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Adaptive Selection of Shifts and Reduced Order in $\mathcal{H}_2$-Pseudo-Optimal Reduction

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Model order reduction (MOR) based on Krylov subspace methods stands out due to its generality and low computational cost, making it a predestined candidate for the reduction of large-scale systems. Even so, the inherent flexibility of the method can lead to quite unsatisfying results as well, attaching even more importance to the careful selection of free design parameters. In this contribution, we show how both the order of the reduced model and the complex frequencies (shifts) at which reduced and original model share the same transfer function values can be chosen adaptively in a cumulative reduction framework.

Adaptive Reduction Framework

Cumulative Reduction (CURE)

Given a SISO system, consider the state space representations of its original and reduced order model (ROM) respectively

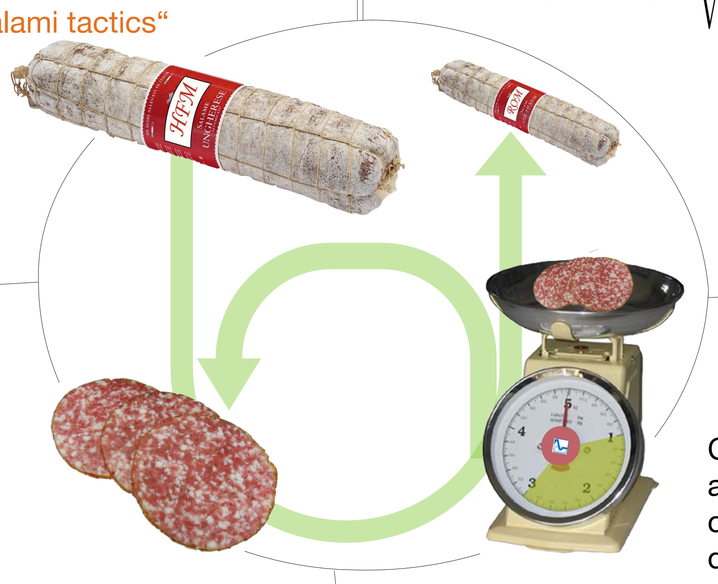
$$G(s) = \left[\begin{array}{c|c} E, A & B \\ \hline C & 0 \end{array}\right], \quad G_r(s) = \left[\begin{array}{c|c} W^\top EV, W^\top AV & W^\top B \\ \hline CV & 0 \end{array}\right], \qquad \begin{array}{c} W, V \in \mathbb{R}^{N\times n} \\ n \ll N \end{array}$$

If V is a basis for the input Krylov subspace, then the error factorizes as

$$G(s) = G_r(s) + G_e(s) = G_r(s) + \underbrace{\left[\begin{array}{c|c} E, A & B_\perp \\ \hline C & 0 \end{array}\right]}_{G_\perp(s)} \cdot \underbrace{\left[\begin{array}{c|c} E_r, A_r & B_r \\ \hline \widetilde{C}_r & I \end{array}\right]}_{\widetilde{G}_r(s)} \qquad (1)$$

Iterative reduction of the $G_\perp(s)$ factor leads to the cumulative „salami tactics“

$$\begin{aligned} G(s) &= G_r(s) + G_\perp(s)\cdot \widetilde{G}_r(s) \\ &= G_r(s) + \left[G_{r,2}(s) + G_{\perp,2}(s)\cdot \widetilde{G}_{r,2}(s)\right]\cdot \widetilde{G}_r(s) \\ &\vdots \\ &= G_r^\Sigma(s) + G_{\perp,k}(s)\cdot \widetilde{G}_r^\Sigma(s) \end{aligned}$$



$\mathcal{H}_2$-Pseudo-Optimality

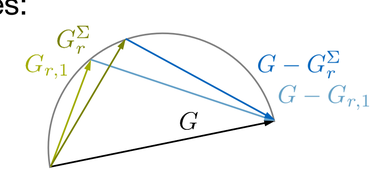
Recalling the Meier-Luenberger first-order necessary conditions for $\mathcal{H}_2$ optimality

$$G(-\overline{\lambda}_{r,i}) = G_r(-\overline{\lambda}_{r,i}) \qquad (2)$$
$$G'(-\overline{\lambda}_{r,i}) = G_r'(-\overline{\lambda}_{r,i}) \qquad (3)$$

a ROM is said to be a global $\mathcal{H}_2$-pseudo-optimal interpolant iff condition (2) holds. Pseudo-optimality denotes optimality in the set of all ROMs sharing the same spectrum.

Focusing on $\mathcal{H}_2$-pseudo-optimal ROMs has several advantages:

- $\widetilde{G}_r(s)$ is an all-pass system
- The $\mathcal{H}_2$-error norm decreases monotonically
- $\min \|G - G_r\|_{\mathcal{H}_2} \;\Rightarrow\; \max \|G_r\|_{\mathcal{H}_2}$
- Stability can be preserved



Accordingly, it appears to be convenient to look for a local $\mathcal{H}_2$ optimum within the set of pseudo-optimal ROMs. An algorithm to compute such ROMs based on a particular Sylvester equation is given by the Pseudo-Optimal Rational Krylov (PORK) algorithm.

Rigorous, Global Error Bounds

CURE allows for an adaptive choice of reduced order. So when should we stop? Due to the high order of the error system, a direct evaluation of its norm is prohibitively expensive. Thus, the employment of some sort of upper bound is inevitable.

Exploiting the error factorization in (1), the error norm can be bounded by

$$\|G_e(s)\| \le \|G_\perp(s)\| \cdot \left\|\widetilde{G}_r(s)\right\|.$$

The key is then to find an upper bound on $\|G_\perp(s)\|$. We propose a global, rigorous, a posteriori bound valid for $A + A^\top < 0$ and $E = E^\top > 0$

$$\|G_\perp(s)\| \le \sqrt{\operatorname{tr}\left[C_\perp \hat{P} C_\perp^\top\right] + \frac{1}{-2\mu_E(A)} \cdot \|L^{-\top} R_c L^{-1}\|_2 \cdot \left\|L^{-\top} C_\perp^\top\right\|_F^2},$$

where

$$R_c = A\hat{P}E^\top + E\hat{P}A^\top + B_\perp B_\perp^\top$$
$$\mu_E(A) = \max_i \lambda_i\left(\frac{A + A^\top}{2}, E\right)$$
$$E = L^\top L$$

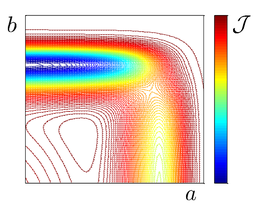
Stability-Preserving, Adaptive Rational Krylov (SPARK)

CURE admits the restriction to reduced models of order $n_k = 2$ at each iteration k. In this case, the search for a pseudo-optimum can be parametrized entirely by two positive real numbers $a, b > 0$ defining the shifts $\sigma_{1,2} = a \pm \sqrt{a^2 - b}$.

The restriction to $\mathcal{H}_2$-pseudo-optimal ROMs simplifies the cost function for minimization to

$$\mathcal{J}(a, b) = -\|G_r\|_{\mathcal{H}_2}^2 = -C_r P_r C_r^\top$$

for which analytical expressions of the gradient and Hessian can be derived.



The resulting greedy algorithm is used to find a locally optimal shift configuration. After convergence, the resulting ROM of order $n_k = 2$ is a Hermite interpolant about the mirror images of its poles and hence a local $\mathcal{H}_2$ optimum.

The computational effort of the algorithm can be reduced by introducing model functions, i.e. approximations of the cost function induced by local ROMs of small to medium size.

Advances and Future Work

DAE Systems

Stability preservation in the reduction of systems

$$\begin{aligned} E\,\dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u \end{aligned} \qquad \text{where} \qquad \det E = 0$$

is under study. If the transfer function is strictly proper

$$\lim_{s\to\infty} G(s) = 0$$

then CUREd SPARK may be applied.

For the special case of index 1, semi-explicit DAEs

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \begin{array}{r} E_{11} = E_{11}^\top > 0 \\ A + A^\top < 0 \end{array}$$

orthogonal reduction $W = V$ can be used.

MIMO Systems

For $\mathcal{H}_2$-optimal reduction of MIMO systems, the interpolatory conditions $(2), (3)$ need to be complemented by left and right tangential directions b_i, c_i

$$G(-\overline{\lambda}_{r,i})b_i = G_r(-\overline{\lambda}_{r,i})b_i$$
$$c_i^\top G(-\overline{\lambda}_{r,i}) = c_i^\top G_r(-\overline{\lambda}_{r,i})$$
$$c_i^\top G'(-\overline{\lambda}_{r,i})b_i = c_i^\top G_r'(-\overline{\lambda}_{r,i})b_i$$

A trust-region-based choice of tangential directions within SPARK, exploiting the advantages of the cumulative setting and $\mathcal{H}_2$-pseudo-optimality, is still topic of research.

2nd Order Structure

Mechanical systems are generally modeled as

$$\begin{aligned} M\,\ddot{z} + D\,\dot{z} + K\,z &= Gu \\ y &= Lz \end{aligned}$$

where M, D, K represent the mass, damping and stiffness properties of the system respectively.

While common approaches transform the model to 1st order form before applying MOR, methods that preserve the 2nd order structure and/or apply directly to the 2nd order formulation are desirable.

Cumulative, $\mathcal{H}_2$-pseudo-optimal reduction might deliver appropriate tools to address this issues.

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