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Separating Invariants of Finite Groups

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Abstract

This thesis studies separating invariants of finite algebraic groups acting on affine varieties through automorphisms. We investigate the question: what restrictions does the existence of a separating set of small size, or a separating algebra with interesting structural properties impose on the group action.

Theorems of Serre, Dufresne, Kac-Watanabe and Gordeev, and Dufresne and Jeffries on invariant rings and separating algebras of linear representations are extended to this more general situation of possibly non-linear actions on varieties that need not be affine spaces.

Under rather mild assumptions on the variety and the group action we prove that polynomial separating algebras can exist only for reflection groups. On a side note this leads to an application to the semigroup problem in multiplicative invariant theory. Then we show that separating algebras which are complete intersections in a certain codimension can exist only for 2-reflection groups. Later we discuss the minimal number γ_{sep} of separating invariants and provide a new proof for its well-known upper bound. Finally we prove that a separating set of a certain size can exist only for k-reflection groups.

Several examples show that most of the assumptions on the group action and the variety that we make cannot be dropped.

Zusammenfassung

Die vorliegende Arbeit behandelt separierende Invarianten endlicher algebraischer Gruppen, die auf affinen Varietäten durch Automorphismen operieren. Wir untersuchen die Frage, welche Einschränkungen die Existenz einer separierenden Menge kleiner Mächtigkeit oder einer separierenden Algebra mit interessanten strukturellen Eigenschaften an die Gruppenoperation stellt.

Sätze von Serre, Dufresne, Kac-Watanabe und Gordeev, und Dufresne und Jeffries über Invariantenringe bzw. separierende Algebren von linearen Darstellungen werden übertragen auf die allgemeinere Situation von möglicherweise nicht-linearen Operationen auf Varietäten, die keine affinen Räume sein müssen.

Unter gewissen Voraussetzungen an die Varietät und die Gruppenoperation zeigen wir, dass separierende Algebren, die isomorph zu Polynomringen sind, nur für Spiegelungsgruppen existieren können. Dies führt in einer Nebenuntersuchung zu einer Anwendung auf das Semigruppenproblem der multiplikativen Invariantentheorie. Danach beweisen wir, dass separierende Algebren, die vollständige Durchschnitte in einer bestimmten Kodimension sind, nur für von 2-Reflektionen erzeugte Gruppen existieren können. Später diskutieren wir die minimale Anzahl γ_{sep} separierender Invarianten und geben einen neuen Beweis für ihre bekannte obere Schranke. Anschließend zeigen wir, dass eine separierende Menge einer bestimmten vorgegebenen Größe nur für k-Reflektionsgruppen existieren kann.

Mehrere Beispiele verdeutlichen, dass die nötigen Voraussetzungen an die Gruppenoperation und die Varietät, die wir treffen, nicht weggelassen werden können.

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Introduction

Invariant theory studies the ring of those polynomial functions on an affine variety X that are fixed under the action of a linear algebraic group G. This thesis considers the case where G is a finite group and the action on X is given through automorphisms. The invariant theory of finite groups is a subject in which there are both many classical results and recent developments. It is presented in-depth in the books of Benson [Ben93], Smith [Smi95], Neusel and Smith [NS10], and Campbell and Wehlau [CW11].

Hilbert's famous finiteness theorem [Hil90] states that invariant rings of so-called linearly reductive groups are always finitely generated algebras. In particular, this applies to finite groups whose order is not divisible by the characteristic of the base field. Noether [Noe26] gave a new proof for the finite generation of the invariant ring that works for all finite groups independently of the characteristic of the field.

Although the minimal number of generators of the invariant ring $K[X]^G$, considered as an algebra over the base field K, is finite by these classical theorems, it can be very large even for small groups and low-dimensional representations (see e.g. [KK12, Table in Section 5]). Derksen and Kemper [DK02] introduced the definition of separating invariants as a more general concept than generating invariants. A set of invariants is called separating if it has the same property of separating the orbits as the whole invariant ring. It has been known for a while that the minimal number γ_{sep} of separating invariants is bounded above by $2n + 1$ where n is the transcendence degree of the invariant ring. In our case of finite groups n is just the dimension of the affine variety whose regular functions we study.

Whenever polynomial invariants are used to distinguish equivalence classes of some objects (and the equivalence relation can be described as lying in the same orbit of a group action), it therefore makes sense to shift the focus on separating sets of invariants rather than possibly much larger and more complicated sets of generating invariants.

This naturally leads to the question if separating sets of the smallest possible size and separating algebras with "good algebraic properties" exist for a given group action. We formulate the following two questions explicitly.

(Q1) When does there exist a separating algebra $A \subseteq K[X]^G$ that is isomorphic to a polynomial ring (which is equivalent to $\gamma_{\text{sep}} = n$)?

(Q2) When does there exist a separating algebra $A \subseteq K[X]^G$ that is a complete intersection (which includes the case $\gamma_{\text{sep}} = n + 1$)?

Previous results.

In the case of a non-modular linear representation $X = V$ of G the theorem of Shephard and Todd [ST54], Chevalley [Che55], and Serre [Ser68] gives a complete answer for $A = K[V]^G$ to the first question. The theorem says that the invariang ring $K[V]^G$ is a polynomial ring if and only if G is generated by 1-reflections (i.e. by elements that act as identity on a subspace of codimension 1 of V), which are just called reflections henceforth. Serre also proved that even in the modular case the invariant ring $K[V]^G$ can only be isomorphic to a polynomial ring if G is a reflection group. With a new proof especially suitable for dealing with separating invariants Dufresne [Duf09] showed that this theorem of Serre remains true if we replace the invariant ring $K[V]^G$ by any separating subalgebra A of $K[V]^G$.

For the second question a similar necessary condition was found by Kac and Watanabe [KW82] and independently by Gordeev [Gor82]. They showed that the invariant ring $K[V]^G$ of a linear representation V of G can only be a complete intersection if G is generated by 2-reflections (which are defined analogously to 1-reflections). This was extended by Dufresne [Duf09] to graded separating subalgebras of $K[V]^G$.

Recently, Dufresne and Jeffries [DJ15] found a remarkable connection, in the case of linear actions on n-dimensional affine spaces, between the size of a separating set of invariants and the property of being a k-reflection group. They proved that if $\gamma_{\text{sep}} = n + k - 1$, then G is generated by k-reflections.

So while the case of linear actions on affine spaces was already covered pretty good, not much was known about the general situation of actions on affine varieties through automorphisms.

Structure of this thesis.

Chapter 1 shortly recalls some concepts of the invariant theory of finite groups which are needed throughout this thesis. We prove the most important properties of the quotient of X by G, define the so-called separating variety, and give different characterizations of separating sets.

Chapter 2 tackles question (Q1). After recalling the concept of "connectedness in codimension k", we relate the connectedness of the separating variety to group elements which act as k-reflections. Using Hartshorne's connectedness theorem we then prove our main theorem concerning the problem of polynomial separating algebras (see main results below). This leads to an application to the semigroup problem in multiplicative invariant theory in Section 2.5.

To gain further results, the technique of completion is used frequently. Therefore, in Chapter 3 we interrupt our study of separating algebras, and recall some properties of complete Noetherian rings and modules which are needed in the later parts of this thesis. A reader familiar with the theory of completions may wish to skip this chapter.

Chapter 4 concerns question (Q2). The simply-connectedness of the quotient is related to the property of being a k-reflection group. Purity theorems are used as a black box to get the main results of this chapter.

In the fifth and final chapter we first provide a new proof for the well-known upper bound $\gamma_{\text{sep}} \leq 2n + 1$. Then we show that the new results of Dufresne and Jeffries carry over to actions on affine varieties as well.

An appendix containing the MAGMA functions for several computations is added so that the examples of this thesis could be reproduced more easily.

Main results.

For a connected, Cohen-Macaulay variety X, and a group action generated by elements having a fixed point we prove the following implication in Theorem 2.20: If there exists a separating algebra $A \subseteq K[X]^G$ that is a polynomial ring, then G is generated by reflections on X. In the setting of multiplicative invariant theory we show in Theorem 2.27 that the invariant ring of an action on a lattice can only be a mixed Laurent polynomial ring if G is generated by reflections.

Now let X be a connected, normal variety, and suppose that the action on X has a fixed point. Under these assumptions we prove in Theorem 4.25: If there exists a separating algebra $A \subseteq K[X]^G$ such that $K[X]^G$ is finite over A and such that A is a complete intersection in codimension $2+\mathrm{cid}(A)$, then G is generated by 2-reflections. For a non-modular (at least 3-dimensional) non-trivial representation $V = X$ of G this gives as a corollary: If $\mathrm{cid}(K[V]^G) \leq n-3$, then $G \setminus \{\mathrm{id}\}$ must contain an $(n-1)$-reflection (see Theorem 4.28). Furthermore, in Theorem 5.14 we prove that the result of Dufresne and Jeffries holds under these assumptions as well: If there exists a separating set of size $n+k-1$, then G is generated by k-reflections.

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Chapter 1

Basics from Invariant Theory

This chapter shortly recalls some of the basic concepts of the invariant theory of finite groups with a clear emphasis on separating invariants. Let us start by fixing some notation. Throughout this thesis we write K for the base field which is always assumed to be algebraically closed (even though some results stay true if we drop this assumption on K).

Furthermore, let G be a finite algebraic group and X an affine variety (both defined over K) on which G acts (on the left) by automorphisms. Whenever it is convenient, we will therefore view the elements of G as morphisms of affine varieties, more precisely, G as a subgroup of $\mathrm{Aut}(X)$. We will refer to this setting briefly by calling X a ***G*-variety**. The case mainly studied in invariant theory is when X is an affine space and G acts linearly. Then of course $X = K^n$ is a linear representation of G and we will usually write V instead of X.

The coordinate ring $K[X]$ is the K-algebra of all regular functions on X. The group action on X induces an action on $K[X]$ through K-algebra automorphisms as follows:

$$\sigma \in G,\, f \in K[X] \quad \Rightarrow \quad \sigma \cdot f := f \circ \sigma^{-1}.$$

The elements fixed by this action form a subalgebra of $K[X]$:

$$K[X]^G := \{f \in K[X] \mid \sigma \cdot f = f \text{ for all } \sigma \in G\},$$

which is usually called the **invariant ring**. The big advantage of a linear representation V of G compared to an arbitrary G-variety is that in this situation the invariant ring $K[V]^G$ inherits the grading of the polynomial ring $K[V] = K[x_1, \ldots, x_n]$.

On several occasions in this thesis we will use the well-known fact that the coordinate ring of X is integral over the invariant ring. As the proof is very short, this might be justification enough to include it here.

Proposition 1.1. *The inclusion $K[X]^G \subseteq K[X]$ is an integral extension of rings.*

Proof. Let $f \in K[X]$. Then f is a zero of the monic polynomial

$$P(t) := \prod_{\sigma \in G} (t - \sigma f) \in (K[X])[t],$$

whose coefficients are the elementary symmetric polynomials in the σf and therefore invariants. Hence $P(t)$ lies in $(K[X]^G)[t]$ and provides an integral equation for f. □

The integrality of $K[X]$ over $K[X]^G$ is key to Noether's proof that the invariant ring of a finite group is always finitely generated.

Theorem 1.2. (Noether [Noe26]) *The invariant ring $K[X]^G$ is a finitely generated K-algebra.*

Since integral extensions of rings preserve the Krull dimension and since the transcendence degree of a finitely generated K-algebra equals its dimension, we also get the following corollary.

Corollary 1.3. *The invariant ring $K[X]^G$ has transcendence degree*

$$\operatorname{trdeg}(K[X]^G) = \dim(X) =: n.$$

Next we collect a few facts about the **quotient** of X by G. Since $K[X]^G$ is a finitely generated K-algebra, it corresponds to an affine variety $X/\!\!/G$, and the inclusion $K[X]^G \subseteq K[X]$ corresponds to a morphism of varieties

$$\pi : X \to X/\!\!/G,$$

which is called the quotient of X by G. Later in this thesis it will be necessary to consider the quotient as a morphism of schemes (see Section 4.2). But if we stay in the concept of affine varieties as Zariski-closed subsets of some affine space here, then of course there is no canonical choice for the quotient $X/\!\!/G$. It depends on the chosen set of generators of $K[X]^G$ as a K-algebra, and is therefore not at all unique. But clearly, a different choice of generators would give an isomorphic variety. So when talking about the quotient, we will always implicitly assume that we fixed generators of $K[X]^G$, say $K[X]^G = K[f_1, \dots, f_m]$, so that $X/\!\!/G$ is then a subvariety of K^m and the morphism $\pi : X \to X/\!\!/G$ is given by $x \mapsto (f_1(x), \dots, f_m(x))$.

Of course, the above definition of $X/\!\!/G$ as an affine variety could also be made for infinite groups if the invariant ring is finitely generated. For reductive groups this always gives a **categorical quotient** of X by G (see [DK15, Section 2.3]). But in general, $X/\!\!/G$ does not correspond to the topological orbit space, which is expressed by saying that it need not be a **geometric quotient**. In our case of a finite group action, however, the map π has these good properties. Let us prove the most important ones in the following proposition.

Proposition 1.4. *(a) The map π is surjective.*

(b) The fibers of π are the orbits of G in X.

(c) The topology on $Y = X/\!\!/G$ is the quotient topology.

Proof. It follows from the lying-over lemma (see [Kem11, Theorem 8.12]) that π is surjective.

Obviously, the map π is G-invariant. Now suppose that two points $x, y \in X$ have the same image. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of $K[X]$ corresponding to x and y, respectively. Take $f \in \mathfrak{m}$, i.e. a regular function on X with $f(x) = 0$. Since the orbit product of f is an invariant, we have

$$\left(\prod_{\sigma \in G} \sigma f\right)(y) = \left(\prod_{\sigma \in G} \sigma f\right)(x) = \prod_{\sigma \in G} (\sigma f)(x) = 0.$$

Hence there exists an element $\sigma \in G$ with $\sigma f \in \mathfrak{n}$. In total, this implies

$$\mathfrak{m} \subseteq \bigcup_{\sigma \in G} \sigma\mathfrak{n},$$

which gives $\mathfrak{m} \subseteq \sigma\mathfrak{n}$ for some $\sigma \in G$, by the prime avoidance lemma (see [Kem11, Lemma 7.7]). We conclude $\mathfrak{m} = \sigma\mathfrak{n}$ by the maximality, and get $x = \sigma y$. This proves (b).

For (c) we need to show that a subset A of Y is closed if and only if $\pi^{-1}(A) \subseteq X$ is closed. One direction is clearly given by continuity of π. Moreover, once again by the lying-over lemma, π is a closed map, as $\mathcal{V}_X(I)$, the subvariety of X defined by an ideal $I \subseteq K[X]$, is mapped onto $\mathcal{V}_Y(I^G)$. So if $\pi^{-1}(A) \subseteq X$ is closed, then

$$A = \pi(\pi^{-1}(A))$$

is closed, too. □

Remark 1.5. The statements of Proposition 1.4 are also true for the scheme morphism $\pi : \operatorname{Spec}(K[X]) \to \operatorname{Spec}(K[X]^G)$, $\mathfrak{p} \mapsto \mathfrak{p}^G$.

Part (b) from Proposition 1.4 can be rephrased as: "The invariants separate the orbits." For if two points $x, y \in X$ satisfy $f(x) = f(y)$ for all $f \in K[X]^G$, i.e. $\pi(x) = \pi(y)$, then they must lie in the same orbit.

The definition of separating sets of invariants was introduced by Derksen and Kemper [DK02, Definition 2.3.8] as a more general concept than generating sets of invariants (by which we always mean a set of K-algebra generators of $K[X]^G$).

Definition 1.6. *A subset $S \subseteq K[X]^G$ of the invariant ring is called* ***separating*** *if for all $x, y \in X$ we have: If there exists an invariant $f \in K[X]^G$ with $f(x) \neq f(y)$, then there exists an invariant $g \in S$ with $g(x) \neq g(y)$.*

Following [Kem09], we write $\boldsymbol{\gamma}_{\mathbf{sep}}$ for the smallest natural number m such that there exists a separating subset of size m.

Proposition 1.7. *Let $A = K[g_1, \dots, g_r] \subseteq K[X]^G$ be a finitely generated subalgebra. So A can be viewed as the coordinate ring of an affine variety $W \subseteq K^r$ and there is an induced morphism $\theta : X /\!\!/ G \to W$. Then A is separating if and only if θ is injective.*

Proof. The composition $\theta \circ \pi$ corresponds to the inclusion $A \subseteq K[X]$. So for all $x \in X$ we have $\theta(\pi(x)) = (g_1(x), \dots, g_r(x))$ and $\pi(x) = (f_1(x), \dots, f_m(x))$ where $f_1, \dots, f_m$ form a set of generating invariants. Hence Definition 1.6 reads as: The subalgebra A is separating if and only if for all points $x, y \in X$ with $\pi(x) \neq \pi(y)$ we have $\theta(\pi(x)) \neq \theta(\pi(y))$. But this condition is equivalent to θ being injective since π is surjective by Proposition 1.4(a). □

Remark 1.8. Injectivity of the map θ in Proposition 1.7 also implies injectivity of the scheme morphism $\operatorname{Spec}(K[X]^G) \to \operatorname{Spec}(A)$ induced by the inclusion $A \subseteq K[X]^G$ (see the proof of [Duf09, Theorem 2.2]).

In addition, Proposition 1.7 together with the general lower bound of fiber dimension (see [Kem11, Corollary 10.6]) leads to the following corollary.

Corollary 1.9. *A finitely generated, separating subalgebra $A \subseteq K[X]^G$ has dimension $n = \dim(X)$. Futhermore, the lower bound $\gamma_{\text{sep}} \geq n$ holds.*

Example 1.10. Let $\operatorname{char}(K) = 0$ and let ζ be a primitive m-th root of unity in K. Then $G := \langle \zeta \rangle \subseteq K^\times$, the cyclic group of order m, acts on $V = K^2$ diagonally:

$$\sigma \in G, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K^2 \quad \Rightarrow \quad \sigma \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \sigma x_1 \\ \sigma x_2 \end{pmatrix}.$$

Here every $\sigma \in G$ does not only act degree-preserving on $K[V]$, but even maps monomials to scalar multiples of themselves. This is probably the easiest case in invariant theory that one can hope for since now the invariant ring is just generated by those monomials that are fixed by the group action. Therefore, a minimal set of homogeneous generators of the invariant ring $K[V]^G \subseteq K[x_1, x_2]$ is given by

$$M := \{x_1^m,\ x_1^{m-1}x_2,\ x_1^{m-2}x_2^2,\ \ldots,\ x_2^m\}.$$

Let us see that the smaller set

$$S := \{x_1^m,\ x_1^{m-1}x_2,\ x_2^m\},$$

which does obviously not generate the invariant ring, is still a separating set. In the field of fractions of $K[V]^G$ we have for all $2 \leq i \leq m-1$:

$$x_1^{m-i}x_2^i = \frac{(x_1^{m-1}x_2)^i}{(x_1^m)^{i-1}}.$$

So the values of those monomials of M that are missing in S at a point $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in V$ can be recovered from S if $a_1 \neq 0$. But if $a_1 = 0$, then these values are 0 anyway. Hence S has exactly the same separating properties as $K[V]^G$. ◁

Definition 1.11. *The **separating variety** $\mathcal{V}_{\text{sep}}$ of the G-action on X is defined to be the following subvariety of $X \times X$:*

$$\mathcal{V}_{\text{sep}} := \{(x, y) \in X \times X \mid f(x) = f(y) \text{ for all } f \in K[X]^G\}.$$

The coordinate ring of $X \times X$ is the tensor product $K[X] \otimes_K K[X]$. The projection $X \times X \to X$ on the first coordinates corresponds to the inclusion homomorphism

$$\iota : K[X] \to K[X] \otimes_K K[X], \quad f \mapsto f \otimes 1. \tag{1.0.1}$$

When studying separating varieties, the following map naturally comes into play:

$$\delta : K[X] \to K[X] \otimes_K K[X], \quad f \mapsto f \otimes 1 - 1 \otimes f. \tag{1.0.2}$$

Following the notation of [Duf09] we will refer to 1.0.2 as the **$\boldsymbol{\delta}$-map** throughout this thesis.

Proposition 1.12. *The δ-map is K-linear, and with the notation as above δ and ι satisfy:*

$$\delta(fg) = \delta(f)\iota(g) + \iota(f)\delta(g) - \delta(f)\delta(g).$$

In particular, $\delta(fg)$ lies in the ideal generated by $\delta(f)$ and $\delta(g)$ in $K[X] \otimes_K K[X]$.

Proof. The K-linearity is clear from the definition. Now for $f,\, g \in K[X]$ an easy calculation shows:

$$\begin{aligned}\delta(fg) + \delta(f)\delta(g) &= (fg)\otimes 1 - 1\otimes (fg) + (f\otimes 1 - 1\otimes f)(g\otimes 1 - 1\otimes g) =\\ &= (fg)\otimes 1 - g\otimes f + (fg)\otimes 1 - f\otimes g =\\ &= (f\otimes 1 - 1\otimes f)(g\otimes 1) + (g\otimes 1 - 1\otimes g)(f\otimes 1) =\\ &= \delta(f)\iota(g) + \iota(f)\delta(g).\end{aligned}$$ □

Proposition 1.12 shows that the images under δ of all generating sets of invariants generate the same ideal in $K[X]\otimes_K K[X]$. Let us write $\boldsymbol{\mathcal{I}_{\mathrm{sep}}}$ for this ideal, i.e. $\mathcal{I}_{\mathrm{sep}} = (\delta(K[X]^G))_{K[X]\otimes_K K[X]}$. Definition 1.11 now reads as

$$\mathcal{V}_{\mathrm{sep}} := \mathcal{V}_{X\times X}(\mathcal{I}_{\mathrm{sep}}) := \{(x,\, y) \in X\times X \mid h(x,\, y) = 0 \text{ for all } h \in \mathcal{I}_{\mathrm{sep}}\}.$$

Furthermore, a set of invariants $S \subseteq K[X]^G$ is now separating if and only if the image of S under the δ-map defines the subvariety $\mathcal{V}_{\mathrm{sep}} \subseteq X\times X$ as its vanishing set. Since the field K is assumed to be algebraically closed, Hilbert's Nullstellensatz tells us that this is equivalent to saying that the radical of the ideal generated by $\delta(S)$ and the radical of $\mathcal{I}_{\mathrm{sep}}$ conincide. This is summarized in the following Proposition.

Proposition 1.13. *Let $S \subseteq K[X]^G$ be a set of invariants. Then the following conditions are equivalent:*

(a) S is separating,

(b) $\mathcal{V}_{X\times X}(\delta(S)) = \mathcal{V}_{\mathrm{sep}}$,

(c) $\sqrt{(\delta(S))} = \sqrt{\mathcal{I}_{\mathrm{sep}}}$.

In particular, this implies in a general context that there always exists a finite separating set: Since the ring $K[X]\otimes_K K[X]$ is Noetherian, there always exists a finite subset $\{\delta(f_1), \ldots, \delta(f_m)\}$ of $\delta(K[X]^G)$ generating the ideal $\mathcal{I}_{\mathrm{sep}}$, which gives the separating set $S := \{f_1, \ldots, f_m\}$.

Of course, for finite groups this observation is superflous because there even is a finite generating set by Theorem 1.2. For non-reductive groups, however, where the invariant ring might not be finitely generated, this statement was part of the original motivation to study separating invariants (see [DK02, Theorem 2.3.15]).

Example 1.14. We want to compare the ideals $\mathcal{I}_{\mathrm{sep}}$, $\sqrt{\mathcal{I}_{\mathrm{sep}}}$ and $J := (\delta(S))$ from Example 1.10 in the case $m = 4$. Let us avoid the tensor product notation here and write $R := K[x_1,\, x_2,\, y_1,\, y_2]$ for the coordinate ring of $V\times V$. The ideal $\mathcal{I}_{\mathrm{sep}}$ is generated by the following set of homogeneous polynomials of degree 4:

$$\delta(M) = \{x_1^4 - y_1^4,\quad x_1^3x_2 - y_1^3y_2,\quad x_1^2x_2^2 - y_1^2y_2^2,\quad x_1x_2^3 - y_1y_2^3,\quad x_2^4 - y_2^4\}.$$

So the polynomial $x_1y_2 - x_2y_1$ does not lie in $\mathcal{I}_{\mathrm{sep}}$, but we have

$$\begin{aligned}(x_1y_2 - x_2y_1)^4 &= (y_2^4 - x_2^4)(x_1^4 - y_1^4) - 3(x_1^2x_2^2 - y_2^2y_1^2)^2\\ &\quad - 4y_1y_2^3(x_1^3x_2 - y_1^3y_2) + 4x_1x_2^3(x_1^3x_2 - y_1^3y_2) \in \mathcal{I}_{\mathrm{sep}}.\end{aligned}$$

Thus, $\mathcal{I}_{\mathrm{sep}}$ is not a radical ideal. In fact, a calculation (for example in MAGMA [BCP97]) shows that $\sqrt{\mathcal{I}_{\mathrm{sep}}} = \mathcal{I}_{\mathrm{sep}} + (x_1y_2 - x_2y_1)$. Since J is generated by

$$\delta(S) = \{x_1^4 - y_1^4, \quad x_1^3x_2 - y_1^3y_2, \quad x_2^4 - y_2^4\},$$

we have strict inclusions in this example

$$J \subsetneq \mathcal{I}_{\mathrm{sep}} \subsetneq \sqrt{\mathcal{I}_{\mathrm{sep}}},$$

while a MAGMA computation shows $\sqrt{J} = \sqrt{\mathcal{I}_{\mathrm{sep}}}$. By Proposition 1.13, this confirms a second time that S is a separating set. $\triangleleft$

Chapter 2

Polynomial Rings as Separating Algebras

The famous theorem of Shephard and Todd [ST54], Chevalley [Che55], and Serre [Ser68] says that in the non-modular case (i.e. $\mathrm{char}(K)$ does not divide $|G|$) the invariant ring of a linear representation V of G is a polynomial ring if and only if G is generated by reflections (i.e. by elements that act as identity on a hyperplane of V).

Serre also proved that even in the modular case the invariant ring $K[V]^G$ can only be isomorphic to a polynomial ring if G is a reflection group. In 2009, Dufresne [Duf09] showed that this theorem remains true if we replace the invariant ring $K[V]^G$ by any separating subalgebra of $K[V]^G$.

An important step in Dufresne's proof is her discovery how the connectedness in codimension 1 of the separating variety implies that the group is a reflection group. The proof relies on Hartshorne's Connectedness Theorem. This chapter extends these methods from linear actions to actions on affine varieties and simultaneously from reflections to k-reflections where k is not necessarily equal to 1.

A preprint containing the main results of this chapter was already posted on arXiv [Rei13].

2.1 Connectedness in Codimension

This section recalls the topological concept of connectedness in a certain codimension as it was introduced by Hartshorne [Har62]. In addition, in Proposition 2.7 we prove that for Noetherian Jacobson rings connectedness in codimension k can be checked at the localizations of the maximal ideals.

The **(Krull) dimension** of a topological space Y is the supremum of all lengths of ascending chains of irreducible closed subsets of Y. For a closed subset A of Y there is also the notion of its codimension in Y, given below.

Definition 2.1. *Let Y be a topological space and $Z \subseteq Y$ a closed irreducible subset. The* ***codimension*** *of Z in Y, written as $\mathrm{codim}_Y(Z)$, is defined to be the supremum of all lengths $n \in \mathbb{N}_0$ of ascending sequences*

$$Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_{n-1} \subsetneq Z_n$$

of closed irreducible subsets Z_i of Y. So $\operatorname{codim}_Y(Z) \in \mathbb{N}_0 \cup \{\infty\}$*. Furthermore, for any non-empty closed subset $A \subseteq Y$ we set*

$$\operatorname{codim}_Y(A) = \min\{\operatorname{codim}_Y(Z) \mid Z \subseteq A \textit{ irreducible and closed } \}.$$

In addition, we will use the convention $\operatorname{codim}_Y(\emptyset) = \dim(Y) + 1$ *(together with* $\dim(\emptyset) = -1$*).*

Remark 2.2. (a) If A is a finite union of closed subsets, say

$$A = \bigcup_{i=1}^{n} A_i$$

with $A_i \subseteq Y$ closed, then the following formulas hold:

$$\dim(A) = \max\{\dim(A_i) \mid i = 1, \dots, n\},$$
$$\operatorname{codim}_Y(A) = \min\{\operatorname{codim}_Y(A_i) \mid i = 1, \dots, n\}.$$

(b) If Y is (the spectrum of the coordinate ring of) an equidimensional affine variety, then for all closed subsets A of Y the formula

$$\dim(A) + \operatorname{codim}_Y(A) = \dim(Y)$$

holds (see [Kem11, Corollary 8.23]).

Definition 2.3. *For a non-negative integer k, a Noetherian topological space Y is called* ***connected in codimension k*** *if for all closed subsets $Z \subseteq Y$ with* $\operatorname{codim}_Y(Z) > k$ *the space $Y \setminus Z$ is connected.*

Of course, being connected in codimension k implies being connected in codimension $k+1$, so we have a chain of properties of Y. The strongest condition, Y being connected in codimension 0, is equivalent to Y being irreducible (as we will see in the next proposition). If $\dim(Y) < \infty$, then being connected in codimension $\dim(Y)$ simply means being connected.

Let us prove an important equivalent condition for connectedness in codimension k, given by Hartshorne in [Har62, Prop. 1.1]), to which we will sometimes refer shortly as "the irreducible components of Y intersect in codimension $\leq k$".

Proposition 2.4. *Let Y be a Noetherian topological space and $k \in \mathbb{N}_0$. Then the following are equivalent:*

(a) the space Y is connected in codimension k,

(b) for all irreducible components Y' and Y'' of Y there exists a finite sequence $Y_0, \dots, Y_r$ of irreducible components of Y with $Y_0 = Y'$, $Y_r = Y''$ and

$$\operatorname{codim}_Y(Y_i \cap Y_{i+1}) \leq k \quad \textit{for } i = 0, \dots, r-1.$$

Proof. Let $\mathcal{M}$ be the set of all irreducible components of Y (which is finite, because Y is a Noetherian space), so that
$$Y = \bigcup_{Y' \in \mathcal{M}} Y'.$$
First assume that Y is connected in codimension k and take $Y' \in \mathcal{M}$. Let $\mathcal{N}$ be the set of all irreducible components Y'' of Y such that the condition in (b) is satisfied for Y' and Y''. Then we write
$$Y = \underbrace{\bigcup_{Y'' \in \mathcal{N}} Y''}_{=:A_1} \cup \underbrace{\bigcup_{\widetilde{Y} \in \mathcal{M} \setminus \mathcal{N}} \widetilde{Y}}_{=:A_2}$$
as the union of two closed subsets A_1, A_2 of Y. In addition, we have
$$A_1 \cap A_2 = \bigcup_{Y'' \in \mathcal{N}, \widetilde{Y} \in \mathcal{M} \setminus \mathcal{N}} Y'' \cap \widetilde{Y}.$$
The intersection of a component $Y'' \in \mathcal{N}$ with a component $\widetilde{Y} \in \mathcal{M} \setminus \mathcal{N}$ must have codimension larger than k. Otherwise we could extend a sequence of irreducible components leading from Y' to Y'' in codimension $\leq k$ to a sequence from Y' to $\widetilde{Y}$ which would contradict $\widetilde{Y} \notin \mathcal{N}$. So by Remark 2.2(a), $Z := A_1 \cap A_2$ has codimension larger than k, as well. Therefore, by assumption, the space $Y \setminus Z$ is connected. But the decomposition
$$Y \setminus Z = (A_1 \setminus Z) \cup (A_2 \setminus Z)$$
into closed, disjoint subsets would contradict the connectedness unless one of them is empty. Obviously, $Y' \subseteq A_1$, so A_2 must be empty, and we get $\mathcal{M} = \mathcal{N}$, as desired.

Now assume that (b) is satisfied and take a closed subset $Z \subseteq Y$ of codimension larger than k. As non-empty open subsets of irreducible spaces are irreducible, too, for all $Y' \in \mathcal{M}$ either $Y' \setminus Z = \emptyset$ or $Y' \setminus Z$ is an irreducible component of $Y \setminus Z$. Assume by contradiction that $Y \setminus Z$ is the union of two closed, disjoint, non-empty subsets:
$$Y \setminus Z = (A_1 \setminus Z) \cup (A_2 \setminus Z) \quad \text{with } A_1,\, A_2 \subseteq Y \text{ closed.} \tag{2.1.1}$$
So for an irreducible component $Y' \setminus Z$ of $Y \setminus Z$ we get
$$Y' \subseteq A_1 \cup A_2 \cup Z,$$
which shows $Y' \subseteq A_i$ for one i (since $Y' \subseteq Z$ would imply $Y' \setminus Z = \emptyset$). Since $A_1 \setminus Z$ and $A_2 \setminus Z$ are non-empty and disjoint, there exist components Y', $Y'' \in \mathcal{M}$ with $Y' \subseteq A_1$ and $Y'' \subseteq A_2$. The intersection $(Y' \setminus Z) \cap (Y'' \setminus Z)$ must then be empty as 2.1.1 is a disjoint union. Now (b) yields the existence of a sequence $Y_0, \dots, Y_r \in \mathcal{M}$ with $Y_0 = Y'$, $Y_r = Y''$ and
$$\operatorname{codim}_Y(Y_i \cap Y_{i+1}) \leq k \quad \text{for } i = 0, \dots, r-1.$$
Since $\operatorname{codim}_Y(Z) > k$, the set $(Y_i \cap Y_{i+1}) \setminus Z$ cannot be empty. Therefore, $Y_i \setminus Z$ and $Y_{i+1} \setminus Z$ must both lie either in A_1 or A_2. But as this holds for all i, it contradicts the assumption $Y_0 = Y' \subseteq A_1$ and $Y_r = Y'' \subseteq A_2$. $\square$

Hartshorne also introduced the term "locally connected in codimension k" by defining local spaces of a general topological space. We will not follow this general path, but restrict ourselves to the case of the spectrum of a ring R, where $\operatorname{Spec}(R)$ will be called **locally connected in codimension k** if for all prime ideals $\mathfrak{p}$ of R the spectrum of the localization $R_\mathfrak{p}$ is connected in codimension k. To show that for a connected space local connectedness in codimension k implies global connectedness in codimension k, we need the following lemma, which is just a paraphrase of [Har62, Lemma 1.2].

Lemma 2.5. *Let R be a Noetherian ring with connected spectrum $Y = \operatorname{Spec}(R)$, and let $Z \subseteq Y$ be a closed subset of codimension ≥ 1 such that for all $\mathfrak{p} \in Z$ the space $\operatorname{Spec}(R_\mathfrak{p}) \setminus \{\mathfrak{p}_\mathfrak{p}\}$ is connected. Then $Y \setminus Z$ is connected.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal prime ideals of R. They cannot lie in Z. Now assume by contradiction that there exist closed subsets A_1, $A_2 \subseteq Y$ with

$$Y \setminus Z = (A_1 \setminus Z) \cup (A_2 \setminus Z), \tag{2.1.2}$$

such that both $A_i \setminus Z$ are non-empty, and satisfy

$$(A_1 \setminus Z) \cap (A_2 \setminus Z) = \emptyset. \tag{2.1.3}$$

Since $\mathfrak{p}_1, \dots, \mathfrak{p}_s \in Y \setminus Z$, we can assume $\mathfrak{p}_1, \dots, \mathfrak{p}_r \in A_1 \setminus Z$ and $\mathfrak{p}_{r+1}, \dots, \mathfrak{p}_s \in A_2 \setminus Z$. And as those are the minimal prime ideals of R, taking closures in 2.1.2 leads to

$$Y = \overline{Y \setminus Z} = (\overline{A_1 \setminus Z}) \cup (\overline{A_2 \setminus Z}) = A_1 \cup A_2. \tag{2.1.4}$$

By assumption, Y is connected, so we get $A_1 \cap A_2 \neq \emptyset$ and by 2.1.3 we have

$$A_1 \cap A_2 \subseteq Z.$$

Let $\mathfrak{q} \in A_1 \cap A_2$ be a prime ideal minimal among all elements of $A_1 \cap A_2$. We will show that $\operatorname{Spec}(R_\mathfrak{q}) \setminus \{\mathfrak{q}_\mathfrak{q}\}$ is disconnected, which would contradict the assumption on Z. Let $B = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{q}\}$, the homeomorphic image of $\operatorname{Spec}(R_\mathfrak{q})$ in Y, then 2.1.4 shows that

$$B \setminus \{\mathfrak{q}\} = \underbrace{(A_1 \cap B) \setminus \{\mathfrak{q}\})}_{=:B_1} \cup \underbrace{(A_2 \cap B \setminus \{\mathfrak{q}\})}_{=:B_2}$$

is a union of closed subsets. Furthermore, the minimality of $\mathfrak{q}$ in $A_1 \cap A_2$ implies that $B_1 \cap B_2 = \emptyset$. In addition, $\mathfrak{q}$ must contain at least one of the minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ (since $\mathfrak{q} \in A_1$), and at least one of the minimal prime ideals $\mathfrak{p}_{r+1}, \dots, \mathfrak{p}_s$ (since $\mathfrak{q} \in A_2$). But $\mathfrak{q}$ is neither of these $\mathfrak{p}_i$ (since $\mathfrak{q} \in Z$). This shows that both B_1 and B_2 are non-empty, and we are done. $\square$

Corollary 2.6. *Let R be a Noetherian ring with connected spectrum $Y = \operatorname{Spec}(R)$, and let k be a non-negative integer such that $\operatorname{Spec}(R)$ is locally connected in codimension k. Then Y is also connected in codimension k.*

Proof. Now let $Z \subseteq Y$ be a closed subset of codimension larger than k (so in particular ≥ 1). Then for all prime ideals $\mathfrak{p} \in Z$ we have

$$\operatorname{codim}_{\operatorname{Spec}(R_\mathfrak{p})}(\{\mathfrak{p}_\mathfrak{p}\}) = \operatorname{ht}_R(\mathfrak{p}) \geq \operatorname{codim}_{\operatorname{Spec}(R)}(Z) > k,$$

where $\operatorname{ht}_R(\mathfrak{p})$ is the height of $\mathfrak{p}$. As $\operatorname{Spec}(R_\mathfrak{p})$ is assumed to be connected in codimension k, it follows that $\operatorname{Spec}(R_\mathfrak{p}) \setminus \{\mathfrak{p}_\mathfrak{p}\}$ is connected. So by Lemma 2.5, the space $Y \setminus Z$ is connected. □

It should certainly not surprise that in Corollary 2.6 only the localizations at the maximal ideals are necessary if R is an affine K-algebra. According to the following proposition this is in fact true, since affine K-algebras are Noetherian Jacobson rings.

Proposition 2.7. *Let R be a Noetherian Jacobson ring with connected spectrum $Y = \operatorname{Spec}(R)$, and let k be a non-negative integer such that for all maximal ideals $\mathfrak{m}$ of R the spectrum of the localized ring $R_\mathfrak{m}$ is connected in codimension k. Then Y is connected in codimension k, too.*

Proof. By Corollary 2.6, it suffices to show that for all prime ideals $\mathfrak{p}$ of R the spectrum of the localized ring $R_\mathfrak{p}$ is connected in codimension k. So let $\mathfrak{p} \in \operatorname{Spec}(R)$.

The irreducible components of $\operatorname{Spec}(R_\mathfrak{p})$ correspond to the minimal prime ideals $\mathfrak{p}_0, \ldots, \mathfrak{p}_r$ of R which are contained in $\mathfrak{p}$. Denote by $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_s$ the minimal prime of R which are not contained in $\mathfrak{p}$. Let $\mathcal{M}$ be the following finite subset of $\operatorname{Spec}(R)$:

$$\begin{aligned}\mathcal{M} = \{\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_s\} \cup \{\mathfrak{q} \in \operatorname{Spec}(R) \mid\ & \mathfrak{q} \text{ is minimal over } \mathfrak{p}_i + \mathfrak{p}_j \\ & \text{for some } i, j \in \{0, \ldots, r\} \text{ and } \mathfrak{q} \not\subseteq \mathfrak{p}\},\end{aligned}$$

and let I be the intersection of all prime ideals in $\mathcal{M}$. So we have $I \not\subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an intersection of maximal ideals, there must exist a maximal ideal $\mathfrak{m}$ of R with $\mathfrak{p} \subseteq \mathfrak{m}$ and $I \not\subseteq \mathfrak{m}$. In particular, $\mathfrak{m}$ contains the same minimal prime ideals of R.

For all minimal prime ideals $\mathfrak{p}_i$ (with $i \in \{0, \ldots, r\}$) let $Z_{i,\mathfrak{p}}$ and $Z_{i,\mathfrak{m}}$ be the corresponding irreducible component of $\operatorname{Spec}(R_\mathfrak{p})$ and $\operatorname{Spec}(R_\mathfrak{m})$, respectively. Then for all i, $j \in \{0, \ldots, r\}$ we have

$$\begin{aligned}\operatorname{codim}_{\operatorname{Spec}(R_\mathfrak{p})}(Z_{i,\mathfrak{p}} \cap Z_{j,\mathfrak{p}}) &= \operatorname{ht}_{R_\mathfrak{p}}((\mathfrak{p}_i + \mathfrak{p}_j)_\mathfrak{p}) \\ &= \operatorname{ht}_{R_\mathfrak{m}}((\mathfrak{p}_i + \mathfrak{p}_j)_\mathfrak{m}) \\ &= \operatorname{codim}_{\operatorname{Spec}(R_\mathfrak{m})}(Z_{i,\mathfrak{m}} \cap Z_{j,\mathfrak{m}}),\end{aligned}$$

where the secound equality is true since $\mathfrak{p}$ and $\mathfrak{m}$ contain the same minimal elements of $\mathcal{V}_{\operatorname{Spec}(R)}(\mathfrak{p}_i + \mathfrak{p}_j) := \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{p}_i + \mathfrak{p}_j \subseteq \mathfrak{q}\}$. So the irreducible components of $\operatorname{Spec}(R_\mathfrak{p})$ intersect in the same codimension as the corresponding components of $\operatorname{Spec}(R_\mathfrak{m})$. Therefore, by Proposition 2.4, connectedness in codimension k of $\operatorname{Spec}(R_\mathfrak{m})$ implies the same property for $\operatorname{Spec}(R_\mathfrak{p})$. □

2.2 Hartshorne's Connectedness Theorem

This section recalls the definition of the Cohen-Macaulay defect and states Hartshorne's connectedness theorem in the form that we want to use.

First let us recall the notion of the depth of an ideal $I \subsetneq R$ in a Noetherian ring R. A sequence of elements $a_1, \dots, a_n \in I$ is called ***R*-regular** if for $i = 1, \dots, n$ we have:

$$a_i \text{ is not a zero divisor on } R/(a_1, \dots, a_{i-1}).$$

It can be shown that all maximal R-regular sequences in I have the same finite length, which is called the ***I*-depth** of R and is denoted by $\mathrm{depth}(I,\, R)$ (see [Mat89, Theorem 16.7]), and that the inequality $\mathrm{depth}(I,\, R) \le \mathrm{ht}_R(I)$ holds (see [Eis95, Proposition 18.2]). If R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, then $\mathrm{depth}(\mathfrak{m},\, R) =: \mathrm{depth}(R)$ is simply called the **depth** of R, and the above inequality reads as $\mathrm{depth}(R) \le \dim(R)$.

In his famous connectedness theorem Hartshorne could relate the connectedness property of the spectrum of a Noetherian ring R to the depth of localizations of R.

Theorem 2.8. (Hartshorne [Har62]) *Let R be a Noetherian ring and $k \in \mathbb{N}_0$. Suppose that $\mathrm{Spec}(R)$ is connected and that for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}_R(\mathfrak{p}) > k$ we have $\mathrm{depth}(R_\mathfrak{p}) \ge 2$. Then $\mathrm{Spec}(R)$ is connected in codimension k.*

Proof. In [Har62, Corollary 2.3] it is shown that $\mathrm{Spec}(R)$ is *locally* connected in codimension k under this hypothesis. As $\mathrm{Spec}(R)$ is assumed to be connected, Corollary 2.6 implies that $\mathrm{Spec}(R)$ is connected in codimension k. $\square$

Definition 2.9. *For a Noetherian local ring $(R, \mathfrak{m})$ the* ***Cohen-Macaulay defect*** *is defined as*

$$\mathbf{cmd}(\boldsymbol{R}) := \dim(R) - \mathrm{depth}(R) \in \mathbb{N}_0.$$

Lemma 2.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I \subsetneq R$ a proper ideal. Then the inequality*

$$\mathrm{ht}_R(I) - \mathrm{depth}(I,\, R) \le \dim(R) - \mathrm{depth}(R)$$

holds. Furthermore, for every prime ideal $\mathfrak{p}$ of R we have

$$\mathrm{cmd}(R_\mathfrak{p}) \le \mathrm{cmd}(R).$$

Proof. Let $k := \mathrm{depth}(R)$ and $r := \dim(R/I)$. Then we get

$$\mathrm{Ext}^i_R(R/I,\, R) = 0 \quad \text{for } i = 0, \dots, k - r - 1$$

by [Mat89, Theorem 17.1]. As $\mathrm{depth}(I,\, R)$ can be obtained from the vanishing of these Ext-modules (again see [Mat89, Theorem 16.7]), the inequality

$$\mathrm{depth}(I,\, R) \ge k - r$$

follows. Together with

$$\mathrm{ht}_R(I) + r = \mathrm{ht}_R(I) + \dim(R/I) \le \dim(R)$$

this gives the first statement.

The second statement follows immediately since $\mathrm{depth}(\mathfrak{p},\, R) \le \mathrm{depth}(R_\mathfrak{p})$ (localizing an R-regular sequence in $\mathfrak{p}$ gives an $R_\mathfrak{p}$-regular sequence in $\mathfrak{p}_\mathfrak{p}$). $\square$

Definition 2.11. *For a Noetherian ring R the Cohen-Macaulay defect is defined as*

$$\operatorname{cmd}(R) := \sup\{\operatorname{cmd}(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \in \mathbb{N}_0 \cup \{\infty\},$$

which is consistent with Definiton 2.9 by Lemma 2.10. Furthermore, R is called ***Cohen-Macaulay*** *if* $\operatorname{cmd}(R) = 0$.

Let us express a different version of Hartshorne's connectedness theorem in terms of the Cohen-Macaulay defect.

Corollary 2.12. *Let R be a Noetherian ring. Suppose that* $\operatorname{Spec}(R)$ *is connected and that $k := \operatorname{cmd}(R)$ is finite. Then* $\operatorname{Spec}(R)$ *is connected in codimension $k + 1$.*

Proof. For $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht}_R(\mathfrak{p}) > k + 1$ we have

$$k \geq \operatorname{ht}_R(\mathfrak{p}) - \operatorname{depth}(R_{\mathfrak{p}}) \geq k + 2 - \operatorname{depth}(R_{\mathfrak{p}}),$$

hence $\operatorname{depth}(R_{\mathfrak{p}}) \geq 2$, and the result follows with Theorem 2.8. □

2.3 The Separating Variety and Reflections

We want to study the separating variety next. So for the entire section let G be a finite group and let X be a G-variety with $n := \dim(X)$. Starting with the irreducible components of $\mathcal{V}_{\text{sep}}$, we will precisely see what its connectedness in codimension k means for X and G.

Proposition 2.13. *Let $X = \bigcup_{i=1}^{r} X_i$ be decomposed into its irreducible components X_i. Then for all i and for all $\sigma \in G$ the subspace*

$$H_{\sigma,i} := \{(x, \sigma x) \mid x \in X_i\} \subseteq X \times X$$

is an irreducible component of $\mathcal{V}_{\text{sep}}$, and $\mathcal{V}_{\text{sep}}$ is the union of all $H_{\sigma,i}$.

Proof. Since G is finite, the invariants separate the orbits (see Proposition 1.4). With Definition 1.11 we see that two points $x, y \in X$ lie in the same orbit if and only if $(x, y) \in \mathcal{V}_{\text{sep}}$. So the separating variety really is the graph of the action of G on X:

$$\mathcal{V}_{\text{sep}} = \{(x, \sigma x) \mid x \in X, \sigma \in G\}.$$

Hence $\mathcal{V}_{\text{sep}}$ is the union of all $H_\sigma := \{(x, \sigma x) \mid x \in X\}$. Each H_σ is an affine variety isomorphic to X, so it decomposes as

$$H_\sigma = \bigcup_{i=1}^{r} H_{\sigma,i}$$

into its irreducible components. □

Remark 2.14. With the notation of Proposition 2.13 we also see:

(a) Each $H_{\sigma,i}$ is isomorphic to X_i. In particular, the separating variety has the same dimension as X.

(b) By Remark 2.2(a), $\operatorname{codim}_{X\times X}(\mathcal{V}_{\text{sep}})$ is the minimum of all $\operatorname{codim}_{X\times X}(H_{\sigma,i})$. Since $\operatorname{codim}_{X\times X}(H_{\sigma,i}) = \operatorname{codim}_{X_i\times\sigma X_i}(H_{\sigma,i}) = \dim X_i$, the formula

$$\operatorname{codim}_{X\times X}(\mathcal{V}_{\text{sep}}) = \min\{\dim(X_i) \mid i = 1,\dots,r\}$$

holds. In particular, if X is equidimensional, then $\operatorname{codim}_{X\times X}(\mathcal{V}_{\text{sep}}) = \dim(X) =: n$.

Definition 2.15. *Let k be a natural number. An element $\sigma \in G$ is called a **k-reflection** if its fixed space $X^\sigma := \{x \in X \mid \sigma x = x\}$ has codimension at most k in X. For $k = 1$ we simply say that σ is a reflection.*

Remark 2.16. For $\sigma \in G$ let I_σ be the ideal in $K[X]$ generated by all $\sigma f - f$ with $f \in K[X]$. Then I_σ defines X^σ as a subvariety of X, hence σ is a k-reflection if and only if $\operatorname{ht}_{K[X]}(I_\sigma) \le k$.

It was shown by Dufresne [Duf09, pf. of Theorem 1.1] that for a linear action connectedness in codimension 1 of $\mathcal{V}_{\text{sep}}$ implies that the group is generated by 1-reflections. In the following theorem we extend this to non-linear actions and to $k > 1$, and we also add a converse, which will be needed in the next section.

Theorem 2.17. *Let k be a natural number. Then the separating variety $\mathcal{V}_{\text{sep}}$ is connected in codimension k if and only if X is connected in codimension k and G is generated by k-reflections.*

Proof. Again, let $X = \bigcup_{i=1}^r X_i$ be decomposed into its irreducible components X_i, which leads to the components $H_{\sigma,i}$ of $\mathcal{V}_{\text{sep}}$ as seen in Proposition 2.13. First, we look at the intersection of two components of $\mathcal{V}_{\text{sep}}$ to see which codimension arises. For σ, $\tau \in G$ and indices i, j we have

$$H_{\sigma,i} \cap H_{\tau,j} = \{(x,y) \mid x \in X_i \cap X_j,\ y = \sigma x = \tau x\} \cong (X_i \cap X_j)^{\tau^{-1}\sigma}.$$

We know from Remark 2.14 that $\dim(X) = n = \dim(\mathcal{V}_{\text{sep}})$. In addition, we get

$$\operatorname{codim}_{\mathcal{V}_{\text{sep}}}(H_{\sigma,i} \cap H_{\tau,j}) = \operatorname{codim}_X((X_i \cap X_j)^{\tau^{-1}\sigma}). \tag{2.3.1}$$

Suppose $\mathcal{V}_{\text{sep}}$ is connected in codimension k. By assumption, for all $\sigma \in G$ and i, j there exists a sequence of irreducible components $H_{\sigma_0,i_0}, \dots, H_{\sigma_s,i_s}$ of $\mathcal{V}_{\text{sep}}$ with $i_0 = i$, $i_s = j$, $\sigma_0 = \iota$ (the neutral element of G), $\sigma_s = \sigma$ and

$$\operatorname{codim}_{\mathcal{V}_{\text{sep}}}\left(H_{\sigma_l,i_l} \cap H_{\sigma_{l+1},i_{l+1}}\right) \le k \quad \text{for } 0 \le l \le s-1. \tag{2.3.2}$$

Putting (2.3.1) and (2.3.2) together leads to the inequality

$$\operatorname{codim}_X\left(X_{i_l} \cap X_{i_{l+1}}\right)^{\sigma_l^{-1}\sigma_{l+1}}) \le k \quad \text{for } 0 \le l \le s-1. \tag{2.3.3}$$

In particular, (2.3.3) shows that $X_{i_l} \cap X_{i_{l+1}}$ has codimension $\le k$. So we have a sequence of irreducible components from $X_{i_0} = X_i$ to $X_{i_s} = X_j$ that intersect in codimension $\le k$, hence X is connected in codimension k by Proposition 2.4.

Moreover, (2.3.3) implies that all $X^{\sigma_l^{-1}\sigma_{l+1}}$ have codimension $\le k$, i.e. each $\sigma_l^{-1}\sigma_{l+1}$ is a k-reflection. Using $\sigma_0 = \iota$ and $\sigma_s = \sigma$ we can write

$$\sigma = \sigma_0^{-1}\sigma_s = (\sigma_0^{-1}\sigma_1)\cdot(\sigma_1^{-1}\sigma_2)\cdot\dots\cdot(\sigma_{s-1}^{-1}\sigma_s)$$

as a product of k-reflections.
So, we have proven the only-if-part by simply splitting 2.3.3 into two weaker conclusions. It may therefore be surprising that the converse holds as well.

To prove it, let us start with indices i, j, and a sequence of components $X_{i_0}, \ldots, X_{i_s}$ with $X_i = X_{i_0}$, $X_j = X_{i_s}$ and $\operatorname{codim}_X(\big(X_{i_l} \cap X_{i_{l+1}}\big) \leq k$. Consequently, for $\sigma \in G$ we know from (2.3.1), that all $H_{\sigma,i_l} \cap H_{\sigma,i_{l+1}}$ have codimension $\leq k$. So we already have a sequence from $H_{\sigma,i}$ to $H_{\sigma,j}$ as desired.

Now take two elements σ', $\sigma'' \in G$. By assumption, there exist k-reflections $\tau_1, \ldots, \tau_s \in G$ with $(\sigma')^{-1}\sigma'' = \tau_1 \cdot \ldots \cdot \tau_s$. Since for all l we have

$$\min\{\operatorname{codim}_X(X_m^{\tau_l}) \mid m = 1, \ldots, r\} = \operatorname{codim}_X(X^{\tau_l}) \leq k,$$

for each τ_l there exists an i_l such that

$$\operatorname{codim}_X(X_{i_l}^{\tau_l}) \leq k. \tag{2.3.4}$$

If we write $\sigma_0 := \sigma'$ and $\sigma_l := \sigma_{l-1}\tau_l$ for $l = 1, \ldots, s$, then

$$\sigma_s = \sigma_0 \cdot \tau_1 \cdot \ldots \cdot \tau_s = \sigma' \cdot (\sigma'^{-1} \cdot \sigma'') = \sigma''.$$

It follows from (2.3.1) together with (2.3.4) that

$$\operatorname{codim}_{\mathcal{V}_{\text{sep}}}(H_{\sigma_{l-1},i_l} \cap H_{\sigma_l,i_l}) = \operatorname{codim}_X((X_{i_l})^{\sigma_{l-1}^{-1}\sigma_l}) = \operatorname{codim}_X((X_{i_l})^{\tau_l}) \leq k.$$

We already saw how to construct a sequence of components from every H_{σ,i_l} to $H_{\sigma,i_{l+1}}$ as desired. Putting these together, for all i, j we can construct a sequence

$$H_{\sigma_0,i}, \ldots, H_{\sigma_0,i_1}, H_{\sigma_1,i_1}, \ldots, H_{\sigma_1,i_2}, H_{\sigma_2,i_2}, \ldots, H_{\sigma_s,i_s}, \ldots, H_{\sigma_s,j},$$

from $H_{\sigma',i}$ to $H_{\sigma'',j}$ such that two successive components intersect in codimension $\leq k$. □

Since $\mathcal{V}_{\text{sep}}$ is required to be connected in the proof of the main theorem of this chapter, Theorem 2.19, we specialize Theorem 2.17 to the case $k = n = \dim(X)$.

Corollary 2.18. *The separating variety $\mathcal{V}_{\text{sep}}$ is connected if and only if X is connected and G is generated by elements having a fixed point.*

2.4 Main Result about Polynomial Separating Algebras

We will now combine Hartshorne's connectedness theorem from Section 2.2 with our results on $\mathcal{V}_{\text{sep}}$ from Section 2.3. Again for the entire section let G be a finite group and X a G-variety with $\dim(X) = n$. With the notation $\mathcal{I}_{\text{sep}}$ from Chapter 1 the coordinate ring of the separating variety is $K[\mathcal{V}_{\text{sep}}] = (K[X] \otimes_K K[X])/\sqrt{\mathcal{I}_{\text{sep}}}$.

Theorem 2.19. *Let X be connected, and let G be generated by elements having a fixed point. Write $R = K[X] \otimes_K K[X]$ and define*

$$k := \min\{\operatorname{cmd}(R/J) \mid J \subseteq R \text{ an ideal with } \sqrt{J} = \sqrt{\mathcal{I}_{\text{sep}}}\}.$$

Then G is generated by $(k+1)$-reflections.

Proof. The assumptions on X and the action of G imply that $\mathcal{V}_{\text{sep}}$ is connected by Corollary 2.18. Let J be an ideal in R with $\sqrt{J} = \sqrt{\mathcal{I}_{\text{sep}}}$ and $k = \text{cmd}(R/J)$. Corollary 2.12 tells us now that $\text{Spec}(R/J)$, which is homeomorphic to $\text{Spec}(K[\mathcal{V}_{\text{sep}}])$, is connected in codimension $k+1$. Of course, it is equivalent to say that $\mathcal{V}_{\text{sep}}$ is connected in codimension $k+1$. Therefore, by Theorem 2.17, G is generated by $(k+1)$-reflections. □

Example 1.14 showed that $\mathcal{I}_{\text{sep}}$ need not be radical. Furthermore, in Example 2.22, we will see that neither $\mathcal{I}_{\text{sep}}$ nor $\sqrt{\mathcal{I}_{\text{sep}}}$ must have the smallest Cohen-Macaulay defect among all ideals $J \subseteq K[X] \otimes_K K[X]$ with $\sqrt{J} = \sqrt{\mathcal{I}_{\text{sep}}}$. In particular, the number k in Theorem 2.19 need not be the Cohen-Macaulay defect of $K[\mathcal{V}_{\text{sep}}]$. Since an ideal is called set-theoretically Cohen-Macaulay if there exists a Cohen-Macaulay ideal with the same radical (cf. [SW07]), we propose to call this number the **set-theoretical Cohen-Macaulay defect** of $\mathcal{V}_{\text{sep}}$.

To the best of my knowledge, no algorithm is known to compute the set-theoretical Cohen-Macaulay defect of a Noetherian ring. This might be the reason why I could not find an example in which this number is not the minimal m such that G is generated by $(m+1)$-reflections. However, there are surprisingly many examples (like Example 2.22) in which these two numbers coincide.

Theorem 2.20. *Let X be connected and Cohen-Macaulay, and let G be generated by elements having a fixed point. If $\gamma_{\text{sep}} = n$ (see Definition 1.6), then G is generated by reflections.*

Proof. Since X is Cohen-Macaulay, it follows that $X \times X$ is Cohen-Macaulay, too. We use [WITO69] as a reference for that. In addition, X is connected, so X and $X \times X$ are also equidimensional, since local Cohen-Macaulay rings are equidimensional (see [Eis95, Corollary 18.11]).

Now let $\{f_1, \ldots, f_n\}$ be a set of separating invariants. Using the δ-map (defined in 1.0.2) this separating set defines the following ideal in $K[X] \otimes_K K[X]$:

$$J := (\delta(f_1), \ldots, \delta(f_n)) \,. \tag{2.4.1}$$

By Proposition 1.13, J has the same radical as $\mathcal{I}_{\text{sep}}$. Hence we have

$$\text{ht}(J) = \text{ht}(\sqrt{\mathcal{I}_{\text{sep}}}) = \text{codim}_{X \times X}(\mathcal{V}_{\text{sep}}) = n, \tag{2.4.2}$$

by Remark 2.14. Since $K[X] \otimes_K K[X]$ is Cohen-Macaulay, 2.4.1 and 2.4.2 imply that J is generated by a $(K[X] \otimes_K K[X])$-regular sequence. Therefore, $(K[X] \otimes_K K[X])/J$ is Cohen-Macaulay as well (see [Eis95, Proposition 18.13]). Now we can use Theorem 2.19 with $k = 0$. □

Remark 2.21. Of course, the assumptions on X and G in Theorem 2.19 and Theorem 2.20 are satisfied if $X = V$ is a linear representation of G.

Dufresne [Duf09] gave an example of a representation for which the invariant ring is not a polynomial ring, but still γ_{sep} equals n. This suggested that the choice of J in Theorem 2.19 matters. The following example illustrates this point as it results in various Cohen-Macaulay defects. It is taken from Kemper's et al. [KKM+01] database of invariant rings.

Example 2.22. [KKM+01, ID 10253] Let $\mathrm{char}(K) = 2$. We look at the following subgroup, isomorphic to $C_2 \times C_2 \times C_2$, of $\mathrm{GL}_4(K)$:

$$G := \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \rangle \subseteq \mathrm{GL}_4(K).$$

Its natural action on $V = K^4$ is generated by reflections. Using the computer algebra system MAGMA [BCP97], we have computed the primary invariants

$$\begin{aligned} f_1 &:= x_1, \\ f_2 &:= x_3, \\ f_3 &:= x_1^2 x_3 x_4 + x_1^2 x_4^2 + x_1 x_3^2 x_4 + x_1 x_3 x_4^2 + x_3^2 x_4^2 + x_4^4, \\ f_4 &:= x_1^3 x_2 + x_1 x_2 x_3^2 + x_1 x_3^2 x_4 + x_1 x_3 x_4^2 + x_2^4 + x_2^2 x_3^2 + x_3^3 x_4 + x_3^2 x_4^2, \end{aligned}$$

and a secondary invariant

$$h := x_1^2 x_2 + x_1 x_2^2 + x_3^2 x_4 + x_3 x_4^2.$$

Hence, the invariant ring

$$K[V]^G = K[x_1, x_2, x_3, x_4]^G = K[f_1, f_2, f_3, f_4, h]$$

is not a polynomial ring. Between the generating invariants there is the relation

$$f_1^3 h + f_1^2 f_3 + f_1 f_2^2 h + f_2^2 f_4 + h^2 = 0.$$

So by defining $g_3 := f_1 h + f_3$ and $g_4 := f_1 h + f_4$, we get $h^2 = f_1^2 g_3 + f_2^2 g_4$. As $\mathrm{char}(K) = 2$, we see from this relation that the values of f_1, f_2, g_3, and g_4 at a point $x \in K^4$ determine $h(x)$. From the definition of g_3 and g_4 it is clear that the values of f_3 and f_4 at x are also determined by this. Hence $S := \{f_1, f_2, g_3, g_4\}$ is separating. In this example $\mathcal{I}_{\mathrm{sep}}$ is not a radical ideal. Let J be the ideal in $R := K[V] \otimes_K K[V]$ generated by $\delta(S)$. Using the graded version of the Auslander-Buchsbaum formula, we calculated the following Cohen-Macaulay defects with MAGMA (see Appendix A for the MAGMA code):

$$\mathrm{cmd}(R/I) = 2, \quad \mathrm{cmd}(R/\sqrt{I}) = 1, \quad \mathrm{cmd}(R/J) = 0.$$

Of course, $\mathrm{cmd}(R/J) = 0$ is not surprising, as it was used in Theorem 2.20. $\triangleleft$

Let us look at the assumptions in Theorem 2.20 more closely. Of course, any example of a (non-trivial) free group action of G on X with an invariant ring isomorphic to a polynomial ring shows that the assumption that G has fixed points cannot be dropped from Theorem 2.20.

Example 2.23. Let the characteristic of K be a prime number p, and let $G = \mathbb{F}_p$ be the cyclic group of order p. When we look at the additive action of $\mathbb{F}_p$ on $V = K$ via $(\sigma, x) \mapsto \sigma + x$, we see that

$$K[V]^G = K[x]^G = K[x^p - x]$$

is a polynomial ring. But a non-zero group element $\sigma \in \mathbb{F}_p$ does not have a fixed point, so in particular, G is not a reflection group. $\triangleleft$

The following very general example serves as a demarcation in our purpose to prove theorems in which nice algebraic properties of a separating subalgebra $A \subseteq K[X]^G$ and/or a small value of γ_{sep} imply that the group is generated by certain reflections. More precisely, it shows that the variety X needs to be at least connected in codimension k, in order to conclude that G is generated by k-reflections.

Example 2.24. Let X be a union of two n-dimensional affine spaces (over K) intersecting in a single point. For example, if $\text{char}(K) \neq 2$, we could define X as the following subvariety of K^{2n}:

$$X = \mathcal{V}(x_1 - x_{n+1}, \dots, x_n - x_{2n}) \cup \mathcal{V}(x_1 + x_{n+1}, \dots, x_n + x_{2n}). \tag{2.4.3}$$

So X is connected (or connected in codimension n, if we want to use that notion), but not connected in codimension k for any $k < n$. The cyclic group of order 2 acts on X by interchanging the two n-dimensional affine spaces and fixing the intersection point. With X defined as in 2.4.3 we make this precise with the following matrix representation of $\mathbb{Z}/2\mathbb{Z}$:

$$G = \left\langle \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \right\rangle \subseteq \text{GL}_{2n}(K).$$

Since X is mapped to itself by this matrix, it is a G-variety. With a single fixed point at the origin this action on X is not generated by k-reflections for any $k < n$. The invariant ring of the representation $V = K^{2n}$ of G can be easily seen to be generated by $x_1, \dots, x_n$ and the monomials of degree 2 in the variables $x_{n+1}, \dots, x_{2n}$:

$$K[V]^G = K[x_1, \dots, x_n,\, x_{n+1}^2,\, x_{n+1}x_{n+2}, \dots, x_{2n}^2].$$

Since we are in a non-modular case, the finite group G is linearly reductive. Therefore, $K[X]^G$ is the quotient ring of $K[V]^G$ modulo the vanishing ideal I of X. For all $i, j \in \{1, \dots, n\}$ we have

$$x_i x_j - x_{n+i} x_{n+j} \in I.$$

Therefore, the invariant ring of this action on X is

$$K[X]^G = K[V]^G / (K[V]^G \cap I) = K[\overline{x}_1, \dots, \overline{x}_n],$$

which is a polynomial ring, so it has the best structure we could hope for. In particular, we have $\gamma_{\text{sep}} = n$, but the group is not generated by $(n-1)$-reflections. ◁

We can use the above example to show that the assumption that X is Cohen-Macaulay cannot be dropped from Theorem 2.20.

Example 2.25. Let X and G be as in Example 2.24 with $n = 2$. So X is a union of two planes that intersect in a single point and G is a cyclic group of order 2 that acts on X by interchanging the two planes. Hartshorne's connectedness theorem in the form of Corollary 2.12 now tells us exactly that X is not Cohen-Macaulay at the intersection point, since it is not connected in codimension 1 there. As in Example 2.24 the invariant ring will be a polynomial ring, hence $\gamma_{\text{sep}} = n$, but in contrast to Theorem 2.20, G is not generated by reflections. ◁

2.5 Application to Muliplicative Invariant Theory

It was suggested to me by Gregor Kemper to check if the results of the previous section can be applied to the so called "semigroup problem" in multiplicative invariant theory. Indeed this is true and it is carried out in this section.

Multiplicative invariant theory is a branch of invariant theory which deals with the action of a matrix with integer coefficients on the monomials of a Laurent polynomial ring. The subject is studied in detail in Martin Lorenz's book [Lor05]; here we will only give a short introduction, similar to [Tes04, Section 1.4].

We start with a lattice L, i.e. a free abelian group of finite rank, and (at first) an arbitrary group G acting on L by automorphisms. The ring whose invariants are studied in this subject is a group ring of L. We will not follow the general path in Lorenz's book where the base ring K is just any commutative ring (as the case $K = \mathbb{Z}$ is also important in multiplicative invariant theory). Instead (as through the entire thesis) we denote by K an algebraically closed field and look at the group ring $K[L]$ of L over K. Since the group law of L should appear multiplicatively in $K[L]$, we will write the elements of $K[L]$ as finite formal sums with the lattice points in the exponents:

$$K[L] = \{\sum_{l\in L} \lambda_l\, x^l \mid \lambda_l \in K\}.$$

The action of G on L induces an action on $K[L]$ by K-algebra automorphisms. This follows from the universal property of the group ring, more precisely, for $\sigma \in G$ we have

$$\sigma \cdot \left(\sum_{l\in L} \lambda_l\, x^l\right) := \sum_{l\in L} \lambda_l\, x^{\sigma\cdot l}.$$

Choosing a $\mathbb{Z}$-basis of L amounts to choosing an isomorphism of L with $\mathbb{Z}^n$ where n is the rank of L. The action of G is then given by a group homomorphism $G \to \mathrm{GL}_n(\mathbb{Z})$, and the group ring of L over K is isomorphic to the ring of Laurent polynomials over K in n indeterminates:

$$K[L] \cong K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

We will view $K[L]$ as this Laurent polynomial algebra, which makes L isomorphic to the group of monomials

$$L \cong \{x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} \mid k_i \in \mathbb{Z}\} \subseteq K[L]^\times,$$

a subgroup of the units of $K[L]$.

Example 2.26. Let G be the following cyclic group of order 2:

$$G = \langle\sigma\rangle = \langle\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\rangle \subseteq \mathrm{GL}_2(\mathbb{Z}).$$

Its generator σ acts on the Laurent polynomial ring $K[x_1^{\pm 1}, x_2^{\pm 1}]$ by sending x_1 to x_1x_2 and x_2 to x_2^{-1}. So for example $f = 3x_1^2 + x_1x_2 - x_2^{-3}$ is mapped to

$$\sigma \cdot f = 3x_1^2x_2^2 + x_1 - x_2^3.$$

In particular, observe that the $\mathbb{Z}$-grading of $K[x_1^{\pm 1}, x_2^{\pm 1}]$ is in no way preserved by the action. ◁

It is a speciality of multiplicative invariants that the computation of $K[L]^G$ reduces to the case of finite groups. More precisely $K[L]^G$ is isomorphic to the invariant ring of a finite quotient group of G acting on a sublattice of L (see [Lor05, Prop. 3.1.1]). This makes the case of finite groups especially important.

Let G now be a finite group. Multiplicative invariant theory can be put in our general setting of an algebraic group acting on an affine variety X by setting

$$X := (\mathbb{G}_m)^n = \mathcal{V}(x_1y_1 - 1, \ldots, x_ny_n - 1) \subseteq K^{2n}, \tag{2.5.1}$$

i.e. X is an n-dimensional algebraic torus. Its coordinate ring $K[X]$ is precisely the Laurent polynomial ring $K[L]$ in n indeterminates. Therefore, the G-action on $K[X]$ by K-algebra automorphisms makes X into a G-variety.

In multiplicative invariant theory an element $\sigma \in G$ is called a ***k*-reflection** if the sublattice $\{\sigma l - l \mid l \in L\}$ has rank at most k (see [Lor05, Section 1.7]). But by [Lor05, Lemma 4.5.1], this is equivalent to the condition that the ideal $(\sigma f - f \mid f \in K[L])$ in $K[L] = K[X]$ has height at most k, so that a reflection on L is exactly a reflection on X (see Remark 2.16).

An analogous result in multiplicative invariant theory to the Shephard-Todd-Chevalley-Serre theorem is [Lor05, Theorem 7.1.1]. It says that under the assumption that $\mathrm{char}(K)$ does not divide $|G|$ the following two statements (among others) are equivalent:

(a) $K[L]^G$ is (isomorphic to) a mixed Laurent polynomial ring, i.e. there exists a $k \in \{0, \ldots, n\}$ with $K[L]^G \cong K[x_1^{\pm 1}, \ldots, x_k^{\pm 1}, x_{k+1}, \ldots, x_n]$,

(b) G is generated by reflections on L and $K[L]^G$ is a unique factorization domain.

Without any assumptions about the characteristic of K, Lorenz [Lor01] proved the following: If G is generated by reflections, then there is a submonoid $M \subseteq K[L]^G$ such that $K[L]^G$ is isormorphic to the semigroup algebra $K[M]$. The question whether the converse of this statement holds is called the "semigroup problem in multiplicative invariant theory" (see [Tes04, Section 1.5]). Some partial converses are given in [Tes04] and [Lor05, Section 10.2]. We add another one to the list by proving that of the above statements (a) implies (b) independently of the characteristic of K.

Theorem 2.27. *Let L be a lattice and let G be a finite group acting on L by automorphisms. If $K[L]^G$ is isomorphic to a mixed Laurent polynomial ring, then G is generated by reflections.*

Proof. Since $\mathbb{G}_m$ is a connected linear algebraic group, the affine variety X in 2.5.1 is irreducible and non-singular (hence Cohen-Macaulay). Furthermore, every $\sigma \in G$ fixes the point $(1, \ldots, 1) \in X$. Hence the prerequisites of Theorem 2.20 are satisfied. By assumption, we have

$$K[X]^G = K[L]^G = K[f_1^{\pm 1}, \ldots, f_k^{\pm 1}, f_{k+1}, \ldots, f_n]$$

with invariants $f_i \in K[X]$ (where obviously $n = \mathrm{trdeg}_K(K[X]^G) = \dim(X)$). But from this generating set of invariants we easily extract the smaller separating set

$$S = \{f_1, \ldots, f_n\},$$

since the inverses of $f_1, \ldots, f_k$ are not needed to separate the orbits. So we have $\gamma_{\mathrm{sep}} = n$ and the result follows with Theorem 2.20. □

Chapter 3

Preliminaries on Completion

This chapter recalls some basic facts about the completion of rings and modules, which are needed in Chapters 4 and 5. The material is taken from the standard textbooks on commutative algebra of Atiyah-Macdonald [AM69], Matsumura [Mat89], Eisenbud [Eis95] and in particular from Singh's "Basic Commutative Algebra" [Sin11]. A reader familiar with the theory of completions may therefore wish to skip this chapter entirely.

We do not aim here to simply state these facts in the shortest possible way, but also to explain a bit of the "idea behind completion". At the beginning we will develop the basic concepts in continuous text, and only later state the results, especially those which are needed elsewhere, in propositions and theorems.

3.1 Filtered Rings and Modules

Let R be a ring. A (descending) **filtration** on R is a sequence $(I_n)_{n\geq 0}$ of ideals of R with the following properties for all m, $n \in \mathbb{N}_0$:

$$\begin{aligned} &\text{(i)} && I_0 = R, \\ &\text{(ii)} && I_{n+1} \subseteq I_n, \\ &\text{(iii)} && I_m \cdot I_n \subseteq I_{m+n}. \end{aligned}$$

Most common are so-called ***I*-adic filtrations** of R, where I is an ideal of R and $I_n := I^n$ is the n-th power of I. A filtration can be used to define a topology on R. The basic idea is that a ring element x should be considered "close to zero" if it is contained in an ideal I_n for a large n.

The construction of the completion with respect to a given filtration will be needed for modules as well. Therefore, we will proceed with an R-module M and define and study filtrations on M next.

Let us assume for the entire section that R is already filtered, this means there is a given filtration $(I_n)_{n\geq 0}$ on R. A filtration on M should always be compatible with the given filtration on R, hence we will call a sequence $(M_n)_{n\geq 0}$ of submodules of M a filtration on M if the following properties hold for all $m,\ n \in \mathbb{N}_0$:

$$\begin{aligned} &\text{(i)} && M_0 = M, \\ &\text{(ii)} && M_{n+1} \subseteq M_n, \\ &\text{(iii)} && I_m \cdot M_n \subseteq M_{m+n}. \end{aligned}$$

For example, we always get a filtration on M by setting $M_n := I_n \cdot M$.

Now for the entire section let M be a **filtered *R*-module** , i.e. there is a given filtration $\mathcal{F} := (M_n)_{n\geq 0}$ on M. Then every submodule $U \subseteq M$ is also filtered, with the filtration $(U \cap M_n)_{n\geq 0}$. We define the **order** or **valuation** of M with respect to $\mathcal{F}$ as the following function:

$$v_{\mathcal{F}} : M \to \mathbb{N}_0 \cup \{\infty\}, \quad x \mapsto \sup\{n \in \mathbb{N}_0 \mid x \in M_n\}.$$

The following basic properties of $v_{\mathcal{F}}$ are easily verified for all $x,\ y \in M$:

$$\begin{aligned} &\text{(i)} && v_{\mathcal{F}}(x) = \infty \quad \Leftrightarrow \quad x \in \bigcap_{n=0}^{\infty} M_n, \\ &\text{(ii)} && v_{\mathcal{F}}(-x) = v_{\mathcal{F}}(x), \\ &\text{(iii)} && v_{\mathcal{F}}(x+y) \geq \min\{v_{\mathcal{F}}(x),\ v_{\mathcal{F}}(y)\}. \end{aligned} \tag{3.1.1}$$

This allows us to define the distance of two elements of M through the following function:

$$d_{\mathcal{F}} : M \times M \to \mathbb{R}_{\geq 0}, \quad (x, y) \mapsto \left(\frac{1}{2}\right)^{v_{\mathcal{F}}(x-y)},$$

where we have used the symbolic convention $\left(\frac{1}{2}\right)^{\infty} = 0$. It should be noted that $\frac{1}{2}$ was somehow an arbitrary choice, this fraction could be replaced by any real number strictly between 0 and 1.

For all $x,\ y,\ z \in M$ we have the following immediate consequences:

$$\begin{aligned} &\text{(i)} && d_{\mathcal{F}}(x, x) = 0, \\ &\text{(ii)} && d_{\mathcal{F}}(x, y) = d_{\mathcal{F}}(y, x), \\ &\text{(iii)} && d_{\mathcal{F}}(x, z) \leq \max\{d_{\mathcal{F}}(x, y),\ d_{\mathcal{F}}(y, z)\}. \end{aligned} \tag{3.1.2}$$

The inequality in (iii) is called ultrametric inequality. Obviously, it is stronger than the usual triangle inequality $d_{\mathcal{F}}(x, z) \leq d_{\mathcal{F}}(x, y) + d_{\mathcal{F}}(y, z)$. So these three properties combined imply that $d_{\mathcal{F}}$ is a **pseudometric** on M. With property (i) of 3.1.1 we see immediately that $d_{\mathcal{F}}$ is a metric if and only if $\bigcap_{n=0}^{\infty} M_n = \{0\}$. (In fact, due to the stronger triangle inequality, $d_{\mathcal{F}}$ is called an ultrametric or pseudoultrametric, respectively.)

But either way, the metric or pseudometric induces a topology on M, where a subset $U \subseteq M$ is called **open** if for each of its points it contains an ε-ball (for some $\varepsilon > 0$) around this point. Since the valuation $v_{\mathcal{F}}$ takes only discrete values, more precisely values in $\mathbb{N}_0 \cup \{\infty\}$, for every ε-ball the ε can be chosen in $\{\frac{1}{2^n} \mid n \in \mathbb{N}_0\}$. These balls can be easily described precisely: For $x \in M$ the residue class $x + M_n$ is the closed ball with center x and radius $\frac{1}{2^n}$, which (for $n \geq 1$) is the same as the open ball with center x and radius $\frac{1}{2^{n-1}}$. So the set

$$\mathcal{B}(x) := \{x + M_n \mid n \in \mathbb{N}_0\} \tag{3.1.3}$$

is a fundamental system of neighbourhoods of x.

To summarize, a filtration on M defines a topology on M in which two elements of M are considered to be close to each other if their difference is contained in many submodules of the filtration. Now assume there is a second filtration $\widetilde{\mathcal{F}} := (\widetilde{M_n})_{n\geq 0}$ on M. When do $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ define the same topology?

In general, suppose we are given two topologies $\mathcal{T}$, $\widetilde{\mathcal{T}}$ on a set X and for each $x \in X$ a fundamental system $\mathcal{B}(x)$ of neighbourhoods of x with respect to $\mathcal{T}$. Then $\widetilde{\mathcal{T}}$ is coarser than $\mathcal{T}$ (by definition this means $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$) if and only if for each $x \in X$ and every $\widetilde{\mathcal{T}}$-neighbourhood U of x there exists a $V \in \mathcal{B}(x)$ with $V \subseteq U$. With the explicit fundamental systems of neighbourhoods given by the filtrations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ on M as in 3.1.3 we see that their induced topologies are the same if and only if the following holds:

$$\text{For all } k \in \mathbb{N} \text{ there exist } n_1, n_2 \in \mathbb{N} \text{ such that } M_{n_1} \subseteq \widetilde{M_k} \text{ and } \widetilde{M_{n_2}} \subseteq M_k. \tag{3.1.4}$$

Let us call the filtrations **equivalent** if 3.1.4 is satisfied.

For the special case of ideal-adic topologies on a ring we put this observation into a proposition, so we can refer to it later.

Proposition 3.1. *Let I, J be ideals of R. The I-adic topology on R equals the J-adic topology if and only if there exist n_1, $n_2 \in \mathbb{N}$ with $I^{n_1} \subseteq J$ and $J^{n_2} \subseteq I$. If R is Noetherian, then this is equivalent to $\sqrt{I} = \sqrt{J}$*

Our next topic is convergence. Recall that in a topological space X a point $x \in X$ is the limit of a sequence $(x_n)_{n\geq 0}$ if for every neighbourhood U of x there is an $N \in \mathbb{N}$ such that $x_N \in U$ for all $n \geq N$. If the space is Hausdorff, then every limit is unique. Obviously, the filtered module M is Hausdorff if and only if $d_{\mathcal{F}}$ is (not only a pseudometric, but) a metric, which we already translated to $\bigcap\limits_{n=0}^{\infty} M_n = \{0\}$ earlier.

Of course, in general there is no concept of Cauchy sequences in a topological space, but here the space is a module M, so here we can look at the difference of the elements of a sequence. (More precisely, R is a topological ring and M is a topological module, i.e. addition and multiplication are continuous.) So a sequence $(x_n)_{n\geq 0}$ in M is called a **Cauchy sequence** if for all $n \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ such that $x_{k_1} - x_{k_2} \in M_n$ for all k_1, $k_2 \geq N$. It is interesting to mention that, due to the ultrametric inequality (iii) in 3.1.2, in order to prove that a sequence is Cauchy it is sufficient to show that the differences of two subsequent members of the sequence converge to zero.

The filtered module M is said to be **complete** if it is Hausdorff and every Cauchy sequence converges. So we include the condition that limits are unique in the definition of complete, as it is usually done (cf. [Mat89, §8] or [Eis95, Chapter 7]).

Example 3.2. The ring $R[\![x_1, \ldots, x_r]\!]$ of formal power series in r indeterminates over R is usually considered with the I-adic topology where I is the ideal generated by $x_1, \ldots, x_r$. A power series f has a formal, unique representation as $f = \sum_{d=0}^{\infty} f_d$ where

$f_d \in R[x_1, \ldots, x_r]$ are homogeneous polynomials of degree d. We have $f \in I^n$ if and only if $f_i = 0$ for all $i < n$, and therefore $\bigcap\limits_{n=0}^{\infty} I^n = \{0\}$.

Now a sequence $(g_n)_{n\geq 0}$ in $R[\![x_1, \ldots, x_r]\!]$, where each g_n is written as $g_n = \sum_{d=0}^{\infty} g_{n,d}$ as above, is a Cauchy sequence if and only if for every d the sequence $(g_{n,d})_{n\geq 0}$ is constant after finitely many terms. So if it is Cauchy, we can use $g_d := \lim_{n\to\infty} g_{n,d}$ to build its limit $g := \sum_{d=0}^{\infty} g_d$. Hence, $R[\![x_1, \ldots, x_r]\!]$ is complete with respect to the I-adic topology. $\triangleleft$

If the module M is not yet complete, it is sometimes helpful to pass from M to its completion $\widehat{M}$. We will define the completion next, justifying the title of this section, and study the properties of the canonical morphism $M \to \widehat{M}$.

For natural numbers $m \geq n$ we have $M_m \subseteq M_n$ in the filtration of M. So the canonical projection $M \to M/M_n$ induces a homomorphism $\varphi_{nm} : M/M_m \to M/M_n$. Obviously, $\varphi_{nn} = \mathrm{id}$ and for $l \geq m$ we have $\varphi_{nl} = \varphi_{nm} \circ \varphi_{ml}$. These two functorial properties mean that $((M/M_n)_{n\geq 0}, (\varphi_{nm})_{m\geq n\geq 0}))$ is an example of an **inverse system** of R-modules. An **inverse limit** $\varprojlim_n M/M_n$ of such an inverse system is an R-module N together with homomorphisms $\pi_n : N \to M/M_n$ satisfying $\pi_n = \varphi_{nm} \circ \pi_m$ for all $m \geq n$ such that N is "universal with these properties", i.e. for every other R-module P with homomorphisms $\psi_n : P \to M/M_n$ satisfying $\psi_n = \varphi_{nm} \circ \psi_m$ for all $m \geq n$ there is a unique homomorphism $\psi : P \to N$ satisfying $\pi_n \circ \psi = \psi_n$. The following diagram illustrates the situation:

$$\begin{tikzcd}
& P \arrow[d, dotted, "\exists!\psi"] \arrow[ddl, bend right, "\psi_m"'] \arrow[ddr, bend left, "\psi_n"] & \\
& N \arrow[dl, "\pi_m"'] \arrow[dr, "\pi_n"] & \\
M/M_m \arrow[rr, "\varphi_{nm}"'] & & M/M_n
\end{tikzcd}$$

This universal property assures the uniqueness upto isomorphism of an inverse limit, more precisely, between two inverse limits of the same system there exists a unique isomorphism. The concept of inverse systems and limits as above can be defined in any category. Of course, inverse limits need not exist in such generality, but in the category of R-modules they always do as there is an explicit description of $\varprojlim_n M/M_n$ as a submodule of the direct product. In our situation here, this inverse limit is called the **completion** $\widehat{M}$ of M with respect to $\mathcal{F}$ and it is the following submodule of the direct product of the M/M_n:

$$\begin{aligned}
\widehat{M} &:= \varprojlim_n M/M_n \\
&:= \{(\overline{x_n})_{n\in\mathbb{N}_0} \in \prod_{n\in\mathbb{N}_0} M/M_n \mid \forall n, \forall m \geq n : \varphi_{nm}(\overline{x_m}) = \overline{x_n}\} \\
&= \{(\overline{x_n})_{n\in\mathbb{N}_0} \in \prod_{n\in\mathbb{N}_0} M/M_n \mid \forall n : x_{n+1} - x_n \in M_n\}.
\end{aligned}$$

(For $x \in M$ we use the notation $\overline{x}$ for the equivalence class $x + M_n$ whenever the context dictates modulo which submodule M_n the residue class is taken.) There is a natural R-linear map $\varepsilon : M \to \widehat{M}$, $x \mapsto (x + M_n)_{n \in \mathbb{N}_0}$. In the case $M = R$, the completion $\widehat{R}$ is also a ring and the natural map $\varepsilon : R \to \widehat{R}$ is a ring homomorphism.

The kernel of the projection $\pi_i : \widehat{M} \to M/M_i$ is the following submodule of $\widehat{M}$:

$$\begin{aligned} \widehat{M_i} := \ker(\pi_i) &= \{(\overline{x_n})_{n \geq 0} \in \widehat{M} \mid x_i \in M_i\} \\ &= \{(\overline{x_n})_{n \geq 0} \in \widehat{M} \mid \forall k \geq i : x_k \in M_i\} \\ &= \{(\overline{x_n})_{n \geq 0} \in \widehat{M} \mid \forall k \leq i : \overline{x_k} = 0\}. \end{aligned} \tag{3.1.5}$$

We see that $\widehat{M_0} = \widehat{M}$ and $\widehat{M_{i+1}} \subseteq \widehat{M_i}$ for all i. Furthermore, for $(\overline{x_n})_{n \geq 0} \in \widehat{M_i}$ and $a \in I_j$ the second line in 3.1.5 shows that $a \cdot x_{j+i} \in M_{j+i}$, so $a \cdot (\overline{x_n})_{n \geq 0} \in \widehat{M_{j+i}}$. So these submodules form a filtration on the R-module $\widehat{M}$. In addition, 3.1.5 also shows that $\widehat{M_i} \cong \varprojlim_n M_i/(M_n \cap M_i)$, so this is really the completion of M_i with its submodule-filtration, hence the notation is justified.

Furthermore, $\widehat{M}$ becomes an $\widehat{R}$-module by defining a multiplication:

$$(\overline{r_n})_{n \geq 0} \in \widehat{R},\ (\overline{x_n})_{n \geq 0} \in \widehat{M} \quad \Rightarrow \quad (\overline{r_n})_{n \geq 0} \cdot (\overline{x_n})_{n \geq 0} := (\overline{r_n x_n})_{n \geq 0}.$$

Let us check that the product is again in $\widehat{M}$ to see how the compatible filtrations come into play (and then leave it to the reader to show the rest of the well-definedness of this multiplication in a very similar way): For all n we have

$$r_{n+1}x_{n+1} - r_n x_n = r_{n+1}(x_{n+1} - x_n) + (r_{n+1} - r_n)x_n \in I_0 M_n + I_n M_0 \subseteq M_n,$$

and therefore $(\overline{r_n x_n})_{n \geq 0} \in \widehat{M}$. Moreover, if $(\overline{r_n})_{n \geq 0} \in \widehat{R_i}$, i.e. $r_n \in I_i$ for all $n \geq i$, and $(\overline{x_n})_{n \geq 0} \in \widehat{M_j}$, i.e. $x_n \in M_j$ for all $n \geq j$, then $r_{i+j}x_{i+j} \in M_{i+j}$, i.e. $(\overline{r_n x_n})_{n \in \mathbb{N}} \in \widehat{M_{i+j}}$. This shows that $\widehat{M}$ is a filtered $\widehat{R}$-module.

The properties collected in the following proposition justify the name "completion".

Proposition 3.3. *(a) The canonical morphism $\varepsilon : M \to \widehat{M}$ is continuous.*

(b) The image of M is dense in $\widehat{M}$.

(c) The filtered module $\widehat{M}$ is complete.

(d) Let N be a complete filtered R-module and $\varphi : M \to N$ a continuous R-linear map. Then there exists a unique continuous R-linear map $\Phi : \widehat{M} \to N$ such that $\Phi \circ \varepsilon = \varphi$.

(e) It is ε an isomorphism if and only if M is complete.

Proof. (a) For all $x \in M$ and all $i \in \mathbb{N}_0$ we have $\varepsilon^{-1}(\varepsilon(x) + \widehat{M_i}) = x + M_i$. So the inverse image of every neighbourhood of $\varepsilon(x)$ in $\widehat{M}$ is a neighbourhood of x in M, hence ε is continuous.

(b) For all $(\overline{x_n})_{n \in \mathbb{N}_0} =: x \in \widehat{M}$ and all $i \in \mathbb{N}_0$ we have $\varepsilon(x_i) \in x + \widehat{M_i}$. So every neighbourhood of $x \in \widehat{M}$ contains an element of $\varepsilon(M)$.

(c) First of all, we see that $\widehat{M}$ is Hausdorff since $\bigcap_{i=0}^{\infty} \widehat{M_i} = \{0\}$. Now let $(y_m)_{m \in \mathbb{N}_0}$ be a Cauchy sequence in $\widehat{M}$ with $y_m = (\overline{x_{m,n}})_{n \in \mathbb{N}_0}$. Fix an index $n \in \mathbb{N}_0$. By assumption, there exists an $f(n) \in \mathbb{N}_0$ such that for all k, $m \geq f(n)$ we have $y_k - y_m \in \widehat{M}_n$, which means that the n-th components of y_k and y_m are the same. So the sequence $(\overline{x_{m,n}})_{m\in\mathbb{N}_0}$ of the n-th components of the y_m is constant for all $m \geq f(n)$. So clearly, its limit

$$\overline{x_n} := \lim_{m\to\infty} \overline{x_{m,n}} \in M/M_n$$

exists, and we claim that the $\overline{x_n}$ form the limit that we need. First we see that $(\overline{x_n})_{n\in\mathbb{N}_0}$ is in $\widehat{M}$ since we have

$$x_{n+1} - x_n = x_{f(n+1),n+1} - x_{f(n),n} = x_{m,n+1} - x_{m,n} \in M_n$$

for any m larger than both $f(n)$ and $f(n+1)$.

And then for all n and for all $m \geq f(n)$ we see that $y_m - x$ lies in $\widehat{M}_n$. Therefore, $\lim_{m\to\infty} y_m = x$.

(d) Uniqueness of such a map Φ follows immediately from its continuity together with part (b). Now let $x = (\overline{x_n})_{n\in\mathbb{N}_0} \in \widehat{M}$. Since $x_{n+1} - x_n \in M_n$, the sequence $(x_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in M. Then $(\varphi(x_n))_{n\in\mathbb{N}_0}$ is a Cauchy sequence in N, which has a unique limit in N. We set $\Phi(x) := \lim\limits_{n\to\infty} \varphi(x_n)$. We leave it to the reader to check that this assignment is well-defined, R-linear, and continuous.

(e) The kernel of ε is $\bigcap_{n=0}^{\infty} M_n$. So ε is injective if and only if M is Hausdorff.

Now let ε be surjective and $(x_m)_{m\in\mathbb{N}_0}$ a Cauchy sequence in M. Then $(\varepsilon(x_m))_{m\in\mathbb{N}_0}$ is a Cauchy sequence in $\widehat{M}$. By (c) it has a unique limit, which by assumption we can write as $\varepsilon(y)$ with $y \in M$. So for all i there exists a natural number N such that $\varepsilon(x_m) - \varepsilon(y) \in \widehat{M}_i$ for all $m \geq N$, but this is equivalent to $x_m - y \in M_i$, so $y = \lim\limits_{m\to\infty} x_m$.

For the converse assume that M is complete and let $(\overline{x_n})_{n\in\mathbb{N}_0} \in \widehat{M}$. Then again $(x_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in M. By assumption, it has a limit $y \in M$. This means that for every n there exists a natural number N such that $y - x_m \in M_n$ for all $m \geq N$. If $n \geq N$, then we have $y - x_n \in M_n$ already. Otherweise, we write

$$y - x_n = (y - x_N) + (x_N - x_{N-1}) + \ldots + (x_{n+1} - x_n) \in M_n + M_{N-1} + \ldots + M_n \subseteq M_n$$

to conclude that $y - x_n \in M_n$. This holds for all n, so $\varepsilon(y) = (\overline{x_n})_{n\in\mathbb{N}_0}$. □

There is an alternative construction of the completion of M which is also worth mentioning. The set of Cauchy sequences in M form a submodule $\mathcal{C}$ of $M^{\mathbb{N}_0}$. Inside $\mathcal{C}$ the sequences that converge to 0 form another submodule $\mathcal{N}$. Given an element $(\overline{x_n})_{n\geq 0}$ of $\widehat{M}$, then $(x_n)_{n\geq 0}$ is Cauchy sequence in M. So there is natural map $\varphi : \widehat{M} \to \mathcal{C}/\mathcal{N}$ which is obviously R-linear. Conversely, given a Cauchy sequence $(x_n)_{n\geq 0}$ in M, we see that for all $k \geq 1$ the sequence of residue classes $(x_n + M_k)_{n\geq 0}$ is constant after finitely many terms. So by taking $y_k \in M$ with $y_k + M_k = \lim_{n\to\infty}(x_n + M_k)$ we can form an element $(y_n + M_n)_{n\geq 0} = (\overline{y_n})_{n\geq 0}$. It is an element in $\widehat{M}$, since: For all $k \geq 1$ there exists $N \geq 0$ such that $x_N - y_k \in M_k$ and $x_N - y_{k+1} \in M_{k+1}$, hence $y_{k+1} - y_k \in M_k$. If $(x_n)_{n\geq 0}$ were a sequence converging to 0, then $(\overline{y_n})_{n\geq 0}$ would equal $0 \in \widehat{M}$. Therefore we get a map $\psi : \mathcal{C}/\mathcal{N} \to \widehat{M}$, which is also R-linear and the inverse to φ.

Remark 3.4. This description of $\widehat{M}$ as $\mathcal{C}/\mathcal{N}$ also shows that equivalent filtrations on M give rise to isomorphic completions.

Example 3.5. Let $S = R[x_1, \ldots, x_r]$ and $I = (x_1, \ldots, x_r)_S$. By Example 3.2, the formal power series ring $R[\![x_1, \ldots, x_r]\!]$ is complete, so by Proposition 3.3(d), there exists a map $\widehat{S} \to R[\![x_1, \ldots, x_r]\!]$ with $\varepsilon(g) = g$ for all polynomials g.

This map has an inverse: To a formal power series $f \in R[\![x_1, \ldots, x_r]\!]$ written as $f = \sum_{d=0}^{\infty} f_d$ with homogeneous polynomials $f_d \in R[x_1, \ldots, x_r]$ we assign the element $(\sum_{d=0}^{n-1} f_d + I^n)_{n \geq 0} \in \widehat{S}$.

Hence the I-adic completion of $R[x_1, \ldots, x_r]$ is isomorphic to $R[\![x_1, \ldots, x_r]\!]$. ◁

3.2 Properties of Adic Completions

This section is about **adic filtrations**, as completions with respect to an ideal are the most important ones. So for the entire section let R be a ring, $I \subseteq R$ an ideal, and any R-module M that appears will be considered with its I-adic filtration $(I^n M)_{n \geq 0}$ and I-adic topology. First we observe that I-adic completion is a functor from the category of R-modules to the category of $\widehat{R}$-modules: Every R-linear map $f : M \to N$ is continuous (as $f(I^n M) \subseteq I^n N$), so for the composition $(\varepsilon \circ f) : M \to \widehat{N}$ there exists a unique R-linear map $\widehat{f} : \widehat{M} \to \widehat{N}$ with $\widehat{f} \circ \varepsilon = \varepsilon \circ f$ by Proposition 3.3(d). Explicitly, it is given by

$$\widehat{f}\,((\overline{x_n}))_{n \geq 0} = (\overline{f(x_n)})_{n \geq 0}, \tag{3.2.1}$$

which makes it easy to see that this map is $\widehat{R}$-linear and that the functorial properties $\widehat{\mathrm{id}_M} = \mathrm{id}_{\widehat{M}}$ and $\widehat{f \circ g} = \widehat{f} \circ \widehat{g}$ are satisfied. Furthermore, $\widehat{f + g} = \widehat{f} + \widehat{g}$ for two R-linear maps f, $g : M \to N$, so completion is an additive functor.

Proposition 3.6. *Let R be Noetherian. Then I-adic completion is an exact functor from the category of finitely generated R-modules to the category of $\widehat{R}$-modules.*

Proof. Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \tag{3.2.2}$$

be an exact sequence of finitely generated R-modules. We need to show that the sequence

$$0 \longrightarrow \widehat{L} \xrightarrow{\widehat{f}} \widehat{M} \xrightarrow{\widehat{g}} \widehat{N} \longrightarrow 0 \tag{3.2.3}$$

is exact, too.

First, we will use the general fact that inverse limits preserve left-exactness. Let us explain what this means, but leave the proof, which is straight-forward, to the reader: Suppose there are three inverse systems (A_n, φ^A_{nm}), (B_n, φ^B_{nm}) and (C_n, φ^C_{nm}) of R-modules and R-linear maps $f_n : A_n \to B_n$ and $g_n : B_n \to C_n$ such that for all n and for all $m \geq n$ the following diagram is commutative with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "f_n"] & B_n \arrow[r, "g_n"] & C_n \arrow[r] & 0 \\
0 \arrow[r] & A_m \arrow[r, "f_m"] \arrow[u, "\varphi^A_{nm}"] & B_m \arrow[r, "g_m"] \arrow[u, "\varphi^B_{nm}"] & C_m \arrow[r] \arrow[u, "\varphi^C_{nm}"] & 0.
\end{tikzcd}$$

Then the induced sequence

$$0 \longrightarrow \varprojlim_{n} A_n \xrightarrow{\widetilde{f}} \varprojlim_{n} B_n \xrightarrow{\widetilde{g}} \varprojlim_{n} C_n ,$$

where e.g. $\widetilde{f}$ is defined by $\widetilde{f}((a_n)_{n\geq 0}) := (f_n(a_n))_{n\geq 0}$, is exact.

Here we will apply it to the following exact sequences which are induced from 3.2.2:

$$0 \longrightarrow L/f^{-1}(I^n M) \xrightarrow{f_n} M/I^n M \xrightarrow{g_n} N/I^n N \longrightarrow 0 .$$

So left-exactness of inverse limits already gives the exact sequence:

$$0 \longrightarrow \varprojlim_{n}(L/f^{-1}(I^n M)) \xrightarrow{\widetilde{f}} \widehat{M} \xrightarrow{\widehat{g}} \widehat{N} . \tag{3.2.4}$$

The next step is to identify the first inverse limit in 3.2.4 with $\widehat{L}$. Since R is Noetherian and M is finitely generated, by the Artin-Rees lemma (see [Sin11, Theorem 8.1.3]) there exists a natural number m such that for all $n \geq m$ the formula

$$I^{n+1}M \cap f(L) = I(I^n M \cap f(L))$$

holds. With the injectivity of f we can translate this to $f^{-1}(I^{n+1}M) = If^{-1}(I^n M)$. On the one hand we have $I^n L \subseteq f^{-1}(I^n M)$, but now on the other hand we have for all $n \geq m$:

$$f^{-1}(I^n M) = I^{n-m} f^{-1}(I^m M) \subseteq I^{n-m} L.$$

So by condition 3.1.4 the filtration $(f^{-1}(I^n M))_{n\geq 0}$ is equivalent to the I-adic filtration on L, hence $\widehat{L} \cong \varprojlim_{n}(L/f^{-1}(I^n M))$ (and $\widehat{f}$ corresponds to $\widetilde{f}$).

The left-exactness in 3.2.3 is proven, and it remains to show that $\widehat{g}$ is surjective. For that let $(\overline{y_n})_{n\geq 0} \in \widehat{N}$. As g is surjective, there is $x_0 \in M$ with $g(x_0) = y_0$. We can now inductively construct $x_n \in M$ (with $n \geq 1$) such that the following hold:

(1.) $g(x_n) = y_n$,

(2.) $x_n + I^{n-1}M = x_{n-1} + I^{n-1}M$.

Assume that x_{n-1} is already chosen. Since

$$y_n - y_{n-1} \in I^{n-1}N = I^{n-1}g(M) = g(I^{n-1}M),$$

there exists an $a \in I^{n-1}M$ with $g(a) = y_n - y_{n-1}$. Then $x_n := x_{n-1} + a$ satisfies (1.) and (2.). So $(\overline{x_n})_{n\geq 0}$ lies in $\widehat{M}$ and maps to $(\overline{y_n})_{n\geq 0}$. □

Remark 3.7. Let $R \to S$ be a ring homomorphism. The I-adic filtration on S (considered as an R-module) is the same as the (IS)-adic filtration on the ring S. So the I-adic completion $\widehat{S}$ is a ring and the map $\widehat{R} \to \widehat{S}$ as in 3.2.1 is a ring homomorphism.

Proposition 3.8. *Let R be Noetherian, $J \subseteq R$ an ideal, and M a finitely generated R-module. Then for the I-adic completions we get $J\widehat{M} \cong \widehat{JM}$ and $\widehat{M/JM} \cong \widehat{M}/J\widehat{M}$. For $M = R$ we get $\widehat{R/J} \cong \widehat{R}/J\widehat{R}$ also as rings.*

Proof. The exact sequence

$$0 \longrightarrow JM \xrightarrow{i} M \longrightarrow M/JM \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \widehat{JM} \xrightarrow{\widehat{i}} \widehat{M} \longrightarrow \widehat{M/JM} \longrightarrow 0$$

by Proposition 3.6. So $\widehat{M/JM} \cong \widehat{M}/\widehat{i}(\widehat{JM})$. Let $J = (a_1, \ldots, a_r)_R$ and let φ be the R-linear map $\varphi : M^r \to M$ that sends $(x_1, \ldots, x_r)$ to $\sum_{i=1}^r a_i x_i$. Then the image of φ is JM, so there is an exact sequence

$$M^r \xrightarrow{\varphi} M \xrightarrow{\psi} M/JM \longrightarrow 0,$$

where ψ is the canonical projection. Again this induces an exact sequence

$$\widehat{M^r} \xrightarrow{\phi} \widehat{M} \xrightarrow{\Psi} \widehat{M/JM} \longrightarrow 0. \tag{3.2.5}$$

Under the isomorphism

$$\widehat{M^r} \cong \widehat{M}^r, \quad (\overline{(x_{1,n}, \ldots, x_{r,n})})_{n\geq 0} \mapsto ((\overline{x_{1,n}})_{n\geq 0}, \ldots, (\overline{x_{r,n}})_{n\geq 0})$$

the map ϕ corresponds to

$$\widetilde{\phi} : \widehat{M}^r \to \widehat{M}, \quad (y_1, \ldots, y_r) \mapsto \sum_{i=1}^r a_i y_i,$$

whose image is $J\widehat{M}$. And one checks that under the isomorphism $\widehat{M/JM} \cong \widehat{M}/\widehat{i}(\widehat{JM})$ the map Ψ corresponds to the canonical projection $\pi : \widehat{M} \to \widehat{M}/\widehat{i}(\widehat{JM})$. So from 3.2.5 we get the following exact sequence

$$\widehat{M}^r \xrightarrow{\widetilde{\phi}} \widehat{M} \xrightarrow{\pi} \widehat{M}/\widehat{i}(\widehat{JM}) \longrightarrow 0.$$

Therefore $J\widehat{M} = \operatorname{im}(\widetilde{\phi}) = \ker(\pi) \cong \widehat{JM}$ and

$$\widehat{M}/J\widehat{M} = \widehat{M}/\ker(\pi) \cong \widehat{M}/\widehat{i}(\widehat{JM}) \cong \widehat{M/JM}. \qquad \square$$

Example 3.9. Let $K[X] = K[x_1, \ldots, x_n]/J$ be the coordinate ring of an affine variety $X \subseteq K^n$ that contains the point at the origin. If I is the maximal ideal $(x_1, \ldots, x_n) \subseteq K[x_1, \ldots, x_n]$, then Proposition 3.8 together with Example 3.5 shows that the I-adic completion of $K[X]$ is

$$\widehat{K[X]} \cong K[\![x_1, \ldots, x_n]\!]/JK[\![x_1, \ldots, x_n]\!]. \qquad \triangleleft$$

We can also use Proposition 3.8 to prove that I-adic completions of Noetherian rings are Noetherian again.

Proposition 3.10. *Let R be Noetherian and $I = (a_1, \dots, a_n)$ with $a_i \in R$. Then*

$$\widehat{R} \cong R[\![x_1, \dots, x_n]\!]/(x_1 - a_1, \dots, x_n - a_n).$$

In particular, $\widehat{R}$ is Noetherian.

Proof. Let $S = R[x_1, \dots, x_n]$, $J = (x_1 - a_1, \dots, x_n - a_n)_S$ and $I' = (x_1, \dots, x_n)_S$. The homomorphism $\psi : S \to R$ that (is the identity on R and) sends x_i to a_i has J as its kernel, hence $S/J \cong R$. Under this isomorphism the I-adic topology on R corresponds to the I'-adic topology on S/J. So with Proposition 3.8 and Example 3.5 we get:

$$\widehat{R} \cong \widehat{S/J} \cong \widehat{S}/J\widehat{S} \cong R[\![x_1, \dots, x_n]\!]/(x_1 - a_1, \dots, x_n - a_n).$$

As rings of formal power series in finitely many indeterminates over R are Noetherian (see [Kem11, Exercise 2.4]), it follows that $\widehat{R}$ is Noetherian, too. $\square$

The next part of this section deals with another important property of I-adic completion, which we need later on, namely (faithful) flatness.

The map $M \times \widehat{R} \to \widehat{M}$, $(x, (\overline{a_n})_{n\geq 0}) \mapsto (\overline{a_n x})_{n\geq 0}$ is well-defined an R-bilinear. So there is an induced R-linear map $\eta_M : M \otimes_R \widehat{R} \to \widehat{M}$ such that $x \otimes (\overline{a_n})_{n\geq 0} \mapsto (\overline{a_n x})_{n\geq 0}$. Moreover, η is a natural transformation of functors from $\_ \otimes_R \widehat{R}$ to $\widehat{(\_)}$: For every R-linear map $f : M \to N$ the induced maps form a commutative diagram (which can be easily seen using 3.2.1):

$$\begin{tikzcd} M \otimes_R \widehat{R} \arrow[r, "f\otimes \mathrm{id}"] \arrow[d, "\eta_M"'] & N \otimes_R \widehat{R} \arrow[d, "\eta_N"] \\ \widehat{M} \arrow[r, "\widehat{f}"] & \widehat{N} \end{tikzcd}.$$

Proposition 3.11. *Let R be Noetherian and M a finitely generated R-module. Then $\eta_M : M \otimes_R \widehat{R} \to \widehat{M}$ is an isomorphism.*

Proof. Obviously, $\eta_R : R \otimes_R \widehat{R} \to \widehat{R}$ is an isomorphism. Let us write $F = \_ \otimes_R \widehat{R}$ and $G = \widehat{(\_)}$ for the two functors. As both are additive, they preserve finite direct sums (see [HS71, I, Prop. 9.5]). This property commutes with a natural transformation, i.e. given finitely many R-modules M_i the diagram

$$\begin{tikzcd} F(\oplus_{i=1}^n M_i) \arrow[r, "\eta_{\oplus F(M_i)}"] \arrow[d, "\cong"'] & G(\oplus_{i=1}^n M_i) \arrow[d, "\cong"] \\ \oplus_{i=1}^n F(M_i) \arrow[r, "(\eta_{M_1}, \dots, \eta_{M_n})"] & \oplus_{i=1}^n G(M_i) \end{tikzcd}$$

is commutative. Therefore, the fact that η_R is an isomorphism implies that for all $n \in \mathbb{N}$ the map η_{R^n} is an isomorphism, too.

Now since M is finitely generated, there exists an epimorphism $f : R^n \to M$ for some n. And since R is Noetherian, the kernel of f is again finitely generated (see [Kem11, Theorem 2.10]). So for some m there is an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

Applying F and G, leads to the commutative diagram

$$\begin{tikzcd} F(R^m) \arrow[r] \arrow[d, "\eta_{R^m}"', "\cong"] & F(R^n) \arrow[r] \arrow[d, "\eta_{R^n}"', "\cong"] & F(M) \arrow[r] \arrow[d, "\eta_M"'] & 0 \\ G(R^m) \arrow[r] & G(R^n) \arrow[r] & G(M) \arrow[r] & 0. \end{tikzcd} \tag{3.2.6}$$

Both completion (of finitely generated modules) and tensoring preserve right-exactness (see Proposition 3.6 for completion, and e.g. [AM69, Prop. 2.18] for the tensor product), so both lines of 3.2.6 are exact. We already now that the two vertical maps on the left in 3.2.6 are isomorphisms. Now the claim that η_M is an isomorphism follows by a usual diagram chase, which is standard but cumbersome to write down, so we leave it to the reader. $\square$

Recall that an R-module M is called **flat** if $\_ \otimes_R M$ is an exact functor. Moreover, M is called **faithfully flat** if for all short sequences of R-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0 \tag{3.2.7}$$

the tensored sequence

$$0 \longrightarrow N' \otimes_R M \longrightarrow N \otimes_R M \longrightarrow N'' \otimes_R M \longrightarrow 0$$

is exact if and only if the original sequence 3.2.7 is exact, too. As the tensor product always preserves right-exactness, the module M is flat if and only if for all injective R-linear maps $N' \to N$ the induced map $N' \otimes_R M \to N \otimes_R M$ is injective, too.

In order to show that the $(I = \mathfrak{m})$-adic completion of a Noetherian local ring $(R, \mathfrak{m})$ is faithfully flat over R, we will use the following characterization of faithful flatness.

Proposition 3.12. *Let M be an R-module. The following three conditions are equivalent:*

(a) M is faithfully flat,

(b) M is flat and for all R-modules $N \neq 0$ we have $N \otimes_R M \neq 0$,

(c) M is flat and for all finitely generated R-modules $N \neq 0$ we have $N \otimes_R M \neq 0$.

Proof. See [Mat89, Theorem 7.2]. $\square$

Proposition 3.13. *Let R be Noetherian. Then $\widehat{R}$ is flat as an R-module. If in addition I is contained in the Jacobson radical of R, then $\widehat{R}$ is faithfully flat.*

Proof. Let $f : M \to N$ be injective and R-linear. We need to show that $M \otimes_R \widehat{R} \to N \otimes_R \widehat{R}$ is injective, too. For this, take $x_i \in M$ and $s_i \in \widehat{R}$ such that

$$\sum_{i=1}^{n} f(x_i) \otimes s_i = 0 \tag{3.2.8}$$

in $N \otimes_R \widehat{R}$. Using the explicit construction of the tensor product, one shows that there exists a finitely generated submodule N_0 of N which contains the $f(x_i)$ such that 3.2.8

holds in $N_0 \otimes_R \widehat{R}$ as well (see e.g. [AM69, Corollary 2.13] for a short proof of this general fact). We restrict f to a map f_0 from $M_0 := \langle x_1, \ldots, x_n \rangle_R \subseteq M$ to N_0. Now for those finitely generated modules the map $f_0 \otimes \mathrm{id}$ corresponds to $\widehat{f_0}$ by Proposition 3.11, and $\widehat{f_0}$ is injective by Proposition 3.6. So $\sum_{i=1}^n f(x_i) \otimes s_i = 0$ implies $\sum_{i=1}^n x_i \otimes s_i = 0$ and the first part is proven.

Now assume $I \subseteq \mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subseteq R$, and let M be a finitely generated R-module. Then we have $\cap_{n=1}^{\infty} I^n M = 0$ by a general version of Krull's intersection theorem ([Eis95, Corollary 5.4]). So the natural map $\varepsilon : M \to \widehat{M}$ is injective. So if $M \neq 0$, then $\widehat{M} \neq 0$ and hence $M \otimes_R \widehat{R} \neq 0$ by Proposition 3.11. By Proposition 3.12 this suffices in order to show that $\widehat{R}$ is faithfully flat. □

As flatness implies going-down (see [Mat89, Theorem 9.5]) we get the following corollary as an important consequence.

Corollary 3.14. *Let R be Noetherian. Then the ring map $\varepsilon : R \to \widehat{R}$ satisfies going-down.*

For the special case of the completion of a Noetherian ring at a maximal ideal we collect the following results.

Proposition 3.15. *Let R be Noetherian and let $I = \mathfrak{m}$ be a maximal ideal. The the following hold:*

(a) The $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a Noetherian local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$.

(b) The $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to the $\mathfrak{m}_{\mathfrak{m}}$-adic completion of the localization $R_{\mathfrak{m}}$.

(c) It is $\dim(\widehat{R}) = \mathrm{ht}_R(\mathfrak{m})$.

Proof. (a) It was $\widehat{\mathfrak{m}}$ defined as the kernel of the surjective projection $\pi_1 : \widehat{R} \to R/\mathfrak{m}$, so $\widehat{R}/\widehat{\mathfrak{m}} \cong R/\mathfrak{m}$ is a field. Furthermore, for $(\overline{a_n})_{n\geq 0} \in \widehat{R} \setminus \widehat{\mathfrak{m}}$ we get $a_n \in R \setminus \mathfrak{m}$ for all n. So a_n is invertible modulo $\mathfrak{m}^n$, this means there exists an element $b_n \in R$ with $a_n b_n - 1 \in \mathfrak{m}^n$. These b_n certainly form the inverse of $(\overline{a_n})_{n\geq 0}$ in $\prod_{n\geq 0} R/\mathfrak{m}^n$. It remains to check that $(\overline{b_n})_{n\geq 0}$ lies in $\widehat{R}$, i.e. for all $n \in \mathbb{N}_0$ we need $b_{n+1} - b_n \in \mathfrak{m}^n$, but this is clear from the following equations in $R/\mathfrak{m}^n$:

$$\overline{a_{n+1}} = \overline{a_n}, \quad \overline{a_{n+1}}\overline{b_{n+1}} = 1, \quad \overline{a_n}\overline{b_n} = 1.$$

This shows that $\widehat{R}$ is local with maximal ideal $\widehat{\mathfrak{m}}$. By the discussion following 3.1.5, $\widehat{\mathfrak{m}}$ is the ($\mathfrak{m}$-adic) completion of $\mathfrak{m}$. Hence Proposition 3.11 gives $\widehat{\mathfrak{m}} = \mathfrak{m} \otimes_R \widehat{R}$. So as an $\widehat{R}$-module $\widehat{\mathfrak{m}}$ is generated by $\mathfrak{m}$, i.e. $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$.

(b) This holds just by $R/\mathfrak{m}^n \cong R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}^n$ and the definition of the completion as an inverse limit over these factor rings.

(c) By our previous results, the canonical map $\varepsilon : R \to \widehat{R}$ is a homomorphism of Noetherian rings which satisfies going-down. In this situation the general fact about fiber dimension (see [Mat89, Theorem 15.1]) together with the results of (a) and (b) gives:

$$\dim(\widehat{R}) = \mathrm{ht}_{\widehat{R}}(\widehat{\mathfrak{m}}) = \mathrm{ht}_R(\mathfrak{m}) + \dim(\widehat{R}/\mathfrak{m}\widehat{R}) = \mathrm{ht}_R(\mathfrak{m}).$$ □

Chapter 4

Complete Intersections as Separating Algebras

When studying the minimal number of separating invariants γ_{sep}, the next step after dealing with separating algebras which are polynomial rings (i.e. $\gamma_{\text{sep}} = n$) is the case $\gamma_{\text{sep}} = n + 1$. Again the question is, what restrictions does the existence of a nice separating algebra, in this case now a **hypersurface** separating algebra, impose on G. We will not approach that question for hypersurfaces alone, in fact a hypersurface algebra is a special case of a **complete intersection** ring (at least if the G-variety X is irreducible, so that the rings in question are domains).

In 1982, it was shown by Kac and Watanabe [KW82] and independently by Gordeev [Gor82] that the invariant ring $K[V]^G$ of a representation V of G can only be a complete intersection if G is generated by 2-reflections (which are also called bireflections). This was extended by Dufresne [Duf09] to graded separating subalgebras of the invariant ring.

This chapter contains a generalization of these theorems to non-linear actions on varieties, and to separating subalgebras that are not complete intersections globally but satisfy some weaker local property.

4.1 Complete Intersections

This section recalls some notions of complete intersections. They all arise from the fact that an ideal I in a Noetherian ring S cannot be generated by less than $\operatorname{ht}_S(I)$ many elements (see [Kem11, Theorem 7.5]). Now I is called a complete intersection ideal if a system of generators of I with exactly $\operatorname{ht}_S(I)$ many elements exists. To be precise, in the literature usually an ideal of S is called a complete intersection ideal if it is generated by a regular sequence (see [Vas85]), which is a stronger condition than our definition here (and equivalent for example when S is a local Cohen-Macaulay ring).

In addition, an affine K-algebra A is called a (global) complete intersection (over K) if there exists a presentation $A \cong K[x_1, \ldots, x_m]/I$ with a complete intersection ideal $I \subseteq K[x_1, \ldots, x_m]$.

A Noetherian local ring $(R, \mathfrak{m})$ is called a (local) complete intersection if its $\mathfrak{m}$-adic completion $\widehat{R}$ has a presentation $\widehat{R} \cong S/I$ with a regular local ring S and a complete intersection ideal $I \subseteq S$ (see [BH93, Def. 2.3.1]). The completion is used because $\widehat{R}$

(unlike R) always has a presentation as the quotient of a regular local ring (by the Cohen structure theorem, [Eis95, Theorem 7.7]).

For a Noetherian local ring $(R, \mathfrak{m})$ there is also an intrinsic definition using the Koszul complex, which shall be summarized briefly here. By Nakayama's lemma, each minimal system of generators of $\mathfrak{m}$ has the same length (called the embedding dimension $\operatorname{edim}(R)$ of R). Moreover, the Koszul complexes of two minimal systems of generators of $\mathfrak{m}$ are isomorphic, and hence just called "the" Koszul complex of R. Now the first deviation $\varepsilon_1(R)$ of R is defined to be the $R/\mathfrak{m}$-vector space dimension of the first homology of the Koszul complex, and R is a complete intersection if and only if $\varepsilon_1(R) = \operatorname{edim}(R) - \dim(R)$ (see [BH93, Theorem 2.3.3]). We have $\varepsilon_1(\widehat{R}) = \varepsilon_1(R)$ for the completion of R because the Koszul homologies are the same. With this intrinsic approach one can also show the following remark.

Remark 4.1. Given a presentation $R \cong S/I$ with a regular local ring S and an ideal I of S, the quotient ring R is a (local) complete intersection if and only if I is a complete intersection ideal (again see [BH93, Theorem 2.3.3]).

Slightly more general is the notion of the complete intersection defect, which again is defined both for affine K-algebras and for Noetherian local rings.

Definition 4.2. *(a) Let A be an affine K-algebra of dimension n. The **(global) complete intersection defect** of A (over K), written as **cid(A)**, is the smallest number $l \in \mathbb{N}_0$ such that there exists an $m \in \mathbb{N}_0$ and a presentation*

$$A \cong K[x_1, \ldots, x_m]/(f_1, \ldots, f_{m-n+l})$$

with polynomials $f_i \in K[x_1, \ldots, x_m]$.

*(b) Let R be a Noetherian local ring. The **complete intersection defect** of R is defined to be the number*

$$\boldsymbol{cid(R)} := \varepsilon_1(R) - \operatorname{edim}(R) + \dim(R) \in \mathbb{N}_0.$$

Remark 4.3. As expected, Remark 4.1 extends to the statement: If $R \cong S/I$ is a regular presentation of the local ring R, then $\operatorname{cid}(R)$ equals the minimal number of generators of I minus $\operatorname{ht}_S(I)$ (see [KK65, Satz 1]).

Finally, for a Noetherian ring R we define its local complete intersection defect.

Definition 4.4. *Let R be a Noetherian ring. The number*

$$\boldsymbol{lcid(R)} := \sup\{\operatorname{cid}(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \in \mathbb{N}_0 \cup \{\infty\} \tag{4.1.1}$$

*is called the **local complete intersection defect** of R.*

Remark 4.5. In 4.1.1, it suffices to take the supremum over all $\operatorname{cid}(R_{\mathfrak{m}})$ with maximal ideals $\mathfrak{m}$ of R by a result of Avramov [Avr98, Corollary 7.4.4]. In particular, for a Noetherian local ring R we get $\operatorname{cid}(R) = \operatorname{lcid}(R)$.

Remark 4.6. For an affine K-algebra $A \cong K[x_1, \dots, x_m]/I$ we have

$$\mathrm{lcid}(A) \le \mathrm{cid}(A). \tag{4.1.2}$$

To see this, take the localization of A at a maximal ideal, which is isomorphic to $K[x_1, \dots, x_m]_{\mathfrak{m}}/I_{\mathfrak{m}}$ with a maximal ideal $I \subseteq \mathfrak{m} \subseteq K[x_1, \dots, x_m]$. Since $\mathrm{ht}(I) \le \mathrm{ht}(I_{\mathfrak{m}})$ and since $I_{\mathfrak{m}}$ can be generated by any ideal basis of I, the inequality 4.1.2 follows.

If the affine algebra A is graded (for example if $A = K[V]^G$ is the invariant ring of a representation), then the localization of A at its unique maximal homogeneous ideal A_+ includes all the information about the complete intersection property, as the following proposition shows.

Proposition 4.7. *Let $A = \oplus_{d=0}^{\infty} A_d$ be a graded Noetherian K-algebra with $A_0 = K$. Then*

$$\mathrm{lcid}(A) = \mathrm{cid}(A_{A_+}) = \mathrm{cid}(A).$$

Moreover, we have

$$\begin{aligned} \mathrm{cid}(A) = \mathrm{dim}(A) - {} & \textit{ minimal number of homogeneous generators of } A \\ + {} & \textit{ minimal number of relations between them.} \end{aligned} \tag{4.1.3}$$

Proof. By 4.1.1 and by Remark 4.6, we have

$$\mathrm{cid}(A_{A_+}) \le \mathrm{lcid}(A) \le \mathrm{cid}(A).$$

Let us use the ad-hoc notation $\mathrm{hcid}(A)$ for the number defined in 4.1.3. Of course, the inequality $\mathrm{cid}(A) \le \mathrm{hcid}(A)$ holds, so it remains to show that $\mathrm{hcid}(A) \le \mathrm{cid}(A_{A_+})$. Let $g_1, \dots, g_m$ be a minimal system of homogeneous generators of A and let $I \subseteq K[x_1, \dots, x_m]$ be the ideal of relations between them. Let us consider $K[x_1, \dots, x_m]$ to be graded with $\deg(x_i) = \deg(g_i)$ so that I becomes a homogeneous ideal.

With $\mathfrak{m} = (x_1, \dots, x_m)$, we have

$$A_{A_+} \cong K[x_1, \dots, x_m]_{\mathfrak{m}}/I_{\mathfrak{m}}. \tag{4.1.4}$$

Assume that $\mathrm{cid}(A_{A_+}) = k$ and $\dim(A) = n$. Since 4.1.4 is a regular presentation of the n-dimensional local ring A_{A_+}, we know by Remark 4.3 that $I_{\mathfrak{m}}$ can be generated by $m - n + k$ many elements. By Nakayama's lemma, $m - n + k$ is the K-vector space dimension of $I_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} I_{\mathfrak{m}}$. But this is the same as $\dim_K(I/\mathfrak{m}I)$, which now by the graded Nakayama lemma is equal to the minimal number of (homogeneous) generators of I. So we have

$$\mathrm{hcid}(A) \le n - m + (m - n + k) = k. \qquad \square$$

Apart from the complete intersection defect, another weakening of the concept of complete intersection rings appears in the literature. We recall this definition next.

Definition 4.8. *Let k be a non-negative integer. A Noetherian ring R is called a* ***complete intersection in codimension k*** *if $R_{\mathfrak{p}}$ is a local complete intersection ring for all prime ideals $\mathfrak{p}$ of R with $\mathrm{ht}_R(\mathfrak{p}) \le k$.*

For the proof of the main theorem of this chapter, it will be necessary to pass to a completion (of an affine K-domain at a maximal ideal). Let us therefore show that this does not destroy the property of Definition 4.8. For our purpose it will be enough to prove this for integral domain quotients of a regular ring, by which we mean a Noetherian ring whose localizations are regular local rings.

Proposition 4.9. *Let S be a regular ring, and let $I \subseteq S$ be a prime ideal. Suppose that $R := S/I$ is a complete intersection in codimension k (for some $k \in \mathbb{N}_0$). Moreover, let $\mathfrak{n}$ be a maximal ideal of S with $I \subseteq \mathfrak{n}$, and let $\mathfrak{m} := \mathfrak{n}/I$. Then the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a complete intersection in codimension k, too.*

Proof. By Proposition 3.8, we can view $\widehat{R}$ as $\widehat{S}/I\widehat{S}$ where $\widehat{S}$ is the $\mathfrak{n}$-adic completion of S. Now let $\mathfrak{q} \in \operatorname{Spec}(\widehat{S})$ with $I\widehat{S} \subseteq \mathfrak{q}$ and $\operatorname{ht}_{\widehat{S}/I\widehat{S}}(\mathfrak{q}/I\widehat{S}) \leq k$. We need to show that $(\widehat{S}/I\widehat{S})_{\mathfrak{q}/I\widehat{S}}$, which is isomorphic to $\widehat{S}_{\mathfrak{q}}/(I\widehat{S})_{\mathfrak{q}}$, is a complete intersection ring.

Since $\widehat{S}_{\mathfrak{q}}$ is a regular local ring, it is precisely to show that $(I\widehat{S})_{\mathfrak{q}}$ is a complete intersection ideal (cf. Remark 4.1). As in Chapter 3, ε_R and ε_S denote the canonical ring maps $R \to \widehat{R}$ and $S \to \widehat{S}$, respectively. Let $\mathfrak{p} := \varepsilon_S^{-1}(\mathfrak{q}) \in \operatorname{Spec}(S)$. Then $I \subseteq \mathfrak{p}$ and $\varepsilon_R^{-1}(\mathfrak{q}/I\widehat{S}) = \mathfrak{p}/I$. Since ε_R satisfies going-down (see Corollary 3.14), we have

$$\operatorname{ht}_R(\mathfrak{p}/I) \leq \operatorname{ht}_{\widehat{S}/I\widehat{S}}(\mathfrak{q}/I\widehat{S}) \leq k.$$

So by assumption on R, it is $R_{\mathfrak{p}/I} \cong S_{\mathfrak{p}}/I_{\mathfrak{p}}$ a complete intersection ring, hence $I_{\mathfrak{p}}$ is a complete intersection ideal. Therefore, there exist $a_1, \dots, a_l \in I$ with $I_{\mathfrak{p}} = (a_1, \dots, a_l)_{S_{\mathfrak{p}}}$ and $l = \operatorname{ht}_{S_{\mathfrak{p}}}(I_{\mathfrak{p}})$. This means that for every $a \in I$ there exists an $s \in S \setminus \mathfrak{p}$ such that $sa \in (a_1, \dots, a_l)_S$. But this also shows that $(I\widehat{S})_{\mathfrak{q}}$ is generated by $\varepsilon_S(a_1), \dots, \varepsilon_S(a_l)$. Using going-down for the ring map ε_S we get:

$$\operatorname{ht}_{\widehat{S}_{\mathfrak{q}}}((I\widehat{S})_{\mathfrak{q}}) \geq \operatorname{ht}_{\widehat{S}}(I\widehat{S}) \geq \operatorname{ht}_S(\varepsilon_S^{-1}(I\widehat{S})) \geq \operatorname{ht}_S(I) = \operatorname{ht}_{S_{\mathfrak{p}}}(I_{\mathfrak{p}}),$$

where the last equality holds since I itself is a prime ideal (with $I \subseteq \mathfrak{p}$). It follows that $(I\widehat{S})_{\mathfrak{q}}$ is a complete intersection ideal. □

4.2 Simply Connected Quotients

Vinberg's lemma [KW82, Lemma 2] is one key to the proof of Kac-Watanabe's theorem. Roughly speaking, it states that if a finite group acts on a sufficiently nice topological space such that the quotient is simply connected, then the group must be generated by elements having a fixed point. A version for the euclidean topology of complex algebraic varieties appears in [PV94, Section 8.3], where it is called "Genghis Khan lemma" to avoid the problem of assigning authorship to a result that appears to have been "folkore" for some time.

When we look at the action of a group G on a scheme X by morphisms, the notion of fixed points is a bit more subtle then for affine varieties. The stabilizer of a point $x \in X$, which is sometimes called the decomposition group of x, written as

$$G_d(x) := G_x = \{\sigma \in G \mid \sigma x = x\},$$

acts on the residue field $\kappa(x)$ of x. It contains as a normal subgroup the **inertia group** of x:

$$G_i(x) := \{\sigma \in G_d(x) \mid \sigma = \mathrm{id} \text{ on } \kappa(x)\}.$$

Now x should really be considered a fixed point of a group element $\sigma \in G$ if and only if $\sigma \in G_i(x)$. For example if $X = \mathrm{Spec}(R)$ is an affine scheme and $\mathfrak{p}$ a prime ideal of R, then we get

$$G_i(\mathfrak{p}) = \{\sigma \in G \mid \sigma f - f \in \mathfrak{p} \text{ for all } f \in R\}.$$

Now suppose that X is a separated scheme (see [GW10, Definition 9.7]). Then the following set of fixed points of an element $\sigma \in G$ is always a closed subscheme of X (see [GW10, Exercise 9.7]):

$$X^\sigma := \{x \in X \mid \sigma x = x \text{ and } \sigma = \mathrm{id} \text{ on } \kappa(x)\} = \{x \in X \mid \sigma \in G_i(x)\}.$$

As defined for the action on affine varieties in 2.15, we now call σ a ***k*-reflection** on X (with $k \in \mathbb{N}_0$) if the closed subset X^σ of X has codimension less or equal to k.

Remark 4.10. If X is an affine G-variety, then we see with Remark 2.16 that an element $\sigma \in G$ is a k-reflection on X if and only if it is a k-reflection (in the above sense) on the scheme $\mathrm{Spec}(K[X])$.

We also need some general facts about the quotient of a scheme X by a finite group G. Following the book "Etale Cohomology Theory" by Lei Fu [Fu11] we will call the action **admissible** if there exists an affine G-invariant morphism $\pi : X \to Y$ such that $\mathcal{O}_Y \cong (\pi_* \mathcal{O}_X)^G$. Then Y is not only the categorical but also the geometric quotient of X by G (see [Fu11, p. 119]). Moreover, every open subset $V \subseteq Y$ is the quotient of $\pi^{-1}(V) \subseteq X$.

Finally, we recall what it means for a scheme Z to be simply connected. An **étale covering** of Z is a scheme Z' together with a finite étale morphism $f : Z' \to Z$. It is called trivial if Z' is a finite disjoint union of open subschemes which are all isomorphic to Z via f. And Z is called **simply connected** if every étale covering of Z is trivial (see [Har77, IV, Example 2.5.3] or [Dan96, Chapter 4, Section 2.2]).

The fact that the quotient morphism by a finite free group action is étale is the basis for Vinberg's lemma. We need a slightly more general version here.

Lemma 4.11. *Let X be a scheme on which a finite group G acts admissibly and let $Y = X/G$ be the quotient of X by G. Suppose X is of finite presentation over Y and let H be the subgroup generated by all inertia subgroups $G_i(x)$ with $x \in X$. Then $X/H \to Y$ is étale.*

Proof. Proposition 3.2.5 in [Fu11] contains the case that all inertia subgroups are trivial (i.e. $H = 1$). We will adapt the proof to this modified version. As explained there, we can reduce to the case that $Y = \mathrm{Spec}(A)$ is a strictly Henselian local ring and $X = \mathrm{Spec}(B)$ is a finite A-algebra with $B^G = A$. This reduction is done by taking a strict Henselization A of the local ring $\mathcal{O}_{Y,y}$ of a point $y \in Y$ and works here as well since $X/H \times_Y \mathrm{Spec}(A) \cong (X \times_Y \mathrm{Spec}(A))/H$ (see the proof of [Fu11, Proposition 3.1.4]).

Now as A is Henselian and B^H is finite over A, by [Fu11, Proposition 2.8.3] we have

$$B^H \cong (B^H)_{\mathfrak{m}_1} \times \ldots \times (B^H)_{\mathfrak{m}_r} \quad \text{via} \quad b \mapsto (\frac{b}{1}, \ldots, \frac{b}{1}) \tag{4.2.1}$$

where $\mathfrak{m}_i$ are all the maximal ideals of B^H. Composing the inclusion $A \subseteq B^H$ with the localization $B^H \to (B^H)_{\mathfrak{m}_1}$ gives a map $\varphi_1 : A \to ((B^H)_{\mathfrak{m}_1})^{G_d(\mathfrak{m}_1)}$.

This map is injective: If $\varphi_1(a) = 0$, then there exists an $s \in B^H \setminus \mathfrak{m}_1$ with $sa = 0 \in B^H$, so the annihilator of a in B^H is not contained in $\mathfrak{m}_1$. But since a is G-invariant and all $\mathfrak{m}_i$ lie in the same G-orbit, it follows that the annihilator of a is not contained in any maximal ideal of B^H, so $a = 0$.

Moreover, this map is surjective: As G acts transitively on the maximal ideals, for every i there exists an element $\sigma_i \in G$ with $\sigma_i \mathfrak{m}_1 = \mathfrak{m}_i$ (let $\sigma_1 = \mathrm{id}$). For $\frac{b}{s} \in ((B^H)_{\mathfrak{m}_1})^{G_d(\mathfrak{m}_1)}$ there exists an $a \in B^H$ with

$$\frac{a}{1} = \frac{\sigma_i b}{\sigma_i s} \in (B^H)_{\mathfrak{m}_i} \quad 1 \leq i \leq r$$

by 4.2.1. So in $(B^H)_{\mathfrak{m}_1}$ we have $\frac{a}{1} = \frac{b}{s} = \frac{\sigma_i^{-1} a}{1}$ for all i. The σ_i form a set of representatives for the left cosets modulo $G_d(\mathfrak{m}_1)$. So every $\sigma \in G$ can be written as $\sigma = \sigma_i \tau$ with $\tau \in G_d(\mathfrak{m}_1)$. Since $\frac{a}{1} = \frac{b}{s}$ was assumed to be $G_d(\mathfrak{m}_1)$-invariant, this shows $\frac{a}{1} = \frac{\sigma a}{1}$, i.e. $\mathrm{Ann}_{B^H}(\sigma a - a) \not\subseteq \mathfrak{m}_1$, for every $\sigma \in G$. Similarly, it follows that $\mathrm{Ann}_{B^H}(\sigma a - a)$ is not contained in any maximal ideal, so that $\sigma a = a$ for all σ. So a is a G-invariant with $\varphi_1(a) = \frac{b}{s}$.

Of course, the analogously defined maps $\varphi_i : A \to ((B^H)_{\mathfrak{m}_i})^{G_d(\mathfrak{m}_i)}$ are isomorphisms, too. Since A is a strictly Henselian local ring, its residue field is separably closed. Hence we get $G_d(\mathfrak{m}_i) = G_i(\mathfrak{m}_i) \subseteq H$. So the ring map $A \to B^H$ has the property that for all maximal ideals $\mathfrak{m}$ of B^H the localized map $A = A_{\mathfrak{m} \cap A} \to (B^H)_{\mathfrak{m}}$ is an isomorphism. Thus, $A \to B^H$ is étale.

The result also appears in the book "Henselsche Ringe und algebraische Geometrie" by Kurke et al. [KPR75, Satz 4.2.1]. □

We will now give a version of Vinberg's Lemma designed for a generalization of Dufrene's and Kac-Watanabe's results. It already includes a step, which is performed in Kac-Watanabe's proof, of reducing from k-reflections on X to elements that have a fixed point.

Lemma 4.12. *Let X be a separated scheme, connected in codimension k, on which a finite group G acts admissibly. Let $\pi : X \to Y$ be the quotient and suppose that X is of finite presentation over Y. Furthermore, assume that Y has the following property:*

$$\begin{array}{c} \text{for all closed subsets } Z \text{ of } Y \text{ with } \operatorname{codim}_Y(Z) > k \\ \text{the space } Y \setminus Z \text{ is simply connected.} \end{array} \tag{4.2.2}$$

Then G is generated by k-reflections on X.

Due to the similarity to Definition 2.3, we will refer to 4.2.2 as **simply connected in codimension** k, although this is not a standard term. In the case $k = 1$, Popov and Vinberg call this property **strongly simply connected** (see [PV94, Proposition 8.3]).

Proof. Since X is separated, the finite union

$$L := \bigcup_{\substack{\sigma \in G \\ \operatorname{codim}_X(X^\sigma) > k}} X^\sigma$$

is a closed subset of X. In addition, L is G-stable (since for all τ, $\sigma \in G$ we have $\tau X^\sigma = X^{\tau\sigma\tau^{-1}}$), and has $\operatorname{codim}_X(L) > k$ by Remark 2.2(a). So by assumption, $\widetilde{X} := X \setminus L$ is connected.

As π is a closed map (see the proof of [Fu11, Proposition 3.1.1]), $\widetilde{Y} := Y \setminus \pi(L)$ is an open subscheme of Y. Moreover, we have $\operatorname{codim}_Y(\pi(L)) > k$. So by assumption 4.2.2, $\widetilde{Y} := Y \setminus \pi(L)$ is simply connected.

Now G acts on $\widetilde{X} := X \setminus L$ with quotient morphism $\widetilde{\pi} : \widetilde{X} \to \widetilde{Y}$. An element $\sigma \in G$ is a k-reflection on X if and only if it has a fixed point in $\widetilde{X}$. So we have to show that G is equal to the subgroup $H := \langle G_i(\widetilde{x}) \mid \widetilde{x} \in \widetilde{X} \rangle$.

By Lemma 4.11, the morphism

$$\varphi : \widetilde{X}/H \to \widetilde{Y}$$

is étale. As $\widetilde{Y}$ is simply connected, $\widetilde{X}/H$ is therefore isomorphic to a disjoint union of finitely many copies of $\widetilde{Y}$. But $\widetilde{X}$ and therefore $\widetilde{X}/H$ are connected, so there is only one copy, and φ is an isomorphism. This shows that the quotients of $\widetilde{X}$ by G and by H are the same.

Now let $\sigma \in G$. To show that σ lies in H, take any point $\widetilde{x} \in \widetilde{X}$. Since the G-orbit of this point is the same as its H-orbit, there exists $\tau \in H$ with $\sigma\tau \in G_d(\widetilde{x})$. For the quotient $\widetilde{\pi}$ of a scheme by a finite group, it is a general fact that the canonical homomorphism from the decomposition group of $\widetilde{x}$ to the Galois group of the extension of the residue class fields of $\widetilde{x}$ and $\widetilde{y} = \widetilde{\pi}(\widetilde{x})$ is surjective, see part (iii) of [Fu11, Proposition 3.1.1]). And this holds now for both the decomposition group in G and in H, hence

$$G_d(\widetilde{x})/G_i(\widetilde{x}) \cong \operatorname{Gal}(\kappa(\widetilde{x})/\kappa(\widetilde{y})) \cong H_d(\widetilde{x})/H_i(\widetilde{x}).$$

Thus for $\sigma\tau \in G_d(\widetilde{x})$ there exists $\mu \in H_d(\widetilde{x})$ with $\sigma\tau\mu \in G_i(\widetilde{x})$, which is contained in H, and hence $\sigma \in H$ follows. $\square$

4.3 Purity Theorems

For a scheme Y let us denote by $\mathbf{FEt}(Y)$ the category of étale coverings of Y. So its objects are finite étale maps $f : X \to Y$ and its morphisms are commutative diagrams

$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd},$$

where g is a morphism of schemes and where f_1 and f_2 are two étale coverings of Y. Any base change of a finite morphism is finite, and any base change of an étale morphism is étale (see [Fu11, Proposition 2.3.1]). So given a scheme morphism $h : Y_1 \to Y_2$, we get the following functor:

$$h^* : \mathbf{FEt}(Y_2) \to \mathbf{FEt}(Y_1), \quad (X \to Y_2) \mapsto ((X \times_{Y_2} Y_1) \to Y_1). \tag{4.3.1}$$

Remark 4.13. Suppose that h^* is an equivalence of categories, and that Y_2 is simply connected. Since disjoint union and fiber product commute, it follows then that Y_1 is simply connected, too.

Definition 4.14. *(a) Let X be a scheme, $Z \subseteq X$ a closed subscheme and $U := X \setminus Z$. Following SGA 2 (see [Gro65, X, Définition 3.1]), we call the pair (X, Z)* ***pure*** *if for all open subschemes V of X the functor i^*, as defined in 4.3.1, induced by the inclusion $i : U \cap V \hookrightarrow V$ is an equivalence of categories.*

(b) Again following [Gro65, X, Définition 3.2], a Noetherian local ring $(R, \mathfrak{m})$ is called pure if the pair $(\mathrm{Spec}(R), \{\mathfrak{m}\})$ is pure.

Remark 4.15. Suppose that (X, Z) is pure, and that X is simply connected. In particular, we get that the functor i^* coming from the inclusion $X \setminus Z \hookrightarrow X$ is an equivalence of categories. So by Remark 4.13, $X \setminus Z$ is simply connected, too.

The following collection of purity theorems will be used as a black box here.

Theorem 4.16. *Let R be a Noetherian local ring that satisfies one of the following four conditions:*

(a) R is regular and $\dim(R) \geq 2$,

(b) R is a complete intersection and $\dim(R) \geq 3$,

(c) R is excellent, a quotient of a regular local ring, equidimensional, a complete intersection in codimension $2 + \mathrm{cid}(R)$, and $\dim(R) \geq 3$,

(d) R is the localization of a normal affine K-algebra, where K is a field of characteristic 0, and $\dim(R) \geq 3 + \mathrm{cid}(R)$.

Then R is pure.

Proof. Part (a) is the Zariski-Nagata purity theorem. Part (b) is due to Grothendieck. We refer to SGA 2 for the proofs of both parts (see [Gro65, Théorème 3.4]).

Parts (c) and (d) were proven by Cutkosky [Cut95, Theorem 19 and Corollary after Theorem 26]. Part (d) relies on results of Goresky and Macpherson that are only available in characteristic 0. $\square$

Corollary 4.17. *Let R be a Noetherian local ring and k a natural number. Suppose that $X = \mathrm{Spec}(R)$ is simply connected and that one of the following four cases holds:*

(a) R is regular and $k = 1$,

(b) R is a complete intersection and $k = 2$,

(c) R is excellent, a quotient of a regular local ring, equidimensional, a complete intersection in codimension $2 + \mathrm{cid}(R)$, and $k = 2$,

(d) R is the localization of a normal affine K-algebra, where K is a field of characteristic 0, and $k = 2 + \mathrm{cid}(R)$.

Then X is simply connected in codimension k.

Proof. Let $Z \subseteq X$ be a closed subscheme of codimension larger than k. If we can show that the pair (X, Z) is pure, then the result follows with Remark 4.15. For that, by [Gro65, Proposition 3.3], it is to show that all local rings $R_\mathfrak{p}$ with $\mathfrak{p} \in Z$ are pure. But for such a prime ideal $\mathfrak{p}$ we have

$$\dim(R_\mathfrak{p}) = \mathrm{ht}_R(\mathfrak{p}) = \mathrm{codim}_X(\overline{\{\mathfrak{p}\}}) \geq \mathrm{codim}_X(Z) > k.$$

The assumptions on R in (a), (b), (c), or (d), respectively, imply that $R_\mathfrak{p}$ satisfies the corresponding part of Theorem 4.16. Hence $R_\mathfrak{p}$ is pure by Theorem 4.16. □

The main theorem of this chapter, Theorem 4.25, uses part (c) of the above Corollary. For our purpose, we can really view (c) as a generalization of (b) to rings R with $\mathrm{cid}(R) > 0$ (since the additional assumptions in (c) are always satisfied for local rings of irreducible varieties).

The properties of R that are assumed in part (a) or (b) of Corollary 4.17 each remain true if we pass from R to its completion $\widehat{R}$. For part (d) it is clear that this does not hold.

We also need to get the assumptions of part (c) for a complete local ring.

Proposition 4.18. *Let A be an affine K-domain that is a complete intersection in codimension $2 + \mathrm{cid}(A)$, where $\mathrm{cid}(A)$ is the global complete intersection defect as in Definition 4.2(a). Let $\mathfrak{m}$ be a maximal ideal of A and $\widehat{A}$ the $\mathfrak{m}$-adic completion of A. Then $\widehat{A}$ satisfies the assumptions of part (c) in Corollary 4.17.*

Proof. It is clear that $\widehat{A}$ is excellent and a quotient of a regular local ring. While it need not be an integral domain (see comments below), it is however equidimensional by [Mat89, Corollary after Theorem 31.5]. Moreover, Proposition 4.9 shows that $\widehat{A}$ is a complete intersection in codimension $2+\mathrm{cid}(A)$. The completion $\widehat{A}$ can be viewed as the completion of the localization $A_\mathfrak{m}$ (see Proposition 3.15), and completion of local rings preserves the (local) complete intersection defect, hence

$$\mathrm{cid}(\widehat{A}) = \mathrm{cid}(A_\mathfrak{m}) \leq \mathrm{lcid}(A) \leq \mathrm{cid}(A).$$

So a fortiori $\widehat{A}$ is a complete intersection in codimension $2 + \mathrm{cid}(\widehat{A})$. □

The observation that a completion of the coordinate ring of an irreducible variety need not be an integral domain will appear in Chapter 5 as well (see Example 5.13). However, we will use the fact that this cannot happen for normal varieties.

Remark 4.19. Let x be a normal point on an affine variety X (i.e. the localization of the coordinate ring R of X at the maximal ideal $\mathfrak{m}$ corresponding to x is a normal domain). Zariski [Zar48] showed that normality implies that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is an integral domain. In fact, he later [Zar50] showed that $\widehat{R}$ is normal, too. This is usually described by calling $R_\mathfrak{m}$ **analytically irreducible** or analytically normal, respectively.

4.4 Completions and the Invariant Ring

As a last step before proving the main theorem of this chapter, we use the results of Chapter 3 to study the transition between the invariant ring and its completion in this section. So again let G be a finite group acting on an affine variety X through automorphisms. Let us write R for the coordinate ring $K[X]$. To get a corresponding group action on a completion of R we need a fixed point. So let $x \in X^G$, and let $\mathfrak{m}$ be the maximal ideal of R corresponding to x. We write $\widehat{R}$ for the $\mathfrak{m}$-adic completion of R, and $\varepsilon : R \to \widehat{R}$ for the canonical map.

For every $\sigma \in G$ we now have $\sigma\mathfrak{m} = \mathfrak{m}$. With Remark 3.7 we see that σ induces a ring homomorphism $\widehat{R} \to \widehat{R}$, which we also call σ, such that

$$\begin{tikzcd} R \arrow[r, "\sigma"] \arrow[d, "\varepsilon"] & R \arrow[d, "\varepsilon"] \\ \widehat{R} \arrow[r, "\sigma"] & \widehat{R} \end{tikzcd}$$

commutes. With the explicit formula $\sigma \cdot (f_n + \mathfrak{m}^n)_{n\geq 0} = (\sigma f_n + \mathfrak{m}^n)_{n\geq 0}$, it is easy to see that this defines a group action of G on $\widehat{R}$.

Lemma 4.20. *Let X be a G-variety, $R = K[X]$, and let $\mathfrak{m} \subseteq R$ be a maximal ideal fixed by G. For every finitely generated, separating subalgebra $A \subseteq R^G = K[X]^G$ the following hold:*

(a) The $\mathfrak{m} \cap A$-adic, the $\mathfrak{m}^G$-adic, and the $\mathfrak{m}$-adic filtrations on R are all equivalent.

(b) The $\mathfrak{m} \cap A$-adic, and the $\mathfrak{m}^G$-adic filtrations on R^G are equivalent.

Proof. Of course, we have $\sqrt{(\mathfrak{m} \cap A)R} \subseteq \sqrt{\mathfrak{m}^G R} \subseteq \mathfrak{m}$. For part (a) we need show that these are actually equalities (see Proposition 3.1). For this let $\mathfrak{p} \in \operatorname{Spec}(R)$ be any prime ideal containing $\mathfrak{m} \cap A$. So in A the inclusion $\mathfrak{m} \cap A \subseteq \mathfrak{p} \cap A$ holds.

It is a fact that preimages of maximal ideals under homomorphisms of finitely generated K-algebras are maximal ideals, too (see [Kem11, Prop. 1.2]), so $\mathfrak{m} \cap A$ is a maximal ideal in A. Hence we get $\mathfrak{m} \cap A = \mathfrak{p} \cap A$, and so in particular $\mathfrak{m}^G \cap A = \mathfrak{p}^G \cap A$. As A is separating, this implies $\mathfrak{m}^G = \mathfrak{p}^G$ by Remark 1.8. As the quotient map is a geometric quotient (see Remark 1.5) this implies that there exists an element $\sigma \in G$ with $\sigma\mathfrak{p} = \mathfrak{m}$. But $\sigma^{-1}\mathfrak{m} = \mathfrak{m}$, so $\mathfrak{p}$ is equal to $\mathfrak{m}$.

So $\mathfrak{m}$ is the only prime ideal containing $(\mathfrak{m} \cap A)R$, which gives $\sqrt{(\mathfrak{m} \cap A)R} = \mathfrak{m}$ since every radical ideal is an intersection of prime ideals.

Part (b) follows with similar reasoning in R^G. $\square$

Theorem 4.21. *Let X be a G-variety, $R = K[X]$, and let $\mathfrak{m} \subseteq R$ be a maximal ideal fixed by G. The invariant ring of the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the $\mathfrak{m}^G$-adic completion of the invariant ring:*

$$(\widehat{R})^G \cong \widehat{R^G}.$$

Proof. For each $\sigma \in G$, the map $R \to R$, $f \mapsto f - \sigma f$ is an R^G-module homomorphism. Hence putting them together gives rise to an R^G-linear map

$$\varphi : R \to \prod_{\sigma \in G} R, \quad f \mapsto (f - \sigma f)_{\sigma \in G},$$

whose kernel is obviously the invariant ring R^G. We get an exact sequence of finitely generated R^G-modules:

$$0 \longrightarrow R^G \xrightarrow{i} R \xrightarrow{\varphi} \prod_{\sigma\in G} R.$$

By Proposition 3.6, the induced sequence of $\mathfrak{m}^G$-adic completions is exact, too. Let us write $\widetilde{R}$ for the $\mathfrak{m}^G$-adic completion of R. We get the following commutative diagram with exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \widehat{R^G} \arrow[r, "\widetilde{i}"] \arrow[d, equal] & \widetilde{R} \arrow[r, "\widetilde{\varphi}"] \arrow[d, equal] & (\widetilde{\prod_{\sigma\in G} R}) \arrow[d, "\cong"] \\ 0 \arrow[r] & \widehat{R^G} \arrow[r, "\widetilde{i}"] & \widetilde{R} \arrow[r, "\psi"] & \prod_{\sigma\in G} \widetilde{R} \end{tikzcd}$$

where ψ is given by $\psi(f) = (f - \sigma f)_{\sigma\in G}$. But by Lemma 4.20 and Remark 3.4, we get

$$\begin{tikzcd} 0 \arrow[r] & \widehat{R^G} \arrow[r, "\widetilde{i}"] \arrow[d, equal] & \widetilde{R} \arrow[r, "\psi"] \arrow[d, "\cong"] & \prod_{\sigma\in G} \widetilde{R} \arrow[d, "\cong"] \\ 0 \arrow[r] & \widehat{R^G} \arrow[r, "\widehat{i}"] & \widehat{R} \arrow[r, "\psi'"] & \prod_{\sigma\in G} \widehat{R} \end{tikzcd}$$

where ψ' is again the map $\psi'(f) = (f - \sigma f)_{\sigma\in G}$ and $\widehat{i}$ is the ring homomorphism induced by the inclusion $i : R^G \to R$. So $(\widehat{R})^G = \ker(\psi') = \operatorname{im}(\widehat{i})$ and this is (as a ring) isomorphic to $\widehat{R^G}$. $\square$

Proposition 4.22. *With the situation as in Theorem 4.21 suppose that $\sigma \in G$ is a k-reflection on* $\operatorname{Spec}(\widehat{K[X]})$ *(for some $k \in \mathbb{N}_0$). Then σ is a k-reflection on X as well.*

Proof. Let I be the ideal in $\widehat{R} = \widehat{K[X]}$ generated by $\{f - \sigma f \mid f \in \widehat{R}\}$. By assumption on σ, we have $\operatorname{ht}_{\widehat{R}}(I) \leq k$. As going-down holds for the homomorphism $\varepsilon : R \to \widehat{R}$ (where $R = K[X]$) by Corollary 3.14, this implies $\operatorname{ht}_R(\varepsilon^{-1}(I)) \leq k$. But $\varepsilon^{-1}(I)$ contains the ideal $J := (f - \sigma f \mid f \in R)$. So J has height less or equal to k, which means that σ is a k-reflection on $\operatorname{Spec}(R)$, too. $\square$

4.5 Main Result about Complete Intersection Separating Algebras

For the entire section let G be a finite group and X a G-variety. It was first observed by Dufresne that the property of being simply connected passes well between the spectrum of a separating algebra (with some mild additional assumptions) and the spectrum of the invariant ring.

Lemma 4.23. *Let X be irreducible, and let $A \subseteq K[X]^G$ be a finitely generated, separating algebra such that $K[X]^G$ is a finite A-module. Then the map*

$$\theta : \mathrm{Spec}(K[X]^G) \to \mathrm{Spec}(A)$$

corresponding to $A \subseteq K[X]^G$ is a universal homeomorphism, and the functor θ^* (defined in Equation 4.3.1) is an equivalence of categories.

Proof. By Remark 1.8, θ is injective. In addition, θ is dominant. In [DK02, Proposition 2.3.10] it is shown that this implies that the extension of the fields of fractions

$$\mathrm{Quot}(A) \subseteq \mathrm{Quot}(K[X]^G)$$

is finite and purely inseparable. By the same argument, for all prime ideals $\mathfrak{p} \subseteq K[X]^G$ the extension $\mathrm{Quot}(A/(\mathfrak{p} \cap A)) \subseteq \mathrm{Quot}(K[X]^G/\mathfrak{p})$ is finite and purely inseparable. By [GW10, Proposition 4.35], this means that θ is universally injective.

By assumption, $K[X]^G$ is a finite A-module, hence integral over A. So θ is finite, surjective and universally injective. Therefore θ is a universal homeomorphism (see [GW10, Exercise 12.32]) and the second claim follows by [Gro63, IX., Théorème 4.10]. □

Remark 4.24. (a) Suppose that $K[X]$ is graded (e.g. if X is the affine cone over a projective variety) and that the action of G on $K[X]$ is degree-preserving (so that $K[X]^G$ is graded as well). Then every graded, finitely generated, separating subalgebra $A \subseteq K[X]^G$ satisfies the assumption of Lemma 4.23, i.e. $K[X]^G$ is a finite A-module. This is essentially shown in [DK02, Theorem 2.3.12].

(b) A situation where $K[X]$ is graded and the action is degree-preserving is the following: Suppose that V is a linear representation of G. For every normal subgroup N of G we get an induced action of G/N on $K[V]^N$ such that $K[V]^G = (K[V]^N)^{G/N}$. So the action of G on $K[V]$ is split into two actions of smaller groups. This can be helpful in order to compute the invariant ring. But now $K[V]^N$ need not be a polynomial ring. However, it inherits the grading of $K[V]$ and the action of G/N on $K[V]^N$ is degree-preserving.

We come to our main result of this chapter. It extends Dufresne's result [Duf09, Theorem 1.3] to non-linear actions on normal and connected varieties (i.e. varieties whose coordinate ring is an integrally closed domain), and to separating algebras that are complete intersections in codimension $2 + \mathrm{cid}(A)$.

Theorem 4.25. *Let X be normal and connected, and let $X^G \neq \emptyset$. Suppose that there exists a finitely generated, separating algebra $A \subseteq K[X]^G$, such that $K[X]^G$ is a finite A-module, and that A is a complete intersection in codimension $2 + \mathrm{cid}(A)$. Then G is generated by 2-reflections.*

Proof. Let $x \in X^G$ and let $\mathfrak{m}$ be the maximal ideal of $K[X]$ corresponding to this point. In each of the three rings A, $K[X]^G$ and $K[X]$, we now have fixed a maximal ideal, namely $\mathfrak{m} \cap A$ in A, $\mathfrak{m}^G := \mathfrak{m} \cap K[X]^G$ in $K[X]^G$ and $\mathfrak{m}$ in $K[X]$. We will use the superscript $\widehat{\ }$ for the completions of the three rings at their specified maximal ideal. By Lemma 4.20 and Remark 3.4, they are isomorphic to the $\mathfrak{m} \cap A$-adic completions, which we will denote with the superscript $\widetilde{\ }$.

The inclusions $i : A \to K[X]^G$ and $j : K[X]^G \to K[X]$ induce ring homomorphisms between the completions of these rings. The necessary results about completions are summarized in the following commutative diagram (see Proposition 3.11 and Theorem 4.21):

$$\begin{tikzcd}
 & \widehat{K[X]}^G \arrow[d, "\cong"] & \\
\widehat{A} \arrow[ur, "=:\varphi"] \arrow[r, "\widehat{i}"] \arrow[d, equal] & \widehat{K[X]^G} \arrow[r, "\widehat{j}"] \arrow[d, "\cong"] & \widehat{K[X]} \arrow[d, "\cong"] \\
\widetilde{A} \arrow[r, "\widetilde{i}"] \arrow[d, "\cong"] & \widetilde{K[X]^G} \arrow[r, "\widetilde{j}"] \arrow[d, "\cong"] & \widetilde{K[X]} \arrow[d, "\cong"] \\
\widetilde{A} \otimes_A A \arrow[r] & \widetilde{A} \otimes_A K[X]^G \arrow[r] & \widetilde{A} \otimes_A K[X].
\end{tikzcd}$$

So the ring map $\varphi : \widehat{A} \to \widehat{K[X]}^G$ corresponds to the ring homomorphism

$$\mathrm{id}_{\widetilde{A}} \otimes i \quad : \quad \widetilde{A} \otimes_A A \longrightarrow \widetilde{A} \otimes_A K[X]^G.$$

Therefore, the scheme morphism $\omega := \mathrm{Spec}(\varphi)$ induced by φ corresponds to a base change of the map $\theta = \mathrm{Spec}(i) : \mathrm{Spec}(K[X]^G) \to \mathrm{Spec}(A)$. This map is a universal homeomorphism by Lemma 4.23. Hence ω is a universal homeomorphism, too.

The completion $\widehat{A}$ is a strictly Henselian local ring, hence its spectrum is simply connected (see [Mil80, I, Example 5.2(c)]). By the complete intersection assumption on A and by Proposition 4.18, $\widehat{A}$ satisfies the condition (c) of Corollary 4.17. So $\mathrm{Spec}(\widehat{A})$ is simply connected in codimension 2.

Next we see that $\mathrm{Spec}(\widehat{K[X]}^G)$ is simply connected in codimension 2 as follows: Let $Z \subseteq \mathrm{Spec}(\widehat{K[X]}^G)$ be a closed subset of codimension > 2. Since

$$\omega : \mathrm{Spec}(\widehat{K[X]}^G) \to \mathrm{Spec}(\widehat{A})$$

is finite, the set $\omega(Z)$ is closed and of codimension > 2 as well, hence $\mathrm{Spec}(\widehat{A}) \setminus \omega(Z)$ is simply connected. The restriction of ω gives a morphism

$$\omega_0 : \mathrm{Spec}(\widehat{K[X]}^G) \setminus Z \to \mathrm{Spec}(\widehat{A}) \setminus \omega(Z),$$

which is also a universal homeomorphism. So as in the proof of Lemma 4.23, it follows by [Gro63, IX., Théorème 4.10] that the functor ω_0^* is an equivalence of categories. By Remark 4.13, $\mathrm{Spec}(\widehat{K[X]}^G) \setminus Z$ is simply connected as well.

Now we are in a position to apply Lemma 4.12 to $\mathrm{Spec}(\widehat{K[X]})$. It is irreducible by Remark 4.19, and its quotient is simply connected in codimension 2. So Lemma 4.12 shows that G is generated by elements that are 2-reflections on $\mathrm{Spec}(\widehat{K[X]})$. By Proposition 4.22, these are 2-reflections on X as well. $\square$

Remark 4.26. (a) Applying the technique of the proof of Theorem 4.25 to part (a) of Corollary 4.17 (instead of parts (b) and (c)), results in another theorem about polynomial rings as separating algebras. We do not write this down explicitly here, since we get a more general result in Theorem 5.14. However, it is interesting to observe that the assumptions on X and the action of G here are almost the same as in Theorem 5.14.

(b) Unfortunately, I could not find a way to apply this proof to part (d) of Corollary 4.17, as completion certainly destroys the property of being the localization of an affine K-algebra A. As this part of the Cororally is about simply-connectedness in codimension $2 + \mathrm{cid}(R)$, one could (at least in characteristic 0) hope for a theorem that G is generated by $(2 + \mathrm{cid}(A))$-reflections (where $\mathrm{cid}(A)$ is the global complete intersection defect of a separating subalgebra with some extra assumptions). Kac and Watanabe were aware of this result for the invariant ring $A = \mathbb{C}[V]^G$ of a representation over the complex numbers (see [KW82, Theorem B]). For $\mathrm{cid}(\mathbb{C}[V]^G) = n-3$ (where $n = \dim(V)$) this means that G has to be generated by $(n-1)$-reflections.

One case in which the assumption for an n-dimensional affine algebra A to be a complete intersection in codimension $2 + \mathrm{cid}(A)$ (as in Theorem 4.25) is certainly satisfied is when A has isolated singularities (i.e. is regular in codimension $n - 1$) and has $\mathrm{cid}(A) \leq n-3$. Cutkosky [Cut95] gives various examples (affine cones over some standard examples in projective geometry) where this is the case (and where A is not a complete intersection). One of them is the following.

Example 4.27. The affine algebra

$$A = K[x_1, \ldots, x_6]/I \text{ with } I = (x_1x_5 - x_2x_4,\ x_1x_6 - x_3x_4,\ x_2x_6 - x_3x_5)$$

has dimension 4 and $\mathrm{cid}(A) = 1$. Using the Jacobian criterion (see [Kem11, Theorem 13.10]) we see that the origin in K^6 is the only singular point of the variety $X = \mathcal{V}(I)$. So A is regular in codimension 3. In particular, A is (not a complete intersection, but) a complete intersection in codimension $2 + \mathrm{cid}(A) = 3$. ◁

Such an example with an invariant ring $A = K[V]^G$ would certainly be a nice addendum to Theorem 4.25. But for invariant rings of representations (at least in the non-modular case) being an isolated singularity is a rather strong condition (see [Ste14, Lemma 2.4]). Instead of an example we get the following result as a corollary.

Theorem 4.28. *Suppose that* $\mathrm{char}(K) \nmid |G|$ *and that* $X = V$ *is a non-trivial linear representation of* G *with* $n = \dim(V) \geq 3$. *If* $\mathrm{cid}(K[V]^G) \leq n-3$, *then* $G \setminus \{\mathrm{id}\}$ *contains an* $(n-1)$*-reflection.*

Proof. Assume that G contains no $(n-1)$-reflections other than the identity element. By [Ste14, Lemma 2.4], these assumptions on G and V then imply that $K[V]^G$ is an isolated singularity. But as discussed above, $\mathrm{cid}(K[V]^G) \leq n-3$ now means that $K[V]^G$ is regular (hence complete intersection) in codimension $2 + \mathrm{cid}(K[V]^G)$. And by Theorem 4.25, G would be generated by 2-reflections, contradicting the assumption. □

Remark 4.29. The last line of Remark 4.26 cited Kac-Watanabe's result: If $\mathrm{cid}(\mathbb{C}[V]^G)$ is equal to $n - 3$, then G is generated by $(n - 1)$-reflections. In Theorem 4.28 we only get the result that G contains $(n-1)$-reflections (but are not restricted to the complex numbers).

After observing the importance of the assumption that $K[X]^G$ is a finite A-module in the proof of Theorem 4.25 (recall that this is always true for graded separating subalgebras by Remark 4.24), I asked Gregor Kemper if this condition might always hold for separating subalgebras $A \subseteq K[X]^G$. He responded with the example $A = K[x] \subseteq K[x, x^{-1}]$ where A is separating for the hyperbola. The following example is built on this.

Example 4.30. Let $X = \mathcal{V}(xy-1) \subseteq K^2$ and $G = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle \subseteq \mathrm{GL}_2(K)$ where K is a field of characteristic $\neq 2$. The coordinate ring of X is

$$K[X] = K[x, y]/(xy-1) = K[\overline{x}, \overline{y}],$$

and the invariant ring is

$$K[X]^G = K[\overline{x}, \overline{y}]^G = K[\overline{x}^2, \overline{xy}, \overline{y}^2] = K[\overline{x}^2, \overline{y}^2].$$

The subalgebra $A = K[\overline{x}^2]$ is separating: Two points $u, v \in X$ are of the form $u = \begin{pmatrix} \lambda \\ \lambda^{-1} \end{pmatrix}$, $v = \begin{pmatrix} \mu \\ \mu^{-1} \end{pmatrix} \in X$ with $\lambda, \mu \in K^\times$. So $\lambda^2 = \mu^2$ implies $u = \pm v$, hence u and v are in the same orbit.

So there exists a separating subalgebra which is even isomorphic to a polynomial ring, but G is not generated by reflections, as all stabilizers G_x, for $x \in X$, are trivial. $\triangleleft$

Finally, we recall that an example where the invariant ring $K[V]^G$ is not a complete intersection, but a smaller separating subalgebra A is a hypersurface, hence a complete intersection, was already given in Example 1.10.

Chapter 5

The Minimal Number of Separating Invariants

5.1 Upper Bound for γ_{sep}

It has been known for a while that the upper bound

$$\gamma_{\mathrm{sep}} \leq 2 \cdot \dim(K[X]^G) + 1 \tag{5.1.1}$$

holds (see the introduction of Kamke and Kemper [KK12]). This inequality is true in our most general setting of invariant theory, i.e. when G is any linear algebraic group acting on an affine variety X through automorphisms. So G may be infinite and $K[X]^G$ does not even have to be finitely generated. Even if this is the case, the Krull dimension of $K[X]^G$ equals its transcendence degree over K (see Kemper [Kem11, Theorem 5.9 and Exercise 5.3]). Of course, when we restrict ourselves to finite groups (over an algebraically closed field) again, this upper bound reads as $\gamma_{\mathrm{sep}} \leq 2n + 1$ where n is the dimension of X as well as the dimension of the invariant ring by Corollary 1.3. To the best of my knowledge there is still no example where $\gamma_{\mathrm{sep}} = 2n+1$ and I cannot provide such an example either. This section contains a new proof for this upper bound in the case of finite groups and some remarks on my attempts to improve it.

It is a result of Kronecker [Kro82] that an affine subvariety of K^n can be written as the intersection of $n+1$ hypersurfaces. In fact, a more general statement holds, namely that an ideal I in a Noetherian ring of dimension n can be generated by $n+1$ elements up to radical. This means that there always exist $a_1, \ldots, a_{n+1} \in I$ with $\sqrt{I} = \sqrt{(a_1, \ldots, a_{n+1})}$. The proof of this theorem (as given in [IR05, Threorem 9.5.1]) relies heavily on the prime avoidance lemma of commutative algebra.

The separating variety is a subvariety of $X \times X$, so it is defined by an ideal in the $2n$-dimensional ring $K[X] \otimes_K K[X]$. Hence by the theorem cited above, set-theoretically it can be defined by $2n + 1$ elements. This already looks like the desired upper bound for γ_{sep}, but we are faced with the problem that generators of an ideal defining $\mathcal{V}_{\mathrm{sep}}$ need not lie in the image of the δ-map defined in 1.0.2.

So a small set of generators of an ideal defining $\mathcal{V}_{\mathrm{sep}}$ cannot always be traced back to a set of separating invariants. However, we can modify the proof of [IR05, Threorem 9.5.1] to obtain the upper bound. The key observation is that the δ-map is at least K-linear (see Proposition 1.12). So altering a set of generators linearly, does not destroy the property of lying in the image of δ. Our modification of the theorem about the number of generators upto radical is motivated by this observation. Instead of the prime avoidance lemma we will use the simple fact that a vector space over an infinite field cannot be written as a finite union of proper subspaces. (As specified in Chapter 1 the base field K is always assumed to be algebraically closed. Nevertheless the next theorems carry the (now unnecessary) assumption that K is infinite, just to emphasize that this property of K is needed at this moment.)

Proposition 5.1. *Let V be a vector space over (the infinite field) K. Suppose we are given finitely many subspaces $U_1, \dots, U_r$ of V with $V = \bigcup_{i=1}^r U_i$. Then there exists an i such that $U_i = V$.*

Proof. We will proceed by induction on r. The case $r = 1$ is clear, so let $r > 1$ now. Assume by contradiction that for all $j \in \{1, \dots, r\}$ we have

$$V \neq \bigcup_{\substack{i=1 \\ i \neq j}}^{r} U_i.$$

So for every j there exists a vector $v_j \in V$ such that $v_j \notin \bigcup_{i \neq j} U_i$. Obviously we have $v_j \in U_j \setminus \{0\}$. Since the field of scalars contains infinitely many elements, the set $\{v_1 + \lambda v_2 \mid \lambda \in K\}$ is infinite as well. As V is written as a finite union of subspaces, we conclude that there exists an index j and different λ, $\mu \in K$ with

$$v_1 + \lambda v_2 \in U_j \quad \text{and} \quad v_1 + \mu v_2 \in U_j. \tag{5.1.2}$$

Subtracting these equations shows $v_2 \in U_j$, which would imply $j = 2$. But then we get $v_1 \in U_j$ as well, which would imply $j = 1$. So the assumption was false and we see that V is already the union of $r - 1$ subspaces. By induction we conclude that not all U_i can be proper. □

This results in the following substitute for the prime avoidance lemma.

Lemma 5.2. *Let R be an algebra over (the infinite field) K. Suppose we are given a finitely generated ideal $I = (a_1, \dots, a_m)_R$, and ideals $J_1, \dots, J_r$ of R such that I is not contained in any of the J_k. Then there exists a K-linear combination of the a_i which is not containd in any of the J_k.*

Proof. Ideals of a K-algebra are linear subspaces as well. We define the vector spaces $V := \langle a_1, \dots, a_m \rangle_K$ and $U_k := J_k \cap V$ for $k = 1, \dots, r$. By assumption, V does not equal any of the U_k. Hence by Proposition 5.1, V is not equal to $\bigcup_{k=1}^r U_k$. So there exists a K-linear combination of the a_i that lies in none of the U_k and therefore in none of the J_k. □

Theorem 5.3. *Let R be an affine algebra over (the infinite field) K. Let $n = \dim(R)$ and let $I = (a_1, \dots, a_m)_R$ be an ideal of R. Then there exist $n+1$ elements $b_1, \dots, b_{n+1} \in R$, which can be chosen as K-linear combinations of the a_i, such that $\sqrt{I} = \sqrt{(b_1, \dots, b_{n+1})}$.*

Proof. Using induction, we will construct elements b_j as linear combinations of the a_i such that they have the following property for $1 \leq j \leq n+1$:

$$\begin{gathered}\text{Every prime ideal of } R \text{ which contains } b_1, \dots, b_j \\ \text{and which has height } \leq j-1 \text{ contains } I \text{, too.}\end{gathered} \tag{5.1.3}$$

Let us start by defining b_1. Since the ring is Noetherian, there are only finitely many minimal prime ideals in R. Among these we denote by $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ those minimal prime ideals that do not contain I. By Lemma 5.2 there exists a K-linear combination b_1 of the a_i which is not contained in any of the $\mathfrak{p}_i$. Since minimal primes ideals are precisely those of height 0, the property (5.1.3) is satisfied for $j = 1$.

Now we assume that $j \geq 2$ and that $b_1, \dots, b_{j-1}$ have already been defined. Once again the Noetherian property implies that there are only finitely many prime ideals that are minimal over the ideal $(b_1, \dots, b_{j-1})$. Among these there might be some prime ideals that do not contain I. We will simply overwrite the above notation and denote them by $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ again. By the induction hypothesis they must have height $\geq j-1$. Lemma 5.2 gives the next element $b_j \in I$ which is a K-linear combination of the a_i and not contained in any of these $\mathfrak{p}_i$.

Let us check that (5.1.3) is satisfied for $b_1, \dots, b_j$. For this let $\mathfrak{p} \subseteq R$ be a prime ideal that contains $b_1, \dots, b_j$, but not I. Of course, $(b_1, \dots, b_{j-1}) \subseteq \mathfrak{p}$. So $\mathfrak{p}$ must sit above some prime ideal minimal over $(b_1, \dots, b_{j-1})$. However $I \not\subseteq \mathfrak{p}$, so it must be one of the $\mathfrak{p}_i$ that is contained in $\mathfrak{p}$. Since the element b_j was defined to avoid all the $\mathfrak{p}_i$, we conclude that one of the $\mathfrak{p}_i$ is strictly contained in $\mathfrak{p}$. Therefore, we have

$$\mathrm{ht}(\mathfrak{p}) > \mathrm{ht}(\mathfrak{p}_i) \geq j-1,$$

as desired.

So by gradually avoiding prime ideals of a certain height we have found elements $b_1, \dots, b_{n+1} \in I$ such that every prime ideal over $(b_1, \dots, b_{n+1})$ is a prime ideal over I as well. The converse is trivial, hence these two ideals have the same prime ideals lying above them. Since every radical ideal is the intersection of prime ideals, we get

$$\sqrt{I} = \sqrt{(b_1, \dots, b_{n+1})}.$$

By replacing the usual prime avoidance lemma with the linear version given as Lemma 5.2 we achieved that the b_i are K-linear combinations of the original generators. □

Corollary 5.4. *Let G be a finite group and X a G-variety of dimension n over (the algebraically closed field) K. Then the upper bound $\gamma_{\mathrm{sep}} \leq 2n+1$ holds.*

Proof. Let R be the coordinate ring of the product variety $X \times X$. Furthermore, let $f_1, \dots, f_m \in K[X]^G$ be a finite generating (or at least separating) set of invariants, which always exists by Theorem 1.2. The separating variety is the subvariety of $X \times X$ defined by the ideal

$$\mathcal{I}_{\mathrm{sep}} := \left(\delta(K[X]^G)\right)_R = (\delta(f_1), \dots, \delta(f_m))_R \quad \text{(cf. Proposition 1.12).}$$

Since K is algebraically closed, we have

$$\dim(R) = \dim(K[X \times X]) = \dim(X \times X) = 2n.$$

By Theorem 5.3 there exist 2n+1 K-linear combinations $b_1, \ldots, b_{2n+1}$ of the $\delta(f_i)$ which generate an ideal having the same radical as $\mathcal{I}_{\text{sep}}$. But since δ is K-linear, these b_i are equal to $\delta(g_i)$ with some linear combinations of the invariants f_i. So we have $2n+1$ invariants $g_1, \ldots, g_{2n+1}$ with

$$\sqrt{\mathcal{I}_{\text{sep}}} = \sqrt{\delta(g_1), \ldots, \delta(g_{2n+1})}.$$

By Proposition 1.13, this means exactly that the g_i form a separating set of size $2n+1$. $\square$

We will give some remarks about the above proofs, which all arose from my hope of improving the upper bound to $\gamma_{\text{sep}} \leq 2n$ (maybe with some additional assumptions, like $X = V$ being a linear representation).

Remark 5.5. (a) The theorem of Eisenbud-Evans-Storch from 1972 is an improvement for polynomial rings of Kronecker's result: Every ideal I in $K[x_1, \ldots, x_n]$ can be defined upto radical by n polynomials (see [IR05, Theorem 9.5.2 and Corollary 9.5.3]). But here the generators cannot be chosen "generic". The case $n = 1$ already shows the difference: A generator of an ideal $(a_1, \ldots, a_m)$ in the principal ideal domain $K[x]$ must be a greatest common divisor of the a_i and thus cannot be taken as a K-linear combination of the a_i in general.

(b) In the first step of the proof of Theorem 5.3 some of the minimal prime ideals of the Noetherian ring R are avoided. If X is irreducible, then the ring in question (when applying this theorem in Corollary 5.4) is the integral domain $R = K[X] \otimes_K K[X]$. So (0) is the only prime ideal that is avoided in the first step, hence we can take any $0 \neq \delta(f_1) \in R$ for this. This translates to the following result for irreducible X:

For every non-constant invariant $f_1 \in K[X]^G$ there exist $2n$ invariants $f_2, \ldots, f_{2n+1}$ such that $f_1, \ldots, f_{2n+1}$ form a separating set.

(c) Let us look again at the interplay of the two proofs above. The separating set $\{f_1, \ldots, f_{2n+1}\}$ is built inductively. In each step an invariant f_j is added such that $\delta(f_j)$ avoids finitely many prime ideals $\mathfrak{p}_i$ with $\mathcal{I}_{\text{sep}} \not\subseteq \mathfrak{p}_i$ for $i = 1, \ldots, r$. This simply means that given finitely many pairs of points $(a_i, b_i) \in X \times X$ with $Ga_i \neq Gb_i$ for all i, we need an invariant f_j that separates a_i from b_i. This led to the question if this can always be done using a homogeneous invariant in the case that $K[X]$ is graded. The answer is negative, as the following example with monomial invariants of different degrees shows.

Example 5.6. Let $\text{char}(K) \neq 2$, and let $i \in K$ be a primitive 4-th root of unity. Let G be the cyclic group of order 4 acting on $X = V = K^2$ via $G \cong \langle \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix} \rangle$. The invariant ring is $K[x, y]^G = K[x^2, y^4, xy^2]$. Let us look at the following two pairs of points (a_j, b_j)

and for each pair we give an invariant g_j such that $g_j(a_j) \neq g_j(b_j)$:

j	1	2
a_j	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}-1\\1\end{pmatrix}$
b_j	$\begin{pmatrix}2\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$
g_j	x^2	xy^2

Claim: There is no homogeneous invariant $f \in K[V]^G$ with $f(a_j) \neq f(b_j)$ for $j = 1, 2$.
Proof: Assume $f = \lambda_0 x^d + \lambda_1 x^{d-1} y + \ldots + \lambda_d y^d \in K[V]^G$ would be separating for both pairs. Due to

$$\lambda_0 = f(a_1) \neq f(b_1) = 2^d \lambda_0$$

we would have $\lambda_0 \neq 0$. Hence the degree d must be even, since $f \in K[x^2, y^4, xy^2]$. But then for every monomial $x^k y^j$ that appears in f the x-degree k must be even, too. Thus f cannot separate a_2 from b_2. $\triangleleft$

The following example of a one-dimensional variety with trivial group action was constructed with the hope of finding an example where $\gamma_{\text{sep}} = 2n + 1$. Instead there is a separating set of size $2n$ here. It is still an interesting example, since no two linear combinations of the $2n+1$ coordinate functions form a separating set. This is in contrast to the situation in the proof of Corollary 5.4, where the $2n + 1$ separating invariants can always be constructed as linear combinations of any given (larger) separating set. I thank Stephan Neupert for discussing examples of this kind with me.

Example 5.7. In the three-dimensional affine space over K we consider the union $X = X_1 \cup X_2 \cup X_3$ of the following three pairwise skew lines:

$$X_1 = \mathcal{V}(y - 1, z),$$
$$X_2 = \mathcal{V}(z - 1, x),$$
$$X_3 = \mathcal{V}(x - 1, y).$$

So the lines are just the three coordinate axes shifted by 1 to make them skew to each other. We make this into an example of invariant theory by considering a trivial group action on X. A separating set of invariants is then just a subset S of the coordinate ring $K[X]$ such that for all $p_1, p_2 \in X$ there exists an $f \in S$ with $f(p_1) \neq f(p_2)$. Since obviously the coordinate functions x, y, z form a separating set, it is natural to look at linear combinations of these for a smaller separating set.
Claim: No two linear combinations of x, y, z form a separating set.
Proof: Assume by contradiction that

$$f = \lambda_1 x + \lambda_2 y + \lambda_3 z, \quad g = \mu_1 x + \mu_2 y + \mu_3 z$$

with $\lambda_i, \mu_i \in K$ are separating. In order to distinguish points on the shifted x-axis X_1, it is clear that one of the coefficients λ_1 or μ_1 has to be non-zero. Without loss of generality we can assume $\lambda_1 \neq 0$.

We will make frequent use of the following general principle: Whenever $f_1, \dots, f_m$ form a separating set and $A = (a_{ij})$ is an invertible $m \times m$-matrix over K, then $g_i := \sum_{j=1}^{m} a_{ij} f_j$ with $i = 1, \dots, m$ form a separating set as well.

So we may replace f by $\lambda_1^{-1} f$ and g by $g - \lambda_1^{-1}\mu_1 f$ to get simpler separating functions of the form

$$f = x + \lambda_2 y + \lambda_3 z, \quad g = \mu_2 y + \mu_3 z,$$

where we have just renamed the coefficients λ_2, λ_3, μ_2 and μ_3. Assume $\mu_2 = 0$ first. Then we would get the following points on X_3 and X_1 that can not be separated:

p	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1-\lambda_2\\1\\0\end{pmatrix}$
$f(p)$	1	1
$g(p)$	0	0

Therefore, μ_2 has to be non-zero. Again we simplify the separating set by replacing g by $\mu_2^{-1} g$ and f by $f - \lambda_2\mu_2^{-1} g$ (and renaming λ_3 and μ_3 once more), which results in:

$$f = x + \lambda_3 z, \quad g = y + \mu_3 z.$$

But then we would find a point on X_1 and a point on X_2 that cannot be distinguished with f and g, ultimately contradicting our separation assumption:

p	$\begin{pmatrix}\lambda_3\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1-\mu_3\\1\end{pmatrix}$
$f(p)$	λ_3	λ_3
$g(p)$	1	1

This ends the proof of the claim. However, as promised before, there are still separating sets of size 2, for instance

$$f = x + y + z, \quad g = z + 2y + 2xy + xz.$$

This can best be seen by evaluating f and g on all three irreducible components individually:

p	$\begin{pmatrix}x\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\y\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\z\end{pmatrix}$
$f(p)$	$x+1$	$y+1$	$z+1$
$g(p)$	$2x+2$	$2y+1$	$2z$

(5.1.4)

So if p_1 and p_2 are two different points in X that are mapped to the same element by f, then, clearly, they must lie in two different components X_i. Furthermore, there exists an $a \in K$ such that

$$p_1, p_2 \in \{ \begin{pmatrix}a\\1\\0\end{pmatrix}, \begin{pmatrix}0\\a\\1\end{pmatrix}, \begin{pmatrix}1\\0\\a\end{pmatrix} \}.$$

But the last line in Table 5.1.4 shows that g separates these three points. $\triangleleft$

5.2 Reflection Groups and γ_{sep}

In 2013 Dufresne and Jeffries proved a new connection, in the case of linear actions on n-dimensional affine spaces, between the size of a separating set of invariants and the property of being a k-reflection group [DJ15, Proposition 3.2 & Theorem 3.4]. Their remarkable new result can be expressed shortly by the implication:

$$\gamma_{\text{sep}} = n + k - 1 \quad \Rightarrow \quad G \text{ is generated by } k\text{-reflections.}$$

They gave two proofs for this result, both relying on local cohomology. The first one uses a Mayer-Vietoris spectral sequence for local cohomology, the second one a connectedness theorem by Grothendieck similar to Hartshorne's, which we stated in Theorem 2.8.

Again we would like to extend this result (as far as possible) to non-linear actions on varieties. For this, we will imitate and adapt their second proof. Using Grothendieck's connectedness theorem we can reuse the results of Section 2.3.

This connectedness theorem is usually formulated with the notion of "connectedness in dimension d", which is dual to Definition 2.3.

Definition 5.8. *For an integer d, a Noetherian topological space Y is called* ***connected in dimension d*** *if for all closed subsets $Z \subseteq Y$ with $\dim(Z) < d$ the space $Y \setminus Z$ is connected.*

As for connectedness in codimension there is a different but equivalent description using the intersections of the irreducible components of Y. Since this is just the obvious analogue of Proposition 2.4, we do not write it down explicitly here but instead give a reference to the book "Joins and Intersections" by Flenner et al. [FOV99, Prop. 3.1.4].

Remark 5.9. Let Y be a Noetherian topological space of finite dimension n.

(a) Suppose that for all closed subspaces $Z \subseteq Y$ the formula

$$\operatorname{codim}_Y(Z) + \dim(Z) = \dim(Y)$$

holds. Then the space Y is connected in dimension d (where d is an integer) if and only if Y is connected in codimension $n - d$. For example, this is satisfied if Y is the spectrum of the coordinate ring R of an equidimensional affine variety. It is then also true for the spectrum of the completion $\widehat{R}$ of R at a maximal ideal.

(b) Let $Y = \bigcup_{i=1}^{r} Y_i$ be decomposed into its irreducible components. Suppose that Y is connected in dimension d with an integer $d < n$. Then all Y_i have dimension $> d$: By way of contradiction, assume that a component, say Y_1, has dimension $\leq d$. Then one could disconnect Y by subtracting $Y_1 \cap \bigcup_{i=2}^{r} Y_i$, which is a closed subset of dimension $\leq d - 1$.

Next we state a local version of Grothendieck's connectedness theorem taken from [FOV99].

Theorem 5.10. *Let R be a complete Noetherian local ring of dimension n such that* $\operatorname{Spec}(R)$ *is connected in dimension d with an integer $d < n$. Furthermore, let $\mathfrak{m}$ be the maximal ideal of R and $f \in \mathfrak{m}$. Then* $\operatorname{Spec}(R/(f))$ *is connected in dimension $d - 1$ (and has dimension $\geq d$).*

Proof. In [FOV99, Theorem 3.1.7] it is shown that $\operatorname{Spec}(R/(f))$ is connected in dimension $d-1$ under these assumptions.

Furthermore, we have $\operatorname{ht}_R((f)) \leq 1$ by the principal ideal theorem. So f lies in a prime ideal $\mathfrak{q} \in \operatorname{Spec}(R)$ with $\operatorname{ht}_R(\mathfrak{q}) = 1$. Let $\mathfrak{p} \subsetneq \mathfrak{q}$ be a minimal prime ideal contained in $\mathfrak{q}$. Since complete Noetherian local rings are catenary (see [BH93, Corollary 2.1.13]), we have $\dim(R/\mathfrak{p}) = \dim(R/\mathfrak{q}) + 1$. Together with Remark 5.9(b) this gives

$$\dim(R/(f)) \geq \dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) - 1 > d - 1. \qquad \square$$

Corollary 5.11. *Let R be a complete Noetherian local ring of dimension n such that $\operatorname{Spec}(R)$ is connected in dimension d with an integer $d < n$. Furthermore, let $\mathfrak{m}$ be the maximal ideal of R and $f_1, \ldots, f_r \in \mathfrak{m}$. Then $\operatorname{Spec}(R/(f_1, \ldots, f_r))$ is connected in dimension $d - r$ (and has dimension $\geq d - r + 1$).*

Proof. Of course, we will use induction on r. The case $r = 0$ is trivial. So let us assume that $r > 0$. By induction hypothesis $\operatorname{Spec}(R/(f_1, \ldots, f_{r-1}))$ is connected in dimension $d - r + 1$ and has dimension $\geq d - r + 2$. By Proposition 3.8, $S := R/(f_1, \ldots, f_{r-1})$ is a complete Noetherian local ring, too. So we use Theorem 5.10 to derive that the spectrum of $S/(f_r) = R/(f_1, \ldots, f_r)$ is connected in dimension $d - r$. $\square$

In order to use Grothendieck's connectedness theorem we need to pass from a localization R of the coordinate ring of an affine variety to its completion $\widehat{R}$. So we need to bring the connectedness properties back from $\operatorname{Spec}(\widehat{R})$ to $\operatorname{Spec}(R)$.

Lemma 5.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $(\widehat{R}, \widehat{\mathfrak{m}})$ its $\mathfrak{m}$-adic completion. If k is a non-negative integer such that $\operatorname{Spec}(\widehat{R})$ is connected in codimension k, then $\operatorname{Spec}(R)$ is connected in codimension k, too.*

Proof. By Proposition 3.13, $\widehat{R}$ is a faithfully flat R-module. So the morphism

$$\varphi : \operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R), \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

corresponding to the inclusion $\varepsilon : R \to \widehat{R}$, is surjective (see [Mat89, Theorem 7.3]).

Now let $\mathfrak{p}'$ and $\mathfrak{p}''$ be minimal prime ideals of R. Because of the surjectivity of φ there are prime ideals $\mathfrak{q}'$ and $\mathfrak{q}''$ of $\widehat{R}$ which are mapped to $\mathfrak{p}'$ and $\mathfrak{p}''$, respectively. Since $\widehat{R}$ is Noetherian (see Proposition 3.10), $\mathfrak{q}'$ contains a minimal prime ideal of $\widehat{R}$, which has to be mapped to $\mathfrak{p}'$, too, because of the minimality of $\mathfrak{p}'$. Therefore, we can assume that $\mathfrak{q}'$ and $\mathfrak{q}''$ are minimal. Now suppose that $\operatorname{Spec}(\widehat{R})$ is connected in codimension k. By Proposition 2.4, this guarantees the existence of a finite sequence of minimal prime ideals $\mathfrak{q}_0 = \mathfrak{q}'$, $\mathfrak{q}_1, \ldots, \mathfrak{q}_r = \mathfrak{q}''$ of $\widehat{R}$ such that

$$\operatorname{ht}_{\widehat{R}}(\mathfrak{q}_i + \mathfrak{q}_{i+1}) = \operatorname{codim}_{\operatorname{Spec}(\widehat{R})}(\mathcal{V}(\mathfrak{q}_i) \cap \mathcal{V}(\mathfrak{q}_{i+1})) \leq k \quad (\text{for } i = 0, \ldots, r-1).$$

With $\mathfrak{p}_i := \mathfrak{q}_i \cap R$ (for all i) we get:

$$\operatorname{ht}_R(\mathfrak{p}_i + \mathfrak{p}_{i+1}) \leq \operatorname{ht}_R((\mathfrak{q}_i + \mathfrak{q}_{i+1}) \cap R) \leq \operatorname{ht}_{\widehat{R}}(\mathfrak{q}_i + \mathfrak{q}_{i+1}) \leq k,$$

where the second inequality follows from going-down (see Corollary 3.14). So there is a finite sequence of irreducible closed subsets $Y_i := \mathcal{V}(\mathfrak{p}_i)$ of $\operatorname{Spec}(R)$ which leads from $Y_0 = \mathcal{V}(\mathfrak{p}')$ to $Y_r = \mathcal{V}(\mathfrak{p}'')$ in a way such that two subsequent subsets intersect in codimension $\leq k$. We conclude with Proposition 2.4 that $\operatorname{Spec}(R)$ is connected in codimension k. $\square$

The converse of Lemma 5.12 is not true, as can be seen already in the case $k = 0$, where connected in codimension 0 just means irreducible. We give an example for that behaviour, which is taken from [Eis95, Section 7.2] and [Kem11, Exercise 13.6].

Example 5.13. Let $\mathrm{char}(K) \neq 2$, and let X be the plane cubic curve defined by the equation $y^2 - x^3 - x^2 = 0$. As $x^3 - x^2$ has no square root in $K[x]$ the polynomial $y^2 - x^3 - x^2 \in K[x, y]$ is irreducible. Therefore, the coordinate ring $K[X]$ and its localization $R := K[X]_{(\overline{x},\overline{y})}$ at the maximal ideal corresponding to the point at the origin are integral domains, so their spectra are irreducible.

By Example 3.9 and Proposition 3.15, the completion of R at its maximal ideal is

$$\widehat{R} \cong K[\![x, y]\!]/(y^2 - x^3 - x^2).$$

But in the formal power series ring $K[\![x]\!]$ there is a square root of $x^3 + x^2$ (see [Eis95, Section 7.2] for details), so the completion is not an integral domain. In fact, $\mathrm{Spec}(\widehat{R})$ decomposes into two irreducible components. $\triangleleft$

In the above example the completion at a singular point of an irreducible curve was not a domain. This cannot happen for the completion at nonsingular points by Remark 4.19.

Theorem 5.14. *Let G be a finite group, and let X be a normal and connected G-variety of dimension n, such that G is generated by elements having a fixed point in X. If $\gamma_{\mathrm{sep}} = n + k - 1$ with $k \in \mathbb{N}$, then G is generated by k-reflections.*

Proof. According to Theorem 2.17, we need to show that the separating variety $\mathcal{V}_{\mathrm{sep}}$ of this action is connected in codimension k. Since X is irreducible and G is generated by elements having a fixed point, we know from Corollary 2.18 that $\mathcal{V}_{\mathrm{sep}}$ is at least connected. Hence it is sufficient to show that $\mathcal{V}_{\mathrm{sep}}$ is locally connected in codimension k in the sense of Proposition 2.7. Let us write

$$r := \gamma_{\mathrm{sep}} = n + k - 1$$

with a natural number k, so that there exists a separating subset $\{f_1, \dots, f_r\} \subseteq K[X]^G$ of size r. Using the δ-map from 1.0.2 we define

$$g_i := \delta(f_i) \in K[X] \otimes_K K[X] =: R, \text{ for } i = 1, \dots, n.$$

Moreover, let J be the ideal in R generated by $g_1, \dots, g_r$. Then, by Proposition 1.13, $\mathcal{V}_{\mathrm{sep}}$ is the subvariety

$$\mathcal{V}_{\mathrm{sep}} = \mathcal{V}_{X \times X}(J) \subseteq X \times X.$$

So $\mathrm{Spec}(R/J)$ is homeomorphic to $\mathrm{Spec}(K[\mathcal{V}_{\mathrm{sep}}])$, and we need to show that this space is locally connected in codimension k.

Take a maximal ideal $\mathfrak{m}$ of $R = K[X] \otimes_K K[X]$ which corresponds to a point $(x, y) \in \mathcal{V}_{\mathrm{sep}}$, and let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. By assumption, X is normal, hence the product variety $X \times X$ is normal, too. Normality of $X \times X$ implies that the spectrum of $\widehat{R}$ is irreducible (see Remark 4.19). With Proposition 3.15 we see that

$$\dim(\widehat{R}) = \mathrm{ht}_R(\mathfrak{m}) = \dim(R) = \dim(X \times X) = 2n,$$

hence $\operatorname{Spec}(\widehat{R})$ is connected in dimension d for all $d \leq 2n$. In particular, we can apply the local version of Grothendieck's connectedness theorem, given in Corollary 5.11, with $d = 2n - 1$ to the complete noetherian local ring $\widehat{R}$. It shows that the spectrum of

$$\widehat{R}/J\widehat{R} = \widehat{R}/(g_1, \ldots, g_r)_{\widehat{R}}$$

is connected in dimension

$$(2n - 1) - r = 2n - 1 - (n + k - 1) = n - k.$$

But by Proposition 3.8, $\widehat{R}/J\widehat{R}$ is the $\mathfrak{m}$-adic completion of R/J. So we have

$$\dim(\widehat{R}/J\widehat{R}) = \operatorname{ht}_{R/J}(\mathfrak{m}/J) = \dim(R/J) = \dim(\mathcal{V}_{\text{sep}}) = n$$

(cf. Remark 2.14(a)). Therefore, the spectrum of $\widehat{R}/J\widehat{R}$ is connected in codimension k (see Remark 5.9(a)).

We get the same completion if we localize first (see Proposition 3.15), so $\widehat{R}/J\widehat{R}$ can also be viewed as the completion of the local ring $(R/J)_{\mathfrak{m}/J}$. Hence Lemma 5.12 shows that $\operatorname{Spec}((R/J)_{\mathfrak{m}/J})$ is connected in codimension k.

As this holds for any maximal ideal $\mathfrak{m}$ of $K[X] \otimes_K K[X]$ which corresponds to a point of $\mathcal{V}_{\text{sep}}$, i.e. any maximal ideal $\mathfrak{m}/J$ of R/J, we conclude with Proposition 2.7 that $\operatorname{Spec}(K[\mathcal{V}_{\text{sep}}]) \cong \operatorname{Spec}(R/J)$ is connected in codimension k. □

It is interesting to compare the case $k = 1$ of the above theorem with Theorem 2.20. In Theorem 5.14 it was necessary to assume that X is normal, while in Theorem 2.20 it was necessary to assume that X is Cohen-Macaulay. Neither assumption is stronger than the other.

Of course, the variety X in Theorem 5.14 is irreducible since normal local rings are integral domains. Hence we can reuse Example 2.25, where X was the union of two planes intersecting in a single point, to see that the assumption that X be normal cannot be dropped from Theorem 5.14.

We finish with an example of a G-variety X that is not an affine space and to which our main theorems of Chapters 2, 4 and 5 apply.

Example 5.15. Let X be the following 3-dimensional irreducible subvariety of $\mathbb{C}^4$:

$$X = \mathcal{V}(\underbrace{x_1^3 + x_2x_3 + x_3x_4 + x_4^2}_{=:h}). \tag{5.2.1}$$

Since $h(x_1, -x_2, -x_3, -x_4) = h(x_1, x_2, x_3, x_4)$, a cyclic group $G = \langle\sigma\rangle$ of order 2 acts on X by

$$\sigma \cdot (x_1, x_2, x_3, x_4) := (x_1, -x_2, -x_3, -x_4). \tag{5.2.2}$$

The invariant ring of this G-variety is generated by 6 elements:

$$\begin{aligned} \mathbb{C}[X]^G &= \mathbb{C}[x_1,\ x_2^2,\ x_3^2,\ x_4^2,\ x_2x_3,\ x_2x_4,\ x_3x_4]/(h) \\ &= \mathbb{C}[\overline{x_1},\ \overline{x_2}^2,\ \overline{x_3}^2,\ \overline{x_2x_3},\ \overline{x_2x_4},\ \overline{x_3x_4}]. \end{aligned} \tag{5.2.3}$$

The origin of $\mathbb{C}^4$ lies in X and is fixed by this action, hence $X^G \neq \emptyset$. But there are no other fixed points in X: A point $(x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ is fixed by the action defined in 5.2.2 if and only if $x_2 = x_3 = x_4 = 0$. But for $x \in X$ we see with 5.2.1 that the vanishing of these three coordinates implies $x_1 = 0$, hence $X^G = \{0\}$. In particular, G is not generated by 2-reflections.

Since X is a hypersurface, it is also Cohen-Macaulay. Thus, Theorem 2.20 gives the lower bound $\gamma_{\mathrm{sep}} \geq n + 1 = 4$.

Of course, we can do better now with Theorem 5.14. We need to check that X is normal first. A quick calculation with the Jacobian criterion shows that the singular locus of X is just $X^{\mathrm{sing}} = \{0\}$. In particular, $\mathbb{C}[X]$ satisfies Serre's condition (R_1). As $\mathbb{C}[X]$ is Cohen-Macaulay, it satisfies a fortiori the condition (S_2). By Serre's criterion for normality (see [BH93, Theorem 2.2.22]), we see that $\mathbb{C}[X]$ is normal. So Theorem 5.14 implies that $\gamma_{\mathrm{sep}} \geq 5$. In addition, Theorem 4.25 shows that no separating subalgebra $A \subseteq \mathbb{C}[X]^G$ over which $\mathbb{C}[X]^G$ is integral is a complete intersection.

A generating set of size 6 was given in 5.2.3. We claim that the following subset of size 5 is separating for the action on X:

$$S = \{\overline{x_1},\ \overline{x_2}^2,\ \overline{x_3}^2,\ \overline{x_2 x_4},\ \overline{x_3 x_4}\}.$$

For the proof suppose that $u, v \in X$ satisfy $g(u) = g(v)$ for all $g \in S$. The invariant $f = \overline{x_2 x_3}$ is the only generator missing in S, so we just need to show that $f(u) = f(v)$. Write $u = (u_1, u_2, u_3, u_4)$. Then we have

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \{ \begin{pmatrix} u_1 \\ \pm u_2 \\ \pm u_3 \\ \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}\}.$$

If $u_2 = 0$, then we get $f(u) = 0 = f(v)$ from this. So assume that $u_2 \neq 0$. With the invariant $\overline{x_2 x_4} \in S$ we get

$$u_2 u_4 = v_2 v_4 = \pm u_2 v_4,$$

hence $v_4 = \pm u_4$. The definition of X in 5.2.1 shows that

$$f = \overline{x_2 x_3} = -\overline{x_1}^3 - \overline{x_3 x_4} - \overline{x_4}^2,$$

and for each of these three summands we now know that it takes the same value at u and v, so $f(u) = f(v)$.

With Theorem 5.14 we could really pin down γ_{sep} to the value of 5 in this example.$\triangleleft$

Appendix A

Magma Code

This section provides some MAGMA [BCP97] functions that were used in the computation of the examples of this thesis.

In MAGMA an invariant ring is represented by an object of the category RngInvar. It is constructed with the functions InvariantRing(G) or InvariantRing(G, K) where G is a matrix or permutation group and K is a field. The delta-map defined in Equation 1.0.2 is a necessary tool to compare two sets of invariants with respect to their separating properties.

```
/* FUNCTION DeltaMap

Input:
- f: An element of an invariant ring R = K[V]^G, which is a subring of a
polynomial ring K[x_1, ..., x_n].

Output:
- delta(f): The element f(x) - f(y) in the polynomial ring K[x_1, ..., x_n,
y_1, ..., y_n].
*/

DeltaMap := function(f)

P := Parent(f);
K := CoefficientRing(P);
n := Rank(P);

m := 2*n;
Q := PolynomialRing(K,m);

firstIndeterminates := [ Q.i : i in [1..n]];
lastIndeterminates := [ Q.i : i in [n+1..m]];

help_map1 := hom < P -> Q | firstIndeterminates>;
help_map2 := hom < P -> Q | lastIndeterminates>;
```

```
delta := map < P -> Q | x :-> help_map1(x) - help_map2(x)>;

return (delta(f));

end function;
```

Let F and S be subsets of an invariant ring $K[V]^G \subseteq K[x_1, \ldots, x_n]$. Following [Kem09, Definition 1.1] the set S is called F-separating if for all $x, y \in K^n$ we have:

$$\text{If } g(x) = g(y) \text{ for all } g \in S, \text{ then } f(x) = f(y) \text{ for all } f \in F.$$

To compare separating properties we made use of Hilberts Nullstellensatz in Proposition 1.13 and hence worked over an algebraically closed field. Consequently, the following MAGMA function returns true if and only if S is F-separating when both sets are regarded as functions $\overline{K}^n \to \overline{K}$ where $\overline{K}$ is an algebraic closure of K.

```
/* FUNCTION IsSeparating

Input:
- F: A list of polynomials in K[x_1, ..., x_n].

- S: A list of polynomials in K[x_1, ..., x_n] whose separating property
is to be checked.

Output:
- true, if S is F-separating over an algebraic closure of K,
- false, otherwise.
*/

IsSeparating := function(F, S)

P := Parent(F[1]);
K := CoefficientRing(P);
n := Rank(P);

result := true;
Q := PolynomialRing(K, 2*n);
S2 := [ Q!DeltaMap(s) : s in S ];
J := ideal < Q | S2 >;

for i := 1 to #F do
if not IsInRadical(Q!DeltaMap(F[i]), J) then
result := false; break;
end if;
end for;
```

```
return result;

end function;
```

The standard MAGMA function IsInRadical(f,I) was used in our implementation. This radical membership test uses an additional indeterminate (see [CLO15, Chapter 4, Proposition 8]). In the case of homogeneous polynomials and ideals a simpler radical membership test appears in [DK15, Lemma 4.3.1]. This leads to a faster separating test for homogeneous invariants.

```
/* FUNCTION IsInRadicalHom

Input:
- f: A homogeneous polynomial in K[x_1, ..., x_n].

- I: A homogeneous ideal in K[x_1, ..., x_n].

Output:
- true, if f lies in the radical of I.
- false, otherwise.
*/

IsInRadicalHom := function(f, I)

P := Parent(f);

result := false;

if 1 in ideal < P | I, 1 - f >
then result := true;
end if;

return result;

end function;


/* FUNCTION IsSeparatingHom

Input:
- F: A list of homogeneous polynomials in K[x_1, ..., x_n].

- S: A list of homogeneous polynomials in K[x_1, ..., x_n] whose separating
property is to be checked.

Output:
- true, if S is F-separating over an algebraic closure of K,
```

```
- false, otherwise.
*/

IsSeparatingHom := function(F, S)

P := Parent(F[1]);
K := CoefficientRing(P);
n := Rank(P);

result := true;
Q := PolynomialRing(K, 2*n);
S2 := [ Q!DeltaMap(s) : s in S ];
J := ideal < Q | S2 >;

for i := 1 to #F do
if not IsInRadicalHom(Q!DeltaMap(F[i]), J) then
result := false; break;
end if;
end for;

return result;

end function;
```

A finite set F of homogeneous K-algebra generators of the invariant ring $R := K[V]^G$ can be constructed with the method FundamentalInvariants(R) in MAGMA. A trivial first step on the way to a separating set of invariants of minimal size is to test if any smaller subsets of F are separating, too.

```
/* FUNCTION MinimalSeparatingSubsets

Input:
- F: A list of homogeneous polynomials in K[x_1, ..., x_n].

Output:
- L: A list of all subsets of F that are F-separating and of minimal size
 among all F-separating subsets of F.
*/

MinimalSeparatingSubsets := function(F)

separatingSetFound := false;

for i := 1 to #F do

SubsetsOfFixedSize := Subsets(SequenceToSet(F), i);
```

```
result := [];

for mset in SubsetsOfFixedSize do
if (IsSeparatingHom(F, SetToSequence(mset))) then
separatingSetFound := true;
result := Append(result, mset);
end if;
end for;

if (separatingSetFound) then
break;
end if;

end for;

return result;

end function;
```

To compute the Cohen-Macaulay defect of quotient rings of $K[V \times V]$ modulo homogeneous ideals whose radical is equal to $\mathcal{I}_{\text{sep}}$ as in Section 2.4, we use the MAGMA function HomologicalDimension(M), which returns the length of a minimal free resolution of a module M, together with the graded Auslander-Buchsbaum formula.

```
/* FUNCTION CohenMacaulayDefect

Input:
- I: A homogeneous ideal in Q = K[x_1, ..., x_m].

Output:
- cmdef(Q/I): The Cohen-Macaulay defect of Q / I.
*/

CohenMacaulayDefect := function(I)

m := Rank(I);

return (HomologicalDimension(GradedModule(I)) - m + Dimension(I));

end function;
```

In this thesis we have compared the minimal number k such that G is generated by k-reflections with properties of the separating variety $\mathcal{V}_{\text{sep}}$ and/or with properties of (separating subalgebras of) the invariant ring $K[X]^G$. We have used examples from the [KKM$^+$01] database of invariant rings sevaral times. As the reflection number k is not in the database (if $k \geq 3$), the following function was used to compute it.

```
/* FUNCTION ReflectionNumber

Input:
- G: A matrix group.

Output:
- The minimal number k such that G is generated by k-reflections.
*/

ReflectionNumber := function(G)

n := Degree(G);
K := CoefficientRing(G);
E := ScalarMatrix(n, 1);
R := MatrixAlgebra(K, n);
k := n;
for i := 1 to n - 1 do
if G eq MatrixGroup< n, K | { s : s in G | Dimension(Kernel(R!s - E)) ge
(n-i) } > then
k := i; break;
end if;
end for;

return k;

end function;
```

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Notation

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