in [Ism09, IO11, KLS10]. The latter reference contains the full (q-)Askey scheme.

Similarly as in Section 1.3, we define $h : \mathbb{N}_0 \to (0,\infty)$ by

$$h(n) := \frac{1}{\int_{\mathbb{R}} P_n^2(x)\,\mathrm{d}\mu(x)} = \begin{cases} 1, & n = 0, \\ \prod_{j=1}^{n} \frac{a_{j-1}}{c_j}, & \text{else} \end{cases}$$

[IO11]. Concerning well-definedness of h, note that, although μ need not be unique if (RW) does not hold, the values of $\int_{\mathbb{R}} P(x)\,\mathrm{d}\mu(x)$, $P(x) \in \mathbb{R}[x]$, are always uniquely determined: expand $P(x)$ in the basis $\{P_n(x) : n \in \mathbb{N}_0\}$ and use orthogonality.

Let $(Q_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ be defined by $Q_n(x) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)P_{n-2k}(x)$ $(n \in \mathbb{N}_0)$. The following calculation makes use of our convention that ($c_0 =$)0 times something undefined shall be 0, which helps to avoid clumsy case differentiations. Due to (4.1), for every $n \geq 2$ we have

$$\begin{aligned}
&(A^2 - x^2)Q_n(x) = \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)(A^2 - a_{n-2k}c_{n-2k+1} - c_{n-2k}a_{n-2k-1})P_{n-2k}(x) \\
&- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)a_{n-2k}a_{n-2k+1}P_{n-2k+2}(x) - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)c_{n-2k}c_{n-2k-1}P_{n-2k-2}(x) = \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)(A^2 - a_{n-2k}c_{n-2k+1} - c_{n-2k}a_{n-2k-1})P_{n-2k}(x) \\
&- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} h(n-2k-2)a_{n-2k-2}a_{n-2k-1}P_{n-2k}(x) - h(n)a_n a_{n+1}P_{n+2}(x) \\
&- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor + 1} h(n-2k+2)c_{n-2k+2}c_{n-2k+1}P_{n-2k}(x) + h(n+2)c_{n+2}c_{n+1}P_n(x).
\end{aligned}$$

Using the product formula for h and the fact that $c_{n-2k-1}c_{n-2k} = 0$ if $k = \lfloor \frac{n}{2} \rfloor$ and $c_{n-2k+2}c_{n-2k+1} = 0$ if $k = \lfloor \frac{n}{2} \rfloor + 1$, we get

$$\begin{aligned}
&(A^2 - x^2)Q_n(x) = \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)(A^2 - a_{n-2k}c_{n-2k+1} - c_{n-2k}a_{n-2k-1})P_{n-2k}(x) \\
&- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)c_{n-2k-1}c_{n-2k}P_{n-2k}(x) - h(n)a_n a_{n+1}P_{n+2}(x) \\
&- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)a_{n-2k+1}a_{n-2k}P_{n-2k}(x) + h(n)a_{n+1}a_n P_n(x).
\end{aligned}$$

Since $A^2 - a_{n-2k}c_{n-2k+1} - c_{n-2k}a_{n-2k-1} - c_{n-2k-1}c_{n-2k} - a_{n-2k+1}a_{n-2k} = 0$ for every $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$, we finally obtain that

$$(A^2 - x^2)Q_n(x) = -h(n)a_n a_{n+1}[P_{n+2}(x) - P_n(x)] \ (n \in \mathbb{N}_0); \tag{4.6}$$

the validity for $n = 0$ and $n = 1$ is obvious. Alternatively, (4.6) could be established by a twofold application of a "non-confluent" version of the Christoffel–Darboux formula (cf. [Chi78, I-Theorem 4.5]) to the quotient $\frac{P_{n+2}(x)-P_n(x)}{A^2-x^2}$.

Now if $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ is a symmetric RWPS, then $(A^2-x^2)P_n^*(x)$ is orthogonal to $P_k(x)$ w.r.t. μ for every $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0\backslash\{n, n+2\}$; this is immediate consequence of the definition, the symmetry of the measures and the equation $\int_{\mathbb{R}}(A^2-x^2)P_n^*(x)P_k(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}}P_n^*(x)P_k(x)\,\mathrm{d}\mu^*(x)$. Hence, (4.6) yields $P_n^*(x) = \frac{Q_n(x)}{Q_n(A)}$ $(n \in \mathbb{N}_0)$, i.e.,

$$P_n^*(x) = \frac{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} h(n-2k)P_{n-2k}(x)}{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} h(n-2k)} \ (n \in \mathbb{N}_0). \tag{4.7}$$

If $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ is a symmetric RWPS, then it would also be possible to establish (4.7) by directly showing that the right hand side defines a sequence of polynomials which is orthogonal w.r.t. μ^*; a similar argument has been used in the author's Master's thesis [Kah12].

We now take (4.7) to extend the definition of $(P_n^*(x))_{n\in\mathbb{N}_0}$ to the general case (i.e., (RW) is not required). This extension also preserves the orthogonality of $(A^2-x^2)P_n^*(x)$ and $P_k(x)$ w.r.t. μ $(n \in \mathbb{N}_0,\ k \in \mathbb{N}_0\backslash\{n, n+2\})$, which now is a consequence of (4.6).

In the following, we do not require $(P_n(x))_{n\in\mathbb{N}_0}$ to be an RWPS unless explicitly stated otherwise.

It is well-known that

$$\frac{\mathrm{d}}{\mathrm{d}x}T_n(x) = nU_{n-1}(x) \ (n \in \mathbb{N}_0) \tag{4.8}$$

[KLS10, (9.8.45)]. Thus, if $P_n(x) = T_n(x)$ $(n \in \mathbb{N}_0)$, then $A = 1$ and $P_n'(x) = P_n'(1)P_{n-1}^*(x)$ $(n \in \mathbb{N})$. Obviously, the latter is just a reformulation of ($A = 1$ and) the constancy of all functions $\kappa_{2n-1}|_{\{0,2,\ldots,2n-2\}}$ and $\kappa_{2n}|_{\{1,3,\ldots,2n-1\}}$ $(n \in \mathbb{N})$, where, as in Section 1.4, for each $n \in \mathbb{N}_0$ we define $\kappa_n : \mathbb{N}_0 \to \mathbb{R}$ by

$$\kappa_n(k) := \int_{\mathbb{R}} P_n'(x)P_k(x)\,\mathrm{d}\mu(x).$$

The Lasser–Obermaier result mentioned above characterizes the symmetric RWPS that share this property [LO08, Lemma 1; Theorem 1]:

Theorem 4.1. *If* (RW) *holds, then the following are equivalent:*

(i) $P_n(x) = P_n^{\left(\frac{1}{2c_1}-\frac{3}{2}\right)}(x)$ $(n \in \mathbb{N}_0)$,

(ii) $A = 1$ *and* $P_n'(x) = P_n'(1)P_{n-1}^*(x)$ $(n \in \mathbb{N})$.

The Ismail–Obermaier analogues for the classes of discrete and continuous q-ultraspherical polynomials mentioned above use suitable q-generalizations of the classical derivative:

- for $q \in (0,\infty)\backslash\{1\}$, the '$q$-difference operator' $D_q : \mathbb{R}[x] \to \mathbb{R}[x]$ is given by $D_qP(x) = \frac{P(x)-P(qx)}{x-qx}$ $(x \neq 0)$,
- for $q \in (0,1)$, the (linear) 'Askey–Wilson operator' $\mathcal{D}_q : \mathbb{R}[x] \to \mathbb{R}[x]$ is defined via a q-extension of (4.8), namely by

$$\mathcal{D}_qT_n(x) = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{\sqrt{q} - \frac{1}{\sqrt{q}}}U_{n-1}(x) \ (n \in \mathbb{N}_0); \tag{4.9}$$

  so if $P_n(x) = T_n(x)$ $(n \in \mathbb{N}_0)$, then $\mathcal{D}_qP_n(x) = \mathcal{D}_qP_n(1)P_{n-1}^*(x)$ $(n \in \mathbb{N})$.

Both D_q and $\mathcal{D}_q$ contain the classical derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ as limiting case $q \to 1$. These basics, as well as further information about D_q and $\mathcal{D}_q$, can be found in [Ism09] or [IO11]. Concerning Theorem 4.2 and Theorem 4.3 below, we note that Theorem 4.2 is contained in [IO11, Theorem 4.1] and that Theorem 4.3 is contained in [IO11, Theorem 5.2].

Theorem 4.2. *Let $q \in (0,1)$, $\alpha \in \left(0, \frac{1}{\sqrt{q}}\right)$, $A = \alpha\sqrt{q}$; moreover, let $c_1 = \alpha\sqrt{q}\frac{1-q}{1-\alpha^2 q^2}$. Then the following are equivalent:*

(i) $P_n(x) = P_n(x;\alpha : q)$ $(n \in \mathbb{N}_0)$,

(ii) $D_{q^{-1}}P_n(x) = D_{q^{-1}}P_n(A)P^*_{n-1}(x)$ $(n \in \mathbb{N})$.

Theorem 4.3. *Let $q \in (0,1)$, $\beta \in \left(0, \frac{1}{\sqrt{q}}\right)$ and $A = \frac{\sqrt{\beta}}{2} + \frac{1}{2\sqrt{\beta}}$; assume $c_1 = \frac{\sqrt{\beta}}{2}\frac{1-q}{1-\beta q}$. Then the following are equivalent:*

(i) $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$,

(ii) $\mathcal{D}_q P_n(x) = \mathcal{D}_q P_n(A)P^*_{n-1}(x)$ $(n \in \mathbb{N})$.

To be precise, we note that the special case $\beta = 1$ (Chebyshev polynomials of the first kind), which occurs very naturally in our normalization but only as limiting case in the "standard" normalization of the continuous q-ultraspherical polynomials [KLS10], has been excluded in the original version of [IO11, Theorem 5.2]—however, the proof given in [IO11] extends to this case.

Remark 4.1. Note that if $P_n(x) = P_n^{(\alpha)}(x)$ $(n \in \mathbb{N}_0)$ with $\alpha > -1$, then, up to normalization, $(P_n'(x))_{n\in\mathbb{N}}$ coincides with $(P_{n-1}^{(\alpha+1)}(x))_{n\in\mathbb{N}}$; in the same way, if $P_n(x) = P_n(x;\alpha : q)$ $(n \in \mathbb{N}_0)$ with $q \in (0,1)$ and $\alpha \in \left(0, \frac{1}{\sqrt{q}}\right)$, then $(D_{q^{-1}}P_n(x))_{n\in\mathbb{N}}$ corresponds to $(P_{n-1}(x;\alpha q : q))_{n\in\mathbb{N}}$, and if $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$ with $q \in (0,1)$ and $\beta \in \left(0, \frac{1}{\sqrt{q}}\right)$, then $(\mathcal{D}_q P_n(x))_{n\in\mathbb{N}}$ corresponds to $(P_{n-1}(x;\beta q|q))_{n\in\mathbb{N}}$. These are the corresponding "forward shifts" (which should also be regarded as some motivation for Theorem 4.1, Theorem 4.2 and Theorem 4.3), cf. (3.16), [IO11, (2.10)] and [KLS10, (14.10.23)]. In each case, we have coincidence with $(P^*_{n-1}(x))_{n\in\mathbb{N}}$ (up to normalization). Consequently, the classes are "preserved" by the corresponding operators (and by the passage from μ to μ^*). If $D_{q^{-1}}$ was replaced by D_q, this would become wrong (because $D_q P(x) = D_{q^{-1}}P(qx)$ for every $P(x) \in \mathbb{R}[x]$ and because it is easy to see that there is no $\widetilde{\alpha} \in \left(0, \frac{1}{\sqrt{q}}\right)$ such that $(P_n(qx;\alpha q : q))_{n\in\mathbb{N}_0} = (P_n(x;\widetilde{\alpha} : q))_{n\in\mathbb{N}_0}$). Furthermore, it is interesting to observe that the orthogonalization measure of $(\mathcal{D}_q P_n(x;\beta|q))_{n\in\mathbb{N}}$ is always absolutely continuous and has support $[-1,1]$ (because $\beta q < 1$).

Characterizing families of orthogonal polynomials by specific properties has a long history, including an extensive literature ([AS90] provides a valuable survey up to 1990). The interesting, "characterizing" direction "(ii) $\Rightarrow$ (i)" in Theorem 4.1 can also be obtained from older characterization results: one possibility is to apply a famous contribution of Hahn which tells that only the 'classical' orthogonal polynomials possess derivatives which form orthogonal polynomial sequences again [Hah35][40]—note that the ultraspherical polynomials are the only classical polynomials such that μ has compact support and μ is symmetric (cf. [Chi78, Chapter V §2]). Similarly, one may also use a related (yet independent) characterization of the classical orthogonal polynomials given by Al-Salam and Chihara [ASC72].[41] This result of Al-Salam–Chihara and the result of Hahn have analogues for the q-difference operator D_q [DG06, Hah49]. However, it seems to be open whether there are also analogues for the Askey–Wilson operator

[40] In fact, this characterization might have been discovered already around 50 years earlier by Sonine, cf. [Ism09, p. 528].

[41] The relationship between Theorem 4.1 and the cited results of Hahn [Hah35] and Al-Salam–Chihara [ASC72] was already observed in our Master's thesis [Kah12].

$\mathcal{D}_q$ [Ism09, Conjecture 24.7.8, Conjecture 24.7.10]. Therefore, Theorem 4.3 "(ii) $\Rightarrow$ (i)" is of particular interest.

In [IS12], Ismail and Simeonov gave extensions to other classes, including symmetric Al-Salam–Chihara, symmetric Askey–Wilson and symmetric Meixner–Pollaczek polynomials (with suitably chosen corresponding operators). Moreover, the author's Master's thesis [Kah12] contains suitable extensions of Theorem 4.1 to the generalized Chebyshev polynomials (quadratic transformations of the Jacobi polynomials in the sense of [Chi78, Chapter V §2 (G)], suitably normalized) and to the Jacobi polynomials themselves (which, of course, involves asymmetry). The extension to the generalized Chebyshev polynomials is in terms of the property "$A = 1$ and $P_{2n}'(x) = P_{2n}'(1)P_{2n-1}^*(x)$ $(n \in \mathbb{N})$" (or even weaker conditions) [Kah12, Theorem 4.1 (ii); Corollary 4.1 (ii)]. The extension to the Jacobi polynomials is motivated by (3.14), (3.16), kernel polynomials and quadratic transformations [Kah12, Theorem 4.1 (iii); Corollary 4.1 (iii)].[42] Furthermore, our Master's thesis [Kah12] contains analogues which are motivated by the (Jacobi) backward shift (3.17) (cf. Remark 4.1 above).

In our Master's thesis [Kah12], under the additional assumption (RW) we showed the following sharpened version of Theorem 4.1:

Theorem 4.4. *Let $\alpha > -1$, $A = 1$ and $c_1 = \frac{1}{2\alpha+3}$. Then the following are equivalent:*

(i) $P_n(x) = P_n^{(\alpha)}(x)$ $(n \in \mathbb{N}_0)$,

(ii) $P_n'(x) = P_n'(1)P_{n-1}^*(x)$ $(n \in \mathbb{N})$,

(iii) $P_{2n-1}'(x) = P_{2n-1}'(1)P_{2n-2}^*(x)$ $(n \in \mathbb{N})$,

(iv) $(1-x^2)P_n'(x)$ *is orthogonal to* $P_0(x), \ldots, P_{n-2}(x)$ $(n \geq 2)$,

(v) $(1-x^2)P_{2n-1}'(x)$ *is orthogonal to* $P_0(x), P_2(x), \ldots, P_{2n-4}(x)$ $(n \geq 2)$,

(vi) *one has*

$$\kappa_{2n+1}(2n-2) = \sigma(2n+1) \ (n \in \mathbb{N}),$$

and for every $n \in \mathbb{N}$ there is a $k \in \{0, \ldots, n-1\}$ such that

$$\kappa_{2n+3}(2k) = \sigma(2n+3).$$

As in Section 2.1, we write $\sigma : \mathbb{N} \to \mathbb{R}\backslash\{0\}$,

$$\sigma(n) := \kappa_n(n-1).$$

Since Theorem 4.4, which corresponds to [Kah16, Theorem 2.1], does not assume (RW) to hold and consequently works with an a priori considerably more general meaning of $(P_n^*(x))_{n\in\mathbb{N}_0}$ ((RW) is obtained as a *consequence* of (i) – (vi) then), it can be seen as an improvement of the result of the Master's thesis [Kah12, Theorem 4.1 (i); Corollary 4.1 (i)]. However, this improvement does not affect the actual proof—hence, the proof is omitted at this stage, and we just make the following few notes: the characterization (iii) shows that in Theorem 4.1 the constancy of the functions $\kappa_{2n}|_{\{1,3,\ldots,2n-1\}}$ $(n \in \mathbb{N})$ is redundant; it suffices to require the constancy for the odd indices, i.e., the constancy of the functions $\kappa_{2n-1}|_{\{0,2,\ldots,2n-2\}}$ $(n \in \mathbb{N})$. (iv) and (v) have the advantage to be "stable" w.r.t. renormalization of $(P_n(x))_{n\in\mathbb{N}_0}$. In (vi), which is the apparently weakest condition and hence provides the strongest characterization, the functions κ_n have to be considered only for odd indices and only at some carefully chosen points.

[42] We think about publishing these results in a future paper.

In contrast to Theorem 4.1, Theorem 4.4 is no longer a consequence of the results of Hahn and Al-Salam–Chihara mentioned above. The proof of Theorem 4.4 can be found in [Kah16].

In this thesis, we present analogous improvements of Theorem 4.2 and Theorem 4.3. Moreover, we shall give a characterization of a subclass of the continuous q-ultraspherical polynomials in terms of the 'averaging operator' $\mathcal{A}_q : \mathbb{R}[x] \to \mathbb{R}[x]$, which, for every $q \in (0,1)$, is given by linearity and the defining equation

$$\mathcal{A}_q T_n(x) = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2} T_n(x) \; (n \in \mathbb{N}_0) \tag{4.10}$$

[IO11]. We shall also see that this characterization in terms of $\mathcal{A}_q$ will not transfer to the whole class of continuous q-ultraspherical polynomials.

4.2. A characterization of discrete q-ultraspherical polynomials in terms of the q-difference operator

Let A and $(P_n(x))_{n\in\mathbb{N}_0}$ be as in Subsection 4.1 (property (RW) is not required), and let $q \in (0,1)$. As analogue to $(\kappa_n)_{n\in\mathbb{N}_0}$, and following [IO11] (yet with different notation), we consider the Fourier coefficients associated with the q-difference operator $D_{q^{-1}}$: for every $n \in \mathbb{N}_0$, let $\kappa_n(. : q) : \mathbb{N}_0 \to \mathbb{R}$ be given by

$$\kappa_n(k : q) := \int_{\mathbb{R}} D_{q^{-1}} P_n(x) P_k(x) \, \mathrm{d}\mu(x).$$

It is obvious that $\kappa_n(k : q) = 0$ if $k \geq n$ or if $n - k$ is even. As analogue to σ, we define $\sigma(. : q) : \mathbb{N} \to \mathbb{R}\backslash\{0\}$,

$$\sigma(n : q) := \kappa_n(n-1 : q).$$

The following theorem [Kah16, Theorem 2.2] is the analogue to Theorem 4.4.

Theorem 4.5. *Under the conditions of Theorem 4.2, the following are equivalent:*

(i) $P_n(x) = P_n(x; \alpha : q) \; (n \in \mathbb{N}_0)$,

(ii) $D_{q^{-1}} P_n(x) = D_{q^{-1}} P_n(A) P^*_{n-1}(x) \; (n \in \mathbb{N})$,

(iii) $D_{q^{-1}} P_{2n-1}(x) = D_{q^{-1}} P_{2n-1}(A) P^*_{2n-2}(x) \; (n \in \mathbb{N})$,

(iv) $(A^2 - x^2) D_{q^{-1}} P_n(x)$ *is orthogonal to* $P_0(x), \ldots, P_{n-2}(x) \; (n \geq 2)$,

(v) $(A^2 - x^2) D_{q^{-1}} P_{2n-1}(x)$ *is orthogonal to* $P_0(x), P_2(x), \ldots, P_{2n-4}(x) \; (n \geq 2)$,

(vi) *one has*

$$\kappa_{2n+1}(2n-2 : q) = \sigma(2n+1 : q) \; (n \in \mathbb{N}),$$

and for every $n \in \mathbb{N}$ *there is a* $k \in \{0, \ldots, n-1\}$ *such that*

$$\kappa_{2n+3}(2k : q) = \sigma(2n+3 : q).$$

Since the details of the proof of Theorem 4.5 were omitted in our paper [Kah16] (because the proof is rather similar to the proof of Theorem 4.4—which, however, has been omitted in this thesis, cf. above), we want to give these details here. We preliminarily note two important (general) relations: on the one hand, the analogue to Lemma 2.4 (i) is

$$\sigma(n : q) = q^{1-n} \frac{1 - q^n}{1 - q} \frac{1}{c_n h(n)}, \; \frac{\sigma(n : q)}{\sigma(n+1 : q)} = q \frac{1 - q^n}{1 - q^{n+1}} \frac{a_n}{c_n} \; (n \in \mathbb{N}) \tag{4.11}$$

[IO11, proof of Theorem 4.1].[43] On the other hand, $(\kappa_n(.:q))_{n\in\mathbb{N}_0}$ satisfies the recurrence relation $\kappa_0(.:q)=0$,

$$a_n\kappa_{n+1}(k:q)+c_n\kappa_{n-1}(k:q)=\frac{1}{q}[a_k\kappa_n(k+1:q)+c_k\kappa_n(k-1:q)]+\frac{\delta_{n,k}}{h(n)}\ (n,k\in\mathbb{N}_0) \quad (4.12)$$

[IO11, proof of Theorem 4.1], which relies on the product rule

$$D_q[PQ](x)=P(x)D_qQ(x)+Q(qx)D_qP(x)\ (P(x),Q(x)\in\mathbb{R}[x])$$

[Ism09, (11.4.15)] for D_q.

We now come to the proof of Theorem 4.5. The direction "(i) ⇒ (ii)" is a part of Theorem 4.2, the implications "(ii) ⇒ (iii)" and "(iii) ⇒ (vi)" are trivial. Moreover, (iv) and (v) are obvious reformulations of (ii) and (iii), respectively. It is left to establish "(vi) ⇒ (i)": we use induction to show that (4.3) is satisfied for all $n\in\mathbb{N}$ if (vi) is assumed to hold. The validity for $n=1$ is an assumption of the theorem. Furthermore, (4.12) implies

$$a_2\sigma(3:q)+c_2\sigma(1:q)=\frac{A}{q}\sigma(2:q),$$

which yields

$$\frac{A}{q}=\frac{1}{q}\frac{1-q^3}{1-q^2}c_2+q\frac{1-q}{1-q^2}\frac{a_1}{c_1}c_2=\frac{1}{q}\frac{1-\alpha^2q^4}{1-q^2}c_2$$

as a consequence of (4.11). Hence, (4.3) is shown for $n=2$. Now let $n\in\mathbb{N}$ be arbitrary but fixed, and assume that $1,\dots,2n$ fulfill (4.3). Choose $k\in\{0,\dots,n-1\}$ such that $\kappa_{2n+3}(2k:q)=\sigma(2n+3:q)$. Since $\kappa_{2n}(.:q)$ and $\kappa_{2n+1}(.:q)$ are uniquely determined by $c_1,\dots,c_{2n}$, we obtain from the direction "(i) ⇒ (ii)" (or from Theorem 4.2) that $\kappa_{2n}(2j-1:q)=\sigma(2n:q)$ $(j\in\{1,\dots,n\})$ and $\kappa_{2n+1}(2j-2:q)=\sigma(2n+1:q)$ $(j\in\{1,\dots,n+1\})$. Then (4.12) yields

$$a_{2n+1}\kappa_{2n+2}(2k+1:q)+c_{2n+1}\sigma(2n:q)=\frac{A}{q}\sigma(2n+1:q), \quad (4.13)$$

$$a_{2n+1}\kappa_{2n+2}(2n-1:q)+c_{2n+1}\sigma(2n:q)=\frac{A}{q}\sigma(2n+1:q), \quad (4.14)$$

$$a_{2n+2}\sigma(2n+3:q)+c_{2n+2}\sigma(2n+1:q)=\frac{a_{2k}}{q}\kappa_{2n+2}(2k+1:q)+\frac{c_{2k}}{q}\kappa_{2n+2}(2k-1:q), \quad (4.15)$$

$$a_{2n+2}\sigma(2n+3:q)+c_{2n+2}\sigma(2n+1:q)=\frac{a_{2n}}{q}\sigma(2n+2:q)+\frac{c_{2n}}{q}\kappa_{2n+2}(2n-1:q). \quad (4.16)$$

We distinguish two cases:
Case 1: $k\neq 0$. (4.12) yields

$$a_{2n+1}\kappa_{2n+2}(2k-1:q)+c_{2n+1}\sigma(2n:q)=\frac{A}{q}\sigma(2n+1:q).$$

[43] With regard to the precise relation between (4.11) and Lemma 2.4 (i), we note that the latter would read

$$\sigma(n)=\frac{n}{c_nh(n)}\ (n\in\mathbb{N})$$

in the present setting—which can be seen in the same way as in Section 2.3, however. Concerning the variant using the Christoffel–Darboux formula (cf. (2.20)), take into account that the Christoffel–Darboux formula reads

$$\frac{1}{c_nh(n)}\sum_{k=0}^{n-1}h(k)P_k^2(x)=P_n'(x)P_{n-1}(x)-P_{n-1}'(x)P_n(x)\ (n\in\mathbb{N},x\in\mathbb{R})$$

now (cf. [Chi78, I-Theorem 4.6]). Comparing the present setting and the setting which was underlying the original version of Lemma 2.4 (i), the role of a_0 is a rather different one: while in the first case $a_0=A$ is the normalization point of the sequence $(P_n(x))_{n\in\mathbb{N}_0}$, in the latter case the normalization point was always 1 and any $a_0\neq 1$ caused asymmetry.

Combining this with (4.13) and (4.14), we get $\kappa_{2n+2}(2k+1:q) = \kappa_{2n+2}(2k-1:q) = \kappa_{2n+2}(2n-1:q)$. Hence, (4.15) reduces to

$$a_{2n+2}\sigma(2n+3:q) + c_{2n+2}\sigma(2n+1:q) = \frac{A}{q}\kappa_{2n+2}(2n-1:q). \tag{4.17}$$

Case 2: $k=0$. Here, (4.15) reads $a_{2n+2}\sigma(2n+3:q) + c_{2n+2}\sigma(2n+1:q) = \frac{A}{q}\kappa_{2n+2}(1:q)$. Since (4.13) and (4.14) imply that $\kappa_{2n+2}(1:q) = \kappa_{2n+2}(2n-1:q)$, this reduces to (4.17), too.

Note that both in Case 1 and Case 2 the former dependence on k has vanished. We now combine (4.16) and (4.17) and obtain that $\kappa_{2n+2}(2n-1:q) = \sigma(2n+2:q)$. Consequently, (4.14) and (4.17) simplify to

$$a_{2n+1}\sigma(2n+2:q) + c_{2n+1}\sigma(2n:q) = \frac{A}{q}\sigma(2n+1:q), \tag{4.18}$$

$$a_{2n+2}\sigma(2n+3:q) + c_{2n+2}\sigma(2n+1:q) = \frac{A}{q}\sigma(2n+2:q). \tag{4.19}$$

Applying (4.11) another time, we can deduce from (4.18) that

$$\frac{A}{q} = \frac{1}{q}\frac{1-q^{2n+2}}{1-q^{2n+1}}c_{2n+1} + q\frac{1-q^{2n}}{1-q^{2n+1}}\frac{a_{2n}}{c_{2n}}c_{2n+1} = \frac{1}{q}\frac{1-\alpha^2q^{4n+2}}{1-q^{2n+1}}c_{2n+1},$$

which shows that (4.3) is satisfied for $2n+1$. Knowing this, we finally can apply (4.11) to (4.19) and obtain

$$\frac{A}{q} = \frac{1}{q}\frac{1-q^{2n+3}}{1-q^{2n+2}}c_{2n+2} + q\frac{1-q^{2n+1}}{1-q^{2n+2}}\frac{a_{2n+1}}{c_{2n+1}}c_{2n+2} = \frac{1}{q}\frac{1-\alpha^2q^{4n+4}}{1-q^{2n+2}}c_{2n+2}.$$

So (4.3) holds true for $2n+2$, too, which finishes the induction.

4.3. A characterization of continuous q-ultraspherical polynomials in terms of the Askey–Wilson operator

Let A and $(P_n(x))_{n\in\mathbb{N}_0}$ be as in Subsection 4.1 again (property (RW) not required), and let $q\in(0,1)$. Following [IO11], we consider as well the Fourier coefficients w.r.t. the Askey–Wilson operator $\mathcal{D}_q$ as the Fourier coefficients w.r.t. the averaging operator $\mathcal{A}_q$ and use the following notation: for every $n\in\mathbb{N}_0$, let $\kappa_n(.|q):\mathbb{N}_0\to\mathbb{R}$ and $\alpha_n(.|q):\mathbb{N}_0\to\mathbb{R}$ be defined by

$$\kappa_n(k|q) := \int_{\mathbb{R}} \mathcal{D}_qP_n(x)P_k(x)\,\mathrm{d}\mu(x),$$
$$\alpha_n(k|q) := \int_{\mathbb{R}} \mathcal{A}_qP_n(x)P_k(x)\,\mathrm{d}\mu(x).$$

Obviously, $\kappa_n(k|q)=0$ if $k\geq n$ or if $n-k$ is even, and $\alpha_n(k|q)=0$ if $k\geq n+1$ or if $n-k$ is odd. As analogue to σ and $\sigma(.:q)$, we write $\sigma(.|q):\mathbb{N}\to\mathbb{R}\backslash\{0\}$,

$$\sigma(n|q) := \kappa_n(n-1|q).$$

We have the following analogue [Kah16, Theorem 2.3] to Theorem 4.4 and Theorem 4.5:

Theorem 4.6. *Under the conditions of Theorem 4.3, the following are equivalent:*

(i) $P_n(x) = P_n(x;\beta|q)$ $(n\in\mathbb{N}_0)$,

(ii) $\mathcal{D}_qP_n(x) = \mathcal{D}_qP_n(A)P_{n-1}^*(x)$ $(n\in\mathbb{N})$,

(iii) $\mathcal{D}_qP_{2n-1}(x) = \mathcal{D}_qP_{2n-1}(A)P_{2n-2}^*(x)$ $(n\in\mathbb{N})$,

(iv) $(A^2 - x^2)\mathcal{D}_q P_n(x)$ *is orthogonal to* $P_0(x), \ldots, P_{n-2}(x)$ $(n \geq 2)$,

(v) $(A^2 - x^2)\mathcal{D}_q P_{2n-1}(x)$ *is orthogonal to* $P_0(x), P_2(x), \ldots, P_{2n-4}(x)$ $(n \geq 2)$,

(vi) *one has*

$$\kappa_{2n+1}(2n-2|q) = \sigma(2n+1|q) \; (n \in \mathbb{N}),$$

and for every $n \in \mathbb{N}$ *there is a* $k \in \{0, \ldots, n-1\}$ *such that*

$$\kappa_{2n+3}(2k|q) = \sigma(2n+3|q).$$

The analogue to Lemma 2.4 (i) and (4.11) is

$$\sigma(n|q) = q^{\frac{1-n}{2}} \frac{1-q^n}{1-q} \frac{1}{c_n h(n)}, \; \frac{\sigma(n|q)}{\sigma(n+1|q)} = \sqrt{q} \frac{1-q^n}{1-q^{n+1}} \frac{a_n}{c_n} \; (n \in \mathbb{N}) \tag{4.20}$$

[IO11, proof of Theorem 5.2].[44] The analogue to (4.12) is

$$a_n \kappa_{n+1}(k|q) + c_n \kappa_{n-1}(k|q) = \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right) [a_k \kappa_n(k+1|q) + c_k \kappa_n(k-1|q)] + \alpha_n(k|q) \\ (n, k \in \mathbb{N}_0) \tag{4.21}$$

[IO11, proof of Theorem 5.2]; the latter relies on the product rule

$$\mathcal{D}_q[PQ](x) = \mathcal{D}_q P(x) \mathcal{A}_q Q(x) + \mathcal{A}_q P(x) \mathcal{D}_q Q(x) \; (P(x), Q(x) \in \mathbb{R}[x])$$

[Ism09, (12.1.22)] for $\mathcal{D}_q$.

The proof of Theorem 4.6 is considerably more involved than the proof of Theorem 4.5 (or Theorem 4.4, which can be established by just “copying” the proof of Theorem 4.5 given in the previous subsection): this is due to the fact that (4.21) is considerably more complicated than (4.12) because (4.21) simultaneously involves the Fourier coefficients w.r.t. $\mathcal{A}_q$, i.e., $(\alpha_n(.|q))_{n \in \mathbb{N}_0}$. To overcome this difficulty, we need several more preliminaries. First, we need the (general) relations

$$\alpha_n(n|q) = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2h(n)} \; (n \in \mathbb{N}_0), \tag{4.22}$$

$$\alpha_n(n-2|q) = \underbrace{\frac{(1-q)\left(q^{\frac{n-2}{2}} - q^{-\frac{n}{2}}\right)}{2}}_{=:D_n} \frac{\frac{n}{4} - \sum_{k=1}^{n-1} a_{k-1} c_k}{h(n-2) a_{n-2} a_{n-1}} \; (n \geq 2) \tag{4.23}$$

[IO11, Lemma 5.1].[45] Next, we introduce auxiliary functions $\beta_n(.|q) : \{0, \ldots, n-2\} \to \mathbb{R}$,

$$\beta_n(k|q) := \int_{\mathbb{R}} \mathcal{A}_q[x P_n(x)] P_k(x) \, \mathrm{d}\mu(x) \; (n \geq 2),$$

and use the following result:

Lemma 4.1. *For each* $n \in \mathbb{N}$, *the recursion coefficients* $c_1, \ldots, c_n$ *determine* $\alpha_{n+1}(.|q)|_{\{0,\ldots,n\}}$ *and* $\beta_{n+1}(.|q)$ *uniquely.*

The proof of Lemma 4.1, which is [Kah16, Lemma 3.1], can be found in our paper [Kah16]. Finally, we shall need the following:

[44] Take into account Footnote 43.

[45] Without going into detail, we note at this stage that the proof of [IO11, Lemma 5.1] given in the cited reference contains a little mistake concerning the case $n = 2$. It can easily be corrected by a short calculation.

Lemma 4.2. *Let* $\beta \in \left(0, \frac{1}{\sqrt{q}}\right)$ *and* $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$. *Then*

$$\frac{\alpha_{n+1}(n+1-2k|q)}{\sigma(n+1|q)} = \frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}$$

for each $n \in \mathbb{N}$ *and* $k \in \left\{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\right\}$.

Our proof of Lemma 4.2, which is [Kah16, Lemma 3.2] and makes use of Theorem 4.3, can be found in [Kah16], too; the special case $k = 1$ is already contained in [IO11, proof of Theorem 5.2].

We now give a brief sketch of the proof of Theorem 4.6; the details can be found in our paper [Kah16]. As in the proof of Theorem 4.5, the non-obvious part is the implication "(vi) $\Rightarrow$ (i)", and we use induction to show that (vi) implies that (4.4) is satisfied for all $n \in \mathbb{N}$. The validity for $n = 1$ is clear. Hence, applying the first part of Lemma 4.1 and also Lemma 4.2, we have

$$\alpha_2(0|q) = \frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\sigma(2|q).$$

Now (4.21) yields

$$a_2\sigma(3|q) + c_2\sigma(1|q) = \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right) A\sigma(2|q) + \alpha_2(0|q) = \frac{1}{2}\left(\sqrt{\frac{\beta}{q}} + \sqrt{\frac{q}{\beta}}\right)\sigma(2|q);$$

applying (4.20), we get

$$\frac{1}{2}\left(\sqrt{\frac{\beta}{q}} + \sqrt{\frac{q}{\beta}}\right) = \frac{1-q^3}{\sqrt{q}(1-q^2)}c_2 + \frac{\sqrt{q}}{1+q}\frac{a_1}{c_1}c_2 = \left(\sqrt{\frac{\beta}{q}} + \sqrt{\frac{q}{\beta}}\right)\frac{1-\beta q^2}{\sqrt{\beta}(1-q^2)}c_2.$$

Consequently, (4.4) holds true for $n = 2$. Let $n \in \mathbb{N}$ be arbitrary but fixed, assume that $1, \ldots, 2n$ satisfy (4.4), and choose $k \in \{0, \ldots, n-1\}$ such that $\kappa_{2n+3}(2k|q) = \sigma(2n+3|q)$. In a similar way as in the proof of Theorem 4.5—and taking into account as well the first part of Lemma 4.1 as Lemma 4.2—we obtain

$$\begin{aligned} &a_{2n+2}\sigma(2n+3|q) + c_{2n+2}\sigma(2n+1|q) = \\ &= \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right)[a_{2n}\sigma(2n+2|q) + c_{2n}\kappa_{2n+2}(2n-1|q)] + \alpha_{2n+2}(2n|q), \end{aligned} \tag{4.24}$$

and

$$a_{2n+2}\sigma(2n+3|q) + c_{2n+2}\sigma(2n+1|q) = \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right) A\kappa_{2n+2}(2n-1|q) + \alpha_{2n+2}(2k|q), \tag{4.25}$$

as analogues to (4.16) and (4.17).

At this stage, an important difference to the proof of Theorem 4.5 occurs: in contrast to (4.17), in (4.25) the dependence on k has *not* vanished. Furthermore, $\alpha_{2n+2}(2k|q)$ in (4.25) is *not* determined by the induction hypothesis (i.e., $c_1, \ldots, c_{2n}$), nor is $\alpha_{2n+2}(2n|q)$ in (4.24). In contrast to the proof of Theorem 4.5 and the equations (4.16) and (4.17), (4.24) and (4.25) do *not* immediately yield the (potentially helpful) conclusion $\kappa_{2n+2}(2n-1|q) = \sigma(2n+2|q)$ (nevertheless, the latter will turn out to be true later). These circumstances are the actual difficulty in the proof of Theorem 4.6. The key idea to overcome these problems consists of the following three ingredients:

- via (4.23), relate $\alpha_{2n+2}(2n|q)$ to $\alpha_{2n+1}(2n-1|q)$, which—due to the first part of Lemma 4.1—is determined by the induction hypothesis;

- consider the auxiliary function $\beta_{2n+1}(.|q)$, apply the induction hypothesis and the second part of Lemma 4.1 to $\beta_{2n+1}(2k|q)$ and relate it to $\alpha_{2n+2}(2k|q)$ via $a_{2n+1}\alpha_{2n+2}(2k|q) = \beta_{2n+1}(2k|q) - c_{2n+1}\alpha_{2n}(2k|q)$;
- simultaneously involve the sequence $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$ of continuous q-ultraspherical polynomials itself—from now on, we write an additional tilde when explicitly referring to $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$ (in particular, $\widetilde{c}_{2n+1}\left(=\frac{\sqrt{\beta}}{2}\frac{1-q^{2n+1}}{1-\beta q^{2n+1}}\right)$).

Without going into detail, we note that this approach yields

$$\frac{a_{2n+1}\alpha_{2n+2}(2n|q)}{\sigma(2n+1|q)} = \underbrace{\frac{(1-q)^2}{2q}a_{2n}}_{=:A_1(n)}c_{2n+1} + \underbrace{\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\frac{D_{2n+2}}{D_{2n+1}}c_{2n} - \frac{(1-q)^2}{8q}}_{=:A_2(n)} \tag{4.26}$$

and

$$\begin{aligned}\frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)} =& \underbrace{-\frac{(\beta-1)(1-q)}{4\sqrt{\beta}}\frac{1-q^{2n}}{1-q^{2n+1}}\frac{a_{2n}}{c_{2n}}}_{=:A_3(n)}c_{2n+1} \\ &+ \underbrace{\frac{(\beta-1)(1-q)}{4\sqrt{\beta}(q-q^{2n+2})}\left[(1-q^{2n+2})+(q-q^{2n+1})\frac{a_{2n}}{c_{2n}}\right]\widetilde{c}_{2n+1}}_{=:A_4(n)}.\end{aligned} \tag{4.27}$$

Moreover (again we skip any details), the difference of (4.24) and (4.25) simplifies to

$$\begin{aligned}&\frac{a_{2n+1}\alpha_{2n+2}(2n|q)}{\sigma(2n+1|q)} - \frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)} = \\ &= \underbrace{-\frac{q+1}{2(q-q^{2n+2})}a_{2n}\left[(1-q^{2n+2})+(q-q^{2n+1})\frac{a_{2n}}{c_{2n}}\right]}_{=:A_5(n)}c_{2n+1} + \underbrace{\frac{(\beta+q)(q+1)}{4q\sqrt{\beta}}a_{2n}}_{=:A_6(n)}.\end{aligned} \tag{4.28}$$

Combining (4.26), (4.27) and (4.28), we obtain

$$[A_1(n) - A_3(n) - A_5(n)]c_{2n+1} = -[A_2(n) - A_4(n) - A_6(n)], \tag{4.29}$$

where the former dependence on k has vanished now. $A_1(n),\ldots,A_6(n)$ are determined by the induction hypothesis, and

$$A_1(n) - A_3(n) - A_5(n) \neq 0 \tag{4.30}$$

(the latter can be seen by a short elementary calculation). Hence, we could compute c_{2n+1} as the quotient $-\frac{A_2(n)-A_4(n)-A_6(n)}{A_1(n)-A_3(n)-A_5(n)}$, which—after tedious calculations—would yield the desired result.

However, also a much faster argument is available at this stage: as an immediate consequence of the already verified direction "(i) ⇒ (vi)" (or of Theorem 4.3), the argument up to this point would remain valid if $(P_n(x))_{n\in\mathbb{N}_0}$ was replaced by $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$; hence, (4.29) would remain valid if c_{2n+1} was replaced by $\widetilde{c}_{2n+1}$. So

$$[A_1(n) - A_3(n) - A_5(n)](c_{2n+1} - \widetilde{c}_{2n+1}) = 0.$$

Consequently, $c_{2n+1} = \widetilde{c}_{2n+1}$, i.e., (4.4) is valid for $2n+1$.[46]

[46] This "replacement argument" has another advantage: it would not even have been necessary to compute $A_2(n)$, $A_4(n)$ and $A_6(n)$ as above; instead, it would have been enough to observe that $\frac{a_{2n+1}\alpha_{2n+2}(2n|q)}{\sigma(2n+1|q)} - A_1(n)c_{2n+1}$, $\frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)} - A_3(n)c_{2n+1}$ and $\frac{a_{2n+1}\alpha_{2n+2}(2n|q)}{\sigma(2n+1|q)} - \frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)} - A_5(n)c_{2n+1}$ are fixed by $c_1,\ldots,c_{2n}$. The values of $A_1(n)$, $A_3(n)$ and $A_5(n)$ (which are also determined by $c_1,\ldots,c_{2n}$) are only needed to show (4.30).

Since besides $c_1, \dots, c_{2n}$ also c_{2n+1} is known now, Lemma 4.1 and Lemma 4.2 imply that both (4.24) and (4.25) reduce to

$$a_{2n+2}\sigma(2n+3|q) + c_{2n+2}\sigma(2n+1|q) = \frac{1}{2}\left(\sqrt{\frac{\beta}{q}} + \sqrt{\frac{q}{\beta}}\right)\sigma(2n+2|q). \tag{4.31}$$

Following the remaining proof of Theorem 4.5, we can deduce from (4.31) that (4.4) is valid for $2n+2$, too, which finishes the induction.

4.4. A characterization of continuous q-ultraspherical polynomials in terms of the averaging operator

Let A and $(P_n(x))_{n\in\mathbb{N}_0}$ be as in Subsection 4.1 another time (property (RW) not required), let $q \in (0,1)$ again, and use the notation of the previous subsection. The following theorem [Kah16, Theorem 2.4] is the announced characterization in terms of the averaging operator $\mathcal{A}_q$:

Theorem 4.7. *Under the conditions of Theorem 4.3 and the additional assumption that $\beta \le 1$, the following are equivalent:*

(i) $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$,

(ii) *the quotient* $\frac{\alpha_{n+1}(n-1|q)}{\sigma(n+1|q)}\left(= \frac{\int_{\mathbb{R}}\mathcal{A}_q P_{n+1}(x)P_{n-1}(x)\,\mathrm{d}\mu(x)}{\int_{\mathbb{R}}\mathcal{D}_q P_{n+1}(x)P_n(x)\,\mathrm{d}\mu(x)}\right)$ *is independent of* $n \in \mathbb{N}$.

If the condition $\beta \le 1$ is dropped, then only "(i) $\Rightarrow$ (ii)" remains valid.

Besides the averaging operator $\mathcal{A}_q$, Theorem 4.7 involves $\sigma(n+1|q)$ (and consequently the Askey–Wilson operator $\mathcal{D}_q$). However, recall that $\sigma(n+1|q)$ can be expressed in terms of the recurrence coefficients in a considerably more convenient way than the analogue concerning $\alpha_{n+1}(n-1|q)$, cf. (4.20) and (4.23). Therefore, we think it is indeed justified to call Theorem 4.7 a characterization "in terms of the averaging operator".

We briefly sketch the proof of Theorem 4.7 (and refer to our paper [Kah16] for the details again): first not imposing the additional condition $\beta \le 1$, the implication "(i) $\Rightarrow$ (ii)" is an obvious consequence of Lemma 4.2, and—using (4.20) and (4.23)—we can reformulate (ii) as

$$\left[a_n + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+2}}{1-q^{n+1}}\right]c_{n+1} = \frac{1}{4} + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+1}}{1-q^n}c_n \ (n\in\mathbb{N}). \tag{4.32}$$

Now assume that (ii) is satisfied, and also that $\beta \le 1$. We use induction to verify (4.4) for all $n \in \mathbb{N}$: this is clear for $n=1$, so let $n \in \mathbb{N}$ be arbitrary but fixed and assume (4.4) to be satisfied for n. Short calculations yield

$$a_n + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+2}}{1-q^{n+1}} = \frac{1-\beta q^{n+1}}{2\sqrt{\beta}(1-q^{n+1})}\frac{2-q-q^{n+1}-(1+q^n-2q^{n+1})\beta}{(1-q)(1-\beta q^n)} \tag{4.33}$$

and

$$\frac{1}{4} + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+1}}{1-q^n}c_n = \frac{2-q-q^{n+1}-(1+q^n-2q^{n+1})\beta}{4(1-q)(1-\beta q^n)}. \tag{4.34}$$

Since $\beta \le 1$, we have

$$2-q-q^{n+1}-(1+q^n-2q^{n+1})\beta \neq 0. \tag{4.35}$$

Hence, combining (4.32), (4.33), (4.34) and (4.35), we obtain that (4.4) is also valid for $n+1$, which finishes the induction.

Concerning the second part of the theorem (i.e., that "(ii) $\Rightarrow$ (i)" is not generally true if the additional assumption $\beta \leq 1$ is dropped), via $A := \frac{9}{4\sqrt{5}}$, $c_1 := \frac{\sqrt{5}}{3}$, $c_2 := \frac{1}{2}$,

$$c_{n+1} := \frac{\sqrt{5} - \frac{2^{n+1}-1}{2^n-1}c_n}{9 - 4\sqrt{5}c_n - \frac{2^{n+2}-1}{2^{n+1}-1}} \ (n \geq 2)$$

one can recursively define a symmetric RWPS $(P_n(x))_{n\in\mathbb{N}_0}$ such that, putting $q := \frac{1}{2}$ and $\beta := \frac{5}{4}$, the conditions of Theorem 4.3 and (ii) are satisfied. However, $(P_n(x))_{n\in\mathbb{N}_0} \neq (P_n(x;\beta|q))_{n\in\mathbb{N}_0}$ because $c_2 = \frac{1}{2} \neq \frac{3\sqrt{5}}{11} = \frac{\sqrt{\beta}}{2}\frac{1-q^2}{1-\beta q^2}$ (cf. (4.4)).

Remark 4.2. Although the condition $\beta \leq 1$ cannot be completely removed, the equivalence "(i) $\Leftrightarrow$ (ii)" of Theorem 4.7 remains valid if $\beta \leq 1$ is replaced by one of the weaker conditions

$$\forall n \in \mathbb{N} : \frac{2-q-q^{n+1}}{1+q^n-2q^{n+1}} \neq \beta,$$
$$\forall n \in \mathbb{N} : \left(\frac{2-q-q^{n+1}}{1+q^n-2q^{n+1}} = \beta \Rightarrow c_{n+1} = \frac{\sqrt{\beta}}{2}\frac{1-q^{n+1}}{1-\beta q^{n+1}}\right),$$

cf. the detailed proof of Theorem 4.7 given in our paper [Kah16]. Since the function $(0,\infty) \to (1,\infty)$, $x \mapsto \frac{2-q-q^{x+1}}{1+q^x-2q^{x+1}}$ is strictly increasing, there exists at most one $n \in \mathbb{N}$ such that $\frac{2-q-q^{n+1}}{1+q^n-2q^{n+1}} = \beta$, and if $\beta \leq 1$, then there is none.

Remark 4.3. The ultraspherical polynomials appear as limiting cases of as well the discrete as the continuous q-ultraspherical polynomials (cf. Subsection 4.1); consequently, it is a natural question to ask whether Theorem 4.5 or Theorem 4.6 implies Theorem 4.4 by just "taking the limit" $q \to 1$. However, there is no obvious reason why the "characterizing" directions "(vi) $\Rightarrow$ (i)" should remain valid if one passes to limits. The same is the case w.r.t. the original results of Lasser–Obermaier and Ismail–Obermaier: the directions "(ii) $\Rightarrow$ (i)" of Theorem 4.1, Theorem 4.2 and Theorem 4.3 are mutually independent from each other.
In this context, we make the following observation w.r.t. Theorem 4.7: let $\alpha > -1$. There exists some $q_0 \in [0,1)$ such that $\frac{2-q-q^{n+1}}{1+q^n-2q^{n+1}} \neq q^{\alpha+\frac{1}{2}}$ for all $n \in \mathbb{N}$ and $q \in (q_0,1)$.[47] Hence, if $q \in (q_0,1)$, if $A = \frac{\sqrt{q^{\alpha+\frac{1}{2}}}}{2} + \frac{1}{2\sqrt{q^{\alpha+\frac{1}{2}}}}$, if $c_1 = \frac{\sqrt{q^{\alpha+\frac{1}{2}}}}{2}\frac{1-q}{1-q^{\alpha+\frac{3}{2}}}$ and if $\frac{\alpha_{n+1}(n-1|q)}{\sigma(n+1|q)}$ is independent of $n \in \mathbb{N}$, then $P_n(x) = P_n(x;q^{\alpha+\frac{1}{2}}|q)$ $(n \in \mathbb{N}_0)$ as a consequence of Theorem 4.7 and Remark 4.2.[48] Despite the fact that $\lim_{q\to 1} P_n(x;q^{\alpha+\frac{1}{2}}|q) = P_n^{(\alpha)}(x)$ $(n \in \mathbb{N}_0, x \in \mathbb{R})$ (4.5), the "limiting conditions" $A = 1$, $c_1 = \frac{1}{2\alpha+3}$ and "0 is independent of $n \in \mathbb{N}$" do *not* enforce that $P_n(x) = P_n^{(\alpha)}(x)$ $(n \in \mathbb{N}_0)$, of course. Since the limiting case of Theorem 4.7 (ii) is a trivial property that is always true (because $\mathcal{A}_q$ becomes the identity as $q \to 1$), one has *no* analogue to Theorem 4.7 for the class of ultraspherical polynomials.

[47] This can be seen as follows: let $u,v : (0,1) \to \mathbb{R}$ be defined by $u(q) := 2 - q - q^{\alpha+\frac{1}{2}}$, $v(q) := q + q^{\alpha+\frac{1}{2}} - 2q^{\alpha+\frac{3}{2}}$. Since $\lim_{q\to 1} u(q) = \lim_{q\to 1} v(q) = 0$, and since u and v are continuously differentiable on $(0,1)$ with $\lim_{q\to 1} u'(q) = \lim_{q\to 1} v'(q) = -\alpha - \frac{3}{2} < 0$, there exists some $q_0^* \in (0,1)$ such that $u(q) > 0$ and $v(q) > 0$ for all $q \in [q_0^*,1)$. Now consider the function $w : [q_0^*,1) \to \mathbb{R}$, $w(q) := \frac{\log\frac{u(q)}{v(q)}}{\log q}$. After several applications of L'Hôpital's rule, one sees that w converges to a real number if $q \to 1$ (in fact, $\lim_{q\to 1} w(q) = -\frac{4\alpha+2}{2\alpha+3}$). So w is bounded; in particular, there exists some $N \in \mathbb{N}$ such that $n > w(q)$ for all $q \in [q_0^*,1)$ and $n \in \mathbb{N}$ with $n > N$. The latter is easily seen to imply that $\frac{2-q-q^{n+1}}{1+q^n-2q^{n+1}} > q^{\alpha+\frac{1}{2}}$ for all $q \in [q_0^*,1)$ and $n \in \mathbb{N}$ with $n > N$. Since each of the functions $(0,1) \to \mathbb{R}$, $q \mapsto \frac{2-q-q^{n+1}}{1+q^n-2q^{n+1}} - q^{\alpha+\frac{1}{2}}$, $n \in \{1,\ldots,N\}$, has at most finitely many zeros, we obtain indeed some $q_0 \in [0,1)$ with the desired property.

[48] In the special case $\alpha \geq -\frac{1}{2}$, the argument considerably simplifies because $q^{\alpha+\frac{1}{2}} \leq 1$ then; consequently, one can apply Theorem 4.7 in its original formulation (and does not have to take into account Remark 4.2), and one may choose $q_0 = 0$.

Outlook

One of the things that make science and research so interesting and exciting is that every solved problem immediately leads to new questions. This short section is devoted to some suggestions for future research. We first present some ideas without going into detail, and then briefly discuss two selected problems from which we think that they are of particular interest—and which we plan to solve in the course of our postdoctoral research.

In Section 2, we considered polynomial hypergroups and obtained a sufficient criterion (Theorem 2.1) and a necessary criterion (Theorem 2.2) for weak amenability of their ℓ^1-algebras $\ell^1(h)$. This enabled us to characterize weak amenability (and also point amenability) for the classes of Jacobi polynomials (Theorem 3.1), symmetric Pollaczek polynomials (Theorem 3.2), associated ultraspherical polynomials (Theorem 3.3), random walk polynomials (in the sense of Section 3; Proposition 3.1) and cosh-polynomials (Proposition 3.2). Also the notions of amenability and right character amenability are solved problems concerning theses classes, cf. Section 3. However, it would be interesting to clarify the situation w.r.t. approximate amenability (which, since $\ell^1(h)$ is commutative and unital, coincides with pseudo-amenability and implies weak amenability), w.r.t. pointwise amenability (which implies approximate amenability at this stage) or w.r.t. other amenability notions for Banach algebras (cf. Section 1.2 and Section 1.4). Recall that the problem of pointwise amenability seems to be open even for L^1-algebras of locally compact groups.

It would also be interesting to study more general structures (for instance, hypergroups which are not located at the fruitful crossing point to the theory of orthogonal polynomials), or to study further examples of polynomial hypergroups. Concerning the latter, there is a very interesting class which comes from q-calculus and for which property (P), as well as right character amenability of $\ell^1(h)$, are known to hold: this is the class of little q-Legendre polynomials, and we shall discuss it in some more detail below (as the first of the two selected problems mentioned above).

Recall that, in accordance with previous research on polynomial hypergroups and their ℓ^1-algebras, we considered point derivations (and also the concepts of α- and φ-amenability) w.r.t. *Hermitian* characters. There are good reasons for this restriction: on the one hand, the general theory and the available harmonic analysis becomes richer (cf. Abelian locally compact groups). On the other hand, for our main examples (Jacobi, symmetric Pollaczek and associated ultraspherical polynomials) all characters *are* Hermitian. Nevertheless, the general question about non-symmetric characters suggests interesting problems: for instance, it is an interesting question to ask whether there holds a more general version—concerning non-symmetric characters—of the full assertion of Proposition 2.1 (i) (whose proof essentially relied on the boundedness of the derivatives of the Chebyshev polynomials of the second kind at each $x \in (-1,1)$): since the set $\left\{\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\frac{1}{2})}(x) : n \in \mathbb{N}_0\right\}$ is unbounded for every $x \in \mathbb{C}\backslash(-1,1)$ (use (3.16) and the fact that $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) = \mathsf{supp}\,\mu = [-1,1]$ for the sequence $\left(P_n^{(\frac{3}{2})}(x)\right)_{n\in\mathbb{N}_0}$), at least the proof of Proposition 2.1 (i) breaks down as soon as one does not restrict oneself to Hermitian characters. Moreover, we cannot draw any conclusion concerning the size of $\mathcal{X}^b(\mathbb{N}_0)\backslash\widehat{\mathbb{N}_0}$ (because the equalities $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) = \mathsf{supp}\ \mu = [-1,1]$ are also valid for the Chebyshev polynomials of the second kind).

With regard to Theorem 4.5, Theorem 4.6 and Theorem 4.7: results of our Master's thesis [Kah12] (cf. Section 4.1) suggest to look for generalizations to suitable q-analogues of the Jacobi and generalized Chebyshev polynomials, and the results of Ismail–Simeonov [IS12] suggest to look for generalizations to symmetric Al-Salam–Chihara, symmetric Askey–Wilson, symmetric Meixner–Pollaczek or more abstract symmetric orthogonal polynomials. In particular, we plan to establish a characterization which is motivated by a certain q-extension of the generalized Chebyshev polynomials considered in [Mej10] (as another possible part of our PostDoc research). Our Master's thesis [Kah12] also suggests research motivated by the "backward shift analogues" (cf. [KLS10, (14.5.9)] and [KLS10, (14.10.25)], and cf. Section 4.1). However, we are even more interested in another type of generalization—which is very different from the situations studied in [IS12] and [Kah12]: this problem concerns sieved orthogonal polynomials and certain "limiting operators" of $\mathcal{D}_q$ and $\mathcal{A}_q$; we shall discuss it in some more detail below (as the second of the two selected problems mentioned above).

We now come to the announced details:

- Concerning both our sufficient criterion Theorem 2.1 and our necessary criterion Theorem 2.2 for weak amenability of ℓ^1-algebras of polynomial hypergroups, absolute continuity of the orthogonalization measure plays a crucial role. For instance, the proof of Theorem 2.1 relied on the fundamental lemma of the calculus of variations, and the proof of Theorem 2.2 made use of an integration by parts argument for the Radon–Nikodym derivatives. Considering purely discrete orthogonalization measures, the proof of Theorem 2.1 breaks down for several reasons; for instance, an application of the essential ingredient [MNT87, Theorem 12.1] on limiting behavior is no longer possible.

  An important example of a polynomial hypergroup with purely discrete orthogonalization measure is provided by the little q-Legendre polynomials: given $q \in (0,1)$, the corresponding sequence $(R_n(x;q))_{n\in\mathbb{N}_0}$ of little q-Legendre polynomials is determined by the orthogonalization measure

$$\mu(x) = \begin{cases} q^n(1-q), & x = 1-q^n \text{ with } n \in \mathbb{N}_0, \\ 0, & \text{else} \end{cases}$$

  and the normalization $R_n(1;q) = 1$ ($n \in \mathbb{N}_0$), or, equivalently, via

$$\begin{aligned} a_0 = \frac{1}{q+1},\ a_n &= q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})}\ (n \in \mathbb{N}), \\ c_n &= q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}\ (n \in \mathbb{N}), \\ b_n &\equiv 1 - a_n - c_n. \end{aligned} \tag{4.36}$$

  Property (P) is always satisfied, so a polynomial hypergroup is induced. One has

$$h(n) = \frac{1}{q^n} \frac{1-q^{2n+1}}{1-q}\ (n \in \mathbb{N}_0) \tag{4.37}$$

  and $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) = \operatorname{supp} \mu = \{1\} \cup \{1-q^n : n \in \mathbb{N}_0\}$. Consequently, Reiter's condition P_2 is satisfied. These basics are taken from [KLS10, Section 14.12.1], [Las05, Section 6] and [Las09b, Example 3]. Applying Lemma 2.4 (i), we obtain

$$\sigma(n) = \frac{(1-q)(1+q^n)}{1-q^n} n\ (n \in \mathbb{N}). \tag{4.38}$$

  As we already have recalled, the corresponding ℓ^1-algebra $\ell^1(h)$ is right character amenable (and consequently point amenable, cf. also [Las09b, Example 3]) yet non-amenable; while right character amenability follows from the fact that the hypergroup is 'of strong compact type' [FLS05], non-amenability is an immediate consequence of (4.37) and Theorem 1.4 (i).

Obviously, our sufficiency criterion Theorem 2.1 cannot be applied to establish weak amenability of $\ell^1(h)$. Moreover, the situation is consistent with our necessary criterion Theorem 2.2: in view of (4.38) and the first part of Theorem 2.2, neither (i) nor (ii) of Theorem 2.2 is violated.

Recall that our strategy for the proof of Theorem 2.1 relied on the sequence $(F_n)_{n\in\mathbb{N}_0} \subseteq c_{00}$ (cf. (2.1) and (2.2)), on a specific certain limit F of this sequence (specific in the sense that the limit contains adequate information about the individual underlying sequence of orthogonal polynomials), and on a density argument concerning the linear span of the set $\{T_m F : m \in \mathbb{N}_0\}$. Concerning the little q-Legendre polynomials, this approach breaks down at least for the following two reasons: on the one hand, (4.36) yields that $(R_n(x;q))_{n\in\mathbb{N}_0}$ is of 'Nevai class $M(1,0)$', i.e., $\alpha_n \to 0$ $(n\to\infty)$ and $\beta_n \to 1$ $(n\to\infty)$ (cf. the recurrence relation for the orthonormal sequence $(p_n(x))_{n\in\mathbb{N}_0}$). Applying another result of Nevai on limiting behavior [Nev79, Lemma 4.2.9, Theorem 4.2.10], this implies that $(F_n)_{n\in\mathbb{N}_0}$ converges pointwise to the trivial character ${}_1\alpha$—however, the trivial character is not specific to $(R_n(x;q))_{n\in\mathbb{N}_0}$ in any way; in other words: passing to the limit is accompanied by a considerable loss of information. On the other hand, and this is the even more serious aspect, $T_m\,{}_1\alpha = {}_1\alpha$ for every $m \in \mathbb{N}_0$ (1.10)—consequently, the linear span of $\{T_m\,{}_1\alpha : m \in \mathbb{N}_0\}$ is only one-dimensional.

For these reasons, we think that the question of whether the little q-Legendre polynomials bear weakly amenable ℓ^1-algebras or not is a rather nontrivial one. We plan to solve this problem with completely different methods—as a possible project for our PostDoc research—and do not go into further detail at this stage; we just mention that we have the conjecture that the ℓ^1-algebras *are* weakly amenable, and that we think that the asymptotics of the little q-Legendre polynomials [IW82] will play a crucial role for the solution.[49]

- Let $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ be a symmetric RWPS (in the sense of Section 4) with normalization point 1, i.e., $(P_n(x))_{n\in\mathbb{N}_0}$ is given by a recurrence relation of the form $P_0(x) = 1$,

$$xP_n(x) = a_n P_{n+1}(x) + c_n P_{n-1}(x) \ (n \in \mathbb{N}_0),$$

where $c_0 := 0$, $(c_n)_{n\in\mathbb{N}} \subseteq (0,1)$ and $a_n := 1 - c_n$ $(n \in \mathbb{N}_0)$. Let $k \in \mathbb{N}$, and let $(P_n(x;k))_{n\in\mathbb{N}_0}$ denote the 'k-sieved RWPS' which corresponds to $(P_n(x))_{n\in\mathbb{N}_0}$, i.e., $P_0(x;k) = 1$ and

$$xP_n(x;k) = a(n;k)P_{n+1}(x;k) + c(n;k)P_{n-1}(x;k) \ (n \in \mathbb{N}_0)$$

with

$$c(n;k) := \begin{cases} c_{\frac{n}{k}}, & k|n, \\ \frac{1}{2}, & \text{else} \end{cases}$$

and $a(n;k) := 1 - c(n;k)$ $(n \in \mathbb{N}_0)$; sometimes the additional expression "of the first kind" is used.[50] Sieved RWPS have been studied in [GVA88] and in [IL92] (the latter being part of a series of papers by Ismail et al.), for instance.

[49]Note that if the conjecture is true, one obtains as an obvious consequence that not all of the conditions (i) – (iv) of our sufficiency criterion Theorem 2.1 are necessary for weak amenability of $\ell^1(h)$ (cf. the notes at the end of Section 3.2).

[50]However, the expression "of the first kind" does also appear with a different meaning (in the context of certain sieved polynomials).

If $P_n(x) = P_n^{(\alpha)}(x)$ ($n \in \mathbb{N}_0$) for some $\alpha > -1$, then $(P_n(x;k))_{n\in\mathbb{N}_0}$ is given by

$$P_n(x;k) = \lim_{s\to 1} P_n\left(x; s^{\alpha k+\frac{k}{2}} | se^{\frac{2\pi i}{k}}\right) \tag{4.39}$$

for every $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$ [ASAA84], and we write $(P_n^{(\alpha)}(x;k))_{n\in\mathbb{N}_0}$ for these 'k-sieved ultraspherical polynomials'. Of course, (4.39) requires a more general definition of the continuous q-ultraspherical polynomials $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$ than considered in Section 4 (in particular, allowing q to be non-real, and allowing the moment functionals to be not necessarily restricted to the positive-definite case; we refrain from going into too much detail at this stage). Observe that (4.39) generalizes (4.5).

Having in mind (4.39), in view of Theorem 4.1 and Theorem 4.3 it is a natural question to ask whether there exists an analogous characterization of the k-sieved ultraspherical polynomials in terms of the "k-sieved Askey–Wilson operator" $\mathrm{D}_k : \mathbb{R}[x] \to \mathbb{R}[x]$ which we introduce via

$$\mathrm{D}_k T_n(x) := \lim_{s\to 1} \frac{\left(\sqrt{se^{\frac{2\pi i}{k}}}\right)^n - \left(\sqrt{se^{\frac{2\pi i}{k}}}\right)^{-n}}{\sqrt{se^{\frac{2\pi i}{k}}} - \frac{1}{\sqrt{se^{\frac{2\pi i}{k}}}}} U_{n-1}(x) = U_{n-1}\left(\left|\cos\frac{\pi}{k}\right|\right) U_{n-1}(x) \ (n \in \mathbb{N}_0)$$

(cf. (4.9) and (4.2)) and linear extension. Moreover, in view of Theorem 4.7 it is an interesting problem whether there exists an analogous characterization of the k-sieved ultraspherical polynomials in terms of the "k-sieved averaging operator" $\mathrm{A}_k : \mathbb{R}[x] \to \mathbb{R}[x]$ which we introduce via

$$\mathrm{A}_k T_n(x) := \lim_{s\to 1} \frac{\left(\sqrt{se^{\frac{2\pi i}{k}}}\right)^n + \left(\sqrt{se^{\frac{2\pi i}{k}}}\right)^{-n}}{2} T_n(x) = T_n\left(\left|\cos\frac{\pi}{k}\right|\right) T_n(x) \ (n \in \mathbb{N}_0)$$

(cf. (4.10) and (2.5)) and linear extension.

For $k \geq 2$, which shall be assumed from now on,[51] at first sight it might indeed seem reasonable to expect such analogous characterizations in terms of D_k and A_k.

However, these "sieved operators" are very different from $\frac{\mathrm{d}}{\mathrm{d}x}$, $\mathcal{D}_q$ and $\mathcal{A}_q$—and also from D_q, and from the operators corresponding to the Ismail–Simeonov characterizations in [IS12]—for the reason that D_k and A_k have *infinite-dimensional kernels* (because $U_{n-1}\left(\left|\cos\frac{\pi}{k}\right|\right) = U_{n-1}\left(\cos\frac{\pi}{k}\right)$ becomes zero for infinitely many $n \in \mathbb{N}_0$ (4.2), as well as $T_n\left(\left|\cos\frac{\pi}{k}\right|\right) = T_n\left(\cos\frac{\pi}{k}\right)$ becomes zero for infinitely many $n \in \mathbb{N}_0$ (2.5)). This important observation gives reason to expect a certain loss of information, or, in other words, an additional degree of freedom. We plan to solve this problem as another possible project for our PostDoc research and hence do not go into further detail at this stage; we just mention that we have the conjecture that (still assuming $k \geq 2$) analogues of Theorem 4.1, Theorem 4.3 and Theorem 4.7 do not yield characterizations of the k-sieved ultraspherical polynomials but characterizations of *arbitrary* k-sieved RWPS.[52]

[51] The case $k = 1$ is obvious because $\mathrm{D}_1 = \frac{\mathrm{d}}{\mathrm{d}x}$ (4.8) and $\mathrm{A}_1 = \mathrm{id}$: while there is a characterization of the 1-sieved ultraspherical polynomials—i.e., the ultraspherical polynomials themselves—in terms of D_1, namely just Theorem 4.1 itself, there is no analogue to Theorem 4.7 in terms of A_1 (cf. also Remark 4.3). Note that the formal limit "D_∞" coincides with D_1, as well as "A_∞" coincides with A_1.

[52] We call a symmetric RWPS 'k-sieved' (without further specification) if it is k-sieved w.r.t. another symmetric RWPS (or, equivalently, if $c_n = \frac{1}{2}$ if $k \nmid n$).

Symbols

In the following, we collect some symbols that have occurred in this thesis. It is not the purpose to give details at this stage, nor is it the purpose to mention any occurring symbol or to mention any possible meaning a symbol may have in the course of the thesis, which would become rather clumsy. For instance, there are various meanings of "$*$" or "$\langle .,. \rangle$", and it would not seem reasonable to collect all of them: on the contrary, various meanings become clear out of context.

The text of the thesis contains the precise definitions and hence can be read independently of the following collection. The purpose of the tables is to just provide a concise presentation which—if desired—may help the reader to recall important notation (with meanings as they are "typical" in important parts of the thesis).

To achieve this aim, we give short, "informal" explanations on the one hand, and refer to the page numbers for corresponding precise information on the other hand.

Polynomial hypergroups and their basic harmonic analysis

Symbol	*Informal*	*Precise*
$P_n(x)$	underlying orthogonal polynomials	p. 9
a_n, b_n, c_n	recurrence coefficients	p. 9
μ	orthogonalization (and Plancherel) measure	p. 9
$\rho_n(x)$	monic version of the polynomials	p. 10
$p_n(x)$	orthonormal version of the polynomials	p. 10
α_n, β_n	recurrence coefficients (cf. monic/orthonormal versions)	p. 10
$g(m, n; k)$	linearization coefficients (products)	p. 9
(P)	property (P)	p. 9
ω	hypergroup convolution	p. 10
$\widetilde{.}$	hypergroup involution	p. 10
T_n	translation/shift operator	p. 10
h	Haar function	p. 10
$\Vert\cdot\Vert_p$	$\Vert . \Vert_p$-norm (w.r.t. Haar measure)	p. 11
$\ell^p(h)$	ℓ^p-space (w.r.t. Haar measure)	p. 11
$\langle f, g \rangle$	duality relation	p. 11
$*$	convolution	p. 11
$\ell^1(h)$	ℓ^1-algebra of the polynomial hypergroup	p. 12
$\mathcal{X}^b(\mathbb{N}_0)$	(homeomorphic to) structure space of $\ell^1(h)$	p. 12
$\widehat{\mathbb{N}_0}$	(homeomorphic to) Hermitian structure space of $\ell^1(h)$	p. 9
${}_x\alpha$	(symmetric/Hermitian) character	p. 12
$\widehat{.}$	Fourier transformation (w.r.t. $\widehat{\mathbb{N}_0}$)	p. 12
$\mathcal{P}$	Plancherel isomorphism	p. 13
D_x	point derivation	p. 15
P_p	Reiter's condition P_p	p. 13
$M(0,1)$	Nevai class $M(0,1)$	p. 15
$M(1,0)$	Nevai class $M(1,0)$	p. 60

(Symmetric) orthogonal and random walk polynomial sequences

Symbol	*Informal*	*Precise*
$P_n(x)$	(symmetric) orthogonal polynomials	p. 44
A	normalization point	p. 44
a_n, c_n	recurrence coefficients	p. 44
μ	orthogonalization measure	p. 44
$h(n)$	$\frac{1}{\int_{\mathbb{R}} P_n^2(x)\,\mathrm{d}\mu(x)}$	p. 46
(RW)	property (RW)	p. 44
RWPS	(symmetric) random walk polynomial sequence	p. 44
μ^*	$(A^2 - x^2)\,\mathrm{d}\mu(x)$	p. 45
$P_n^*(x)$	orthogonal polynomials w.r.t. μ^*; $\frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)P_{n-2k}(x)}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)}$	p. 45; p. 47
$P_n(x;k)$	sieved polynomials	p. 60
$a(n;k), c(n;k)$	sieved recurrence coefficients	p. 60

Spaces of functions/measures

Symbol	*Informal*	*Precise*
$C(K)$	(complex-valued) continuous functions	p. 7
$C_b(K)$	bounded continuous functions	p. 7
$C_c(K)$	continuous functions with compact support	p. 7
$C_E(K)$	$f \in C_c(K)$ with $\mathrm{supp}\, f \subseteq E$	p. 7
$M(K)$	(complex) Radon measures	p. 7
$M_+(K)$	positive Radon measures	p. 7
$M^b(K)$	bounded Radon measures	p. 7
$M^1(K)$	probability measures	p. 7
c_0	null sequences	p. 11
c_{00}	sequences with finite support	p. 11
$\Delta(A)$	character/structure space	p. 4; p. 4
$\Delta_s(A)$	Hermitian character/structure space	p. 6

Operators and expansions

Symbol	*Informal*	*Precise*
ϵ_n	Fourier coefficients associated with $P_n(x)$	p. 11
f_n	Fourier coefficients associated with $p_n^2(x)$	p. 19
F_n	arithmetic mean of $f_0, \dots, f_n$	p. 19
κ_n	Fourier coefficients associated with $P_n'(x)$	p. 16; p. 47
$\sigma(n)$	κ_n evaluated at $n-1$	p. 19; p. 49
$\tau(n)$	κ_n evaluated at 0	p. 24
D_q	q-difference operator	p. 47
$\kappa_n(. : q)$	Fourier coefficients associated with $D_{q^{-1}}P_n(x)$	p. 50
$\sigma(n : q)$	$\kappa_n(. : q)$ evaluated at $n-1$	p. 50
$\mathcal{D}_q$	Askey–Wilson operator	p. 47
$\kappa_n(.\vert q)$	Fourier coefficients associated with $\mathcal{D}_q P_n(x)$	p. 52

$\sigma(n\|q)$	$\kappa_n(.\|q)$ evaluated at $n-1$	p. 52
$\mathcal{A}_q$	averaging operator	p. 50
$\alpha_n(.\|q)$	Fourier coefficients associated with $\mathcal{A}_q P_n(x)$	p. 52
$\beta_n(.\|q)$	Fourier coefficients associated with $\mathcal{A}_q[xP_n(x)]$	p. 53
D_k	sieved Askey–Wilson operator	p. 61
A_k	sieved averaging operator	p. 61

Specific classes of orthogonal polynomials

Symbol	*Informal*	*Precise*
$R_n^{(\alpha,\beta)}(x)$	Jacobi	p. 29
D_P	parameter region property (P) holds for Jacobi	p. 29
Δ	suitable subregion of D_P	p. 33
ϕ_n	$R_n^{(\alpha,\beta)}(x)$ evaluated at $x=-1$	p. 33
$P_n^{(\alpha)}(x)$	ultraspherical	p. 16
$T_n(x)$	Chebyshev of the first kind	p. 17
ω_T	Chebyshev weight	p. 19
$U_n(x)$	Chebyshev of the second kind	p. 45
$Q_n^{(\alpha,\lambda)}(x)$	symmetric Pollaczek	p. 29
D_Q	parameter region property (P) holds for symmetric Pollaczek	p. 30
$A_n^{(\alpha,\nu)}(x)$	associated ultraspherical	p. 30
$P_n(x;\alpha:q)$	discrete q-ultraspherical	p. 45
$P_n(x;\beta\|q)$	continuous q-ultraspherical	p. 45
$P_n^{(\alpha)}(x;k)$	sieved ultraspherical	p. 61
$S_n^{(a,b)}(x)$	random walk (in the sense of Section 3)	p. 37
$\Psi_n^{(a)}(x)$	cosh	p. 42
$R_n(x;q)$	little q-Legendre	p. 59

Miscellaneous

Symbol	*Informal*	*Precise*
$\bullet, \circ$	module actions	p. 4
$\mathbb{C}_\varphi$	$\mathbb{C}$ as module via φ (character)	p. 4
$\mathcal{C}(K)$	nonvoid compact subsets	p. 7
conv	convex hull	p. 8
$\chi_A(x)$	1 if $x \in A$, 0 else	p. 16

$\mathbb{R}[x]$ denotes the polynomial ring over $\mathbb{R}$ (and the corresponding $\mathbb{R}$-linear space).

As usual, $(.)_p$ and ${}_pF_q$ denote the Pochhammer symbol (or shifted factorial) and the hypergeometric series (or hypergeometric function), respectively.

As usual, δ and a subscript means a Dirac measure (point mass) or a Dirac function (it becomes clear out of context whether a Dirac *measure* or a Dirac *function* shall be meant). We also use the well-known Kronecker delta.

For the sake of completeness and to avoid any confusion, we note that the set $\mathbb{N}$ shall not contain 0 (if 0 shall be contained, we write $\mathbb{N}_0$).

A. Publication [Kah15]

A.1. Summary

In our paper [Kah15] (sole author), we study two amenability properties for ℓ^1-algebras of polynomial hypergroups: on the one hand, we are interested in 'weak amenability' of these ℓ^1-algebras, i.e., in the nonexistence of nonzero bounded derivations into ℓ^∞; on the other hand, we consider the nonexistence of nonzero bounded point derivations w.r.t. symmetric characters (regarded as a global property, 'point amenability'). Originally coming from abstract harmonic and functional analysis (cohomology), it is known that these properties correspond to questions concerning the derivatives of the underlying orthogonal polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ and concerning certain associated Fourier coefficients (Lasser 2007, 2009). Despite these rather concrete correspondencies, deciding whether a given polynomial hypergroup bears a weakly or point amenable ℓ^1-algebra requires often deep knowledge about $(P_n(x))_{n\in\mathbb{N}_0}$.

Polynomial hypergroups are interesting for several reasons. In our opinion, the most important motivation for considering these structures is that they offer an elegant possibility to study orthogonal polynomials via methods from functional and harmonic analysis—in particular, via the machinery provided by Gelfand theory; for instance, the Hermitian characters can be identified with a compact subset of $\mathbb{R}$ which contains the support of the orthogonalization measure. On the other side of the coin, polynomial hypergroups provide an abundance of interesting hypergroups (and corresponding L^1-algebras) which are based on the theory of orthogonal polynomials and special functions, and which are very different from any locally compact group. For instance, even point amenability—which is the weaker of the two amenability properties we focus on in [Kah15]—is often not satisfied for ℓ^1-algebras of polynomial hypergroups (in contrast to the group case). The results of [Kah15] can be divided into two parts: in Section 2, we give some general results on point and weak amenability of ℓ^1-algebras of polynomial hypergroups, whereas from Section 3 on we consider concrete examples and, in particular, obtain characterizations of these amenability properties for the classes of Jacobi polynomials (Theorem 3.1), symmetric Pollaczek polynomials (Theorem 4.1) and associated ultraspherical polynomials (Theorem 5.1) as main results—by identifying the corresponding parameter regions in each case. In doing so, we shall also obtain examples of polynomial hypergroups with weakly amenable but non-amenable ℓ^1-algebra (e.g., certain ultraspherical polynomials), which solves a problem which has been open for some years.

The main results of Section 2 are a necessary criterion (Theorem 2.2) and a sufficient criterion (Theorem 2.3) for weak amenability; later in the paper, we will apply these to situations where more explicit computations seem to be out of reach. The criteria and their proofs use shift operators, certain regularity and smoothness conditions on the orthogonalization measures, growth conditions (e.g., on the Haar measures), certain limiting behavior of orthogonal polynomials, the Plancherel isomorphism and several further ingredients such as suitable approximation, expansion and convergence arguments or the fundamental lemma of the calculus of variations. Concerning Theorem 3.1, Theorem 4.1 and Theorem 5.1, every result requires its own specific analytical techniques from the theory of orthogonal polynomials and special functions. Theorem 3.1 is based on our sufficient criterion Theorem 2.3, suitable transformation formulas for hypergeometric series, the Stolz–Cesàro theorem, the asymptotics of the Jacobi polynomials, and inheritance via homomorphisms. Theorem 4.1 uses a transformation of the Pollaczek polynomials into a related system whose asymptotics are easier to handle ('random walk polynomials'). Theorem 5.1 relies on our necessary criterion Theorem 2.2, on the interplay between hypergeometric and Fourier series, on Pringsheim's theorem, and on the location of the zeros of hypergeometric functions. Besides these main results, [Kah15] contains necessary criteria for point amenability of ℓ^1-algebras of polynomial hypergroups. Moreover, we characterize both point and weak amenability for the classes of random walk polynomials (cf. above) and cosh-polynomials.

A.2. "Copyright Transfer Statement" Springer and included publication

Our paper "Orthogonal polynomials and point and weak amenability of ℓ^1-algebras of polynomial hypergroups" [Kah15] was first published in Constructive Approximation in Constr. Approx. 42 (2015), no. 1, 31–63, MR3361450, DOI: http://dx.doi.org/10.1007/s00365-014-9246-2, published by Springer Science+Business Media New York. The author respects the following "Copyright Transfer Statement". Springer holds the copyright of [Kah15]. However, the "Copyright Transfer Statement" explicitly permits the author to include the final published journal article in other publications (such as his dissertation). The author gratefully acknowledges the possibility provided by Springer Science+Business Media New York to include [Kah15] in his dissertation.

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Die bibliographischen Angaben von [Kah15] und die DOI betreffend verweise ich auf S. vii, S. 68 und das Literaturverzeichnis am Ende der Dissertation.

B. Publication [Kah16]

B.1. Summary

In our paper [Kah16] (sole author), we find new characterizations for the classes of ultraspherical, discrete q-ultraspherical and continuous q-ultraspherical polynomials. We consider orthogonal polynomial sequences $(P_n(x))_{n\in\mathbb{N}_0}$ which are assumed to be normalized such that $P_n(A) = 1$ $(n \in \mathbb{N}_0)$ for some $A > 0$ and to be orthogonal w.r.t. a symmetric probability (Borel) measure μ on $\mathbb{R}$ with $|\mathsf{supp}\ \mu| = \infty$. Writing $h : \mathbb{N}_0 \to (0, \infty)$, $h(n) := \frac{1}{\int_{\mathbb{R}} P_n^2(x)\,\mathrm{d}\mu(x)}$, we define $(P_n^*(x))_{n\in\mathbb{N}_0}$ by the expansions

$$P_n^*(x) = \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k) P_{n-2k}(x)}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)} \ (n \in \mathbb{N}_0);$$

if $\mathsf{supp}\ \mu \subseteq [-A, A]$ (which we do not generally assume), then $\mathrm{d}\mu^*(x) := (A^2 - x^2)\,\mathrm{d}\mu(x)$ defines a measure and $(P_n^*(x))_{n\in\mathbb{N}_0}$ is orthogonal w.r.t. μ^*. If $A = 1$ and $\mathsf{supp}\ \mu \subseteq [-1, 1]$, then a result of Lasser and Obermaier (2008) yields that $(P_n(x))_{n\in\mathbb{N}_0}$ belongs to the class of ultraspherical polynomials if and only if $P_n'(x) = P_n'(1)P_{n-1}^*(x)$ $(n \in \mathbb{N})$. Later, Ismail and Obermaier (2011) found a suitable extension to the class of discrete q-ultraspherical polynomials in terms of the 'q-difference operator' D_q instead of the classical derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ (more precisely, in terms of $D_{q^{-1}}$), and they found an extension to the class of continuous q-ultraspherical polynomials in terms of the 'Askey–Wilson operator' $\mathcal{D}_q$, which is another q-analogue of $\frac{\mathrm{d}}{\mathrm{d}x}$ $(q \in (0, 1)$; these Ismail–Obermaier results did not presume that $\mathsf{supp}\ \mu \subseteq [-A, A]$).

In Theorem 2.1, which is the first main result of [Kah16], we sharpen the Lasser–Obermaier result and show that the characterization remains valid if "$P_n'(x) = P_n'(1)P_{n-1}^*(x)$ $(n \in \mathbb{N})$" is replaced by the apparently weaker condition "$P_{2n-1}'(x) = P_{2n-1}'(1)P_{2n-2}^*(x)$ $(n \in \mathbb{N})$"; in fact, it suffices to require a constancy property of certain, carefully chosen Fourier coefficients belonging to the derivatives $P_{2n-1}'(x)$ $(n \in \mathbb{N})$ (furthermore, the condition "$\mathsf{supp}\ \mu \subseteq [-1, 1]$" can be weakened to "$x > 1 \Rightarrow P_2(x) \neq 0$"). While the original Lasser–Obermaier result can also be obtained from older, more "classical" results of Hahn or Al-Salam–Chihara, this is no longer the case for our sharpening.

The further main results of [Kah16] are Theorem 2.2, Theorem 2.3 and Theorem 2.4. Theorem 2.2 and Theorem 2.3 provide analogous sharpenings of the Ismail–Obermaier characterizations of the discrete and continuous q-ultraspherical polynomials, respectively. In fact, we shall see that the same redundancy holds. However, while Theorem 2.2 can be established by more or less "copying" the induction argument of Theorem 2.1, the proof of Theorem 2.3 requires considerably more effort because the product formula for the Askey–Wilson operator $\mathcal{D}_q$ additionally involves the 'averaging operator' $\mathcal{A}_q$ and because the proof of Theorem 2.3 requires to simultaneously tackle (determinacy problems concerning) the Fourier coefficients w.r.t. $\mathcal{A}_q$. We shall overcome this problem by studying determinacy of Fourier coefficients belonging to the functions $n \mapsto \mathcal{A}_q[xP_n(x)]$, and by a kind of "simultaneous involvement" of the continuous q-ultraspherical polynomials themselves (this approach will also help to avoid some tedious calculations).

Finally, Theorem 2.4 provides a characterization via $\mathcal{A}_q$ of those continuous q-ultraspherical polynomials whose corresponding orthogonalization measure is absolutely continuous (w.r.t. the Lebesgue–Borel measure on $\mathbb{R}$). Theorem 2.4 also shows that this characterization in terms of $\mathcal{A}_q$ does not extend to the whole class of continuous q-ultraspherical polynomials (we shall give an explicit counterexample).

B.2. "Consent to publish" AMS and included publication

Our paper "Characterizations of ultraspherical polynomials and their q-analogues" [Kah16] was first published in Proceedings of the American Mathematical Society in Proc. Amer. Math. Soc. 144 (2016), no. 1, 87–101, MR3415579, electronically published on September 4, 2015, DOI: http://dx.doi.org/10.1090/proc/12640 (to appear in print), published by American Mathematical Society. The author electronically signed the following "Consent to Publish". The American Mathematical Society holds the copyright of [Kah16]. However, the "Consent to Publish" explicitly permits the author to use (part or all of) the work in his own future publications. The author gratefully acknowledges the possibility provided by the American Mathematical Society, Providence, RI to include [Kah16] in his dissertation.

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