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Sturm-Liouville Operator Functions

A General Concept of Multiplicative Operator Functions on Hypergroups

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Abstract

Many special functions are solutions of both, a differential and a functional equation. We use this duality to solve a large class of abstract Sturm-Liouville equations, initiating a theory of Sturm-Liouville operator functions; cosine, Bessel, and Legendre operator functions are contained as special cases. This is part of a general concept of operator functions being multiplicative with respect to convolution of a hypergroup – containing all representations of (hyper)groups, and further abstract Cauchy problems.

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Zusammenfassung

Viele spezielle Funktionen lösen sowohl eine Differential- als auch eine Funktionalgleichung. Wir verwenden diese Dualität um eine große Klasse von abstrakten Sturm-Liouville-Gleichungen zu lösen. Hierfür wird eine Theorie von Sturm-Liouville-Operatorfunktionen angestoßen; Kosinus-, Bessel- und Legendre-Operatorfunktionen sind als Spezialfälle enthalten. Dies ist Teil eines allgemeinen Konzepts von Operatorfunktionen, die multiplikativ bezüglich einer Hypergruppen-Faltung sind; alle Darstellungen von (Hyper)gruppen und weitere abstrakte Cauchy-Probleme sind darin enthalten.

Contents

Preface

It has become a standard method of analysis to treat several partial differential equations by means of C_0-semigroups. The basic idea is to consider an abstract Cauchy problem of first order as an ordinary differential equation in some function space. The solution of this equation is thought to be an operator-valued exponential function and the exponential functional equation is used for the definition of a C_0-semigroup.

Analogously, abstract Cauchy problems of second order correspond to cosine operator functions. This is where we start our introduction in Chapter 1. By the way, we give a new, direct proof of M. Sova's original result that C_0-regularity implies strong continuity. The proof can also be transferred to the related operator functions of cosine step response and cosine cumulative output.

Content of the thesis. Motivated by C_0-semigroups and cosine operator functions, it is natural to investigate whether functional equations of further special functions can be used to define operator functions solving abstract Cauchy problems. A major difficulty lies in the fact that these functional equations are much more complicated. In a first attempt, T. Dietmair considered in his diploma thesis, Dietmair (1985), the special case of Bessel functions and Gegenbauer's product formula (cf. Definition 1.2.6). The methods employed are quite elementary, the abstract Cauchy problem is not considered at all.

The major contribution of this thesis is, more generally, to solve and investigate abstract Sturm-Liouville equations by means of Sturm-Liouville operator functions. We define these operator functions to be solutions of functional equations provided by associated Sturm-Liouville hypergroups. This approach is promising since it includes several important examples and we have the rich theory of hypergroups at our disposal. We give short introductions to hypergroups and Sturm-Liouville hypergroups in Sections 2.1 and 4.1, respectively. Actually, the theory of hypergroups has been developed in view of abstract harmonic analysis and theoretical probability. So we also contribute a new aspect to the theory of hypergroups. Therefore, several times some pioneering work is necessary. To begin with, we transfer in Chapter 2 some preliminary results to Banach space valued functions. One has to be careful concerning integration in Banach spaces; the usual notions and prerequisites imposed on the Bochner integral are not appropriate to deal with hypergroups in a concise way, thus we have included Appendix A.

When working in this general setting, there is no reason to restrict immediately to special hypergroups. So we define in Chapter 3 a multiplicative operator function to be an operator function on an arbitrary hypergroup which is multiplicative with respect to convolution. This notion is very general, we show that it comprises all representations of locally compact groups and all representations of hypergroups. Thereby we investigate the relationship between measurability, weak, strong, and uniform continuity. A prominent example is provided by translation operator functions on homogeneous Banach spaces; we show that K-weakly stationary processes (K a hypergroup), as investigated by M. Leitner and R. Lasser, fit into this setting. In the final section of this chapter, we show that a multiplicative operator function on a commutative hypergroup with associated integral equation solves the corresponding abstract Cauchy problem. In the discrete setting of polynomial hypergroups everything collapses to linear difference equations as considered by K. Ey and R. Lasser; the case of translation operator functions on (compact) dual Jacobi polynomial hypergroups has been investigated by A. Weinmann and R. Lasser. From the perspective of the theories of C_0-semigroups and cosine operator functions, the most interesting example, however, seems to be provided by Sturm-Liouville hypergroups on the non-negative real line.

So the central part of this thesis is to initiate a theory of Sturm-Liouville operator functions. Chapter 4 is to prepare the tools concerning Sturm-Liouville hypergroups. In particular, we investigate the asymptotic behaviour of multiplicative functions and the principal solutions of the associated Sturm-Liouville equation. The results are strong enough to redetermine the dual space of a Sturm-Liouville hypergroup and to reprove the Laplace representation theorem, thereby giving a new asymptotic interpretation. In Chapter 5 a Sturm-Liouville operator function is defined to be a multiplicative operator function with respect to a Sturm-Liouville hypergroup on the non-negative real line. This definition is justified: A Sturm-Liouville operator function solves the abstract Sturm-Liouville equation; we give a second proof which is more in the spirit of M. Sova. Conversely, an operator function solving the abstract Sturm-Liouville equation is a Sturm-Liouville operator function. Basic properties of the generator are determined, uniformly continuous Sturm-Liouville operator functions are characterized, and a spectral inclusion theorem is shown. All these results do not need an exponential norm bound. The answer to the question whether such a bound exists is two-fold: We prove existence if the underlying hypergroup is a Levitan hypergroup and give a counterexample for Chébli-Trimèche hypergroups. Under the assumption of an exponential bound we present the resolvent formula. We show that each Sturm-Liouville operator function generates a C_0-semigroup and, conversely, a C_0-group (more generally, a cosine operator function) generates a Sturm-Liouville operator function. The last three sections concern special classes of Sturm-Liouville operator functions. We investigate the relation between Bessel-Kingman, hyperbolic, and Jacobi operator functions of varying order, determine the generator of translation operator functions, and consider multiplication operator functions as a source of examples and counterexamples.

Related topics. Although our approach seems to be completely new, there exists of course a vast literature about abstract Cauchy problems in general; these are usually required to be well-posed in some sense or other. We refer to Xiao and Liang (1998) for abstract Cauchy problems of higher order. An interesting and well-developed concept generalizing the access to first and second order abstract Cauchy problems is also provided by the theory of abstract Volterra equations, see Prüss (1993).

Besides, there are intensive studies of A. V. Glushak and collaborators about Bessel and Legendre operator functions. These are closest to our considerations, and we have included short expositions in Sections 1.2 and 1.3. Unfortunately, to the best of our knowledge there are no published proofs of the basic results in Glushak (1997b). Anyway, our approach is based, independently, on the functional equation instead of the abstract Cauchy problem, and it will only appear later, that cosine, Bessel, and Legendre operator functions are contained in our notion of a Sturm-Liouville operator function as special cases.

Concerning operator functions solving some functional equations there are many single contributions. We mention, just to give examples, Buche (1975), Chojnacki (1988), Piskarev and Shaw (1997), and Stetkær (2005).

Notation. Throughout X denotes a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of bounded linear operators $T : X \to X$ with operator norm $\|T\| := \sup_{\|x\|\leq 1} \|Tx\|$ and unit I. The dual space of X is denoted by X^*.

Our notation concerning hypergroups follows Bloom and Heyer (1995). We denote by $\mathbb{N}$ the set of all natural numbers $n = 1, 2, \ldots,$ and by $\mathbb{N}_0$ the natural numbers including zero. Further, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}_+$ the subset of non-negative real numbers, and $\mathbb{R}_+^\times$ the subset of (strictly) positive real numbers. The set of complex numbers is denoted by $\mathbb{C}$.

Further notation is introduced successively, mostly at the beginning of each chapter, and is collected at the end of this thesis.

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Prof. Dr. Rupert Lasser acquainted me with the fundamental idea of this thesis. Although his idea sounded great to me, its realization turned out to be rather sophisticated, a major difficulty being to get the theory of hypergroups into effective application; at an initial stage of this work the Bessel-Kingman hypergroup has been considered only. I want to thank Prof. Lasser for his encouragement, his patience and optimism, his interest in the progress of my work, and numerous discussions.

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Garching near Munich, October 2015 Felix Früchtl

Chapter 1

Cosine, Bessel, and Legendre Operator Functions

This chapter is mainly for introduction, to motivate the main ideas of this thesis, and to describe the plan of the following chapters. Essentially, we give a brief exposition of operator functions solving some abstract Cauchy problems of second order. We begin with the well-established theory of cosine operator functions and then proceed to Bessel and Legendre operator functions about which much less is known. Roughly speaking, these theories have in common that the operator functions are in some sense operator-valued generalizations of special functions. We will also see that they satisfy certain functional equations. This is where hypergroups come in, more precisely these are very special one-dimensional hypergroups, the cosine, the Bessel-Kingman, and the hyperbolic hypergroup, respectively. Nevertheless it is possible to consider such operator functions on arbitrary hypergroups, see Chapter 3; we will see that further abstract Cauchy problems are contained in this setting. After that we focus on the non-negative real line. In Chapter 5, we initiate a theory about Sturm-Liouville operator functions, a notion which unifies the three types of operator functions considered in the present chapter.

1.1 Cosine Operator Functions

Operator functions solving the cosine functional equation were already investigated in the late '50s by S. Kurepa, see Kurepa (1960a,b, 1962), see also Kurepa (1982). The core of these papers lies on questions of measurability and (uniform) continuity. The modern theory started with the work of Sova (1966) and Da Prato and Giusti (1967), who established, independently, a generation theorem of Hille-Yosida type. Further research was also influenced by important contributions due to Fattorini (1969a,b).

For an introduction to cosine operator functions we refer to Arendt et al. (2011), Sections 3.14–3.16. A frequently cited older and short exposition can be found in Goldstein (1985), see Section 8 of Chapter II. For detailed treatments see the monograph Fattorini (1985) and the encyclopedic survey articles Vasil′ev et al. (1991) and Vasil′ev and Piskarev (2004).

1.1.1 A Brief Introduction for Motivation

The following exposition relies on the basic work by M. Sova, see Sova (1966). In some points, the presentation in Früchtl (2012) is similar.

Recall from the Preface that X always denotes a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of bounded linear operators on X.

Definition 1.1.1. A *cosine operator function* is a transformation $C : \mathbb{R}_+ \to \mathcal{L}(X)$ such that

(i) $C(0) = I$,

(ii) $C(t)C(s) = \frac{1}{2}C(t+s) + \frac{1}{2}C(|t-s|)$ for all t, $s \in \mathbb{R}_+$,

(iii) $\lim_{t\to 0^+} C(t)x = x$ for each $x \in X$.

Usually, cosine operator functions are defined on the real line, that is $C : \mathbb{R} \to \mathcal{L}(X)$, and (ii) is replaced by $C(t+s) + C(t-s) = 2C(t)C(s)$ for all t, $s \in \mathbb{R}$; the latter is called the cosine or d'Alembert's functional equation. It is easily checked that setting $C(-t) := C(t)$, $t \in \mathbb{R}_+$ above these two definitions coincide. Our notation emphasizes the structure of the underlying cosine hypergroup (see Example 4.5.5). We always assume condition (iii), that is to say we consider C_0- (or regular) cosine operator functions.

Theorem 1.1.2. *Let C be a cosine operator function. Then C is exponentially bounded, that is there exist $M \geq 0$ and $\omega \geq 0$ such that*

$$\|C(t)\| \leq Me^{\omega t}$$

for all $t \in \mathbb{R}_+$.

This theorem is due to Sova (1966), Proposition 2.4. The proof proceeds by induction. To begin with, there exists $t_0 > 0$ such that $M := \sup_{t\in[0,t_0]} \|C(t)\| < \infty$; this follows from regularity at 0 and the uniform boundedness principle, see Lemma 3.2.1 for an extension of this idea to topological spaces with countable bases. Then the cosine relation $C((n+1)t) = 2C(nt)C(t) - C((n-1)t)$, $n \in \mathbb{N}$ is applied. This strategy will be used again in the proof of Theorem 5.4.4.

M. Sova observed that Theorem 1.1.2 is equivalent to the existence of $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq M\cosh(\omega t)$$

for all $t \in \mathbb{R}_+$. This equivalence persists in the setting of Sturm-Liouville operator functions, see Section 5.4. For the following theorem see Sova (1966), Theorem 2.10, see Theorem 5.4.5 below for a generalization with proof using the underlying hypergroup structure.

Theorem 1.1.3. *Let C be a cosine operator function and $M \geq 1$, $\omega \geq 0$ such that $\|C(t)\| \leq M\cosh(\omega t)$ for all $t \in \mathbb{R}_+$. Then for $n \in \mathbb{N}$ and all $t_1, t_2, \ldots, t_n \in \mathbb{R}_+$*

$$\|C(t_1)C(t_2)\ldots C(t_n)\| \leq M\cosh(\omega t_1)\cosh(\omega t_2)\ldots\cosh(\omega t_n).$$

Theorem 1.1.4. *Let C be a cosine operator function. Then C is strongly continuous, that is for $x \in X$ the mapping $C(\cdot)x : t \mapsto C(t)x$, $\mathbb{R}_+ \to X$ is continuous.*

This theorem was shown by Sova (1966), Theorem 2.7. See the ensuing discussion in Subsection 1.1.2 for a new proof.

Definition 1.1.5. Let C be a cosine operator function. Then its generator $\mathbb{A}$ is defined by

$$\mathbb{A}x := 2 \cdot \lim_{t \to 0^+} \frac{C(t)x - x}{t^2}$$

with domain

$$D(\mathbb{A}) := \{x \in X : \lim_{t \to 0^+} \frac{C(t)x - x}{t^2} \text{ exists}\}.$$

Proposition 1.1.6. *Let C be a cosine operator function with generator $\mathbb{A}$ and $x \in D(\mathbb{A})$. Then $C(t)x \in D(\mathbb{A})$ and $\mathbb{A}C(t)x = C(t)\mathbb{A}x$ for all $t \in \mathbb{R}_+$.*

This proposition is an immediate consequence of Definition 1.1.5 since cosine operators commute.

Theorem 1.1.7. *Let C be a cosine operator function. Then for $x \in X$ and $t > 0$*

$$\int_0^t \int_0^s C(r)x \text{ dr ds} \in D(\mathbb{A})$$

and

$$C(t)x - x = \mathbb{A}\left(\int_0^t \int_0^s C(r)x \text{ dr ds}\right).$$

This was shown by Sova (1966), Fundamental Lemma 2.14. In fact, Sova's calculation shows, that for fixed $t > 0$ and $0 < \varepsilon < t - \varepsilon < t$

$$2\frac{C(\varepsilon) - I}{\varepsilon^2} \int_0^t \int_0^s C(r)x \text{ dr ds} = \int_0^{t+\varepsilon} k(\varepsilon, r)C(r)x \text{ dr}, \tag{1.1}$$

$x \in X$, where

$$k(\varepsilon, r) = \begin{cases} -\frac{2}{\varepsilon^2}(\varepsilon - r) & \text{if } 0 < r < \varepsilon \\ 0 & \text{if } \varepsilon < r < t - \varepsilon \\ \frac{1}{\varepsilon^2}(r - (t - \varepsilon)) & \text{if } t - \varepsilon < r < t \\ \frac{1}{\varepsilon^2}((t + \varepsilon) - r) & \text{if } t < r < t + \varepsilon, \end{cases} \tag{1.2}$$

and $\int_0^\varepsilon k(\varepsilon, r) \text{ dr} = -1$, $\int_{t-\varepsilon}^{t+\varepsilon} k(\varepsilon, r) \text{ dr} = 1$.

In Chapter 5 we generalize Theorem 1.1.7 and its proof (including formulas (1.1) and (1.2)) to Sturm-Liouville operator functions, see Theorem 5.2.1.

Corollary 1.1.8. *The generator $\mathbb{A}$ of a cosine operator function is densely defined and closed.*

The following theorem states that cosine operator functions solve the second order abstract Cauchy problem. It follows from Theorem 1.1.7 by differentiation using that for $x \in D(\mathbb{A})$ the generator and the integral commute.

As usual, we denote by $C^k(\mathcal{I}, X)$, $\mathcal{I}$ some real interval, the space of k-times ($k = 1, 2, \ldots, \infty$) continuously differentiable functions from $\mathcal{I}$ to X.

Theorem 1.1.9. *Let C be a cosine operator function and $x \in D(\mathbb{A})$. Then $C(\cdot)x \in C^2(\mathbb{R}_+, X)$ taking values in $D(\mathbb{A})$ and solving the abstract second order Cauchy problem*

$$\begin{aligned} C''(t)x &= \mathbb{A}C(t)x, \quad t \geq 0, \\ C(0)x &= x, \quad C'(0)x = 0. \end{aligned}$$

Remark 1.1.10. Given a cosine operator function C, the *sine operator function* S is defined by $S(t)x := \int_0^t C(s)x \, \mathrm{d}s$, $x \in X$. Then for $x, y \in D(\mathbb{A})$ the unique solution of

$$\begin{aligned} u''(t) &= \mathbb{A}u(t), \quad t \geq 0, \\ u(0) &= x, u'(0) = y \end{aligned}$$

is given by $u = C(\cdot)x + S(\cdot)y$.

Theorem 1.1.11. *Let C be a cosine operator function and $M \geq 1$, $\omega \geq 0$ such that $\|C(t)\| \leq M\cosh(\omega t)$ for all $t \in \mathbb{R}_+$. Then the resolvent $R(\lambda^2, \mathbb{A}) = (\lambda^2 - \mathbb{A})^{-1}$ exists for $\mathrm{Re}(\lambda) > \omega$ and for any $x \in X$*

$$\lambda R(\lambda^2, \mathbb{A})x = \int_0^\infty e^{-\lambda s} C(s)x \, \mathrm{d}s,$$

that is $\lambda R(\lambda^2, \mathbb{A})x$ is the Laplace transform of $C(\cdot)x$.

The corresponding formula in the Sturm-Liouville setting will be established in Theorem 5.4.6.

The following generation theorem of Hille-Yosida type is due to Sova (1966), a similar variant was shown, independently, by Da Prato and Giusti (1967). See also the references in Vasil′ev et al. (1991), 3.1.15 Theorem 1 and 3.1.16.

As usual we denote by $\rho(\mathbb{A})$ and $\sigma(\mathbb{A})$ the resolvent set and the spectrum of a linear operator $\mathbb{A}$, respectively.

Theorem 1.1.12. *Given a linear operator $\mathbb{A}$ defined on a subspace $D(\mathbb{A})$ of a Banach space X and constants $M \geq 1$, $\omega \geq 0$, the following properties are equivalent.*

(i) $\mathbb{A}$ generates a cosine operator function with norm bound

$$\|C(t)\| \leq Me^{\omega t}$$

for all $t \in \mathbb{R}_+$.

(ii) $\mathbb{A}$ is closed, densely defined and for every $\lambda > \omega$, $\lambda^2 \in \rho(\mathbb{A})$ and for all $n \in \mathbb{N}_0$

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda R(\lambda^2, \mathbb{A})) \right\| \leq \frac{Mn!}{2} \left(\frac{1}{(\lambda - \omega)^{n+1}} + \frac{1}{(\lambda + \omega)^{n+1}} \right).$$

1.1.2 Continuity of Cosine and Related Operator Functions

Cosine operator functions are strongly continuous. The original proof of this fact is due to Sova (1966), Theorem 2.7. His proof is indirect, using sequences, and is based on an idea of Van der Lyn (1940) for real cosine functions.

We present here a second, direct proof which seems to be new. It gives an intimate connection between continuity at 0 and t.

Theorem 1.1.13. *Let C be a cosine operator function. Then C is strongly continuous, i. e. $t \mapsto C(t)x$ is continuous for each $x \in X$. If C is uniformly continuous at 0, i. e. $\lim_{t\to 0} \|C(t) - I\| = 0$, then C is uniformly continuous.*

More precisely, suppose $M \geq 1$, $\omega \geq 0$ are chosen such that $\|C(t)\| \leq Me^{\omega|t|}$ for all $t \in \mathbb{R}$ (here we employ, as mentioned before, the usual extension of C to the real line, that is $C(-t) = C(t)$). Then for $t \in \mathbb{R}$, $|h| < 1$ and $x \in X$

$$\|C(t+2h)x - C(t)x\| \leq M(t) \left(\sqrt{|h|}\|x\| + 2 \sup_{0<s<\sqrt{|h|}} \|C(s)x - x\| \right)$$

with $M(t) := 2Me^{\omega(|t|+3)}$.

Proof. Our proof needs the exponential bound from Theorem 1.1.2 and the cosine-related formula

$$C(t+h) - C(t-h) = \frac{1}{2}(C(t+2h) - C(t-2h)) - (C(t+h) - C(t-h))(C(h) - I) \quad (1.3)$$

where t, $h \in \mathbb{R}$ are arbitrary real numbers. By induction we derive for $n \in \mathbb{N}$

$$C(t+h) - C(t-h) = \frac{1}{2^n}(C(t+2^n h) - C(t-2^n h)) - \sum_{k=0}^{n-1} \frac{1}{2^k}(C(t+2^k h) - C(t-2^k h))(C(2^k h) - I).$$

Suppose $x \in X$, then the exponential bound gives

$$\|C(t+h)x - C(t-h)x\| \leq 2Me^{\omega(|t|+2^n|h|)} \left(\frac{1}{2^n}\|x\| + \sum_{k=0}^{n-1} \frac{1}{2^k}\|C(2^k h)x - x\| \right)$$

and shifted by the substitution $t' = t + h$

$$\|C(t+2h)x - C(t)x\| \leq 2Me^{\omega(|t+h|+2^n|h|)} \left(\frac{1}{2^n}\|x\| + \sum_{k=0}^{n-1} \frac{1}{2^k}\|C(2^k h)x - x\| \right).$$

Suppose $0 < |h| < 1$. Then there exists $n = n(h) \in \mathbb{N}$ such that $2^{-n} \leq \sqrt{|h|} < 2^{-(n-1)}$. So we get

$$\|C(t+2h)x - C(t)x\| \leq 2Me^{\omega(|t+h|+2\sqrt{|h|})} \left(\sqrt{|h|}\|x\| + 2 \sup_{0<s<\sqrt{|h|}} \|C(s)x - x\| \right).$$

The statement about uniform continuity is obvious. $\square$

The preceding proof of continuity of trajectories can be transferred to families of C_0-cosine step response and C_0-cosine cumulative output. These are defined as follows, see Piskarev and Shaw (1997), Definition 1.1.

Definition 1.1.14. Let C be a cosine operator function. A transformation $F : \mathbb{R} \to \mathcal{L}(X)$ is called C_0-*cosine step response* if

(i) $F(0) = 0$,

(ii) $F(t+s) - 2F(t) + F(t-s) = 2C(t)F(s)$ for all $t, s \in \mathbb{R}$,

(iii) $\lim_{t\to 0} F(t)x = 0$ for each $x \in X$.

A transformation $G : \mathbb{R} \to \mathcal{L}(X)$ is called C_0-*cosine cumulative output* if

(i) $G(0) = 0$,

(ii) $G(t+s) - 2G(t) + G(t-s) = 2G(s)C(t)$ for all $t, s \in \mathbb{R}$,

(iii) $\lim_{t\to 0} G(t)x = 0$ for each $x \in X$.

Setting $t = 0$ reveals that F and G are even functions.

Proposition 1.1.15. *Any C_0-cosine step response F or C_0-cosine cumulative output G is exponentially bounded.*

The proof is similar to that of Theorem 1.1.2 above, see Piskarev and Shaw (1997), Proposition 3.1(ii).

Theorem 1.1.16. *Consider a C_0-cosine step response F and a C_0-cosine cumulative output G. Then F and G are strongly continuous. If F or G is uniformly continuous at 0 then F or G is uniformly continuous on $\mathbb{R}$, respectively.*
More precisely, suppose $M \geq 1$, $\omega \geq 0$ are chosen such that $\|F(t)\| \leq Me^{\omega|t|}$ and $\|G(t)\| \leq Me^{\omega|t|}$ for all $t \in \mathbb{R}$. Then for $t \in \mathbb{R}$, $|h| < 1$ and $x \in X$

$$\|F(t+2h)x - F(t)x\| \leq M(t)\left(\sqrt{|h|}\|x\| + 2 \sup_{0<s<\sqrt{|h|}} \|F(s)x\|\right)$$

and

$$\|G(t+2h)x - G(t)x\| \leq M(t)\left(\sqrt{|h|}\|x\| + 2 \sup_{0<s<\sqrt{|h|}} \|C(s)x - x\|\right)$$

with $M(t) := 2Me^{\omega(|t|+3)}$.

The first part of this theorem was shown by Piskarev and Shaw (1997), Theorem 2.2 using the technique of Sova's original proof of Theorem 1.1.4.

Proof. See the proof of Theorem 1.1.13, formula (1.3) replaced by

$$F(t+h) - F(t-h) = \frac{1}{2}(F(t+2h) - F(t-2h)) - (C(t+h) - C(t-h))F(h)$$

and

$$G(t+h) - G(t-h) = \frac{1}{2}(G(t+2h) - G(t-2h)) - (G(t+h) - G(t-h))(C(h) - I),$$

respectively. □

We close with a short discussion of the relationship between measurability and continuity.

Theorem 1.1.17. *Let C be a cosine operator function. Then the following conditions are equivalent.*

(i) $C : \mathbb{R}_+ \to \mathcal{L}(X)$ *is $\mathscr{B}(\mathbb{R}_+)$-measurable ("Borel measurable"; see Appendix A).*

(ii) $C : \mathbb{R}_+ \to \mathcal{L}(X)$ *is uniformly continuous.*

(iii) $\lim_{t\to 0^+} C(t) = I$ *in uniform operator topology.*

(iv) There exists $\mathbb{A} \in \mathcal{L}(X)$ *such that*

$$C(t) = \cosh(\sqrt{\mathbb{A}}t) := \sum_{n=0}^{\infty} \frac{\mathbb{A}^n t^{2n}}{(2n)!} \tag{1.4}$$

for all $t \geq 0$.

The equivalence of (i) and (iv) was shown in Kurepa (1962), Theorem 1 for operators satisfying the cosine functional equation in a Banach algebra. Sova observed that with his notion of cosine operator functions, Kurepa's proof also shows that (iii) implies (iv), see Sova (1966), Fundamental Theorem 3.4. For cosine step responses and cosine cumulative outputs it is shown in Piskarev and Shaw (1997), Theorem 2.1 that uniform measurability implies uniform continuity. For Bessel operator functions (see Section 1.2) the equivalence of (ii) to (iv) is stated in Glushak (1997b), Theorem 7. We will consider the corresponding problem in the Sturm-Liouville setting, see Theorem 5.3.5 and Example 5.8.9.

Remark 1.1.18. In the setting of C_0-semigroups conditions (ii)-(iv) are also equivalent where, of course, (1.4) has to be replaced by $T(t) = \exp(\mathbb{A}t) := \sum_{n=0}^{\infty} \frac{\mathbb{A}^n t^n}{n!}$; in this case T can be extended to a C_0-group. Condition (i) for C_0-semigroups implies only uniform continuity on $]0, \infty[$. These facts are contained in Hille and Phillips (1957), Theorems 9.4.2 and 9.3.1. For analytic semigroups (see, e. g., Pazy, 1983, Section 2.5) the mapping $T :]0, \infty[\to \mathcal{L}(X)$ is infinitely differentiable, but all interesting examples are not uniformly continuous at zero. Further differences between cosine operator functions and C_0-semigroups can be found in Bobrowski and Chojnacki (2013).

1.2 Bessel Operator Functions

These are by definition operator functions solving the abstract Euler-Poisson-Darboux equation. They were introduced and examined in a series of papers by A. V. Glushak and collaborators, see Glushak (1997b) and in chronological order Glushak et al. (1986), Glushak and Shmulevich (1992), Glushak (1996, 1997b,a, 1999b,a), Vorob'eva and Glushak (2001), Glushak (2003, 2006), and Glushak and Popova (2008). For related and previous work see also the references therein. It is not possible to give a comprehensive survey within a reasonable amount of space. We restrict ourselves to the mere definition and a generation theorem. Several further properties are similar to cosine operator functions. We will cite those results separately in Chapter 5 whenever they occur.

Let X be a Banach space, u a function defined on $\mathbb{R}_+^\times$ with values in X and $\mathbb{A}$ a closed, densely defined linear operator with domain $D(\mathbb{A})$. Consider the *abstract Euler-Poisson-Darboux equation*

$$u''(t) + \frac{2\alpha+1}{t} u'(t) = \mathbb{A}u(t), \quad t > 0, \tag{1.5}$$

$$u(0) = x, \quad u'(0) = 0 \tag{1.6}$$

for some parameter $\alpha > -\frac{1}{2}$. (In the notation of A. V. Glushak $k = 2\alpha + 1$. We follow the common notation for Bessel-Kingman hypergroups.) By a *solution* of (1.5) we mean a twice continuously differentiable function u defined on $\mathbb{R}_+^\times$ with values in $D(\mathbb{A})$ such that (1.5) holds for all $t > 0$.

Definition 1.2.1. Problem (1.5), (1.6) is called *uniformly correct* if there is an operator function $Y_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ such that each $Y_\alpha(t)$, $t \in \mathbb{R}_+$ commutes with $\mathbb{A}$ and for each $x \in D(\mathbb{A})$ the function $Y_\alpha(\cdot)x$ is the unique solution of (1.5), (1.6). Moreover it is assumed that there is an exponential bound, i. e. there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|Y_\alpha(t)\| \leq M \exp(\omega t)$$

for all $t \geq 0$. Then Y_α is called the *Bessel operator function*; the set of operators for which problem (1.5), (1.6) is uniformly correct is denoted by $\mathcal{G}_\alpha$.

The following important Hille-Yosida Type Theorem 1.2.4 and Corollary 1.2.5 are taken from Glushak (1997b). To the best of our knowledge there are no published proofs of these results.

Let K_α denote the MacDonald function (also known as the modified Bessel function of the third kind) of order α.

Theorem 1.2.2. *Suppose $\mathbb{A} \in \mathcal{G}_\alpha$. Then for $\lambda > \omega$ it is $\lambda^2 \in \rho(\mathbb{A})$ and*

$$\lambda^{\frac{1}{2}-\alpha} R(\lambda^2, \mathbb{A})x = 2^{-\alpha}/\Gamma(\alpha+1) \cdot \int_0^\infty \sqrt{\lambda t} K_\alpha(\lambda t) t^{\alpha+\frac{1}{2}} Y_\alpha(t)x \, \mathrm{d}t.$$

The following generation theorem for Bessel operator functions relies on the special second order differential operator

$$S_{\alpha,\lambda} = \lambda^{-\alpha-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\lambda} \lambda^{2\alpha+1} \frac{\mathrm{d}}{\mathrm{d}\lambda} \lambda^{-\alpha-\frac{1}{2}}$$

and the next definition.

Definition 1.2.3. Suppose $\mathbb{A}$ is a linear operator with $\omega \geq 0$ such that $\lambda^2 \in \rho(\mathbb{A})$ for all $\lambda > \omega$. We say the corresponding *K-transformation of order α is invertible* if there exists $M \geq 1$ such that for any $x \in X$, $\|x\| \leq 1$ and $x^* \in X^*$, $\|x^*\| \leq 1$, there exists a measurable function $\Phi_\alpha(\cdot, x, x^*)$ with

$$|\Phi_\alpha(t, x, x^*)| \leq M \exp(\omega t)$$

for all $t \geq 0$ such that

$$P_\alpha(\lambda, x, x^*) = 2^{-\alpha}/\Gamma(\alpha+1) \cdot \int_0^\infty \sqrt{\lambda t} K_\alpha(\lambda t) t^{\alpha+\frac{1}{2}} \Phi_\alpha(t, x, x^*)\,\mathrm{d}t$$

for all $\lambda > \omega$ where $P_\alpha(\lambda, x, x^*) = x^*(\lambda^{\frac{1}{2}-\alpha} R(\lambda^2, \mathbb{A})x)$.

Theorem 1.2.4. *A densely defined, closed linear operator $\mathbb{A}$ generates a Bessel operator function of order $\alpha > -\frac{1}{2}$ if and only if there exist $M \geq 1$ and $\omega \geq 0$ such that for $\lambda > \omega$ it is $\lambda^2 \in \rho(\mathbb{A})$, the K-transformation of order α is invertible, and*

$$\left\| S^n_{\alpha,\lambda}(\lambda^{\frac{1}{2}-\alpha} R(\lambda^2, \mathbb{A})) \right\| \leq M \frac{\Gamma(2n+\alpha+3/2)}{(\lambda-\omega)^{2n+\alpha+3/2}}$$

for all $n \in \mathbb{N}_0$. In this case

$$Y_\alpha(t)x = \left(\frac{2}{t}\right)^{\alpha+\frac{1}{2}} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}} \lim_{n\to\infty} \frac{1}{(2n)!} \left(\frac{2n}{t}\right)^{2n+1} S^n_{\alpha,\lambda}(\lambda^{\frac{1}{2}-\alpha} R(\lambda^2, \mathbb{A}))x|_{\lambda=2n/t}$$

for any $t > 0$.

Corollary 1.2.5. *A Bessel operator function Y_α satisfies*

$$Y_\alpha(t) Y_\alpha(s)x = c_\alpha \int_0^\pi Y_\alpha(\sqrt{t^2+s^2-2ts\cos\theta})x \sin^{2\alpha}\theta\,\mathrm{d}\theta \tag{1.7}$$

for all $t, s \in \mathbb{R}_+$ and $x \in X$ where

$$c_\alpha := \Gamma(\alpha+1)/(\Gamma(\alpha+1/2)\Gamma(1/2)). \tag{1.8}$$

Glushak (1997b) observes that the right hand side of (1.7) corresponds to the generalized shift operator (see Levitan, 1951) of the Euler-Poisson-Darboux equation.

The integral representation (1.7) can also be regarded as Gegenbauer's product formula (see (4.54)) for Bessel operator functions. At least for half-integer values of α it can be interpreted as a radial random walk on spheres, see Kingman (1963). The argument of Y_α on the right hand side comes from the (euclidean) law of cosines.

It is known that this structure gives rise to a hypergroup, the so-called Bessel-Kingman hypergroup, see Example 4.5.2 for more details. In this context the right hand side of (1.7) corresponds to hypergroup convolution. This access does not depend on a specific differential equation – in the present setting the Euler-Poisson-Darboux equation – and is open to massive generalization, see Chapter 3. The following definition is by analogy to cosine operator functions. As far as we know it has only been considered before by Dietmair (1985) (cf. the Preface above).

Definition 1.2.6. An operator function $Y_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ is called a *Bessel-Kingman operator function* of order $\alpha > -\frac{1}{2}$ if $t \mapsto Y_\alpha(t)x$, $\mathbb{R}_+ \to X$ is continuous for each $x \in X$ and

(i) $Y_\alpha(0) = I$,

(ii) $Y_\alpha(t)Y_\alpha(s)x = c_\alpha \int_{-1}^{1} Y_\alpha(\sqrt{t^2 + s^2 - 2tsr})x\,(1 - r^2)^{\alpha - \frac{1}{2}} \mathrm{d}r$ for all $t, s \in \mathbb{R}_+$ and any $x \in X$ with c_α as in (1.8),

(iii) $\lim_{t \to 0^+} Y_\alpha(t)x = x$ for each $x \in X$.

Taking the limit $\alpha \to -\frac{1}{2}^+$ in the functional equation (ii) (analogously in the abstract Euler-Poisson-Darboux equation (1.5), (1.6)), cosine operator functions are contained as a limiting case. We will see that a Bessel-Kingman operator function Y_α solves the abstract Euler-Poisson-Darboux equation (1.5), (1.6). Thus a Bessel operator function is a Bessel-Kingman operator function with the property that solutions are unique and the operator function is exponentially bounded (see Remark 5.2.10).

1.3 Legendre Operator Functions

These were introduced in Glushak (2001) and compared to Bessel operator functions.

Let $\mathbb{A}$ be a closed, densely defined linear operator with domain $D(\mathbb{A})$. Consider the *abstract Legendre equation*

$$u''(t) + (2\alpha + 1)\coth(t)u'(t) + (\alpha + \tfrac{1}{2})^2 u(t) = \mathbb{A}u(t), \quad t > 0, \tag{1.9}$$

$$u(0) = x, \quad u'(0) = 0 \tag{1.10}$$

where $\alpha > -\frac{1}{2}$. Comparing with the notation in Glushak (2001) it is $k = 2\alpha + 1$ and $\gamma = 1$. The parameter $\gamma > 0$ in Glushak (2001) is introduced to show that for $\gamma \to 0^+$ the abstract Legendre equation approaches the Euler-Poisson-Darboux equation. For our purposes it is not necessarily needed, see Remarks 4.1.7 and 5.2.7.

The Legendre equation occurs in solving the Laplace equation in prolate spheroidal coordinates. For connections to hypergroups we refer to Connett et al. (1993) and Connett et al. (1999).

Glushak's definition of a Legendre operator function is by analogy to Definition 1.2.1.

Definition 1.3.1. Problem (1.9), (1.10) is called *uniformly correct* if there is an operator function $P_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ such that each $P_\alpha(t)$, $t \in \mathbb{R}_+$ commutes with $\mathbb{A}$ and for each $x \in D(\mathbb{A})$ the function $P_\alpha(\cdot)x$ is the unique solution of (1.9), (1.10). Moreover it is assumed that there is an exponential bound, i. e. there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|P_\alpha(t)\| \leq M \exp(\omega t)$$

for all $t \geq 0$. Then P_α is called the *Legendre operator function*; the set of operators for which problem (1.9), (1.10) is uniformly correct is denoted by $\mathcal{G}_\alpha^1$ (the exponent refers to $\gamma = 1$ in Glushak, 2001).

We will see that for the Legendre operator function the hyperbolic hypergroup (see Example 4.5.4) plays the same role as the Bessel-Kingman hypergroup for the Bessel operator function. For motivation let us note that the Sturm-Liouville function $A(t) = t^{2\alpha+1}$ of a Bessel-Kingman hypergroup satisfies $A'(t)/A(t) = (2\alpha+1)/t$ whereas the Sturm-Liouville function $A(t) = \sinh^{2\alpha+1} t$ of a hyperbolic hypergroup satisfies $A'(t)/A(t) = (2\alpha + 1)\coth(t)$. This leads us to the following definition.

Definition 1.3.2. An operator function $P_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ is called a *hyperbolic operator function* of order $\alpha > -\frac{1}{2}$ if $t \mapsto P_\alpha(t)x$, $\mathbb{R}_+ \to X$ is continuous for each $x \in X$ and

(i) $P_\alpha(0) = I$,

(ii) $P_\alpha(t)P_\alpha(s)x = c_\alpha \int_{-1}^{1} P_\alpha(\operatorname{arcosh}(\cosh t \cosh s - r \sinh t \sinh s))x\ (1 - r^2)^{\alpha-\frac{1}{2}} \mathrm{d}r$ for all $t, s \in \mathbb{R}_+$ and any $x \in X$ with c_α as in (1.8),

(iii) $\lim_{t\to 0^+} P_\alpha(t)x = x$ for each $x \in X$.

The argument of P_α on the right hand side of (ii) corresponds to the hyperbolic law of cosines.

We will see that a hyperbolic operator function P_α solves the abstract Legendre equation (1.5), (1.6). Thus a Legendre operator function is a hyperbolic operator function with the property that solutions are unique and the operator function is exponentially bounded (see Remark 5.2.10).

Chapter 2

Hypergroups and Banach Space Valued Functions

The intention of the present chapter is, first of all, to give a short but concise introduction to hypergroups which provides the reader with all the necessary knowledge needed in subsequent chapters. The second section extends some basic theorems about convolution to Banach space valued functions. These technical means are needed in Chapter 3 for the notion of a multiplicative operator function.

2.1 Hypergroups – a Slightly Different Presentation

We introduce hypergroups in the axiomatic of Jewett (1975) which is widely accepted. Similar approaches around the same time are Dunkl (1973) and Spector (1975). Surveys were given by Heyer (1984) and Litvinov (1987). A different concept is the notion of hypercomplex systems, see Berezansky and Kalyuzhnyi (1998). Many discussions and a broad view on the applications of hypergroups can be found in the proceedings of the first conference on hypergroups, held in Seattle in 1993, see Connett et al. (1995). See also the proceedings of a succeeding conference, held in Delhi in 1995 (Ross et al., 1998). Our main source of reference is the monograph by Bloom and Heyer (1995). The ensuing exposition has been written on the basis of this monograph, including notation. Some parts are also influenced by Lasser (2016).

Hypergroups are a generalization of locally compact groups. The idea is to associate to a pair of elements a probability measure instead of just one single element. The concept of a locally compact group is then contained by identifying single elements with point measures. The following presentation of hypergroups is intended to emphasize this analogy. Many concepts, ideas, and proofs for hypergroups are inherited from the group case. There exists a vast literature about locally compact Abelian groups. We refer to Rudin (1962), Chapter 1 for a brief introduction, Kaniuth (2009) for some illustrations, and the monumental treaties Hewitt and Ross (1979, 1970).

To avoid to get immediately overall technical, we content ourselves for the moment with a minimum of notation. Several further notions and measure theoretical details are explained below.

Let K be a non-void locally compact Hausdorff space. Let $M^1(K)$ denote the space of probability, and $M^b(K)$ the space of all bounded complex Borel measures on K. Both, $M^1(K)$ and $M^b(K)$ are endowed with the weak topology induced by $C_b(K)$, the space of continuous and bounded functions. We remark in advance that a measure is usually regarded as a continuous linear functional on $C_c(K)$, the space of continuous functions with compact support.

Suppose

$$*:\ K \times K \to M^1(K)$$

is a continuous mapping. Identifying $t \in K$ with its point measure ε_t, its positive-continuous extension

$$*:\ M^b(K) \times M^b(K) \to M^b(K),$$

called *convolution*, is given by

$$(\mu * \nu)(f) = \int_K \int_K (\varepsilon_t * \varepsilon_s)(f)\, \mu(\mathrm{d}t)\nu(\mathrm{d}s), \quad f \in C_c(K) \tag{2.1}$$

for μ, $\nu \in M^b(K)$.

Definition 2.1.1. Let K be a non-void locally compact Hausdorff space with convolution as above. The triple $(M^b(K), +, *)$ will be called a *hypergroup* if the following conditions are satisfied.

H1 The convolution $*$ is *associative*.

H2 There exists a *neutral element*, that is $e \in K$ such that $\varepsilon_e * \varepsilon_t = \varepsilon_t * \varepsilon_e = \varepsilon_t$ for all $t \in K$.

H3 There exists an *involution*, that is a self-inverse homeomorphism $^-$ on K such that $e \in \mathrm{supp}(\varepsilon_t * \varepsilon_s)$ if and only if $t = s^-$, and $(\varepsilon_t * \varepsilon_s)^- = \varepsilon_{s^-} * \varepsilon_{t^-}$ for all $t, s \in K$ where μ^- denotes the image of μ under $^-$.

H4 For every t, $s \in K$, $\mathrm{supp}(\varepsilon_t * \varepsilon_s)$ is compact, and the mapping $(t, s) \to \mathrm{supp}(\varepsilon_t * \varepsilon_s)$ of $K \times K$ into $\mathscr{C}(K)$ is continuous, where $\mathscr{C}(K)$ denotes the collection of non-void compact subsets of K, endowed with the Michael topology (see (2.3)).

A hypergroup is called *commutative* if the algebra $(M^b(K), +, *)$ is commutative, and *hermitian* (or *symmetric*) if the involution is the identity mapping.

For abbreviation $(K, *)$, or simply K, is called a hypergroup if there is a corresponding measure algebra $(M^b(K), +, *)$ which is a hypergroup in the sense of Definition 2.1.1. Further notions like "commutative" or "hermitian" are transferred analogously. Nevertheless, harmonic analysis of hypergroups is mostly done in the corresponding measure algebra.

We note that for associativity it is sufficient to consider point measures, i. e. $\varepsilon_t * (\varepsilon_s * \varepsilon_r) = (\varepsilon_t * \varepsilon_s) * \varepsilon_r$ for all $t, s, r \in K$. It is easy to see that the neutral element e and the involution $^-$ are unique. Every hermitian hypergroup is commutative by H3.

The construction of hypergroups allows the following basic observation.

Theorem 2.1.2. *Let* $(K, *)$ *be a hypergroup. Then* $(M^b(K), +, *)$ *is a Banach* $\sim$*-algebra with convolution* $*$*, involution* $\mu^\sim := \overline{\mu^-}$*, and unit* ε_e*.*

As announced we introduce now several further notions following Bloom and Heyer (1995), Section 1.1.1, and give some measure theoretical background.

Notation and preliminaries: Let K be a non-void locally compact Hausdorff space.

Functions. Let $C(K)$ denote the space of continuous complex-valued functions on K, and $C_b(K)$, $C_0(K)$, $C_c(K)$ the subspaces consisting of bounded functions, those vanishing at infinity (a function $f \in C(K)$ is said to vanish at infinity if for every $\varepsilon > 0$ there exists a compact set C such that $|f(t)| < \varepsilon$ for all $t \notin C$), and those with compact support. The spaces $C_b(K)$, $C_0(K)$ are endowed with the uniform norm $\|\cdot\|_\infty$. The topology of $C_c(K)$ is described by the property that a net (f_ι) in $C_c(K)$ converges to $f \in C_c(K)$ if and only if there exists a compact set $C \subset K$ and ι_0 such that $\operatorname{supp}(f_\iota) \subset C$ for all $\iota \geq \iota_0$ and $\lim_\iota \|f - f_\iota\|_\infty = 0$.

Measures. A *complex Radon measure* μ on K is a continuous linear functional on $C_c(K)$. Thus for each compact set $C \subset K$ there exists a constant $\alpha_C \geq 0$ such that $|\mu(f)| \leq \alpha_C \|f\|_\infty$ for all $f \in C_c(K)$ with $\operatorname{supp}(f) \subset C$. The set of complex Radon measures on K is denoted by $M(K)$. The *support* $\operatorname{supp}(\mu)$ of a measure $\mu \in M(X)$ is defined as the complement of the largest open subset U of K such that $\mu(f) = 0$ for all $f \in C_c(K)$ with support contained in U.

We consider the following subspaces of $M(K)$. The subspace of positive linear functionals in $M(K)$ is denoted by $M_+(K)$, that is the collection of those $\mu \in M(K)$ such that $\mu(f) \geq 0$ for all $f \in C_c(K)$ with $f \geq 0$. The subspace of bounded measures $M^b(K)$ is defined as the collection of those $\mu \in M(K)$ such that

$$\|\mu\| := \sup\{|\mu(f)|,\, f \in C_c(K),\, \|f\|_\infty \leq 1\} < \infty.$$

$M^b_+(K)$ denotes the intersection of $M^b(K)$ and $M_+(K)$. Finally, $M^1(K)$ denotes the probability measures in $M^b_+(K)$, that is those $\mu \in M^b_+(K)$ with $\|\mu\| = 1$.

According to a generalized Riesz representation theorem, each complex Radon measure $\mu \in M(K)$ can be identified with a unique set function μ° of the form $\mu^\circ := \mu_1^\circ - \mu_2^\circ + i\mu_3^\circ - i\mu_4^\circ$, defined for all relatively compact Borel sets, and with non-negative regular Borel measures μ_i°, $i = 1 \ldots 4$ such that

$$\mu^\circ(f) = \int_K f \,\mathrm{d}\mu^\circ \tag{2.2}$$

for all $f \in C_c(K)$, see Berg et al. (1984), Chapter 2, Theorem 2.5. To be precise, a non-negative Borel measure (that is a measure defined on the Borel σ-algebra $\mathscr{B}(K)$ of K with values in $\mathbb{R}_+$) is said to be *regular* if it is regular in the sense of Hewitt and Ross (1979) (and many other authors), that is the open Borel sets are inner regular, and every Borel set is outer regular. Recall that given a non-negative Borel measure ν, a set $A \in \mathscr{B}(K)$ is called *inner regular* if

$$\nu(A) = \sup\{\nu(C) :\ C \subset A,\, C \text{ compact}\},$$

and *outer regular* if

$$\nu(A) = \inf\{\nu(U) : U \supset A, U \text{ open}\}.$$

If ν is regular in the sense above, then each Borel set $A \in \mathscr{B}(K)$ with $\nu(A) < \infty$ is inner regular.

If $m \in M_+(K)$ then we are in the situation of the classical Riesz representation theorem and the (unique) extension of m° to all Borel sets is the so-called *principal measure*. There is also a unique non-negative Borel measure $m_\circ$, called the *essential measure*, which satisfies (2.2) and for which every Borel set is inner regular. Both coincide on open and compact sets. Under various circumstances they are equal, see Bauer (1992), §§ 28–29. In the sequel we always choose m° (more generally μ°) and also denote it by m (and μ, respectively).

The space of bounded measures $M^b(K)$ with norm $\|\cdot\|$ as above is isometrically isomorphic to the Banach space of continuous linear functionals on $C_0(K)$, which in turn is isometrically isomorphic by the Riesz representation theorem to the Banach space of (bounded) complex Borel measures with the norm of total variation, see Rudin (1987), Chapter 6 for complex measures. Every complex Borel measure μ is regular in the sense that all Borel sets are inner and outer regular with respect to the total variation measure $|\mu|$.

Hypergroup specifics. If not stated differently, $M^b(K)$ will always be endowed with the weak topology $\sigma(M^b(K), C_b(K))$ induced by $C_b(K)$ (however, $M^b(K)$ is in general a proper subset of the dual space of $C_b(K)$, see Elstrodt, 2008, Aufgabe VIII.2.7). The subspaces $M^b_+(K)$ and $M^1(K)$ are also endowed with the weak topology, that is the relative topology induced by $M^b(K)$. The point measure in $t \in K$ is denoted by ε_t.

To clarify notation in H3 and Theorem 2.1.2, given a Borel measure μ we denote by μ^- the measure defined by $\mu^-(A) := \mu(A^-)$ for all Borel sets A where $A^- := \{t^- : t \in A\}$, and given a function f we define f^- by $f^-(t) := f(t^-)$ for all $t \in K$. The complex conjugate of a measure μ is denoted by $\overline{\mu}$, that is $\overline{\mu}(A) = \overline{\mu(A)}$ for all Borel sets A, and given a function f, $\overline{f}$ is defined by $\overline{f}(t) = \overline{f(t)}$ for all $t \in K$.

The collection of non-void compact subsets of K is denoted by $\mathscr{C}(K)$ and is given the *Michael topology*, that is the topology generated by the subbasis of all

$$\mathscr{C}_U(V) := \{C \in \mathscr{C}(K) : C \cap U \neq \varnothing \text{ and } C \subset V\} \tag{2.3}$$

with U and V open subsets of K, which makes $\mathscr{C}(K)$ a locally compact Hausdorff space.

A *neighbourhood* of a point $t \in K$ is by definition any open set containing t. The closure of a subset A of K is denoted by $\mathrm{cl}(A)$.

Let $(K, *)$ be a hypergroup and suppose A and B are subsets of K. Then convolution of subsets is defined by

$$A * B := \bigcup_{t\in A,\, s\in B} \mathrm{supp}(\varepsilon_t * \varepsilon_s).$$

For the following properties about convolution of subsets we refer to Jewett (1975), Subsections 3.2, 4.1, and Bloom and Heyer (1995), pp. 16–17.

Lemma 2.1.3. *If A, B are compact subsets of K then $A * B$ is also compact, and given an open set U containing $A * B$ there exist open sets V and W such that $A \subset V$, $B \subset W$, and $V * W \subset U$.*

Lemma 2.1.4. *Let A, B and C be subsets of K.*

(i) *The set operation $*$ is associative, i. e. $(A * B) * C = A * (B * C)$.*

(ii) $(A * B)^- = B^- * A^-$.

(iii) $(A * B) \cap C \neq \varnothing$ *iff* $(A^- * C) \cap B \neq \varnothing$ *iff* $(C * B^-) \cap A \neq \varnothing$.

Comparing with Jewett (1975), 4.1B we have added to (iii) the second equivalence, which follows from the first one by involution and (ii).

Proposition 2.1.5. *Suppose μ, $\nu \in M^b(K)$ and μ, $\nu \geq 0$. Then*

$$\operatorname{supp}(\mu * \nu) = \operatorname{cl}(\operatorname{supp}(\mu) * \operatorname{supp}(\nu)).$$

If additionally μ and ν have compact support, then so has $\mu * \nu$ and

$$\operatorname{supp}(\mu * \nu) = \operatorname{supp}(\mu) * \operatorname{supp}(\nu).$$

For a proof see Bloom and Heyer (1995), Proposition 1.2.12.

For a (measurable) function f and $t, s \in K$ we set

$$f(t * s) := \int_K f \, \mathrm{d}(\varepsilon_t * \varepsilon_s)$$

whenever this expression makes sense. The *left translate* is defined by

$$(T^t f)(s) := f(t * s)$$

and the *right translate* by

$$(T_t f)(s) := f(s * t).$$

We will use this notation also in the more general context of Banach space valued functions and will show that it is well-defined for locally integrable functions, see Remark 2.2.10.

Definition 2.1.6. A non-zero measure $m \in M_+(K)$ is called *left invariant* or a *left Haar measure* if

$$\int_K T^t f \, \mathrm{d}m = \int_K f \, \mathrm{d}m$$

for all $t \in K$ and $f \in C_c(K)$.

It has long been known that a Haar measure exists for compact and commutative hypergroups. For arbitrary hypergroups this has been a long standing conjecture which was shown only recently by Chapovsky (2012).

In the sequel we always suppose that m is a left Haar measure. Then it follows by involution that m^- is a *right Haar measure*, that is

$$\int_K T_t f \, \mathrm{d}m^- = \int_K f \, \mathrm{d}m^-$$

for all $t \in K$ and $f \in C_c(K)$.

It is known that a left Haar measure is unique up to scaling by a positive constant. Further, it is easily seen that with m also $(m * \varepsilon_t)(f) = \int_K T_t f \, \mathrm{d}m$, $f \in C_c(K)$, is a left invariant measure, hence there exists a positive constant $\Delta(t) > 0$ such that $m * \varepsilon_t = \Delta(t)m$. This defines the *right modular function* $\Delta : K \to \mathbb{R}_+^\times$ which is continuous, satisfies $\Delta\Delta^- = 1$, and gives the relation

$$m = \Delta m^- \tag{2.4}$$

between the left Haar measure m and the right Haar measure m^-. If $\Delta \equiv 1$ then K is called *unimodular*.

It follows from the translation property that m has full support, that is $\mathrm{supp}(m) = K$. We denote by $L^p(K, m)$ the space of p-integrable functions with respect to m with norm $\|\cdot\|_p$, see Appendix A for the general setting of Banach space valued functions.

Definition 2.1.7. A locally bounded Borel measurable function $\chi : K \to \mathbb{C}$ is called *multiplicative function* if

(i) $\chi(e) = 1$,

(ii) $\chi(t)\chi(s) = \chi(t * s)$ for all $t, s \in K$,

(iii) $\lim_{t \to e} \chi(t) = 1$.

If in addition $\chi(t^-) = \overline{\chi(t)}$ for all $t \in K$ then χ is called a *semicharacter*. A bounded semicharacter is called a *character*.

Proposition 2.1.8. *Every multiplicative function is continuous.*

We postpone the proof, since later on we will show an operator-valued generalization, see Theorem 3.2.6.

If condition (iii) is omitted, as done by Bloom and Heyer (1995), then even characters may be discontinuous. For an example, consider the Bessel-Kingman hypergroup $K = \mathbb{R}_+$ (see Example 4.5.2) and the function χ which is equal to 1 for $t = e = 0$ and 0 otherwise (see Bloom and Heyer, 1995, p. 47). To guarantee continuity, Bloom and Heyer (1995), Proposition 1.4.33 considers instead of (iii) the additional condition that χ is *not locally null*, that is there exist $\varepsilon > 0$ and a compact set $C \subset K$ with $m(C) > 0$ such that $|\chi(t)| > \varepsilon$ for all $t \in C$. A slight modification of its proof shows that the restriction to semicharacters is not necessary, see Theorem 3.2.10 for the operator-valued analogue.

We continue with a collection of some preliminaries about commutative hypergroups, see Bloom and Heyer (1995), Section 2.2. Let K be a commutative hypergroup. In this case, the set of characters is denoted by $\widehat{K}$ and given the compact-open topology (that is the topology of uniform convergence on compact subsets of K), which makes it a locally compact Hausdorff space, called the *dual space*. For $\mu \in M^b(K)$ the *Fourier-Stieltjes transform* $\hat{\mu}$ is defined on $\widehat{K}$ by

$$\hat{\mu}(\chi) := \int_K \overline{\chi}\,\mathrm{d}\mu.$$

One can show that there exists a *Plancherel measure*, that is the unique $\pi \in M_+(\widehat{K})$ such that

$$\int_K |f|^2\,\mathrm{d}m = \int_{\widehat{K}} |\hat{f}|^2\,\mathrm{d}\pi$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. In contrast to the situation for locally compact Abelian groups, $\operatorname{supp}(\pi)$ is in general a proper subset of $\widehat{K}$.

Theorem 2.1.9 (Uniqueness)**.** *Let K be a commutative hypergroup. Suppose μ, $\nu \in M^b(K)$ and $\hat{\mu}(\chi) = \hat{\nu}(\chi)$ for all $\chi \in \operatorname{supp}(\pi)$. Then $\mu = \nu$.*

In other words, Theorem 2.1.9 states that the Fourier transform is injective on $M^b(K)$.

The following observation is taken from Lasser (2016) where it is used in the proof of Theorem 2.1.11 below. For strong hypergroups the assertion can be deduced from the Stone-Weierstrass theorem, see Bloom and Heyer (1995), Theorem 2.4.5.

Theorem 2.1.10. *Let F be a compact subset of a commutative hypergroup K. Then $\operatorname{Res}_F \widehat{K}$, the set of all characters restricted to F, spans a uniformly dense subspace of $C(F)$.*

Proof. Let $T(F)$ denote the linear span of the functions in $\operatorname{Res}_F \widehat{K}$, and $\operatorname{cl}_{\|\cdot\|_\infty} T(F)$ its closure in $C(F)$ with respect to the uniform norm $\|\cdot\|_\infty$.

Assume $f \in C(F) \setminus \operatorname{cl}_{\|\cdot\|_\infty} T(F)$. The Riesz representation theorem states that $M^b(F)$ endowed with the norm of total variation is the Banach space dual of $C(F)$ (see, e.g., Rudin, 1987, Theorem 6.19). By the Hahn-Banach theorem there exists $\nu \in M^b(F)$ such that $\nu(f) \neq 0$ and $\nu(\chi) = 0$ for all $\chi \in \widehat{K}$. Identifying ν with its trivial extension to a complex measure on K, Theorem 2.1.9 yields $\nu = 0$ which contradicts $\nu(f) \neq 0$. □

We have seen that to each commutative hypergroup corresponds a set of characters. In some sense also the converse is true. The following theorem can be traced back to Wolfenstetter (1984), Satz 2.1.1, see Lasser (1983), Proposition 1 for a dual space version. We follow the strengthened version in Lasser (2016).

Theorem 2.1.11. *Let K be a non-void locally compact Hausdorff space and $S \subset C_b(K)$ a family of continuous and bounded functions which satisfies the uniqueness property from Theorem 2.1.9, that is given $\mu \in M^b(K)$ with $\int_K \chi\,\mathrm{d}\mu = 0$ for all $\chi \in S$ it is $\mu = 0$. Further, suppose that the following conditions hold.*

F1 *For each* $t, s \in K$ *there exists a measure in* $M^1(K)$ *denoted by* $\varepsilon_t * \varepsilon_s$ *such that*

$$\chi(t)\chi(s) = \int_K \chi \, \mathrm{d}(\varepsilon_t * \varepsilon_s)$$

for all $\chi \in S$.

F2 *There exists* $e \in K$ *such that* $\chi(e) = 1$ *for all* $\chi \in S$.

F3 *There exists a homeomorphism* $^-$ *on* K *such that for any* $t, s \in K$ *it is* $e \in \operatorname{supp}(\varepsilon_t * \varepsilon_s)$ *if and only if* $t = s^-$, *and* $\chi(t^-) = \overline{\chi(t)}$ *for all* $\chi \in S$.

F4 *For every* t, $s \in K$, $\operatorname{supp}(\varepsilon_t * \varepsilon_s)$ *is compact, and the mapping* $(t, s) \to \operatorname{supp}(\varepsilon_t * \varepsilon_s)$ *of* $K \times K$ *into* $\mathscr{C}(K)$ *is continuous.*

Then K *furnished with* $*$ *as convolution,* $^-$ *as involution, and* e *as unit element, is a commutative hypergroup and* $S \subset \widehat{K}$.

Proof. Suppose the prerequisites are satisfied. First of all, the mapping $* : K \times K \to M^1(K)$ defined by $(t, s) \mapsto \varepsilon_t * \varepsilon_s$ is weakly continuous, that is $(t, s) \mapsto (\varepsilon_t * \varepsilon_s)(f)$ is continuous for each $f \in C_b(K)$. Indeed, by F4 it suffices to consider $f \in C_c(K)$ and by Theorem 2.1.10 (its proof only depends on the uniqueness property) it is then enough to consider finite linear combinations of functions in S, but for those continuity is clear by F1. So $*$ has a positive-continuous extension $* : M^b(K) \times M^b(K) \to M^b(K)$ defined as in (2.1).

Suppose $\mu, \nu, \eta \in M^b(K)$. By definition of $*$ and F1 we have

$$(\mu * \nu)(\chi) = \int_K \chi \, \mathrm{d}\mu \int_K \chi \, \mathrm{d}\nu$$

for any $\chi \in S$. Using this identity gives

$$\begin{aligned} ((\mu * \nu) * \eta)(\chi) &= (\int_K \chi \, \mathrm{d}\mu \int_K \chi \, \mathrm{d}\nu) \int_K \chi \, \mathrm{d}\eta \\ &= \int_K \chi \, \mathrm{d}\mu (\int_K \chi \, \mathrm{d}\nu \int_K \chi \, \mathrm{d}\eta) = (\mu * (\nu * \eta))(\chi) \end{aligned}$$

for all $\chi \in S$, thus $(\mu * \nu) * \eta = \mu * (\nu * \eta)$ by the uniqueness property. Analogously one shows all remaining properties using $\chi(e) = 1$ for all $\chi \in S$, and given $t \in K$ $\chi(t^-) = \overline{\chi(t)}$ for all $\chi \in S$. □

2.2 Translations and Convolutions of Banach Space Valued Functions

Let $(K, *)$ be a hypergroup with left Haar measure m. In this section we extend some results for scalar-valued functions of Jewett (1975) (see also Bloom and Heyer, 1995) to Banach space valued functions. These will be needed in the sequel, particularly in the proof of Theorem 3.2.6. For some preliminaries on integration in Banach spaces see Appendix A.

We begin with Banach space valued continuous functions. The notations $C(K, X)$, $C_c(K, X)$, and $C_0(K, X)$ are self-explanatory (cf. Section 2.1).

Lemma 2.2.1. *Suppose $f \in C(K,X)$. Then given a compact set C and an open set U with $C \subset U \subset K$ there exists $g \in C_c(K,X)$ with $\|g(r)\|_X \leq \|f(r)\|_X$ for all $r \in K$, $g = f$ on C and $g = 0$ on $K \backslash U$.*

Proof. Urysohn's lemma (see e. g. Bauer, 1992, Korollar 27.3) gives a continuous function $0 \leq \varphi \leq 1$ with $\varphi = 1$ on C and support contained in U; set $g = \varphi \cdot f$. □

Lemma 2.2.2. *Suppose $\mu_\iota, \mu \in M^b(K)$, $\tau_v - \lim_\iota \mu_\iota = \mu$ vaguely, that is $\lim_\iota \int_K \varphi \,\mathrm{d}\mu_\iota = \int_K \varphi \,\mathrm{d}\mu$ for each $\varphi \in C_c(K)$, and $\limsup_\iota \|\mu_\iota\| < \infty$. Then for each $f \in C_c(K,X)$*

$$\lim_\iota \int_K f \,\mathrm{d}\mu_\iota = \int_K f \,\mathrm{d}\mu. \tag{2.5}$$

Proof. Suppose the prerequisites are satisfied. Take $f \in C_c(K,X)$ and set $C = \mathrm{supp}(f)$. Then $f(C)$ is compact in X. Given $\varepsilon > 0$ there exist $x_1, \ldots, x_n \in f(C)$, $n \in \mathbb{N}$ such that the open balls $(B_\varepsilon(x_i))$ (with centers x_i and radii ε) form an open cover of $f(C)$. Then $(f^{-1}(B_\varepsilon(x_i)))$ is an open cover of C. Let (φ_i) be a corresponding partition of unity (see Bauer, 1992, Satz 27.2) and set

$$\tilde{f}(r) := \sum_{i=1}^{n} \varphi_i(r) x_i.$$

Then $\tilde{f} \in C_c(K,X)$ and $\sup_{r \in K} \|f(r) - \tilde{f}(r)\| < \varepsilon$ by construction. The prerequisite of vague convergence gives

$$\lim_\iota \int_K \tilde{f} \,\mathrm{d}\mu_\iota = \int_K \tilde{f} \,\mathrm{d}\mu.$$

Since $\limsup_\iota \|\mu_\iota\| < \infty$ and $\varepsilon > 0$ has been chosen arbitrarily this implies (2.5). □

Proposition 2.2.3. *Suppose $f \in C(K,X)$. Then the mapping*

$$\begin{aligned} K \times K &\to X \\ (t,s) &\mapsto f(t * s) \end{aligned}$$

is continuous.

Proof. Since convolution of two relatively compact neighbourhoods is contained in a compact set (see Lemma 2.1.3), we may assume without loss of generality $f \in C_c(K,X)$ by Lemma 2.2.1. Axiom H1 states that the mapping $(t,s) \mapsto \varepsilon_t * \varepsilon_s$ from $K \times K$ to $M^1(K)$ is vaguely continuous, hence the assertion follows from Lemma 2.2.2. □

Corollary 2.2.4. *Suppose $f \in C_0(K,X)$. Then for any $t \in K$, $T^t f \in C_0(K,X)$, and the mapping*

$$\begin{aligned} K &\to C_0(K,X) \\ t &\mapsto T^t f \end{aligned}$$

is $\|\cdot\|_\infty$-continuous. If $f \in C_c(K,X)$ then $T^t f \in C_c(K,X)$.

The scalar case is contained in Bloom and Heyer (1995), Proposition 1.2.16(iii),(iv), see also the proof of Proposition 1.2.28.

Proof. Suppose $f \in C_0(K, X)$. Then for any $\varepsilon > 0$ there exists a compact set $C \subset K$ such that $\|f\|_X < \varepsilon$ on $K \backslash C$. Take $g \in C_c(K, X)$ as stated in Lemma 2.2.1. Then for all $t, r \in K$

$$\|T^t f(r) - T^t g(r)\|_X = \|T^t(f - g)(r)\|_X \leq \|f - g\|_\infty \leq 2\varepsilon.$$

So without loss of generality we may assume $f \in C_c(K, X)$.

Take $t_0 \in K$ and $\varepsilon > 0$. Note that $T^{t_0} f$ is a continuous function (see Proposition 2.2.3) with compact support $\mathrm{supp}(T^{t_0} f) \subset \{t_0^-\} * \mathrm{supp}(f)$ (see Lemma 2.1.4(iii)).

Let V_0 be a neighbourhood of t_0 with compact closure. Set $C^* = (\mathrm{cl}(V_0))^- * \mathrm{supp}(f)$. Then C^* is compact and $\mathrm{supp}(T^t f) \subset C^*$ for all $t \in V_0$. Choose $r \in C^*$. According to Proposition 2.2.3 there exist neighbourhoods V_r of t_0 and W_r of r such that

$$\|f(t * r') - f(t_0 * r')\| < \varepsilon$$

for all $t \in V_r$ and $r' \in W_r$. Since $(W_r)_{r \in C^*}$ is an open cover of C^* there exist $r_1, \ldots, r_n \in C^*$ such that $C^* \subset \bigcup_{i=1}^n W_{r_i}$. Set $V_{t_0} = V_0 \cap \bigcap_{i=1}^n V_{r_i}$. Then V_{t_0} is a neighbourhood of t_0 and

$$\|f(t * r) - f(t_0 * r)\| < \varepsilon$$

for all $t \in V_{t_0}$ and $r \in K$. □

In the remaining part we consider Bochner integrable functions. Appendix A is presupposed, we begin directly with some special features of integration on topological spaces.

Definition 2.2.5. A function $f : K \to X$ is called *locally m-measurable* if for any $t \in K$ there exists a neighbourhood U of t such that $1_U f$ is m-measurable.

Definition 2.2.6. A *local m-null set* is a subset $N \subset K$ such that every $t \in K$ has a neighbourhood U for which $N \cap U$ is a m-null set. A property $P(t)$ defined for every $t \in K$ is said to hold *locally m-almost everywhere* if it holds outside of a local m-null set. The space of *locally bounded measurable functions* $L^\infty_{loc}(K, m, X)$ is defined as the space of (equivalence classes of) functions $f : K \to X$ such that every $t \in K$ has a neighbourhood U with $1_U f \in L^\infty(K, m, X)$.

Note that a subset of K is a local m-null set iff its intersection with any compact set is a m-null set. The last characterization is used in Hewitt and Ross (1979), Definition (11.26). Analogously, $f \in L^\infty_{loc}(K, m, X)$ implies $1_C f \in L^\infty(K, m, X)$ for any compact set C.

Lemma 2.2.7. *The space $C_c(K, X)$ of continuous, Banach space valued functions with compact support lies dense in $L^p(K, m, X)$ for $1 \leq p < \infty$.*

Proof. Suppose $1 \leq p < \infty$. Then the space of m-step functions lies dense in $L^p(K, m, X)$ (see Proposition A.10). Thus it suffices to show that any indicator function $1_A x$ with $A \in \mathscr{B}(K)$, $m(A) < \infty$ and $x \in X$ may be approximated in $L^p(K, m, X)$ by continuous, compactly supported functions of the form $\phi \cdot x$ with $\phi \in C_c(K)$. This is in fact a scalar assertion, which is easily shown using that sets of finite measure are inner regular (see the discussion in Section 2.1) and Urysohn's lemma (see Bauer, 1992, Korollar 27.3). □

The following proposition is basic for our considerations in Chapter 3.

Proposition 2.2.8. *Suppose* $1 \leq p \leq \infty$ *and* $f \in L^p(K, m, X)$. *Then for any* $t \in K$, $T^t f \in L^p(K, m, X)$, *and*

$$\|T^t f\|_p \leq \|f\|_p. \tag{2.6}$$

Remark 2.2.9. For the construction of $T^t f = f(t * \cdot)$ we use a m-version of f which is $\mathscr{B}(K)$-measurable. The so defined function $T^t f \in L^p(K, m, X)$ is independent of our choice of m-version, which follows a posteriori from (2.6). Thus, given $t \in K$ we associate to each equivalence class of functions $f \in L^p(K, m, X)$ the corresponding equivalence class $T^t f \in L^p(K, m, X)$.

Remark 2.2.10. The spaces $L^p_{loc}(K, m, X)$ are invariant under translation, that is if $f \in L^p_{loc}(K, m, X)$, $t \in K$ then $T^t f \in L^p_{loc}(K, m, X)$. This is an immediate consequence of Proposition 2.2.8 and Lemma 2.2.1 above.

Proof of Proposition 2.2.8. For positive, Borel measurable functions Proposition 2.2.8 is content of 3.3B in Jewett (1975). We use his ideas in Steps 1 and 3 of this proof.

Suppose $1 \leq p \leq \infty$, $f \in L^p(K, m, X)$ and $t \in K$. As indicated above, we fix for the proof a representative $f : K \to X$ which is everywhere defined and $\mathscr{B}(K)$-measurable. Then $\|f\|_X$ is $\mathscr{B}(K)$-measurable.

1. Suppose $1 \leq p < \infty$. Then

$$\|f\|_X(t * r) = \int_K \|f\|_X \cdot 1 \, \mathrm{d}(\varepsilon_t * \varepsilon_r) \leq \left(\int_K \|f\|_X^p \, \mathrm{d}(\varepsilon_t * \varepsilon_r) \right)^{\frac{1}{p}}$$

for all $r \in K$ according to Hölder's inequality (with possibly ∞ on both sides). Since $\|f\|_X$ is $\mathscr{B}(K)$-measurable the same holds true for the translation $r \mapsto \|f\|_X(t * r)$ (see Jewett, 1975, 3.1D). The mapping $r \mapsto \|f\|_X(t * r)$ is σ-finite and

$$\int_K (\|f\|_X(t * r))^p \, m(\mathrm{d}r) \leq \int_K \int_K \|f\|_X^p \, \mathrm{d}(\varepsilon_t * \varepsilon_r) \, m(\mathrm{d}r) = \int_K \|f\|_X^p \, \mathrm{d}m < \infty \tag{2.7}$$

since m is a left Haar measure on K, see Jewett (1975), 3.3F. It follows that there exists a m-null set $N \in \mathscr{B}(K)$ such that $\|f\|_X(t * r) < \infty$ for all $r \in K \backslash N$. Thus $f(t * r)$ exists for all $r \in K \backslash N$ by Definition A.8.

2. We show that the m-almost everywhere defined function $f(t * \cdot)$ is strongly m-measurable. Therefore we set $f(t * \cdot)$ to zero on N and show that the so defined m-version $f(t * \cdot) : K \to X$ is strongly $\mathscr{B}(K)$-measurable. This is done by the Pettis measurability theorem, see Theorem A.6. Since f is strongly $\mathscr{B}(K)$-measurable by assumption its range is contained in a separable closed linear subspace Y of X and for any $x^* \in X^*$ the mapping $x^*(f)$ is $\mathscr{B}(K)$-measurable, so is $\mathcal{J}x^*(f)$ where $\mathcal{J}$ denotes the operation of taking positive/negative real or imaginary part.

For any $x^* \in X^*$ it is $x^*(f(t*\cdot)) = (x^*f)(t*\cdot)$ (and $\mathcal{J}x^*(f(t*\cdot)) = (\mathcal{J}x^*f)(t*\cdot)$ respectively) on $K\backslash N$ and $x^*(f(t*\cdot)) = 0$ on N. Since the translates of positive Borel measurable functions are Borel measurable (see Jewett, 1975, 3.1D), we deduce that $x^*(f(t*\cdot))$ is $\mathscr{B}(K)$-measurable. Thus the Pettis measurability theorem finishes the proof of strong $\mathscr{B}(K)$-measurability of $f(t*\cdot)$.

3. Now in the case $1 \le p < \infty$ (2.7) yields

$$\int_K \|f(t*\cdot)\|_X^p \,\mathrm{dm} \le \int_K (\|f\|_X(t*r))^p \; m(\mathrm{d}r) \le \int_K \|f\|_X^p \,\mathrm{dm},$$

i. e. $\|f(r*\cdot)\|_p \le \|f\|_p$ as desired.

Suppose $p = \infty$. Then $f = g+h$ with $g := f\,1_{\{\|f\|_X \le \|f\|_\infty\}}$ and $h := f\,1_{\{\|f\|_X > \|f\|_\infty\}}$ strongly $\mathscr{B}(K)$-measurable. It is $h = 0$ m-almost everywhere and

$$\int_K \|h\|_X(t*r)\, m(\mathrm{d}r) = \int_K \|h\|_X \,\mathrm{dm} = 0,$$

thus $\|h\|_X(t*r) = 0$ m-almost everywhere. Consequently,

$$\|f\|_X(t*r) \le \|g\|_X(t*r) \le \|g\|_\infty \le \|f\|_\infty$$

m-almost everywhere. The m-almost everywhere defined function $f(t*\cdot)$ is strongly m-measurable (see Step 2) and we conclude $\|f(t*\cdot)\|_\infty \le \|f\|_\infty$.

□

Corollary 2.2.11. *Suppose* $1 \le p < \infty$ *and* $f \in L^p(K,m,X)$. *Then the mapping*

$$\begin{aligned} K &\to L^p(K,m,X) \\ t &\mapsto T^t f \end{aligned}$$

is continuous.

Proof. Suppose $1 \le p < \infty$, $f \in L^p(K,m,X)$, $t_0 \in K$, and $\varepsilon > 0$. First of all there exists a function $g \in C_c(K,X)$ such that $\|f-g\|_p < \frac{\varepsilon}{3}$, see Lemma 2.2.7. Proposition 2.2.8 states that

$$\|T^t(f-g)\|_p \le \|f-g\|_p < \frac{\varepsilon}{3} \tag{2.8}$$

for every $t \in K$. Let V be a relatively compact neighbourhood of t_0. Then $\mathrm{supp}(T^t g)$ is contained in $C = \mathrm{cl}(V^-)*\mathrm{supp}(g)$ and $\|T^t g - T^{t_0} g\|_p \le m(C)^{\frac{1}{p}} \|T^t g - T^{t_0} g\|_\infty$ for all $t \in V$. Corollary 2.2.4 gives a neighbourhood W of t_0 such that $m(C)^{\frac{1}{p}}\|T^t g - T^{t_0} g\|_\infty < \frac{\varepsilon}{3}$ for all $r \in W$. Thus

$$\|T^t g - T^{t_0} g\|_p < \frac{\varepsilon}{3} \tag{2.9}$$

for all $t \in V \cap W$. So (2.8) and (2.9) yield

$$\|T^t f - T^{t_0} f\|_p < \varepsilon$$

for all $r \in V \cap W$. □

Suppose X, Y, and Z are Banach spaces, and X operates on Y in the sense that $X \times Y \to Z$ is a bilinear continuous mapping such that $\|xy\|_Z \leq \|x\|_X \|y\|_Y$.

The following generalization of Hölder's inequality can be found in Dinculeanu (2002), see 2.3.36, or Dinculeanu (1966), p. 221, Corollary 1.

Proposition 2.2.12 (Hölder's inequality)**.** *Suppose $p, q \in [1, \infty]$ are conjugate numbers, i. e. $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(K, m, X)$, $g \in L^q(K, m, Y)$. Then $f \cdot g \in L^1(K, m, Z)$ where*

$$(f \cdot g)(t) := f(t)g(t) \in Z$$

and

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. Suppose f and g as stated. It is routine to show that $f \cdot g$ is m-measurable. Thus the Hölder inequality for scalar-valued functions yields

$$\|f \cdot g\|_1 = \int_K \|fg\|_Z \,\mathrm{d}m \leq \int_K \|f\|_X \|g\|_Y \,\mathrm{d}m \leq \|f\|_p \|g\|_q < \infty.$$

□

Proposition 2.2.13 (Young type inequality)**.** *Suppose $p, q \in [1, \infty]$ are conjugate numbers, $f \in L^p(K, m, X)$, $g \in L^q(K, m, Y)$. Then for any $t \in K$*

$$(f * g^-)(t) := \int_K f(t * r)g(r)\, m(\mathrm{d}r) \in Z \tag{2.10}$$

is well-defined and

$$\sup_{t \in K} \|(f * g^-)(t)\|_Z \leq \|f\|_p \|g\|_q. \tag{2.11}$$

Proof. This is clear in view of Propositions 2.2.8 and 2.2.12. □

Theorem 2.2.14. *Suppose $p, q \in [1, \infty]$ are conjugate numbers and $f \in L^p(K, m, X)$, $g \in L^q(K, m, Y)$. Then for any $t \in K$*

$$\int_K T^t f(r) g(r)\, m(\mathrm{d}r) = \int_K f(r) T^{t^-} g(r)\, m(\mathrm{d}r). \tag{2.12}$$

Proof. For non-negative, Borel measurable functions f and g such that either f or g is σ-finite with respect to m, formula (2.12) is content of Theorem 5.1D in Jewett (1975). It is transferred to the Banach space valued case by approximation.

Fix an arbitrary $t \in K$. Then (2.12) is clear for f and g m-step functions taking only one value in X and Y except zero, respectively; so it holds for all m-step functions f and g by linearity. As in Proposition 2.2.13 it is shown that the left and the right hand side of (2.12) are bilinear continuous mappings from $L^p(K,m,X) \times L^q(K,m,Y)$ to Z.

Thus if $1 < p,q < \infty$ use that m-step functions are dense in $L^p(K,m,X)$ and $L^q(K,m,Y)$, respectively.

So assume $p = 1$ and $q = \infty$ (the case $p = \infty$ and $q = 1$ is symmetric). Since the m-step functions lie dense in $L^1(K,m,X)$ it follows that (2.12) holds for $f \in L^1(K,m,X)$ and g an m-step function. Let $f \in L^1(K,m,X)$ and $g \in L^\infty(K,m,Y)$ be $\mathscr{B}(K)$-measurable representatives and suppose g is bounded on K. Then $\|f\|_X$ is $\mathscr{B}(K)$-measurable and σ-finite (see Proposition A.10). Thus $F = \{y \in K : \|f\|(t * r) > 0\} \in \mathscr{B}(K)$ is σ-finite (see 3.3F in Jewett, 1975). According to Theorem 5.1D in Jewett (1975)

$$\int_K \|f\|_X(t*r)h(r)\,m(\mathrm{d}r) = \int_K \|f\|_X(r)h(t^- * r)\,m(\mathrm{d}r)$$

for $h = \|g\|_Y$, $h = \|1_F g\|_Y$ and $h = \|1_{K\setminus F} g\|_Y$. Since $\|g\|_Y = \|1_F g\|_Y + \|1_{K\setminus F} g\|_Y$ this implies

$$\int_K f(t*r)g(r)\,m(\mathrm{d}r) = \int_K f(t*r)(1_F g)(r)\,m(\mathrm{d}r)$$

and

$$\int_K f(r)g(t^- * r)\,m(\mathrm{d}r) = \int_K f(r)(1_F g)(t^- * r)\,m(\mathrm{d}r),$$

that is, we may assume additionally that g is σ-finite. Then there is a sequence of m-step functions (g_n) such that $\|g_n\|_Y \le \|g\|_Y$ for all $n \in \mathbb{N}$ and $g_n \to g$ pointwise on K (see Theorem A.2). The dominated convergence theorem yields $g_n(t^- * \cdot) \to g(t^- * \cdot)$ pointwise on K. Further applications of the dominated convergence theorem to the left and the right hand side of (2.12) complete the proof. □

Theorem 2.2.15. *Suppose $p,q \in [1,\infty]$ are conjugate numbers and $f \in L^p(K,m,X)$, $g \in L^q(K,m,Y)$. Then $f * g^- \in C_b(K,Z)$. If $1 < p < \infty$ then $f * g^- \in C_0(K,Z)$.*

The scalar version of this theorem is stated in Jewett (1975), 5.5D and 5.5P.

Proof. *1.* Take an arbitrary $t_0 \in K$. Then for all $t \in K$

$$\|(f*g^-)(t) - (f*g^-)(t_0)\|_Z \le \|T^t f - T^{t_0} f\|_p \|g\|_q \tag{2.13}$$

by Hölder's inequality and analogously with Theorem 2.2.14

$$\|(f*g^-)(t) - (f*g^-)(t_0)\|_Z \le \|f\|_p \|T^{t^-} g - T^{t_0^-} g\|_q. \tag{2.14}$$

Thus continuity of $f*g^-$ follows from Corollary 2.2.11 and (2.13) in case of $1 \le p < \infty$ and from (2.14) in case of $p = \infty$. Clearly, $f * g^-$ is bounded by Proposition 2.2.8.

2. Note that convolution is a continuous bilinear mapping from $L^p(K,m,X) \times L^q(K,m,Y)$ to $C_b(K,Z)$ by Step 1 and (2.11). Suppose $1 < p < \infty$. Then f and g may be approximated by continuous functions with compact support (see Lemma 2.2.7). Further, the convolution of two continuous functions with compact support has compact support by Lemma 2.1.4(iii). Thus $f * g^- \in C_0(K,Z)$.

□

The following definition is a modification of Definition 1.4.25 in Bloom and Heyer (1995). Recall that $L^\infty_{loc}(K, m, X)$ denotes the space of (equivalence classes of) locally bounded measurable X-valued functions (see Definition 2.2.6).

Definition 2.2.16. A function $f \in L^\infty_{loc}(K, m, X)$ is called *right locally m-uniformly continuous* at $t_0 \in K$ if there exists a neighbourhood U of t_0 such that for every $\varepsilon > 0$ there exists a neighbourhood V of the identity e such that for all $t \in U$

$$\|f(t * r) - f(t)\|_X < \varepsilon$$

for m-almost all $r \in V$.

Note that this notion is well-defined in context of Remark 2.2.10. Without loss of generality one may assume that U and V are relatively compact; the convolution of compact sets is compact.

Theorem 2.2.17. *Suppose $f \in L^\infty_{loc}(K, m, X)$ is right locally m-uniformly continuous at $t_0 \in K$. Then f is continuous in a neighbourhood of t_0.*

Proof. This is a Banach space valued variant of Corollary 1.4.28 in Bloom and Heyer (1995). Choose U as stated in Definition 2.2.16 such that its closure is compact. Take an arbitrary $\varepsilon > 0$. Then there exists a corresponding neighbourhood V of e such that V is symmetric (i. e. $V = V^-$) and relatively compact. Set $g = 1_C f$ where $C = \mathrm{cl}(U) * \mathrm{cl}(V)$ is compact and $k = m(V)^{-1} 1_V$ ($m(V) > 0$ since $\mathrm{supp}(m) = K$).

Then $g \in L^\infty(K, m, X)$ and $k = k^- \in L^1(K, m, \mathbb{C})$, so $g * k \in C_b(K, X)$ by Theorem 2.2.15. Further for all $t \in U$

$$\|(g * k)(t) - f(t)\|_X = \left\| \int_K f(t * r) k(r^-) \, m(\mathrm{d}r) - f(t) \int_K k(r) \, m(\mathrm{d}r) \right\|_X$$

$$= \left\| \int_K (f(t * r) - f(t)) k(r) \, m(\mathrm{d}r) \right\|_X \leq \int_K \|f(t * r) - f(t)\|_X k(r) \, m(\mathrm{d}r) < \varepsilon$$

since f is right locally m-uniformly continuous. Thus f is continuous on U. □

Chapter 3

Multiplicative Operator Functions

In the present chapter we introduce the central notion of a multiplicative operator function in the general setting of arbitrary hypergroups. This is motivated by a brief outline of some basic ideas of representation theory. Several continuity theorems are shown. Examples of multiplicative operator functions are provided by translation operators on homogeneous Banach spaces. Finally, we show that multiplicative operator functions on commutative hypergroups with associated integral equation solve abstract Cauchy problems. This is deepened in Chapter 5 in the Sturm-Liouville setting.

3.1 Definition in the Framework of Representation Theory

Representation theory for locally compact groups is a large field. For an introduction we refer to Hewitt and Ross (1979), Sections 21 and 22, and Lyubich (1988).

Definition 3.1.1. A *representation* of a locally compact group G is a mapping $T : G \to \mathcal{L}(X)$ such that

(i) $T(e) = I$,

(ii) $T(t)T(s) = T(ts)$ for all $t, s \in G$,

(iii) for each $x \in X$, $x^* \in X^*$ the mapping $t \mapsto x^*T(t)x$ is continuous.

Theorem 3.1.2. *Every representation of a locally compact group is strongly continuous, that is for each $x \in X$ the mapping $t \mapsto T(t)x$ is continuous.*

This theorem was first published by de Leeuw and Glicksberg (1965), Theorem 2.8, see Lyubich (1988), pp. 89–90. In particular, Definition 3.1.1 coincides with the common definition in Lyubich (1988).

For hypergroups, representation theory was initiated by Jewett (1975), Subsection 11.3.

We only present the definition of a hypergroup representation together with some basic properties, see Bloom and Heyer (1995), Section 2.1 for a further development of harmonic analysis upon this definition.

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{L}(H)$ the Banach $\sim$-algebra of bounded linear operators on H. Recall that $M^b(K)$ is a Banach $\sim$-algebra with $\sim$ as in Theorem 2.1.2.

Definition 3.1.3. A *representation* of a hypergroup K on H is a mapping $D : M^b(K) \to \mathcal{L}(H)$ such that

(i) $D(\varepsilon_e) = I$,

(ii) D is a $\sim$-representation of the Banach $\sim$-algebra $M^b(K)$,

(iii) for each $x, y \in H$, $\mu \mapsto \langle D(\mu)x, y\rangle$ is continuous on $M^b_+(K)$ with respect to the weak topology (recall that this is the relative topology on $M^b_+(K)$ induced by the weak topology $\sigma(M^b(K), C_b(K))$).

It is known that $\|D(\mu)\| \leq \|\mu\|$ for all $\mu \in M^b(K)$. The following continuity property is easily derived from the definition of a hypergroup representation, see Jewett (1975), 11.3B.

Theorem 3.1.4. *Every representation of a hypergroup is strongly continuous in the sense that for each $x \in H$ the mapping $\mu \mapsto D(\mu)x$, $M^b_+(K) \to H$ is continuous where $M^b_+(K)$ bears the weak topology.*

For abbreviation we write $D(t)$ for $D(\varepsilon_t)$. Suppose $x \in H$. We know from above that the mapping $t \mapsto D(t)x$, $K \to H$ is continuous and $\|D(t)\| \leq 1$ for all $t \in K$. This leads us to the observation that

$$D(\mu)x = \int_K D(t)x\,\mu(\mathrm{d}t) \tag{3.1}$$

for all $\mu \in M^b(K)$ (cf. Theorems (22.3) and (22.5) in Hewitt and Ross, 1979). Indeed, first of all, the right hand side exists and the identity holds for finitely supported μ by linearity. Next, given $\mu \in M^b_+(K)$ there exists a net (μ_ι) of finitely supported measures in $M^b_+(K)$ which converges weakly to μ (see Jewett, 1975, Lemma 2.2A). Hence by Definition 3.1.3(iii) for each $y \in H$

$$\begin{aligned} \langle D(\mu)x, y\rangle &= \lim_\iota \langle D(\mu_\iota)x, y\rangle = \lim_\iota \langle \int_K D(t)x\,\mu_\iota(\mathrm{d}t), y\rangle \\ &= \lim_\iota \int_K \langle D(t)x, y\rangle\,\mu_\iota(\mathrm{d}t) = \int_K \langle D(t)x, y\rangle\,\mu(\mathrm{d}t) = \langle \int_K D(t)x\,\mu(\mathrm{d}t), y\rangle, \end{aligned}$$

so (3.1) holds for all $\mu \in M^b_+(K)$. Finally, it holds for all $\mu \in M^b(K)$ by linearity.

It can be seen from (3.1) that the mapping $\mu \mapsto D(\mu)$ from $M^b(K)$ to $\mathcal{L}(H)$, both endowed with norm topology, is continuous. This is a general property of $\sim$ representations of Banach $\sim$-algebras, see Hewitt and Ross (1979), Theorem (21.22).

Inspired by the preceding glimpse into representation theory, we introduce now the central notion of a “multiplicative operator function”. The prerequisites of the definition are chosen strong enough to guarantee strong continuity. They are very similar to those imposed on groups in Hewitt and Ross (1979), Theorem (22.8). However, due to the different structure of a hypergroup, involution is not involved. This makes the notion of a multiplicative operator function much more general than the notion of a hypergroup representation, as we will see in Theorem 3.1.7.

Definition 3.1.5. Suppose $(K, *)$ is a hypergroup with Haar measure m. A function $S : K \to \mathcal{L}(X)$ is called *multiplicative operator function* if the following conditions are satisfied.

(i) $S(e) = I$.

(ii) For any $x \in X$ it is $S(\cdot)x \in L^{\infty}_{loc}(K, m, X)$ and for all $t \in K$

$$S(t)S(s)x = S(t * s)x$$

for locally m-almost every $s \in K$.

(iii) For any $x \in X$, $x^* \in X^*$ there exists a local m-null set N^* such that

$$\lim_{\substack{t \to e \\ t \notin N^*}} x^*(S(t)x) = x^*(x).$$

A few words about this definition. First of all $S(t * s)x = (S(\cdot)x)(t * s)$ is well-defined by Remarks 2.2.9 and 2.2.10, for the definitions of “local m-null set”, “locally m-almost everywhere”, and $L^{\infty}_{loc}(K, m, X)$ see Definition 2.2.6 and Appendix A. These technicalities turn out to be useful in the succeeding Section 3.2, see in particular Remark 3.2.7 and Lemma 3.2.9.

We point out that the notion of a multiplicative operator function does not depend on whether we consider the left Haar measure m or the right Haar measure m^- since involution is a homeomorphism and null sets are preserved (as can be seen from (2.4)).

Further, we remark that the measurability condition in Definition 3.1.5(ii) is satisfied if K has the property that each point has a neighbourhood which is second-countable and for each $x \in X$, $x^* \in X^*$ the mapping $x^*S(\cdot)x$ is continuous, see the note in Hille and Phillips (1957) following Corollary 2 of Theorem 3.5.3 on page 73 for real intervals. If X is separable it is equivalent to the condition that for each $x \in X$, $x^* \in X^*$ the mapping $x^*S(\cdot)x : K \to \mathbb{C}$ is locally Borel measurable; this is content of the Pettis Measurability Theorem A.6.

As one would expect, if $X = \mathbb{C}$ a multiplicative operator function can be identified with a multiplicative function and vice versa (cf. Definition 2.1.7 and see Theorem 3.1.6 below).

The following characterization of multiplicative operator functions could, for a shortcut, also serve as a definition.

Theorem 3.1.6. *A transformation $S : K \to \mathcal{L}(X)$ is a multiplicative operator function iff $S(\cdot)x$ is continuous for each $x \in X$ and* *(i)* $S(e) = I$,

(ii) $S(t)S(s)x = S(t * s)x$ *for all* $t, s \in K$ *and any* $x \in X$,

(iii) $\lim_{t \to e} S(t)x = x$ *for each* $x \in X$.

We postpone the proof to Section 3.2.

One can immediately see from Theorem 3.1.6 that if K is commutative then $S(t)S(s) = S(s)S(t)$ for all $t, s \in K$. Further, two multiplicative operator functions $S_1 : K \to \mathcal{L}(X)$ and $S_2 : K \to \mathcal{L}(Y)$, X, Y some Banach spaces, can be combined to a multiplicative operator function $\mathcal{S}$ from K to $\mathcal{L}(Z)$, $Z = X \times Y$, defined by

$$\mathcal{S}(t) := \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix}.$$

Of course, the concept of equivalence classes persists, that is given a multiplicative operator function $S : K \to \mathcal{L}(X)$ and any isomorphism $T \in \mathcal{L}(X)$, the mapping $t \mapsto T^{-1}S(t)T$, $K \to \mathcal{L}(X)$ is also a multiplicative operator function.

It is clear by Theorem 3.1.6 and Theorem 3.1.2 that the notion of a multiplicative operator function generalizes the notion of a representation of a locally compact group.

The following theorem shows that it also generalizes the notion of a representation of a hypergroup. So the notion of a multiplicative operator function is in fact very general.

Theorem 3.1.7. *Suppose* D *is a representation of a hypergroup* K. *Then the restriction* $D|_K = D|_{\{\varepsilon_t, t \in K\}}$ *is a multiplicative operator function.*

Conversely, a multiplicative operator function $S : K \to \mathcal{L}(H)$, H *a Hilbert space, is the restriction of a hypergroup representation* D *if and only if* S *is uniformly bounded and* $S(t^-) = S(t)^\sim$ *for all* $t \in K$. *In this case* D *is given by*

$$D(\mu)x = \int_K S(t)x \, \mu(\mathrm{d}t), \quad x \in H \tag{3.2}$$

for all $\mu \in M^b(K)$.

Proof. It only remains to show that given a multiplicative operator function S which is uniformly bounded and satisfies $S(t^-) = S(t)^\sim$ for all $t \in K$ formula (3.2) defines a hypergroup representation. Therefore, suppose $\mu, \nu \in M^b(K)$ and $x \in H$. Then

$$\begin{aligned} D(\mu * \nu)x &= \int_K \int_K \int_K S(u)x \, (\varepsilon_t * \varepsilon_s)(\mathrm{d}u)\mu(\mathrm{d}t)\nu(\mathrm{d}s) \\ &= \int_K \int_K S(t)S(s)x \, \mu(\mathrm{d}t)\nu(\mathrm{d}s) = D(\mu)D(\nu)x \end{aligned}$$

and for each $y \in H$ one can show similarly using $S(t^-) = S(t)^\sim$ that $\langle D(\mu^\sim)x, y\rangle = \overline{\langle x, D(\mu)y\rangle}$, thus $D(\mu^\sim) = D(\mu)^\sim$.

Finally, continuity of the mapping $\mu \mapsto \langle D(\mu)x, y\rangle$ defined on $M_+^b(K)$ follows from $\langle D(\mu)x, y\rangle = \int_K \langle S(t)x, y\rangle \, \mu(\mathrm{d}t)$ and the definition of the weak topology. □

3.2 Strong and Uniform Continuity

First of all, we show that multiplicative operator functions are strongly continuous. Then we conclude that uniform continuity in e implies uniform continuity. Finally, we consider the problem whether measurability of the operator function implies uniform continuity. It turns out, that this is only the case under some additional conditions.

For semigroups of operators and cosine operator functions (on the non-negative real line) one of the first observations is that the C_0-regularity condition implies local uniform boundedness at 0. The following lemma is an immediate generalization of the corresponding idea to the setting of topological spaces with countable bases.

Lemma 3.2.1. *Suppose K is a topological space and $t_0 \in K$ has a countable neighbourhood basis. Let $(T(t))_{t\in K} \subset \mathcal{L}(X, Y)$ be a family of bounded linear operators from a Banach space X to a normed space Y. If for every $x \in X$ there exists a neighbourhood U_x of t_0 and $M_x \geq 0$ such that $\|T(t)x\| \leq M_x$ for all $t \in U_x$ then there exists a neighbourhood U of t_0 and $M \geq 0$ such that $\|T(t)\| \leq M$ for all $t \in U$.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a neighbourhood basis of t_0 with $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. Suppose the assertion does not hold. Then $\sup_{t\in U_n} \|T(t)\| = \infty$ for each $n \in \mathbb{N}$. Choose a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \in U_n$ and $\|T(t_n)\| \geq n$ for all $n \in \mathbb{N}$. Thus, by the Banach-Steinhaus theorem there exists $x \in X$ such that $\sup_{n\in\mathbb{N}} \|T(t_n)x\| = \infty$, which is a contradiction. □

Note that Lemma 3.2.1 is in fact a topological equivalent of the Banach-Steinhaus theorem. To see that Lemma 3.2.1 implies the Banach-Steinhaus theorem put $K = I$ for an arbitrary given set I and give K the indiscrete topology, that is $\varnothing$ and I are the only open sets.

Remark 3.2.2. In the previous lemma, the prerequisite of a countable neighbourhood basis is essential. For illustration, we construct an example where $K = G$ is a compact topological group, X a separable Banach space, $Y = \mathbb{C}$ and $\lim_{g\to e} T(g)x = 0$ for all $x \in X$, but $\sup_{g\in U'} \|T(g)\| = \infty$ for every neighbourhood U' of e. In particular, this example shows that the C_0-regularity condition of a multiplicative operator function is not sufficient in itself for local uniform boundedness in e.

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ and consider the direct product $G = P_{r\in\mathbb{R}}\mathbb{T}_r$ endowed with the Cartesian product topology. It is a topological group and compact by the Tychonoff theorem (see e.g. Hewitt and Ross, 1979, Section 6, pp. 52–53). Choose $X = C_0(\mathbb{R})$ and $Y = \mathbb{C}$. (We remark that $\mathbb{C}$ could be regarded as a subspace of $C_0(\mathbb{R})$ and $\mathcal{L}(C_0(\mathbb{R}), \mathbb{C})$ is isometrically isomorphic to $M^b(\mathbb{R})$.) Let F be a bijection from $\mathbb{R}$ to the set of monotonically increasing sequences of positive numbers $(c_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} c_n = \infty$. Indeed, the existence of such a bijection is a consequence of the Schröder-Bernstein theorem and $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$. Suppose $g \in G$. If $P = \{r \in \mathbb{R} : |g_r - 1| < 1\}$ is finite and if for every $r \in P$ there exists $n_r \in \mathbb{N}$ with $\frac{1}{n_r+1} \leq |g_r - 1| < \frac{1}{n_r}$, then with $n = \max_{r\in P} n_r$ and $c = \min_{r\in P}(F(r))_n$ define $T(g)f := (c\,\varepsilon_n)(f) = c\,f(n)$ for every $f \in C_0(\mathbb{R})$; otherwise set $T(g) = 0$.

Suppose $f \in C_0(\mathbb{R})$ and $\varepsilon > 0$. Then there exists a monotonically increasing sequence of positive numbers $(c_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} c_n f(n) = 0$.

Take $N \in \mathbb{N}$ such that $|c_n f(n)| < \varepsilon$ for all $n \geq N$ and let r be the real number with $F(r) = (c_n)_{n\in\mathbb{N}}$. Then $U = \{g \in G : |g_r - 1| < \frac{1}{N}\}$ is an open neighbourhood of $e = (1)_{r\in\mathbb{R}}$ and $|T(g)f| < \varepsilon$ for all $g \in U$.

Suppose U' is a neighbourhood of e. Then U' contains a finite intersection of sets of the form $\{g \in G : |g_r - 1| < \frac{1}{n}\}$ with $r \in \mathbb{R}$ and $n \in \mathbb{N}$. Denote the finite collection of these $r \in \mathbb{R}$ by P'. Since $\lim_{n\to\infty} \min_{r\in P'}(F(r))_n = \infty$ there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in U' such that $\lim_{n\to\infty} \|T(g_n)\| = \infty$. Thus $\sup_{g\in U'} \|T(g)\| = \infty$.

The following lemma is in some sense an analogue of Lemma 3.2.1 which relies on a measure space instead of a topological space. Its proof is based on the Baire category theorem; a similar application can be found in Arendt et al. (2011), Lemma 3.2.14.

Lemma 3.2.3. *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Suppose $(T(\omega))_{\omega\in\Omega} \subset \mathcal{L}(X, Y)$ is a family of bounded linear operators from a Banach space X to a normed space Y and for each $x \in X$ there exists $M_x \geq 0$ such that $\|T(\omega)x\| \leq M_x$ almost everywhere. Then*

(i) there exists $M_0 \geq 0$ such that for each $x \in X$ with $\|x\| \leq 1$ it is $\|T(\omega)x\| \leq M_0$ almost everywhere.

(ii) If X is separable then $\|T(\omega)\| \leq M_0$ almost everywhere with M_0 from (i).

Proof. Set $A_M = \{x \in X : \|T(\omega)x\| \leq M$ almost everywhere$\}$ for $M \in \mathbb{N}$. It is straightforward to show that A_M is closed. The union of all A_M is equal to X by assumption. Hence, the Baire category theorem yields $M_1 \in \mathbb{N}$ such that A_{M_1} has non-empty interior. Choose $\varepsilon > 0$ and $x_0 \in X$ such that the open ball $B_\varepsilon(x_0)$ (with center x_0 and radius $\varepsilon > 0$) is a subset of A_{M_1}. Then for each $x \in B_\varepsilon(x_0)$ it is $\|T(\omega)(x - x_0)\| \leq 2M_1$ almost everywhere. Thus (i) holds with $M_0 = 2M_1/\varepsilon$.

Suppose X is separable and $(x_n)_{n\in\mathbb{N}}$ is a sequence dense in $B_1(0)$. Then $\|T(\omega)\| = \sup_{n\in\mathbb{N}} \|T(\omega)x_n\|$. Hence, if $M_0 \geq 0$ fulfills (i), then $\|T(\omega)\| \leq M_0$ almost everywhere. □

Remark 3.2.4. In Lemma 3.2.3 part (ii) fails if the assumption of separability is omitted. Consider $\Omega =]0, 1[$ with λ_B, the Lebesgue measure restricted to Borel sets, $X = M^b(]0, 1[)$ with norm, and $Y = \mathbb{C}$. Define $T(\omega)\nu = \frac{1}{\omega}\nu(\{\omega\})$ for $\omega \in]0, 1[$ and any $\nu \in M^b(]0, 1[)$. Then for each $\nu \in M^b(]0, 1[)$, $T(\cdot)\nu$ is zero except for a set which is at most countable, but $\|T(\omega)\| = \frac{1}{\omega}$ for arbitrary $\omega \in]0, 1[$.

We begin with the proof that the weak regularity condition in Definition 3.1.5 can equivalently be replaced by a corresponding strong regularity condition. The idea of this lemma and its proof (see Step 2) are based on a related result for semigroups of operators on the non-negative real line, see Engel and Nagel (2000), Theorem I.5.8, pp. 40–41. Similar techniques are used in the context of representation theory, see e. g. Lyubich (1988), pp. 89–90.

Lemma 3.2.5. *Suppose a function $S : K \to \mathcal{L}(X)$ satisfies condition* (ii) *of Definition 3.1.5. Then for each $x \in X$, weak almost continuity in e, that is for any $x^* \in X^*$ there exists a local m-null set N^* such that*

$$\lim_{\substack{t\to e\\ t\notin N^*}} x^*(S(t)x) = x^*(x), \tag{3.3}$$

implies strong almost continuity in e, that is there exists a local m-null set N such that

$$\lim_{\substack{t\to e\\ t\notin N}} S(t)x = x. \tag{3.4}$$

Proof. Suppose the prerequisites of the theorem are satisfied for $x_0 \in X$. Let W be a relatively compact neighbourhood of e.

1. Note that $1_W S(\cdot)x_0$ is m-measurable by Definition 3.1.5(ii). Thus the Pettis measurability theorem (Theorem A.6) yields a m-null set N_0 such that $\{S(t)x_0,\, t \in W\backslash N_0\} \cup \{x_0\}$ is contained in a separable Banach space $X_0 \subset X$. Further, according to Definition 3.1.5(ii), $(1_W S(\cdot))x \in L^\infty(K, m, X)$ for all $x \in X_0$, thus Lemma 3.2.3(ii) gives a m-null set N and $M > 0$ such that $\|S(t)\|_{\mathcal{L}(X_0,X)} \leq M$ for all $t \in W\backslash N$.

2. Let C be a compact subset of W with $m(C) > 0$. Set

$$x_C := \frac{1}{m(C)} \int_C S(r)x_0 \, m(\mathrm{d}r) \in X_0.$$

Then x_0 is in the weak closure of

$$D := \{x_C : \ C \subset W \text{ compact with } m(C) > 0\}$$

in X_0 by (3.3). For each $x_C \in D$ set $k = \frac{1}{m(C)} 1_C$. Then for all $t \in W$

$$\begin{aligned} S(t)x_C - x_C = S(t) \int_K k(r)S(r)x_0 \, m(\mathrm{d}r) - \int_K k(r)S(r)x_0 \, m(\mathrm{d}r) \\ = \int_K k(r)(S(t)S(r)x_0 - S(r)x_0) \, m(\mathrm{d}r) = \int_K k(r)(S(t*r)x_0 - S(e*r)x_0) \, m(\mathrm{d}r) \\ = \int_K k(r)(g(t*r) - g(e*r)) \, m(\mathrm{d}r) \end{aligned}$$

where $g = 1_{\mathrm{cl}(W)*C} S(\cdot)x_0 \in L^1(K, m, X)$. Hölder's inequality yields

$$\|S(t)x_C - x_C\| \leq \frac{1}{m(C)} \|g(t*\cdot) - g(e*\cdot)\|_1$$

which tends to zero as $t \to e$ by Corollary 2.2.11. So $\lim_{t\to e} S(t)x = x$ for all $x \in \mathrm{conv}(D)$, the convex hull of D. Observe that

$$E := \{x \in X_0 : \lim_{\substack{t\to e\\ t\notin N}} S(t)x = x\}$$

is closed in X_0 since $\|S(\cdot)\|_{\mathcal{L}(X_0,X)}$ is bounded on $W\backslash N$ by Step 1. Thus $\lim_{\substack{t\to e\\ t\notin N}} S(t)x = x$ for all $x \in \mathrm{cl}_{\|\cdot\|_{X_0}}(\mathrm{conv}(D))$. For convex sets in Banach spaces the weak and the norm closure coincide (see, e. g., Rudin, 1991, Theorem 3.12). Consequently, since x_0 is in the weak closure of D in X_0, it belongs to $\mathrm{cl}_{\|\cdot\|_{X_0}}(\mathrm{conv}(D))$. □

Theorem 3.2.6. *Every multiplicative operator function S is strongly continuous.*

Proof. Choose $x_0 \in X$ and $t_0 \in K$. Let U be a relatively compact neighbourhood of t_0 and W a relatively compact neighbourhood of e.

1. According to Step 1 in the proof of Lemma 3.2.5 there exists a m-null set N_0 such that $\{S(t)x_0,\, t \in W\backslash N_0\} \cup \{x_0\}$ is contained in a separable Banach space $X_0 \subset X$. Further, there exists a m-null set N_1 and $M > 0$ such that $\|S(t)\|_{\mathcal{L}(X_0,X)} \leq M$ for all $t \in U\backslash N_1$.

2. Take an arbitrary $\varepsilon > 0$. From (3.4) we get a relatively compact neighbourhood V of e and a m-null set N_e such that $M \cdot \|S(s)x_0 - x_0\| < \varepsilon$ for all $s \in V\backslash N_e$. By Definition 3.1.5(ii) for every $t \in U$ there exists a m-null set N_t such that $S(t)S(s)x_0 = S(t*s)x_0$ for all $s \in V\backslash N_t$. Then for every $t \in U\backslash N_1$ and any $s \in (V\cap W)\backslash(N_0 \cup N_e \cup N_t)$

$$\begin{aligned}\|(S(\cdot)x_0)(t*s) - (S(\cdot)x_0)(t)\| &= \|S(t)S(s)x_0 - S(t)x_0\| \\ = \|S(t)(S(s)x_0 - x_0)\| &\leq \|S(t)\|_{\mathcal{L}(X_0,X)}\|S(s)x_0 - x_0\| \\ &\leq M\|S(s)x_0 - x_0\| < \varepsilon.\end{aligned}$$

The proof of Theorem 2.2.17 shows that there is a sequence of continuous functions from K to X converging to $S(\cdot)x_0$ uniformly on $U\backslash N_1$. Note that $U\backslash N_1$ is dense in U since the Haar measure of any non-empty open set is strictly positive. Thus, there is a continuous function h from U to X such that $h(t) = S(t)x_0$ for all $t \in U\backslash N_1$.

Setting h to zero on $K\backslash U$ the functions h and $1_U S(\cdot)x_0$ are m-versions of each other in $L^\infty(K, m, X)$.

3. Take $t' \in U$. Then by Lemma 2.1.3 there exists a relatively compact neighbourhood V' of e such that $\{t'\} * V' \subset U$. So there exists a m-null set N' such that

$$(S(\cdot)x_0)(t' * s) = h(t' * s)$$

for all $s \in V'\backslash N'$ by Remark 2.2.9. The idea is to take the limit $s \to e$ on both sides. Concerning the left hand side it is shown as before that given $\varepsilon > 0$

$$\|(S(\cdot)x_0)(t' * s) - (S(\cdot)x_0)(t')\| \leq \|S(t')\|_{\mathcal{L}(X_0,X)}\|S(s)x_0 - x_0\| < \varepsilon$$

for m-almost all s in an appropriate neighbourhood of e. The right hand side tends to $h(t')$ as $s \to e$ which is clear using Lemma 2.1.3 once again. Thus $S(t')x_0 = h(t')$ and since $t' \in U$ was chosen arbitrarily $S(\cdot)x_0$ is equal to h on U. □

Remark 3.2.7. Note that we did not use $S(e) = I$ in the proof; so Definition 3.1.5(i) is in fact superfluous since it follows from Theorem 3.2.6. It is important that Definition 3.1.5(ii) holds for all $t \in K$ and not just for locally m-almost every $t \in K$. Otherwise, the notion of a "multiplicative operator function" would in general not depend on local m-null sets excluding e. In particular, Theorem 3.2.6 would fail. Theorem 3.2.6 also requires the regularity condition 3.1.5(iii). These assertions can be seen from the Bessel-Kingman hypergroup, consult the example following Proposition 2.1.8, and Example 5.8.9.

Proof of Theorem 3.1.6. This is a consequence of Theorem 3.2.6. Concerning Theorem 3.1.6(ii), note that the complement of any local m-null set is dense in K and use Proposition 2.2.3. □

Corollary 3.2.8. *Suppose S is a multiplicative operator function. Then $\|S(\cdot)\|$ is locally bounded.*

Proof. Let C be a compact subset of K. Then for any $x \in X$ Theorem 3.2.6 states $S(\cdot)x$ continuous and thus bounded on C. Hence, $\|S(\cdot)\|$ is bounded on C by the Banach-Steinhaus theorem. □

We turn now to uniform continuity. Our starting lemma states that under the assumption of measurability the basic functional equation of Definition 3.1.5(ii) can be read in $\mathcal{L}(X)$.

Lemma 3.2.9. *Suppose a multiplicative operator function $S : K \to \mathcal{L}(X)$ is locally m-measurable. Then for each $t \in K$*

$$S(t)S(s) = S(t * s)$$

for locally m-almost every $s \in K$ where the right hand side is a $\mathcal{L}(X)$-valued Bochner integral.

Proof. Suppose $S : K \to \mathcal{L}(X)$ is a locally m-measurable multiplicative operator function. Then $S \in L^{\infty}_{loc}(K, m, \mathcal{L}(X))$ by Corollary 3.2.8 and thus for each $t \in K$ the $\mathcal{L}(X)$-valued Bochner integral $S(t*s) = \int_K S(r)\,\mathrm{d}(\varepsilon_t * \varepsilon_s)(r)$ exists for locally m-almost every $s \in K$, see Remark 2.2.10. If the last integral exists, then $S(t)S(s) = S(t * s)$ by Theorem 3.1.6(ii) and Hille's theorem (Theorem A.12). □

Let S be a multiplicative operator function. We say S is *somewhere invertible-integrable* if there exists a compact set $C \subset K$ with $m(C) > 0$ such that $S(t)^{-1}$ exists for all $t \in C$ and $1_C S(t)^{-1} \in L^1(K, m, \mathcal{L}(X))$. This is a generalization of the notion "not locally null" in Bloom and Heyer (1995), Proposition 1.4.33. Recall Definition 2.1.7 and the discussion of Proposition 2.1.8.

The following theorem is presented by analogy to Theorem 1.1.17 for cosine operator functions.

Theorem 3.2.10. *Let S be a multiplicative operator function. Then the following conditions are equivalent.*

(i) S is uniformly continuous.

(ii) There exists a local m-null set N such that $\lim_{\substack{t \to e \\ t \notin N}} S(t) = I$ in uniform operator topology.

(iii) $S : K \to \mathcal{L}(X)$ is locally m-measurable and for each $L \in (\mathcal{L}(X))^*$ there exists a local m-null set N^* such that $\lim_{\substack{t \to e \\ t \notin N^*}} L(S(t)) = L(I)$.

(iv) $S : K \to \mathcal{L}(X)$ is locally m-measurable and somewhere invertible-integrable.

Proof. *1.* (ii)⇒(i). Let U be a relatively compact neighbourhood of some point $t_0 \in K$. Then $M = \sup_{t\in U} \|S(t)\| < \infty$ by Corollary 3.2.8. For any $\varepsilon > 0$ there exists a symmetric and relatively compact neighbourhood V of e such that $M\,\|S(t) - I\| < \varepsilon$ for all $t \in V \backslash N$. Thus by Theorem 3.1.6

$$\|(S(\cdot)x)(t * s) - (S(\cdot)x)(t)\| = \|S(t)(S(s)x - x)\| \leq M\,\|S(s)x - x\| < \varepsilon$$

for all $t \in U$, $s \in V\backslash N$ and $x \in X$ with $\|x\| \leq 1$. In other words, $S(\cdot)x$ is right locally m-uniformly continuous, uniformly for $x \in X$ with $\|x\| \leq 1$. This uniform behaviour carries over to the proof of Theorem 2.2.17 and backwards to the proof of continuity of convolution in Theorem 2.2.15, see inequality (2.14) where it is used that $\|S(\cdot)\|$ is bounded on $C = \mathrm{cl}(U) * \mathrm{cl}(V)$ by Corollary 3.2.8. Thus, $S(\cdot)$ is uniformly continuous on U.

2. (iii)⇒(i). Define a mapping $\mathcal{S} : K \to \mathcal{L}(\mathcal{X})$, $\mathcal{X} := \mathcal{L}(X)$ by $\mathcal{S}(t)T = S(t)T$ for all $t \in K$ and $T \in \mathcal{X}$. The trick to deduce a uniform result on some Banach space X from pointwise considerations on $\mathcal{L}(X)$ is also used in the context of Katznelson-Tzafriri type theorems, see Vũ (1992), Theorems 2.2, 3.2 and 4.2, or the survey Vũ (1997).

We show that $\mathcal{S} : K \to \mathcal{L}(\mathcal{X})$ is a multiplicative operator function. Obviously, $\mathcal{S}(e) = I$. Further, $\mathcal{S} \in L^\infty_{loc}(K, m, \mathcal{L}(\mathcal{X}))$ and thus for each $t \in K$ holds $\mathcal{S}(t)\mathcal{S}(s) = \mathcal{S}(t*s)$ for locally m-almost every $s \in K$ by Lemma 3.2.9. By assumption, for each $L \in \mathcal{X}^*$ there exists a local m-null set N^* such that $\lim_{\substack{t\to e\\ t\notin N^*}} L(\mathcal{S}(t)I) = L(I)$. So Lemma 3.2.5 yields a local m-null set N such that $\lim_{\substack{t\to e\\ t\notin N}} \mathcal{S}(t)I = I$ and thus $\lim_{\substack{t\to e\\ t\notin N}} \mathcal{S}(t)T = T$ for all $T \in \mathcal{X}$. In conclusion, $\mathcal{S}(t) : K \to \mathcal{L}(\mathcal{X})$ is a multiplicative operator function. Now Theorem 3.2.6 states that the mapping $S(\cdot) = \mathcal{S}(\cdot)I$ from K to $\mathcal{X}$ is continuous.

3. (iv)⇒(i) is derived from Bloom and Heyer (1995), Proposition 1.4.33.

Suppose $S : K \to \mathcal{L}(X)$ is locally m-measurable and somewhere invertible-integrable. Let $C \subset K$ be a compact set with $m(C) > 0$ such that $S(t)^{-1}$ exists for $t \in C$ and $1_C(t)S(t)^{-1} \in L^1(K, m, \mathcal{L}(X))$. Suppose $t_0 \in K$ and U is a relatively compact neighbourhood of t_0. Then $f = 1_{\mathrm{cl}(U)*C}S \in L^\infty(K, m, \mathcal{L}(X))$ and $g = m(C)^{-1}1_C S^{-1} \in L^1(K, m, \mathcal{L}(X))$. For every $t \in U$

$$\begin{aligned}(f * g^-)(t) &= \int_K f(t*r)g(r)\,m(\mathrm{d}r) = \int_K S(t*r)m(C)^{-1}1_C(r)S(r)^{-1}\,m(\mathrm{d}r)\\ &= \int_K S(t)S(r)m(C)^{-1}1_C(r)S(r)^{-1}\,m(\mathrm{d}r) = S(t).\end{aligned}$$

According to Theorem 2.2.15, $f * g^-$ is a continuous function from K to $\mathcal{L}(X)$.

4. (i)⇒(iv) needs only to check that if $S : K \to \mathcal{L}(X)$ is uniformly continuous then S is somewhere invertible-integrable. In fact, there exists a neighbourhood U of e such that $\|I - S(t)\| < \frac{1}{2}$ for all $t \in U$, thus $S(t)^{-1}$ exists, $t \mapsto S(t)^{-1}$, $K \to \mathcal{L}(X)$ is continuous, and $\|S(t)^{-1}\| \leq 2$ for all $t \in U$. Since open sets are inner regular, we also find a compact set $C \subset U$ with $m(C) > 0$.

□

Note that condition (ii) is always satisfied for all $L \in \{x^*(\,\cdot\, x) : x^* \in X^*,\, x \in X\} \subset \mathcal{L}(X)^*$, which is a norming set for $\mathcal{L}(X)$. However, measurability of $S : K \to \mathcal{L}(X)$ itself is not sufficient for uniform continuity, see Example 5.8.9.

The situation is different if $K = G$ is a locally compact group; here the additional condition "somewhere invertible-integrable" is superfluous since $S(t)^{-1} = S(t^{-1})$.

Corollary 3.2.11. *If X is finite dimensional, then every (matrix-valued) multiplicative operator function $S : K \to \mathcal{L}(X)$ is uniformly continuous.*

Proof. Clearly $\lim_{t\to 0^+} \|S(t) - I\| = 0$ because of $\lim_{t\to 0^+} S(t)x = x$ uniformly for all elements x of a finite basis and equivalence of norms. □

3.3 Translation Operator Functions on Homogeneous Banach Spaces

Elementary but important examples of C_0-groups and cosine operator functions are provided by translations of functions. More generally, we show that translation operator functions on homogeneous Banach spaces are examples of multiplicative operator functions.

First of all, we define the notion of a homogeneous Banach space with respect to an arbitrary hypergroup in the spirit of Katznelson (2004). Our notion is more general than the notion introduced in Fischer and Lasser (2005) for the dual Jacobi polynomial hypergroup. Lasser (2016) considers homogeneous Banach spaces on weak dual structures induced by polynomial hypergroups.

To obtain multiplicative operator functions it turns out to be necessary to consider right translations. We use throughout that several results about left translations can be transferred to right translations by involution (and vice versa). So instead of the left Haar measure m we have to use the right Haar measure m^-.

Definition 3.3.1. A linear subspace $B \subset L^1_{loc}(K, m^-)$ with norm $\|\cdot\|_B$ is called *homogeneous Banach space* if the following conditions are satisfied.

(i) B is complete with respect to $\|\cdot\|_B$ and for each compact set $C \subset K$ there exists $L \geq 0$ such that
$$\| f|_C \|_1 \leq L\|f\|_B$$
for all $f \in B$.

(ii) For each $f \in B$, $t \in K$ it is $T_t f \in B$ and for each compact set $C \subset K$ there exists $M \geq 0$ such that
$$\|T_t f\|_B \leq M\|f\|_B$$
for all $f \in B$ and $t \in C$.

(iii) For each $f \in B$ the mapping $t \mapsto T_t f$, $K \to B$ is continuous.

Remark 3.3.2. A simple generalization of this definition is to allow Banach space valued functions, that is considering $B \subset L^1_{loc}(K, m^-, Y)$ with Y some Banach space; this is used in Example 3.3.9. All subsequent results and proofs concerning homogeneous Banach spaces can immediately be transferred to this more general setting, using the results of Section 2.2 and Appendix A.

The following theorem introduces the aforementioned class of multiplicative operator functions. Its proof relies on Lemma 3.3.6 and will be conducted afterwards.

Theorem 3.3.3. *Let $X = B$ be a homogeneous Banach space. Then $S : t \mapsto T_t$, $K \to \mathcal{L}(B)$ is a multiplicative operator function.*

Definition 3.3.4. We call such a multiplicative operator function a *translation operator function.*

Proposition 3.3.5. *Given $t, s \in K$ and $f \in C_b(K)$ we have for all $u \in K$*

$$(T_t T_s f)(u) = \int_K T_r f(u)\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r).$$

This proposition is a simple consequence of associativity of convolution and is taken from Lasser (2016), Proposition 1.1.8(?).

Lemma 3.3.6. *Suppose $f \in L^1_{loc}(K, m^-)$ and $t, s \in K$. Then*

$$T_t T_s f = \int_K T_r f\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r)$$

in $L^1_{loc}(K)$ where the right hand side is to be read in the sense of distributions, that is

$$\left\langle \int_K T_r f\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r),\, \varphi \right\rangle = \int_K \langle T_r f,\, \varphi\rangle\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r)$$

for all $\varphi \in C_c(K)$ where $\langle \cdot, \varphi\rangle = \int_K \cdot\, \varphi\,\mathrm{d}m^-$.

Proof. Suppose $f \in L^1_{loc}(K, m^-)$, $t, s \in K$ and choose an arbitrary $\varphi \in C_c(K)$. The space $L^1_{loc}(K, m^-)$ is invariant under right translations, right translation is a continuous operation on $L^1(K, m^-)$ and thus the mapping $r \mapsto \int_K (T_r f)(u)\varphi(u)\,m^-(\mathrm{d}u)$, $K \to \mathbb{C}$ is continuous (see Proposition 2.2.8, Remark 2.2.10 and Lemma 2.1.3). Note that functions in $L^1_{loc}(K, m^-)$ are determined uniquely through $\langle \cdot, \varphi\rangle$, $\varphi \in C_c(K)$. So it remains to show that

$$\int_K (T_t T_s f)(u)\varphi(u)\,m^-(\mathrm{d}u) = \int_K \int_K (T_r f)(u)\varphi(u)\,m^-(\mathrm{d}u)\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r). \tag{3.5}$$

Therefore we may assume without loss of generality $f \in L^1(K, m^-)$. If $f \in C_b(K)$ then (3.5) holds true by Proposition 3.3.5 and Fubini's theorem. Finally, use that $C_c(K)$ is dense in $L^1(K, m^-)$. □

Proof of Theorem 3.3.3. The only thing to prove is the functional equation. Suppose $f \in B$ and $t, s \in K$. Then $T_t T_s f \in B$, the B-valued Bochner integral $\int_K T_r f\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \in B$ exists, and we have to show that they are equal in $L^1_{loc}(K)$. Therefore note that $\langle \cdot, \varphi \rangle = \int_K \cdot\, \varphi \,\mathrm{d}m^- \in B^*$ for all $\varphi \in C_c(K)$ by Definition 3.3.1(i), then apply Lemma 3.3.6. □

Remark 3.3.7. Left translations do in general not form a multiplicative operator function. Consider the group G of automorphisms on a finite dimensional Banach space X endowed with the uniform operator topology (see, e. g., Lyubich, 1988, p. 46, Example 5). Provided X is at least two dimensional, there exist $t, s \in G$ such that $ts \neq st$. Set $B = C_0(G)$ and choose $f \in C_0(G)$ with $f(ts) \neq f(st)$; it is $(T^t T^s f)(e) = f(st)$ and $(\int_K T^r f\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r))(e) = f(ts)$.

We give some examples of homogeneous Banach spaces. Let $C_{ub}(K) \subset C_b(K)$ denote the set of uniformly continuous and bounded functions on K. By uniformly continuous we mean β-uniformly continuous in the sense of Bloom and Heyer (1995), Definition 1.2.26(ii), that is, for each $t_0 \in K$ and $\varepsilon > 0$ there exists a neighbourhood U_{t_0} of t_0 such that $\|T_t f - T_{t_0} f\|_\infty < \varepsilon$ for all $t \in U_{t_0}$. Note that $C_{ub}(K)$ endowed with $\|\cdot\|_\infty$ is a Banach space. Indeed, $C_{ub}(K)$ is a closed linear subspace of $C_b(K)$ since $\|T_t f\|_\infty \leq \|f\|_\infty$ for all $f \in C_b(K)$.

Proposition 3.3.8. *The spaces $C_0(K)$, $C_{ub}(K)$ with $\|\cdot\|_\infty$ and $L^p(K, m^-)$, $1 \leq p < \infty$ with $\|\cdot\|_p$ are homogeneous Banach spaces. In these cases translations are contractions.*

Proof. Consider the Banach space $C_0(K)$ with norm $\|\cdot\|_\infty$. For each compact set $C \subset K$, $\|f|_C\|_1 \leq m^-(C)\|f\|_\infty$ for all $f \in C_0(K)$. For each $f \in C_0(K)$ and $t \in K$, $T_t f \in C_0(K)$, and for every $t \in K$, $\|T_t f\|_\infty \leq \|f\|_\infty$ for all $f \in C_0(K)$. For each $f \in C_0(K)$ the mapping $t \mapsto T_t f$, $K \to C_0(K)$ is continuous by Corollary 2.2.4.

Consider the Banach space $L^p(K, m^-)$ with norm $\|\cdot\|_p$ for some $1 \leq p < \infty$. For each compact set $C \subset K$ Hölder's inequality states $\|f|_C\|_1 \leq \|1_C\|_q \|f\|_p$ for all $f \in L^p(K, m^-)$ with $\frac{1}{p} + \frac{1}{q} = 1$. For each $f \in L^p(K, m^-)$ and $t \in K$, $T_t f \in L^p(K, m^-)$, and for every $t \in K$, $\|T_t f\|_p \leq \|f\|_p$ for all $f \in L^p(K, m^-)$, see Proposition 2.2.8. For each $f \in L^p(K, m^-)$ the mapping $t \mapsto T_t f$, $K \to L^p(K, m^-)$ is continuous by Corollary 2.2.11.

The space $C_{ub}(K)$ with $\|\cdot\|_\infty$ is complete, see above. Suppose $f \in C_{ub}(K)$ and $t \in K$. It only remains to show that $T_t f \in C_{ub}(K)$. If K is a commutative hypergroup, translations commute and this is clear by definition. For the general case we note that for $s \in K$

$$T_s T_t f = \int_K T_u f\,(\varepsilon_s * \varepsilon_t)(\mathrm{d}u)$$

read in the Banach space $C_b(K)$. Indeed, the $C_b(K)$-valued Bochner integral on the right hand side exists since the integrand is continuous by choice of f, and the equality holds pointwise by Proposition 3.3.5. In other words

$$T_s T_t f = g(s * t)$$

where $g = T_\bullet f \in C(K, Y)$ with $Y = C_b(K)$. Thus Proposition 2.2.3 yields $s \mapsto T_s(T_t f)$, $K \to Y$ continuous, that is $T_t f \in C_{ub}(K)$ by definition. □

Examples of translation operator functions are provided by K-weakly stationary processes, as introduced by R. Lasser and M. Leitner, see Lasser and Leitner (1989) and Leitner (1991), or Leitner (1989), see also Bloom and Heyer (1995), Section 8.2.

Example 3.3.9. Let K be a hypergroup and $(\Omega, \mathscr{F}, P)$ a probability space. A family $(\mathcal{X}_t)_{t\in K} \subset L^2(\Omega, \mathscr{F}, P)$ is called K-*weakly stationary process* if the following conditions are satisfied.

(i) The means are constant, i. e. there exists a constant $c \in \mathbb{C}$ such that $E[\mathcal{X}_t] = c$ for all $t \in K$.

(ii) The covariance function

$$\begin{aligned} d : K \times K &\to \mathbb{C} \\ (t, s) &\mapsto E[(\mathcal{X}_t - c)\overline{(\mathcal{X}_s - c)}] \end{aligned}$$

is continuous and bounded and satisfies

$$d(t, s) = \int_K d(r, e)\, (\varepsilon_t * \varepsilon_{s^-})(\mathrm{d}r)$$

for all $t, s \in K$.

In the following we always assume that K is a commutative hypergroup and any K-weakly stationary process is *centered*, i. e. c=0.

Let $(\mathcal{X}_t)_{t\in K} \subset L^2(\Omega, \mathscr{F}, P)$ be a K-weakly stationary process. Leitner (1991), Section 2, introduces the notion of a *translation operator* T_t^{ws} *for* $t \in K$ *on* $(\mathcal{X}_s)_{s\in K}$. For a shortcut, his definition is equivalent to

$$T_t^{\mathrm{ws}} \mathcal{X}_s := \int_K \mathcal{X}_r\, (\varepsilon_t * \varepsilon_s)(\mathrm{d}r) = (T_t \mathcal{X})(s) \tag{3.6}$$

where

$$\begin{aligned} \mathcal{X} : K &\to L^2(\Omega, \mathscr{F}, P) \\ t &\mapsto \mathcal{X}_t \end{aligned} \tag{3.7}$$

is a continuous transformation, see Leitner (1991), Theorem 2, 8., using 6. and $\mathcal{X}_t = T_t^{\mathrm{ws}} \mathcal{X}_e$. Further, following Leitner (1991), Theorem 2, $\|T_t^{\mathrm{ws}}\| \le 1$ for all $t \in K$, thus the translation operators T_t^{ws} are extended to

$$H := \mathrm{cl}_{\|\cdot\|_{L^2(\Omega,\mathscr{F},P)}} \mathrm{lin}\{\mathcal{X}_s,\ s \in K\},$$

the closure of the linear span of $\{\mathcal{X}_s,\ s \in K\}$ in $L^2(\Omega, \mathscr{F}, P)$; then the mapping

$$t \mapsto T_t^{\mathrm{ws}},\ K \to \mathcal{L}(H) \tag{3.8}$$

is continuous,

$$\begin{aligned} T_e^{\mathrm{ws}} &= I, \\ T_t^{\mathrm{ws}} T_s^{\mathrm{ws}} &= \int_K T_r^{\mathrm{ws}}\, (\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \end{aligned}$$

for all $t, s \in K$, and

$$(T_t^{\mathrm{ws}})^{\sim} = T_{t^-}^{\mathrm{ws}} \tag{3.9}$$

for all $t \in K$.

Stated in our terminology, the translation operator function

$$\begin{aligned} S : K &\to \mathcal{L}(H) \\ t &\mapsto T_t^{\mathrm{ws}} \end{aligned}$$

is a uniformly continuous multiplicative operator function. It can be extended to a $\sim$ representation by (3.9), see Theorem 3.1.7.

Conversely, suppose $H \subset L^2(\Omega, \mathscr{F}, P)$ is a Hilbert space such that $E[\mathcal{X}] = 0$ for all $\mathcal{X} \in H$ and let $S : K \to \mathcal{L}(H)$ be a uniformly continuous multiplicative operator function which is the restriction of a $\sim$ representation (in the sense of Theorem 3.1.7). Then for each $\mathcal{X} \in H$,

$$\mathcal{X}_t := S(t)\mathcal{X}$$

defines a K-weakly stationary process $(\mathcal{X}_t)_{t\in K}$, compare Leitner (1991), p. 325.

Finally, given a K-weakly stationary process $(\mathcal{X}_t)_{t\in K}$, associate $\mathscr{X}$ as defined in (3.7), then $\mathscr{X} \in C_{ub}(K, L^2(\Omega, \mathscr{F}, P))$, see (3.6), (3.8), and note that $\|\mathcal{X}_t\| = \|T_t^{\mathrm{ws}}\mathcal{X}_e\| \leq \|\mathcal{X}_e\|$ for all $t \in K$. In particular, looking at (3.6) once again, and with Remark 3.3.2 in mind, each K-weakly stationary process $(\mathcal{X}_t)_{t\in K}$ can be identified with the orbit $S(\cdot)\mathscr{X}$ of the translation operator function S on the (generalized) homogeneous Banach space $B = C_{ub}(K, L^2(\Omega, \mathscr{F}, P))$.

3.4 Abstract Cauchy Problems

In this section we establish the abstract Cauchy problem on arbitrary commutative hypergroups with associated integral equation. Some examples are discussed. It would go beyond the scope of this thesis to consider all of them in detail; the remaining part of this thesis is devoted to Sturm-Liouville hypergroups and operator functions on $\mathbb{R}_+$. The approach of the present section constitutes a unifying abstract framework.

Let K be a commutative hypergroup. Let $\mathfrak{J} \subset M_+^b(K)$ be a family of non-negative non-zero measures with compact support, and suppose that for any neighbourhood U of e in K there exists $\mathcal{J} \in \mathfrak{J}$ such that $\mathrm{supp}(\mathcal{J}) \subset U$. Further, suppose that for each $\mathcal{J} \in \mathfrak{J}$ there exists $\delta_{\mathcal{J}} \in M^b(K)$ with compact support such that for each $\chi \in \mathrm{supp}(\pi) \subset \widehat{K}$ there exists a constant c_χ such that

$$\int_K \chi \,\mathrm{d}\delta_{\mathcal{J}} = c_\chi \int_K \chi \,\mathrm{d}\mathcal{J}$$

for all $\mathcal{J} \in \mathfrak{J}$. In this situation, we say K is a hypergroup with *associated integral equation*.

We remark in advance that every commutative hypergroup has an associated integral equation, namely, its functional equation, see Example 3.4.8.

Definition 3.4.1. Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation. Then the *universal generator* $\mathbb{A}_0$ is defined by

$$\mathbb{A}_0 x := \lim_{\substack{\operatorname{supp}(\jmath)\to\{e\} \text{ in } \mathscr{C}(K) \\ \jmath\in\mathfrak{J}}} {}_{\jmath} \frac{\int_K S(\cdot)x \,\mathrm{d}\delta_{\jmath}}{\jmath(K)}$$

with domain

$$D(\mathbb{A}_0) := \{x \in X : \lim \dots \text{ exists}\}.$$

The name "universal generator" emphasizes that its definition does not depend on the properties of a concrete integral equation or abstract Cauchy problem; however, for specific examples, the notion of an "adapted generator", briefly called "generator", may be more convenient, see Example 3.4.9, (3.14), and Definition 5.1.2.

Proposition 3.4.2. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation, and $\mathbb{A}_0$ its universal generator. Suppose $x \in D(\mathbb{A}_0)$. Then $S(t)x \in D(\mathbb{A}_0)$ and $\mathbb{A}_0 S(t)x = S(t)\mathbb{A}_0 x$ for all $t \in K$.*

Proof. The values of S commute since K is commutative. So the assertion is clear by Definition 3.4.1 and Hille's Theorem A.12. □

Theorem 3.4.3. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation, and $\mathbb{A}_0$ its universal generator. Suppose $x \in X$. Then $\int_K S(\cdot)x \,\mathrm{d}\jmath \in D(\mathbb{A}_0)$ and*

$$\int_K S(\cdot)x \,\mathrm{d}\delta_{\jmath} = \mathbb{A}_0 \int_K S(\cdot)x \,\mathrm{d}\jmath \tag{3.10}$$

for all $\jmath \in \mathfrak{J}$.

Proof. *1.* Suppose $j, \jmath \in \mathfrak{J}$. Then for all $\chi \in \operatorname{supp}(\pi)$

$$c_\chi \int_K \chi \,\mathrm{d}j \int_K \chi \,\mathrm{d}\jmath = \int_K \chi \,\mathrm{d}\delta_j \int_K \chi \,\mathrm{d}\jmath = \int_K \int_K \int_K \chi \,\mathrm{d}(\varepsilon_t * \varepsilon_s)\, \delta_j(\mathrm{d}t)\, \jmath(\mathrm{d}s) \tag{3.11}$$

and

$$\int_K \chi \,\mathrm{d}j \; c_\chi \int_K \chi \,\mathrm{d}\jmath = \int_K \chi \,\mathrm{d}j \int_K \chi \,\mathrm{d}\delta_{\jmath} = \int_K \int_K \int_K \chi \,\mathrm{d}(\varepsilon_t * \varepsilon_s)\, j(\mathrm{d}t)\, \delta_{\jmath}(\mathrm{d}s). \tag{3.12}$$

We see that the measures defined by the right hand sides of (3.11) and (3.12) coincide by Uniqueness Theorem 2.1.9.

2. Suppose $x \in X$. Reading Step 1 backwards, as far as possible, with the Banach space valued function $S(\cdot)x$ in place of χ, we arrive at

$$\int_K S(\cdot) \,\mathrm{d}\delta_j \int_K S(\cdot)x \,\mathrm{d}\jmath = \int_K S(\cdot) \,\mathrm{d}j \int_K S(\cdot)x \,\mathrm{d}\delta_{\jmath}, \tag{3.13}$$

where we have used that S is multiplicative (and Hille's Theorem A.12); the operator-valued integrals are defined in the strong sense. Dividing (3.13) by $j(K) > 0$, and taking the limit $\operatorname{supp}(j) \to \{e\}$ in $\mathscr{C}(K)$, $j \in \mathfrak{J}$, the right hand side gives

$$\frac{1}{j(K)} \int_K S(\cdot)\,\mathrm{d}j \int_K S(\cdot)x\,\mathrm{d}\delta_{\jmath} = \int_K S(s)\,\frac{1}{j(K)} \int_K S(t)x\,j(\mathrm{d}t)\,\delta_{\jmath}(\mathrm{d}s)$$
$$\to \int_K S(s)x\,\delta_{\jmath}(\mathrm{d}s),$$

where we have used that j are non-negative, non-zero measures. Thus the left hand side of (3.13) yields $\int_K S(\cdot)x\,\mathrm{d}\jmath \in D(\mathbb{A}_0)$ and

$$\mathbb{A}_0 \int_K S(\cdot)x\,\mathrm{d}\jmath = \int_K S(\cdot)x\,\mathrm{d}\delta_{\jmath}.$$

□

The following conclusions of Theorem 3.4.3 are almost copies of those in the Sturm-Liouville setting, see Chapter 5; in the cosine setting the ideas can be traced back to M. Sova and S. Kurepa. The proofs are included for the sake of completeness.

Remark 3.4.4. For $x \in D(\mathbb{A}_0)$, the universal generator $\mathbb{A}_0$ and the integral commute, that is

$$\mathbb{A}_0 \int_K S(\cdot)x\,\mathrm{d}\jmath = \int_K S(\cdot)\mathbb{A}_0 x\,\mathrm{d}\jmath.$$

Indeed, this can be seen from (3.13), using

$$\int_K S(\cdot)\,\mathrm{d}\delta_j \int_K S(\cdot)x\,\mathrm{d}\jmath = \int_K S(s) \int_K S(t)x\,\delta_j(\mathrm{d}t)\,\jmath(\mathrm{d}s).$$

Theorem 3.4.5. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation. Then its universal generator $\mathbb{A}_0$ is densely defined and closed.*

Proof. To show that $\mathbb{A}_0$ is densely defined, choose an arbitrary $x \in X$ and $\varepsilon > 0$. Then there exists $j \in \mathfrak{J}$ close to $\{e\}$ in $\mathscr{C}(K)$ such that $\|x - x_j\| < \varepsilon$ where

$$x_j := (j(K))^{-1} \int_K S(\cdot)x\,\mathrm{d}j.$$

Theorem 3.4.3 yields $x_j \in D(\mathbb{A}_0)$ and

$$\mathbb{A}_0 x_j = (j(K))^{-1} \int_K S(\cdot)x\,\mathrm{d}\delta_j.$$

Hence $\mathbb{A}_0$ is densely defined.

To show that $\mathbb{A}_0$ is closed, assume $(x_n)_{n\in\mathbb{N}} \subset D(\mathbb{A}_0)$, $x, y \in X$ and $x_n \to x$, $\mathbb{A}_0 x_n \to y$ as $n \to \infty$. Applying Theorem 3.4.3 to x_n, $n \in \mathbb{N}$, using Remark 3.4.4, and taking the limit $n \to \infty$, we get for any $j \in \mathfrak{J}$

$$\int_K S(\cdot)x\,\mathrm{d}\delta_j = \int_K S(\cdot)y\,\mathrm{d}j.$$

It follows from Definition 3.4.1 that $x \in D(\mathbb{A}_0)$ and $\mathbb{A}_0 x = y$. □

Remark 3.4.6. The proof above also shows, by iteration, that $D(\mathbb{A}_0^n)$ is dense in X for all $n \in \mathbb{N}$.

Theorem 3.4.7. *Let $S : K \to \mathcal{L}(X)$ be a multiplicative operator function on a hypergroup K with associated integral equation, and $\mathbb{A}_0$ its universal generator. Then $\lim_{t\to 0^+} S(t) = I$ in uniform operator topology if and only if S is uniformly continuous: In this case $\mathbb{A}_0$ is bounded.*

Proof. The first equivalence is content of Theorem 3.2.10. So suppose S is uniformly continuous. According to (3.10) it is sufficient to show that there exists $j \in \mathfrak{J}$ such that the operator $\int_K S(\cdot)\,\mathrm{d}j$, defined as Bochner integral in $\mathcal{L}(X)$, is invertible. Therefore take $j \in \mathfrak{J}$ close to $\{e\}$ in $\mathscr{C}(K)$ such that

$$\left\| I - (j(K))^{-1} \int_K S(\cdot)\,\mathrm{d}j \right\| \leq (j(K))^{-1} \int_K \|I - S(\cdot)\|\,\mathrm{d}j < \frac{1}{2}.$$

□

In the general setting of Theorem 3.4.7, the converse assertion is not true, that is if $\mathbb{A}_0$ is bounded then S may or may not be uniformly continuous, see the following example.

Example 3.4.8 (Functional equation)**.** Let K be a commutative hypergroup and suppose $t_0 \in K$. Set $\mathfrak{J} := \{\varepsilon_s,\ s \in K\}$ and $\delta_{\varepsilon_s} := \varepsilon_{t_0} * \varepsilon_s$ for each $\varepsilon_s \in \mathfrak{J}$. Further, given $\chi \in \widehat{K}$ set $c_\chi := \chi(t_0)$. This mimics the functional equation $\chi(t_0 * s) = \chi(t_0)\chi(s)$, i. e.

$$\int_K \chi\,\mathrm{d}\delta_{\varepsilon_s} = c_\chi \int_K \chi\,\mathrm{d}\varepsilon_s$$

for all $\varepsilon_s \in \mathfrak{J}$.

Let S be a multiplicative operator function on K. Then for each $x \in X$,

$$\mathbb{A}_0 x = \lim_{\substack{\{s\}\to\{e\} \text{ in } \mathscr{C}(K) \\ s \in K}} \frac{\int_K S(\cdot)x\,\mathrm{d}\delta_{\varepsilon_s}}{\varepsilon_s(K)} = \lim_{s\to e} \frac{S(t_0)S(s)x}{1} = S(t_0)x,$$

that is the universal generator is given by

$$\mathbb{A}_0 = S(t_0) \in \mathcal{L}(X),$$

and the corresponding abstract Cauchy problem states that for each $x \in X$

$$S(t_0 * s)x = \mathbb{A}_0 S(s)x$$

for all $s \in K$.

Example 3.4.9 (Polynomial hypergroups)**.** This is a special sub-example of Example 3.4.8 in the discrete setting, extracted from Ey and Lasser (2007).

We start with a quick introduction to polynomial hypergroups, see Lasser (1983, 1994, 2016) or the survey in Bloom and Heyer (1995). Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$

be sequences of non-negative real numbers such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $a_n, c_n > 0$ for all $n \in \mathbb{N}$, and suppose $a_0 > 0$, $b_0 \in \mathbb{R}$, and $a_0 + b_0 = 1$. Let $(R_n)_{n\in\mathbb{N}_0}$ be the sequence of polynomials defined recursively by

$$\begin{aligned} R_0(t) = 1, \quad R_1(t) = \frac{1}{a_0}(t - b_0) \quad \text{and} \\ R_1(t)R_n(t) = a_n R_{n+1}(t) + b_n R_n(t) + c_n R_n(t), \quad n \in \mathbb{N} \end{aligned}$$

with $t \in \mathbb{R}$. By construction $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. According to Favard's theorem, $(R_n)_{n\in\mathbb{N}_0}$ is orthogonal with respect to some measure $\pi \in M^1(\mathbb{R})$. From the orthogonality one can deduce that

$$R_n(t)R_m(t) = \sum_{k=|n-m|}^{n+m} g(n, m; k)R_k(t)$$

for all $n, m \in \mathbb{N}_0$ and $t \in \mathbb{R}$, where $g(n, m; k) \in \mathbb{R}$ for all $k = |n - m| \dots n + m$ and $g(n, m; |n - m|) \neq 0$, $g(n, m; n + m) \neq 0$. Many important and well-known examples of orthogonal polynomials satisfy the crucial condition

$$g(n, m; k) \geq 0 \text{ for all } k = |n - m| \dots n + m.$$

In this case,

$$\varepsilon_n * \varepsilon_m = \sum_{k=|n-m|}^{n+m} g(n, m; k)\varepsilon_k, \quad n, m \in \mathbb{N}_0$$

defines the convolution of point measures of a hypergroup $K = \mathbb{N}_0$, called *polynomial hypergroup* with respect to $(R_n)_{n\in\mathbb{N}_0}$, and denoted by $(\mathbb{N}_0, *(R_n))$. The neutral element is 0, involution is the identity map, and the dual space $\widehat{\mathbb{N}_0}$ is homeomorphic to

$$D_S := \{t \in \mathbb{R} : \ |R_n(t)| \leq 1 \text{ for all } n \in \mathbb{N}\} \subset [1 - 2a_0, 1].$$

Let $(\mathbb{N}_0, *(R_n))$ be a polynomial hypergroup. We consider

$$\begin{aligned} R_1(t) = c_t R_0(t) \quad \text{for } n = 0, \\ a_n R_{n+1}(t) + b_n R_n(t) + c_n R_{n-1}(t) = c_t R_n(t) \quad \text{for all } n \in \mathbb{N} \end{aligned}$$

as the associated integral equation; here $\mathfrak{J} = \{\varepsilon_n,\ n \in \mathbb{N}_0\}$, and $\delta_{\varepsilon_0} = \varepsilon_1$, $\delta_{\varepsilon_n} = a_n\varepsilon_{n+1} + b_n\varepsilon_n + c_n\varepsilon_{n-1}$ for all $n \in \mathbb{N}$, and $t \in D_S$, $c_t = R_1(t)$.

Let S be a multiplicative operator function on $(\mathbb{N}_0, *(R_n))$. Then its universal generator is given by

$$\mathbb{A}_0 = \frac{\int_{\mathbb{N}_0} S(\cdot)\,\mathrm{d}\delta_{\varepsilon_0}}{\varepsilon_0(\mathbb{N}_0)} = S(1) \in \mathcal{L}(X),$$

and the corresponding abstract Cauchy problem is given by

$$\begin{aligned} S(1) = \mathbb{A}_0 S(0) \quad \text{for } n = 0, \\ a_n S(n + 1) + b_n S(n) + c_n S(n - 1) = \mathbb{A}_0 S(n) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

In this setting it is convenient to define the *(adapted) generator* $\mathbb{A}$ through

$$\mathbb{A}_0 = R_1(\mathbb{A}), \tag{3.14}$$

that is

$$\mathbb{A} = a_0\mathbb{A}_0 + b_0 = a_0 S(1) + b_0;$$

then it is easy to see that

$$S(n) = R_n(\mathbb{A})$$

for all $n \in \mathbb{N}_0$, compare Ey and Lasser (2007), Theorem 1.

Example 3.4.10 (Dual Jacobi polynomial hypergroups)**.** This example is inspired by Weinmann and Lasser (2011), Section 3 where the case of translation operator functions on homogeneous Banach spaces is investigated.

To begin with, let us collect some facts and notation for dual Jacobi polynomial hypergroups. Let $R_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$, denote the Jacobi polynomials with parameters $(\alpha, \beta) \in J$, where $J = \{(\alpha', \beta') \in \mathbb{R}^2 : \alpha' \geq \beta' \geq -\frac{1}{2} \ \vee \ (\alpha' \geq \beta' > -1 \wedge \alpha' + \beta' \geq 0)\}$, normalized by $R_n^{(\alpha,\beta)}(1) = 1$. These are orthogonal with respect to $\pi^{(\alpha,\beta)}$, the probability measure on $\mathcal{S} = [-1, 1]$ with Lebesgue density $w(s) = c_{\alpha,\beta}(1-s)^\alpha(1+s)^\beta$, $c_{\alpha,\beta} = 2^{-\alpha-\beta-1}\Gamma(\alpha+\beta+2)\Gamma(\alpha+1)^{-1}\Gamma(\beta+1)^{-1}$. It has been shown by Gasper (1972) that there exists a positive linearization formula on $\mathcal{S} = [-1, 1]$, and by Lasser (1983), Section 4, that $\mathcal{S}$ becomes a hypergroup with dual space $\widehat{\mathcal{S}} = \{R_n^{(\alpha,\beta)}, n \in \mathbb{N}_0\}$. Its neutral element is 1, and involution is the identity map. In the sequel let $(\alpha, \beta) \in J$ be fixed; we drop its notation.

It is well-known that Jacobi polynomials satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(w(t)(1-t^2)\,\frac{\mathrm{d}}{\mathrm{dt}}\,R_n(t)\right) = -n(n+\alpha+\beta+1)w(t)R_n(t).$$

Integration gives the integral equation

$$R_n(t) - R_n(1) = -n(n+\alpha+\beta+1)\int_t^1 \frac{1}{w(s)(1-s^2)}\int_s^1 R_n(r)w(r)\ \mathrm{d}r\ \mathrm{d}s,$$

$t \in]-1, 1]$; after integration by parts it takes the form

$$R_n(t) - R_n(1) = -n(n+\alpha+\beta+1)\int_t^1 \theta(t,s)R_n(s)\,\pi(\mathrm{d}s)$$

where

$$\theta(t,s) := \int_t^s \frac{1}{w(r)(1-r^2)}\ \mathrm{d}r\ 1_{(t,1)}(s)$$

and $t \in]-1, 1]$, see Weinmann and Lasser (2011).

Suppose now that S is a multiplicative operator function on $\mathcal{S} = [-1, 1]$. It is easily checked that Theorem 3.4.3 is applicable, thus for every $x \in X$, $\int_t^1 \theta(t, s)S(s)x\,\pi(\mathrm{d}s) \in D(\mathbb{A}_0)$ and

$$S(t)x - x = \mathbb{A}_0 \int_t^1 \theta(t, s)S(s)x\,\pi(\mathrm{d}s)$$

for all $t \in]-1, 1]$; see Weinmann and Lasser (2011), Lemma 3.4 for the special case of a translation operator function on a homogeneous Banach space, and a proof using Fourier analysis.

Example 3.4.11 (Sturm-Liouville hypergroups of compact type)**.** Each dual Jacobi polynomial hypergroup from Example 3.4.10 is isomorphic to a Sturm-Liouville hypergroup of compact type. So, more generally, one could consider this class of hypergroups; examples are provided by Achour-Trimèche, Zeuner, and Fourier-Bessel hypergroups. For the first statement, and the examples, see Bloom and Heyer (1995), 3.5.80–3.5.88. No attempt has been made to elaborate these examples.

Example 3.4.12 (Sturm-Liouville hypergroups on $\mathbb{R}_+$)**.** These are of primary interest to us since we will get generalizations of cosine, Bessel, and Legendre operator functions. The remaining part of this thesis is devoted to this class of examples. The aim is to initiate a theory of "Sturm-Liouville operator functions"; this will be done in Chapter 5.

Chapter 4

Sturm-Liouville Hypergroups and Asymptotics

Sturm-Liouville hypergroups are of particular interest to us since their multiplicative functions are solutions of associated Sturm-Liouville equations. Often, these are special functions like cosine, Bessel, and Jacobi functions, see Chapter 1 for an introduction to the corresponding theories of operator functions. In Chapter 5 we consider, more generally, multiplicative operator functions on Sturm-Liouville hypergroups.

In the present chapter we introduce Sturm-Liouville hypergroups and investigate the asymptotic behaviour of solutions of the associated Sturm-Liouville equations. The results are used to reprove some basic theorems for Sturm-Liouville hypergroups. Moreover, they are for preparation of Chapter 5.

4.1 Sturm-Liouville Hypergroups

In this section we collect some basic facts about Sturm-Liouville hypergroups. We consider only the non-compact case $K = \mathbb{R}_+$. The notion of a Sturm-Liouville hypergroup was introduced by Hm. Zeuner in Zeuner (1989a) and Zeuner (1992), unifying the concepts of Chébli-Trimèche and Levitan hypergroups. For a survey on Sturm-Liouville hypergroups see Bloom and Heyer (1995), Section 3.5. We take some material from there.

Definition 4.1.1. A *Sturm-Liouville function* is a continuous mapping $A : \mathbb{R}_+ \to \mathbb{R}$ such that $A(t) > 0$ for all $t \in \mathbb{R}_+^\times$ and $A|_{\mathbb{R}_+^\times} \in C^1(\mathbb{R}_+^\times)$.

We say a function $y : \mathcal{I} \to \mathbb{C}$, $\mathcal{I} \subset \mathbb{R}$ some interval, is *locally absolutely continuous*, in signs $y \in AC_{loc}(\mathcal{I})$, if y is absolutely continuous on compact subintervals of $\mathcal{I}$.

Definition 4.1.2. Given a Sturm-Liouville function A, the *Sturm-Liouville operator* L is defined for $y \in AC_{loc}(\mathbb{R}_+^\times)$ with $Ay' \in AC_{loc}(\mathbb{R}_+^\times)$ by

$$Ly := A^{-1}(Ay')'.$$

These functions y are said to be in the expression domain of L (cf. Zettl, 2005, p. 172).

We note that for $y \in C^2(\mathbb{R}_+^\times)$ we may write $Ly = y'' + \frac{A'}{A}y'$, which is, except for a minus sign, exactly the definition of a Sturm-Liouville operator in Bloom and Heyer (1995).

Following Zeuner (1989a) it is no restriction to suppose that a hypergroup on $K = \mathbb{R}_+$ is *normalized* in the sense that $\min \operatorname{supp}(\varepsilon_t * \varepsilon_s) = |t-s|$ and $\max \operatorname{supp}(\varepsilon_t * \varepsilon_s) = t+s$. In this case

$$\operatorname{supp}(\varepsilon_t * \varepsilon_s) \subset [|t-s|, t+s]. \tag{4.1}$$

Definition 4.1.3. A *Sturm-Liouville hypergroup* is a normalized hypergroup $K = \mathbb{R}_+$ together with a Sturm-Liouville function A such that for every restriction f of an even non-negative function in $C^\infty(\mathbb{R})$ to $\mathbb{R}_+$ the function $u_f \in C(\mathbb{R}_+ \times \mathbb{R}_+)$ defined by

$$u_f(t,s) := \int_{\mathbb{R}_+} f \, \mathrm{d}(\varepsilon_t * \varepsilon_s)$$

is two times differentiable and satisfies the partial differential equation

$$L_t u_f(t,s) = L_s u_f(t,s), \quad (u_f)_t(0,s) = 0 \quad \text{for all } t,s \in \mathbb{R}_+^\times.$$

We denote a Sturm-Liouville hypergroup with Sturm-Liouville function A by $(\mathbb{R}_+, *(A))$.

It is known that a one-dimensional hypergroup $K = \mathbb{R}_+$ is always commutative, the neutral element is 0 and involution is the identity. A Haar measure of a Sturm-Liouville hypergroup is given by $A\lambda_{\mathbb{R}_+}$, where $\lambda_{\mathcal{I}}$ denotes, more generally, the Lebesgue measure on some real interval $\mathcal{I}$.

The following conditions, imposed on a Sturm-Liouville function A, guarantee existence of a corresponding Sturm-Liouville hypergroup, see Theorem 4.1.4 below.

SL1 Suppose that

$$\frac{A'(t)}{A(t)} = \frac{\alpha_0}{t} + \alpha_1(t) \tag{4.2}$$

for all $t \in \mathbb{R}_+^\times$ in a neighbourhood of 0 where either

SL1.1 (Singularity at 0) $\alpha_0 > 0$ and $\alpha_1 \in C^\infty(\mathbb{R})$ is an odd function (which implies that $A(0) = 0$)

or

SL1.2 (Regularity at 0) $\alpha_0 = 0$ and $\alpha_1 \in C^1(\mathbb{R}_+)$ (which implies that $A(0) > 0$).

SL2 There exists $\beta \in C^1(\mathbb{R}_+)$ such that $\beta(0) \geq 0$, $\frac{A'}{A} - \beta$ is non-negative and decreasing on $\mathbb{R}_+^\times$, and $q := \frac{1}{2}\beta' - \frac{1}{4}\beta^2 + \frac{A'}{2A}\beta$ is decreasing on $\mathbb{R}_+^\times$.

For our further considerations it is important that given a Sturm-Liouville function A satisfying SL2 the limit

$$\rho := \frac{1}{2} \lim_{t \to \infty} \frac{A'(t)}{A(t)} \geq 0, \tag{4.3}$$

called *index*, exists and since $\frac{A'}{A} \geq 0$ we know that

$$A \text{ is increasing on } \mathbb{R}_+.$$

Theorem 4.1.4. *Let A be a Sturm-Liouville function satisfying SL1 and SL2. Then a Sturm-Liouville hypergroup $(\mathbb{R}_+, *(A))$ exists.*

Two classes of such hypergroups are provided by Chébli-Trimèche and Levitan functions, see Section 4.5 for examples.

The proof of Theorem 4.1.4 (Theorem 3.11 in Zeuner, 1992) needs the following lemma which strengthens Definition 4.1.3.

Lemma 4.1.5. *Let A be a Sturm-Liouville function satisfying SL1 and SL2. Suppose $f \in C^2(\mathbb{R}_+)$ and $f'(0) = 0$. Set $u_f(t,s) = f(t * s)$ as in Definition 4.1.3. Then u_f is two times differentiable and satisfies the partial differential equation*

$$(L_t u_f)(t,s) = (L_s u_f)(t,s) = (u_{Lf})(t,s) \quad \text{for all } t,s \in \mathbb{R}_+^\times$$

and $(u_f)_s(t,0) = 0$ for all $t \in \mathbb{R}_+^\times$.

For the boundary condition see the proof of Lemma 3.12 in Zeuner (1992).

The following proposition is the basis for our considerations of operator functions on Sturm-Liouville hypergroups.

Proposition 4.1.6. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying $SL1$ and $SL2$. Then the multiplicative functions are exactly the solutions $y \in C^2(\mathbb{R}_+)$ of*

$$Ly(t) = a_0 y(t), \quad t > 0, \tag{4.4}$$

$$y(0) = 1, \quad y'(0) = 0 \tag{4.5}$$

with $a_0 \in \mathbb{C}$.

Note that for $A \equiv 1$ the differential equation and its solution $y(t) = \cosh(\sqrt{a_0}t)$ correspond to cosine operator functions.

Remark 4.1.7. Let A be a Sturm-Liouville function satisfying $SL1$, $SL2$ and suppose $\gamma > 0$. Then $A_\gamma := A(\gamma\,\cdot)$ is also a Sturm-Liouville function satisfying SL1, SL2 (see Bloom and Heyer, 1995, 3.5.16). Let y_{a_0} denote the solution of (4.4), (4.5); analogously let $y^\gamma_{a_0}$ denote the solution of (4.4), (4.5) with respect to the Sturm-Liouville function A_γ. It is easily seen from the differential equation that

$$y^\gamma_{\gamma^2 a_0} = y_{a_0}(\gamma\,\cdot). \tag{4.6}$$

Let $*_\gamma$ denote the convolution of the Sturm-Liouville hypergroup $(\mathbb{R}_+, *(A_\gamma))$. Then for all $t,s \in \mathbb{R}_+$

$$\varepsilon_t *_\gamma \varepsilon_s = (\varepsilon_{\gamma t} * \varepsilon_{\gamma s})(\gamma\,\cdot),$$

which follows readily from (4.6) and Uniqueness Theorem 2.1.9.

4.2 Power Series Expansions of Sturm-Liouville Solutions

As we have pointed out above, the characterization of multiplicative functions in Proposition 4.1.6 is basic for our considerations of Sturm-Liouville operator functions. Here we discuss power series expansions of multiplicative functions and Sturm-Liouville solutions in general. This presentation is, with slight modifications, a summary of well-known facts scattered in the literature.

Definition 4.2.1. Given a Sturm-Liouville function A satisfying SL2, let $J : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be the Sturm-Liouville integral operator defined by

$$(Jy)(t) := \int_0^t A(s)^{-1} \int_0^s A(r)y(r) \text{ dr ds}$$

for $y \in C(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$. Further, let J^0 denote the identity operator on $C(\mathbb{R}_+)$ and define recursively $J^k = J \circ J^{k-1}$ for $k \in \mathbb{N}$.

The integral operator J satisfies

$$|(J^k y)(t)| \leq \frac{t^{2k}}{(2k)!} \sup_{s\in[0,t]} |y(s)| \tag{4.7}$$

for all $k \in \mathbb{N}_0$, in particular J is well-defined. Indeed, suppose the estimate is true for some $k-1$, $k \in \mathbb{N}$. Then, using that A is monotonically increasing,

$$\begin{aligned}
|(J^k y)(t)| &\leq \int_0^t A(s)^{-1} \int_0^s A(r) \frac{r^{2(k-1)}}{(2(k-1))!} \text{ dr ds} \sup_{s\in[0,t]} |y(s)| \\
&\leq \int_0^t A(s)^{-1} A(s) \int_0^s \frac{r^{2(k-1)}}{(2(k-1))!} \text{ dr ds} \sup_{s\in[0,t]} |y(s)| \\
&= \frac{t^{2k}}{(2k)!} \sup_{s\in[0,t]} |y(s)|.
\end{aligned}$$

Obviously, J is in some sense the inverse of L, the next lemma makes this more precise.

Lemma 4.2.2. *Let A be a Sturm-Liouville function satisfying SL1 and SL2. Then a function f is in $C^2(\mathbb{R}_+)$ with $f'(0) = 0$ iff there exists $\varphi \in C(\mathbb{R}_+)$ such that*

$$f(t) - f(0) = (J\varphi)(t)$$

for all $t \in \mathbb{R}_+$. In this case $Lf = \varphi$, $f - f(0) = J(Lf)$ and $L(J\varphi) = \varphi$.

Moreover, $f''(0) = \frac{1}{\alpha_0+1}\varphi(0)$ and $\lim_{t\to 0^+} \frac{A'(t)}{A(t)} f'(t) = \frac{\alpha_0}{\alpha_0+1}\varphi(0)$.

Proof. Suppose $f(t) - f(0) = (J\varphi)(t)$ for some $\varphi \in C(\mathbb{R}_+)$. Then obviously $f \in C^2(\mathbb{R}_+^\times)$ and $(Lf)(t) = \varphi(t)$ for all $t > 0$. Concerning the derivative of f in $t = 0$ we use that A is monotonically increasing and get $|\frac{1}{t}(f(t) - f(0))| \leq \frac{1}{t}(J|\varphi|)(t) \leq \frac{1}{t}\int_0^t \int_0^s |\varphi(r)| \text{ dr ds} \to 0$ $(t \to 0^+)$ and $|f'(t)| \leq A(t)^{-1}\int_0^t A(r)|\varphi(r)| \text{ dr} \leq \int_0^t |\varphi(r)| \text{ dr} \to$

0 $(t \to 0^+)$, thus $f \in C^1(\mathbb{R}_+)$ and $f'(0) = 0$. The difference quotient of f' in zero may be written as

$$\frac{1}{t}(f'(t) - f'(0)) = (tA(t))^{-1} \int_0^t A(r) \text{ dr } \varphi(0) + (tA(t))^{-1} \int_0^t A(r)(\varphi(r) - \varphi(0)) \text{ dr}\,.$$

Concerning the first term note that

$$\lim_{t\to 0^+} \frac{tA(t)}{\int_0^t A(r) \text{ dr}} = \lim_{t\to 0^+} \frac{A(t) + tA'(t)}{A(t)} = \lim_{t\to 0^+} (1 + t(\frac{\alpha_0}{t} + \alpha_1(t))) = \alpha_0 + 1$$

by L'Hôpital's rule and (4.2). The second term tends to zero as $t \to 0^+$ since A is monotonically increasing and φ is continuous in zero. Thus $f''(0) = \frac{1}{\alpha_0+1}\varphi(0)$ exists. We still need to show $\lim_{t\to 0^+} f''(t) = f''(0)$. Indeed, for small values of $t > 0$

$$\frac{A'(t)}{A(t)} f'(t) = (\frac{\alpha_0}{t} + \alpha_1(t))f'(t) = \alpha_0 \frac{1}{t}(f'(t) - f'(0)) + \alpha_1(t) f'(t)$$

by (4.2) and thus using $(Lf)(t) = \varphi(t)$, $t > 0$,

$$\lim_{t\to 0^+} f''(t) = \lim_{t\to 0^+} (\varphi(t) - \frac{A'(t)}{A(t)} f'(t)) = \varphi(0) - \frac{\alpha_0}{\alpha_0 + 1}\varphi(0) = \frac{1}{\alpha_0 + 1}\varphi(0) = f''(0).$$

Conversely, suppose $f \in C^2(\mathbb{R}_+)$ and $f'(0) = 0$. Setting $\varphi := Lf$, it follows immediately from (4.2) that $\varphi \in C(\mathbb{R}_+)$. Integration yields $f(t) - f(0) = (J\varphi)(t)$ for all $t \in \mathbb{R}_+$. □

Proposition 4.2.3. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL1 and SL2. Then the multiplicative functions are exactly the solutions $y \in C(\mathbb{R}_+)$ of the integral equations*

$$y(t) - 1 = a_0(Jy)(t), \quad t \in \mathbb{R}_+$$

with $a_0 \in \mathbb{C}$.

Proof. This is an immediate consequence of Proposition 4.1.6 and Lemma 4.2.2. □

Proposition 4.2.4. *Let A be a Sturm-Liouville function satisfying SL2. Consider the Sturm-Liouville equation*

$$Ly(t) = a_0 y(t), \quad t > 0 \tag{4.8}$$

for some $a_0 \in \mathbb{C}$.

Then the solution of (4.8) with initial conditions

$$y(0) = 1, \quad y'(0) = 0, \tag{4.9}$$

that is the corresponding multiplicative function (cf. Proposition 4.1.6), is given by

$$y(t) = \sum_{k=0}^{\infty} a_0^k (J^k 1)(t) \tag{4.10}$$

and

$$|y(t)| \leq \cosh(\sqrt{|a_0|}\, t)$$

for all $t \in \mathbb{R}_+$.

If 0 *is a regular endpoint, i. e.* $1/A \in L^1_{loc}(\mathbb{R}_+, \lambda_{\mathbb{R}_+})$ *(see Zettl, 2005), then there exists a second, linearly independent solution of* (4.8) *with*

$$y(0) = 0, \quad A(t)y'(t)\big|_{t=0} = 1 \tag{4.11}$$

given by

$$y(t) = \sum_{k=0}^{\infty} a_0^k (J^k \int_0^{\bullet} A(s)^{-1} \,\mathrm{d}s)(t). \tag{4.12}$$

The inhomogeneous problem

$$Ly(t) = a_0 y(t) + f(t), \quad t > 0 \tag{4.13}$$

with $a_0 \in \mathbb{C}$, $f \in C(\mathbb{R}_+)$ *and*

$$y(0) = 0, \quad y'(0) = 0 \tag{4.14}$$

has solution

$$y(t) = \sum_{k=0}^{\infty} a_0^k (J^{k+1} f)(t).$$

The power series for Sturm-Liouville solutions are well-known, see e. g. (2), (4), (7) in the seminal paper Weyl (1910). At least heuristically, (4.10) may be regarded as a special case of Delsarte's generalized Taylor formula

$$(T^t f)(s) = \sum_{k=0}^{\infty} (J^k 1)(t)(L^k f)(s), \tag{4.15}$$

see Chébli (1995), Subsection 2.1 and the references therein.

Proof. The proof is by the method of successive approximation. Applying J to (4.8) with initial values (4.9) yields

$$y = 1 + a_0 J y$$

and by induction we get

$$y = \sum_{k=0}^{n-1} a_0^k J^k 1 + a_0^n J^n y$$

for all $n \in \mathbb{N}$. Taking the limit $n \to \infty$ yields (4.10), uniformly for $t \in \mathbb{R}_+$ in compacta. It is easily checked that y constructed in this way is a solution of (4.8) and, using that A is monotonically increasing, $y'(0) = 0$.

Applying J to (4.8) with initial values (4.11) yields

$$y = \int_0^{\bullet} A(s)^{-1} \,\mathrm{d}s + a_0 J y$$

where we need the assumption that $1/A \in L^1_{loc}(\mathbb{R}_+, \lambda_{\mathbb{R}_+})$. Similarly as above, we get (4.12), the series converging uniformly for $t \in \mathbb{R}_+$ in compacta.

The same procedure works for (4.13) with initial values (4.14), here

$$y = a_0 Jy + Jf.$$

□

Remark 4.2.5. The k-th coefficient in (4.10), regarded as a power series in a_0, multiplied by $k!$, $m_k(t) = k!(J^k 1)(t)$, $k \in \mathbb{N}_0$ is a moment function of order k (see Definition 4.2.6), which follows from multiplicativity, see e. g. the proof of Proposition 2.1 in Berezansky (1998). Two general characterizations of moment function sequences on Sturm-Liouville hypergroups are presented in Székelyhidi (2013), Section 4.3.

Definition 4.2.6. Let K be a hypergroup. A continuous function $\varphi : K \to \mathbb{C}$ is called *moment function of order* n, $n \in \mathbb{N}_0$, if there exist continuous functions $m_k : K \to \mathbb{C}$, $k = 0..n$ such that $m_0 = 1$, $m_n = \varphi$ and

$$m_k(t * s) = \sum_{j=0}^{k} \binom{k}{j} m_j(t) m_{k-j}(s)$$

for $k = 0..n$ and all $t, s \in K$. In this case the sequence m_k, $k = 0..n$ is called a *moment function sequence of order* n.

For an introduction to moment functions on hypergroups see Székelyhidi (2013), Section 1.8, Bloom and Heyer (1995), Section 7.2, and Zeuner (1992).

Remark 4.2.7. If $1/A \notin L^1_{loc}(\mathbb{R}_+, \lambda_{\mathbb{R}_+})$, any second solution of (4.8) which is linearly independent of the solution (4.10) has a singularity at 0, see Lemma 4.3.5 below. We remark that, under the assumption of SL1, $1/A \in L^1_{loc}(\mathbb{R}_+, \lambda_{\mathbb{R}_+})$ is equivalent to $0 \le \alpha_0 < 1$. Indeed, suppose (4.2) holds for $0 < t \le \delta$, then (4.33) yields $1/A(t) = t^{-\alpha_0} \delta^{\alpha_0} A(\delta)^{-1} \exp\left(\int_t^\delta \alpha_1(s)\ \mathrm{ds}\right)$.

Let A be a Sturm-Liouville function satisfying SL2. Then with $\rho \ge 0$ as in (4.3) we denote by Φ_λ, $\lambda \in \mathbb{C}$ the solution $\Phi \in C^2(\mathbb{R}_+)$ of

$$L\Phi = \Phi'' + \frac{A'}{A}\Phi' = (\lambda^2 - \rho^2)\Phi \tag{4.16}$$

$$\Phi(0) = 1, \quad \Phi'(0) = 0. \tag{4.17}$$

Our definition of L and Φ_λ is adapted to fit the notation of Sturm-Liouville operator functions and its special cases, cosine, Bessel, and Legendre operator functions, see 5. The function ϕ_λ, $\lambda \in \mathbb{C}$ in Zeuner (1992), 4.1 (cf. Bloom and Heyer, 1995, p. 223) is defined as the solution $\phi \in C^2(\mathbb{R}_+)$ of

$$-\phi'' - \frac{A'}{A}\phi' = (\lambda^2 + \rho^2)\phi \tag{4.18}$$

$$\phi(0) = 1, \quad \phi'(0) = 0, \tag{4.19}$$

so $\Phi_\lambda = \phi_{i\lambda}$, in other words Φ and ϕ are connected to each other by rotation of the spectral set by 90 degrees. Further, we frequently use $\Phi_{-\lambda} = \Phi_\lambda$.

With this notation in mind, the Laplace representation theorem states, see Bloom and Heyer (1995), Theorem 3.5.58, that for each $t \in \mathbb{R}_+$ there exists $\nu_t \in M^1([-t,t])$ such that

$$\Phi_\lambda(t) = \int_{-t}^{t} e^{(-\rho+\lambda)r} \ \nu_t(\mathrm{d}r) \tag{4.20}$$

for all $\lambda \in \mathbb{C}$. We give a proof based on asymptotic results later, see Theorem 4.4.4. Here we draw some conclusions from (4.20) which are of own interest and useful in the next section to treat some exceptional cases. These conclusions are independent from the proof of Theorem 4.4.4.

A first simple consequence of (4.20) is that

$$|\Phi_\lambda(t)| \leq e^{|-\rho+\mathrm{Re}(\lambda)|t} \tag{4.21}$$

for all $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}_+$.

Further, for fixed $t \in \mathbb{R}_+$ the Taylor series of $\Phi_\lambda(t)$ in λ is given by

$$\Phi_\lambda(t) = \sum_{k=0}^{\infty} c_k(t)\lambda^{2k} \tag{4.22}$$

with non-negative coefficients

$$c_k(t) := \frac{1}{(2k)!} \frac{\mathrm{d}^{2k}}{\mathrm{d}\lambda^{2k}} \Phi_\lambda(t)\bigg|_{\lambda=0} = \frac{1}{(2k)!} \int_{-t}^{t} r^{2k} e^{-\rho r} \ \nu_t(\mathrm{d}r) \geq 0.$$

These coefficients, multiplied by $k!$, form also a moment function sequence of any order (cf. Remark 4.2.5, see also Bloom and Heyer, 1995, 7.2.3).

Remark 4.2.8. The multiplicative functions $\Phi_\lambda(t)$, $\lambda \in \mathbb{C}$ of a Sturm-Liouville hypergroup form an "exponential family", a notion which can be found in Székelyhidi (2013), Section 1.6, see Definition 4.2.9 below and compare with the power series (4.10). This important property will enable us to define uniformly continuous multiplicative operator functions on Sturm-Liouville hypergroups via the holomorphic functional calculus, see Theorem 5.3.5.

Definition 4.2.9. Let K be a commutative hypergroup. An *exponential family* is a function $\Phi : K \times \mathbb{C}^n \to \mathbb{C}$, $n \in \mathbb{N}$ such that the following properties are satisfied.

(i) For each $\lambda \in \mathbb{C}^n$, $t \mapsto \Phi(t,\lambda)$ is a multiplicative function (called *exponential* in Székelyhidi, 2013).

(ii) For fixed $t \in K$, $\lambda \mapsto \Phi(t,\lambda)$ is an entire function.

(iii) For each multiplicative function χ there exists a unique $\lambda \in \mathbb{C}^n$ such that $\chi(t) = \Phi(t,\lambda)$ for all $t \in K$.

4.3 The Asymptotic Behaviour of Sturm-Liouville Solutions

In this section we prove asymptotic formulas for specific Sturm-Liouville solutions in the context of Sturm-Liouville hypergroups. These results will be needed in Chapter 5 starting with Section 5.4. Our considerations are based on asymptotic results for the general Sturm-Liouville equation in Eastham (1989), Chapter 2.

We begin with a brief review of some known results. Our selection is subjective and focused on Sturm-Liouville hypergroups. Some asymptotic results are also known for hypercomplex systems constructed for the Sturm-Liouville equation, see Berezansky and Kalyuzhnyi (1998), Subsection 4.4 of Chapter 2, pp. 302–310. Of course, extensive literature exists about the asymptotics of special functions, we refer to Olver (1997).

For Chébli-Trimèche functions, satisfying certain conditions, several asymptotic formulas and estimates for multiplicative functions and derivatives were shown by Bloom and Xu (1995), Sections 2 and 3, and Bloom and Xu (1999), Section 2, see also Chébli (1995), Section 3, Trimèche (1997), Section 6.I, and the references therein.

We extract the following result which is derived from Langer (1935), see Bloom and Xu (1999), Lemma 2.2. The assumptions imposed are clearly satisfied for Jacobi functions (see Example 4.5.3). Here $A(t) = \sinh^{2\alpha+1} t \cosh^{2\beta+1} t$ for some parameters $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$. Then there exist constants C_A, c_1, c_2 and $R_1, R_2 > 0$ such that

$$\Phi_\lambda(t) = C_A A(t)^{-\frac{1}{2}} (i\lambda)^{-(\alpha+\frac{1}{2})} (c_1 e^{\lambda t} + c_2 e^{-\lambda t}) \cdot [1 + \mathcal{O}((i\lambda)^{-1}) + \mathcal{O}((i\lambda t)^{-1})] \quad (4.23)$$

for $|\lambda t| > R_2$, $|\lambda| > R_1$ and $|\mathrm{ph}(i\lambda)| \leq \pi - \delta$ for the principal branch of the logarithm, where ph denotes the phase and δ an arbitrary positive constant. Note that $\Phi_{-\lambda}(t) = \Phi_\lambda(t)$. For $\mathrm{Re}(\lambda) > 0$ the asymptotic behaviour of Jacobi functions is also known as

$$\Phi_\lambda(t) = c(-i\lambda) e^{(\lambda-\rho)t} (1 + o(1))$$

as $t \to \infty$, see Koornwinder (1984), (2.19).

We remark that the asymptotic behaviour of spherical Bessel functions (see Example 4.5.2) can be deduced from the asymptotics of Bessel functions of the first kind. Indeed, Hankel's expansion gives asymptotic expansions of any order, see DLMF (2015), Equation 10.17.3. Using DLMF (2015), Equation 10.7.8 we get

$$\Phi_\lambda(t) = j_\alpha(i\lambda t) = C_\alpha (i\lambda t)^{-(\alpha+\frac{1}{2})} (\cos(i\lambda t - (\alpha + \frac{1}{2})\frac{\pi}{2}) + e^{|\mathrm{Re}(\lambda t)|} o(1)) \quad (4.24)$$

as $|\lambda t| \to \infty$, $|\mathrm{ph}(i\lambda)| \leq \pi - \delta$, where $C_\alpha = \Gamma(\alpha+1) 2^{\alpha+\frac{1}{2}} / \Gamma(\frac{1}{2})$.

The following properties of Φ_λ for $\lambda \in \mathbb{R} \cup i\mathbb{R}$ were shown by Zeuner (1992), Propositions 4.2 and 4.3, and used to determine the dual space of a Sturm-Liouville hypergroup. We follow Bloom and Heyer (1995), Proposition 3.5.49. In advance, we remark that knowledge of the asymptotic behaviour of Φ_λ is also sufficient to determine the dual space, see Theorem 4.4.1.

Proposition 4.3.1. *The functions* Φ_λ *are*

(i) *(strictly) positive for* $\lambda \in \mathbb{R}_+$,

(ii) *strictly increasing for* $\lambda \in]\rho, \infty[$ *and, if* $\rho > 0$, *strictly decreasing for* $\lambda \in [0, \rho[$,

(iii) *bounded for* $\lambda \in i\mathbb{R}_+ \cup]0, \rho]$, *if* A *is not constant and* $\lambda \neq \rho$ *then*

$$\limsup_{t\to\infty} \Phi_\lambda(t) < 1.$$

The following example gives an impression of the asymptotic behaviour of multiplicative functions on Sturm-Liouville hypergroups. In the sequel, we will show that the asymptotic behaviour is quite similar in the general case.

Example 4.3.2. Consider the function $A(t) = e^{2\rho t}$ for some $\rho \geq 0$. It is easily checked that it is a Sturm-Liouville function satisfying SL1 and SL2. The corresponding Sturm-Liouville equation

$$y''(t) + 2\rho y'(t) = (\lambda^2 - \rho^2)y(t), \quad t > 0 \tag{4.25}$$

has the physical interpretation of a damped simple harmonic motion (see e. g. Walter, 2000, 20.III). Two linearly independent solutions of (4.25) with $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$ and $\lambda \neq 0$ are given by

$$\begin{aligned} y_+(t) &= \exp(\{-\rho + \lambda\}t) \\ y_-(t) &= \exp(\{-\rho - \lambda\}t). \end{aligned}$$

In this case, the initial values $y(0) = 1$, $y'(0) = 0$ give

$$\Phi_\lambda(t) = \left(\frac{1}{2} + \frac{\rho}{2\lambda}\right) y_+(t) + \left(\frac{1}{2} - \frac{\rho}{2\lambda}\right) y_-(t).$$

If $\lambda = 0$ then

$$\begin{aligned} y_+(t) &= t\exp(-\rho t) \\ y_-(t) &= \exp(-\rho t) \end{aligned}$$

and

$$\Phi_\lambda(t) = \rho y_+(t) + y_-(t).$$

The following notions of the Lagrange sesquilinear form and Green's formula can be found in Zettl (2005), p. 172. They will become an important tool in the following investigations. We formulate Green's formula in our special setting, emphasizing the Haar measure $A\lambda_{\mathbb{R}_+}$.

Let A be a Sturm-Liouville function. For y and z in the expression domain of L (see Definition 4.1.2) the Lagrange sesquilinear form $[\cdot,\cdot]$ is defined by

$$[y, z] = yA\bar{z}' - Ay'\bar{z}.$$

Lemma 4.3.3 (Green's Formula)**.** *Let A be a Sturm-Liouville function. For any y, z in the expression domain of L and $\delta, \gamma \in \mathbb{R}_+^\times$ we have*

$$\int_\delta^\gamma y(s)(\overline{Lz})(s)\,A(s)\mathrm{d}s - \int_\delta^\gamma (Ly)(s)\bar{z}(s)\,A(s)\mathrm{d}s = [y,z](\gamma) - [y,z](\delta).$$

Proof. This is a simple application of integration by parts. Indeed,

$$\begin{aligned}\int_\delta^\gamma y(s)(\overline{Lz})(s)\,A(s)\mathrm{d}s &= \int_\delta^\gamma y(s)(A(s)\bar{z}'(s))'\,\mathrm{d}s \\ &= y(s)A(s)\bar{z}'(s)\big|_{s=\delta}^{s=\gamma} - \int_\delta^\gamma y'(s)A(s)\bar{z}'(s)\,\mathrm{d}s\end{aligned}$$

and analogously

$$\int_\delta^\gamma (Ly)(s)\bar{z}(s)\,A(s)\mathrm{d}s = A(s)y'(s)\bar{z}(s)\big|_{s=\delta}^{s=\gamma} - \int_\delta^\gamma A(s)y'(s)\bar{z}'(s)\,\mathrm{d}s\,.$$

□

Lemma 4.3.4. *Let A be a Sturm-Liouville function satisfying SL2. Then for* $\mathrm{Re}(\lambda) > 0$ *and $\lambda = 0$, the function $\Phi_\lambda : \mathbb{R}_+ \to \mathbb{C}$ has no zeros.*

Proof. If $\lambda^2 = \rho^2$, then $\Phi_\lambda \equiv 1$. For $\rho > 0$ and $\lambda \in [0, \rho[$, Φ_λ is positive and decreasing, see Proposition 4.3.1. So it is sufficient to show that $\Phi_\lambda : \mathbb{R}_+ \to \mathbb{C}$ has no zeros for $\lambda \in \mathbb{C}\backslash(i\mathbb{R} \cup [-\rho, \rho])$.

Lommel's theorem states that all zeros of Bessel functions of the first kind $J_\alpha(z)$, $\alpha > -1$, are real. In Watson (1995), § 15·25 this is shown using the identity

$$(\lambda^2 - \mu^2)\int_0^t J_\alpha(\lambda s)J_\alpha(\mu s)\,\mathrm{d}s = t\left[J_\alpha(\lambda t)\frac{\mathrm{d}J_\alpha(\mu t)}{\mathrm{d}t} - J_\alpha(\mu t)\frac{\mathrm{d}J_\alpha(\lambda t)}{\mathrm{d}t}\right]$$

for $\lambda, \mu \in \mathbb{C}$ and $t > 0$. We transfer this idea to solutions of the Sturm-Liouville equation. In fact, taking the limit $\delta \to 0^+$ in Green's formula, see Lemma 4.3.3, we get

$$(\lambda^2 - \mu^2)\int_0^t \Phi_\lambda(s)\Phi_\mu(s)\,A(s)\mathrm{d}s = \Phi_\mu(t)A(t)\Phi_\lambda'(t) - \Phi_\lambda(t)A(t)\Phi_\mu'(t). \tag{4.26}$$

If $\lambda \in \mathbb{R}\backslash[-\rho, \rho]$ then $\Phi_\lambda \geq 1$ by its power series. So suppose $\lambda \in \mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R})$ and assume $\Phi_\lambda(t) = 0$ for some $t > 0$. Then $\lambda^2 \neq \bar{\lambda}^2$ and $\Phi_{\bar{\lambda}}(t) = \overline{\Phi_\lambda(t)} = 0$, which contradicts the last formula. □

Lemma 4.3.5. *Let A be a Sturm-Liouville function satisfying SL2. Then for* $\mathrm{Re}(\lambda) > 0$ *a second, linearly independent solution of Φ_λ solving*

$$Ly(t) = (\lambda^2 - \rho^2)y(t), \quad t > 0,$$

is given by

$$\Psi_\lambda(t) := \Phi_\lambda(t)\int_t^\infty \frac{1}{A(s)\Phi_\lambda(s)^2}\,\mathrm{d}s, \quad t > 0; \tag{4.27}$$

it satisfies

$$\lim_{t\to 0^+} \Psi_\lambda(t) = \int_0^\infty \frac{1}{A(s)\Phi_\lambda(s)^2}\,\mathrm{ds} \quad \textit{if } 0 \textit{ is a regular endpoint, i. e. } \frac{1}{A} \in L^1_{loc}(\mathbb{R}_+, \lambda_{\mathbb{R}_+}), \tag{4.28}$$

$$\lim_{t\to 0^+} |\Psi_\lambda(t)| = \infty \quad \textit{if } 0 \textit{ is a singular endpoint, i. e. } \frac{1}{A} \notin L^1_{loc}(\mathbb{R}_+, \lambda_{\mathbb{R}_+}), \tag{4.29}$$

and

$$\lim_{t\to 0^+} A(t)\Psi'_\lambda(t) = -1. \tag{4.30}$$

Moreover,

$$\lim_{t\to 0^+} A(t)\Psi_\lambda(t) = \begin{cases} A(0)\displaystyle\int_0^\infty \frac{1}{A(s)\Phi_\lambda(s)^2}\,\mathrm{ds} & \textit{if } A(0) > 0 \\ 0 & \textit{if } A(0) = 0, \end{cases} \tag{4.31}$$

in particular $\limsup_{t\to 0^+} A(t)|\Psi_\lambda(t)| < \infty$.

Formula (4.27) is known for real solutions from oscillation theory, see Zettl (2005), (6.2.4). In this context, Ψ_λ is called the *principal solution* for its property that the quotient of Ψ_λ and any other solution which is not a multiple of it tends to zero.

Proof. For (4.27), note that the integral exists by Lemma 4.3.4, and Lemma 4.3.6, Theorem 4.3.9 below (the proofs are conducted independently). We verify that Ψ_λ is a solution. Indeed, for $t > 0$

$$A(t)\Psi'_\lambda(t) = A(t)\Phi'_\lambda(t)\int_t^\infty \frac{1}{A(s)\Phi_\lambda(s)^2}\,\mathrm{ds} - \frac{1}{\Phi_\lambda(t)} \tag{4.32}$$

and thus

$$\begin{aligned}(A(t)\Psi'_\lambda(t))' = (A(t)\Phi'_\lambda(t))'\int_t^\infty \frac{1}{A(s)\Phi_\lambda(s)^2}\,\mathrm{ds} - \frac{\Phi'_\lambda(t)}{\Phi_\lambda(t)^2} + \frac{\Phi'_\lambda(t)}{\Phi_\lambda(t)^2} \\ = (\lambda^2 - \rho^2)A(t)\Phi_\lambda(t)\int_t^\infty \frac{1}{A(s)\Phi_\lambda(s)^2}\,\mathrm{ds} \\ = (\lambda^2 - \rho^2)A(t)\Psi_\lambda(t).\end{aligned}$$

Concerning the asymptotic behaviour in $t = 0$, (4.28) and (4.29) are clear. The limit (4.31) follows from (4.27), if $A(0) = 0$ we may apply the dominated convergence theorem using that A is monotonically increasing. Finally, for (4.30) use (4.32),

$$A(t)\Phi'_\lambda(t)\int_t^\infty \frac{1}{A(s)\Phi_\lambda(s)^2}\,\mathrm{ds} = \Phi'_\lambda(t)\Phi_\lambda(t)^{-1}A(t)\Psi_\lambda(t)$$

and (4.31). □

Next we investigate the asymptotic behaviour for $t \to \infty$. We start with a simple lemma (cf. Bloom and Xu, 1995, Lemma 3.28).

Lemma 4.3.6. *Let A be a Sturm-Liouville function satisfying SL2 and suppose $z \in \mathbb{C}$. Then*

$$A(t)^z = \exp(\{2\rho z + o(1)\}t)$$

as $t \to \infty$.

Proof. Take an arbitrary constant $\delta > 0$, then

$$A(t) = A(\delta) \exp\left(\int_\delta^t \frac{A'(s)}{A(s)} \, \mathrm{ds} \right) \tag{4.33}$$

for all $t > 0$ since $A'/A = (\log(A))'$. Thus given $z \in \mathbb{C}$

$$A(t)^z = A(\delta)^z \exp\left(z \int_\delta^t \frac{A'(s)}{A(s)} \, \mathrm{ds} \right) = ce^{2\rho z t} \exp\left(z \int_\delta^t \frac{A'(s)}{A(s)} - 2\rho \, \mathrm{ds} \right) \tag{4.34}$$

with a constant c depending only on δ, z and ρ, in particular

$$A(t)^z = \exp(\{2\rho z + o(1)\}t)$$

as $t \to \infty$ by definition of ρ. □

Lemma 4.3.7. *Let A be a Sturm-Liouville function satisfying SL2. Then for $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and $\lambda^2 \neq \rho^2$*

$$Ly = (\lambda^2 - \rho^2)y, \quad t > 0$$

has two linearly independent solutions y_+, y_- such that

$$\begin{aligned} y_+(t) &= \exp(\{-\rho + \lambda + o(1)\}\, t) \\ A(t){y_+}'(t) &= \exp(\{\rho + \lambda + o(1)\}\, t) \\ y_-(t) &= \exp(\{-\rho - \lambda + o(1)\}\, t) \\ A(t){y_-}'(t) &= -\exp(\{\rho - \lambda + o(1)\}\, t) \end{aligned} \tag{4.35}$$

as $t \to \infty$.

In Eastham (1989), Chapter 2 the general Sturm-Liouville equation

$$(p(t)y'(t))' - q(t)y(t) = 0, \quad t > 0$$

is reduced to the first order system

$$\begin{bmatrix} y(t) \\ p(t)y'(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1/p(t) \\ q(t) & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ p(t)y'(t) \end{bmatrix}, \quad t > 0, \tag{4.36}$$

which in turn, depending on the properties of p and q, can be transformed into an asymptotically diagonal system. Finally, asymptotic results are derived from Levinson's theorem (see also Remark 4.3.13). The following proof is an application of two of those results.

Proof of Lemma 4.3.7. In the following we always take the principal branch of the square root. Without loss of generality we may assume $\mathrm{Re}(\lambda) \geq 0$ and $t > \delta$ for an arbitrary constant $\delta > 0$. We set $r = \frac{1}{2}\frac{A'}{A}$ and use throughout that $r(t) \to \rho$ as $t \to \infty$ by (4.3).

1. Suppose $\rho = 0$ and $\lambda \neq 0$. In this case we may apply Theorem 2.2.1 in Eastham (1989), a generalization of the Liouville-Green asymptotic formulas, with $p = A$ and $q = \lambda^2 A$. Indeed, all conditions are satisfied, we mention that p, q are everywhere unequal to zero, $4r = o(1)$ at infinity and $(4r)' \in L^1([\delta,\infty), \lambda_{[\delta,\infty)})$ by Lemma 4.3.11 below (see also Remark 4.3.13). By choice of the square root we have $Re(q/p+r^2)^{\frac{1}{2}} \geq 0$ in $[\delta,\infty[$. Theorem 2.2.1 in Eastham (1989) gives solutions y_+ and y_- such that

$$
\begin{aligned}
y_+(t) &\sim (\lambda^2 A^2)^{-\frac{1}{4}} \exp(I(t)) \qquad (4.37)\\
A(t){y_+}'(t) &\sim (\lambda^2 A^2)^{\frac{1}{4}} \exp(I(t))\\
y_-(t) &\sim (\lambda^2 A^2)^{-\frac{1}{4}} \exp(-I(t)) \qquad (4.38)\\
A(t){y_-}'(t) &\sim -(\lambda^2 A^2)^{\frac{1}{4}} \exp(-I(t))
\end{aligned}
$$

with

$$I(t) = \int_\delta^t \left(\lambda^2 + r(s)^2\right)^{\frac{1}{2}} \,\mathrm{ds} = -\lambda\delta + \lambda t + \int_\delta^t \left(\lambda^2 + r(s)^2\right)^{\frac{1}{2}} - \lambda \,\mathrm{ds}\,. \qquad (4.39)$$

In particular $I(t) = \lambda t + o(1)t$ since r tends to zero, the asserted asymptotic formulas follow from Lemma 4.3.6.

2. Suppose $\rho > 0$, $\lambda \neq 0$ and $\lambda^2 \neq \rho^2$. In this case we may apply Theorem 2.6.1 in Eastham (1989) with $p = A$ and $q = (\lambda^2 - \rho^2)A$. Again, all conditions are satisfied, we mention that p, q are everywhere unequal to zero,

$$4r = \kappa(\lambda^2 - \rho^2)^{\frac{1}{2}}(1 + h)$$

where $\kappa = 4\rho/(\lambda^2 - \rho^2)^{\frac{1}{2}}$ ($\kappa \neq 0$, $\kappa^2 \neq -16$) and $h = \frac{1}{\rho}r - 1 = o(1)$. Moreover, $h' = \frac{1}{\rho}r' \in L^1([\delta,\infty), \lambda_{[\delta,\infty)})$, see Lemma 4.3.11 (again see also Remark 4.3.13) and $Re(q/p + r^2)^{\frac{1}{2}} \geq 0$ in $[\delta,\infty[$. Thus Theorem 2.6.1 in Eastham (1989) states that there exist solutions y_+ and y_- such that

$$
\begin{aligned}
y_+(t) &\sim ((\lambda^2 - \rho^2)A^2)^c \exp(I(t)) \qquad (4.40)\\
A(t){y_+}'(t) &\sim c\kappa((\lambda^2 - \rho^2)A^2)^{\frac{1}{2}+c} \exp(I(t))\\
y_-(t) &\sim ((\lambda^2 - \rho^2)A^2)^{-\frac{1}{2}-c} \exp(-I(t))\\
A(t){y_-}'(t) &\sim -(c\kappa)^{-1}((\lambda^2 - \rho^2)A^2)^{-c} \exp(-I(t))
\end{aligned}
$$

with

$$c = -\frac{1}{4} + \frac{1}{4}(1 + 16\kappa^{-2})^{\frac{1}{2}} = -\frac{1}{4} + \frac{1}{4}\frac{(\lambda^2)^{\frac{1}{2}}}{\rho} = -\frac{1}{4} + \frac{1}{4}\frac{\lambda}{\rho}$$

and

$$I(t) = \int_\delta^t r(s)H(s)\,\mathrm{ds} \tag{4.41}$$

where

$$H = -16\kappa^{-2}(1+16\kappa^{-2})^{-\frac{1}{2}}h + \mathcal{O}(h^2) = -\frac{\lambda^2-\rho^2}{\rho\lambda}h + \mathcal{O}(h^2).$$

The stated asymptotic formulas follow now again from Lemma 4.3.6, note that $\exp(I(t)) = \exp(o(1)t)$ and the leading constants may be included in the term $\exp(o(1)t)$.

□

Remark 4.3.8. Suppose $\rho > 0$ and $\lambda^2 = \rho^2$. Then $q = 0$ and any solution is a linear combination of $y_+ = 1$ and $y_-(t) = \int_t^\infty \frac{1}{A(s)}\,\mathrm{ds}$. We have $y_+(t) = 1$, $A(t){y_+}'(t) = 0$, $y_-(t) = \exp(\{-2\rho + o(1)\}t)$ by Lemma 4.3.6, and $A(t){y_-}'(t) = -1$.

If $\rho = 0$ and $\lambda = 0$ consider $y_-(t) = -\int_\delta^t \frac{1}{A(s)}\,\mathrm{ds}$ for some $\delta > 0$, then $y_-(t) = -\exp(o(1)t)$ and $A(t){y_-}'(t) = -1$.

Theorem 4.3.9. *Let A be a Sturm-Liouville function satisfying SL2. Then for* $\mathrm{Re}(\lambda) > 0$ *and* Φ_λ, Ψ_λ *as above*

$$\begin{aligned} \Phi_\lambda(t) &= \exp(\{-\rho+\lambda+o(1)\}\,t) \\ A(t)\Phi_\lambda'(t) &= \exp(\{\rho+\lambda+o(1)\}\,t) \\ \Psi_\lambda(t) &= \exp(\{-\rho-\lambda+o(1)\}\,t) \\ A(t)\Psi_\lambda'(t) &= -\exp(\{\rho-\lambda+o(1)\}\,t) \end{aligned} \tag{4.42}$$

as $t \to \infty$.

Proof. First suppose $\lambda > 0$. Then $\Phi_\lambda(t) \geq e^{-\rho t}$ for all $t \in \mathbb{R}_+$ by (4.22) (if $\lambda \geq \rho$ then we even have $\Phi_\lambda(t) \geq 1$ for $t \in \mathbb{R}_+$ by (4.10)). Comparing with Lemma 4.3.7 yields that the asymptotic formulas for y_+ and $A\,{y_+}'$ also hold true for Φ_λ and $A\,\Phi_\lambda'$.

Suppose now $\mathrm{Re}(\lambda) > 0$ but λ not real. We show that Φ_λ is not a scalar multiple of y_- as stated in Lemma 4.3.7. Green's formula states in the form (4.26) and with $\mu = \overline{\lambda}$

$$(\lambda^2 - \overline{\lambda}^2)\int_0^t |\Phi_\lambda(s)|^2 A(s)\mathrm{d}s = \overline{\Phi_\lambda(t)}A(t)\Phi_\lambda'(t) - \Phi_\lambda(t)\overline{A(t)\Phi_\lambda'(t)}. \tag{4.43}$$

Assume $\Phi_\lambda = c\,y_-$ for some constant $c \in \mathbb{C}\backslash\{0\}$. Then (4.35) yields

$$\overline{\Phi_\lambda(t)}A(t)\Phi_\lambda'(t) = \exp(\{-\rho-\overline{\lambda}+o(1)\}\,t)\exp(\{\rho-\lambda+o(1)\}\,t) = \exp(\{-\overline{\lambda}-\lambda+o(1)\}\,t)$$

as $t \to \infty$. So the right hand side of (4.43) tends to zero whereas the left hand side obviously does not, which is a contradiction. As in the case $\lambda > 0$ we conclude that the asymptotic formulas for y_+ and $A\,{y_+}'$ also hold true for Φ_λ and $A\,\Phi_\lambda'$.

Concerning the asymptotic of Ψ_λ, take an arbitrary $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. We may plug in the definition of Ψ_λ, see (4.27), the asymptotics of Φ_λ and A (see Lemma 4.3.6). Estimating the absolute value of Ψ_λ and comparing with Lemma 4.3.7 yields that Ψ_λ must be a scalar multiple of y_-. □

Remark 4.3.10. We note that (4.42) is also valid for $\lambda = 0$. Indeed, for $\lambda > 0$, $e^{-\rho t} \leq \Phi_0(t) \leq \Phi_\lambda(t)$ for all $t \in \mathbb{R}_+$ by (4.22), the statement follows readily.

Lemma 4.3.11. *Suppose A is a Sturm-Liouville function satisfying SL2. Then the quotient A'/A is of bounded variation on $[\delta, \infty)$ for any $\delta > 0$, in signs $A'/A \in BV([\delta, \infty))$. Thus this quotient is differentiable Lebesgue almost everywhere and*

$$(A'/A)' \in L^1([\delta, \infty), \lambda_{[\delta,\infty)})$$

for any $\delta > 0$.

Proof. Suppose $\delta > 0$. Condition SL2 states that there exists $\beta \in C^1(\mathbb{R}_+)$ such that $\frac{A'}{A} - \beta =: \widetilde{\alpha} \geq 0$ and $q := \frac{1}{2}\beta' - \frac{1}{4}\beta^2 + \frac{A'}{2A}\beta$ are decreasing on $\mathbb{R}_+^\times$. So $\frac{A'}{A} = \beta + \widetilde{\alpha}$ and thus $\frac{A'}{A} \in BV([\delta, \infty))$ is equivalent to $\beta \in BV([\delta, \infty))$. We show that $\beta \in BV([\delta, \infty))$. Therefore we may assume without loss of generality $\beta \geq 0$, see Bloom and Heyer (1995), 3.5.15. Our starting point is that $q = \frac{1}{2}\beta' - \frac{1}{4}\beta^2 + \frac{1}{2}(\widetilde{\alpha} + \beta)\beta = \frac{1}{2}\beta' + \frac{1}{4}\beta^2 + \frac{1}{2}\widetilde{\alpha}\beta$ is monotonically decreasing, that is for $t \geq s$

$$\begin{aligned} q(t) - q(s) &= \frac{1}{2}(\beta'(t) - \beta'(s)) + \frac{1}{4}(\beta(t) + \beta(s))(\beta(t) - \beta(s)) \\ &\quad + \frac{1}{2}\widetilde{\alpha}(t)(\beta(t) - \beta(s)) + \frac{1}{2}(\widetilde{\alpha}(t) - \widetilde{\alpha}(s))\beta(s) \\ &= \frac{1}{2}(\beta'(t) - \beta'(s)) + \frac{1}{4}(\beta(t) - \beta(s))^2 \\ &\quad + \frac{1}{2}(\beta(s) + \widetilde{\alpha}(t))(\beta(t) - \beta(s)) + \frac{1}{2}\beta(s)(\widetilde{\alpha}(t) - \widetilde{\alpha}(s)) \\ &\leq 0. \end{aligned}$$

Here we see, that we may assume without loss of generality $\beta > 0$. Indeed, $\beta \geq 0$ by assumption, so suppose $\beta(s_0) = 0$ for some $s_0 \in [\delta, \infty)$. Then it must be $\beta'(s_0) = 0$ and the inequality above implies $\beta'(t) \leq 0$ for $t \geq s_0$, thus $\beta(t) = 0$ for $t \geq s_0$.

Now suppose $\delta \leq s_0 < t_0$, $\beta'(s_0) = 0$ and β is monotonically increasing on $[s_0, t_0]$. Then the inequality above implies $\frac{1}{2}\beta(s_0)(\beta(t) - \beta(s_0)) \leq -\frac{1}{2}\beta(s_0)(\widetilde{\alpha}(t) - \widetilde{\alpha}(s_0))$ and multiplying by $\frac{2}{\beta(s_0)} > 0$ we obtain

$$\beta(t) - \beta(s_0) \leq -(\widetilde{\alpha}(t) - \widetilde{\alpha}(s_0)) \tag{4.44}$$

for $t \in [s_0, t_0]$.

The central theorem of calculus states that for $t \geq \delta$ we have

$$\beta(t) - \beta(\delta) = \int_\delta^t \beta'(s)\,\mathrm{ds} = \int_\delta^t 1_{\{\beta'>0\}}(s)\beta'(s)\,\mathrm{ds} + \int_\delta^t 1_{\{\beta'\leq 0\}}(s)\beta'(s)\,\mathrm{ds}\,. \tag{4.45}$$

The set $\{s > \delta : \beta'(s) > 0\}$ is open since $\beta \in C^1(\mathbb{R}_+)$. Thus it is the union of a countable family of open intervals $\{(s_i, t_i), s_i < t_i, i \in I\}$. Without loss of generality we may assume that these intervals are disjoint. Then β is monotonically increasing on each of the intervals $[s_i, t_i]$ and $\beta'(s_i) = \beta'(t_i) = 0$. Thus we get from (4.44)

$$\begin{aligned} \int_\delta^t 1_{\{\beta'>0\}}(s)\beta'(s)\,\mathrm{ds} &\leq \int_\delta^\infty 1_{\{\beta'>0\}}(s)\beta'(s)\,\mathrm{ds} = \sum_{i\in I}\int_{s_i}^{t_i}\beta'(s)\,\mathrm{ds} \\ &= \sum_{i\in I}(\beta(t_i) - \beta(s_i)) \leq \sum_{i\in I} -(\widetilde{\alpha}(t_i) - \widetilde{\alpha}(s_i)) \leq \widetilde{\alpha}(\delta). \end{aligned}$$

This, (4.45) and $\beta \geq 0$ imply

$$\int_\delta^t 1_{\{\beta' \leq 0\}}(s)\beta'(s)\ \mathrm{ds} \geq -\beta(\delta) - \widetilde{\alpha}(\delta)$$

for all $t > \delta$. Now (4.45) states that $\beta(t) - \beta(\delta)$ is the sum of two monotonous and bounded functions, thus $\beta \in BV(\mathbb{R}_+)$.

Finally, it is well-known that the second statement of the lemma follows from the first one (see, e. g., Rudin, 1987, Chapter 7, Exercise 13(e)). □

Remark 4.3.12. Writing $A'/A = \beta + \widetilde{\alpha}$ (4.33) states that for an arbitrary constant $\delta > 0$

$$A(t) = A(\delta) \exp\left(\int_\delta^t \beta(s) + \widetilde{\alpha}(s)\ \mathrm{ds}\right). \tag{4.46}$$

for all $t > 0$. Conversely, suppose $\widetilde{\alpha}$ is a continuous non-negative and decreasing function defined on $\mathbb{R}_+^\times$ which vanishes at infinity, suppose $\beta \equiv 2\rho$ for some constant $\rho \geq 0$, and let $A(\delta)$ denote an arbitrary positive constant. Then (4.46) defines a Sturm-Liouville function A, $A'/A = \beta + \widetilde{\alpha}$ and SL2 is satisfied. If $\widetilde{\alpha}$ is chosen carefully then also SL1 is satisfied. If we choose $\widetilde{\alpha}$ to behave somewhere like a Cantor type function, this example shows that A'/A is in general not locally absolutely continuous (cf. Rudin, 1987, Example 7.16(b)).

Remark 4.3.13. The proof of Lemma 4.3.7 relies on Theorem 2.2.1 and Theorem 2.6.1 in Eastham (1989). Using the notation there, it is required that p and q have locally absolutely continuous first derivatives in $[a, \infty)$. However, we do not have to assume the same property for A since the quoted theorems are also true if p' and q' are only assumed to be of bounded variation in $[a, \infty)$. This is justified in the following. All stated references refer to Eastham (1989).

In Eastham (1989) the proofs of Theorem 2.2.1 and Theorem 2.6.1 are based on Theorem 1.6.1, which in turn uses Theorem 1.3.1, the Levinson theorem. The condition that p' and q' are locally absolutely continuous is only used in the proof of Theorem 1.6.1, see the bottom line on page 25 where the central theorem of calculus is applied. However, this part of the proof is not needed in the situation of Theorem 2.2.1 and Theorem 2.6.1 since the diagonal form (1.6.6) can be computed explicitly.

In the proof of Theorem 2.2.1 the original system

$$Y' = AY$$

with

$$Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{1}{p} \\ q & 0 \end{bmatrix}$$

is transformed by $Z = T_0 Y$ with

$$T_0 := \begin{bmatrix} 1 & 1 \\ (pq)^{\frac{1}{2}} & -(pq)^{\frac{1}{2}} \end{bmatrix}$$

to

$$Z' = (\Lambda + R)Z$$

where

$$\Lambda = (q/p)^{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R = -(q/p)^{\frac{1}{2}} s \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

with

$$s := \frac{1}{4}(pq)'/p^{\frac{1}{2}}q^{\frac{3}{2}} = (p/q)^{\frac{1}{2}} r$$

and $r = \frac{1}{4}(pq)'/pq$; by assumption $s = o(1)$ and $s' \in L(a,\infty)$. Then Theorem 1.6.1 is applied to the system $Z' = (\Lambda + R)Z$. The eigenvalues of $\Lambda + R$ are given by

$$\mu_1 = (q/p)^{\frac{1}{2}}(-s + (1+s^2)^{\frac{1}{2}}), \quad \mu_2 = (q/p)^{\frac{1}{2}}(-s - (1+s^2)^{\frac{1}{2}}),$$

see (2.2.7). After scaling by $(q/p)^{-\frac{1}{2}}$ it is easy to check that the diagonal form (1.6.6)

$$T^{-1}(\Lambda + R)T = \Lambda_1$$

with $\Lambda_1 = \mathrm{dg}(\mu_1, \mu_2)$ is satisfied by

$$T = \begin{bmatrix} \frac{1}{2}(1+s^2)^{\frac{1}{2}} + \frac{1}{2} & -\frac{1}{2}s \\ \frac{1}{2}s & \frac{1}{2}(1+s^2)^{\frac{1}{2}} + \frac{1}{2} \end{bmatrix}$$

and the difference $Q = T - I$ satisfies the conditions $Q(x) \to 0$ as $x \to \infty$ and $Q' \in L(a,\infty)$ since $s = o(1)$ and $s' \in L(a,\infty)$.

Concerning the proof of Theorem 2.6.1 the system $Y' = AY$ is transformed in the same way to $Z' = (\Lambda + R)Z$. Here by assumption $s = \frac{1}{4}\kappa(1+\phi)$ with $\phi = o(1)$ and $\phi' \in L(a,\infty)$, hence $\Lambda + R = (q/p)^{\frac{1}{2}}(C + R_1)$ with

$$C = \begin{bmatrix} 1 - \frac{1}{4}\kappa & \frac{1}{4}\kappa \\ \frac{1}{4}\kappa & -1 - \frac{1}{4}\kappa \end{bmatrix}, \quad R_1 = -\frac{1}{4}\kappa\phi \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The proposed way is to diagonalize C by

$$T_1 = \begin{bmatrix} \frac{1}{4}\kappa & -\frac{1}{4}\kappa \\ -1 + (1+\kappa^2/16)^{\frac{1}{2}} & 1 + (1+\kappa^2/16)^{\frac{1}{2}} \end{bmatrix}.$$

On the other hand we already know from the setting of Theorem 2.2.1 above that

$$T^{-1}(C + R_1)T = (q/p)^{-\frac{1}{2}}\Lambda_1.$$

Taking the limit $x \to \infty$ we obtain $T(x) \to T_\infty$ with

$$T_\infty := \begin{bmatrix} \frac{1}{2}(1+\kappa^2/16)^{\frac{1}{2}} + \frac{1}{2} & -\frac{1}{8}\kappa \\ \frac{1}{8}\kappa & \frac{1}{2}(1+\kappa^2/16)^{\frac{1}{2}} + \frac{1}{2} \end{bmatrix}$$

and

$$(q/p)^{-\frac{1}{2}}\Lambda_1 \to \Lambda_\infty := \mathrm{dg}(-\frac{1}{4}\kappa + (1+\frac{\kappa^2}{16})^{\frac{1}{2}}, -\frac{1}{4}\kappa - (1+\frac{\kappa^2}{16})^{\frac{1}{2}}),$$

hence $T_\infty^{-1}CT_\infty = \Lambda_\infty$. Instead of taking the further transformation $Z = T_1 W$, as done in (2.6.11), we consider the transformation $Z = T_\infty W$. Then the role of "T" in (1.6.6) is played by $\widetilde{T} = T_\infty^{-1}T$ with T as stated above, in particular $\widetilde{Q} = \widetilde{T} - I$ satisfies $\widetilde{Q}(x) \to 0$ as $x \to \infty$ and $\widetilde{Q}' \in L(a,\infty)$. The inverse transformations in the proof of Theorem 2.6.1 are not affected.

4.4 Some Asymptotic Proofs for Sturm-Liouville Hypergroups

The asymptotic results from above allow us to determine the dual space of Sturm-Liouville hypergroups. This was done before, by Zeuner (1992), Section 4 on the basis of Proposition 4.3.1, see Bloom and Heyer (1995), Theorem 3.5.50.

Theorem 4.4.1. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL1 and SL2. Then the characters are given by Φ_λ with $\lambda \in i\mathbb{R}_+ \cup [0, \rho]$.*

We remark that the following proof does not depend on the Laplace Representation Theorem 4.4.4.

Proof. The multiplicative functions are given by Φ_λ, $\lambda \in \mathbb{C}$, see Proposition 4.1.6. The semicharacters are those functions that satisfy additionally $\overline{\Phi_\lambda} = \Phi_{\lambda^-}$. The last condition is equivalent to $\Phi_{\overline{\lambda}} = \Phi_\lambda$ since $\overline{\Phi_\lambda} = \Phi_{\overline{\lambda}}$ and involution is the identity, thus to $\lambda^2 = \overline{\lambda}^2$ and $\lambda \in \mathbb{R} \cup i\mathbb{R}$ (cf. Bloom and Heyer, 1995, Theorem 3.5.50).

For $\lambda > \rho$, Φ_λ is unbounded by Theorem 4.3.9. If $\rho = 0$, then $\Phi_0 \equiv 1$ and for $\lambda \in i\mathbb{R}\backslash\{0\}$ we get Φ_λ bounded by (4.37) and (4.38). If $\rho > 0$, then $\Phi_\rho \equiv 1$ and for $\lambda \in i\mathbb{R}_+ \cup [0, \rho]\backslash\{0, \rho\}$, Φ_λ tends to zero by Lemma 4.3.7. These are bounded multiplicative functions, thus their absolute value must be bounded by 1, taking the limit $\lambda \to 0$ yields $|\Phi_0| \leq 1$. Finally, note that $\Phi_{-\lambda} = \Phi_\lambda$. □

Proposition 4.4.2. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL1 and SL2. Suppose $\lambda \in i\mathbb{R}_+ \cup [0, \rho[$. Then*

$$\lim_{t\to\infty} \Phi_\lambda(t) = 0$$

if and only if A is unbounded.

Compare this result with Proposition 4.3.1(iii).

Proof. If $\rho > 0$ then A is unbounded and the statement follows from Theorem 4.3.9, Remark 4.3.10, and Lemma 4.3.7. In case of $\rho = 0$ consider (4.37) and (4.38). □

Theorem 4.4.1 states that $\widehat{\mathbb{R}_+} = \{\Phi_\lambda, \lambda \in D\}$ where $D := i\mathbb{R}_+ \cup [0, \rho]$. Note that $\widehat{\mathbb{R}_+}$ bears the compact-open topology whereas D is endowed with the Euclidean topology. The following proposition is basic for harmonic analysis on $\widehat{\mathbb{R}_+}$ and is used in the literature without further notice. For hypercomplex systems constructed for the Sturm-Liouville equation a proof can be found in Berezansky and Kalyuzhnyi (1998), see Theorem 4.4 of Chapter 2.

Proposition 4.4.3. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL2. Then the spaces $\widehat{\mathbb{R}_+}$ and D are topologically isomorphic.*

Proof. We show that the mapping $\lambda \mapsto \Phi_\lambda$, $D \to \widehat{\mathbb{R}_+}$ is a homeomorphism. It is easy to see that the stated mapping is continuous, explicitly, apply the mean value theorem to $\lambda \mapsto \Phi_\lambda(t)$, see (4.10) and use (4.7).

We deduce continuity of the inverse mapping from the resolvent formula for multiplicative functions, see Theorem 5.4.6. It gives that for a fixed $\lambda_0 > \rho$

$$(\lambda_0^2 - \lambda^2)^{-1} = \int_0^\infty \Psi_{\lambda_0}(s)\Phi_\lambda(s)\, A(s)\mathrm{d}s \tag{4.47}$$

for all $\lambda \in D$. Since $\Psi_{\lambda_0} \in L^1(\mathbb{R}_+, A(s)\mathrm{d}s)$ it is easy to see from the definition of the compact-open topology that the mapping $\Phi_\lambda \mapsto \int_0^\infty \Psi_{\lambda_0}(s)\Phi_\lambda(s)\, A(s)\mathrm{d}s$, $\widehat{\mathbb{R}_+} \to \mathbb{C}$ is continuous. (More generally, the compact-open topology on $\widehat{\mathbb{R}_+}$ is topologically isomorphic to a subspace of the structure space $\Delta(L^1(\mathbb{R}_+, A(s)\mathrm{d}s))$ endowed with the Gelfand topology, see Bloom and Heyer, 1995, pp. 81–82, Proposition 2.2.7.) Thus, $\Phi_\lambda \mapsto \lambda$, $\widehat{\mathbb{R}_+} \to D$ is continuous by (4.47). □

As announced at the end of Section 4.2, we give here an asymptotic proof of the Laplace representation theorem (cf. Bloom and Heyer, 1995, Theorem 3.5.58). The proof reveals that for Sturm-Liouville hypergroups the centered translation converges in distribution, see (4.48) and the examples below.

Theorem 4.4.4. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL1 and SL2. Then for each $t \in \mathbb{R}_+$ there exists $\nu_t \in M^1([-t,t])$ such that*

$$\lim_{T\to\infty} (\varepsilon_T * \varepsilon_t)(T + \cdot\,) = \nu_t \tag{4.48}$$

in distribution and

$$\Phi_\lambda(t) = \int_{-t}^{t} e^{(-\rho+\lambda)r}\, \nu_t(\mathrm{d}r) \tag{4.49}$$

for all $\lambda \in \mathbb{C}$.

Proof. Suppose $t > 0$ and $T > t$. In the following we always suppose $\mathrm{Re}(\lambda) > 0$ and $\lambda \notin]0, \rho[$; for this range of λ the proofs of the preceding section do not use the Laplace Representation Theorem 4.4.4.

1. Our starting point is the product formula

$$\Phi_\lambda(T)\Phi_\lambda(t) = \int_{T-t}^{T+t} \Phi_\lambda(r)(\varepsilon_T * \varepsilon_t)(\mathrm{d}r),$$

which we may rewrite as

$$\Phi_\lambda(t) = \int_{T-t}^{T+t} \Phi_\lambda(T)^{-1}\Phi_\lambda(r)(\varepsilon_T * \varepsilon_t)(\mathrm{d}r) = \int_{-t}^{t} \Phi_\lambda(T)^{-1}\Phi_\lambda(T+r)(\varepsilon_T * \varepsilon_t)(T + \mathrm{d}r) \tag{4.50}$$

since Φ_λ does not possess any zeros by Lemma 4.3.4.

Looking at the proofs from the preceding section once again, we see from (4.34), (4.37) with (4.39) in case of $\rho = 0$ and (4.40) with (4.41) in case of $\rho > 0$, and the proof of Theorem 4.3.9 that given $\delta > 0$ there exists a constant $c(\lambda, \delta) \neq 0$ and a function $h = o(1)$ such that

$$\Phi_\lambda(r) = c(\lambda, \delta) e^{(-\rho+\lambda)r} \exp\left(\int_\delta^r h(u)\,\mathrm{d}u\right) \cdot (1 + o(1))$$

as $r \to \infty$. In particular

$$\Phi_\lambda(T)^{-1}\Phi_\lambda(T+r) = e^{(-\rho+\lambda)r}(1+o(1))$$

as $T \to \infty$, uniformly for $r \in [-t, t]$, and (4.50) yields

$$\Phi_\lambda(t) = \lim_{T\to\infty} \int_{-t}^{t} e^{(-\rho+\lambda)r}(\varepsilon_T * \varepsilon_t)(T + \mathrm{d}r). \tag{4.51}$$

2. If $\rho > 0$ then (4.51) contains that

$$\Phi_{\rho+i\omega}(t) = \lim_{T\to\infty} \int_{-t}^{t} e^{i\omega r}(\varepsilon_T * \varepsilon_t)(T + \mathrm{d}r)$$

for all $\omega \in \mathbb{R}$, the Lévy continuity theorem gives $\nu_t \in M^1(\mathbb{R})$ such that

$$\lim_{T\to\infty} (\varepsilon_T * \varepsilon_t)(T + \,\cdot\,) = \nu_t$$

in distribution and

$$\Phi_{\rho+i\omega}(t) = \int_{-t}^{t} e^{i\omega r}\nu_t(\mathrm{d}r)$$

for all $\omega \in \mathbb{R}$. The left and right hand side of this equation can be extended to the complex plane and after transforming back we get

$$\Phi_\lambda(t) = \int_{-t}^{t} e^{(-\rho+\lambda)r}\,\nu_t(\mathrm{d}r)$$

for all $\lambda \in \mathbb{C}$.

3. If $\rho = 0$ take $\varepsilon > 0$, then (4.51) gives

$$\Phi_{\varepsilon+i\omega}(t) = \lim_{T\to\infty} \int_{-t}^{t} e^{i\omega r}e^{\varepsilon r}(\varepsilon_T * \varepsilon_t)(T + \mathrm{d}r)$$

for all $\omega \in \mathbb{R}$. The Lévy continuity theorem gives $\nu_{t,\varepsilon} \in M_+^b(\mathbb{R})$ such that

$$\lim_{T\to\infty} e^{\varepsilon r}(\varepsilon_T * \varepsilon_t)(T + \mathrm{d}r) = \nu_{t,\varepsilon}$$

in distribution and

$$\Phi_{\varepsilon+i\omega}(t) = \int_{-t}^{t} e^{i\omega r}\nu_{t,\varepsilon}(\mathrm{d}r)$$

for all $\omega \in \mathbb{R}$. Setting $\nu_t := e^{-\varepsilon r}\nu_{t,\varepsilon}$ it follows readily that

$$\lim_{T\to\infty} (\varepsilon_T * \varepsilon_t)(T + \,\cdot\,) = \nu_t$$

in distribution and

$$\Phi_{\varepsilon+i\omega}(t) = \int_{-t}^{t} e^{(\varepsilon+i\omega)r} \nu_t(\mathrm{d}r)$$

for all $\omega \in \mathbb{R}$. As in the case $\rho > 0$ we get

$$\Phi_\lambda(t) = \int_{-t}^{t} e^{\lambda r}\, \nu_t(\mathrm{d}r)$$

for all $\lambda \in \mathbb{C}$, the case $\lambda = 0$ gives $\nu_t \in M^1(\mathbb{R})$. □

Remark 4.4.5. Since $\Phi_{i\lambda} = \Phi_{-i\lambda}$ the Laplace representation (4.49) gives

$$\int_{-t}^{t} \sin(\lambda r)\, e^{-\rho r} \nu_t(\mathrm{d}r) = 0$$

for all $\lambda \in \mathbb{R}_+$. Thus $e^{-\rho\,\cdot}\nu_t$ is an even measure and setting

$$\mu_t(\mathrm{d}r) := (2 \cdot 1_{]0,t]}(r) + 1_{\{0\}}(r))e^{-\rho r}\nu_t(\mathrm{d}r)$$

the Laplace representation (4.49) may be rewritten as

$$\Phi_\lambda(t) = \int_0^t \cosh(\lambda r)\, \mu_t(\mathrm{d}r).$$

In special cases the explicit form of the measure ν_t is well-known. In some cases, the structure of convolution of point measures is known, then (4.48) provides a direct method to determine ν_t. We list here some prominent examples; see Section 4.5 for definitions and basic properties of specific Sturm-Liouville hypergroups.

Let us start with the cosine hypergroup. Here obviously

$$\nu_t = \frac{1}{2}\varepsilon_t + \frac{1}{2}\varepsilon_{-t}.$$

For the cosh hypergroup $\rho = 1$ and

$$e^{-\,\cdot}\nu_t = \frac{1}{2\cosh t}\varepsilon_t + \frac{1}{2\cosh(-t)}\varepsilon_{-t},$$

which is easily derived from the simple form of convolution of point measures (see (4.55)) and (4.48).

For a Bessel-Kingman hypergroup of order $\alpha > -\frac{1}{2}$ the Laplace representation (4.49) is exactly the Poisson integral (5.34), so ν_t is described by its kernel

$$\nu_t(\mathrm{d}r) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})t^{2\alpha}}(t^2 - r^2)^{\alpha-\frac{1}{2}}\lambda_{(-t,t)}(\mathrm{d}r).$$

It can also be regarded as the limiting distribution of radial random walks on spheres, see Kingman (1963).

Finally, consider a Jacobi hypergroup of order (α, β) with $\alpha > \beta > -\frac{1}{2}$. Its Laplace representation (4.49) is a generalized Mehler-Dirichlet type integral, explicitly

$$e^{-\rho r}\nu_t(\mathrm{d}r) = \frac{1}{2}A_{\alpha,\beta}(|r|, t)\lambda_{(-t,t)}(\mathrm{d}r)$$

for $t > 0$ where

$$A_{\alpha,\beta}(r,t) = \frac{\Gamma(\alpha+1)2^{\alpha-2\beta+\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\frac{\sinh 2t}{\sinh^{2\alpha+1} t\cosh^{2\beta+1} t}$$
$$\times\int_r^t (\cosh 2t - \cosh 2q)^{\beta-\frac{1}{2}}(\cosh q - \cosh r)^{\alpha-\beta-1}\sinh q\,\mathrm{d}q \times 1_{(0,t)}(r),$$

see (2.16) and (2.18) in Koornwinder (1975) where this formula is developed to prove a Paley-Wiener type theorem for Jacobi functions.

In the special case of a hyperbolic hypergroup, that is if $\alpha > \beta = -\frac{1}{2}$, it is

$$e^{-\rho r}\nu_t(\mathrm{d}r) = \frac{1}{2}A_{\alpha,-\frac{1}{2}}(r,t)\lambda_{(-t,t)}(\mathrm{d}r)$$

for $t > 0$ where

$$A_{\alpha,-\frac{1}{2}}(r,t) = \frac{\Gamma(\alpha+1)2^{\alpha+\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}(\sinh t)^{-2\alpha}(\cosh t - \cosh r)^{\alpha-\frac{1}{2}} \times 1_{(-t,t)}(r), \tag{4.52}$$

see Zeuner (1986), 2.3, or Theorem 5.6.4 where it is contained as a special case.

4.5 Examples

For reasons of self-containedness we include here a list of prominent examples following Bloom and Heyer (1995), pp. 234–240. We only mention facts which are interesting in our context. Essentially, there are two classes of Sturm-Liouville hypergroups, Chébli-Trimèche and Levitan hypergroups.

Definition 4.5.1. A *Chébli-Trimèche function* is a Sturm-Liouville function A of type SL1.1 such that the quotient $\frac{A'}{A} \geq 0$ is decreasing, A is increasing and $\lim_{t\to\infty} A(t) = \infty$. In this case SL2 is satisfied with $\beta := 0$. A *Levitan function* is a Sturm-Liouville function of type SL1.2 with the additional assumption that $A \in C^2(\mathbb{R}_+)$. Further we suppose that SL2 is satisfied. (This is the case if $A'(0) \geq 0$ and $q := \frac{1}{2}\beta' + \frac{1}{2}\beta^2$ is decreasing where $\beta = \frac{A'}{A}$.) The corresponding hypergroups are called *Chébli-Trimèche* and *Levitan hypergroups*, respectively.

Before we start, let us remind that the notion of a Sturm-Liouville hypergroup is not restricted to single examples, as the following list might suggest. In fact any Sturm-Liouville function A satisfying SL1 and SL2 gives rise to a Sturm-Liouville hypergroup, see Theorem 4.1.4. In many cases conditions SL1 and SL2 are easily checked, see e. g. Remark 4.3.12 and Example 4.3.2. Moreover, A is open to scaling, that is if A is a Sturm-Liouville function satisfying SL1 and SL2 then the same is true for $c\,A(\gamma\,\cdot)$ where c and γ are arbitrary positive constants, see Remark 4.1.7 (cf. Remark 5.2.7 for operator functions).

4.5.1 Chébli-Trimèche hypergroups

There are two principal examples, Bessel-Kingman and Jacobi hypergroups.

Example 4.5.2 (Bessel-Kingman hypergroups)**.** A *Bessel-Kingman hypergroup* of order $\alpha > -\frac{1}{2}$ is a Sturm-Liouville hypergroup with respect to the Chébli-Trimèche function

$$A(t) = t^{2\alpha+1},$$

hence $\rho = 0$. Convolution of point measures is given for $t, s \in \mathbb{R}_+$ by

$$\varepsilon_t * \varepsilon_s = c_\alpha \int_0^\pi \varepsilon_{\sqrt{t^2+s^2-2ts\cos\theta}} \sin^{2\alpha}\theta \, \mathrm{d}\theta \tag{4.53}$$

where $c_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}$. This convolution can be interpreted as a radial random walk with spherical symmetry, see Kingman (1963). Further contributions to probability theory are due to Finckh (1986). Convolution (4.53) may be rewritten by a change of variables as

$$\varepsilon_t * \varepsilon_s = \begin{cases} w_\alpha(t,s,\cdot)\lambda_{\mathbb{R}_+} & \text{if } t,s>0 \\ \varepsilon_t & \text{if } s=0 \\ \varepsilon_s & \text{if } t=0 \end{cases}$$

where $w_\alpha : \mathbb{R}_+^\times \times \mathbb{R}_+^\times \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$w_\alpha(t,s,r) := \begin{cases} c_\alpha \frac{2^{2\alpha-1}(A(t,s,r))^{2\alpha-1}}{(tsr)^{2\alpha}} r^{2\alpha+1} & \text{if } |t-s| < r < t+s, \\ 0 & \text{else} \end{cases}$$

with c_α as above and

$$A(t,s,r) = \frac{1}{4}[(r^2-(t-s)^2)((t+s)^2-r^2)]^{\frac{1}{2}}$$

the area of a triangle with sides of length t, s and r, see Trimèche (2001), Subsection 1.II.4 and Finckh (1986), Section I.1, pp. 11–15.

The multiplicative functions have the particular form

$$\phi_\lambda(t) = j_\alpha(\lambda t)$$

with $\lambda \in \mathbb{C}$ (in the common notation from (4.18), (4.19)) and j_α the spherical Bessel function defined by

$$j_\alpha(z) := \Gamma(\alpha+1)(\tfrac{1}{2}z)^{-\alpha} J_\alpha(z) = \sum_{k=0}^\infty (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+k+1)} \left(\frac{z}{2}\right)^{2k}$$

where J_α denotes the Bessel function of the first kind and order α. Expressed in terms of the hypergeometric function ${}_0F_1$ it is $j_\alpha(z) = {}_0F_1(—; \alpha+1; -\frac{z^2}{4})$ (see, e. g., DLMF, 2015, Equation 10.16.9). These functions satisfy Gegenbauer's product formula, that is for $\lambda \in \mathbb{C}$ and all $t, s \in \mathbb{R}_+$

$$j_\alpha(\lambda t) j_\alpha(\lambda s) = c_\alpha \int_0^\pi j_\alpha(\lambda\sqrt{t^2+s^2-2ts\cos\theta}) \sin^{2\alpha}\theta \, \mathrm{d}\theta$$

or equivalently, substituting $r = \cos\theta$

$$j_\alpha(\lambda t) j_\alpha(\lambda s) = c_\alpha \int_{-1}^{1} j_\alpha(\lambda\sqrt{t^2+s^2-2tsr})\,(1-r^2)^{\alpha-\frac{1}{2}}\mathrm{d}r \tag{4.54}$$

and they solve the Euler-Poisson-Darboux equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} j_\alpha(\lambda t) + \frac{2\alpha+1}{t}\frac{\mathrm{d}}{\mathrm{d}t} j_\alpha(\lambda t) = -\lambda^2 j_\alpha(\lambda t), \quad t > 0,$$
$$j_\alpha(\lambda\cdot 0) = 1, \quad \frac{\mathrm{d}}{\mathrm{d}t} j_\alpha(\lambda t)|_{t=0} = 0.$$

We remark that for half-integer values of α, j_α can be expressed explicitly, for example

$$j_{-\frac{1}{2}}(t) = \cos t, \quad j_{\frac{1}{2}}(t) = \frac{\sin t}{t} \quad \text{and} \quad j_{\frac{3}{2}}(t) = \frac{3}{t^3}(\sin t - t\cos t),$$

see DLMF (2015), Equations 10.49.2 and 10.49.3.

Example 4.5.3 (Jacobi hypergroups)**.** A *Jacobi hypergroup* of order (α,β) with $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$ is a Sturm-Liouville hypergroup with respect to the Chébli-Trimèche function

$$A(t) = \sinh^{2\alpha+1} t \cosh^{2\beta+1} t,$$

hence $\rho = \alpha+\beta+1$. Its multiplicative functions are given by the Jacobi functions

$$\phi_\lambda(t) = {}_2F_1(\frac{1}{2}(\alpha+\beta+1-i\lambda), \frac{1}{2}(\alpha+\beta+1+i\lambda); \alpha+1; -\sinh^2 t)$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function. Convolution of point measures can be stated explicitly. The functions ϕ_λ fulfill the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi_\lambda(t) + ((2\alpha+1)\coth t + (2\beta+1)\tanh t)\frac{\mathrm{d}}{\mathrm{d}t}\phi_\lambda(t) = -(\lambda^2+(\alpha+\beta+1)^2)\phi_\lambda(t),$$
$$t > 0,$$
$$\phi_\lambda(0) = 1, \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi_\lambda(t)|_{t=0} = 0.$$

For a profound discussion see Koornwinder (1984).

Example 4.5.4 (Hyperbolic hypergroups)**.** These are the special Jacobi hypergroups with $\beta = -\frac{1}{2}$. In this case convolution of point measures takes the form

$$\varepsilon_t * \varepsilon_s = c_\alpha \int_{-1}^{1} \varepsilon_{\mathrm{arcosh}(\cosh t\cosh s - r\sinh t\sinh s)}\,(1-r^2)^{\alpha-\frac{1}{2}}\mathrm{d}r$$

for all $t, s \in \mathbb{R}_+$ with $c_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}$ as above. This hypergroup has been investigated in Zeuner (1986).

A further sub-example of hyperbolic hypergroups is the *Naimark hypergroup* where $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, see Bloom and Heyer (1995), Example 3.5.66.

4.5.2 Levitan hypergroups

We present the cosine and the cosh hypergroup. They are the only hypergroups on $\mathbb{R}_+$ which have the property that $\operatorname{supp}(\varepsilon_t * \varepsilon_s)$ contains at most two points for all $t, s \in \mathbb{R}_+$, see Bloom and Heyer (1995), Theorem 3.4.28.

Example 4.5.5 (Cosine hypergroup)**.** The *cosine hypergroup* has Sturm-Liouville function $A \equiv 1$, convolution of point measures is given for $t, s \in \mathbb{R}_+$ by

$$\varepsilon_t * \varepsilon_s = \frac{1}{2}\varepsilon_{t+s} + \frac{1}{2}\varepsilon_{|t-s|}$$

and the cosine functions $\cos(\lambda\,\cdot)$, $\lambda \in \mathbb{C}$ are its multiplicative functions. Explicitly, for any $\lambda \in \mathbb{C}$

$$\cos(\lambda t)\cos(\lambda s) = \frac{1}{2}\cos(\lambda(t+s)) + \frac{1}{2}\cos(\lambda(t-s))$$

for all $t, s \in \mathbb{R}_+$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(\lambda t) = -\lambda^2\cos(\lambda t), \quad t > 0,$$
$$\cos(\lambda \cdot 0) = 1, \; \frac{\mathrm{d}}{\mathrm{dt}}\cos(\lambda t)|_{t=0} = 0.$$

In Bloom and Heyer (1995) the cosine hypergroup is called “symmetric hypergroup”, a notion which is also used for hermitian hypergroups in general. Lasser (2016) calls it the “Chebyshev hypergroup”, thus extending the notion referring to the polynomial hypergroup in one variable induced by Chebyshev polynomials of the first kind, see Bloom and Heyer (1995), Example 3.1.15 for the general setting in several variables.

Example 4.5.6 (cosh hypergroup)**.** The *cosh hypergroup* is defined by $A(t) = \cosh^2 t$. Convolution of point measures has the form

$$\varepsilon_t * \varepsilon_s = \frac{\cosh(t+s)}{2\cosh(t)\cosh(s)}\varepsilon_{t+s} + \frac{\cosh(t-s)}{2\cosh(t)\cosh(s)}\varepsilon_{|t-s|} \tag{4.55}$$

for all $t, s \in \mathbb{R}_+$ and its multiplicative functions are given by

$$\phi_\lambda(t) = \frac{\cos(\lambda t)}{\cosh t}$$

where $\lambda \in \mathbb{C}$. For further properties see also Zeuner (1989b).

Bloom and Heyer (1995) consider also the *square hypergroup* which has Sturm-Liouville function $A(t) = (1+t)^2$.

Chapter 5

Sturm-Liouville Operator Functions

This chapter is the most important one in this thesis. We bring multiplicative operator functions to fruitful applications on Sturm-Liouville hypergroups. Paralleling the theories of cosine, Bessel, and Legendre operator functions, see Chapter 1, we initiate a theory of Sturm-Liouville operator functions, solving abstract Sturm-Liouville equations. This justifies the heuristic approach of Chapter 1, and the notion of a multiplicative operator function in general.

5.1 Definition and Generator

Throughout Chapter 5 we suppose that $(\mathbb{R}_+, *(A))$ is an arbitrary but fixed Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL1 and SL2.

Definition 5.1.1. A *Sturm-Liouville operator function* is a multiplicative operator function $S : \mathbb{R}_+ \to \mathcal{L}(X)$ with respect to $(\mathbb{R}_+, *(A))$.

In terms of Theorem 3.1.6 an operator function $S : \mathbb{R}_+ \to \mathcal{L}(X)$ is a Sturm-Liouville operator function iff $t \mapsto S(t)x$, $\mathbb{R}_+ \to X$ is continuous for each $x \in X$ and

(i) $S(0) = I$,

(ii) $S(t)S(s)x = S(t * s)x$ for all $t, s \in \mathbb{R}_+$ and any $x \in X$,

(iii) $\lim_{t\to 0^+} S(t)x = x$ for each $x \in X$.

Each Sturm-Liouville hypergroup $(\mathbb{R}_+, *(A))$ corresponds to a specific class of Sturm-Liouville operator functions. Recall the list of examples of Sturm-Liouville hypergroups in Section 4.5. If $(\mathbb{R}_+, *(A))$ is the cosine, a Bessel-Kingman, or a hyperbolic hypergroup, then the notion of a Sturm-Liouville operator function coincides with that of a cosine, a Bessel-Kingman, or a hyperbolic operator function as defined in Chapter 1, respectively, see Definitions 1.1.1, 1.2.6 and 1.3.2. In the same way several further operator functions are defined. Except for Section 5.6 (and some (counter)examples), we always investigate these operator functions in a unified way.

The following definition is motivated by the integral equation associated to multiplicative functions, see Proposition 4.2.3.

Definition 5.1.2. Let S be a Sturm-Liouville operator function.
Its *universal generator* $\mathbb{A}_0$ is given by

$$\mathbb{A}_0 x = \lim_{t\to 0^+} \frac{S(t)x - x}{(J1)(t)}$$

with $(J1)(t) = \int_0^t A(s)^{-1} \int_0^s A(r)\ \mathrm{dr}\ \mathrm{ds}$ (cf. Definition 4.2.1) and domain

$$D(\mathbb{A}_0) := \{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{(J1)(t)} \text{ exists}\},$$

see Definition 3.4.1 and Proposition 4.2.3.
Here, the *adapted generator* $\mathbb{A}$, briefly called *generator*, is defined as

$$\mathbb{A} = \rho^2 + \mathbb{A}_0$$

with domain $D(\mathbb{A}) = D(\mathbb{A}_0)$.

Proposition 5.1.3. *Let S be a Sturm-Liouville operator function. The definition of $\mathbb{A}_0$ from above is equivalent to*

$$\mathbb{A}_0 x = \lim_{t\to 0^+} 2(\alpha_0 + 1)\frac{S(t)x - x}{t^2}$$

with domain

$$D(\mathbb{A}_0) = \{x \in X : \lim_{t\to 0^+} 2(\alpha_0 + 1)\frac{S(t)x - x}{t^2} \text{ exists}\}.$$

This is an immediate consequence of the following lemma.

Lemma 5.1.4. *Let A be a Sturm-Liouville function satisfying SL1. Then*

$$\frac{2(\alpha_0 + 1)}{t^2}(J1)(t) = 1 + \mathcal{O}(t^j) \tag{5.1}$$

as $t \to 0^+$ where $j = 1$ if $\alpha_1(0) > 0$ and $j = 2$ if $\alpha_1(0) = 0$.

Proof. Integration by parts and (4.2) give

$$\begin{aligned}(J1)(t) &= \int_0^t A(s)^{-1} \int_0^s A(r)\ \mathrm{dr}\ \mathrm{ds} = \int_0^t A(s)^{-1}(sA(s) - \int_0^s r\frac{A'(r)}{A(r)}A(r)\ \mathrm{dr})\ \mathrm{ds} \\ &= \frac{t^2}{2} - \alpha_0 \int_0^t A(s)^{-1} \int_0^s A(r)\ \mathrm{dr}\ \mathrm{ds} - \int_0^t A(s)^{-1} \int_0^s r\alpha_1(r)A(r)\ \mathrm{dr}\ \mathrm{ds},\end{aligned}$$

thus

$$\frac{2(\alpha_0 + 1)}{t^2}(J1)(t) = 1 - \frac{2}{t^2}\int_0^t A(s)^{-1} \int_0^s r\alpha_1(r)A(r)\ \mathrm{dr}\ \mathrm{ds}$$

where, according to SL1.2 and SL1.1, $\alpha_1(r) = \mathcal{O}(1)$ if $\alpha_1(0) > 0$ and $\alpha_1(r) = \mathcal{O}(r)$ if $\alpha_1(0) = 0$. Using that A is monotonically increasing (cf. the proof of (4.7)) we get the stated asymptotic expansion. ☐

Proposition 5.1.5. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$ and $x \in D(\mathbb{A})$. Then $S(t)x \in D(\mathbb{A})$ and $\mathbb{A}S(t)x = S(t)\mathbb{A}x$ for all $t \in \mathbb{R}_+$.*

Proof. The values of the Sturm-Liouville operator function commute since Sturm-Liouville hypergroups are commutative. So this is clear by Definition 5.1.2. □

5.2 The Abstract Sturm-Liouville Equation

In the sequel we will apply the operators L and J from Definition 4.1.2 and Definition 4.2.1, respectively, also to Banach space valued functions. To keep notation simple, we write $LS(t)x$ instead of $(L\,S(\cdot)x)(t)(= S''(t)x + \frac{A'(t)}{A(t)}S'(t)x)$, whenever the derivatives exist, and $JS(t)x$ instead of $(J\,S(\cdot)x)(t)(= \int_0^t A(s)^{-1} \int_0^s A(r)S(r)x \text{ dr ds})$.

Theorem 5.2.1. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$ and suppose $x \in X$. Then $JS(t)x \in D(\mathbb{A})$ for all $t \in \mathbb{R}_+$ and $S(\cdot)x$ solves the* abstract Sturm-Liouville integral equation

$$S(t)x - x = (\mathbb{A} - \rho^2)(JS(t)x), \quad t \geq 0. \tag{5.2}$$

The solutions of (5.2) are referred to as *mild solutions*, see Theorem 5.2.5 for classical solutions.

Theorem 5.2.1 is a special case of Theorem 3.4.3, see Proposition 4.2.3. We give a second proof which is more in the spirit of M. Sova, see Theorem 1.1.7.

Second proof. Suppose $x \in X$ and $t > 0$. We show that for $\varepsilon > 0$, $0 < \varepsilon < t - \varepsilon < t$

$$\frac{S(\varepsilon) - I}{(J1)(\varepsilon)} \int_0^t A(s)^{-1} \int_0^s A(r)\, S(r)x \text{ dr ds} = \int_0^{t+\varepsilon} k(\varepsilon, s) S(s)x\, A(s)\text{d}s \tag{5.3}$$

with

$$k(\varepsilon, s) \begin{cases} \leq 0 & \text{if } 0 < s < \varepsilon \\ = 0 & \text{if } \varepsilon < s < t - \varepsilon \\ \geq 0 & \text{if } t - \varepsilon < s < t + \varepsilon \end{cases}$$

and

$$\int_0^\varepsilon k(\varepsilon, s)\, A(s)\text{d}s = -1, \quad \int_{t-\varepsilon}^{t+\varepsilon} k(\varepsilon, s)\, A(s)\text{d}s = 1.$$

To begin with, integration by parts gives (using A monotonically increasing)

$$\begin{aligned} \int_0^t A(s)^{-1} \int_0^s A(r)S(r)x \text{ dr ds} &= \int_0^t \int_s^t A(r)^{-1} \text{ dr}\, A(s)S(s)x \text{ ds} \\ &= \int_{\mathbb{R}_+} 1_{(0,t)}(s) h(s)\, S(s)x\, A(s)\text{d}s \end{aligned} \tag{5.4}$$

with

$$h(s) = \int_s^t A(r)^{-1} \text{ dr}\,.$$

So we get

$$\begin{aligned}S(\varepsilon)\int_0^t A(s)^{-1}\int_0^s A(r)\,S(r)x\;\mathrm{dr}\;\mathrm{ds} &= \int_{\mathbb{R}_+} 1_{(0,t)}(s)h(s)\,S(\varepsilon)S(s)x\,A(s)\mathrm{d}s\\ &= \int_{\mathbb{R}_+}(1_{(0,t)}h)(s)\,S(\varepsilon * s)x\,A(s)\mathrm{d}s = \int_{\mathbb{R}_+}(1_{(0,t)}h)(\varepsilon * s)S(s)x\,A(s)\mathrm{d}s\end{aligned}$$

by Theorem 2.2.14. Thus (5.3) is satisfied with

$$k(\varepsilon,s) = \frac{1}{(J1)(\varepsilon)}((1_{(0,t)}h)(\varepsilon * s) - (1_{(0,t)}h)(s)).$$

Note that $(Lh)(s) = A(s)^{-1}(A(s)h'(s))' = 0$ for $s > 0$. Lemma 5.2.3 shows that $h(\varepsilon * s) = h(\varepsilon)$ for $0 < s < \varepsilon$ and $h(\varepsilon * s) = h(s)$ for $s > \varepsilon$. This implies for $0 < s < \varepsilon$

$$k(\varepsilon,s) = \frac{1}{(J1)(\varepsilon)}(h(\varepsilon) - h(s)) = -\frac{1}{(J1)(\varepsilon)}\int_s^\varepsilon A(r)^{-1}\;\mathrm{dr} \le 0,$$

further $k(\varepsilon,s) = 0$ for $\varepsilon < s < t-\varepsilon$,

$$\begin{aligned}k(\varepsilon,s) &= \frac{1}{(J1)(\varepsilon)}((1_{(0,t)}h)(\varepsilon * s) - h(\varepsilon * s)) = \frac{1}{(J1)(\varepsilon)}(-1_{[t,\infty)}h)(\varepsilon * s)\\ &= \frac{1}{(J1)(\varepsilon)}\left(1_{[t,\infty)}\int_t^{\bullet} A(r)^{-1}\;\mathrm{dr}\right)(\varepsilon * s) \ge 0\end{aligned}$$

for $t-\varepsilon < s < t$ and

$$k(\varepsilon,s) = \frac{1}{(J1)(\varepsilon)}(1_{(0,t)}h)(\varepsilon * s) = \frac{1}{(J1)(\varepsilon)}\left(1_{(0,t)}\int_{\bullet}^t A(r)^{-1}\;\mathrm{dr}\right)(\varepsilon * s) \ge 0$$

for $t < s < t+\varepsilon$. Finally, we conclude that

$$\int_0^\varepsilon k(\varepsilon,s)\,A(s)\mathrm{d}s = -\frac{1}{(J1)(\varepsilon)}\int_0^\varepsilon\int_s^\varepsilon A(r)^{-1}\;\mathrm{dr}\,A(s)\;\mathrm{ds} = -1$$

reading (5.4) backwards with ε in place of t and with $S \equiv I$. Keeping $S \equiv I$, (5.3) yields

$$\int_{t-\varepsilon}^{t+\varepsilon} k(\varepsilon,s)\,A(s)\mathrm{d}s = -\int_0^\varepsilon k(\varepsilon,s)\,A(s)\mathrm{d}s = 1.$$

□

Remark 5.2.2. In (5.2), for $x \in D(\mathbb{A})$, the universal generator $\mathbb{A}_0 = \mathbb{A} - \rho^2$ and the integral operator J commute, that is

$$(\mathbb{A}-\rho^2)(JS(t)x) = JS(t)(\mathbb{A}-\rho^2)x,$$

see (5.3), apply Hille's Theorem A.12, and observe that $S(r)$ and $S(\varepsilon)$ commute.

Lemma 5.2.3. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL1 and SL2 and suppose $f \in C^2(\mathbb{R}_+^\times)$. Then the following conditions are equivalent.*

(i) For all $t > 0$
$$(Lf)(t) = 0.$$

(ii) There exist constants $b, c \in \mathbb{C}$ *such that*
$$f(t) = b \int_1^t A(s)^{-1} \,\mathrm{d}s + c$$
for all $t > 0$.

(iii) For each $t > 0$
$$f(t * r) = f(t)$$
for all $0 < r < t$.

Proof. Integration and differentiation show that (i) and (ii) are equivalent.

Concerning (i) implies (iii) suppose $0 < r_0 < t$. Since $\operatorname{supp}(\varepsilon_t * \varepsilon_r) \subset [|t - r_0|, t + r_0]$ for $0 < r < r_0$ we may assume without loss of generality $f'(0) = 0$. Then $L_r(f(t*r)) = (Lf)(t*r) = 0$ for all $0 < r < r_0$ by Lemma 4.1.5. Integrating this identity with respect to r (with boundary condition as in Lemma 4.1.5) one obtains $f(t * r_0) = f(t)$.

Conversely, suppose $t > 0$ and $f(t * r) = f(t)$ for small positive r. Again, we may assume without loss of generality $f'(0) = 0$. Then $(Lf)(t * r) = L_r(f(t * r)) = L_r(f(t)) = 0$ for small positive r according to Lemma 4.1.5. Taking the limit $r \to 0^+$ we get $(Lf)(t) = 0$. □

Remark 5.2.4. Lemma 5.2.3 is in some sense about harmonic functions. Let $(\mathbb{R}_+, *(A))$ be a Bessel-Kingman hypergroup of half-integer order $\alpha = \frac{n}{2} - 1$ with $n \in \mathbb{N}$, $n \geq 2$. Then (ii) implies that
$$f(r) := \begin{cases} \log(r) & \text{if } n = 2 \\ \frac{1}{r^{n-2}} & \text{if } n \geq 3, \end{cases}$$
defined for $r > 0$, satisfies (iii). Using the interpretation of the Bessel-Kingman convolution as a random walk, see Kingman (1963), we get for $x \in \mathbb{R}^n$, $x \neq 0$ and $0 < r < \|x\|$
$$\frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(\xi) \,\mathrm{d}S(\xi) = u(x)$$
where $u(x) = f(\|x\|)$, $x \in \mathbb{R}^n$ (the integral denotes the spherical mean on the surface of the ball $B_r(x)$ with center x and radius r).

From the perspective of partial differential equations, see, e. g., Evans (1998), Section 2.2, pp. 20–26, u is a radially symmetric, harmonic function on $\mathbb{R}^n \backslash \{0\}$ and a scalar multiple of the fundamental solution of Laplace's equation; the last formula is the mean-value property of harmonic functions.

We remark that for arbitrary hypergroups a related concept of "σ-harmonic functions" has been considered by Amini and Chu (2011), see Definition 2.6. A Borel function f defined on a hypergroup $(K, *)$ is called *σ-harmonic* with respect to a probability measure σ on K if the convolution $f * \sigma$ exists and $f = f * \sigma$.

Theorem 5.2.5. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$ and suppose $x \in D(\mathbb{A})$. Then $S(t)x \in D(\mathbb{A})$ for all $t \in \mathbb{R}_+$ and $S(\cdot)x \in C^2(\mathbb{R}_+, X)$ solves the* abstract Sturm-Liouville equation

$$LS(t)x = (\mathbb{A} - \rho^2)S(t)x, \quad t > 0, \tag{5.5}$$
$$S(0)x = x, \quad S'(0)x = 0. \tag{5.6}$$

This theorem is in fact a corollary of Theorem 5.2.1. Using Remark 5.2.2, the proof runs as in the scalar case, see Lemma 4.2.2, which also shows that

$$\lim_{t\to 0^+} S''(t)x = \frac{1}{\alpha_0 + 1}(\mathbb{A} - \rho^2)x \quad \text{and} \quad \lim_{t\to 0^+} \frac{A'(t)}{A(t)} S'(t)x = \frac{\alpha_0}{\alpha_0 + 1}(\mathbb{A} - \rho^2)x.$$

Remark 5.2.6. Setting $S(-t) = S(t)$ and $A(-t) = A(t)$ solutions of (5.5), (5.6) may be extended to the real line. This is by analogy with cosine operator functions.

Remark 5.2.7. Let S be a Sturm-Liouville operator function on $(\mathbb{R}_+, *(A))$ with generator $\mathbb{A}$. Suppose $\gamma > 0$. Then $S_\gamma := S(\gamma\, \cdot)$ is a Sturm-Liouville operator function on $(\mathbb{R}_+, *(A_\gamma))$ where $A_\gamma := A(\gamma\cdot)$, see Remark 4.1.7. Its generator is given by $\mathbb{A}_\gamma = \gamma^2\mathbb{A}$, which follows from the definition of the generator and Proposition 5.1.3 (note that α_0 does not depend on γ). Explicitly, S_γ satisfies for each $x \in D(\mathbb{A})$

$$S''_\gamma(t)x + \gamma\frac{A'(\gamma t)}{A(\gamma t)}S'_\gamma(t)x = (\gamma^2\mathbb{A} - (\gamma\rho)^2)S_\gamma(t)x, \quad t > 0,$$
$$S_\gamma(0)x = x, \quad S'_\gamma(0) = 0.$$

Also the converse of the theorem above is true.

Theorem 5.2.8. *Let $S : \mathbb{R}_+ \to \mathcal{L}(X)$ be a locally uniformly bounded transformation. Suppose $\mathbb{A}$ is a densely defined linear operator commuting with all $S(t)$ on $D(\mathbb{A})$ and for each $x \in D(\mathbb{A})$ it is $S(\cdot)x \in C^2(\mathbb{R}_+, X)$ and*

$$LS(t)x = (\mathbb{A} - \rho^2)S(t)x, \quad t > 0, \tag{5.7}$$
$$S(0)x = x, \quad S'(0)x = 0 \tag{5.8}$$

where $L = L_A$ with underlying Sturm-Liouville function A satisfying SL1 and SL2. Then S is a Sturm-Liouville operator function and its generator $\mathbb{A}_S$ is an extension of $\mathbb{A}$.

Proof. This statement is a generalization of the corresponding implication in the scalar case, see Proposition 4.1.6. The proof runs as in the scalar case, see Zeuner (1989a), Proposition 4.3(b) and Székelyhidi (2013), Theorem 4.2.

Suppose S is given as stated above. Clearly, $S(0) = I$ and $S(\cdot)x$ is continuous for each $x \in X$ since by assumption $D(\mathbb{A})$ is dense in X and $S(\cdot)$ is locally uniformly bounded. To show that S is a Sturm-Liouville operator function it remains to prove the functional equation. Therefore, take an arbitrary $x \in D(\mathbb{A})$ and $x^* \in X^*$. Setting $\varphi(t) = x^*S(t)x$, it is $\varphi \in C^2(\mathbb{R}_+)$ and $\varphi'(0) = 0$ by assumption and according to Lemma 4.1.5, $u_\varphi(t, s) = \varphi(t * s)$ satisfies the partial differential equation defining a

Sturm-Liouville hypergroup. On the other hand $\psi(t,s) = x^*S(t)S(s)x$ also satisfies $L_t\psi(t,s) = L_s\psi(t,s)$ since $\mathbb{A}$ and $S(\cdot)$ commute on $D(\mathbb{A})$ and $\psi_s(t,0) = \psi_t(0,s) = 0$ by assumption. Thus $u_\varphi = \psi$ and S is a Sturm-Liouville operator function.

Let $\mathbb{A}_S$ denote the generator of S. Given $x \in D(\mathbb{A})$, integrating (5.7), (5.8) gives $S(t)x - x = JS(t)(\mathbb{A} - \rho^2)x$, thus $x \in D(\mathbb{A}_S)$ and $\mathbb{A}_S x = \mathbb{A}x$ by Definition 5.1.2, that is $\mathbb{A}_S \supset \mathbb{A}$. □

Remark 5.2.9. Since $\mathbb{A}_S \supset \mathbb{A}$ and $\mathbb{A}_S$ is closed (see Theorem 5.3.1) it is clear that $\mathbb{A}_S \supset \overline{\mathbb{A}}$ where $\overline{\mathbb{A}}$ denotes the closure of $\mathbb{A}$. Under the additional assumption that S is exponentially bounded the proof of Theorem 5.4.6 shows that $\lambda^2 - \overline{\mathbb{A}}$ is bijective for any $\lambda > \omega$. Since $\lambda^2 - \mathbb{A}_S \supset \lambda^2 - \overline{\mathbb{A}}$ and $\lambda^2 - \mathbb{A}_S$ is bijective by the same theorem we get $\mathbb{A}_S = \overline{\mathbb{A}}$.

Remark 5.2.10. We already know from Theorem 5.2.5 that a Bessel-Kingman and a hyperbolic operator function solve the abstract Euler-Poisson-Darboux and Legendre equation, respectively. Conversely, Theorem 5.2.8 shows that any Bessel or Legendre operator function is a Bessel-Kingman or hyperbolic operator function, respectively (this also reproves Glushak's Corollary 1.2.5). Thus a Bessel or Legendre operator function is a Bessel-Kingman or hyperbolic operator function, respectively, with the property that solutions are unique and the operator function is exponentially bounded; for uniqueness in the Bessel case see Glushak (2006), pp. 622–623, for the topic of exponential boundedness consult Section 5.4.

5.3 The Generator: Basic Properties and Relations

We continue our series of conclusions from Theorem 5.2.1.

Theorem 5.3.1. *Let S be a Sturm-Liouville operator function. Then its generator $\mathbb{A}$ is densely defined and closed.*

Proof. The proof runs as in the cosine setting, see Sova (1966), Theorems 2.17 and 2.20.

To show that $\mathbb{A}_0$ is densely defined, choose an arbitrary $x \in X$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\|x - x_\delta\| < \varepsilon$ where

$$x_\delta := (J1)(\delta)^{-1} \int_0^\delta A(s)^{-1} \int_0^s A(r)S(r)x \,\mathrm{dr}\,\mathrm{ds}\,.$$

Theorem 5.2.1 yields $x_\delta \in D(\mathbb{A}_0)$ and

$$\mathbb{A}_0 x_\delta = (J1)(\delta)^{-1}(S(\delta)x - x).$$

Hence $\mathbb{A}_0$ is densely defined.

To show that $\mathbb{A}_0$ is closed, assume $(x_n)_{n\in\mathbb{N}} \subset D(\mathbb{A}_0)$, $x, y \in X$ and $x_n \to x$, $\mathbb{A}_0 x_n \to y$ as $n \to \infty$. Applying Theorem 5.2.1 to x_n, $n \in \mathbb{N}$, using Remark 5.2.2, and taking the limit $n \to \infty$, we get for any $t > 0$

$$S(t)x - x = \int_0^t A(s)^{-1} \int_0^s A(r)S(r)y \,\mathrm{dr}\,\mathrm{ds}\,.$$

It follows from the definition of the universal generator that $x \in D(\mathbb{A}_0)$ and $\mathbb{A}_0 x = y$. □

Remark 5.3.2. The proof above also shows, by iteration, that $D(\mathbb{A}^n)$ is dense in X for all $n \in \mathbb{N}$.

Theorem 5.3.3. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$ and suppose A satisfies SL1.1 or SL1.2 with $\alpha_1 \in C^\infty(\mathbb{R}_+^\times)$. Then*

$$D(\mathbb{A}^\infty) := \bigcap_{n \in \mathbb{N}} D(\mathbb{A}^n)$$

is dense in X.

For Bessel operator functions this result was stated in Glushak (1997b), Theorem 8.

Proof. The basic idea of the following proof is taken from Sova (1966), see Theorem 2.26. The prerequisites imposed on A guarantee that A is infinitely differentiable in an open (non-void) interval touching zero. Suppose $x \in X$ and $\varepsilon > 0$. Then there exists $\delta > 0$ (sufficiently small such that A is smooth in $(0, 2\delta)$) such that $\|S(t)x - x\| < \varepsilon$ for all $0 < t < \delta$. Choose $\varphi \in C^\infty(\mathbb{R}_+)$ with $\operatorname{supp}(\varphi) \subset (\frac{\delta}{2}, \delta)$, $\varphi \geq 0$, $\int_{\frac{\delta}{2}}^{\delta} \varphi(s)\, A(s) \mathrm{d}s = 1$ and set

$$y := \int_{\frac{\delta}{2}}^{\delta} \varphi(s) S(s) x \, A(s) \mathrm{d}s.$$

Then $\|y - x\| \leq \int_{\frac{\delta}{2}}^{\delta} \varphi(s) \|S(s)x - x\| \, A(s) \mathrm{d}s < \varepsilon$ and we show $y \in \bigcap_{n \in \mathbb{N}} D(\mathbb{A}_0^n)$.

According to Green's formula, see Lemma 4.3.3, we have

$$y = \int_{\frac{\delta}{2}}^{\delta} \varphi(s) S(s) x \, A(s) \mathrm{d}s = \int_{\frac{\delta}{2}}^{\delta} L\varphi(s) J S(s) x \, A(s) \mathrm{d}s,$$

thus Theorem 5.2.1 and Hille's Theorem A.12 yield $y \in D(\mathbb{A}_0)$ and

$$\begin{aligned} \mathbb{A}_0 y = \int_{\frac{\delta}{2}}^{\delta} L\varphi(s) \mathbb{A}_0 J S(s) x \, A(s) \mathrm{d}s &= \int_{\frac{\delta}{2}}^{\delta} L\varphi(s) (S(s)x - x) \, A(s) \mathrm{d}s \\ &= \int_{\frac{\delta}{2}}^{\delta} L\varphi(s) S(s) x \, A(s) \mathrm{d}s. \end{aligned}$$

Inductively, we obtain $y \in D(\mathbb{A}_0^n)$ for all $n \in \mathbb{N}$ and

$$\mathbb{A}_0^n y = \int_{\frac{\delta}{2}}^{\delta} L^n \varphi(s) S(s) x \, A(s) \mathrm{d}s.$$

□

We also obtain a Taylor type theorem. Similar formulas as below are well-known for several operator functions, see e. g. Glushak (2001), Theorem 6 for Legendre operator functions.

Theorem 5.3.4. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$ and $x \in D(\mathbb{A}^n)$ for some $n \in \mathbb{N}$. Then*

$$S(t)x = \sum_{k=0}^{n} (J^k 1)(t)\,(\mathbb{A} - \rho^2)^k x + o(t^{2n})$$

as $t \to 0^+$.

Proof. The Sturm-Liouville integral equation in Theorem 5.2.1 states that

$$S(\cdot)x = x + (\mathbb{A} - \rho^2)(JS(\cdot)x)$$

for $x \in X$. Proceeding as in the proof of Proposition 4.2.4 we derive for $n \in \mathbb{N}$ and $x \in D(\mathbb{A}^n)$

$$S(\cdot)x = \sum_{k=0}^{n-1} (\mathbb{A} - \rho^2)^k (J^k x) + (\mathbb{A} - \rho^2)^n (J^n S(\cdot)x),$$

that is

$$S(t)x = \sum_{k=0}^{n} (J^k 1)(t)\,(\mathbb{A} - \rho^2)^k x + (J^n\{S(\cdot)(\mathbb{A} - \rho^2)^n x - (\mathbb{A} - \rho^2)^n x\})(t)$$

where

$$\|(J^n\{S(\cdot)(\mathbb{A} - \rho^2)^n x - (\mathbb{A} - \rho^2)^n x\})(t)\| \le \frac{t^{2n}}{(2n)!} \sup_{s\in[0,t]} \|S(s)(\mathbb{A} - \rho^2)^n x - (\mathbb{A} - \rho^2)^n x\|.$$

□

The next theorem characterizes uniformly continuous Sturm-Liouville operator functions. For Bessel operator functions it is stated in Glushak (1997b), Theorem 7. For cosine operator functions it is due to Kurepa (1962), Theorem 1 (see Theorem 1.1.17 above). We use his ideas in the following proof. We note that for cosine operator functions there is a second proof by Lutz (1982), see Theorem 2.18, based on holomorphic functional calculus.

Theorem 5.3.5. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$. Then $\lim_{t\to 0^+} S(t) = I$ in uniform operator topology if and only if $\mathbb{A}$ is bounded.*

In this case S is uniformly continuous and

$$S(t) = \Phi_{\sqrt{\mathbb{A}}}(t) := \sum_{k=0}^{\infty} (J^k 1)(t)\,(\mathbb{A} - \rho^2)^k \tag{5.9}$$

for all $t \in \mathbb{R}_+$,

$$LS(t) = (\mathbb{A} - \rho^2)S(t), \quad t > 0, \tag{5.10}$$

$$S(0) = I, \quad S'(0) = 0 \tag{5.11}$$

in uniform operator topology and

$$\|S(t)\| \le \Phi_{\sqrt{\|\mathbb{A}\|}}(t) \tag{5.12}$$

for all $t \in \mathbb{R}_+$.

Conversely, if a function $S : \mathbb{R}_+ \to \mathcal{L}(X)$ *admits a representation of the form* (5.9) *or is a solution of* (5.10) *and* (5.11) *for some bounded linear operator* $\mathbb{A}$, *then* S *is a Sturm-Liouville operator function with generator* $\mathbb{A}$.

Proof. Suppose S is a Sturm-Liouville operator function with generator $\mathbb{A}$ and $\lim_{t\to 0^+} \|S(t) - I\| = 0$. Theorem 5.2.1 states that for $x \in D(\mathbb{A})$

$$S(t)x - x = \left(\int_0^t A(s)^{-1} \int_0^s A(r)S(r) \text{ dr ds}\right)(\mathbb{A} - \rho^2)x \tag{5.13}$$

where the double integral in brackets, defined in the strong sense, stands for a bounded linear operator. We show that it has a bounded inverse. Indeed,

$$\begin{aligned} &\left\|\left(I - (J1)(t)^{-1} \int_0^t A(s)^{-1} \int_0^s A(r)S(r) \text{ dr ds}\right) x\right\| \\ &= \left\|(J1)(t)^{-1} \int_0^t A(s)^{-1} \int_0^s A(r)(I - S(r))x \text{ dr ds}\right\| \\ &\le (J1)(t)^{-1} \int_0^t A(s)^{-1} \int_0^s A(r)\|x - S(r)x\| \text{ dr ds} < 1 \end{aligned}$$

for $t > 0$ sufficiently small, uniformly for $x \in X$ with $\|x\| \le 1$. Thus the inverse exists and since $\mathbb{A}$ is densely defined and closed by Theorem 5.3.1 we get from (5.13) that $\mathbb{A}$ is a bounded linear operator defined on $D(\mathbb{A}) = X$. Now, the proof of Theorem 5.3.4 shows that

$$S(t) = \sum_{k=0}^{\infty} (J^k 1)(t)\,(\mathbb{A} - \rho^2)^k, \tag{5.14}$$

in particular, S is uniformly continuous. Thus,

$$S(t) - I = (\mathbb{A} - \rho^2) \int_0^t A(s)^{-1} \int_0^s A(r)S(r) \text{ dr ds} \tag{5.15}$$

in $\mathcal{L}(X)$, the proof of Theorem 5.2.5 shows that S is in $C^2(\mathbb{R}_+, \mathcal{L}(X))$ and solves (5.10), (5.11).

Concerning estimate (5.12), we get from (4.22)

$$\begin{aligned} \|\Phi_{\sqrt{\mathbb{A}}}(t)\| &= \left\| \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{\mathrm{d}^{2k}}{\mathrm{d}\lambda^{2k}} \Phi_\lambda(t)\Big|_{\lambda=0} \mathbb{A}^k \right\| \\ &\le \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{\mathrm{d}^{2k}}{\mathrm{d}\lambda^{2k}} \Phi_\lambda(t)\Big|_{\lambda=0} \|\mathbb{A}\|^k = \Phi_{\sqrt{\|\mathbb{A}\|}}(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$.

Conversely, suppose $S \in C^2(\mathbb{R}_+, \mathcal{L}(X))$ is a function which solves (5.10), (5.11) for some $\mathbb{A} \in \mathcal{L}(X)$. By integration we get (5.15). The proof of Theorem 5.3.4 shows (5.14), since $\mathbb{A}$ is bounded, the estimate of the remainder term does not require that $\mathbb{A}$ and $S(t)$ commute. It can be derived from (5.14) and Remark 4.2.5 that S satisfies the functional equation $S(t*s) = S(t)S(s)$ for all $t, s \in \mathbb{R}_+$. From an abstract perspective this is an application of holomorphic functional calculus; for $t, s \in \mathbb{R}_+$ fixed $(\Phi_{\sqrt{\lambda}}(t)\Phi_{\sqrt{\lambda}}(s))(\mathbb{A}) = \Phi_{\sqrt{\mathbb{A}}}(t)\Phi_{\sqrt{\mathbb{A}}}(s)$ with the notation from (5.9) and $(\int_{\mathbb{R}_+} \Phi_{\sqrt{\lambda}}(r)\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r))(\mathbb{A}) = \int_{\mathbb{R}_+} \Phi_{\sqrt{\mathbb{A}}}(r)\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r)$ by definition of the holomorphic functional calculus via Cauchy's integral formula and Fubini's theorem (cf. Rudin, 1991, pp. 258–267, Definition 10.26 and Theorem 10.27, see also Heuser, 1975, §§46–48). In conclusion, S is a Sturm-Liouville operator function. □

Remark 5.3.6. If X is finite dimensional any Sturm-Liouville operator function $S : \mathbb{R}_+ \to \mathcal{L}(X)$ is uniformly continuous (cf. Corollary 3.2.11) and hence given by (5.9).

Next, we prove a spectral inclusion theorem for Sturm-Liouville operator functions. For cosine operator functions it states that $\cosh(\sqrt{\sigma(\mathbb{A})}t) \subset \sigma(C(t))$ for $t \in \mathbb{R}_+$ and can be traced back to Nagy (1974). His proof is based on the following formula, which is easily shown through integration by parts. Written in our notation, it states

$$(\cosh(\lambda t) - C(t))x = (\lambda^2 - \mathbb{A}) \int_0^t \gamma(t,s)C(s)x \;\mathrm{d}s$$

for $t \in \mathbb{R}_+$ and $x \in D(\mathbb{A})$ where

$$\gamma(t,s) = \begin{cases} \frac{1}{\lambda}\sinh(\lambda(t-s)) & \text{if } \lambda \neq 0 \\ t-s & \text{if } \lambda = 0. \end{cases}$$

Formula (5.16) below is a generalization for Sturm-Liouville operator functions.

For Bessel operator functions the spectral inclusion theorem is due to Glushak and Popova (2008), see Theorem 1.5. Their proof is based on the cosine setting using techniques similar to Theorem 5.6.1.

Theorem 5.3.7. *Suppose S is a Sturm-Liouville operator function with generator $\mathbb{A}$. Then*

$$\Phi_{\sqrt{\sigma(\mathbb{A})}}(t) \subset \sigma(S(t))$$

for each $t \in \mathbb{R}_+$.

Proof. We show that for $t > 0$, $\lambda \in \mathbb{C}$ and all $x \in X$

$$(\Phi_\lambda(t) - S(t))x = (\lambda^2 - \mathbb{A}) \int_0^t \gamma(t,s)S(s)x\,A(s)\mathrm{d}s \tag{5.16}$$

where $\gamma(t,s) = -\Psi_\lambda(t)\Phi_\lambda(s) + \Phi_\lambda(t)\Psi_\lambda(s)$ is a fundamental solution (cf. Walter, 2000, 26.IV and V). This identity shows that $\Phi_\lambda(t) \in \rho(S(t))$ implies $\lambda^2 \in \rho(\mathbb{A})$, thus finishing the proof.

For a formal proof of (5.16), we may assume without loss of generality $x \in D(\mathbb{A})$. Then by Green's formula, see Lemma 4.3.3,

$$
\begin{aligned}
&(\lambda^2 - \mathbb{A}) \int_0^t \gamma(t,s) S(s)x\, A(s) \mathrm{d}s \\
&= \int_0^t (\lambda^2 - \rho^2) \gamma(t,s) S(s)x\, A(s) \mathrm{d}s - \int_0^t \gamma(t,s) \mathbb{A}_0 S(s)x\, A(s)\mathrm{d}s \\
&= \int_0^t (L_s \gamma(t,s)) S(s)x\, A(s)\mathrm{d}s - \int_0^t \gamma(t,s)(L_s S(s)x)\, A(s)\mathrm{d}s \\
&= [S(\cdot)x, \overline{\gamma(t,\cdot)}](t) - \lim_{\varepsilon \to 0^+} [S(\cdot)x, \overline{\gamma(t,\cdot)}](\varepsilon) \\
&= A(t)\gamma_s(t,t)S(t)x - \gamma(t,t)A(t)S'(t)x - \\
&- \lim_{\varepsilon \to 0^+} \{A(\varepsilon)\gamma_s(t,\varepsilon)S(\varepsilon)x - \gamma(t,\varepsilon)A(\varepsilon)S'(\varepsilon)x\} \\
&= (\Phi_\lambda(t) - S(t))x
\end{aligned}
$$

where we have used $\gamma(t,t) = 0$, Lemma 4.3.5 and $A(t)\gamma_s(t,t) = -1$, note that $A(t)\gamma_s(t,t)$ is the Wronskian of (4.36), which is constant. □

5.4 Exponential Bounds and the Resolvent

For cosine operator functions, the first thing shown by Sova (1966) is the existence of an exponential bound, see Theorem 1.1.2. In contrast, as we will see, a Sturm-Liouville operator function does in general not possess such a bound. This is why we have postponed this topic as long as possible. Note that, starting with the general definition of multiplicative operator functions, all results up to now only need local boundedness. However, at the end of this section we need an exponential bound to prove existence of the resolvent $R(\lambda, \mathbb{A})$.

Definition 5.4.1. Let S be a Sturm-Liouville operator function. We say S admits a *multiplicative bound* if there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq M\Phi_\omega(t)$$

for all $t \in \mathbb{R}_+$. Besides, S is called *exponentially bounded* if there exist constants $M' \geq 1$ and $\omega' \geq 0$ such that

$$\|S(t)\| \leq M' e^{\omega' t}$$

for all $t \in \mathbb{R}_+$.

Note that existence of a multiplicative bound and exponential boundedness are equivalent, see the asymptotic behaviour of Φ_λ stated in Theorem 4.3.9. The first notion is more convenient in the context of Sturm-Liouville operator functions, see e. g. Theorem 5.4.5 and its application in the proof of Proposition 5.5.2, whereas the second notion, of course, is much more common. For cosine operator functions this ambivalence was observed by M. Sova, see Subsection 1.1.1. By the way, we remark that an arbitrary operator function $V : \mathbb{R}_+ \to \mathcal{L}(X)$ is exponentially bounded if and only if $V(\cdot)x$ is exponentially bounded for every $x \in X$, see Arendt et al. (2011), Lemma 3.2.14; thus a Sturm-Liouville operator function S is exponentially bounded iff all mild solutions $S(\cdot)x$, $x \in X$ are exponentially bounded.

The following theorem, taken from Bloom and Heyer (1995), Theorem 3.5.48, characterizes fundamental differences between convolution of point measures for Chébli-Trimèche and Levitan hypergroups. These lie at the core of our problem.

Theorem 5.4.2. *Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup.*

(i) If $(\mathbb{R}_+, *(A))$ is a Chébli-Trimèche hypergroup, then for every $t, s \in \mathbb{R}_+^\times$ the convolution $\varepsilon_t * \varepsilon_s$ is $m = A\lambda_{\mathbb{R}_+}$-absolutely continuous.

(ii) If $(\mathbb{R}_+, *(A))$ is a Levitan hypergroup with $A \in C^2(\mathbb{R}_+)$, then for every $t, s \in \mathbb{R}_+^\times$ there exists an $m = A\lambda_{\mathbb{R}_+}$-absolutely continuous measure $\nu_{t,s} \in M_+^b(\mathbb{R}_+)$ such that

$$\varepsilon_t * \varepsilon_s = \frac{1}{2}\left(\frac{A(|t-s|)A(0)}{A(t)A(s)}\right)^{\frac{1}{2}} \varepsilon_{|t-s|} + \nu_{t,s} + \frac{1}{2}\left(\frac{A(t+s)A(0)}{A(t)A(s)}\right)^{\frac{1}{2}} \varepsilon_{t+s}.$$

We begin with the negative result for Chébli-Trimèche hypergroups.

Theorem 5.4.3. *Let $(\mathbb{R}_+, *(A))$ be a Bessel-Kingman or Jacobi hypergroup. Then there exists a Sturm-Liouville operator function S on $(\mathbb{R}_+, *(A))$ which does not have an exponential bound.*

For the construction of such Sturm-Liouville operator functions see Example 5.8.10.

The following positive result for Levitan hypergroups generalizes the corresponding result for cosine operator functions.

Theorem 5.4.4. *Let S be a Sturm-Liouville operator function defined on a Levitan hypergroup $(\mathbb{R}_+, *(A))$ with $A \in C^2(\mathbb{R}_+)$. Then there exist constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\| \leq M e^{\omega t}$$

for all $t \in \mathbb{R}_+$.

Proof. The strategy of the proof is to isolate in Theorem 5.4.2(ii) a positive weight at $t+s$. This generalizes the proof for cosine operator functions, see Sova (1966), Proposition 2.4.

1. We start with the weights at $|t-s|$ and $t+s$. Since $A(0) > 0$ (A is a Levitan function) we have by (4.33)

$$\frac{A(|t-s|)A(0)}{A(t)A(s)} = \exp\left(-\int_{|t-s|}^{s} \frac{A'(r)}{A(r)}\,\mathrm{dr} - \int_0^t \frac{A'(r)}{A(r)}\,\mathrm{dr}\right)$$

and

$$\frac{A(t+s)A(0)}{A(t)A(s)} = \exp\left(\int_s^{t+s} \frac{A'(r)}{A(r)}\,\mathrm{dr} - \int_0^t \frac{A'(r)}{A(r)}\,\mathrm{dr}\right).$$

We know that $\frac{A'}{A}$ is bounded since $A(0) > 0$, $A \in C^2(\mathbb{R}_+)$ and $\lim_{r\to\infty} \frac{A'(r)}{A(r)} = 2\rho$, thus

$$\lim_{t\to 0^+} \frac{A(|t-s|)A(0)}{A(t)A(s)} = 1 \quad \text{and} \quad \lim_{t\to 0^+} \frac{A(t+s)A(0)}{A(t)A(s)} = 1,$$

uniformly for $s \in \mathbb{R}_+^\times$. Further, Theorem 5.4.2(ii) gives

$$1 = \frac{1}{2}\left(\frac{A(|t-s|)A(0)}{A(t)A(s)}\right)^{\frac{1}{2}} + \|\nu_{t,s}\| + \frac{1}{2}\left(\frac{A(t+s)A(0)}{A(t)A(s)}\right)^{\frac{1}{2}},$$

hence $\|\nu_{t,s}\| \to 0$ as $t \to 0^+$, uniformly for $s \in \mathbb{R}_+^\times$.

2. In terms of Theorem 5.4.2(ii), the functional equation of S reads

$$S(t)S(s)x = \frac{1}{2}\left(\frac{A(|t-s|)A(0)}{A(t)A(s)}\right)^{\frac{1}{2}} S(|t-s|)x + \int_{|t-s|}^{t+s} S(r)x\,\nu_{t,s}(\mathrm{d}r) + \frac{1}{2}\left(\frac{A(t+s)A(0)}{A(t)A(s)}\right)^{\frac{1}{2}} S(t+s)x$$

for all $t, s \in \mathbb{R}_+^\times$ and any $x \in X$, thus

$$\|S(t+s)x\| \le 2\left(\frac{A(t+s)A(0)}{A(t)A(s)}\right)^{-\frac{1}{2}} \|\nu_{t,s}\| \sup_{r\in[|t-s|,t+s]} \|S(r)x\| + \left(\frac{A(t+s)}{A(|t-s|)}\right)^{-\frac{1}{2}} \|S(|t-s|)x\| + 2\left(\frac{A(t+s)A(0)}{A(t)A(s)}\right)^{-\frac{1}{2}} \|S(t)\|\|S(s)x\|.$$

We choose $t_0 > 0$ sufficiently small such that

$$2\left(\frac{A(t_0+s)A(0)}{A(t_0)A(s)}\right)^{-\frac{1}{2}} \|\nu_{t_0,s}\| < \frac{1}{2} \quad \text{and} \quad 2\left(\frac{A(t_0+s)A(0)}{A(t_0)A(s)}\right)^{-\frac{1}{2}} < 3$$

for all $s \in \mathbb{R}_+^\times$. Since A is monotonically increasing we get for $s \in \mathbb{R}_+^\times$

$$\|S(s+t_0)x\| \le \frac{1}{2} \sup_{r\in[|s-t_0|,s+t_0]} \|S(r)x\| + \|S(|t_0-s|)x\| + 3\|S(t_0)\|\|S(s)x\|. \tag{5.17}$$

3. Taking the supremum of (5.17) over $s \in [0,t]$ for any $t \ge t_0$ we obtain

$$\sup_{s\in[0,t+t_0]} \|S(s)x\| \le \frac{1}{2} \sup_{s\in[0,t+t_0]} \|S(s)x\| + \sup_{s\in[0,t]} \|S(s)x\| + 3\|S(t_0)\| \sup_{s\in[0,t]} \|S(s)x\|,$$

that is

$$\sup_{s\in[0,t+t_0]} \|S(s)x\| \le (2 + 6\|S(t_0)\|) \sup_{s\in[0,t]} \|S(s)x\|.$$

Since this holds for any $x \in X$ we have shown

$$\sup_{s\in[0,t+t_0]} \|S(s)\| \le (2 + 6\|S(t_0)\|) \sup_{s\in[0,t]} \|S(s)\|$$

for all $t \geq t_0$. Setting $M := \sup_{s\in[0,t_0]} \|S(s)\| < \infty$ (cf. Corollary 3.2.8) we get by induction

$$\sup_{s\in[0,nt_0]} \|S(s)\| \leq M(2 + 6\|S(t_0)\|)^{n-1}$$

for all $n \in \mathbb{N}$. Setting $\omega = \frac{1}{t_0}\log(2 + 6\|S(t_0)\|)$ it is $M(2 + 6\|S(t_0)\|)^{n-1} \leq Me^{\omega s}$ for $(n-1)t_0 \leq s \leq nt_0$ and $n \in \mathbb{N}$, hence

$$\sup_{s\in[0,t]} \|S(s)\| \leq Me^{\omega t}$$

for all $t \in \mathbb{R}_+$.

□

Whenever needed, the assumption of an exponential bound will be stated explicitly.

Theorem 5.4.5. *Let S be a Sturm-Liouville operator function. Suppose there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\| \leq M\Phi_\omega(t)$ for all $t \in \mathbb{R}_+$. Then*

$$\|S(t_1)S(t_2)\ldots S(t_n)\| \leq M\Phi_\omega(t_1)\Phi_\omega(t_2)\ldots\Phi_\omega(t_n)$$

for all $n \in \mathbb{N}$ and all $t_1, t_2 \ldots t_n \in \mathbb{R}_+$.

For the cosine setting see Theorem 1.1.3. For bounded Bessel-Kingman operator functions this has been shown by Dietmair (1985), Satz (2.2.1). The idea of his proof is the same as below.

Proof. We prove the assertion by induction on n. For $n = 1$ the lemma holds true by assumption. So assume it is correct for $n - 1 \in \mathbb{N}$. Then we have for $t_1, t_2 \ldots t_n \in \mathbb{R}_+$ and any $x \in X$ with $\|x\| \leq 1$

$$\begin{aligned}
&\|S(t_1)\ldots S(t_{n-1})S(t_n)x\| \\
&= \left\| S(t_1)\ldots S(t_{n-2}) \int_{\mathbb{R}_+} S(r)x\,(\varepsilon_{t_{n-1}} * \varepsilon_{t_n})(\mathrm{d}r) \right\| \\
&= \left\| \int_{\mathbb{R}_+} S(t_1)\ldots S(t_{n-2})S(r)x\,(\varepsilon_{t_{n-1}} * \varepsilon_{t_n})(\mathrm{d}r) \right\| \\
&\leq \int_{\mathbb{R}_+} \|S(t_1)\ldots S(t_{n-2})S(r)x\|\,(\varepsilon_{t_{n-1}} * \varepsilon_{t_n})(\mathrm{d}r) \\
&\leq M\Phi_\omega(t_1)\Phi_\omega(t_2)\ldots\Phi_\omega(t_{n-2}) \int_{\mathbb{R}_+} \Phi_\omega(r)\,(\varepsilon_{t_{n-1}} * \varepsilon_{t_n})(\mathrm{d}r) \\
&= M\Phi_\omega(t_1)\Phi_\omega(t_2)\ldots\Phi_\omega(t_n).
\end{aligned}$$

□

We finish this section with the important theorem about existence of the resolvent. The resolvent formula (5.18) is similar to Laplace transformation in the sense that Ψ_λ is the substitute of the decaying exponential and the integral is taken with respect to the Haar measure $A\lambda_{\mathbb{R}_+}$, see Theorem 1.1.11 for the cosine, and Theorem 1.2.2 for the Bessel setting.

Theorem 5.4.6. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$ and suppose that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\| \leq M\Phi_\omega(t)$ for all $t \in \mathbb{R}_+$. Then the resolvent $R(\lambda^2, \mathbb{A}) = (\lambda^2 - \mathbb{A})^{-1}$ exists for $\mathrm{Re}(\lambda) > \omega$ and is given by*

$$R(\lambda^2, \mathbb{A})x = \int_0^\infty \Psi_\lambda(s)S(s)x\, A(s)\mathrm{d}s \tag{5.18}$$

for $x \in X$.

Proof. The following calculation is based on Green's formula, see Lemma 4.3.3. All occurring integrals and limits exist by the asymptotic results worked out in Theorem 4.3.9 and Lemma 4.3.6; for the boundary conditions at zero see (4.31) and (4.30). Suppose $x \in D(\mathbb{A})$, then for $\mathrm{Re}(\lambda) > \omega$

$$\begin{aligned}
&(\lambda^2 - \mathbb{A}) \int_0^\infty \Psi_\lambda(s)S(s)x\, A(s)\mathrm{d}s \\
&= \int_0^\infty (\lambda^2 - \rho^2)\Psi_\lambda(s)S(s)x\, A(s)\mathrm{d}s - \int_0^\infty \Psi_\lambda(s)\mathbb{A}_0 S(s)x\, A(s)\mathrm{d}s \\
&= \lim_{t\to\infty} \{A(s)\Psi_\lambda'(s)S(s)x - \Psi_\lambda(s)A(s)S'(s)x\}\big|_{s=1/t}^{s=t} \\
&= x.
\end{aligned}$$

Note that an appropriate norm bound of $A(s)S'(s)x$ at infinity can be deduced from the integrated abstract Sturm-Liouville equation.

Setting $R_\lambda = \int_0^\infty \Psi_\lambda(s)S(s)\, A(s)\mathrm{d}s$, we have shown that $(\lambda^2 - \mathbb{A})R_\lambda x = R_\lambda(\lambda^2 - \mathbb{A})x = x$ for all $x \in D(\mathbb{A})$. Since $\mathbb{A}$ is densely defined and closed it is $R_\lambda x \in D(\mathbb{A})$ for all $x \in X$, that is R_λ is the bounded inverse of $\lambda^2 - \mathbb{A}$. □

Remark 5.4.7. The condition $|\mathrm{Re}(\lambda)| > \omega$ is equivalent to $\mathrm{Re}(\lambda^2) > \omega^2 - \frac{(\mathrm{Im}(\lambda^2))^2}{4\omega^2}$ if $\omega > 0$ and $\lambda^2 \in \mathbb{C}\setminus]-\infty, 0]$ if $\omega = 0$. In particular, the spectrum $\sigma(\mathbb{A})$ of $\mathbb{A}$ is contained in the parabola $\{\xi + i\eta,\ \eta \in \mathbb{R},\ \xi \leq \omega^2 - \eta^2/4\omega^2\}$ if $\omega > 0$ and in $]-\infty, 0]$ if $\omega = 0$, cf. Proposition 3.14.18 in Arendt et al. (2011).

5.5 The Relation to Regular (Semi)groups

The following theorem shows that the generator of a Sturm-Liouville operator function is also the generator of a C_0-semigroup. The proof is based on Gaussian convolution semigroups and a central limit theorem for Sturm-Liouville hypergroups, see Bloom and Heyer (1995), Definition 5.4.25, Examples 7.3.18(c), (d), and Theorem 7.4.1.

Let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Sturm-Liouville function A satisfying SL2. Then for each $t \in \mathbb{R}_+$ there exists a unique measure $\gamma_t \in M^1(\mathbb{R}_+)$, called the *Gaussian measure*, such that

$$\widehat{\gamma_t}(\lambda) = \exp\left(\frac{t}{2}(\lambda^2 - \rho^2)\right)$$

for all $\lambda \in \widehat{\mathbb{R}_+}$ (with our parametrization; cf. (4.16), (4.17) and Theorem 4.4.1). The family $(\gamma_t)_{t\geq 0}$ is called *Gaussian convolution semigroup* and satisfies $\gamma_t * \gamma_s = \gamma_{t+s}$ for all $t, s \in \mathbb{R}_+$ and $\lim_{t\to 0^+} \gamma_t = \varepsilon_0$ in distribution (cf. *Schoenberg correspondence*, Theorem 5.2.15(b) in Bloom and Heyer (1995), note that Proposition 4.4.3 is needed at this point).

If $(\mathbb{R}_+, *(A))$ is the cosine hypergroup, then γ_{2t}, $t > 0$ has $\lambda_{\mathbb{R}_+}$-density g_{2t} given by

$$g_{2t}(r) = \frac{1}{\sqrt{\pi t}} e^{-\frac{r^2}{4t}}$$

and Theorem 5.5.1 states that cosine operator functions satisfy the *Weierstrass formula*

$$T(t)x = \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-\frac{r^2}{4t}} C(r)x \, \mathrm{d}r\,.$$

This is a well-known result due to Fattorini (1969a), Remark 5.11, see Arendt et al. (2011), Theorem 3.14.17, second proof.

If $(\mathbb{R}_+, *(A))$ is a Bessel-Kingman hypergroup of order $\alpha > -\frac{1}{2}$, then γ_{2t}, $t > 0$ is a Rayleigh distribution; its $\lambda_{\mathbb{R}_+}$-density g_{2t} is given by

$$g_{2t}(r) = \frac{1}{2^{2\alpha+1}\Gamma(\alpha+1)t^{\alpha+1}} r^{2\alpha+1} e^{-\frac{r^2}{4t}},$$

see Bloom and Heyer (1995), Example 7.3.18(d). The corresponding assertion of Theorem 5.5.1 for Bessel operator functions is stated in Glushak (1997b), Theorem 6, see also Theorem 5 for an inverse result.

A related formula for Legendre operator functions can be found in Glushak (2001), Theorem 12.

Theorem 5.5.1. *Let S be a Sturm-Liouville operator function with generator $\mathbb{A}$ and suppose that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\| \leq M\Phi_\omega(t)$ for all $t \in \mathbb{R}_+$. Then $\mathbb{A}$ is the generator of a C_0-semigroup $T : \mathbb{R}_+ \to \mathcal{L}(X)$ given by*

$$T(t)x := e^{\rho^2 t} \int_0^\infty S(r)x \, \gamma_{2t}(\mathrm{d}r)$$

and $\|T(t)\| \leq Me^{\omega^2 t}$ for all $t \in \mathbb{R}_+$.

Note that the rescaled C_0-semigroup $\widetilde{T}(t) = \int_0^\infty S(r)x \, \gamma_{2t}(\mathrm{d}r)$ has universal generator $\mathbb{A}_0$ (cf. Engel and Nagel, 2000, II.2.2).

Proof. *1.* Suppose $t > 0$ and $a \in \mathbb{C}$. Let μ^n denote the n-fold convolution of a bounded measure μ. Our starting point is a central limit theorem for Sturm-Liouville hypergroups, see Bloom and Heyer (1995), Theorem 7.4.1, from which we deduce (using Proposition 7.1.6 loc. cit.) that

$$\lim_{n\to\infty}\left(\varepsilon_{\sqrt{\frac{2(\alpha_0+1)t}{n}}}\right)^n = \gamma_{2t}$$

in distribution. On the other hand

$$\begin{aligned}\lim_{n\to\infty}\left(\varepsilon_{\sqrt{\frac{2(\alpha_0+1)t}{n}}}\right)^n(\Phi_{\sqrt{a}}) &= \lim_{n\to\infty}\left(\Phi_{\sqrt{a}}\left(\sqrt{\frac{2(\alpha_0+1)t}{n}}\right)\right)^n \\ &= \lim_{n\to\infty}\left(1+(a-\rho^2)(J1)\left(\sqrt{\frac{2(\alpha_0+1)t}{n}}\right)+\mathcal{O}\left(\frac{1}{n^2}\right)\right)^n \\ &= \lim_{n\to\infty}\left(1+(a-\rho^2)\frac{t}{n}+o\left(\frac{1}{n}\right)\right)^n \\ &= e^{(a-\rho^2)t} \end{aligned} \tag{5.19}$$

where we have used Lemma 5.1.4. So we see that $\int_0^\infty \Phi_{\omega'}(r)\,\gamma_{2t}(\mathrm{d}r) < \infty$ for each $\omega' \geq 0$ and

$$\int_0^\infty \Phi_{\sqrt{a}}(r)\,\gamma_{2t}(\mathrm{d}r) = e^{(a-\rho^2)t}. \tag{5.20}$$

2. Now, we may define

$$\widetilde{T}(t)x = \int_0^\infty S(r)x\,\gamma_{2t}(\mathrm{d}r)$$

for all $t \in \mathbb{R}_+$ and $x \in X$. In fact, (5.20) yields $\|\widetilde{T}(t)\| \leq Me^{(\omega^2-\rho^2)t}$ for all $t \geq 0$. Since $(\gamma_t)_{t\geq 0}$ is a Gaussian convolution semigroup we get for $t, s \in \mathbb{R}_+$ and $x \in X$

$$\begin{aligned}\widetilde{T}(t)\widetilde{T}(s)x &= \int_0^\infty S(r)\,\gamma_{2t}(\mathrm{d}r)\int_0^\infty S(q)x\,\gamma_{2s}(\mathrm{d}q) = \int_0^\infty\int_0^\infty S(r)S(q)x\,\gamma_{2t}(\mathrm{d}r)\,\gamma_{2s}(\mathrm{d}q) \\ &= \int_0^\infty\int_0^\infty S(r*q)x\,\gamma_{2t}(\mathrm{d}r)\,\gamma_{2s}(\mathrm{d}q) = \int_0^\infty S(u)x\,(\gamma_{2t}*\gamma_{2s})(\mathrm{d}u) \\ &= \int_0^\infty S(u)x\,\gamma_{2(t+s)}(\mathrm{d}u) = \widetilde{T}(t+s)x.\end{aligned}$$

We show that $\lim_{t\to 0^+}\widetilde{T}(t)x = x$ for any given $x \in X$. If S is uniformly bounded this follows from $\lim_{t\to 0^+}\gamma_t = \varepsilon_0$ in distribution. We treat the general case by estimating

$$\widetilde{T}(t)x - x = \int_0^\infty S(r)x - x\,\gamma_{2t}(\mathrm{d}r). \tag{5.21}$$

Suppose $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\|S(r)x - x\| \leq \varepsilon$ for all $0 \leq r < \delta$. Further,

$$\|S(r)x - x\| \leq \|S(r)\| + 1 \leq M\Phi_\omega(r) + 1 \leq (M+1)\Phi_\omega(r)$$

for all $r \geq 0$, where we have assumed w.l.o.g. $\|x\| \leq 1$ and $\omega \geq \rho$. Looking at the power series (4.10), we find $\omega' \geq \omega$ such that $(M+1)\Phi_\omega(r) \leq \varepsilon\Phi_{\omega'}(r)$ for all $r \geq \delta$, in conclusion $\|S(r)x - x\| \leq \varepsilon\Phi_{\omega'}(r)$ for all $r \geq 0$. So we get from (5.21) using (5.20)

$$\|\widetilde{T}(t)x - x\| \leq \int_0^\infty \|S(r)x - x\|\,\gamma_{2t}(\mathrm{d}r) \leq \varepsilon \int_0^\infty \Phi_{\omega'}(r)\,\gamma_{2t}(\mathrm{d}r) = \varepsilon e^{(\omega'^2-\rho^2)t},$$

thus $\|\widetilde{T}(t)x - x\| \leq 2\varepsilon$ for $t > 0$ sufficiently small.

3. Let $\mathbb{A}_{\widetilde{T}}$ denote the generator of the C_0-semigroup $\widetilde{T}$, and $\mathbb{A}$ the generator of S. Suppose $\sqrt{a} \in \widehat{\mathbb{R}_+}$, that is $a \in]-\infty, \rho^2]$, and $\lambda > \rho$. In the scalar setting Theorem 5.4.6 states

$$(\lambda^2 - a)^{-1} = \int_0^\infty \Psi_\lambda(r)\Phi_{\sqrt{a}}(r)\,A(r)\mathrm{d}r \tag{5.22}$$

and the resolvent formula for C_0-semigroups states

$$(\lambda^2 - a)^{-1} = \int_0^\infty e^{-\lambda^2 r}e^{ar}\ \mathrm{dr} = \int_0^\infty e^{-(\lambda^2-\rho^2)r}\int_0^\infty \Phi_{\sqrt{a}}(q)\,\gamma_{2r}(\mathrm{d}q)\ \mathrm{dr}, \tag{5.23}$$

using (5.20). Define $\mu_\lambda, \nu_\lambda \in M^b(\mathbb{R}_+)$ by

$$\mu_\lambda(f) = \int_0^\infty \Psi_\lambda(r)f(r)\,A(r)\mathrm{d}r$$
$$\nu_\lambda(f) = \int_0^\infty e^{-(\lambda^2-\rho^2)r}\int_0^\infty f(q)\,\gamma_{2r}(\mathrm{d}q)\ \mathrm{dr}$$

for all $f \in C_0(\mathbb{R}_+)$. Then $\mu_\lambda = \nu_\lambda$ by (5.22), (5.23) and Uniqueness Theorem 2.1.9. So we get for $\lambda > \max(\omega, \rho)$

$$(\lambda^2 - \rho^2 - \mathbb{A}_{\widetilde{T}})^{-1}x = \int_0^\infty e^{-(\lambda^2-\rho^2)r}\widetilde{T}(r)x\ \mathrm{dr} = \int_0^\infty \Psi_\lambda(r)S(r)x\,A(r)\mathrm{d}r = (\lambda^2 - \mathbb{A})^{-1}x, \tag{5.24}$$

thus $\rho^2 + \mathbb{A}_{\widetilde{T}} = \mathbb{A}$.

□

The usage of the central limit theorem in the proof of Theorem 5.5.1, see (5.19), has an interesting operator-valued analogon.

Proposition 5.5.2. *Let S and $\widetilde{T}$ be as in Theorem 5.5.1. Then for each $x \in X$ and $t_0 > 0$*

$$\lim_{n\to\infty}\left[S\left(\sqrt{\frac{2(\alpha_0+1)t}{n}}\right)\right]^n x = \widetilde{T}(t)x \tag{5.25}$$

as $n \to \infty$, uniformly for $t \in [0, t_0]$. For $x \in D(\mathbb{A}^2)$ the order of convergence is $\mathcal{O}(\frac{1}{\sqrt{n}})$ if $\alpha_1(0) > 0$ and $\mathcal{O}(\frac{1}{n})$ if $\alpha_1(0) = 0$.

The limiting relation (5.25) is a special case of Chernoff's product formula: Set $V : t \mapsto S(\sqrt{2(\alpha_0+1)t})$, $\mathbb{R}_+ \to \mathcal{L}(X)$ and apply Corollary III.5.3 in Engel and Nagel (2000); the prerequisites are satisfied, $\|[V(t)]^k\| \le M(\Phi_\omega(\sqrt{2(\alpha_0+1)t}))^k \le Me^{k(\rho+\omega)\sqrt{2(\alpha_0+1)t}}$ for all $t \ge 0$, $k \in \mathbb{N}$ by Theorem 5.4.5 and (4.21), $\lim_{t\to 0^+} \frac{V(t)x-x}{t} = \mathbb{A}_0 x$ for $x \in D(\mathbb{A}_0)$ by Proposition 5.1.3, $D(\mathbb{A}_0)$ is dense in X, and $(\lambda_0 - \mathbb{A}_0)(D(\mathbb{A}_0))$ is dense in X for an arbitrary constant $\lambda_0 > \omega^2$ by Theorem 5.4.6.

For cosine operator functions some generalizations of (5.25) have been shown in Goldstein (1982). Also in the cosine setting, asymptotic expansions of order one and two for (5.25) have been considered in Früchtl (2012), see also Früchtl (2009).

The following proof of Proposition 5.5.2 uses basic ideas in the context of the Lax equivalence theorem, see Lax and Richtmyer (1956) or Richtmyer and Morton (1967), Section 3.5. We mention that the Lax equivalence theorem (with orders) can be used to prove the central limit theorem and the weak law of large number (both with orders), see Butzer et al. (1979) (consult Goldstein (1985), 9.19 of Chapter 1 and the notes thereto).

Proof. Suppose $t_0 > 0$ and define for abbreviation $s_n := \sqrt{\frac{2(\alpha_0+1)t}{n}}$, $t_n := \frac{t}{n}$ where $n \in \mathbb{N}$ and $t \in [0, t_0]$. Referring to Richtmyer and Morton (1967), Section 3.5, we have to verify the "stability condition" and "consistency".

To check the "stability condition", we may assume without loss of generality $\omega \ge \rho$, that is Φ_ω monotonically increasing (cf. Proposition 4.3.1). Then for $t \in [0, t_0]$ and $0 \le k \le n$ we obtain from Theorem 5.4.5 and (5.19)

$$\|[S(s_n)]^k\| \le M(\Phi_\omega(s_n))^k \le M(\Phi_\omega(s_n^{\max}))^n \le C$$

with $s_n^{\max} = \sqrt{\frac{2(\alpha_0+1)t_0}{n}}$ and $C > 0$ a constant depending only on t_0 (cf. Proposition 2.5 in Früchtl (2012)).

Suppose $x \in D(\mathbb{A}^2)$. Concerning "consistency" we know that as $n \to \infty$, uniformly for $t \in [0, t_0]$

$$\widetilde{T}(t_n)x = x + \frac{t}{n}\mathbb{A}_0 x + \mathcal{O}(\frac{1}{n^2})$$

and

$$S(s_n)x = x + \frac{t}{n}\mathbb{A}_0 x + \mathcal{O}(\frac{1}{n^j})$$

by Theorem 5.3.4 and Lemma 5.1.4, where $j = 3/2$ if $\alpha_1(0) > 0$ and $j = 2$ if $\alpha_1(0) = 0$. Thus the difference of these two expansions is $\mathcal{O}(\frac{1}{n^j})$. Since $S(s_n)$ and $\widetilde{T}(t)$ commute by Theorem 5.5.1 (cf. Hille's Theorem A.12), we have shown a consistency condition with rates,

$$\|[S(s_n) - \widetilde{T}(t_n)]\widetilde{T}(t)x\| = \mathcal{O}(\frac{1}{n^j})$$

as $n \to \infty$, uniformly for $t \in [0, t_0]$.

Combining consistency and stability we arrive at

$$[S(s_n)]^n x - \widetilde{T}(t)x = \sum_{k=0}^{n-1} [S(s_n)]^k \left(S(s_n) - \widetilde{T}(t_n)\right) \widetilde{T}((n-1-k)t_n)x = \mathcal{O}(\frac{1}{n^{j-1}})$$

as $n \to \infty$, uniformly for $t \in [0, t_0]$.

Finally, note that $D(\mathbb{A}^2)$ is dense in X by Remark 5.3.2. □

We continue with a Banach space valued generalization of the Laplace Representation Theorem 4.4.4.

Theorem 5.5.3. *Let C be a cosine operator function with generator $\mathbb{A}_C$ and let $(\mathbb{R}_+, *(A))$ be a Sturm-Liouville hypergroup with Laplace representation measure ν_t as stated in Theorem 4.4.4. Then*

$$S(t)x := \int_{-t}^{t} C(r)x\, e^{-\rho r} \nu_t(\mathrm{d}r), \tag{5.26}$$

$t \in \mathbb{R}_+$ and $x \in X$, defines a Sturm-Liouville operator function on $(\mathbb{R}_+, *(A))$ with generator $\mathbb{A}_S = \mathbb{A}_C$.

Proof. In the following we write the Laplace representation (4.49) and (5.26) in the form

$$\Phi_\lambda(t) = \int_{\mathbb{R}_+} \cosh(\lambda r)\, \mu_t(\mathrm{d}r) \quad \text{and} \quad S(t)x = \int_{\mathbb{R}_+} C(r)x\, \mu_t(\mathrm{d}r) \tag{5.27}$$

where

$$\mu_t(\mathrm{d}r) := (2 \cdot 1_{]0,t]}(r) + 1_{\{0\}}(r)) e^{-\rho r} \nu_t(\mathrm{d}r),$$

see Remark 4.4.5.

1. Suppose $x \in X$. Concerning strong continuity of S, the scalar side of (5.27) gives $t \mapsto \int_{\mathbb{R}_+} \cosh(\lambda r)\, \mu_t(\mathrm{d}r)$ continuous for every λ in $i\mathbb{R}_+$, the dual space of the cosine hypergroup, hence a Lévy continuity theorem for commutative hypergroups (see Bloom and Heyer (1995), Theorem 4.2.11) yields that the mapping $t \mapsto \mu_t$ from $\mathbb{R}_+$ to $M_+^b(\mathbb{R}_+)$ is vaguely continuous. Hence $t \mapsto S(t)x$, $\mathbb{R}_+ \to X$ is continuous by Lemma 2.2.2, here we use that, locally in t, the supports of μ_t are contained in a compact set (see also Lemma 2.2.1). Further (5.27) gives

$$S(t)x - x = \int_{\mathbb{R}_+} C(r)x - x\, \mu_t(\mathrm{d}r) + (\Phi_0(t) - 1)x, \tag{5.28}$$

thus $\lim_{t \to 0^+} S(t)x = x$.

2. Concerning the functional equation suppose $t, s \in \mathbb{R}_+$. To avoid misunderstandings, we add to the notation of convolution $*$, the integral operator J, and the generator $\mathbb{A}$ subscripts C and S whenever they rely to the cosine or Sturm-Liouville hypergroup, respectively. Then

$$\begin{aligned}
\Phi_\lambda(t)\Phi_\lambda(s) &= \int_{\mathbb{R}_+} \cosh(\lambda r)\; \mu_t(\mathrm{d}r) \int_{\mathbb{R}_+} \cosh(\lambda q)\, \mu_s(\mathrm{d}q) \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \cosh(\lambda r) \cosh(\lambda q)\; \mu_t(\mathrm{d}r)\, \mu_s(\mathrm{d}q) \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \cosh(\lambda(r *_C q))\; \mu_t(\mathrm{d}r)\, \mu_s(\mathrm{d}q) \\
&= \int_{\mathbb{R}_+} \cosh(\lambda r)\, (\mu_t *_C \mu_s)(\mathrm{d}r)
\end{aligned}$$

and

$$\Phi_\lambda(t *_S s) = \int_{\mathbb{R}_+} \Phi_\lambda(r)\,(\varepsilon_t *_S \varepsilon_s)(\mathrm{d}r) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \cosh(\lambda q)\,\mu_r(\mathrm{d}q)(\varepsilon_t *_S \varepsilon_s)(\mathrm{d}r),$$

hence

$$\int_{\mathbb{R}_+} f(r)\,(\mu_t *_C \mu_s)(\mathrm{d}r) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(q)\,\mu_r(\mathrm{d}q)(\varepsilon_t *_S \varepsilon_s)(\mathrm{d}r) \tag{5.29}$$

for all $f \in C_b(\mathbb{R}_+)$ by Uniqueness Theorem 2.1.9 applied to the cosine hypergroup. Equation (5.29) also holds for $f = C(\cdot)x$ by the Hahn-Banach theorem; reading the lines above backwards we get $S(t)S(s)x = S(t * s)x$.

3. Suppose $x \in D(\mathbb{A}_C)$. Theorem 5.3.4 states that

$$C(r)x - x = (J_C 1)(r)\,\mathbb{A}_C x + o(r^2)$$

as $r \to 0^+$, plugging this into (5.28) gives

$$\frac{S(t)x - x}{(J_S 1)(t)} = \frac{\int_{\mathbb{R}_+} (J_C 1)(r)\,\mu_t(\mathrm{d}r)}{(J_S 1)(t)}\,\mathbb{A}_C x + \frac{\Phi_0(t) - 1}{(J_S 1)(t)} x + o(1),$$

as $t \to 0^+$, where the summand $o(1)$ is justified by Lemma 5.1.4. We know that the (scalar) coefficient of the second summand tends to $-\rho^2$; from the scalar setting $S = \Phi_\lambda$, $\lambda \neq 0$ an arbitrary constant, we infer that the (scalar) coefficient of the first summand tends to 1, hence $\mathbb{A}_S x = \mathbb{A}_C x$, that is we have shown $\mathbb{A}_S \supset \mathbb{A}_C$.

4. Finally, we know that C is exponentially bounded by Theorem 1.1.2. Hence there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq M\cosh(\omega t)$ for all $t \in \mathbb{R}_+$, which implies $\|S(t)\| \leq M\Phi_\omega(t)$ for all $t \in \mathbb{R}_+$ by (5.27), that is S is exponentially bounded (recall the discussion at the beginning of Section 5.4). So $\lambda^2 - \mathbb{A}_C$ and $\lambda^2 - \mathbb{A}_S$ are bijective for large $\lambda > 0$ by Theorem 5.4.6. Since we already know $\lambda^2 - \mathbb{A}_S \supset \lambda^2 - \mathbb{A}_C$ this implies $\mathbb{A}_S = \mathbb{A}_C$.

□

Recall that the Laplace representation measure ν_t is known for many interesting examples, see the discussion following Theorem 4.4.4, so (5.26) can be stated explicitly in these cases.

Example 5.5.4. Consider in Theorem 5.5.3 the special case where $(\mathbb{R}_+, *(A))$ is the cosh hypergroup. Then

$$\mathrm{Cosh}(t)x := \int_{-t}^{t} C(r)x\,e^{-r}\nu_t(\mathrm{d}r) = \frac{C(t)x}{\cosh t}$$

is a cosh operator function (a Sturm-Liouville operator function with respect to the cosh hypergroup) with generator $\mathbb{A}_{\mathrm{Cosh}} = \mathbb{A}_C$. Conversely, given a cosh operator function Cosh with generator $\mathbb{A}_{\mathrm{Cosh}}$ define

$$C(t)x := \cosh t\,\cdot\,\mathrm{Cosh}(t)x.$$

It is easily checked that C is a cosine operator function with generator $\mathbb{A}_C = \mathbb{A}_{\mathrm{Cosh}} + I$ (use Proposition 5.1.3). Thus $\mathbb{A}_{\mathrm{Cosh}}$ itself is the generator of a cosine operator function, see, e. g., Arendt et al. (2011), Corollary 3.14.10 or Fattorini (1985), Lemma III.4.1. In particular, each generator of a cosh operator function is also the generator of a cosine operator function.

The following corollary restricts to C_0-groups and is the inverse counterpart to Theorem 5.5.1.

Corollary 5.5.5. *Let T be a C_0-group with generator $\mathbb{A}_T$ and $(\mathbb{R}_+, *(A))$ a Sturm-Liouville hypergroup with Laplace representation measure ν_t as stated in Theorem 4.4.4. Then*

$$S(t)x := \int_{-t}^{t} T(r)x\, e^{-\rho r} \nu_t(\mathrm{d}r),$$

$t \in \mathbb{R}_+$ and $x \in X$, defines a Sturm-Liouville operator function on $(\mathbb{R}_+, *(A))$ with generator $\mathbb{A}_S = \mathbb{A}_T^2$.

Given a C_0-group $T : \mathbb{R} \to \mathcal{L}(X)$ with generator $\mathbb{A}_T$,

$$C(t) := \frac{1}{2}(T(t) + T(-t)) \tag{5.30}$$

defines a cosine operator function with generator $\mathbb{A}_C = \mathbb{A}_T^2$, as already shown by Sova (1966), Theorem 4.12, see Arendt et al. (2011), Example 3.14.15. Hence Corollary 5.5.5 is an immediate consequence of Theorem 5.5.3 and symmetry of the Laplace representation. We note that Corollary 5.5.5 itself is a generalization of (5.30) for Sturm-Liouville operator functions.

The question whether a cosine operator function C admits a representation of the form (5.30) was investigated by several authors. We refer to Kisyński (1971, 1972) for the basic problem as well as examples and counterexamples, and to Haase (2009), where it is shown that for uniformly bounded cosine functions on UMD-spaces such a representation always exists (cf. also Cioranescu and Keyantuo, 2001) and the associated C_0-group is uniformly bounded.

5.6 Relations between Bessel-Kingman, Hyperbolic, and Jacobi Operator Functions

We have seen in Theorem 5.5.3 that the (scalar) Laplace representation theorem gives rise to Sturm-Liouville operator functions. In this section we use this observation to generalize integral transformations for Bessel, Legendre, and Jacobi functions to corresponding operator functions.

Recall the definition of spherical Bessel functions j_α in Example 4.5.2. Suppose $\alpha > \beta \geq -\frac{1}{2}$. Then Sonine's first finite integral states that for each $\lambda \in \mathbb{R}_+$ and all

$t \geq 0$

$$j_\alpha(\lambda t) = \frac{\Gamma(\alpha+1)\cdot 2}{\Gamma(\beta+1)\Gamma(\alpha-\beta)} \int_0^1 j_\beta(\lambda tr)\,(1-r^2)^{\alpha-\beta-1} r^{2\beta+1} \mathrm{d}r \tag{5.31}$$

$$= \frac{\Gamma(\alpha+1)\cdot 2}{\Gamma(\beta+1)\Gamma(\alpha-\beta)t^{2\alpha}} \int_0^t j_\beta(\lambda r)\,(t^2-r^2)^{\alpha-\beta-1} r^{2\beta+1} \mathrm{d}r; \tag{5.32}$$

in fact this is easily derived from Watson (1995), § 12·1 (1), p. 373, see Finckh (1986), (3.2.11). The special case $\beta = -\frac{1}{2}$ leads to the Poisson integral

$$j_\alpha(\lambda t) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} \int_{-1}^1 e^{i\lambda tr}\,(1-r^2)^{\alpha-\frac{1}{2}} \mathrm{d}r \tag{5.33}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})t^{2\alpha}} \int_{-t}^t e^{i\lambda r}\,(t^2-r^2)^{\alpha-\frac{1}{2}} \mathrm{d}r, \tag{5.34}$$

see, e. g., Watson (1995), § 3·3 (4), p. 48.

The following theorem is a generalization of Sonine's first finite integral (5.31) for Bessel-Kingman operator functions. It was published in Glushak (1996), Theorem 2 with a proof relying on differential equations (for the special case $\beta = -\frac{1}{2}$ see also Glushak et al., 1986, Lemma 1). Recall that a Bessel-Kingman operator function is a Sturm-Liouville operator function defined on a Bessel-Kingman hypergroup (cf. Definition 1.2.6).

Theorem 5.6.1. *Let Y_β be a Bessel-Kingman operator function (or a cosine operator function) of order $\beta \geq -\frac{1}{2}$ with generator $\mathbb{A}_\beta$. Then for $\alpha > \beta$*

$$Y_\alpha(t)x := \frac{\Gamma(\alpha+1)\cdot 2}{\Gamma(\beta+1)\Gamma(\alpha-\beta)} \int_0^1 Y_\beta(tr)x\,(1-r^2)^{\alpha-\beta-1} r^{2\beta+1} \mathrm{d}r, \tag{5.35}$$

$t \in \mathbb{R}_+$ and $x \in X$, defines a Bessel-Kingman operator function of order α with generator $\mathbb{A}_\alpha \supset \mathbb{A}_\beta$. If Y_β is exponentially bounded, then Y_α is also exponentially bounded and $\mathbb{A}_\beta = \mathbb{A}_\alpha$.

The proof of Theorem 5.5.3 can be transferred word by word.

We remark that, conversely, the generator of a Bessel-Kingman operator function is in general not necessarily the generator of a cosine operator function, see the example in Glushak (1997b), p. 104.

Proposition 5.6.2. *Theorem 5.6.1 provides an equivalence relation between Bessel-Kingman operator functions sharing one underlying Banach space X. To be precise, given two Bessel-Kingman operator functions (or cosine operator functions) Y_α and Y_β, $\alpha, \beta \geq -\frac{1}{2}$ we write $Y_\alpha \sim Y_\beta$ if one of the following conditions holds.*

(i) $\alpha = \beta$ *and* $Y_\alpha(t) = Y_\beta(t)$ *for all* $t \in \mathbb{R}_+$.

(ii) $\alpha > \beta$ *and* (5.35) *is satisfied for all* $t \in \mathbb{R}_+$.

(iii) $\alpha < \beta$ *and* (5.35)*, with interchanged roles of* α *and* β*, is satisfied for all* $t \in \mathbb{R}_+$.

Proof. We only have to show that $\sim$ is transitive. Again, this can be deduced from the scalar setting. In the following we always suppose $\alpha > \beta > \gamma \geq -\frac{1}{2}$. Let $\mu_t^{\alpha,\beta}$ be defined for $t > 0$ by

$$\mu_t^{\alpha,\beta}(\mathrm{d}r) := \frac{\Gamma(\alpha+1)\cdot 2}{\Gamma(\beta+1)\Gamma(\alpha-\beta)t^{2\alpha}}(t^2-r^2)^{\alpha-\beta-1}r^{2\beta+1}\lambda_{[0,t)}(\mathrm{d}r)$$

and set $\mu_0^{\alpha,\beta}(\mathrm{d}r) := \varepsilon_0$. Sonine's first finite integral (5.32) states that for $\lambda \in \mathbb{R}_+$ and $t \geq 0$

$$j_\alpha(\lambda\, t) = \int_{\mathbb{R}_+} j_\beta(\lambda\, r)\, \mu_t^{\alpha,\beta}(\mathrm{d}r) \tag{5.36}$$

and

$$j_\beta(\lambda\, t) = \int_{\mathbb{R}_+} j_\gamma(\lambda\, r)\, \mu_t^{\beta,\gamma}(\mathrm{d}r), \tag{5.37}$$

thus

$$j_\alpha(\lambda\, t) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} j_\gamma(\lambda\, r)\, \mu_s^{\beta,\gamma}(\mathrm{d}r)\, \mu_t^{\alpha,\beta}(\mathrm{d}s) \tag{5.38}$$

and

$$j_\alpha(\lambda\, t) = \int_{\mathbb{R}_+} j_\gamma(\lambda\, r)\, \mu_t^{\alpha,\gamma}(\mathrm{d}r). \tag{5.39}$$

Applying Uniqueness Theorem 2.1.9 to the right hand sides of (5.38), (5.39) yields

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(r)\, \mu_s^{\beta,\gamma}(\mathrm{d}r)\, \mu_t^{\alpha,\beta}(\mathrm{d}s) = \int_{\mathbb{R}_+} f(r)\, \mu_t^{\alpha,\gamma}(\mathrm{d}r) \tag{5.40}$$

for all $f \in C_b(\mathbb{R}_+)$ and $t \geq 0$.

Suppose $x \in X$. In the remaining part of the proof we replace $j_\alpha(\lambda\,\cdot)$, $j_\beta(\lambda\,\cdot)$ and $j_\gamma(\lambda\,\cdot)$ successively by $Y_\alpha(\cdot)x$, $Y_\beta(\cdot)x$ and $Y_\gamma(\cdot)x$, respectively. We have to distinct three cases.

First, suppose $Y_\alpha \sim Y_\beta$ and $Y_\beta \sim Y_\gamma$. Then (5.36) and (5.37) hold in the vector setting, so does (5.38) and hence (5.39) by (5.40), that is $Y_\alpha \sim Y_\gamma$.

Second, suppose $Y_\alpha \sim Y_\gamma$ and $Y_\beta \sim Y_\gamma$. So (5.39) holds in the vector setting, use (5.40) to get (5.38) and replace the inner integral by the vector analogue of (5.37), thus $Y_\alpha \sim Y_\beta$.

Third, suppose $Y_\alpha \sim Y_\beta$ and $Y_\alpha \sim Y_\gamma$. Similarly, we get

$$\int_{\mathbb{R}_+} Y_\beta(r)x\, \mu_t^{\alpha,\beta}(\mathrm{d}r) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y_\gamma(r)x\, \mu_s^{\beta,\gamma}(\mathrm{d}r)\, \mu_t^{\alpha,\beta}(\mathrm{d}s).$$

Taking the difference of the left and the right hand side and applying the Hahn-Banach theorem we may conclude $Y_\beta \sim Y_\gamma$ from the associated scalar integral transform. Therefore, define for $f \in C_b(\mathbb{R}_+)$ and $t > 0$

$$\begin{aligned}\mathcal{I}_{\beta,\alpha-\beta}f(t) :=& \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)}\int_{\mathbb{R}_+} f(r)\, \mu_t^{\alpha,\beta}(\mathrm{d}r)\\ =& \frac{2}{\Gamma(\alpha-\beta)t^{2\alpha}}\int_0^t f(r)\,(t^2-r^2)^{\alpha-\beta-1}r^{2\beta+1}\mathrm{d}r.\end{aligned}$$

$\mathcal{I}_{\beta,\alpha-\beta}$ is an Erdélyi-Kober operator,, which is known to be invertible, see Sneddon (1975), (2.1) and (2.13). Consequently,

$$\mathcal{I}_{\beta,\alpha-\beta} f(t) = 0 \quad \text{for all } t > 0 \tag{5.41}$$

implies $f = 0$. We note that (5.41) can easily be reduced to a Riemann-Liouville fractional integral. Indeed, $\mathcal{I}_{\beta,\alpha-\beta} f(t) = 0$ for all $t > 0$ is equivalent to $\int_0^1 (1 - s^2)^{\alpha-\beta-1} s g(t^2 s^2)\, \mathrm{ds} = 0$ for all $t > 0$ where $g(u) := u^\beta f(\sqrt{u})$. Clearly, this is the same as $\int_0^1 (1 - s^2)^{\alpha-\beta-1} s g(ts^2)\, \mathrm{ds} = 0$ for all $t > 0$. Substituting $r = ts^2$ yields

$$(\mathcal{R}_{\alpha-\beta}\, g)(t) := \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-r)^{\alpha-\beta-1} g(r)\, \mathrm{dr} = 0$$

for all $t > 0$. Thus $g = 0$ and so $f = 0$ (see, e. g., Samko et al., 1993 for fractional calculus). □

Proposition 5.6.3. *Let Y_β be a Bessel-Kingman operator function (or a cosine operator function) of order $\beta \geq -\frac{1}{2}$ and let Y_α for $\alpha > \beta$ be defined as in Theorem 5.6.1. Then for $t_0 > 0$ and $x \in X$*

$$\lim_{\alpha\to\infty} Y_\alpha(t)x = x, \tag{5.42}$$

uniformly for $t \in [0, t_0]$, and

$$\lim_{\alpha\to\beta^+} Y_\alpha(t)x = Y_\beta(t)x, \tag{5.43}$$

uniformly for $t \in [0, t_0]$.

For $\beta = -\frac{1}{2}$ limit (5.43) is content of Glushak (1999a), Theorem 2, for the corresponding limit for Legendre operator functions see Glushak (2001), Theorem 10.

Proof. Suppose $\varepsilon > 0$. Concerning (5.42) choose $\delta > 0$ such that $\|Y_\beta(tr)x - x\| \leq \varepsilon$ for all $0 \leq r \leq \delta$ and $t \in [0, t_0]$, concerning (5.43) choose $\delta > 0$ such that $\|Y_\beta(tr)x - Y_\beta(t)x\| \leq \varepsilon$ for all $1 - \delta \leq r \leq 1$ and $t \in [0, t_0]$.

Let $w_{\alpha,\beta}$ be defined by

$$w_{\alpha,\beta}(r) := \frac{\Gamma(\alpha+1)\cdot 2}{\Gamma(\beta+1)\Gamma(\alpha-\beta)} (1-r^2)^{\alpha-\beta-1} r^{2\beta+1} 1_{[0,1)}(r).$$

We know that $w_{\alpha,\beta}$ is the density of a probability measure (cf. e. g. (5.31) with $\lambda = 0$). Further $w_{\alpha,\beta}(r) \to 0$ as $\alpha \to \infty$, uniformly for $\delta \leq r < 1$ since $\Gamma(\alpha+1)/\Gamma(\alpha-\beta) \sim \alpha^{\beta+1}$, and $w_{\alpha,\beta}(r) \to 0$ as $\alpha \to \beta^+$, uniformly for $0 \leq r \leq 1 - \delta$.

Now

$$\|Y_\alpha(t)x - x\| \leq \int_0^\delta \|Y_\beta(tr)x - x\|\, w_{\alpha,\beta}(r)\mathrm{d}r + \int_\delta^1 \|Y_\beta(tr)x - x\|\, w_{\alpha,\beta}(r)\mathrm{d}r$$

gives that for α large enough $\|Y_\alpha(t)x - x\| \leq 2\varepsilon$ for all $t \in [0, t_0]$ and

$$\begin{aligned} \|Y_\alpha(t)x - Y_\beta(t)x\| \leq & \int_0^{1-\delta} \|Y_\beta(tr)x - Y_\beta(t)x\|\, w_{\alpha,\beta}(r)\mathrm{d}r \\ & + \int_{1-\delta}^1 \|Y_\beta(tr)x - Y_\beta(t)x\|\, w_{\alpha,\beta}(r)\mathrm{d}r \end{aligned}$$

implies that for $\alpha > \beta$ close to β it is $\|Y_\alpha(t)x - Y_\beta(t)x\| \leq 2\varepsilon$ for all $t \in [0, t_0]$. □

The following theorem is due to Glushak (2001), see Theorem 1. Recall that a hyperbolic operator function is a Sturm-Liouville operator function with respect to a hyperbolic hypergroup (cf. Definition 1.3.2).

Theorem 5.6.4. *Let P_β be a hyperbolic operator function (or a cosine operator function) of order $\beta \geq -\frac{1}{2}$ with generator $\mathbb{A}_\beta$. Then for $\alpha > \beta$*

$$P_\alpha(t)x := \frac{\Gamma(\alpha+1)2^{\alpha-\beta}}{\Gamma(\alpha-\beta)\Gamma(\beta+1)}(\sinh t)^{-2\alpha}\int_0^t P_\beta(r)x\,(\cosh t - \cosh r)^{\alpha-\beta-1}(\sinh r)^{2\beta+1}\,\mathrm{d}r,$$

$t > 0$ and $x \in X$, defines a hyperbolic operator function of order α with generator $\mathbb{A}_\alpha \supset \mathbb{A}_\beta$. If P_β is exponentially bounded, then P_α is also exponentially bounded and $\mathbb{A}_\alpha = \mathbb{A}_\beta$.

For a proof we take the scalar-valued assertion for granted, see the references to Theorem 1 in Glushak (2001), the case $\beta = -\frac{1}{2}$ is included as a limiting case, see also (4.52). Then one can proceed again word by word as in the proof of Theorem 5.5.3.

We state one more result of this type. It is based on a generalized Mehler-Dirichlet type integral for Jacobi functions due to Koornwinder (1975), see (2.14) therein. By analogy to previous notation, a Jacobi operator function of order (α, β) is a Sturm-Liouville operator function on a Jacobi hypergroup of order (α, β).

Theorem 5.6.5. *Let $P_{(\alpha,\beta)}$ be a Jacobi operator function of order (α, β), $\alpha > \beta > -\frac{1}{2}$ with generator $\mathbb{A}_{(\alpha,\beta)}$. Then for $\mu > 0$*

$$\begin{aligned} P_{(\alpha+\mu,\beta+\mu)}(t)x := & \frac{\Gamma(\alpha+\mu+1)2^{-\mu+1}}{\Gamma(\alpha+1)\Gamma(\mu)} \frac{\sinh 2t}{\sinh^{2(\alpha+\mu)+1} t \cosh^{2(\beta+\mu)+1} t} \\ & \times \int_0^t P_{(\alpha,\beta)}(r)x\,(\cosh 2t - \cosh 2r)^{\mu-1}\sinh^{2\alpha+1} r \cosh^{2\beta+1} r\,\mathrm{d}r, \end{aligned}$$

$t > 0$ and $x \in X$, defines a Jacobi operator function of order $(\alpha + \mu, \beta + \mu)$ with generator $\mathbb{A}_{(\alpha+\mu,\beta+\mu)} \supset \mathbb{A}_{(\alpha,\beta)}$. If $P_{(\alpha,\beta)}$ is exponentially bounded, then $P_{(\alpha+\mu,\beta+\mu)}$ is also exponentially bounded and $\mathbb{A}_{(\alpha+\mu,\beta+\mu)} = \mathbb{A}_{(\alpha,\beta)}$.

The proof follows, once again, the lines of the proof of Theorem 5.5.3.

As we have seen, the results above depend only on the scalar setting. So we expect that by means of hypergroup theory many more formulas for special functions can be shown to admit operator-valued generalizations.

5.7 Translation Operator Functions on Homogeneous Banach Spaces Revisited

In Section 3.3 we have introduced the notion of a translation operator function. Here, in the Sturm-Liouville setting, the generator can be stated explicitly. By a *Sturm-Liouville translation operator function* we mean a translation operator function on a

Sturm-Liouville hypergroup $(\mathbb{R}_+, *(A))$ with Sturm-Liouville function A satisfying SL1 and SL2. (cf. Definition 5.1.1).

For the cosine setting see Gessinger (2001) who considers a variety of modified cosine translations on selected spaces, including $C_{ub}(\mathbb{R}_+)$ and $L^p(\mathbb{R}_+, \lambda_{\mathbb{R}_+})$, $1 \le p < \infty$.

Recall that $AC_{loc}(\mathbb{R}_+^\times)$ denotes the set of functions on $\mathbb{R}_+^\times$ which are absolutely continuous on compact subintervalls of $\mathbb{R}_+^\times$ (cf. Definition 4.1.2).

Theorem 5.7.1. *Let S be a Sturm-Liouville translation operator function on a homogeneous Banach space B with generator $\mathbb{A}$. Then*

$$\mathbb{A}_0 f = Lf \tag{5.44}$$

with

$$D(\mathbb{A}_0) = \{f \in B : f \in AC_{loc}(\mathbb{R}_+^\times),\ Af' \in AC_{loc}(\mathbb{R}_+),\ A(t)f'(t)\big|_{t=0} = 0 \text{ and } Lf \in B\}.$$

Heuristically, (5.44) follows from the abstract Sturm-Liouville equation, the special form of S and Lemma 4.1.5. One can also regard it as a consequence of Delsarte's generalized Taylor formula (4.15). Conversely, Theorem 5.7.1 together with Theorem 5.3.4 can be used to give conditions such that the generalized Taylor formula converges in B.

See also Remark 5.7.3, and the examples in Glushak (1996) and Glushak (1997b).

Proof. *1.* Suppose $f \in D(\mathbb{A}_0)$. The abstract Sturm-Liouville integral equation, see Theorem 5.2.1, gives

$$S(t)f - S(t_0)f = \int_{t_0}^t A(s)^{-1} \int_0^s A(r)S(r)\mathbb{A}_0 f \text{ dr ds}$$

for $t, t_0 \in \mathbb{R}_+$. Successively, we apply the continuous linear functionals $\langle \cdot, \varphi_n \rangle = \int_{\mathbb{R}_+} \cdot\, \varphi_n(s)\, A(s) \mathrm{d}s \in B^*$ with $\varphi_n \in C_c(\mathbb{R}_+)$, $\varphi_n \ge 0$, $\operatorname{supp}(\varphi_n) \subset [0, \frac{1}{n}]$, and $\|\varphi_n\| = 1$ in $L^1(\mathbb{R}_+, A\lambda_{\mathbb{R}_+})$, $n \in \mathbb{N}$. Using the translation property $S(t)f = f(\cdot * t)$ we get

$$(f * \varphi_n)(t) - (f * \varphi_n)(t_0) = \int_{t_0}^t A(s)^{-1} \int_0^s A(r)((\mathbb{A}_0 f) * \varphi_n)(r) \text{ dr ds}. \tag{5.45}$$

The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is a bounded approximate unit for $L^1(\mathbb{R}_+, A\lambda_{\mathbb{R}_+})$, see Theorem 1.6.15 in Bloom and Heyer (1995), that is $\|g * \varphi_n - g\|_1 \to 0$ as $n \to \infty$ for all $g \in L^1(\mathbb{R}_+, A\lambda_{\mathbb{R}_+})$. So we may choose a subsequence, also denoted by $(\varphi_n)_{n\in\mathbb{N}}$, such that

$$(f * \varphi_n)(t) \to f(t)$$

as $n \to \infty$ for (Lebesgue) almost every $t \in \mathbb{R}_+$. Let $t_0 > 0$ be such a Lebesgue point. Then for each $t > 0$

$$\begin{aligned} &\left| \int_{t_0}^t A(s)^{-1} \int_0^s A(r)((\mathbb{A}_0 f) * \varphi_n - (\mathbb{A}_0 f))(r) \text{ dr ds} \right| \\ &= \left| \int_{t_0}^t A(s)^{-1} \int_0^s A(r)(((\mathbb{A}_0 f)|_{[0,t+1]}) * \varphi_n - (\mathbb{A}_0 f)|_{[0,t+1]})(r) \text{ dr ds} \right| \\ &\le \int_{t_0}^t A(s)^{-1} \text{ ds} \cdot \|((\mathbb{A}_0 f)|_{[0,t+1]}) * \varphi_n - \mathbb{A}_0 f|_{[0,t+1]}\|_1 \to 0 \end{aligned}$$

as $n \to \infty$. Thus, (5.45) gives

$$f(t) - f(t_0) = \int_{t_0}^{t} A(s)^{-1} \int_0^s A(r)(\mathbb{A}_0 f)(r) \text{ dr ds} \tag{5.46}$$

for almost all $t \in \mathbb{R}_+$. So a representative of f lies in $AC_{loc}(\mathbb{R}_+^\times)$ and

$$A(t)f'(t) = \int_0^t A(r)(\mathbb{A}_0 f)(r) \text{ dr},$$

thus $Af' \in AC_{loc}(\mathbb{R}_+)$ with $A(t)f'(t)|_{t=0} = 0$ and $Lf = \mathbb{A}_0 f$.

2. Conversely, suppose $f \in B$ with $f \in AC_{loc}(\mathbb{R}_+^\times)$, $Af' \in AC_{loc}(\mathbb{R}_+)$, $A(t)f'(t)|_{t=0} = 0$ and $Lf \in B$. We show that for $t > 0$

$$S(t)f - f = \int_0^t A(s)^{-1} \int_0^s A(r)S(r)(Lf) \text{ dr ds}$$

in B. Then by definition of the universal generator $f \in D(\mathbb{A}_0)$ and $\mathbb{A}_0 f = Lf$.

The following calculation is straightforward, for details see below. Suppose $t > 0$ and $\varphi \in C_c(\mathbb{R}_+)$. Then

$$\begin{aligned}
&\left\langle \int_0^t A(s)^{-1} \int_0^s A(r)S(r)(Lf) \text{ dr ds}, \varphi \right\rangle \\
&= \int_0^t A(s)^{-1} \int_0^s A(r)\langle (Lf)(\cdot * r), \varphi\rangle \text{ dr ds} \\
&= \int_0^t A(s)^{-1} \int_0^s A(r)\langle Lf, \varphi(\cdot * r)\rangle \text{ dr ds} \\
&= \left\langle Lf, \int_0^t A(s)^{-1} \int_0^s A(r)\varphi(\cdot * r) \text{ dr ds} \right\rangle \\
&= \langle Lf, (J\varphi)(\cdot * t) - J\varphi\rangle \\
&= \langle f, \varphi(\cdot * t) - \varphi\rangle \\
&= \langle S(t)f - f, \varphi\rangle.
\end{aligned}$$

First of all, we have used $\langle \cdot, \varphi\rangle \in B^*$, Theorem 2.2.14 and Fubini's theorem; Theorem 2.2.14 is used a second time in the last step. So it remains to justify equalities four and five.

The fourth equality uses

$$(J_r(\varphi(u * r)))(t) = (J\varphi)(u * t) - (J\varphi)(u),$$

which follows from $\varphi(u * r) = (L(J\varphi))(u * r) = L_r((J\varphi)(u * r))$, see Lemmata 4.2.2 and 4.1.5. This expression vanishes for $u \geq T := t + \sup(\operatorname{supp}(\varphi))$ as can be seen from the left hand side, using the support property (4.1).

The fifth equality is an application of Green's formula (Lemma 4.3.3). First note that $L_u((J\varphi)(u * t) - (J\varphi)(u)) = \varphi(u * t) - \varphi(u)$ by Lemmata 4.1.5 and 4.2.2. For the moment we may assume without loss of generality $0, t \notin \operatorname{supp}(\varphi)$. Then we get for $\varepsilon > 0$

$$\int_\varepsilon^T (Lf)(u)((J\varphi)(u*t) - (J\varphi)(u))\, A(u)\mathrm{d}u - \int_\varepsilon^T f(u)(\varphi(u*t) - \varphi(u))\, A(u)\mathrm{d}u$$

$$= 0 - \Bigg(A(\varepsilon)f'(\varepsilon)\, ((J\varphi)(\varepsilon * t) - (J\varphi)(\varepsilon)) - f(\varepsilon)\ A(u)\frac{\partial}{\partial u}((J\varphi)(u*t) - (J\varphi)(u))\Big|_{u=\varepsilon} \Bigg).$$

Taking the limit $\varepsilon \to 0^+$, the first term in brackets tends to zero since $A(t)f'(t)|_{t=0} = 0$ by assumption, the second is equal to zero for small $\varepsilon > 0$ since $0 \notin \operatorname{supp}(\varphi)$ implies $(J\varphi)(u) = 0$ for small u, and $t \notin \operatorname{supp}(\varphi)$ gives $L(J\varphi)(v) = 0$ in a neighbourhood of $v = t$, thus $(J\varphi)(u * t) = (J\varphi)(t)$ for small u, see the proof of Lemma 5.2.3.

□

Remark 5.7.2. If $X = C_0(\mathbb{R}_+)$, then

$$D(\mathbb{A}_0) = \{f \in C_0(\mathbb{R}_+) :\ f \in C^2(\mathbb{R}_+),\ f'(0) = 0 \text{ and } Lf \in C_0(\mathbb{R}_+)\}.$$

This is a consequence of (5.46), stating

$$f(t) - f(0) = \int_0^t A(s)^{-1} \int_0^s A(r)(\mathbb{A}_0 f)(r)\ \mathrm{dr}\ \mathrm{ds},$$

and Lemma 4.2.2. Note that for $X = C_0(\mathbb{R}_+)$ the proof above can be simplified considerably.

Remark 5.7.3. Although our intention is to investigate Sturm-Liouville operator functions, the following observation should be of independent interest. Consider the Gaussian convolution semigroup $(\gamma_t)_{t\geq 0}$ introduced at the beginning of Section 5.5. Combining Theorem 5.7.1 with Theorem 5.5.1 we are able to determine its infinitesimal generator. For reasons of simplicity let us consider again $X = C_0(\mathbb{R}_+)$. Then

$$T(t)f = \gamma_t * f \tag{5.47}$$

defines a C_0-semigroup of contractions and its generator is given by

$$\mathbb{A}_T f = \frac{1}{2}Lf = \frac{1}{2}(f'' + \frac{A'}{A}f')$$

with domain

$$D(\mathbb{A}_T) = \{f \in C_0(\mathbb{R}_+) :\ f \in C^2(\mathbb{R}_+),\ f'(0) = 0 \text{ and } Lf \in C_0(\mathbb{R}_+)\}.$$

Some known facts about the generator can be found in Rentzsch and Voit (2000), Section 6, see also the references therein. For semigroups of the form (5.47), in a general setting, see, e. g., Bloom and Heyer (1995), Section 6.5, p. 427.

5.8 Multiplication Operator Functions

In this section we consider Sturm-Liouville operator functions induced by multiplication operators on some function spaces. This is by analogy with multiplication semigroups. In the theory of semigroups of operators, they provide a rich source of examples and counterexamples. Here, we construct an example of a Sturm-Liouville operator function which is not exponentially bounded.

The content of this section is based on the treatment of multiplication semigroups in Engel and Nagel (2000), Section I.4, pp. 24–33 and Section II.2.9, p. 65. The statements and proofs in the Sturm-Liouville setting are quite similar. As done in Engel and Nagel (2000), we first consider the space $X = C_0(\Omega)$, Ω a locally compact Hausdorff space, and then $X = L^p(\Omega, \mathscr{A}, \mu)$, $1 \le p < \infty$ where $(\Omega, \mathscr{A}, \mu)$ is a σ-finite measure space. The corresponding basic propositions about multiplication operators are independent of the Sturm-Liouville setting and therefore left without proof. In Engel and Nagel (2000) one can also find some hints and references on how to use the specific spaces and results for applications.

Let Ω be a locally compact Hausdorff space and consider $X = C_0(\Omega)$, the space of continuous functions vanishing at infinity with supremum norm.

Let $q : \Omega \to \mathbb{C}$ be a continuous function. We define the *multiplication operator* M_q by

$$\begin{aligned} M_q : C_0(\Omega) &\to C_0(\Omega) \\ (M_q f)(\omega) &:= q(\omega) \cdot f(\omega) \end{aligned}$$

with domain

$$D(M_q) = \{f \in C_0(\Omega) : q \cdot f \in C_0(\Omega)\}.$$

Proposition 5.8.1. *The multiplication operator* $M_q : C_0(\Omega) \to C_0(\Omega)$ *satisfies the following conditions.*

(i) M_q *is densely defined and closed.*

(ii) M_q *is bounded if and only if* q *is bounded. In this case*

$$\|M_q\| = \|q\|_\infty = \sup_{\omega \in \Omega} |q(\omega)|.$$

(iii) $\sigma(M_q) = \mathrm{cl}(q(\Omega))$.

In the following we always assume that $q : \Omega \to \mathbb{C}$ is a continuous function such that for any compact set $C \subset \mathbb{R}_+$

$$\sup_{t \in C,\, \omega \in \Omega} |\Phi_{\sqrt{q(\omega)}}(t)| < \infty.$$

In this case define for $t \in \mathbb{R}_+$ and $f \in C_0(\Omega)$

$$(S_q(t)f)(\omega) := \Phi_{\sqrt{q(\omega)}}(t) \cdot f(\omega)$$

for all $\omega \in \Omega$. Then S_q is called a *(Sturm-Liouville) multiplication operator function* with $X = C_0(\Omega)$. This terminology is justified by the following proposition.

Proposition 5.8.2. *A Sturm-Liouville multiplication operator function S_q with $X = C_0(\Omega)$ is a Sturm-Liouville operator function.*

Proof. The operator function $S_q : \mathbb{R}_+ \to \mathcal{L}(C_0(\Omega))$ is well-defined, $S_q(0) = I$ and S_q is locally uniformly bounded, that is for each compact set $C \subset \mathbb{R}_+$, $\sup_{t\in C} \|S_q(t)\| = \sup_{t\in C,\,\omega\in\Omega} |\Phi_{\sqrt{q(\omega)}}(t)| < \infty$ by assumption.

In order to show that S_q is strongly continuous we may assume without loss of generality $f \in C_c(\Omega)$. Then $\sup_{\omega\in\mathrm{supp}(f)} |q(\omega)| < \infty$ since q is continuous and we see from the power series of $\Phi_\lambda(t)$ (cf. (4.10))that $S_q(\cdot)f$ is continuous.

So it remains to check the functional equation. Suppose $t, s \in \mathbb{R}_+$ and $f \in C_0(\Omega)$. Then

$$(S_q(t)S_q(s)f)(\omega) = \Phi_{\sqrt{q(\omega)}}(t)\Phi_{\sqrt{q(\omega)}}(s)f(\omega) = \Phi_{\sqrt{q(\omega)}}(t * s)f(\omega)$$

and

$$\begin{aligned}(S_q(t * s)f)(\omega) &= (\int_{\mathbb{R}_+} S_q(r)f\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r))(\omega) = \int_{\mathbb{R}_+} (S_q(r)f)(\omega)\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r) \\ &= \int_{\mathbb{R}_+} \Phi_{\sqrt{q(\omega)}}(r)f(\omega)\,(\varepsilon_t * \varepsilon_s)(\mathrm{d}r) = \Phi_{\sqrt{q(\omega)}}(t * s)f(\omega)\end{aligned}$$

for all $\omega \in \Omega$, that is $S_q(t)S_q(s)f = S_q(t * s)f$. □

Proposition 5.8.3. *Let $S_q : \mathbb{R}_+ \to \mathcal{L}(C_0(\Omega))$ be a Sturm-Liouville multiplication operator function. Then $t \mapsto S_q(t)$ is uniformly continuous if and only if q is bounded. In this case the generator is given by $\mathbb{A} = M_q$.*

Proof. Suppose q is bounded. Then M_q is bounded by Proposition 5.8.1 and it is easy to see that $S_q(t) = \Phi_{\sqrt{M_q}}(t)$, written in the notation of Theorem 5.3.5.

Conversely, suppose S_q is uniformly continuous. According to Theorem 5.3.5 there exists $\mathbb{A} \in \mathcal{L}(C_0(\Omega))$ such that $S_q(t) = \Phi_{\sqrt{\mathbb{A}}}(t)$. For each $f \in C_0(\Omega)$ and $\omega \in \Omega$ we have

$$\lim_{t\to 0^+} \frac{S_q(t)f - f}{(J1)(t)}(\omega) = \lim_{t\to 0^+} \frac{\Phi_{\sqrt{q(\omega)}} - 1}{(J1)(t)} f(\omega) = (q(\omega) - \rho^2)f(\omega). \tag{5.48}$$

Thus $\mathbb{A}_0 = M_q - \rho^2$, that is $\mathbb{A} = M_q$ is bounded, so q is bounded by Proposition 5.8.1. □

Next, we consider Sturm-Liouville multiplication operator functions where $X = L^p(\Omega, \mathscr{A}, \mu)$, $1 \leq p < \infty$ where $(\Omega, \mathscr{A}, \mu)$ is an arbitrary σ-finite measure space. The results and proofs are very similar, but some slight changes are necessary.

Let $q : \Omega \to \mathbb{C}$ be a measurable function. The set

$$q_{ess}(\Omega) := \{\lambda \in \mathbb{C} : \mu(\{\omega \in \Omega : |q(\omega) - \lambda| < \varepsilon\}) \neq 0 \text{ for all } \varepsilon > 0\}$$

is called its *essential range.* The corresponding *multiplication operator* M_q is defined by

$$\begin{aligned} M_q : L^p(\Omega, \mathscr{A}, \mu) &\to L^p(\Omega, \mathscr{A}, \mu) \\ (M_q f)(\omega) &:= q(\omega) \cdot f(\omega) \end{aligned}$$

with domain

$$D(M_q) = \{f \in L^p(\Omega, \mathscr{A}, \mu) : q \cdot f \in L^p(\Omega, \mathscr{A}, \mu)\}.$$

Proposition 5.8.4. *The multiplication operator* $M_q : L^p(\Omega, \mathscr{A}, \mu) \to L^p(\Omega, \mathscr{A}, \mu)$ *satisfies the following conditions.*

(i) M_q *is densely defined and closed.*

(ii) M_q *is bounded if and only if* q *is essentially bounded, i. e. the set* $q_{ess}(\Omega)$ *is bounded in* $\mathbb{C}$*. In this case*

$$\|M_q\| = \|q\|_\infty := \sup\{|\lambda| : \lambda \in q_{ess}(\Omega)\}.$$

(iii) $\sigma(M_q) = q_{ess}(\Omega)$.

In the following we always assume that $q : \Omega \to \mathbb{C}$ is a measurable function such that for any compact set $C \subset \mathbb{R}_+$

$$\sup_{t \in C,\, \lambda \in q_{ess}(\Omega)} |\Phi_{\sqrt{\lambda}}(t)| < \infty.$$

In this case define for $t \in \mathbb{R}_+$ and $f \in L^p(\Omega, \mathscr{A}, \mu)$

$$(S_q(t)f)(\omega) := \Phi_{\sqrt{q(\omega)}}(t) \cdot f(\omega)$$

for μ-a.e. $\omega \in \Omega$. Then S_q is called a *(Sturm-Liouville) multiplication operator function* with $X = L^p(\Omega, \mathscr{A}, \mu)$.

Proposition 5.8.5. *A Sturm-Liouville multiplication operator function* S_q *with* $X = L^p(\Omega, \mathscr{A}, \mu)$ *is a Sturm-Liouville operator function.*

Proof. The proof is the same as for Proposition 5.8.2 but differs in two technicalities. Concerning strong continuity of S_q we may assume without loss of generality $f = 1_E$ with $E \in \mathscr{A}$, $\mu(E) < \infty$ and $q(E)$ bounded in $\mathbb{C}$ (for the last restriction note that q is measurable and $\mathbb{C}$ can be covered by a sequence of compact sets). For the functional equation one may use the duality $(L^p(\Omega, \mathscr{A}, \mu))^* = L^q(\Omega, \mathscr{A}, \mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and apply Fubini's theorem since $(\Omega, \mathscr{A}, \mu)$ is a σ-finite measure space. □

Proposition 5.8.6. *Let* S_q *be a Sturm-Liouville multiplication operator function with* $X = L^p(\Omega, \mathscr{A}, \mu)$*. Then* $t \mapsto S_q(t)$ *is uniformly continuous if and only if* q *is essentially bounded. In this case the generator is given by* $\mathbb{A} = M_q$.

Proof. See the proof of Proposition 5.8.3. □

In the following we treat the cases $X = C_0(\Omega)$ and $X = L^p(\Omega, \mathscr{A}, \mu)$ simultaneously.

Proposition 5.8.7. *Suppose $X = C_0(\Omega)$ or $X = L^p(\Omega, \mathscr{A}, \mu)$ and let $S_q : \mathbb{R}_+ \to \mathcal{L}(X)$ be a Sturm-Liouville multiplication operator function with generator $\mathbb{A}$. Then*

$$\mathbb{A}f = M_q f = q \cdot f$$

on $D(\mathbb{A}) \subset D(M_q) = \{f \in X : \; q \cdot f \in X\}$.

If S_q admits an exponential bound, then $D(\mathbb{A}) = D(M_q)$.

Proof. We consider only the case $X = C_0(\Omega)$, the proof for $X = L^p(\Omega, \mathscr{A}, \mu)$ is almost the same. From (5.48) we see that $f \in D(\mathbb{A})$ implies $q \cdot f \in C_0(\Omega)$ and $\mathbb{A}f = q \cdot f$, thus $\mathbb{A} \subset M_q$.

Suppose now that S_q is exponentially bounded. On the one hand $\lambda - M_q$ is invertible for large positive λ since $\sigma(M_q) = \mathrm{cl}(q(\Omega))$ by Proposition 5.8.1(iii) and $\sup_{\omega \in \Omega} |\Phi_{\sqrt{q(\omega)}}(t)| < \infty$ for an arbitrary $t > 0$ by assumption. On the other hand $\lambda - \mathbb{A}$ is invertible for large positive λ by Theorem 5.4.6. Since we already know $\mathbb{A} \subset M_q$, this implies $\mathbb{A} = M_q$. □

Example 5.8.8. Suppose $\Omega = \mathbb{N}$ and $(q_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers such that for each compact set $C \subset \mathbb{R}_+$

$$\sup_{t \in C,\, n \in \mathbb{N}} |\Phi_{\sqrt{q_n}}(t)| < \infty. \tag{5.49}$$

Suppose $X = c_0(\mathbb{N})$, the space of null sequences with supremum norm, or $X = \ell^p(\mathbb{N})$, $1 \leq p < \infty$, the space of p-summable sequences with norm $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$. Then

$$S_q(t)x := (\Phi_{\sqrt{q_n}}(t) x_n)_{n \in \mathbb{N}} \tag{5.50}$$

with $t \in \mathbb{R}_+$, $x = (x_n)_{n \in \mathbb{N}} \in X$ defines a Sturm-Liouville multiplication operator function. It is uniformly continuous if and only if the sequence $(q_n)_{n \in \mathbb{N}}$ is bounded.

If the sequence $(q_n)_{n \in \mathbb{N}}$ consists solely of negative real numbers, the functions $\Phi_{\sqrt{q_n}}$ are characters and the example gets particularly simple. If additionally $(q_n)_{n \in \mathbb{N}}$ is unbounded, this is probably the most elementary example of a Sturm-Liouville operator function which is not uniformly continuous. For cosine operator functions such an example was considered in Früchtl (2009), for Bessel-Kingman operator functions several variants of it were investigated in Dietmair (1985).

Here is a sub-example which shows that several results in Section 3.2 are in some sense best possible. Compare also with Theorem 1.1.17.

Example 5.8.9. Let $(\mathbb{R}_+, *(A))$ be a Bessel-Kingman hypergroup, a Jacobi hypergroup, or a Levitan hypergroup as in Example 4.3.2 with $\rho > 0$. Consider Example 5.8.8 with $\sqrt{q_n} = \lambda_n = in$. These Sturm-Liouville hypergroups have in common that for each $t_0 > 0$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\Phi_{in}(t)| \leq \varepsilon$ for all $t \geq t_0$ and $n \geq N$, see (4.23) and (4.24) for the Bessel-Kingman and the Jacobi hypergroup, respectively.

So it is easy to see that S is uniformly continuous on $\mathbb{R}_+^\times$, hence it cannot be uniformly continuous in $t = 0$ by Proposition 5.8.3 and Proposition 5.8.6, respectively. In particular, S is an example of a multiplicative operator function which is locally $m = A\lambda_{\mathbb{R}_+}$-measurable but not uniformly continuous.

Further, $S(t)S(s) = S(t * s)$ for all $t, s \in \mathbb{R}_+$ in the sense of Lemma 3.2.9. Define $\mathcal{S} : \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ with $\mathcal{X} = \mathcal{L}(X)$ by $\mathcal{S}(t)T = S(t)T$ for all $t \in \mathbb{R}_+$ and $T \in \mathcal{X}$ (cf. Step 2 in the proof of Theorem 3.2.10). Then $\mathcal{S}$ is locally m-measurable, $\mathcal{S}(0) = I$, $\mathcal{S}(t)\mathcal{S}(s) = \mathcal{S}(t * s)$ for all $t, s \in \mathbb{R}_+$ and $\mathcal{S}(\cdot)I$ is continuous on $\mathbb{R}_+^\times$, but not in $t = 0$ (see Chander and Singh, 1981 for the cosine setting).

Finally, suppose that $X = H = \ell^2(\mathbb{N})$. Then the operators $S(t)$ are self-adjoint contractions and S can be extended through $D(\mu) := \int_{\mathbb{R}_+} S(t)\,\mu(\mathrm{d}t)$ to a representation from $M^b(\mathbb{R}_+)$ to $\mathcal{L}(H)$, see Theorem 3.1.7.

Here is another sub-example, which shows the assertion of Theorem 5.4.3.

Example 5.8.10. Let $(\mathbb{R}_+, *(A))$ be a Bessel-Kingman or Jacobi hypergroup. Consider Example 5.8.8 with $\sqrt{q_n} = \lambda_n = n + ie^{n^2}$. (This choice has some similarity with Counterexample II.4.33(i) in Engel and Nagel, 2000, although the purpose is different.) We show that assumption (5.49) is satisfied and S_q, as defined in (5.50), does not admit an exponential bound.

Proof. First, let us check condition (5.49). The asymptotic formulas (4.24) and (4.23) give constants $C > 0$ and $R_2, R_1 > 0$ such that

$$|\Phi_\lambda(t)| \le C \frac{1}{(|\lambda|t)^{\alpha+\frac{1}{2}}} e^{|\operatorname{Re}(\lambda)|t}$$

for $|\lambda|t > R_2$, $|\lambda| > R_1$. Take $t_0 > 0$ and choose $R_3 \ge \max(1, R_2, R_1 t_0)$. Then for $t \in [0, t_0]$

$$|\Phi_\lambda(t)| \le C \left| \frac{\operatorname{Re}(\lambda)}{\lambda} \right|^{\alpha+\frac{1}{2}} e^{|\operatorname{Re}(\lambda)|t_0}$$

if $|\operatorname{Re}(\lambda)|t > R_3$ and, moreover, $|\Phi_\lambda(t)| \le e^{\rho t_0 + R_3}$ if $|\operatorname{Re}(\lambda)|t \le R_3$ by (4.21). In particular, with the choice of λ_n, $n \in \mathbb{N}$, as above, this shows $\sup_{t\in[0,t_0],\, n\in\mathbb{N}} |\Phi_{\lambda_n}(t)| < \infty$.

Second, no exponential bound can exist since $\|S_q(t)e_n\|_X = |\Phi_{\lambda_n}(t)|$, e_n the n-th unit vector in X, and $|\Phi_{\lambda_n}(t)| = \exp(\{-\rho + \operatorname{Re}(\lambda_n) + o(1)\}t)$ as $t \to \infty$ by Theorem 4.3.9. ∎

Appendix A

Integration in Banach Spaces

The theory of hypergroups necessarily needs integration theory on locally compact spaces. Even for topological groups several delicate techniques are needed, see the treatment in Hewitt and Ross (1979), Chapter III, pp. 117–184 which is also basic for hypergroups. Some preliminaries for hypergroups are introduced at the beginning of Section 2.1.

Here we introduce a general setting for measurability and the Bochner integral which is presupposed in the main text for integration of Banach space valued functions on hypergroups. Standard references for integration in Banach spaces are Diestel and Uhl (1977), Hille and Phillips (1957), and Dunford and Schwartz (1958), see also Amann and Escher (2001). However, the Haar measure of a hypergroup is in general neither complete nor σ-finite. Further, we want to employ different measures on the same measure space, which means that there should be a notion of measurability relying on a measurable space but being independent of a specific measure. In fact, it is possible to initiate integration theory from this point of view, although this approach is not standard in the literature. The ensuing introduction follows Dinculeanu (2000), § 1, pp. 1–19, see also the survey Dinculeanu (2002). The approach in Van Neerven (2008), Chapter 1 is similar.

Although Dinculeanu (2000), § 1 is very close to what we need, some modifications are necessary. Therefore we explicitly refer to the proofs in Dinculeanu (2000). It is easily checked that we may choose the field of complex numbers instead of real numbers. After discussing measurability we construct the Bochner integral on an arbitrary measure space $(\Omega, \mathscr{A}, \mu)$, μ a non-negative measure. We do not assume that μ satisfies the "finite measure property" (a generalization of σ-finiteness). This becomes necessary since this condition is in general not satisfied by the Haar measure of a hypergroup, even not for locally compact Abelian groups, see Hewitt and Ross (1979), Note (11.33). We will see that this property is not needed for the construction of the Bochner integral. Anyway, this does not make too much of a difference since all integrable functions are σ-finite.

Let $(\Omega, \mathscr{A})$ be a measurable space (Ω a set, $\mathscr{A} \subset 2^{\Omega}$ a σ-algebra) and X a complex Banach space. The indicator function of a subset A of Ω is denoted by 1_A. A *$\mathscr{A}$-step* *function* (or *$\mathscr{A}$-simple function*) is a function $f : \Omega \to X$ of the form

$$f = \sum_{i=1}^{n} 1_{A_i} x_i$$

with $A_i \in \mathscr{A}$, $x_i \in X$ and $n \in \mathbb{N}$.

Denote by $\overline{\mathbb{R}}$ the extended set of real numbers, that is $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. Given any topological space K we denote by $\mathscr{B}(K)$ its Borel σ-algebra. A function $f : \Omega \to \overline{\mathbb{R}}$ is called *$\mathscr{A}$-measurable* if $f^{-1}(B) \in \mathscr{A}$ for every Borel set $B \in \mathscr{B}(\overline{\mathbb{R}})$. A classical result states that a function $f : \Omega \to \overline{\mathbb{R}}$ is $\mathscr{A}$-measurable if and only if there is a sequence (f_n) of $\mathbb{R}$-valued $\mathscr{A}$-step functions such that $f_n \to f$ pointwise on Ω. This equivalent formulation is used for the definition of measurability for Banach space valued functions.

Definition A.1. A function $f : \Omega \to X$ is called *$\mathscr{A}$-measurable* if there is a sequence (f_n) of X-valued $\mathscr{A}$-step functions such that $f_n \to f$ pointwise on Ω.

We state some basic theorems which confirm that this definition makes sense.

Theorem A.2. *Suppose $f : \Omega \to X$ is $\mathscr{A}$-measurable. Then there is a sequence of $\mathscr{A}$-step functions $f_n : \Omega \to X$ such that $f_n \to f$ pointwise and $\|f_n\|_X \leq \|f\|_X$ for every n.*

Theorem A.3. *A function $f : \Omega \to X$ is $\mathscr{A}$-measurable if and only if it has separable range and $f^{-1}(B) \in \mathscr{A}$ for every Borel set $B \in \mathscr{B}(X)$.*

Theorem A.4. *If (f_n) is a sequence of X (or $\overline{\mathbb{R}}$)-valued, $\mathscr{A}$-measurable functions, converging pointwise to a function f, then the limit f is also $\mathscr{A}$-measurable.*

Next, we state a useful characterization of $\mathscr{A}$-measurability.

Definition A.5. A function $f : \Omega \to X$ is called *weakly $\mathscr{A}$-measurable* if for every $x^* \in X^*$ the scalar function $x^*(f)$ is $\mathscr{A}$-measurable.

In view of this definition the term "*strongly $\mathscr{A}$-measurable*" is often used in place of "$\mathscr{A}$-measurable" to stress different kinds of measurability.

Theorem A.6 (Pettis measurability theorem)**.** *A function $f : \Omega \to X$ is (strongly) $\mathscr{A}$-measurable if and only if it is weakly $\mathscr{A}$-measurable and has separable range.*

To define the Bochner integral let μ be a non-negative measure on $(\Omega, \mathscr{A})$, that is $(\Omega, \mathscr{A}, \mu)$ a measure space.

A property $P(\omega)$ defined for every $\omega \in \Omega$ is said to hold *μ-almost everywhere* (μ-a.e.) if there exists a μ-null set N such that P holds for all $\omega \in \Omega \backslash N$.

Definition A.7. A function $f : \Omega \to X$ or $\overline{\mathbb{R}}$ is called *μ-measurable* if it is equal μ-a.e. to a $\mathscr{A}$-measurable function. If f is defined μ-a.e. with μ-measurable extension to the whole space Ω, then f is also said to be μ-measurable.

Note that if a μ-a.e. defined function f has a μ-measurable extension then any extension is μ-measurable. If f is μ-measurable then $\|f\|_X$ is also μ-measurable. Functions which are equal μ-a.e. are called *μ-versions* of each other and will often be identified. We remark, since a particular μ-measurable function f may be chosen arbitrarily on a set of measure zero it is in general neither separably valued nor is $f^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}(X)$. The last defect disappears if we assume the measure space $(\Omega, \mathscr{A}, \mu)$ to be complete.

Definition A.8. A function $f : \Omega \to X$ is said to be *Bochner μ-integrable*, if f is μ-measurable and $\|f\|_X$ is μ-integrable. The space of equivalence classes of Bochner μ-integrable functions is denoted by $L^1(\Omega, \mu, X)$.

The Bochner integral is constructed in the following way:

A *μ-step function* (or *μ-simple function*) is a function $f : \Omega \to X$ of the form

$$f = \sum_{i=1}^{n} 1_{A_i} x_i$$

with $A_i \in \mathscr{A}$ such that $\mu(A_i) < \infty$, $x_i \in X$ and $n \in \mathbb{N}$. Without loss of generality the A_i may be chosen mutually disjoint. Then

$$\|f\|_X = \sum_{i=1}^{n} 1_{A_i} \|x_i\|_X \tag{A.1}$$

and we define

$$\int_\Omega f \,\mathrm{d}\mu := \sum_{i=1}^{n} \mu(A_i) x_i \in X,$$

hence

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\|_X = \left\| \sum_{i=1}^{n} \mu(A_i) x_i \right\|_X \leq \sum_{i=1}^{n} \mu(A_i) \|x_i\|_X = \int_\Omega \|f\|_X \,\mathrm{d}\mu =: \|f\|_1. \tag{A.2}$$

The μ-step functions are dense in $L^1(\Omega, \mu, X)$ with respect to the norm $\|\cdot\|_1$, see Proposition A.10 below. Hence this integral can be extended to the whole space $L^1(\Omega, \mu, X)$ by continuity. Inequality (A.2) extends to all $f \in L^1(\Omega, \mu, X)$,

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\|_X \leq \int_\Omega \|f\|_X \,\mathrm{d}\mu\,.$$

Remark A.9. Many authors say a function $f : \Omega \to X$ is Bochner μ-integrable if there exists a sequence of μ-step functions $f_n : \Omega \to X$ such that $f_n \to f$ μ-a.e. and $\int_\Omega \|f_n - f\|_X \,\mathrm{d}\mu \to 0$. Then $\int_\Omega f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu$ (with $\int_\Omega f_n \,\mathrm{d}\mu$ as above) is well-defined. It is straightforward to show that both definitions coincide, see also the proof of Proposition A.10 below.

If $1 \leq p < \infty$ the space $L^p(\Omega, \mu, X)$ is defined as the space of equivalence classes of μ-measurable functions $f : \Omega \to X$ such that $\|f\|_X^p$ is μ-integrable. If $p = \infty$ the space $L^\infty(\Omega, \mu, X)$ is the space of equivalence classes of μ-measurable functions $f : \Omega \to X$ such that $\|f\|_X$ is bounded μ-almost everywhere. Endowed with the norm

$$\|f\|_p = \|\|f\|_X\|_p = \left(\int_\Omega \|f\|_X^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

and the essential supremum norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\omega \in \Omega} \|f(\omega)\|_X,$$

respectively, they are Banach spaces (proof as in the scalar case, see e. g. Elstrodt, 2008, Satz VI.2.5). These spaces are also called *Lebesgue-Bochner spaces*.

For scalar-valued functions we write $L^p(\Omega, \mathscr{A}, \mu)$ or shortly $L^p(\Omega, \mu)$.

Proposition A.10. *Suppose $1 \leq p < \infty$. Then the μ-step functions are dense in $L^p(\Omega, \mu, X)$ and all functions $f \in L^p(\Omega, \mu, X)$ are σ-finite. Both properties are in general not true for $p = \infty$.*

A set $B \in \mathscr{A}$ is said to be *σ-finite* if there is a sequence $(B_n)_{n \geq 1}$ of sets in $\mathscr{A}$ with $\mu(B_n) < \infty$ for all n and $B = \bigcup_{n=1}^\infty B_n$. A function $f : \Omega \to X$ is σ-finite if $\{\omega \in \Omega : f(\omega) \neq 0\}$ is contained in a σ-finite set. Note that this property is independent of the values of f on a set of measure zero.

Proof. Suppose $1 \leq p < \infty$ and let $f : \Omega \to X$ be a $\mathscr{A}$-measurable representative of an equivalence class in $L^p(\Omega, \mu, X)$. Theorem A.2 gives a sequence (f_n) of $\mathscr{A}$-step functions corresponding to f. In fact, all f_n are μ-step functions since they may be written in the form (A.1), $\|f_n\|_X \leq \|f\|_X$ and $\|f\|_X$ is μ-integrable by assumption. Moreover, $\|f_n - f\|_X \to 0$ pointwise and $\|f_n - f\|_X \leq 2\|f\|_X$, thus the dominated convergence theorem yields $\int_\Omega \|f_n - f\|_X^p \, \mathrm{d}\mu \to 0$. This also shows that $\{\omega \in \Omega : f(\omega) \neq 0\}$ is contained in the countable union of μ-finite sets $\{\omega \in \Omega : f_n(\omega) \neq 0\}$.

For the negative assertion concerning $p = \infty$ suppose Ω is not σ-finite and consider $f = 1_\Omega$. □

Theorem A.11 (Dominated Convergence Theorem)**.** *Suppose $1 \leq p < \infty$ and let (f_n) be a sequence in $L^p(\Omega, \mu, X)$ converging μ-a.e. to a function $f : \Omega \to X$. If there exists a function $g \in L^p(\Omega, \mu, \mathbb{R})$ such that $\|f_n\|_X \leq g$ for each n then $f \in L^p(\Omega, \mu, X)$ and $f_n \to f$ in $L^p(\Omega, \mu, X)$.*

Proof. Observe that f is μ-measurable by Theorem A.4. So $\|f_n - f\|_X$ is μ-measurable, $\|f_n - f\|_X \to 0$ pointwise μ-a.e. and $\|f_n - f\|_X \leq 2g$ μ-a.e. Thus the assertion follows from the classical dominated convergence theorem. □

The following theorem is basic and is used frequently in the main text.

Theorem A.12 (Hille)**.** *Let $\mathbb{A}$ be a closed linear operator with domain $D(\mathbb{A})$ in X and taking values in a Banach space Y. Suppose $f \in L^1(\Omega, \mu, X)$ with values in $D(\mathbb{A})$ μ-a.e. and $\mathbb{A}f \in L^1(\Omega, \mu, Y)$. Then $\int_\Omega f \, \mathrm{d}\mu \in D(\mathbb{A})$ and*

$$\mathbb{A} \int_\Omega f \, \mathrm{d}\mu = \int_\Omega \mathbb{A}f \, \mathrm{d}\mu \, .$$

Proof. The usual proof runs with approximation of f and $\mathbb{A}f$ by countably-valued μ-measurable functions, see Hille and Phillips (1957), Theorem 3.7.12, or Diestel and Uhl (1977), pp. 47–48, Theorem 6.

We follow the more elegant proof in Van Neerven (2008), Theorem 1.19, the same idea is also used in Arendt et al. (2011), Proposition 1.1.7. Suppose the prerequisites of the theorem are fulfilled. Observe that the function $g = (f, \mathbb{A}f) = (f, 0) + (0, \mathbb{A}f)$ with values in $X \times Y$ is Bochner μ-integrable with

$$\int_\Omega g \, \mathrm{d}\mu = (\int_\Omega f \, \mathrm{d}\mu, \int_\Omega \mathbb{A}f \, \mathrm{d}\mu).$$

Moreover, the range of g is contained in the graph $G(\mathbb{A}) = \{(x, \mathbb{A}x), \, x \in D(\mathbb{A})\}$ of $\mathbb{A}$ which is a closed linear subspace of $X \times Y$ by assumption. Thus $\int_\Omega g \, \mathrm{d}\mu \in G(\mathbb{A})$. Combining both facts yields the stated result. □

Theorem A.13. *Suppose $f \in L^1(\Omega, \mu, X)$ and let $A \in \mathscr{A}$ be a set of finite, non-zero measure. Then the mean of f on A is contained in the closed convex hull of $f(A)$, in signs*

$$\frac{1}{\mu(A)} \int_A f \, \mathrm{d}\mu \in \mathrm{cl}(\mathrm{conv}(f(A))).$$

For a proof see Diestel and Uhl (1977), pp. 48–49, Corollary 8.

Up to now we have considered general measure spaces.

Theorem A.14 (Fubini)**.** *Suppose $(\Omega_i, \mathscr{A}_i, \mu_i)$, $i = 1, 2$ are σ-finite measure spaces and $f \in L^1(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2, X)$. Then $f_1(t) = \int_{\Omega_2} f(t,s) \, \mu_2(\mathrm{d}s)$ and $f_2(s) = \int_{\Omega_1} f(t,s) \, \mu_1(\mathrm{d}t)$ are defined almost everywhere in Ω_1 and Ω_2 respectively and*

$$\int_{\Omega_1 \times \Omega_2} f \, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_1} f_1(t) \, \mu_1(\mathrm{d}t) = \int_{\Omega_2} f_2(s) \, \mu_2(\mathrm{d}s).$$

Proof. See the proof in Hille and Phillips (1957), Theorem 3.7.13. Note that all ingredients of the proof do not require completeness of the measure spaces. □

We mention that several results can be transferred to (bounded) complex measures by the Hahn decomposition theorem.

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Notation

H1,...,H4	hypergroup axioms	14
F1,...,F4	axioms equivalent to H1,...,H4 (for commutative hypergroups)	20
SL1,...,SL2	conditions imposed on a Sturm-Liouville function A	52
$\mu * \nu$	convolution of measures	14
$\mu^{\sim}$	$= \overline{\mu^{-}}$, adjoint of μ	15
μ^{-}	image of μ under involution	16
A^{-}	$= \{t^{-} : t \in A\}$	16
f^{-}	f transferred by involution, $f^{-}(t) := f(t^{-})$	16
$\overline{\mu}$	complex conjugate of μ	16
$\overline{f}$	complex conjugate of f	16
$\mathrm{cl}(A)$	closure of a set A in a topological space	16
$A * B$	$= \bigcup\{\mathrm{supp}(\varepsilon_t * \varepsilon_s) : t \in A,\, s \in B\}$	16
$f(t * s)$	$= \int_K f \,\mathrm{d}(\varepsilon_t * \varepsilon_s)$	17
χ	multiplicative function	18
$\hat{\mu}$	Fourier-Stieltjes transform of μ	19
$A \backslash B$	complement of B in A	19
$f * g^{-}$	convolution of (X, Y-valued) functions f and g	25
$f\|_A$	restriction of f to the set A	32
$T^{\sim}$	adjoint of the operator T in $\mathcal{L}(H)$	30
1_A	indicator function of a set A	113
$\\| \cdot \\|_X$	norm of the Banach space X	21
$\\| \cdot \\|_p$	p-norm of $L^p(\Omega, \mu, X)$	116
$\\| \cdot \\|_\infty$	essential supremum norm of $L^\infty(\Omega, \mu, X)$ or uniform norm of $C_b(K, X)$	116
A	Sturm-Liouville function	51
$\mathbb{A}$	(adapted) generator; $= \mathbb{A}_0 + \rho^2$ for a Sturm-Liouville operator function	48, 78
$\mathbb{A}_0$	universal generator	44, 78
$AC_{loc}(\mathcal{I})$	locally absolutely continuous functions on a real interval $\mathcal{I}$	51
α_0	constant describing the singular part of A'/A at 0	52
α_1	function describing the nonsingular part of A'/A in a neighbourhood of 0	52

B	homogeneous Banach space	39
$\mathscr{B}(K)$	Borel subsets of K	15, 114
$B_r(x)$	open ball with centre x and radius r	21, 34, 81
$BV(\mathcal{I})$	functions of bounded variation on a real interval $\mathcal{I}$	66
C	cosine operator function	2
$\mathbb{C}$	complex numbers	ix
$C(K)$, $C(K, X)$	continuous functions on K, those with values in X	15, 20
$C_b(K)$, $C_b(K, X)$	bounded continuous functions on K	15, 20
$C_{ub}(K)$, $C_{ub}(K, X)$	bounded uniformly continuous functions on K	41
$C_0(K)$, $C_0(K, X)$	continuous functions on K vanishing at infinity	15, 20
$C_c(K)$, $C_c(K, X)$	continuous functions on K with compact support	15, 20
$C^k(\mathcal{I}, X)$	space of k-times continuously differentiable functions from a real interval $\mathcal{I}$ to X	4
$\mathscr{C}(K)$	nonvoid compact subsets of K; given the Michael topology	16
$\mathrm{conv}(A)$	convex hull of a subset A of X	35, 117
D	$\sim$-representation of K	30
$D(\mathbb{A})$	domain of $\mathbb{A}$	78
$\delta_{\mathcal{J}}$	measure associated to $\mathcal{J} \in \mathfrak{J}$; left hand side of an associated integral equation	43
e	neutral element of K	14
ε_t	point (Dirac) measure at t	16
$(\gamma_t)_{t \geq 0}$	Gaussian convolution semigroup	93
H	complex Hilbert space	30
I	identity operator in $\mathcal{L}(X)$	ix
J	integral operator, inverse of L	54
$\mathcal{J}$	measure belonging to $\mathfrak{J}$	43
$\mathfrak{J}$	family of measures building the right hand side of an associated integral equation	43
j_α	spherical Bessel function of order α	59, 74, 99
K	$= (K, *)$ hypergroup	14
$\widehat{K}$	set of characters, dual space of K	18

L	Sturm-Liouville operator	51
$\lambda_{\mathcal{I}}$	Lebesgue measure on some real interval $\mathcal{I}$	52
$\mathcal{L}(X)$, $\mathcal{L}(X,Y)$	bounded linear operators on X, those from X to Y	ix, 33
$L^p(\Omega,\mathscr{A},\mu)$, $L^p(\Omega,\mu)$	Lebesgue space of p-integrable functions on $(\Omega,\mathscr{A},\mu)$	18, 108, 116
$L^p(\Omega,\mu,X)$	Lebesgue-Bochner space of p-integrable, X-valued functions on Ω	116
$L^1_{loc}(K,m^-)$	locally integrable functions with respect to m^-	39
$L^\infty_{loc}(K,m,X)$	locally bounded m-measurable functions from K to X	22
m	left Haar measure on K	18
m^-	involution of m, right Haar measure on K	18
$M(K)$	complex Radon measures on K	15
$M^b(K)$	bounded complex measures on K	15
$M_+(K)$	non-negative Radon measures on K	15
$M^b_+(K)$	non-negative bounded measures on K	15
$M^1(K)$	probability measures on K	15
M_q	multiplication operator associated to a function q	107, 109
$\mathbb{N}$	set of natural numbers $n=1,2,\ldots$	ix
$\mathbb{N}_0$	set of natural numbers including zero $n=0,1,2,\ldots$	ix
$(\Omega,\mathscr{A},\mu)$	some measure space	108, 114
P_α	Legendre/hyperbolic operator function	10, 103
$P_{(\alpha,\beta)}$	Jacobi operator function	103
ϕ_λ	multiplicative function, common parameterization	57
Φ_λ	$=\phi_{i\lambda}$	57
π	Plancherel measure corresponding to m	19
Ψ_λ	principal solution, related to Φ_λ	61
$\mathbb{R}$	real line	ix
$R(\lambda,\mathbb{A})$	resolvent of $\mathbb{A}$ in λ	4, 8, 92
$\mathbb{R}_+$	$=[0,\infty[$	ix
$(\mathbb{R}_+,*(A))$	Sturm-Liouville hypergroup with Sturm-Liouville function A	52
$\widehat{\mathbb{R}_+}$	dual space of $(\mathbb{R}_+,*(A))$	69
$\mathbb{R}_+^\times$	$=]0,\infty[$	ix

$\overline{\mathbb{R}}$	extended real line, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$	114
ρ	index (of a Sturm-Liouville hypergroup)	52
$\rho(\mathbb{A})$	resolvent set of a linear operator $\mathbb{A}$	4
R_n	polynomial of degree n	47
S	multiplicative operator function or Sturm-Liouville operator function	31
S_q	Sturm-Liouville multiplication operator function associated to a function q	107, 109
$\sigma(\mathbb{A})$	spectrum of a linear operator $\mathbb{A}$	4, 87, 92
$\mathrm{supp}(\mu)$	support of the measure μ	15
T	C_0-semigroup or group	93, 99
$\mathbb{T}$	unit circle	33
$T^t f,\ T_t f$	left and right translate of f	17
X	complex Banach space	ix
X^*	dual space of X	ix
Y_α	Bessel/Bessel-Kingman operator function	8, 100

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