

Homogeneous Polynomial Lyapunov Functions for Robust Local Synchronization with Time-varying Uncertainties

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Abstract

This paper studies robust local synchronization in multi-agent systems with time-varying parametric uncertainties constrained in a polytope. In contrast to existing methods with non-convex conditions via using quadratic Lyapunov function (QLF), a new criteria is proposed based on using homogeneous polynomial Lyapunov functions (HPLFs) where the original system is suitably approximated by an uncertain polytopic system. Furthermore, corresponding tractable conditions of linear matrix inequalities (LMIs) have been provided by exploiting squares matrix representation (SMR). Then, polytopic synchronization margin problem is, for the first time, proposed and investigated via handling generalized eigenvalue problems (GEVPs). Lastly, numerical examples illustrate the usefulness of the proposed method.

I. INTRODUCTION

Collective motions of multi-agent systems are appearing in a widespread field, stimulating a tremendous upsurge of research efforts toward the mechanism behind the phenomena. As a key problem, synchronization has attracted particular research attentions due to its emerging broad range of applications in various fields, like biology, electronics, sociology, to name just a few [1]–[4]. Interestingly, synchronization problem of complex networks shares common features with another academic focus: consensus of multi-agent system, especially in the case of identical nodes with linear dynamics [5]–[8]. Common examples can be easily found in rendezvous problems where a certain manner-distance is stabilized with each agent communicating with the nearest neighbours [9].

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For fixed topology, synchronization of coupled networks is extensively investigated by a master stability function (MSF) method which calculates the maximum Lyapunov exponent of variational equation for the nonlinear coupled networks [10], [11]. The local synchronization of a linearized system can be successfully guaranteed by using MSF, thus triggering a particular interest in how synchronization depends on structural factors, such as clustering coefficient, coupling strength, average distance [12]. Besides the eigenvalue analysis of the coupling matrix for assorted synchronization schemes, Belykh et al. have proposed an alternative approach called the Connection Graph Stability (CGS) method, which combines the stability theory with graph theory [13].

However, most of existing results establish on the assumption that the models of dynamical networks are accurate. Such an assumption can not always be applicable when it meets multitudinous applications in practice. For a simple instance in electronic circuits, the values of resistance and capacitance as communication weights of networks are not constant, while displaying fluctuations in different circumstances. Thus, more recently, robust synchronization of systems with uncertainties has been given critical attentions in this field [14]–[17]. For uncertain adjacency matrix (or called uncertain coupling matrix), MSFs and eigenvalue analysis can hardly be applied, making Lyapunov stability theory as a main approach for robust synchronization. In [14] a decentralized hybrid feedback scheme is applied into a robust global synchronization problem with relative-attitude error. In [15], by using quadratic Lyapunov Functions (QLFs) an impulsive control scheme is proposed for robust synchronization of coupled neural networks with bounded coupling force. In [16], by using QLFs, robust 2-D synchronization is investigated with time-invariant polytopic parameter uncertainties. In [17], robust synchronization performance is analyzed by QLF with control gains disturbed by square integrable bounded time-varying uncertainties.

Contrast with the literatures, this paper considers robust local synchronization with time-varying parametric uncertainties and provides synchronization conditions by employing HPLFs which are much less conservative comparing with QLFs. Specifically, in contrary to non-convex approaches, the original system is approximated by a polytopic system whose asymptotical stability is properly guaranteed through a non-conservatism approach by using HPLFs which can be tackled by solving an LMI feasibility test. Furthermore, it is shown that polytopic synchronization margin can be searched by solving GEVPs. Lastly, the usefulness of proposed

method is proved by numerical examples.

II. PRELIMINARIES

Notations: \mathbb{N}, \mathbb{R} : natural and real number sets; $\mathbf{0}_n$: origin of \mathbb{R}^n ; $\mathbb{R}_0^n: \mathbb{R}^n \setminus \{\mathbf{0}_n\}$; A' : transpose of A ; $\mathbf{1}_n$: ones vector of \mathbb{R}^n ; I_n : $n \times n$ identity matrix; $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A ; $A \otimes B$: Kronecker product of matrices A and B ; $\text{he}(A)$: $A + A'$, with $A \in \mathbb{R}^{n \times n}$; $\text{co}\{X_1, \dots, X_p\}$: convex hull of matrices $X_1, \dots, X_p \in \mathbb{R}^{m \times n}$; $X^{[i]}$: i -th Kronecker power, i.e.

$$X^{[i]} = \begin{cases} X \otimes X^{[i-1]} & \text{if } i > 1 \\ 1 & \text{if } i = 0. \end{cases}$$

In graph theory, a weighted and directed graph $\mathcal{G} = (\mathcal{A}, \mathcal{E}, G)$ consists of a finite nonempty node set $\mathcal{A} = \{A_1, \dots, A_N\}$, a directed edge set $\mathcal{E} \subseteq \mathcal{A} \times \mathcal{A}$, and a weighted adjacency matrix $G \in \mathbb{R}^{N \times N}$. A directed edge from A_j to A_i is described by G_{ij} which means information can be transmitted from the j -th node to the i -th node but not conversely.

In this paper, robustness of local synchronization is considered for time-varying parametric uncertainties. In particular, it is supposed that the weighted adjacency matrix G is affected by uncertain parameters $\theta(t) \in \mathbb{R}^a$, denoting the time-varying perturbations from environment to the system dynamics [16], [18], [19]. And $\theta(t)$ satisfies

$$\theta(t) \in \Omega. \quad (1)$$

In this paper, we consider Ω as follows.

$$\Omega = \text{co}\{\theta^{(1)}, \dots, \theta^{(v)}\} \quad (2)$$

for some given vectors $\theta^{(1)}, \dots, \theta^{(v)} \in \mathbb{R}^a$. Then, let us introduce the uncertain multi-agent systems with time-varying uncertainties by

$$\dot{x}_i(t) = f(x_i(t)) - c \sum_{j=1}^N L_{ij}(\theta(t)) \Gamma x_j(t), \quad i, j = 1, \dots, N \quad (3)$$

where $x_i \in \mathbb{R}^n$ is the state of i -th agent, N is the number of agents, c is the coupling weight, $f(x_i) \in \mathbb{R}^n$ is a nonlinear function, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^{n \times n}$ is a diagonal matrix where $\gamma_i > 0$ stands for the agents communicating through their i -th states. $L_{ij}(\theta(t))$ is the ij -th entry of the uncertain Laplacian matrix $L(\theta(t)) \in \mathbb{R}^{N \times N}$ given by $L_{ij}(\theta(t)) = -G_{ij}(\theta(t))$ for all $i \neq j$ and by $L_{ii}(\theta(t)) = -\sum_{j=1, j \neq i}^N L_{ij}(\theta(t))$.

Linear perturbation in communication network is widely adopted in literatures where $G_{ij}(\theta(t))$ is a linear function [16], [17], [20]. Thus the uncertain Laplacian matrix can be expressed as

$$L(\theta(t)) = L_0 + \sum_{i=1}^a \theta_i(t) L_i.$$

Nonlinear perturbations and corresponding approaches will also be discussed in Section IV. Let us introduce the uncertain multi-agent dynamical system (3) in compact form

$$\dot{x}(t) = g(x(t)) - c(L(\theta(t)) \otimes \Gamma)x(t) \quad (4)$$

where $x(t) = (x_1(t)', \dots, x_N(t)')'$ and $g(x(t)) = (f(x_1(t))', \dots, f(x_N(t))')'$. Let $s(t) \in \mathbb{R}^n$ be a solution of an isolated node, i.e.

$$\dot{s}(t) = f(s(t)). \quad (5)$$

Let us observe that $s(t)$ can be an equilibrium point, a periodic orbit, or a chaotic orbit, etc. Then, the robust local synchronization problem is proposed as follows.

Problem 1: To establish if the uncertain multi-agent dynamical system (4) achieves robust local synchronization, i.e. for any ϵ there exist $\kappa(\epsilon)$ and $T > 0$ such that $\|x_i(0) - x_j(0)\| \leq \kappa(\epsilon)$ implies $\|x_i(t) - x_j(t)\| \leq \epsilon$ for all $\theta(t) \in \Omega$, $t > T$ and $i, j = 1, \dots, N$.

An extending problem of great interests is the synchronization margin problem, which will be proposed and investigated in Section V.

III. SYSTEM TRANSFORMATION

First, let us introduce the following assumptions on $f(x_i)$.

Assumption 1: The function $f(x_i)$ is continuously differentiable in a neighbourhood of the solution $s(t)$.

Remark 1: This assumption just requires that the continuity of first derivative of the vector field is guaranteed in a neighbourhood of the solution of interest.

Let $\theta(t) \in \Omega$ defined by (2).

Remark 2: The uncertain parameter $\theta(t)$ is constrained in a polytope which is a very typical form both for time-varying uncertain system and for time-invariant uncertain system in robust synchronization and robust control [20]–[22].

Observe $\sum_{j=1}^N L_{ij}(\theta(t))\Gamma s(t) = 0$, by subtracting (5) from (3), we get the system

$$\dot{y}_i(t) = f(x_i(t)) - f(s(t)) - c \sum_{j=1}^N L_{ij}(\theta(t))\Gamma y_j(t) \quad (6)$$

where $y_i = x_i - s$, $i = 1, \dots, N$. For local synchronization, one can use the dynamics of the system locally about $s(t)$ in the case without uncertainty [23], [10], [24]. For the uncertain system (6), it can also be expressed as

$$\dot{y}(t) = (I_N \otimes Df(s(t)))y(t) - c(L(\theta(t)) \otimes \Gamma)y(t) \quad (7)$$

where $y(t) = (y_1(t)', \dots, y_N(t)')$ and $Df(s(t)) \in \mathbb{R}^{n \times n}$ is the Jacobin matrix of $f(x_i)$ evaluated for $x_i = s(t)$. Observe $\mathbf{1}_N$ is the right eigenvector of $L(\theta(t))$ corresponding to eigenvalue zero, let $\eta' = (\eta_1, \dots, \eta_N) \in \mathbb{R}^{1 \times N}$ be the left eigenvector of the uncertain Laplacian matrix $L(\theta(t))$ corresponding to eigenvalue zero, and $\sum_{i=1}^N \eta_i = 1$. A new disagreement variable can be introduced as follows:

$$z(t) = y(t) - ((\mathbf{1}_N \eta') \otimes I_n)y(t) \quad (8)$$

where $z(t) \in \mathbb{R}^{nN}$ satisfies $(\eta' \otimes I_n)z(t) = \mathbf{0}_n$. Define

$$M = (I_N - \mathbf{1}_N \eta') \otimes I_n. \quad (9)$$

Observe matrix M commutes with matrices $I_N \otimes Df(s(t))$ and $c(L(\theta(t)) \otimes \Gamma)$, then one can get an uncertain disagreement system as follows:

$$\dot{z}(t) = (I_N \otimes Df(s(t)) - cL(\theta(t)) \otimes \Gamma)z(t). \quad (10)$$

Lemma 1: Suppose that Assumption 1 holds. The robust local synchronization of system (7) can be achieved if and only if system (10) is asymptotically stable.

Proof (Necessity) From the definition of y_i one has that the robust local synchronization of system (7) can be achieved if $|y_i - y_j| \rightarrow \mathbf{0}_n$ whenever the initial condition for y lies in a neighborhood of the equilibrium characterized by $y_i^* = y_j^*$ for all i, j . Assume $\lim_{t \rightarrow \infty} y(t) \rightarrow (\tau(t)', \dots, \tau(t)')' = \mathbf{1}_N \otimes \tau(t)$. One has

$$\begin{aligned} \lim_{t \rightarrow \infty} z(t) &= ((I_N - \mathbf{1}_N \eta') \otimes I_n) \times (\mathbf{1}_N \otimes \tau(t)) \\ &= ((I_N - \mathbf{1}_N \eta') \mathbf{1}_N) \otimes \tau(t) = \mathbf{0}_{nN}. \end{aligned}$$

(Sufficiency) According to the structure of $L(\theta(t))$, there exist matrices $\Upsilon \in \mathbb{R}^{N \times (N-1)}$ and $\Psi \in \mathbb{R}^{(N-1) \times N}$ such that

$$\begin{pmatrix} \eta' \\ \Psi \end{pmatrix} L(\theta(t)) (\mathbf{1}_N \quad \Upsilon) = \begin{pmatrix} 0 & \mathbf{0}'_{N-1} \\ \mathbf{0}_{N-1} & \Xi(\theta(t)) \end{pmatrix}$$

where $\Xi \in \mathbb{R}^{(N-1) \times (N-1)}$ is a matrix function in $\theta(t)$. For system (7), pre-multiplying by $\begin{pmatrix} \eta' \\ \Psi \end{pmatrix} \otimes I_n$, the first n rows generate that

$$\dot{\xi} = Df(s(t))\xi(t) \quad (11)$$

where $\xi(t) \in \mathbb{R}^n$. Suppose system (10) is asymptotically stable, it is clear that

$$y(t) \rightarrow (\xi(t)', \xi(t)', \dots, \xi(t)')' = \mathbf{1}_N \otimes \xi(t).$$

This completes the proof. \square

Corollary 1: Since (10) is a linear time-varying system, one has following equivalent conditions.

- system (10) is asymptotically stable.
- system (10) is exponentially stable.
- robust local synchronization of system (7) can be achieved.
- robust local exponential synchronization of system (7) can be achieved.

Lemma 2: Under Assumption 1, the robust local synchronization of system (4) can be achieved if the following polytopic system is asymptotically stable.

$$\begin{cases} \dot{z}(t) = \widehat{A}(p(t))z(t) \\ p(t) \in \mathcal{P} \end{cases} \quad (12)$$

where $p(t) \in \mathbb{R}^q$ is an uncertain parameter vector, \mathcal{P} is the polytope defined by

$$\mathcal{P} = \text{co}\{p^{(1)}, \dots, p^{(w)}\}$$

and $\widehat{A}(p(t))$ is given by

$$\widehat{A}(p(t)) = \widehat{A}_0 + \sum_{i=1}^q p_i(t) \widehat{A}_i$$

for some $\widehat{A}_0, \widehat{A}_1, \dots, \widehat{A}_q \in \mathbb{R}^{q \times q}$.

Proof Let us define

$$D(t) = I_N \otimes Df(s(t)).$$

One can choose any suitable bounds $b_{ij}, c_{ij} \in \mathbb{R}$ satisfying

$$b_{ij} \leq D_{ij}(t) \leq c_{ij} \quad \forall t \geq 0$$

for all $i, j = 1, \dots, k$ and $k = nN$. Clearly, such bounds always exist since $Df(s(t))$ is continuous. Then define $\iota(t) \in \mathbb{R}^b$ satisfying

$$\iota \in \mathcal{S} = \text{co}\{\iota^{(1)}, \dots, \iota^{(c)}\}$$

a parameter $\iota_l(t)$ is assigned to each entry of $D_{ij}(t)$ choosing

$$\begin{cases} \widehat{D}_{0,ij} &= b_{ij} \\ \widehat{D}_{l,ij} &= c_{ij} - b_{ij} \end{cases}$$

in order to guarantee that $D(t)$ is included by the uncertain polytopic system. Obviously, for entries of $D_{ij}(t)$ that are linearly dependent, merely one parameter $\iota_l(t)$ is needed. Then system (10) can be expressed as

$$\dot{z}(t) = A \left(\sum_{i=1}^b D_i \iota_i(t), \sum_{i=1}^a L_i \theta_i(t) \right) z(t) \quad (13)$$

where function A is linear on $\iota_i(t)$, for all $i = 1, \dots, b$ and also linear on $\theta_i(t)$, for all $i = 1, \dots, a$. One can have a new time-varying variable $\hat{p}(t) \in \mathbb{R}^{a+b}$ constrained in $\widehat{\mathcal{P}} = \text{co}\{\hat{p}^{(1)}, \dots, \hat{p}^{(a+b)}\}$ such that system (13) can be further equivalently expressed as

$$\dot{z}(t) = \widehat{A}(\hat{p}(t))z(t).$$

Thus the proof completes. □

Remark 3: In literatures, local synchronization conditions are proposed based on the manifold $s(t)$, thus making it a non-convex condition which is not tractable. This lemma gives an essential transformation which provides a useful way to make conditions of robust local synchronization solvable by convex approaches given by Section IV. Nevertheless, it is admitted that conservatism generates from the gap between the polytope \mathcal{S} and the manifold $s(t)$. Approaches without employing this transformation will also be discussed in Section IV.

IV. MAIN RESULTS

Based on the transformation introduced by Lemma 2, robust local synchronization problem changes to a robust stability problem of (12), which can be appropriately investigated by HPLFs, a non-conservative class of Lyapunov functions. More importantly, by including $s(t)$ in a polytope, robust local synchronization conditions can be checked by solving an LMI feasibility test.

A. Conditions via HPLF

Let us first introduce the definition of HPLF.

Definition 1: Let $v : \mathbb{R}^{nN} \rightarrow \mathbb{R}$ be a homogeneous polynomial of degree $2m$ satisfying

$$\begin{cases} v(z) > 0, \quad \forall z \in \mathbb{R}_0^{nN} \\ \dot{v}(z) < 0, \quad \forall z \in \mathbb{R}_0^{nN} \text{ and } \forall p \in \mathcal{P} \end{cases} \quad (14)$$

where

$$\dot{v}(z) = \left. \frac{dv(z)}{dt} \right|_{\dot{z} = \widehat{A}(p)z}.$$

Then $v(z)$ is called a *HPLF* of degree $2m$ for the system (12).

Theorem 1: Under Assumption 1, if there exists a continuously differentiable homogeneous function $v(z)$ satisfying

$$\begin{cases} 0 < v(z) \\ 0 < -\mu_i(z) \quad \forall i = 1, \dots, w, \end{cases} \quad \forall z \neq 0 \quad (15)$$

where

$$\mu_i(z) = \dot{v}(z, p) \Big|_{p=p^{(i)}}$$

and

$$\dot{v}(z, p) = \left(\frac{dv(z)}{dz} \right)' \left(\widehat{A}(p)z \right).$$

Then, function $v(z)$ is a HPLF for (12) and the robust local synchronization of (3) can be achieved.

Proof Since $p(t) \in \mathbb{R}^q$ and \mathcal{P} is a polytope described by $\mathcal{P} = \text{co}\{p^{(1)}, \dots, p^{(w)}\}$, one can find $d_1(p), \dots, d_w(p) \in \mathbb{R}$ such that

$$\widehat{A}(p(t)) = \sum_{i=1}^w d_i(p) \widehat{A}(p^{(i)})$$

where $d_1(p), \dots, d_w(p) \in \mathbb{R}$ are such that

$$\begin{cases} \sum_{i=1}^w d_i(p) p^{(i)} = p \\ d_i(p) \geq 0 \quad \forall i = 1, \dots, w \\ \sum_{i=1}^w d_i(p) = 1. \end{cases}$$

on the condition that (15) holds. Accordingly, one has that

$$\begin{aligned} \dot{v}(z, p) &= \left(\frac{dv(z)}{dz} \right)' \left(\sum_{i=1}^w d_i(p) \widehat{A}(p^{(i)})z \right) \\ &= \sum_{i=1}^w d_i(p) \left(\frac{dv(z)}{dz} \right)' \left(\widehat{A}(p^{(i)})z \right) \\ &= \sum_{i=1}^w d_i(p) \mu_i(z) \end{aligned}$$

which implies that

$$\dot{v}(z, p) < 0 \quad \forall z \neq 0$$

Hence, for all $p \in \mathcal{P}$, $v(z)$ is a HPLF for (12). Therefore, (12) is robustly asymptotically stable, and robust local synchronization of (3) can be achieved. \square

Remark 4: For Theorem 1, it is worthy to note that

- Theorem 1 provides conditions for robust local synchronization, and significantly it makes free of calculating all the eigenvalues of Laplacian matrix as required in the literatures. Moreover, HPLF is used and gives a less conservative condition than QLFs widely adopted by literatures, thus proposing a promising way to combine with graph theory to obtain some topological conditions.
- For nonlinear time-varying uncertainties, the approach of HPLF can not be adopted. However, under an assumption that $G(\theta)$ is polynomial function of θ , sufficient conditions can be derived by using polynomial parameter-dependent Homogeneous Lyapunov function (PPD-HLF), i.e., searching a Lyapunov function which is a polynomial function of uncertain parameter θ .
- Sufficient conditions can also be proposed by PPD-HLF for the case without the transformation introduced by Lemma 2. Nevertheless, this approach can hardly provide solvable

conditions such as LMI conditions since $s(t)$ is engaged in. Furthermore, in order to give some solvable conditions, various assumptions are needed while the conservatism level increases, such as assuming $|s(t)|_\infty < c$, where c is a positive constant.

B. SMR Conditions

One effective way for checking whether a homogeneous polynomial is nonnegative consists of checking whether it is a SOS polynomial, which can be equivalently expressed as an LMI feasibility test [25].

Indeed, let $x \in \mathbb{R}^r$ and let $h(x)$ be a homogeneous polynomial with all the monomials of degree $2m$. And let $x^{\{m\}} \in \mathbb{R}^{\sigma(r,m)}$ be a vector containing all monomials of degree m where

$$\sigma(r, m) = \frac{(r + m - 1)!}{(r - 1)!m!}. \quad (16)$$

Accordingly, $h(x)$ can be written in the form of SMR as

$$\begin{aligned} h(x) &= x^{\{m\}'}(H + E(\delta))x^{\{m\}} \\ &\triangleq \Lambda(H + E(\delta), m, r) \end{aligned} \quad (17)$$

where $H \in \mathbb{R}^{\sigma(r,m) \times \sigma(r,m)}$ is a symmetric matrix, and $E(\delta)$ stands for a linear parametrization of the linear subspace

$$\mathcal{E}_{r,m} = \{E \in \mathbb{R}^{\sigma(r,m) \times \sigma(r,m)} : \Lambda(E, m, r) = 0\}. \quad (18)$$

By using representation (17), one can establish whether a homogeneous polynomial is SOS polynomial via LMIs.

Definition 2: $h(x)$ is SOS if there exist polynomials $h_1(x), h_2(x), \dots$ such that

$$h(x) = \sum_i h_i(x)^2 \quad (19)$$

and this condition holds if and only if there exists a δ such that the following LMI holds:

$$H + E(\delta) \geq 0. \quad (20)$$

For more details to obtain $E(\delta)$, interested readers can refer to [25] and references therein.

According to Definition 1, we can express the HPLF $v(z)$ via SMR as

$$v(z) = \Lambda(V, m, r)$$

where $V \in \mathbb{R}^{\sigma(nN,m) \times \sigma(nN,m)}$ is a symmetric matrix. Before deriving the LMIs condition, let us first introduce the following definition.

Definition 3: Define matrix $\widehat{A}^\#$ to be an *extended matrix* of \widehat{A} if it is satisfied that

$$\frac{dz^{\{m\}}}{dt} = \frac{\partial z^{\{m\}}}{\partial z} \widehat{A}z = \widehat{A}^\# z^{\{m\}}. \quad (21)$$

Lemma 3: [26] Let $z^{\{m\}}$ be the m -th Kronecker power of z (This notation given in Section II), and K_m be the matrix satisfying $z^{\{m\}} = K_m z^{\{m\}}$. Then, the extended matrix $\widehat{A}^\#$ can be obtained by

$$\widehat{A}^\# = (K_m' K_m)^{-1} K_m' \left(\sum_{i=0}^{m-1} I_{m-1-i} \otimes \widehat{A} \otimes I_i \right) K_m.$$

Note that

$$\widetilde{A}_i = \widehat{A}(p^{(i)})$$

and let $\widetilde{A}_i^\#$ be the extended matrix of \widetilde{A}_i . Now we can propose the LMI condition for robust local synchronization.

Theorem 2: Under Assumption 1, the robust local synchronization of (3) can be achieved if there exist a symmetric matrix V and $\delta^{(1)}, \dots, \delta^{(w)}$ such that

$$\begin{cases} 0 < V \\ 0 > \text{he} \left(V \widetilde{A}_i^\# \right) + E \left(\delta^{(i)} \right) \quad \forall i = 1, \dots, w. \end{cases} \quad (22)$$

Proof on the condition that (22) holds, via Pre- and post-multiplying the first inequality in (22) by $z^{\{m\}'}$ and $z^{\{m\}}$, respectively, one has that

$$\begin{aligned} 0 &< \Lambda(V, m, r) \\ &= v(z) \end{aligned}$$

which directly follows that $v(z)$ is positive definite since the square of power vector $z^{\{m\}'}$ $z^{\{m\}} > 0$ for all $z \neq 0$. On the other hand, from (21) one can obtain that

$$\begin{aligned} \mu_i(z) &= z^{\{m\}'} \left(V \widetilde{A}_i^\# + \left(V \widetilde{A}_i^\# \right)' \right) z^{\{m\}} \\ &= \Lambda \left(\text{he} \left(V \widetilde{A}_i^\# \right), m, r \right) \end{aligned}$$

and according to the second LMI one can have that

$$\mu_i(z) < 0.$$

Thus, by condition (22), $v(z)$ is guaranteed to be a HPLF for (12). Therefore, from Theorem 1, the robust local synchronization of (3) can be achieved which completes the proof. \square

Remark 5: One can systematically establish if there exist a symmetric matrix V and $\delta^{(1)}, \dots, \delta^{(w)}$ such that (22) holds. In fact, this is an LMI condition, which amounts to solving a convex optimization problem.

V. POLYTOPIC SYNCHRONIZATION MARGIN

Section IV answers how the robust local synchronization with polytopic uncertainties can be achieved. Another question comes naturally that what is the largest level of polytopic uncertainties on which the robustness of local synchronization maintains. In order to answer this question, let us first introduce following definitions.

Definition 4: ζ_{2m}^P is called *2m-HPLF polytopic synchronization margin* for system (3) if there exists a HPLF v with degree $2m$ for system (3) such that

$$\zeta_{2m}^P = \sup \left\{ \zeta \in \mathbb{R} : \theta(t) \in \text{co} \left\{ \zeta \theta^{(1)}, \dots, \zeta \theta^{(v)} \right\} \right\}.$$

Of special usefulness is another definition which comes from a special instance of above denotation, concerning on the polytope Ω as the unit ℓ_∞ box.

Definition 5: ζ_{2m}^∞ is called *2m-HPLF ℓ_∞ synchronization margin* for system (3) if there exists a HPLF v with degree $2m$ for (3) such that

$$\zeta_{2m}^\infty = \sup \left\{ \zeta \in \mathbb{R} : \|\theta(t)\|_\infty \leq \zeta \right\}.$$

For ease of description, we consider the problem of estimating ζ_{2m}^∞ as follows.

Problem 2: (*2m-HPLF ℓ_∞ synchronization margin problem*) To search for the lower bound of ζ_{2m}^∞ if there exists a HPLF v with degree $2m$ for (3).

First let us rewrite system (12) with $\theta(t) = p(t) \in \mathbb{R}^a$ and $\Omega = \mathcal{P}$ as follows.

$$\begin{cases} \dot{z}(t) &= \widehat{A}(\theta(t))z(t) \\ \theta(t) &\in \Omega. \end{cases} \quad (23)$$

Let us denote the vertices of the unit ℓ_∞ ball by $\nu^{(1)}, \dots, \nu^{(2^a)}$, and define

$$\bar{A}_i = \widehat{A}(\theta^{(i)}) - \widehat{A}_0, \quad i = 1, \dots, 2^a,$$

and denote $\bar{A}_i^\#$, $i = 1, \dots, 2^a$, to be the corresponding extended matrix of \bar{A}_i (please refer to Definition 3). Next result proposes a desirable way which consists of a quasi-convex optimization to check the $2m$ -HPLF ℓ_∞ synchronization margin.

Theorem 3: Let us define

$$\hat{\zeta}_{2m}^\infty = \frac{1}{\phi^*} \quad (24)$$

where integer $m \geq 1$, ϕ^* is the solution of

$$\phi^* = \inf_{\phi, V, \delta^{(0)}, \dots, \delta^{(2^a)}} \phi$$

$$s.t. \begin{cases} 0 < \phi \\ 0 < V \\ 0 < -\text{he}(V\widehat{A}_0^\#) - E(\delta^{(0)}) \\ 0 < \phi \left(-\text{he}(V\widehat{A}_0^\#) - E(\delta^{(0)}) \right) \\ \quad -\text{he}(V\bar{A}_i^\#) - E(\delta^{(i)}) \quad \forall i = 1, \dots, 2^a \end{cases} \quad (25)$$

and $E(\cdot)$ is a linear parametrization of $\mathcal{E}_{nN,m}$. Then $\hat{\zeta}_{2m}^\infty$ is the lower bound of ζ_{2m}^∞ , i.e. $\hat{\zeta}_{2m}^\infty \leq \zeta_{2m}^\infty$.

Proof Suppose that (25) holds. Pre- and post-multiplying the second LMI in (25) by $z^{\{m\}'}$ and $z^{\{m\}}$, respectively, one has that

$$0 < \Lambda(V, m, r)$$

hence implying $v(z)$ is positive definite since $z^{\{m\}' } z^{\{m\}} > 0$ for all $z \neq 0$. Moreover, the time derivative of $v(z)$ for $\theta = \phi^{-1}\nu^{(i)}$ is given by

$$\begin{aligned} \dot{v}(z)|_{\theta=\phi^{-1}\nu^{(i)}} &= z^{\{m\}' } \text{he} \left(V \left(\widehat{A}_0^\# + \phi^{-1}\bar{A}_i^\# \right) \right) z^{\{m\}} \\ &= \phi^{-1} z^{\{m\}' } \left(\phi \text{he} \left(V\widehat{A}_0^\# \right) + \text{he} \left(V\bar{A}_i^\# \right) \right) \\ &= \phi^{-1} z^{\{m\}' } \left(\phi \left(\text{he} \left(V\widehat{A}_0^\# \right) + E \left(\delta^{(0)} \right) \right) \right. \\ &\quad \left. + \text{he} \left(V\bar{A}_i^\# \right) + E \left(\delta^{(i)} \right) \right) z^{\{m\}}. \end{aligned} \quad (26)$$

Thus, due to the last constraint in (25) one has

$$\dot{v}(z)|_{\theta=\phi^{-1}\nu^{(i)}} < 0 \quad \forall i = 1, \dots, 2^a.$$

Based on this, one can also have that $\dot{v}(z)$ is negative definite for all $\theta(t)$ in following set

$$\left\{ \theta(t) \in \mathbb{R}^a : \|\theta(t)\|_\infty \leq \phi^{-1} \right\}.$$

Therefore, one has $\hat{\zeta}_{2m}^\infty \leq \zeta_{2m}^\infty$ which completes this proof. \square

Remark 6: Theorem 3 provides an lower bound for $2m$ -HPLF ℓ_∞ synchronization margin ζ_{2m}^∞ . Specially, one has $\hat{\zeta}_{2m}^\infty = \zeta_{2m}^\infty$ when $(nN, 2m)$ is in certain sets, e.g. $\{(nN, 2) : nN \in \mathbb{N}\}$, $\{(2, 2m) : m \in \mathbb{N}\}$ and $\{(3, 4)\}$ [26]. These sets are strongly related with the Hilberts 17th problem which concerns on the gap between SOS polynomials and positive polynomials.

A simple result can be obtained directly from Theorem 3 when we consider $a = 1$ and $\theta \in [0, \psi]$. Paralleled with ζ_{2m}^∞ , we define ψ_{2m}^∞ for the case of system (12) with scalar uncertainty widely adopted in literatures.

Corollary 2: Let us define

$$\hat{\psi}_{2m}^\infty = \frac{1}{\phi^*} \quad (27)$$

where integer $m \geq 1$, ϕ^* is the solution of

$$\begin{aligned} \phi^* &= \inf_{\phi, V, \delta^{(1)}, \delta^{(2)}} \phi \\ \text{s.t.} \quad &\begin{cases} 0 < V \\ 0 < -\text{he}\left(V\hat{A}_0^\#\right) - E\left(\delta^{(1)}\right) \\ 0 < \phi\left(-\text{he}\left(V\hat{A}_0^\#\right) - E\left(\delta^{(1)}\right)\right) \\ \quad -\text{he}\left(V\hat{A}_1^\#\right) - E\left(\delta^{(2)}\right) \end{cases} \end{aligned} \quad (28)$$

and $E(\cdot)$ is a linear parametrization of $\mathcal{E}_{nN, m}$. Then $\hat{\psi}_{2m}^\infty$ is the lower bound of ψ_{2m}^∞ , i.e. $\hat{\psi}_{2m}^\infty \leq \psi_{2m}^\infty$.

VI. NUMERICAL EXAMPLES

To illustrate our proposed approach, two deliberately simple examples are provided by using MATLAB.

A. Example 1

In this case, we consider a coupled jet engines of Moore-Greitzer model [27]. $f(x)$ in (6) describes the intrinsic dynamics of each jet engine as

$$f(x_i) = \begin{pmatrix} -0.5x_{i1}^3 - 1.5x_{i1}^2 - x_{i2} \\ 3x_{i1} - x_{i2} \end{pmatrix}$$

where $x_i = (x_{i1}, x_{i2})'$, $i = 1, 2$. For this jet engine model, a no-stall equilibrium is translated to the origin by following transformation.

$$\begin{cases} x_{i1} = \tilde{x}_{i1} - 1 \\ x_{i2} = \tilde{x}_{i2} - x_{co} - 2. \end{cases}$$

Here, we briefly introduce the practical meaning of each parameter: \tilde{x}_{i1} is the mass flow, \tilde{x}_{i2} is the pressure rise and x_{co} is a constant. The communications between these two jet engines are disturbed by a time-varying uncertainty $\theta(t)$ where the uncertain weighted adjacency matrix $G(\theta(t))$ is given as

$$G(\theta(t)) = \begin{pmatrix} 1 & 2 - \theta(t) \\ 1 & 1 \end{pmatrix}.$$

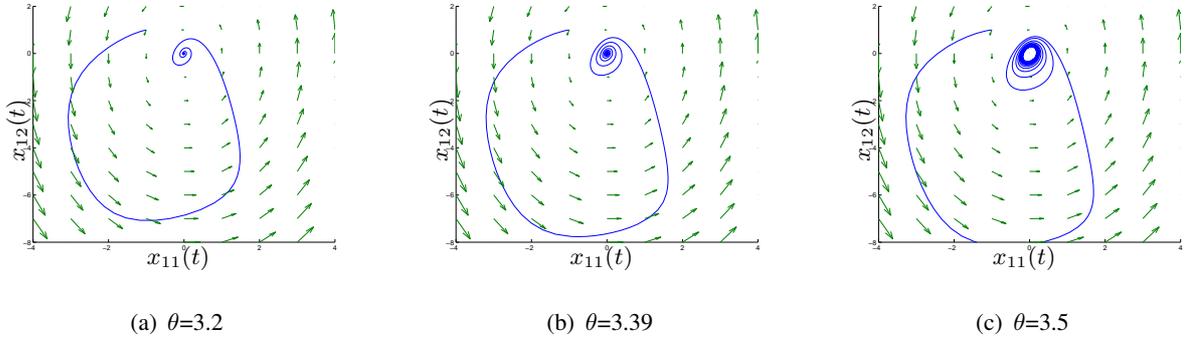


Fig. 1. Hopf bifurcation of coupled M-G jet engines.

Around $\theta = 3.392$, a Hopf bifurcation takes place as shown in Fig. 1 and robust local synchronization can not be achieved when $\theta > 3.392$. Thus let us assume $\theta \in \Omega = \text{co}\{0, 3.0\}$ and we want to establish whether there exists a QLF or HPLF such that robust local synchronization can be achieved for this given uncertainty range.

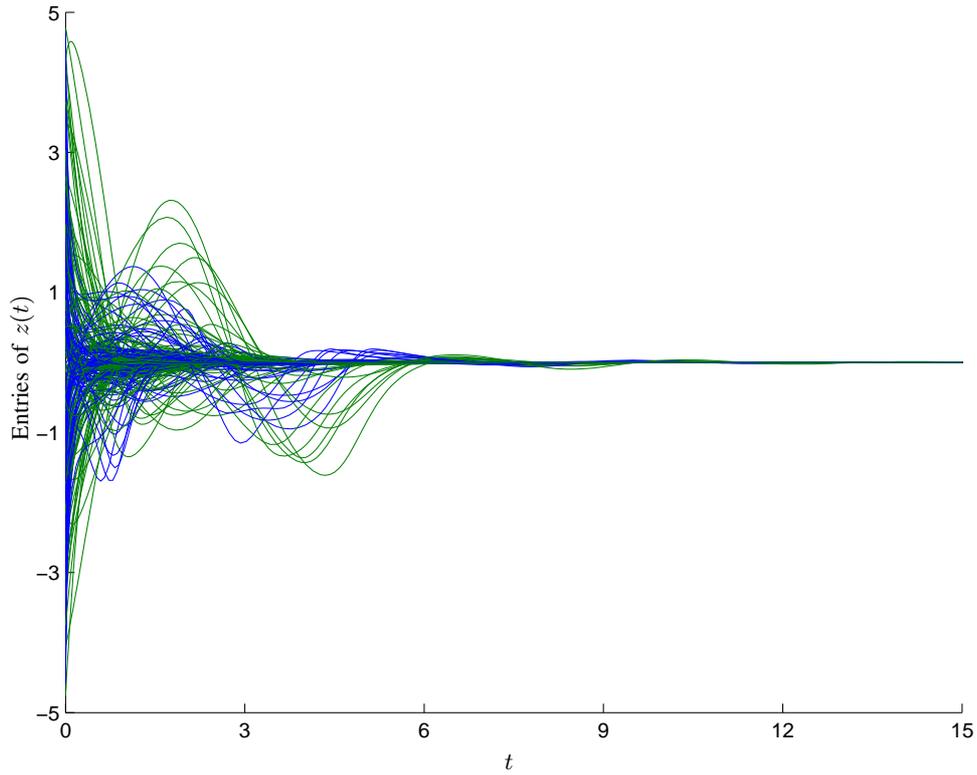


Fig. 2. Trajectories of robust local synchronization.

Results show that one can not find a QLF such that the coupled M-G jet engines are able to achieve robust local synchronization where $m = 1$. However, by using a HPLF where $m = 2$, one can obtain that the LMIs (22) hold and hence robust local synchronization can be achieved according to Theorem 2. In particular, a HPLF for this case is given by $v(z) = z^{\{2\}'} I_z^{\{2\}}$ with $m = 2$. Figure 2 shows 100 trajectories of $z(t)$ with the initial conditions $x(0)$ randomly chosen in $[-5, 5]^4$, and $\theta(t)$ randomly chosen in Ω .

B. Example 2

Let us consider (3) with $N = 3$, $n = 1$, $c = 1$, $\Gamma = 1$ and nonlinear function $f(x)$ is given by

$$f(x) = -x - x^3 - x^5.$$

The uncertain weighted adjacency matrix $G(\theta)$ is given by

$$G(\theta) = \begin{pmatrix} 1 & 2 + \theta & \theta \\ -2 - \theta & 1 & 5 \\ \theta & -3 & 1 \end{pmatrix}$$

where $\theta(t) \in \text{co}\{0, 1\}$. One has that (5) holds with $s(t) = (0, 0)'$. By choosing $p_1 = \theta(t)$, it follows that $\hat{A}(p)$ in (12) can be obtained as

$$\hat{A}(p) = \begin{pmatrix} -3 - 2p_1 & 2 + p_1 & p_1 \\ -2 - p_1 & -4 + p_1 & 5 \\ p_1 & -3 & 2 - p_1 \end{pmatrix}.$$

We find that the LMIs (22) hold and hence robust local synchronization can be achieved according to Theorem 2. For this case, the lower bound provided by (25) is tight, i.e., $\hat{\psi}_{2m}^\infty = \psi_{2m}^\infty$. By applying QLFs, i.e., $m = 1$, one has $\psi_2^\infty = 8.9458$. By contrast, via solving the GEVP (28) and using a HPLF, we can obtain that robust synchronization margin has been significantly expanded, as shown in Table I. By using bisection method, we obtain that the maximal synchronization margin is 13.000 which means by using a HPLF merely with $m = 2$ one can get a very desirable result for this case.

TABLE I
SYNCHRONIZATION MARGIN COMPARISON

	$m=1$	$m=2$	$m=3$	$m=4$
ψ_{2m}^∞	8.9458	12.9397	12.9532	12.9698

VII. CONCLUSIONS

We have investigated robust local synchronization in multi-agent systems with time-varying parametric uncertainties. A novel convex approach has been proposed based on the transformation from the original system to an uncertain polytopic system and on the use of HPLFs. Corresponding LMI-based conditions are obtained by using SMR technique. Polytopic synchronization margin has also been investigated by a convex optimization consisting of GEVPs. As a nature extension, future works will focus on robust global synchronization with time-varying

uncertainties both for the case in a bounded-rate polytope and for the case in a semialgebraic set. Another interesting and promising extension is H_∞ synthesis for robust synchronization of polynomial nonlinear system where pioneering work has already been done in [28].

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