Bernoulli and tail-dependence matrices: A simple numerical test

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Abstract

In a recent paper [Embrechts et al., 2015], the question "*When is a given matrix [...] the matrix of pairwise (either lower or upper) tail-dependence coefficients?*" is investigated and a link to Bernoulli-compatible matrices is provided. This question is interesting, e.g., for model building and stress testing in the financial industry. As part of their conclusions, the authors state that "*[...] an interesting open question is how one can (theoretically or numerically) determine whether a given arbitrary non-negative, square matrix is a tail-dependence or Bernoulli-compatible matrix. To the best of our knowledge there are no corresponding algorithms available.*" Such an algorithm is provided in this paper and a stochastic model based on its solution is constructed as a corollary. The theoretical foundation of these results stems from [Fiebig et al., 2014], who investigate the geometry of tail-dependence matrices in quite some detail.

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1 Introduction

Motivated by industry requests, [Embrechts et al., 2015] imposed the question, "*When is a given matrix in $[0,1]^{d\times d}$ the matrix of pairwise (either lower or upper) tail-dependence coefficients?*" In what follows, we call such a matrix a tail-dependence matrix. As a main result, [Embrechts et al., 2015] established a connection between tail-dependence matrices and so called Bernoulli-compatible matrices, i.e. matrices equal to the expected value of a random matrix $\mathbf{X}\mathbf{X}^T$ with $\mathbf{X}$ a d-dimensional vector whose univariate components are Bernoulli-distributed, see also [Fiebig et al., 2014]. In particular, they show that a matrix with diagonal entries being 1 is a tail-dependence matrix if and only if it is a Bernoulli-compatible matrix multiplied by a constant.

[Fiebig et al., 2014] derive essentially the same result, however, from a more theoretical angle, working with max-stable and Tawn–Molchanov processes and results for convex polytopes. They furthermore find that the set of all tail correlation functions coincides with the set of tail correlation functions stemming from a subclass of max-stable processes. [Fiebig et al., 2014] show that, on a finite dimensional space, the set of tail correlation functions forms a convex polytope which can be characterized by finite-dimensional inequalities. They compute its vertices and facet inducing inequalities up to dimension 6. Another related work is [Strokorb et al., 2015]. They derive different results for tail correlation functions in the context of max-stable processes. An early investigation on these matters, leading to a stochastic model based on extreme-value distributions for a subclass of all tail-dependence matrices, is provided in [Falk, 2005].

[Embrechts et al., 2015] conclude with the open research question: "*Concerning future research, an interesting open question is how one can (theoretically or numerically) determine whether a given arbitrary non-negative, square matrix is a tail-dependence or Bernoulli-compatible matrix. To the best of our knowledge there are no corresponding algorithms available.*"

In this paper we provide a surprisingly simple answer to this question and state an algorithm to decide whether a given square matrix is a Bernoulli-compatible or tail-dependence matrix. To this end, we introduce the mathematical idea of reformulating the question as a linear optimization problem (LP). We furthermore construct an explicit stochastic model for a given Bernoulli-compatible and tail-dependence matrix, respectively, and link a given tail-dependence matrix to a corresponding Bernoulli-compatible matrix. We describe the numerical implementation using examples, including a discussion on computational times and tractable dimensions.

2 Translation to a linear program

A Bernoulli vector is a random vector $\mathbf{X} = (X_1, \dots, X_d)$ supported by $\{0,1\}^d$ for some $d \in \mathbb{N}$, the set of all d-Bernoulli vectors is denoted by ν_d. Assume you are given a matrix $B \in [0,1]^{d\times d}$ and your task is to construct a Bernoulli vector $\mathbf{X}$ such that $B = \mathbb{E}\left[\mathbf{X}\mathbf{X}^T\right]$.

Obviously, B has to be symmetric as a necessary requirement, and the marginal laws are easily matched to the diagonal of B. Still, it is not obvious how to select the dependence structure to ultimately arrive at B. So from a probabilistic point of view, this is a quite involved problem.

Another path is to stress the geometry of the set of all $d \times d$ Bernoulli-compatible matrices, denoted by $\mathcal{B}_d$, and to see if B is included in this set. For an advanced analysis of the geometrical structure see [Fiebig et al., 2014, Chapter 3]. It is easily seen that a mixture model of two Bernoulli vectors corresponds to a mixture of their Bernoulli matrices, so $\mathcal{B}_d$ is convex. It is also not difficult to identify some extremal points of this set, consider the 2^d deterministic vectors $\mathbf{p} \in \{0,1\}^d$ and compute their Bernoulli matrices. These are extreme, since obtaining 0 (or 1, respectively) as the arithmetic average of values in $[0,1]$ is only possible if all values are 0 (or 1, respectively). Compactness of this subset of matrices is also immediate. By the Krein–Milman Theorem we know that every point in this set can be obtained as a weighted average of the set's extremal points (see, e.g., [Krein and Milman, 1940]). Finally, it is established in [Embrechts et al., 2015] that $\mathcal{B}_d$ has the following characterization:

$$\mathcal{B}_d = \left\{ \sum_{i=0}^{2^d-1} a_i \cdot \mathbf{p}_i \mathbf{p}_i^T : \mathbf{p}_i \in \{0,1\}^d, a_i \geq 0, i = 0, \ldots, 2^d - 1, \sum_{i=0}^{2^d-1} a_i = 1 \right\}, \qquad (1)$$

i.e. $\mathcal{B}_d$ is the convex hull of $\left\{ \mathbf{p}\mathbf{p}^T : \mathbf{p} \in \{0,1\}^d \right\}$. Based on these considerations, only a few technical, but rather simple steps are left to introduce the linear problem that solves the initially raised question. Among these technical issues are the efficient representation of matrices as vectors and the binary representation of natural numbers. Having introduced the linear program for Bernoulli-compatible matrices, a similar, but different linear program for tail-dependence matrices is derived using the main result of [Embrechts et al., 2015].

2.1 Bernoulli-compatible matrices

A $d \times d$ matrix B is a Bernoulli-compatible matrix if $B = \mathbb{E}\left[\mathbf{X}\mathbf{X}^T\right]$ for some Bernoulli vector $\mathbf{X}$, i.e. for some $\mathbf{X} \in \nu_d$. It is worth mentioning that the question of whether a Bernoulli vector $\mathbf{X}$ can be found such that $B = \mathbb{E}\left[\mathbf{X}\mathbf{X}^T\right]$ for a given matrix B is closely related to the characterization of Bernoulli-compatible covariance matrices. This is due to the relation $\mathbb{E}\left[\mathbf{X}\mathbf{X}^T\right] = \mathrm{Cov}\left(\mathbf{X}\right) + \mathbb{E}\left[\mathbf{X}\right]\mathbb{E}\left[\mathbf{X}\right]^T$.

Taking Equation (1) as starting point, given a $d \times d$ matrix B it holds that B is a Bernoulli-compatible matrix if and only if a vector $\mathbf{a} := \left(a_0, a_1, \ldots, a_{2^d-1}\right)^T \in \mathbb{R}^{2^d}$

exists that fulfills the following conditions:

$$\begin{cases} a_i \geq 0, & i = 0, \dots, 2^d - 1, \\ \sum_{i=0}^{2^d-1} a_i = 1, & \\ B = \begin{pmatrix} B_{11} & \cdots & B_{1d} \\ \vdots & \ddots & \vdots \\ B_{d1} & \cdots & B_{dd} \end{pmatrix} = \sum_{i=0}^{2^d-1} a_i \cdot \mathbf{p}_i \mathbf{p}_i^T, & \mathbf{p}_i \in \{0,1\}^d. \end{cases} \tag{2}$$

With $\mathbf{p}_i := \big(\mathbf{p}_i(1), \dots, \mathbf{p}_i(d)\big)^T \in \{0,1\}^d$ it obviously holds:

$$\mathbf{p}_i \mathbf{p}_i^T = \begin{pmatrix} \mathbb{1}_{\{\mathbf{p}_i(1)=1\}} & \mathbb{1}_{\{\mathbf{p}_i(1)=\mathbf{p}_i(2)=1\}} & \cdots & \mathbb{1}_{\{\mathbf{p}_i(1)=\mathbf{p}_i(d)=1\}} \\ \mathbb{1}_{\{\mathbf{p}_i(1)=\mathbf{p}_i(2)=1\}} & \mathbb{1}_{\{\mathbf{p}_i(2)=1\}} & \cdots & \mathbb{1}_{\{\mathbf{p}_i(2)=\mathbf{p}_i(d)=1\}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{1}_{\{\mathbf{p}_i(1)=\mathbf{p}_i(d)=1\}} & \mathbb{1}_{\{\mathbf{p}_i(2)=\mathbf{p}_i(d)=1\}} & \cdots & \mathbb{1}_{\{\mathbf{p}_i(d)=1\}} \end{pmatrix}.$$

Let a square matrix B be given. If $B \neq B^T$, the third condition in (2) can obviously not be fulfilled as $a_i \cdot \mathbf{p}_i \mathbf{p}_i^T$ is symmetric for all $i = 0, \dots, 2^d - 1$. If $B = B^T$, conditions (2) can be translated to the following linear equation system with linear inequality constraint:

$$P \cdot \mathbf{a} = \mathbf{b}, \quad \mathbf{a} \geq 0, \tag{3}$$

whereby $\mathbf{b}$ is a $\left(\frac{d(d+1)}{2} + 1\right)$-dimensional column vector and $P \in \{0,1\}^{\left(\frac{d(d+1)}{2}+1\right) \times 2^d}$ with

$$\mathbf{b} := \begin{pmatrix} \tilde{\mathbf{b}} \\ 1 \end{pmatrix}, \quad P := \begin{pmatrix} \tilde{P}_{:,0} & \cdots & \tilde{P}_{:,2^d-1} \\ 1 & \cdots & 1 \end{pmatrix},$$

$$\tilde{\mathbf{b}} := \begin{pmatrix} B_{1,1} \\ B_{1,2} \\ \vdots \\ B_{1,d} \\ B_{2,2} \\ B_{2,3} \\ \vdots \\ B_{d-1,d} \\ B_{d,d} \end{pmatrix}, \quad \tilde{P}_{:,i} := \begin{pmatrix} \mathbb{1}_{\{\mathbf{p}_i(1)=1\}} \\ \mathbb{1}_{\{\mathbf{p}_i(1)=\mathbf{p}_i(2)=1\}} \\ \vdots \\ \mathbb{1}_{\{\mathbf{p}_i(1)=\mathbf{p}_i(d)=1\}} \\ \mathbb{1}_{\{\mathbf{p}_i(2)=1\}} \\ \mathbb{1}_{\{\mathbf{p}_i(2)=\mathbf{p}_i(3)=1\}} \\ \vdots \\ \mathbb{1}_{\{\mathbf{p}_i(d-1)=\mathbf{p}_i(d)=1\}} \\ \mathbb{1}_{\{\mathbf{p}_i(d)=1\}} \end{pmatrix}, \quad i = 0, \dots, 2^d - 1.$$

Therefore, if $B = B^T$, B is a Bernoulli-compatible matrix if and only if the system (3) has a solution.

Remark that $P \cdot \mathbf{a} = \mathbf{b}$ represents the second and third condition in (2). The second corresponds to the 1-entry in $\mathbf{b}$ and the 1-vector in the last row of P. Going rowwise through B, starting on the diagonal element in each row, the third condition can be translated to the linear equation system $\tilde{P} \cdot \mathbf{a} = \tilde{\mathbf{b}}$.

Example 2.1 (Bernoulli matrices in dimension $d = 3$)
In dimension $d = 3$ we choose:

$$\mathbf{p}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$
$$\mathbf{p}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{p}_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{p}_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{p}_7 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore, a symmetric 3×3 matrix B is a Bernoulli-compatible matrix if and only if the solution set of the following linear equation system with linear inequality constraints is not empty:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} B_{1,1} \\ B_{1,2} \\ B_{1,3} \\ B_{2,2} \\ B_{2,3} \\ B_{3,3} \\ 1 \end{pmatrix}, \quad a_i \geq 0,\ i = 0, \ldots, 7. \tag{4}$$

Remark 2.2 (Binary representation)
The concrete form of P only depends on the dimension d and the bijection

$$p : \{0, \ldots, 2^d - 1\} \to \{0, 1\}^d, \quad p(i) = \mathbf{p}_i.$$

In particular, it is independent of B. We choose the binary representation to get an explicit expression for P, i.e. we define $\mathbf{p}_i$ such that

$$i = \sum_{j=0}^{d-1} \mathbf{p}_i(j+1) \cdot 2^j, \quad i = 0, \ldots, 2^d - 1. \tag{5}$$

In dimension $d = 3$ this precisely leads to the representation chosen above.

2.2 Tail-dependence matrices

We start with properly defining the notion of lower (resp. upper) tail-dependence. Denoting the copula of (X_1, X_2) by C and its survival copula by $\hat{C}$, then

$$\lambda_L := \lim_{u \downarrow 0} \frac{C(u,u)}{u}, \quad \lambda_U := \lim_{u \downarrow 0} \frac{\hat{C}(1-u, 1-u)}{u},$$

are called the lower (resp. upper) tail-dependence coefficients, provided the limits exist. Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random vector. Assume the lower (upper) tail-dependence coefficient to exist for all pairs X_i and X_j, $i, j = 1, \ldots, d$ and denote these by λ_{ij}. Then the lower (upper) tail-dependence matrix of $\mathbf{X}$ is $\Lambda = (\lambda_{ij})_{i,j=1,\ldots,d}$. The set of all lower and upper tail-dependence matrices is denoted by $\mathcal{T}_d$. We call a matrix tail-dependence matrix if and only if it is in $\mathcal{T}_d$. [Embrechts et al., 2015, Theorem 3.3] provides the following link to Bernoulli-compatible matrices: "A square matrix with diagonal entries being 1 is a tail-dependence matrix if and only if it is a Bernoulli-compatible matrix multiplied by a constant". Using a different language, [Fiebig et al., 2014] state the same result in Theorem 6.c.

If $T = \theta \cdot B$ with θ a constant, T a tail-dependence matrix and B a Bernoulli-compatible matrix, then $\theta \geq 1$. This holds, since the diagonal entries of a Bernoulli-compatible matrix are obviously smaller than or equal to 1. The diagonal entries of a tail-dependence matrix are equal to 1. Therefore, multiplying a Bernoulli-compatible matrix by a constant θ can only deliver a tail-dependence matrix if $\theta \geq 1$. Based on our considerations in Section 2.1, we now get the following test.

Theorem 2.3 (Testing for tail-dependence matrices via a LP)
Let a square matrix T with entries from $[0, 1]$ be given. If $T \neq T^T$ or $\text{diag}(T) \neq (1, \ldots, 1)$, T is obviously not a tail-dependence matrix. If $T = T^T$ and $\text{diag}(T) = (1, \ldots, 1)$, T is a tail-dependence matrix if and only if the solution set of the following LP is not empty:

$$\tilde{P} \cdot \tilde{\mathbf{a}} = \tilde{\mathbf{b}}, \quad \tilde{\mathbf{a}} \geq 0, \quad \sum_{i=0}^{2^d-1} \tilde{a}_i \geq 1, \tag{6}$$

whereby $\tilde{P}$ and $\tilde{\mathbf{b}}$ are defined as in Section 2.1 (replace B_{kl} by T_{kl} in the definition of $\tilde{\mathbf{b}}$). If a solution $\tilde{\mathbf{a}}^*$ exists, the corresponding Bernoulli-compatible matrix B^* is given by

$$B^* = \frac{1}{\theta} \cdot T = \sum_{i=0}^{2^d-1} a_i^* \cdot \mathbf{p}_i \mathbf{p}_i^T, \quad \theta := \sum_{i=0}^{2^d-1} \tilde{a}_i^*, \quad \mathbf{a}^* := \frac{1}{\theta} \tilde{\mathbf{a}}^*. \tag{7}$$

Proof
Let a symmetric matrix T with $\text{diag}(T) = (1, \ldots, 1)$ be given. According to results by [Embrechts et al., 2015] it is a tail-dependence matrix if and only if a Bernoulli-compatible matrix B and a constant $\theta \geq 1$ exist such that $T = \theta \cdot B$. B is a Bernoulli-

compatible matrix if and only if

$$
\begin{aligned}
& \exists \mathbf{a} \in \mathbb{R}^{2^d}: \quad B = \sum_{i=0}^{2^d-1} a_i \cdot \mathbf{p}_i \mathbf{p}_i^T = \frac{1}{\theta} \cdot T, \quad \mathbf{a} \geq 0, \quad \sum_{i=0}^{2^d-1} a_i = 1 \\
\Leftrightarrow \quad & \exists \tilde{\mathbf{a}} := \theta \cdot \mathbf{a} \in \mathbb{R}^{2^d}: \quad T = \sum_{i=0}^{2^d-1} \tilde{a}_i \cdot \mathbf{p}_i \mathbf{p}_i^T, \quad \tilde{\mathbf{a}} \geq 0, \quad \sum_{i=0}^{2^d-1} \tilde{a}_i = \theta \geq 1 \\
\Leftrightarrow \quad & \exists \tilde{\mathbf{a}} \in \mathbb{R}^{2^d}: \quad \tilde{P} \cdot \tilde{\mathbf{a}} = \tilde{\mathbf{b}}, \quad \tilde{\mathbf{a}} \geq 0, \quad \sum_{i=0}^{2^d-1} \tilde{a}_i \geq 1,
\end{aligned}
$$

with $\tilde{P}$ as in Section 2.1 and $\tilde{\mathbf{b}} := (T_{1,1}, T_{1,2}, \ldots, T_{1,d}, T_{2,2}, T_{2,3}, \ldots, T_{d-1,d}, T_{d,d})^T$. □

2.3 Stochastic model

After having introduced a one-to-one correspondence between finding an answer to the question whether a given square matrix is a Bernoulli-compatible matrix (resp. tail-dependence matrix) and solving a LP, we will now give an explicit stochastic model for a Bernoulli-compatible matrix (resp. tail-dependence matrix) based on the solution of the LP. Given a Bernoulli-compatible matrix B and a 2^d-dimensional vector $\mathbf{a}^*$ solving LP (3), we define the Bernoulli vector $\mathbf{X}^*$ as a mixture model. Interpret a_i^* as the probability that the (degenerate) vector $\mathbf{p}_i$ is drawn, i.e.

$$
\mathbb{P}\left(\mathbf{X}^* = \mathbf{p}_i\right) := a_i^*, \quad i = 0, \ldots, 2^d - 1. \tag{8}
$$

This stochastic model fulfills $\mathbb{E}\left[\mathbf{X}^* {\mathbf{X}^*}^T\right] = B$.

Proof
Let $k, l \in \{1, \ldots, d\}$. Then:

$$
\begin{aligned}
\left(\mathbb{E}\left[\mathbf{X}^* {\mathbf{X}^*}^T\right]\right)_{k,l} &= \mathbb{E}\left[X_k^* \cdot X_l^*\right] = \mathbb{P}\big(X_k^* = 1, X_l^* = 1\big) = \mathbb{P}\big(\mathbf{X}^* = p_i,\ \mathbf{p}_i(k) = \mathbf{p}_i(l) = 1\big) \\
&= \sum_{\substack{i=0, \\ \mathbf{p}_i(k) = \mathbf{p}_i(l) = 1}}^{2^d-1} a_i^* = \sum_{\substack{i=0, \\ \mathbf{p}_i(k) = \mathbf{p}_i(l) = 1}}^{2^d-1} a_i^* \cdot \mathbf{p}_i(k) \cdot \mathbf{p}_i(l) = \sum_{i=0}^{2^d-1} a_i^* \cdot \mathbf{p}_i(k) \cdot \mathbf{p}_i(l) \\
&= \left(\sum_{i=0}^{2^d-1} a_i^* \cdot \mathbf{p}_i \mathbf{p}_i^T\right)_{k,l} = B_{k,l}.
\end{aligned}
$$
□

Remark that the second equality is based on the fact that $\mathbf{X}^*$ is obviously a Bernoulli vector as $\mathbb{P}\Big(\mathbf{X}^* \in \{0,1\}^d\Big) = 1$. The last equality is based on the fact that $\mathbf{a}^*$ is a solution of LP (3).

Given a $d \times d$ dimensional tail-dependence matrix T and a 2^d-dimensional vector $\tilde{\mathbf{a}}^*$ solving LP (6), the d-dimensional random variable $\widetilde{\mathbf{Y}}^*$ defined as

$$\widetilde{\mathbf{Y}}^* := \frac{1}{\theta} U \cdot \mathbf{X}^* + \left(\frac{1}{\theta} + \left(1 - \frac{1}{\theta} \right) V \right) \cdot (\mathbf{1} - \mathbf{X}^*), \quad \theta := \sum_{i=0}^{2^d - 1} \tilde{a}_i^*, \quad U, V \sim \mathcal{U}[0,1],$$

has T as its lower tail-dependence matrix. $\mathbf{X}^*$ is defined as in Equation (8), a Bernoulli vector with $\mathbb{E}\left[\mathbf{X}^* \mathbf{X}^{*T}\right] = B^*$ with B^* and $\mathbf{a}^*$ as in Equation (7). Furthermore U, V, and $\mathbf{X}^*$ are independent of each other.

Proof
Analogously to [Embrechts et al., 2015, proof of Theorem 3.3]. $\square$

3 Numerical implementation and examples

To evaluate numerically whether a given symmetric $d \times d$-matrix B (resp. T) is a Bernoulli-compatible matrix (resp. tail-dependence matrix), we set up the following LPs:

1. **Bernoulli-compatible matrix:**

$$\text{(LP Bernoulli)} \begin{cases} \mathbf{f}^T \cdot \mathbf{a} \longrightarrow \min \\ P \cdot \mathbf{a} = \mathbf{b}, \\ \mathbf{a} \geq 0, \end{cases} \tag{9}$$

with P and $\mathbf{b}$ as in Section 2.1 and $\mathbf{f}$ an arbitrary 2^d-dimensional vector. Later, we will discuss the choice of $\mathbf{f}$ to obtain specific solutions.

2. **Tail-dependence matrix:**

$$\text{(LP tail-dependence)} \begin{cases} \mathbf{g}^T \cdot \tilde{\mathbf{a}} \longrightarrow \min \\ \tilde{P} \cdot \tilde{\mathbf{a}} = \tilde{\mathbf{b}}, \\ \tilde{\mathbf{a}} \geq 0, \end{cases} \tag{10}$$

with $\tilde{P}$ and $\tilde{\mathbf{b}}$ as in Theorem 2.3 and $\mathbf{g}$ an arbitrary 2^d-dimensional vector with non-negative entries. Later on we will scrutinize sensible choices of $\mathbf{g}$.

B (resp. T) is a Bernoulli-compatible matrix (resp. tail-dependence matrix) if and only if the set of constraints is not empty, which is equivalent to the existence of a solution to LP (9) (resp. LP (10)). Note that for $d \geq 3$ the solution (provided it exists) is not unique, as the equation system $P \cdot \mathbf{a} = \mathbf{b}$ (resp. $\tilde{P} \cdot \tilde{\mathbf{a}} = \tilde{\mathbf{b}}$) is under-determined. Thus, a solution of LP (9) (resp. LP (10)) is one particular element of the set of points fulfilling the constraints. Below, we discuss how to choose $\mathbf{f}$ (resp. $\mathbf{g}$) such that the solution of the LP has properties in ones' favor.

A linear optimization problem as (9) or (10) can be solved with standard routines, like e.g. the "Simplex algorithm". That way, not only the question whether a given $d \times d$ matrix is a Bernoulli-compatible matrix (resp. tail-dependence matrix) can be answered, but also an explicit vector $\mathbf{a}^*$ (resp. $\tilde{\mathbf{a}}^*$) is found such that

$$B = \sum_{i=0}^{2^d-1} a_i^* \cdot \mathbf{p}_i \mathbf{p}_i^T \quad \text{or} \quad T = \sum_{i=0}^{2^d-1} \tilde{a}_i^* \cdot \mathbf{p}_i \mathbf{p}_i^T,$$

respectively. We used MATLAB to program a function returning $\mathbf{b}$ (resp. $\tilde{\mathbf{b}}$) and P (resp. $\tilde{P}$) explicitly for any matrix $B = B^T \in \mathbb{R}^{d \times d}$ (resp. $T = T^T \in \mathbb{R}^{d \times d}$). Thereby we applied bijection (5) to get an explicit expression for $\mathbf{p}_i$, $i = 0, \dots, 2^d - 1$ with the MATLAB function `dec2bin`. We then used the function `linprog` to find a solution to LP (9) (resp. (10)). If `linprog` does not deliver a solution, the solution set of the side constraints is empty and B (resp. T) is not a Bernoulli-compatible matrix (resp. tail-dependence matrix).

The solution of the optimization problems (9) and (10) not only provide a representation, but exactly the one minimizing the objection function in use. In that way we can even control what sort of vector we get that fulfills conditions (2).

Example 3.1 (Dimension $d = 3$)
For example, the matrix

$$B_1 := \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

is a Bernoulli-compatible matrix. Two possible solutions of (2) are

$$\mathbf{a}^{*,1} = \begin{pmatrix} 0 \\ 1/4 \\ 1/4 \\ 0 \\ 1/4 \\ 0 \\ 0 \\ 1/4 \end{pmatrix}, \quad \text{and} \quad \mathbf{a}^{*,2} = \begin{pmatrix} 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \end{pmatrix}.$$

The MATLAB function `linprog` returns $\mathbf{a}^{*,1}$ for $\mathbf{f} = (1, 0, \dots, 0)^T$, meaning that we want to find an element in the solution set $L_{B_1} = \left\{ \mathbf{a} \in \mathbb{R}^{2^d} \mid P \cdot \mathbf{a} = \mathbf{b},\ \mathbf{a} \geq 0 \right\}$ with minimal weight of

$$\mathbf{p}_0 \mathbf{p}_0^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{in representation} \quad B_1 = \sum_{i=0}^{2^d-1} a_i^{*,1} \cdot \mathbf{p}_i \mathbf{p}_i^T.$$

We find $\mathbf{a}^{*,2}$ as solution when choosing $\mathbf{f} = (0, \ldots, 0, -1)$, meaning that we seek to maximize the weight of

$$\mathbf{p}_7\mathbf{p}_7^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{in representation} \quad B_1 = \sum_{i=0}^{2^d-1} a_i^{*,2} \cdot \mathbf{p}_i\mathbf{p}_i^T.$$

In this particular case, a probabilistic proof for the Bernoulli-compatibility of B_1 is quite obvious. Take $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$, $Y_i \sim B(1, p)$ with $p = 1/2$ and Y_i, $i = 1, 2, 3$, independent. Then $\mathbb{E}\left[\mathbf{Y}\mathbf{Y}^T\right] = B_1$. If we define the random vector $\mathbf{X}^{(1)}$ according to the stochastic model in Section 2.3 with $\mathbf{a}^* = \mathbf{a}^{*,1}$, we get:

$$\mathbb{P}\left(\mathbf{X}^{(1)} = \mathbf{p}_1\right) = \mathbb{P}\left(\mathbf{X}^{(1)} = \mathbf{p}_2\right) = \mathbb{P}\left(\mathbf{X}^{(1)} = \mathbf{p}_4\right) = \mathbb{P}\left(\mathbf{X}^{(1)} = \mathbf{p}_7\right) = 1/4,$$

with

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{p}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{p}_7 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $k, l = 1, 2, 3, \ k \neq l$, it obviously holds

$$\mathbb{P}\left(X_k^{(1)} = 1\right) = 1/2, \quad \mathbb{P}\left(X_k^{(1)} = X_l^{(1)} = 1\right) = 1/4.$$

Thus $\mathbb{E}\left[\mathbf{X}^{(1)}\mathbf{X}^{(1)^T}\right] = B_1$, however, $\mathbf{X}^{(1)}$ is in distribution not equal to $\mathbf{Y}$, as

$$\mathbb{P}\left(\mathbf{X}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = 0 \quad < \quad 1/8 = \mathbb{P}\left(\mathbf{Y} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right).$$

If we define $\mathbf{X}^{(2)}$ according to the stochastic model in Section 2.3 with $\mathbf{a}^* = \mathbf{a}^{*,2}$, we get the same distribution as of $\mathbf{Y}$, i.e. $\mathbf{X}^{(2)} \stackrel{d}{=} \mathbf{Y}$. Some more examples are given in the Appendix.

3.1 Computational time and viable dimensions

To give an overview of what dimensions are doable with the proposed methodology and the computational time required, we performed the following analysis on a standard personal computer (processor: Intel Core i7, 2.60GHz, RAM: 16GB).

- Time required for evaluating whether a given symmetric matrix in $[0, 1]^d$ is a Bernoulli-compatible matrix is measured as follows. We randomly generate 100 independent, symmetric test matrices, whereby the diagonal entries are chosen as independent realizations of a $\mathcal{U}[0, 1]$-distributed random variable, rounded on one decimal place. Thus, the diagonal entries are distributed independently with the following probability density:

$$\mathbb{P}(B_{kk} = 0.1) = \mathbb{P}(B_{kk} = 0.2) = \ldots = \mathbb{P}(B_{kk} = 0.9) = 0.1,$$
$$\mathbb{P}(B_{kk} = 0.0) = \mathbb{P}(B_{kk} = 1.0) = 0.05.$$

Obviously, $B_{kl} \leq min\{B_{kk}, B_{ll}\}$ is a necessary condition for a matrix being Bernoulli-compatible. Therefore, the off-diagonal entries are chosen independently as the minimum of the two corresponding diagonal entries multiplied by a $\mathcal{U}\,[0,1]$-distributed random variable, rounded on one decimal place:

$$B_{kl} := \frac{1}{10} \cdot \lfloor 10 \cdot \min\{B_{kk}, B_{ll}\} \cdot U_{kl} \rfloor, \quad U_{kl} \sim \mathcal{U}\,[0,1]\,.$$

We choose this discrete probability distribution over a continuous distribution to increase the likelihood of actually ending up with a Bernoulli-compatible matrix. After generating these random matrices, every matrix is analyzed regarding Bernoulli-compatibility with the described numerical routine. We measure the computational time needed and count how many of the randomly generated matrices are indeed proper Bernoulli-compatible matrices.

- Time required for evaluating whether a given symmetric matrix in $[0,1]^d$ with 1-entries on the diagonal is a tail-dependence matrix is measured as follows. We use the same approach for tail-dependence matrices as for Bernoulli-compatible matrices. However, we obviously generate different random matrices: We generate 100 random symmetric matrices with 1-entries on their diagonal, whereby the off-diagonal entries are distributed according to the following probability density:

$$\mathbb{P}(X = 0.1) = \ldots = \mathbb{P}(X = 0.9) = 0.1,\ \mathbb{P}(X = 0.0) = \mathbb{P}(X = 1.0) = 0.05.$$

Numerical results for both, testing Bernoulli-compatibility as well as tail-dependence compatibility, can be found in Table 1. The maximum dimension analyzed is $d = 15$, because for larger dimensions the implemented procedure is not doable on our laptop as the maximum array size in MATLAB is exceeded.

As the relative number of Bernoulli-compatible matrices and tail-dependence matrices is 0 for the generated random matrices in $d \geq 10$, we decided to do another runtime analysis, however, only for matrices which are known to be tail-dependence matrices. The relative number of tail-dependence matrices is then obviously 100%, the required time is expected to be lower. [Embrechts et al., 2015, Proposition 4.1] delivers that the following matrices are tail-dependence matrices in any dimension $d \geq 2$:

1. Equicorrelation matrix with parameter α in $[0,1]$:
   $T_{kl} = \mathbb{1}_{\{k=l\}} + \alpha \mathbb{1}_{\{k \neq l\}},\ k,l = 1,\ldots,d.$
2. $AR(1)$ matrix with parameter α in $[0,1]$: $T_{kl} = \alpha^{|k-l|},\ k,l = 1,\ldots,d.$

Table 1 Density and average runtime for random matrices

Dimension		3	5	10	15
Bernoulli-compatible	*Density*	39%	2%	0%	0%
	Avg. CPU in sec.	0.02	0.02	0.39	84.37
Tail-Dependence	*Density*	76%	20%	0%	0%
	Avg. CPU in sec.	0.01	0.02	0.36	111.41

3. $MA(1)$ matrix with parameter α in $[0, 1/2]$:
   $T_{kl} = \mathbb{1}_{\{k=l\}} + \alpha \mathbb{1}_{\{|k-l|=1\}}, \ k, l = 1, \ldots, d.$

[Falk, 2005, Theorem 1.3] constructs for any dimension $d \geq 2$ an explicit extreme-value copula which has the following matrix T with parameters $\theta \geq 1$ and $0 \leq a_1 \leq \cdots \leq a_d \leq 1$ as tail-dependence matrix:

4. $diag(T) = (1, \ldots, 1)\,, \quad T_{kl} = 2 - (1 + a_l - a_k)^{1/\theta}\,, \ 1 \leq k < l \leq d, \quad T = T^T.$

Table 2 displays the average CPU time for assessing whether a given matrix is a tail-dependence matrix. Thereby we iterate over the four different kinds of tail-dependence matrices given above. For each of them we iterate over 10 arbitrarily chosen parameter sets. In every iteration for the first three kinds of tail-dependence matrices the parameter α is chosen as a sample of the uniform distribution over $\{0, 0.05, 0.1, \ldots, 1.0\}$ and $\{0, 0.05, 0.1, \ldots, 0.5\}$, respectively. For the fourth kind of tail-dependence matrix the parameter θ is chosen arbitrarily from a uniform distribution over $\{1, \ldots, 10\}$. The parameters $a_1, \ldots, a_d$ are chosen as the ordered vector of d independent samples of the uniform distribution over $\{0, 1/10, \ldots, 9/10, 1\}$. As expected, the average runtime is smaller than in the above analysis for random matrices. All tested matrices are recognized correctly as tail-dependence matrices with the exception of one matrix in dimension $d = 15$. This exception comes from a parameter set with numercially difficult entries in the matrix. For example, for the second kind of tail-dependence matrix and $\alpha = 0.01$ the upper-right and lower-left entry of the matrix are equal to 10^{-15} which is very (respectively too) close to 0.

Table 2 Average runtime for tail-dependence matrices

Dimension	3	5	10	15
Avg. CPU in sec.	0.01	0.01	0.36	21.20

4 Conclusion

Inspired by an open question in [Embrechts et al., 2015], we were able to transform the question whether a given matrix is a Bernoulli-compatible matrix to the search for a solution of a linear optimization problem. For moderate dimensions, a solution to this LP can be easily found via standard routines. If the given matrix is a Bernoulli-compatible matrix, a solution of the LP exists and provides an explicit representation of the given matrix as a convex combination of all extremal matrices $\mathbf{p}_i\mathbf{p}_i^T$, $\mathbf{p}_i \in \{0,1\}^d$. Building upon a link between Bernoulli-compatible matrices and tail-dependence matrices established in [Embrechts et al., 2015] and [Fiebig et al., 2014], we could also transfer the question whether a given matrix is a tail-dependence matrix to a related LP. If a solution exists, the given matrix is a tail-dependence matrix and the solution explicitly provides the associated Bernoulli-compatible matrix. Furthermore, we presented a stochastic model compatible to a given tail-dependence matrix (resp. Bernoulli-compatible matrix).

An analysis for different dimensions and the associated computational time of the proposed numerical procedures was provided. An interesting open research question is whether and if so, how a matrix can be analyzed regarding Bernoulli-compatibility and tail-dependence compatibility with an optimization problem of polynomial effort instead of $\mathcal{O}\left(2^d\right)$. In line with the concluding remarks of [Fiebig et al., 2014], we conjecture that this is not possible.

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5 Appendix

Interestingly, the matrix B_2 is not a Bernoulli-compatible matrix, where

$$B_2 := \begin{pmatrix} 3/5 & 1/4 & 1/4 \\ 1/4 & 3/5 & 1/4 \\ 1/4 & 1/4 & 3/5 \end{pmatrix},$$

i.e. the MATLAB solver `linprog` does not find a solution for LP (9) with $B = B_2$. The matrix

$$T_1 := 2 \cdot B_1 = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

is a tail-dependence matrix by Example 3.1 and the connection to Bernoulli-compatible matrices established in [Embrechts et al., 2015]. This is confirmed by our numerical procedure, with $\mathbf{g} = (1, 0, \ldots, 0)^T$ it delivers

$$\theta = 1.9926, \quad B_1^* = \frac{1}{\theta} \cdot T_1 = \begin{pmatrix} 0.5019 & 0.2509 & 0.2509 \\ 0.2509 & 0.5019 & 0.2509 \\ 0.2509 & 0.2509 & 0.5019 \end{pmatrix},$$

$$\tilde{\mathbf{a}}^{*,1} = \begin{pmatrix} 0 \\ 0.4926 \\ 0.4926 \\ 0.0074 \\ 0.4926 \\ 0.0074 \\ 0.0074 \\ 0.4926 \end{pmatrix}, \quad T_1 = \sum_{i=0}^{2^d-1} \tilde{a}_i^{*,1} \cdot \mathbf{p}_i \mathbf{p}_i^T.$$

In perfect correspondence with [Embrechts et al., 2015, Proposition 4.7] our numerical procedure delivers that

$$T_2 := \begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}$$

is not a tail-dependence matrix, but

$$T_3 := \begin{pmatrix} 1 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1 \end{pmatrix}$$

is a proper tail-dependence matrix.