Conditional risk measures in a bipartite market structure

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Abstract

In this paper we study the effect of network structure between agents and objects on measures for systemic risk. We model the influence of sharing large exogeneous losses to the financial or (re)insuance market by a bipartite graph. Using Pareto-tailed losses and multivariate regular variation we obtain asymptotic results for conditional risk measures based on the Value-at-Risk and the Conditional Tail Expectation. These results allow us to assess the influence of an individual institution on the systemic or market risk and vice versa through a collection of conditional risk measures. For large markets Poisson approximations of the relevant constants are provided. Differences of the conditional risk measures for an underlying homogeneous and inhomogeneous random graph are illustrated by simulations.

1 Introduction

Quantitative assessments of financial risk and of (re)insurance risk have to take the interwoven web of agents and business relations into account in order to capture systemic risk phenomena. Measuring such risks while accounting for this complex system of agents is an ongoing area of research, see for example [7, 10, 13, 14, 15, 17, 21]. An economic model involving conditional systemic risk measures has been developed in [1]; and an econometric model which uses conditional systemic risk measures can be found in [9]. This paper joins the discussion by adapting conditional systemic risk measures on a bipartite graph model for an agent-object market structure. This specific market structure, which we have proposed in [20], has not been investigated in the context of conditional risk measures before.

    Conditional risk measures originate in the seminal paper [2], where a quantile-based conditional systemic risk measure, the so-called CoVaR, was introduced. Since then this idea has

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been developed much further; in the survey paper [8] the entire Section E is dedicated to *Cross-Sectional Measures* with the CoVaR as prominent example. Following the observation in [2] that ΔCoVaR is a "tail dependency measure" (p. 2), we carry out a probabilistic study in the framework of heavy-tailed loss distributions, explicitly allowing for tail dependence between exposures of financial agents.

Applications of the bipartite graph model include the insurance market considered in [20] as well as investments in overlapping portfolios [16]. The bipartite model has also been successfully applied to operational risk data in [19].

We extend results derived in [20] to a conditional setting. In [20] two marginal risk measures have been considered; the *Value-at-Risk* (VaR), which is defined for a random variable X at confidence level $1 - \gamma$ as

$$\text{VaR}_{1-\gamma}(X) := \inf\{t \geq 0 : \mathbb{P}\left(X > t\right) \leq \gamma\}, \quad \gamma \in (0, 1),$$

and the *Conditional Tail Expectation* (CoTE) at confidence level $1 - \gamma$, based on the corresponding VaR, defined as

$$\text{CoTE}_{1-\gamma}(X) := \mathbb{E}[X \mid X > \text{VaR}_{1-\gamma}(X)], \quad \gamma \in (0, 1). \tag{1.1}$$

The conditional systemic risk measures in this paper are conditional versions of the VaR and the CoTE, and they are motivated by the following observations. For a systemic risk approach it is of interest to quantify not only the risk of single agents, but also the aggregated market risk, which is of high relevance to regulators. Moreover, it is natural to investigate an agent's risk based on market risk; see e.g. Theorem 2.4 of [24]. Consequently, we will study conditional systemic risk measures, where the conditioning event involves the aggregated market risk, as well as its influence on one specific agent. In the same way, it is of interest to evaluate the market risk conditioned on the event that one agent faces high losses. Such ideas lead to a classification of conditional systemic risk measures as in Table 1.1 (inspired by [11]) which will be defined in Definition 1.1.

marginal risk measure	agent | system	system | agent	agent | agent
VaR	ICoVaR	SCoVaR	MCoVaR
CoTE	ICoTE	SCoTE	MCoTE

Table 1.1: Classifying conditional systemic risk measures: "I" stands for *individual* indicating the risk of an individual agent given high market risk; "S" stands for *system* indicating the risk of the system given high risk of an agent; and "M" stands for *mutual* indicating the risk measure of one agent given high risk in another agent.

In [7, 10, 14, 21] an axiomatic framework for systemic risk has been developed. This general framework assumes that a conditional systemic risk measure ρ of a multivariate risk $X = (X_1, \ldots, X_n)$ can be represented as the composition of a univariate (single-agent) risk measure

ρ_0 with an *aggregation function* $h : \mathbb{R}^n \to \mathbb{R}$, so that $\rho = \rho_0 \circ h$. Here, ρ_0 is usually assumed to be convex as well as monotone and positively 1-homogeneous, so that $\rho(ax) = a\rho(x)$ for $a > 0$. While the conditions on h vary, there is consensus that h should be (positively) 1-homogeneous. We deviate from [10] in that we do not assume that $h((1, \ldots, 1)^\top) = n$. Examples for such aggregation functions are $h(x) = \|x\| = (\sum_{i=1}^n |x_i|^r)^{\frac{1}{r}}$, which is a norm for $r \geq 1$ and a quasi-norm for $0 < r < 1$, and $h(x) = x_i$, the projection onto one coordinate.

The fact that we do not require $h((1, \ldots, 1)^\top) = n$ has consequences in terms of system size: Assuming that ρ_0 is monotone, the inequalities $n < \rho_0(\|(1, \ldots, 1)^\top\|_r)$ for $0 < r < 1$ as well as $n > \rho_0(\|(1, \ldots, 1)^\top\|_r)$ for $1 < r \leq \infty$ hold. Therefore, systemic risk may increase faster or increase slower, respectively, as the number of individual risks grows compared to systemic risk with respect to a normalized aggregation function. It is well-known for insurance portfolios that the risks of a larger market is not necessarily linearly related to the risks of a smaller market, because of balancing of risks. In addition, we argue that in a small and risky market the regulator may well strive for more risk capital than the sum of risks. Also moral hazard from different agents is well-known, and the regulator may guard against this hazard by choosing a conditional systemic risk measure which is larger than the sum of the individual risks in the market, as a quasi-norm would imply. Whatever type of aggregation function is chosen, in practice this is an economic decision. Our framework provides considerable variability in the choice of the aggregation function.

In this paper we relate market risk to individual risk in the mathematical framework of multivariate regular variation. This framework allows us to assess conditional systemic risk measures as in Table 1.1 asymptotically in a precise way; cf. [1, 2, 9].

Definition 1.1. [Conditional systemic risk measures] Let $F = (F_1, \ldots, F_q)$ be the random exposure vector. For $\gamma_i, \gamma \in (0, 1)$ referring to agent i and the market, respectively, the conditional systemic risk measures from Table 1.1 are defined as follows:

(a) *Individual Conditional Value-at-Risk*
$$\text{ICoVaR}_{1-(\gamma_i|\gamma)}(F_i \mid h(F)) := \inf\{t \geq 0 : \mathbb{P}(F_i > t \mid h(F) > \text{VaR}_{1-\gamma}(h(F))) \leq \gamma_i\},$$

(b) *Systemic Conditional Value-at-Risk*
$$\text{SCoVaR}_{1-(\gamma|\gamma_i)}(h(F) \mid F_i) := \inf\{t \geq 0 : \mathbb{P}(h(F) > t \mid F_i > \text{VaR}_{1-\gamma_i}(F_i)) \leq \gamma\},$$

(c) *Mutual Conditional Value-at-Risk*
$$\text{MCoVaR}_{1-(\gamma_i|\gamma_k)}(F_i \mid F_k) := \inf\{t \geq 0 : \mathbb{P}(F_i > t \mid F_k > \text{VaR}_{1-\gamma_k}(F_k)) \leq \gamma_i\},$$

(d) *Individual Conditional Tail Expectation*
$$\text{ICoTE}_{1-\gamma}(F_i \mid h(F)) := \mathbb{E}[F_i \mid h(F) > \text{VaR}_{1-\gamma}(h(F))],$$

(e) *Systemic Conditional Tail Expectation*
$$\text{SCoTE}_{1-\gamma}(h(F) \mid F_i) := \mathbb{E}[h(F) \mid F_i > \text{VaR}_{1-\gamma}(F_i)],$$

(f) *Mutual Conditional Tail Expectation*
$$\text{MCoTE}_{1-\gamma}(F_i \mid F_k) := \mathbb{E}[F_i \mid F_k > \text{VaR}_{1-\gamma}(F_k)].$$

For the risk measures (d)-(f) finite first moments of the underlying random variables are required. $\square$

Note that the Conditional Value-at-Risk measures in (a)-(c) are quantiles of the conditional distributions, whereas, the Conditional Tail Expectations in (d)-(f) measure the average or
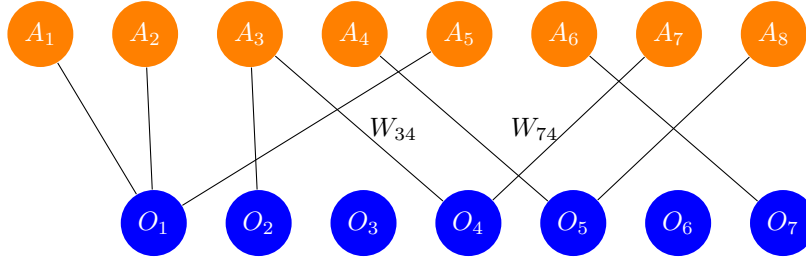
Figure 1: The hierarchical structure of the market as a bipartite graph with agents $A_1, \ldots, A_8$ and risky objects $O_1, \ldots, O_7$ including exemplarily the weights W_{34} and W_{74}.

expected behavior of an agent's exposure, when some extreme event affects another agent, or groups of agents.

To model the complex interaction between economic agents and objects we use a bipartite network; cf. Figure 1. Each agent may cover a random amount or proportion of an object, modelled by a random *weight matrix* $W = (W_{ij})_{i,j=1}^{q,d}$ (we abbreviate with this notation a matrix with row index $i = 1, \ldots, q$ and column index $j = 1, \ldots, d$). We assume that $W_{ij} > 0$ for all (i,j) such that agent i is connected to object j, which we denote by $i \sim j$. The random variable $\mathbb{1}(i \sim j)$ equals 1 whenever agent i holds a contractual relationship to object j, and 0 otherwise. The amount or proportion of object j which affects agent i is represented by $W_{ij}\mathbb{1}(i \sim j)$. Then $F_i := \sum_{j=1}^d W_{ij}\mathbb{1}(i \sim j)$ denotes the *exposure of agent* i and $F = (F_1, \ldots, F_q)^\top$ is the vector of the joint exposures of the agents in the market. Hence, the *weighted adjacency matrix* $A : \Omega \to \mathbb{R}^{q \times d}$ representing the market structure is given by

$$A_{ij} = W_{ij}\mathbb{1}(i \sim j). \tag{1.2}$$

Throughout this paper we assume that the objects, which are large claims or losses, incur a random amount if the claim or loss occurs; these amounts differ between objects and are modelled by random variables V_j for $j = 1, \ldots, d$ with Pareto-tails such that, for possibly different scale parameters $K_j > 0$ and tail index $\alpha > 0$,

$$\mathbb{P}(V_j > t) \sim K_j t^{-\alpha}, \quad t \to \infty, \quad j = 1, \ldots, d. \tag{1.3}$$

(For two functions f and g we write $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t \to \infty} f(t)/g(t) = 1$.) We summarize all objects in the vector $V = (V_1, \ldots, V_d)^\top$ and assume that V is independent of the random graph construction, while $V_1, \ldots, V_d$ may not be independent of each other. The vector F of agent exposures is the matrix-vector product

$$F = AV.$$

In Section 4 we shall see that the network is of considerable importance for the asymptotic behaviour of the conditional systemic risk measures.

Example 1.2. *As a first example, assume that agents are reinsurance companies and objects can generate catastrophic claims, see [20]. Under the simplified assumption that claims are split*

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into equal proportions among all agents which insure this risk, the market matrix A is

$$A_{ij} = \frac{\mathbb{1}(i \sim j)}{\deg(j)}, \quad with \quad \frac{0}{0} := 0, \tag{1.4}$$

where $\deg(j)$ denotes the number of agents that insure object j. In this case $W_{ij} = \deg(j)^{-1}$ for all $i = 1, \ldots, q$ and $j = 1, \ldots, d$. As it can happen that $\deg(j) = 0$, namely when no agent insures claim j, in that case we define the entries $A_{ij}, i = 1, \ldots q$ in the market matrix as 0; this is meant by the convention that $\frac{0}{0} := 0$.

Example 1.3. *As a second example, assume that agents are investors and objects are investment opportunities which can result in large losses, as investments in land banking, some venture capital trust, unregulated collective investment schemes, high risk real estate trusts, or cat bonds of some reinsurance company; see also [16]. Each agent i has a certain amount of capital to invest, say $c_i > 0$. Again for simplicity, we assume that the agents split their investment capital equally among all the assets they have chosen to invest in. This results in a market matrix A given by*

$$A_{ij} = c_i \frac{\mathbb{1}(i \sim j)}{deg(i)}, \quad with \quad \frac{0}{0} := 0, \tag{1.5}$$

where $deg(i)$ denotes the number of different assets agent i invests in. In this case $W_{ij} = c_i \deg(i)^{-1}$ for all $j = 1, \ldots, d$ and $i = 1, \ldots, q$. As it can happen that $\deg(i) = 0$, namely when an agent does not invest at all, in this case we define the entries $A_{ij}, j = 1, \ldots d$ in the market matrix as 0.

We consider risk measures of the exposure vector $F = AV$, where the random matrix A models the network structure of the market. When attributing a risk measure to an agent's exposure or to the market exposure, we write as abbreviation *an agent's risk* or *the market risk*. In Corollaries 3.6 and 3.7 of [20] it was shown that under the assumption of regularly varying exposure vectors the asymptotic behaviour of the VaR and the CoTE can be described using the constants

$$C_{\text{ind}}^i = C_{\text{ind}}^i(A) := \sum_{j=1}^d K_j \mathbb{E} A_{ij}^\alpha, \ i = 1, \ldots, q, \ \text{ and } \ C_{\text{ind}}^S = C_{\text{ind}}^S(A) = \sum_{j=1}^d K_j \mathbb{E} \|Ae_j\|^\alpha, \tag{1.6}$$

as well as

$$C_{\text{dep}}^i = C_{\text{dep}}^i(A) := \mathbb{E}(AK^{1/\alpha}\mathbb{1})_i^\alpha, \ i = 1, \ldots, q, \ \text{ and } \quad C_{\text{dep}}^S = C_{\text{dep}}^S(A) = \mathbb{E}\|AK^{1/\alpha}\mathbb{1}\|^\alpha, \tag{1.7}$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^q$, e_j is the j-th unit vector in $\mathbb{R}^d$ with entry 1 at its j-th component and 0 elsewhere, and $\mathbb{1}$ is the $d-$dimensional vector with entries all equal 1. Moreover, with tail index α and scale parameter K_j as in (1.3), $K^{1/\alpha} = \text{diag}(K_1^{1/\alpha}, \ldots, K_d^{1/\alpha})$ is a $d \times d$ diagonal matrix. The subscripts "ind" and "dep" in (1.6) and (1.7) refer to asymptotically independent or asymptotically fully dependent components of the V_j's, respectively. Furthermore, the superscript i indicates the individual setting of agent i, whereas S refers to the systemic setting.

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The applications we envisage concern objects which can have very high amounts V_j associated with them, such as be catastrophic insurance claims, high risk investments, or operational risk cells. Furthermore, we consider high risk as modelled by extreme quantiles like 99% or even 99.9%, which are required, for instance, for operational risk assessment by the regulator. Such quantile-based risk measures cannot be estimated empirically from real data, simply because there are too few data points available for empirical estimation. Extreme value theory provides methods to estimate such high risk based on the asymptotics for $\gamma \to 0$ as in for example in (1.8) below.

The individual Value–at–Risk of agent $i \in \{1, \ldots, q\}$ shows the asymptotic behaviour

$$\mathrm{VaR}_{1-\gamma}(F_i) \sim C^{1/\alpha} \gamma^{-1/\alpha}, \quad \gamma \to 0, \tag{1.8}$$

with either $C = C^i_{\mathrm{ind}}$ or $C = C^i_{\mathrm{dep}}$, respectively, in the case that $V_1, \ldots, V_d$ are asymptotically independent or asymptotically fully dependent, respectively. The market Value–at–Risk of the aggregated vector $\|F\|$ satisfies

$$\mathrm{VaR}_{1-\gamma}(\|F\|) \sim C^{1/\alpha} \gamma^{-1/\alpha}, \quad \gamma \to 0, \tag{1.9}$$

with either $C = C^S_{\mathrm{ind}}$ or $C = C^S_{\mathrm{dep}}$, respectively, in the case that $V_1, \ldots, V_d$ are asymptotically independent or asymptotically fully dependent, respectively. Analogous statements hold for the CoTE; see [20].

For the asymptotic behaviour of the Value-at-Risk and of the Conditional Tail Expectation the underlying network model enters only through the constants (1.6) and (1.7). Many underlying networks may hence give rise to the same asymptotic behaviour, including even networks for which the adjacency matrix is deterministic. In general, small constants are more desirable, indicating a smaller risk. The case of fully dependent objects is equivalent to having a single source of risk, but with the loss possibly unevenly distributed among the agents.

As indicated in [20] these two extreme dependence cases give rise to risk bounds (cf. [18]) which are determined via the constants given in (1.6) and (1.7). These constants also play a major role in the asymptotic behaviour of the conditional risk measures in this paper.

Our paper is organised as follows. Section 2 summarizes the necessary results from regular variation. Here we also present the asymptotic results of conditional probabilities and conditional expectations. Whereas we formulate our results in the general context of regular variation with arbitrary dependence structure, we single out two cases for the dependence among the loss variables, namely asymptotic independence and asymptotic complete dependence. In Section 3 we discuss the asymptotic behaviour of the conditional systemic risk measures in our network model. When introducing conditional systemic risk measures, for the individual risk of every agent in the market we focus on the one-dimensional projections of the exposure vector, and take norms and quasinorms as appropriate aggregation functions.

Calculating the network-dependent quantities which determine the asymptotic behaviour of the conditional systemic risk measures is in many cases not straightforward. On the one hand they can be approximated through functions of Poisson variables, and on the other hand they can be found by numerical algorithms. In Section 4 we provide a Poisson approximation for the example of a portfolio of large insurance claims as in (1.4) with bounds on the total variation

distance. Simulations for a homogeneous and inhomogeneous bipartite model for the example of a portfolio of risky investments as in (1.5) illustrate the numerical approach and the differences between these two random graphs.

2   Asymptotic results from multivariate regular variation

Our framework is that of regular variation of the random vector of exposures $F = (F_1, \ldots, F_q)$, which follows from the Pareto-tailed claims and the dependence structure introduced by the bipartite graph; cf. [20]. There are several equivalent definitions of multivariate regular variation; cf. Theorem 6.1 of [22] and Ch. 2.1 of [4]. Also notions like *one point uncompactification* and *vague convergence* are defined there, we refer to [22], Section 6.1.3, for more background.

For $d \in \mathbb{N}$, let $\mathbb{S}_+^{d-1} = \{x \in \mathbb{R}_+^d : \|x\| = 1\}$ denote the positive unit sphere in $\mathbb{R}^d$ with respect to an arbitrary norm $\| \cdot \|$ on $\mathbb{R}^d$ so that $\|e_j\| = 1$ for all unit vectors e_j. Furthermore, we shall use the notation $\mathcal{E} := \overline{\mathbb{R}}_+^d \setminus \{0\}$ with $\overline{\mathbb{R}}_+ = [0, \infty]$, 0 is the d-dimensional vector with entries all equal to 0, and $\mathcal{B} = \mathcal{B}(\mathcal{E})$ denotes the Borel σ-algebra with respect to the so-called one point uncompactification.

Definition 2.1. A random vector X with state space $\mathcal{E}$ is called *multivariate regularly varying* if there is a Radon measure $\mu \not\equiv 0$ on $\mathcal{B}(\mathcal{E})$ with $\mu(\overline{\mathbb{R}}_+^d \setminus \mathbb{R}_+^d) = 0$ and

$$\frac{\mathbb{P}(X \in t\cdot)}{\mathbb{P}(\|X\| > t)} \xrightarrow{v} \mu(\cdot), \quad t \to \infty, \tag{2.1}$$

where $\xrightarrow{v}$ denotes vague convergence. In this case there exists some $\alpha > 0$ such that the limit measure is homogeneous of order $-\alpha$:

$$\mu(uS) = u^{-\alpha}\mu(S)$$

for every $S \in \mathcal{B}(\mathcal{E})$ satisfying $\mu(\partial S) = 0$. The measure μ is called *intensity measure of X*.

The *tail index* $\alpha > 0$ is also called *index of regular variation* of X, and we write $X \in \mathcal{R}(-\alpha)$.
□

Remark 2.2. Vague convergence is a useful concept in the framework of multivariate regular variation as explained in [22]. We do not want to go into topological details, but it may help to understand the concept by the following equivalences for regular variation in $\mathbb{R}_+$; cf. Theorem 3.6 of [22]. Suppose that X is a nonnegative random variable with distribution function G. Then the following are equivalent:

(i) $1 - G =: \overline{G} \in \mathcal{R}(-\alpha)$ for $\alpha > 0$.

(ii) There exists a sequence $b_n \to \infty$ such that $\lim_{n \to \infty} n\overline{G}(b_n x) = x^{-\alpha}$ for $x > 0$.

(iii) There exists a sequence $b_n \to \infty$ such that $\mu_n(\cdot) = n\mathbb{P}\left(\frac{X}{b_n} \in \cdot\right) \xrightarrow{v} \mu(\cdot)$, where $\mu((x, \infty]) = x^{-\alpha}$.

The set of intervals $\{(x, \infty] : x > 0\}$ generate the Borel σ-algebra $\mathcal{B}(\mathcal{E})$ from Definition 2.1 for $d = 1$, and the measure μ is defined on $\mathcal{B}(\mathcal{E})$. The dot in (iii) stands for an arbitrary Borel set in $\mathcal{B}(\mathcal{E})$. The implication (iii)$\Rightarrow$(ii) follows immediately from evaluating $\mu_n(\cdot)$ for the interval $(x, \infty]$ for $x > 0$. $\qquad\square$

We shall use that regular variation of V implies regular variation of F under the Breiman condition

$$\mathbb{E}[\|W\|^{\alpha+\delta}] < \infty \text{ for some } \delta > 0$$

on the weight matrix W from (1.2), see Proposition 2.3.

To obtain asymptotic results as in Corollaries 3.6 and 3.7 of [20] for the conditional systemic risk measures from Definition 1.1 in a more general framework, we first extend classical results of regular variation to continuous 1-homogeneous functions. Examples for such continuous 1-homogeneous functions are projections of the vector $F = (F_1, \ldots, F_q)^\top$ on the i-th coordinate F_i and a norm or quasinorm of the vector F.

We shall use the following result, which is based on Proposition A.1 in [5].

Proposition 2.3. *Let $V := (V_1, \ldots, V_d)^\top \in \mathcal{R}(-\alpha)$ having components with Pareto-tails $\mathbb{P}(V_j > t) \sim K_j t^{-\alpha}$ as $t \to \infty$ for $K_j, \alpha > 0$ as in (1.3) with intensity measure μ as in (2.1). Furthermore, let the weight matrix $W : \Omega \to \mathbb{R}_+^{q \times d}$ satisfy $\mathbb{E}[\|W\|^{\alpha+\delta}] < \infty$ for some $\delta > 0$. Then the random vector $F = AV$ with A as in (1.2) and independent of V belongs to $\mathcal{R}(-\alpha)$ if there is a relatively compact set $C \subseteq \mathbb{R}^q \setminus \{0\}$ with $\mathbb{E}\mu \circ A^{-1}(C) > 0$. Let $h : \overline{\mathbb{R}}^q \setminus \{0\} \to \overline{\mathbb{R}}^k \setminus \{0\}$ for $k \in \mathbb{N}$ be a continuous 1-homogeneous function. Then we have on $\mathcal{B}(h(\overline{\mathbb{R}}_+^q) \setminus \{0\})$:*

$$\frac{\mathbb{P}(h(F) \in t\cdot)}{\mathbb{P}(\|V\| > t)} \xrightarrow{v} \mathbb{E}\mu\{x \in \mathbb{R}_+^d : h(Ax) \in \cdot\}, \quad t \to \infty. \tag{2.2}$$

Proof. Vague convergence of F is given by Proposition A.1 in [5] and is equivalent to

$$\frac{\mathbb{P}(F \in tC)}{\mathbb{P}(\|V\| > t)} \to \mathbb{E}\mu\{x \in \mathbb{R}_+^d : Ax \in C\}, \quad t \to \infty,$$

for all relatively compact sets $C \in \mathcal{B}(\overline{\mathbb{R}}_+^q \setminus \{0\})$. This implies $F \in \mathcal{R}(-\alpha)$ due to Corollary 2.1.9 in [4]. Furthermore, by 1-homogeneity of h, for $t > 0$ and $B \in \mathcal{B}(\overline{\mathbb{R}}_+^k \setminus \{0\})$ we have $\{h(F) \in tB\} = \{F \in th^{-1}(B)\}$. Note also that every B which is bounded away from zero is relatively compact in the topology we use. Since $h^{-1}(B)$ is bounded away from zero, by continuity of h and the fact that $h(0) = 0$, $h^{-1}(B)$ is also relatively compact. Putting all this together, for every relatively compact set $B \in \mathcal{B}(\overline{\mathbb{R}}_+^k \setminus \{0\})$ we have, as $t \to \infty$,

$$\frac{\mathbb{P}(h(F) \in tB)}{\mathbb{P}(\|V\| > t)} = \frac{\mathbb{P}(F \in th^{-1}(B))}{\mathbb{P}(\|V\| > t)} \to \mathbb{E}\mu\{x \in \mathbb{R}_+^d : Ax \in h^{-1}(B)\} = \mathbb{E}\mu\{x \in \mathbb{R}_+^d : h(Ax) \in B\}.$$

This is equivalent to vague convergence in (2.2). $\qquad\square$

When the vector V has asymptotically independent components, the limit measure μ has support on the axes, whereas for V with asymptotically fully dependent components it is supported on the line $\{sK^{1/\alpha}1 : s > 0\}$. This difference in support is reflected in the difference between (1.6) and (1.7) and affects the behaviour of aggregated exposures.

Proposition 2.4. *Assume the situation of Proposition 2.3. For the aggregated exposures $h(F)$ we obtain*

$$\mathbb{P}(h(F) > t) \sim C^h t^{-\alpha}, \quad t \to \infty,$$

with

$$C^h = C^h_{\text{ind}} = \sum_{j=1}^{d} \mathbb{E}h^\alpha(AK^{1/\alpha}e_j) \quad and \quad C^h = C^h_{\text{dep}}(h) = \mathbb{E}h^\alpha(AK^{1/\alpha}1), \tag{2.3}$$

if $V_1, \ldots, V_d$ are asymptotically independent or asymptotically fully dependent, respectively.

Proof. The assertion can be shown in an analogous way to Theorem 3.4 of [20]. □

The following result gives limit relations in the most general situation, without any restriction on the dependence in the exposure vector.

Theorem 2.5. *Let $g, h : \mathbb{R}_+^q \to \mathbb{R}_+$ be continuous $1-$homogeneous functions and assume the situation of Proposition 2.3. Then for $u \in (0, \infty)$, the following assertions hold:*

(a) $\lim\limits_{t\to\infty} \mathbb{P}\left(g(F) > t \mid h(F) > ut\right) = u^\alpha \dfrac{\mathbb{E}\mu \circ A^{-1}(\{x \in \mathbb{R}_+^q : h(x) > u, g(x) > 1\})}{\mathbb{E}\mu \circ A^{-1}(\{x \in \mathbb{R}_+^q : h(x) > 1\})}.$

(b) If $\alpha > 1$ and g is additionally bounded or has compact support on $\overline{\mathbb{R}}_+^d \setminus \{0\}$, then

$$\lim_{t\to\infty} \mathbb{E}[g(F) \mid h(F) > t] = \frac{t}{\tilde{\mu}(\{h(x) > 1\})} \int_{h(x)>1} g(x)\tilde{\mu}(dx), \tag{2.4}$$

where $\tilde{\mu}(\cdot) = \mathbb{E}\mu \circ A^{-1}(\cdot)$.

Proof. (a) We use Proposition 2.3 to obtain

$$\mathbb{P}\left(g(F) > t \mid h(F) > ut\right) = \int_{\substack{h(F)>ut \\ g(F)>t}} \frac{d\mathbb{P}}{\mathbb{P}\left(h(F) > ut\right)}$$

$$= \int_{\substack{h(x)>u \\ g(x)>1}} \frac{\mathbb{P}\left(F \in tdx\right)}{\mathbb{P}\left(\|V\| > t\right)} \frac{\mathbb{P}\left(\|V\| > t\right)}{\mathbb{P}\left(h(F) > ut\right)}.$$

The second ratio converges by Proposition 2.3(a) and also the first, when taking there for h the identity function. The result follows then by vague convergence.

(b) Using 1-homogeneity of g and Proposition 2.3,

$$\mathbb{E}[g(F) \mid h(F) > t] = \frac{1}{\mathbb{P}(h(F) > t)} \int_{h(x)>t} g(x)\mathbb{P}(F \in dx)$$

$$= \frac{1}{\mathbb{P}(h(F) > t)} \int_{h(x)>1} g(tx)\mathbb{P}(F \in tdx)$$

$$= \frac{\mathbb{P}(\|V\| > t)}{\mathbb{P}(h(F) > t)} \int_{h(x)>1} g(tx)\frac{\mathbb{P}(F \in tdx)}{\mathbb{P}(\|V\| > t)}$$

$$\sim \frac{t}{\tilde{\mu}(\{h(x) > 1\})} \int_{h(x)>1} g(x)\tilde{\mu}(dx), \quad t \to \infty. \tag{2.5}$$

Recall that the sequence of bounded measures in (2.2) converges to a bounded measure vaguely if and only if it converges weakly to this measure; see Theorem 2.1.4 in [4] for further details. Hence, either assumption on g in (b) is sufficient to achieve convergence in (2.5). □

Corollary 2.6. *Let $u \in (0, \infty)$ and assume the situation of Theorem 2.5. Recall the constants C_{ind}^h and C_{dep}^h from (2.3).*

(a) If $V_1, \ldots, V_d$ are asymptotically independent, then

$$\lim_{t \to \infty} \mathbb{P}\left(g(F) > t \mid h(F) > ut\right) = (C_{\text{ind}}^h)^{-1} \sum_{j=1}^d \mathbb{E}\min\{h^\alpha(AK^{1/\alpha}e_j), u^\alpha g^\alpha(AK^{1/\alpha}e_j)\}. \qquad (2.6)$$

(b) If $V_1, \ldots, V_d$ are asymptotically fully dependent, then

$$\lim_{t \to \infty} \mathbb{P}\left(g(F) > t \mid h(F) > ut\right) = (C_{\text{dep}}^h)^{-1} \mathbb{E}\min\{h^\alpha(AK^{1/\alpha}\mathbb{1}), u^\alpha g^\alpha(AK^{1/\alpha}\mathbb{1})\}. \qquad (2.7)$$

Proof. The proof is similar to the proof of Theorem 3.4 of [20]. For asymptotically independent claims $V_1, \ldots, V_d$ we obtain by Theorem 2.5(a) for the numerator

$$\mathbb{E}\mu \circ A^{-1}(\{h(x) > u, g(x) > 1\}) = \left(\sum_{j=1}^d K_j\right)^{-1} \sum_{j=1}^d K_j \mathbb{E}\min\{u^{-\alpha}h^\alpha(Ae_j), g^\alpha(Ae_j)\},$$

and the expression in the denominator is

$$\left(\sum_{j=1}^d K_j\right)^{-1} \sum_{j=1}^d K_j \mathbb{E}\{u^{-\alpha}h^\alpha(Ae_j)\} = \left(\sum_{j=1}^d K_j\right)^{-1} C_{\text{ind}}^h,$$

which yields (2.6). In the case of asymptotically fully dependent claims we get

$$\mathbb{E}\mu \circ A^{-1}(\{h(x) > u, g(x) > 1\}) = \|K^{1/\alpha}\mathbb{1}\|^{-\alpha} \mathbb{E}\min\{u^{-\alpha}h^\alpha(AK^{1/\alpha}\mathbb{1}), g^\alpha(AK^{1/\alpha}\mathbb{1})\},$$

giving with corresponding nominator relation equation (2.7). $\qquad \square$

Remark 2.7. In bivariate extreme value theory the *tail dependence coefficient* is usually defined for two possibly dependent random variables X_1, X_2 with distribution functions H_1, H_2 by means of their generalized inverse functions $H_1^\leftarrow, H_2^\leftarrow$ as

$$\lim_{x \uparrow 1} \mathbb{P}(X_2 > H_1^\leftarrow(x) \mid X_1 > H_2^\leftarrow(x)), \qquad (2.8)$$

provided that this limit exists; e.g. [6], p. 343, eq. (9.75). If X_1, X_2 are multivariate regularly varying and asymptotically independent, then the tail dependence coefficient equals 0, while the tail dependence coefficient equals 1 in case of asymptotically fully dependent variables X_1, X_2. As the VaR of a random variable acts as its generalized inverse distribution function,

$$\lim_{\gamma \to 0} \mathbb{P}\left(g(F) > \text{VaR}_{1-\gamma}(g(F)) \mid h(F) > \text{VaR}_{1-\gamma}(h(F))\right),$$

is of the form (2.8). As a consequence of regular variation the limits of the following conditional probabilities can be computed explicitly: if $\gamma_g/\gamma_h \to 1$, then

$$\mathbb{P}\left(g(F) > \text{VaR}_{1-\gamma_g}(g(F)) \mid h(F) > \text{VaR}_{1-\gamma_h}(h(F))\right)$$
$$\sim \mathbb{P}\left(h(F) > \text{VaR}_{1-\gamma_h}(h(F)) \mid g(F) > \text{VaR}_{1-\gamma_g}(g(F))\right).$$

The conditional probabilities in Definition 1.1 (a)-(c) are defined via such quantities. In addition, we allow for asymmetry in the sense that the confidence levels of the VaRs do not need to be asymptotically equivalent, we only require $\gamma_g/\gamma_h \to \kappa$ for some $\kappa \in (0, \infty)$. $\qquad \square$

Corollary 2.8. *Let $u \in (0, \infty)$, $\alpha > 1$, and assume the situation of Proposition 2.3. Recall the constants from* (2.3).

(a) If $V_1, \ldots, V_d$ are asymptotically independent, we find

$$\mathbb{E}[g(F) \mid h(F) > t] \sim \frac{\alpha}{\alpha - 1}(C_{\mathrm{ind}}^h)^{-1} \sum_{j=1}^{d} \mathbb{E}g(AK^{1/\alpha}e_j)h^{\alpha-1}(AK^{1/\alpha}e_j)\, t, \quad t \to \infty. \qquad (2.9)$$

(b) If $V_1, \ldots, V_d$ are asymptotically fully dependent, we find

$$\mathbb{E}[g(F) \mid h(F) > t] \sim \frac{\alpha}{\alpha - 1}(C_{\mathrm{dep}}^h)^{-1}\mathbb{E}g(AK^{1/\alpha}1)h^{\alpha-1}(AK^{1/\alpha}1)\, t, \quad t \to \infty. \qquad (2.10)$$

(c) For $g = h$ we obtain the classical Conditional Tail Expectation (1.1).

Proof. (a) We evaluate the integral in (2.4) as

$$\mathbb{E}\int_{h(x)>1} g(x)\mu \circ A^{-1}(dx)$$

$$= (\sum_{j=1}^{d} K_j)^{-1}\mathbb{E}\sum_{j=1}^{d}\int_{h(x)>1, x \in \{uAK^{1/\alpha}e_j : u>0\}} g(x)\nu^*(\{se_j \in \mathbb{R}^d : sAK^{1/\alpha}e_j \in dx\}),$$

where the measure ν^*, called the canonical exponent measure, is related to the exponent measure ν of the vector $V = (V_1, \ldots, V_d)$ by $\nu = \nu^* \circ K^{-1/\alpha}$, see Lemma 2.2 in [20]. For independent components ν^* is concentrated on the axes. We take into account that, whenever $x \in \{uAK^{1/\alpha}e_j : u > 0\}$, the equality

$$\nu^*(\{se_j \in \mathbb{R}^d : sAK^{1/\alpha}e_j \in dx\}) = \alpha u^{-\alpha-1}du$$

holds. Integration over the set $\{u > 1/h(AK^{1/\alpha}e_j)\}$ yields

$$\int_{1/h(AK^{1/\alpha}e_j)}^{\infty} \alpha g(AK^{1/\alpha}e_j)u^{-\alpha}du = \frac{\alpha}{\alpha - 1}g(AK^{1/\alpha}e_j)h^{\alpha-1}(AK^{1/\alpha}e_j),$$

implying

$$\int_{h(x)>1} g(x)\mathbb{E}\mu \circ A^{-1}(dx) = \frac{\alpha}{\alpha - 1}(\sum_{j=1}^{d} K_j)^{-1}\sum_{j=1}^{d}\mathbb{E}[g(AK^{1/\alpha}e_j)h^{\alpha-1}(AK^{1/\alpha}e_j)].$$

Since $\tilde{\mu}(\{h(x) > 1\}) = \int_{h(x)>1} \mathbb{E}\mu \circ A^{-1}(dx) = (\sum_{j=1}^{d} K_j)^{-1}C_{\mathrm{ind}}^h$, we get (2.9).

(b) To show (2.10), recall that for asymptotically fully dependent components the canonical exponent measure ν^* is concentrated on the diagonal $\{u1 \in \mathbb{R}^d : u > 0\}$ and connected to the exponent measure ν of V by $\nu = \nu^* \circ K^{-1}$, see also Lemma 4.2 in [20]. Hence,

$$\int_{h(x)>1} g(x)\mathbb{E}\mu \circ A^{-1}(dx)$$

$$= \|K^{1/\alpha}1\|^{-\alpha}\mathbb{E}\int_{h(x)>1, x \in \{uAK^{1/\alpha}1:u>0\}} g(x)\nu^*(\{s1 \in \mathbb{R}^d : sAK^{1/\alpha}1 \in dx\}).$$

For $x \in \{uAK^{1/\alpha}1 : \; u > 0\}$, we have $\mathbb{E}\nu(\{sK^{1/\alpha}1 \in \mathbb{R}^d : sAK^{1/\alpha}1 \in dx\}) = \alpha u^{-\alpha-1}du$, which yields

$$
\begin{aligned}
\int_{h(x)>1} g(x)\mathbb{E}\mu \circ A^{-1}(dx) &= \|K^{1/\alpha}1\|^{-\alpha}\mathbb{E}\int_{1/h(AK^{1/\alpha}1)}^{\infty} \alpha u^{-\alpha}g(AK^{1/\alpha}1)du \\
&= \|K^{1/\alpha}1\|^{-\alpha}\frac{\alpha}{\alpha-1}\mathbb{E}h^{\alpha-1}(AK^{1/\alpha}1)g(AK^{1/\alpha}1).
\end{aligned}
$$

This leads to (2.10). $\qquad\square$

Remark 2.9. (i) For two risks (X_1, X_2) which are multivariate regularly varying, the Conditional Tail Expectation of the form $\mathbb{E}[X_1 \mid X_2 > t]$ has been investigated in [12]. In particular, an estimation procedure has been developed, which also details the case of asymptotically independent risks under hidden regular variation.

(ii) A statistical methodology which is based on the spectral measure of regular variation has been developed for estimating unconditional and conditional risk measures, and has been applied in [19] to operational risk data. $\qquad\square$

3   The conditional systemic risk measures

We are now ready to investigate the conditional systemic risk measures from Definition 1.1 of a financial or insurance market based on the bipartite graph represented by the random matrix $A = (A_{ij})_{i,j=1}^{q,d}$ as in (1.2) with q agents and d objects.

First, we assess to which extent the risk of agent i is affected by high market losses. Second, we evaluate the influence of an individual agent's risk to the market risk, reflecting the systemic importance of an individual agent. Third, we consider the influence of the risk of agent k on the risk of agent i.

Throughout this section we assume that the loss variables $V_1, \ldots, V_d$ are asymptotically independent. The asymptotically fully dependent case can be tackled similarly to the asymptotically independent case, with the modification that the intensity measure of the loss vector V is concentrated on the line $\{sK1^{1/\alpha 1} : s > 0\}$.

The assumption of losses being asymptotically independent is sensible, for example, when modelling risks in reinsurance markets, where $V_1, ..., V_d$ represent large claims of different type and spread over different geographic regions—it is appropriate to assume the claims arising from a storm in the Gulf of Mexico to be independent from the claims arising from an earthquake in Turkey. A similar argument applies to examples for high-risk investment such as a venture capital trust or cat bonds of a reinsurance company. There is also evidence for asymptotic independence situations in operational risk modelling, where $V_1, \ldots, V_d$ represent losses of different event types with arbitrary dependence structure; see [19].

In this section we return to the conditional risk measures from Definition 1.1 applied to appropriate aggregation functions; we take $g(F)$ as the projection on some component and $h(F) = \|F\|$ as a norm or quasinorm. In particular this norm can be different to the reference norm in the definition of regular variation in (2.1).

The following result determines the probability of joint large losses for individual agents and the system in different conditional situations.

Proposition 3.1. *Let $V_1, \ldots, V_d$ be asymptotically independent and $u \in (0, \infty)$. Assume that the conditions of Proposition 2.3 are satisfied. Moreover, assume that $\kappa \in (0, \infty)$ and $\gamma \to 0$. Then*

$$\mathbb{P}\left(F_i > \text{VaR}_{1-\gamma\kappa}(F_i) \mid \|F\| > \text{VaR}_{1-\gamma}(\|F\|)\right) \; \to \; \sum_{j=1}^{d} K_j \mathbb{E} \min\left\{ \frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa \frac{A_{ij}^\alpha}{C_{\text{ind}}^i} \right\}, \quad (3.1)$$

$$\mathbb{P}\left(\|F\| > \text{VaR}_{1-\kappa\gamma}(\|F\|) \mid F_i > \text{VaR}_{1-\gamma}(F_i)\right) \; \to \; \sum_{j=1}^{d} K_j \mathbb{E} \min\left\{ \kappa \frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \frac{A_{ij}^\alpha}{C_{\text{ind}}^i} \right\}, \quad (3.2)$$

$$\mathbb{P}\left(F_i > \text{VaR}_{1-\gamma\kappa}(F_i) \mid F_k > \text{VaR}_{1-\gamma}(F_k)\right) \; \to \; \sum_{j=1}^{d} K_j \mathbb{E} \min\left\{ \kappa \frac{A_{ij}^\alpha}{C_{\text{ind}}^i}, \frac{A_{kj}^\alpha}{C_{\text{ind}}^k} \right\}. \quad (3.3)$$

Moreover, for the Conditional Tail Expectations, if $\alpha > 1$, then

$$\text{ICoTE}_{1-\gamma}(F_i \mid \|F\|) \; \sim \; \frac{\alpha}{\alpha - 1} (C_{\text{ind}}^S)^{1/\alpha - 1} \sum_{j=1}^{d} K_j \mathbb{E}[A_{ij} \|Ae_j\|^{\alpha-1}] \gamma^{-1/\alpha}, \quad (3.4)$$

$$\text{SCoTE}_{1-\gamma}(\|F\| \mid F_i) \; \sim \; \frac{\alpha}{\alpha - 1} (C_{\text{ind}}^i)^{1/\alpha - 1} \sum_{j=1}^{d} K_j \mathbb{E}[A_{ij}^{\alpha-1} \|Ae_j\|] \gamma^{-1/\alpha}, \quad (3.5)$$

$$\text{MCoTE}_{1-\gamma}(F_i \mid F_k) \; \sim \; \frac{\alpha}{\alpha - 1} (C_{\text{ind}}^k)^{1/\alpha - 1} \sum_{j=1}^{d} K_j \mathbb{E}[A_{kj}^{\alpha-1} A_{ij}] \gamma^{-1/\alpha}. \quad (3.6)$$

Proof. We show the following, slightly more general result: Let

$$g, h \in \{f : \mathbb{R}_+^q \to \mathbb{R}_+ \; : \; f(x) = \|x\| \text{ and } f_k : \mathbb{R}_+^q \to \mathbb{R}_+; f_k(x) = x_k, \, k = 1, \ldots, q\},$$

then, under the assumptions of this proposition,

$$\mathbb{P}\left(g(F) > \text{VaR}_{1-\gamma\kappa}(g(F)) \mid h(F) > \text{VaR}_{1-\gamma}(h(F))\right)$$

$$\to \; \sum_{j=1}^{d} \mathbb{E} \min\left\{ \frac{h^\alpha(AK^{1/\alpha} e_j)}{C_{\text{ind}}^h}, \frac{\kappa g^\alpha(AK^{1/\alpha} e_j)}{C_{\text{ind}}^g} \right\}, \quad \gamma \to 0. \quad (3.7)$$

To show (3.7) we set $\text{VaR}_{1-\gamma\kappa}(g(F)) = t$ and $\text{VaR}_{1-\gamma}(h(F)) = ut$. Now recall that by (1.8) and (1.9)

$$\text{VaR}_{1-\gamma}(F_i) \sim (C_{\text{ind}}^i)^{1/\alpha} \gamma^{-1/\alpha} \quad \text{and} \quad \text{VaR}_{1-\gamma}(\|F\|) \sim (C_{\text{ind}}^S)^{1/\alpha} \gamma^{-1/\alpha}, \quad \gamma \to 0.$$

This implies that

$$u = \frac{\text{VaR}_{1-\gamma}(h(F))}{\text{VaR}_{1-\gamma\kappa}(g(F))} = \frac{(C_{\text{ind}}^h)^{1/\alpha} \gamma^{-1/\alpha}}{(C_{\text{ind}}^g)^{1/\alpha} (\gamma\kappa)^{-1/\alpha}} (1 + o(1)), \quad \gamma \to 0$$

such that

$$u^\alpha = \frac{C_{\text{ind}}^h}{C_{\text{ind}}^g} \kappa (1 + o(1)), \quad \gamma \to 0.$$

Corollary 2.6 now gives that (3.7) holds.

The analogous expressions for the Conditional Tail Expectation follow similarly from Corollary 2.8. $\qquad \square$

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Remark 3.2. For simplicity we refer to the first contribution of the sum on the right-hand side of (3.1), $\|Ae_j\|^\alpha/C^S_{\text{ind}}$, as the *systemic constant*, and to the second contribution, $\kappa A^\alpha_{ij}/C^i_{\text{ind}}$, as the *individual constant* of (3.1).

With these notions we give an interpretation of (3.1). First note that for any market matrix A with bounded components such that $\mathbb{P}(A_{ij} > 0) > 0$ for all $i = 1,\ldots,q$ and $j = 1,\ldots,d$, there is some $\delta > 1$ such that

$$\mathbb{E}\min\left\{\frac{\|Ae_j\|^\alpha}{C^S_{\text{ind}}}, \kappa\delta^\alpha\frac{A^\alpha_{ij}}{C^i_{\text{ind}}}\right\} = \mathbb{E}\frac{\|Ae_j\|^\alpha}{C^S_{\text{ind}}}$$

for all $j = 1,\ldots,d$. Suppose that there is a crisis situation worse than presumed by the regulator; i.e., instead of the event $\{\|F\| > \text{VaR}_{1-\gamma}(\|F\|)\}$ the more extreme event $\{\|F\| > \delta\text{VaR}_{1-\gamma}(\|F\|)\}$ for some $\delta > 1$ is observed. Then by (2.6) the factor κ in the limit of (3.1) changes to $\delta^\alpha\kappa$.

Consider the situation where the minimum in (3.1) has been attained by the individual constant under the previous market condition that $\|F\| > \text{VaR}_{1-\gamma}(\|F\|)$. Assume that the occurrence of a severe crisis event, which results in the change of κ to $\delta^\alpha\kappa$ for some $\delta > 1$, is so severe that δ becomes so large that the minimum in (3.1) is attained for the systemic contribution part. Then, as a consequence, for such large δ, by (1.6), the tail dependence coefficient between F_i and $\|F\|$ equals 1, so that there will be full asymptotic dependence between the single financial agent and the market. In order to avoid this dangerous full dependence of a single agent on the market in a future crisis situation of the same extent, the regulator might impose capital restrictions on all agents in the market by adjusting the exposure weights W_{ij} to W_{ij}/δ. Clearly, this adjustment must be carried out smoothly in order to avoid provoking fire sales or other shocks to the market. Then

$$\kappa\delta^\alpha\frac{A^\alpha_{ij}}{C^i_{\text{ind}}} = \kappa\frac{W^\alpha_{ij}}{C^i_{\text{ind}}}\mathbb{1}(i \sim j),$$

so that the dependence on δ vanishes. Then with positive probability, the minimum (3.1) will be attained by the individual constant and full asymptotic dependence between the single financial agent and the market is avoided.

The other limit relations have analogous interpretations. $\qquad\square$

For each of the expressions in Proposition 3.1 the limiting behaviour for $\kappa \to 0$ is linear, as is made precise in the next Proposition 3.3. For this result we assume that there exist constants w and W which do not depend on the network, such that for all $i = 1,\ldots q$ and $j = 1,\ldots,d$,

$$0 < w \le W_{ij} \le W. \tag{3.8}$$

For the reinsurance market with $W_{ij} = \deg(j)^{-1}$ (cf. (1.4)) we can take $w = 1/(1+d)$ and $W = 1$. Then—with the matrix A given by (1.2)—there are constants b, B which do not depend on the network so that

$$0 < b \le \|Ae_j\| \le B, \tag{3.9}$$

whenever $\mathbb{1}(i \sim j) = 1$ for at least one i in $1,\ldots,q$. We set

$$\kappa_0 = \kappa_0(i) = \frac{b^\alpha}{C^S_{\text{ind}}}\frac{C^i_{\text{ind}}}{W^\alpha}, \quad \kappa_1 = \kappa_1(i) = \frac{C^S_{\text{ind}}}{C^i_{\text{ind}}}\frac{b^\alpha}{W^\alpha}, \quad \text{and} \quad \kappa_2 = \kappa_2(i,k) = \frac{C^i_{\text{ind}}}{C^k_{\text{ind}}}\frac{w^\alpha}{W^\alpha}. \tag{3.10}$$

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Moreover, we define

$$\tau(i) = \sum_{j=1}^{d} K_j \mathbb{E}\left[\mathbb{1}(i \sim j)\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}\right] \quad \text{and} \quad \tau(i,k) = \sum_{j=1}^{d} K_j \mathbb{E}\left[\mathbb{1}(k \sim j)\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right], \qquad (3.11)$$

and note that $\tau(i) \leq 1$ and $\tau(i,k) \leq 1$ through the definitions of C_{ind}^S and C_{ind}^i, respectively. If i and k do not share an object then $\tau(i,k) = 0$.

With these notations we can give more precise expressions for the right-hand side of (3.1), (3.2) and (3.3).

Proposition 3.3. *Assume that the conditions of Proposition 3.1 as well as (3.8) and thus (3.9) hold.*

(a) *For $\kappa \leq \kappa_0(i)$,*

$$\sum_{j=1}^{d} K_j \mathbb{E}\min\left\{\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} = \kappa \sum_{j=1}^{d} K_j \mathbb{E}\frac{A_{ij}^\alpha}{C_{\text{ind}}^i} = \kappa. \qquad (3.12)$$

(b) *For $\kappa \leq \kappa_1(i)$,*

$$\sum_{j=1}^{d} K_j \mathbb{E}\left\{\min\left\{\kappa\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\}\right\} = \kappa\tau(i). \qquad (3.13)$$

(c) *For $\kappa \leq \kappa_2(i,k)$,*

$$\sum_{j=1}^{d} K_j \mathbb{E}\min\left\{\kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}, \frac{A_{kj}^\alpha}{C_{\text{ind}}^k}\right\} = \kappa\tau(i,k). \qquad (3.14)$$

Proof. To show (3.12) we start with (3.1). Consider the expression

$$\min\left\{\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} = \mathbb{1}(i \sim j)\min\left\{\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa\frac{W_{ij}^\alpha}{C_{\text{ind}}^i}\right\}.$$

If $i \nsim j$ then the minimum is 0, and if $i \sim j$ we can choose

$$\kappa < \frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}\frac{C_{\text{ind}}^i}{W_{ij}^\alpha}. \qquad (3.15)$$

While this expression can be random, κ_0 is not random, and for $\kappa \leq \kappa_0$, the inequality (3.15) is satisfied for every realisation of the network. Hence,

$$\mathbb{E}\min\left\{\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} = \kappa\mathbb{E}\left[\mathbb{1}(i \sim j)\frac{W_{ij}^\alpha}{C_{\text{ind}}^i}\right].$$

Summing over $j = 1, \ldots, d$ and recalling the definition of C_{ind}^i gives (3.12).

To show (3.13) we start with (3.2); the argument is similarly straightforward. Consider the expression

$$\min\left\{\kappa\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} = \mathbb{1}(i \sim j)\min\left\{\kappa\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \frac{W_{ij}^\alpha}{C_{\text{ind}}^i}\right\}.$$

If

$$\kappa \leq \frac{W_{ij}^\alpha}{C_{\text{ind}}^i} \frac{C_{\text{ind}}^S}{\|Ae_j\|^\alpha},$$

then

$$\mathbb{1}(i \sim j)\min\left\{\kappa\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \frac{W_{ij}^\alpha}{C_{\text{ind}}^i}\right\} = \mathbb{1}(i \sim j)\kappa\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}.$$

In particular, this equation holds for $\kappa \leq \kappa_1$ with κ_1 given in (3.10). Again summing over all j gives (3.13).

To show (3.14) we use (3.3) Consider the expression

$$\min\left\{\kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}, \frac{A_{kj}^\alpha}{C_{\text{ind}}^k}\right\} = \mathbb{1}(i \sim j)\mathbb{1}(k \sim j)\min\left\{\kappa\frac{W_{ij}^\alpha}{C_{\text{ind}}^i}, \frac{W_{kj}^\alpha}{C_{\text{ind}}^k}\right\}.$$

For $\kappa \leq \kappa(i,k)$,

$$\mathbb{1}(i \sim j)\mathbb{1}(k \sim j)\min\left\{\kappa\frac{W_{ij}^\alpha}{C_{\text{ind}}^i}, \frac{W_{kj}^\alpha}{C_{\text{ind}}^k}\right\} = \mathbb{1}(k \sim j)\kappa\frac{W_{ij}^\alpha}{C_{\text{ind}}^i} = \kappa\mathbb{1}(k \sim j)\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}$$

for $\alpha > 0$. Summing over j gives the assertion (3.14). $\qquad\square$

Continuing from (3.12), (3.13) and (3.14) we can now assess the limiting behaviour of ICoVaR, SCoVaR and MCoVaR from Definition 1.1, specified for the aggregation function $h(f) = \|F\|$ of the exposure vector. For $\gamma_i, \gamma \in (0,1)$ referring to agent i and the market, respectively, we consider the following conditional systemic risk measures:

(a) *Individual Conditional Value-at-Risk*
$$\text{ICoVaR}_{1-(\gamma_i|\gamma)}(F_i \mid \|F\|) := \inf\{t \geq 0 : \mathbb{P}\left(F_i > t \mid \|F\| > \text{VaR}_{1-\gamma}(\|F\|)\right) \leq \gamma_i\},$$
(b) *Systemic Conditional Value-at-Risk*
$$\text{SCoVaR}_{1-(\gamma|\gamma_i)}(\|F\| \mid F_i) := \inf\{t \geq 0 : \mathbb{P}\left(\|F\| > t \mid F_i > \text{VaR}_{1-\gamma}(F_i)\right) \leq \gamma\},$$
(c) *Mutual Conditional Value-at-Risk*
$$\text{MCoVaR}_{1-(\gamma_i|\gamma_k)}(F_i \mid F_k) := \inf\{t \geq 0 : \mathbb{P}\left(F_i > t \mid F_k > \text{VaR}_{1-\gamma_k}(F_k)\right) \leq \gamma_i\}.$$

Theorem 3.4. *Assuming (3.8) and (3.9), we observe the following asymptotic behaviour of the different versions of the conditional risk measures (a) - (c):*

(a) *As $\gamma \to 0$ for $\gamma_i \leq \kappa_0(i)$,*

$$\text{ICoVaR}_{1-(\gamma_i|\gamma)}(F_i \mid \|F\|) \sim \text{VaR}_{1-\gamma_i\gamma}(F_i) \sim (C_{\text{ind}}^i)^{\frac{1}{\alpha}}(\gamma_i\gamma)^{-\frac{1}{\alpha}}. \qquad (3.16)$$

(b) *As $\gamma_i \to 0$ for $\gamma \leq \kappa_1(i)\tau(i)$,*

$$\text{SCoVaR}_{1-(\gamma|\gamma_i)}(\|F\| \mid F_i) \sim \text{VaR}_{1-\frac{\gamma_i\gamma}{\tau(i)}}(\|F\|) \sim (C_{\text{ind}}^S)^{\frac{1}{\alpha}}\left\{\frac{\gamma_i\gamma}{\tau(i)}\right\}^{-\frac{1}{\alpha}}. \qquad (3.17)$$

(c) *If $\tau(i,k) \neq 0$, then as $\gamma_k \to 0$, for $\gamma_i \leq \kappa_2(i,k)\tau(i,k)$, we have*

$$\text{MCoVaR}_{1-(\gamma_i|\gamma_k)}(F_i \mid F_k) \sim \text{VaR}_{1-\frac{\gamma_i\gamma_k}{\tau(i,k)}}(F_i) \sim (C_{\text{ind}}^i)^{\frac{1}{\alpha}}\left\{\frac{\gamma_i\gamma_k}{\tau(i,k)}\right\}^{-\frac{1}{\alpha}}; \qquad (3.18)$$

and, if $\tau(i,k) = 0$, then as $\gamma_i \to 0$,

$$\text{MCoVaR}_{1-(\gamma_i|\gamma_k)}(F_i \mid F_k) \sim \text{VaR}_{1-\gamma_i}(F_i) \sim (C_{\text{ind}}^i)^{\frac{1}{\alpha}}\gamma_i^{-\frac{1}{\alpha}}.$$

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Proof. First, from (3.1) and (3.12), for $\kappa \leq \kappa_0 = \kappa_0(i)$ as $\gamma \to 0$,

$$\mathbb{P}\left(F_i > \mathrm{VaR}_{1-\gamma\kappa}(F_i) \mid \|F\| > \mathrm{VaR}_{1-\gamma}(\|F\|)\right) \quad \to \quad \sum_{j=1}^{d} K_j \mathbb{E} \min\left\{\frac{\|Ae_j\|^\alpha}{C_{\mathrm{ind}}^S}, \kappa\frac{A_{ij}^\alpha}{C_{\mathrm{ind}}^i}\right\} = \kappa.$$

Hence, for $\gamma_i \leq \kappa_0$, as $\gamma \to 0$,

$$\mathbb{P}\left(F_i > \mathrm{VaR}_{1-\gamma\gamma_i}(F_i) \mid \|F\| > \mathrm{VaR}_{1-\gamma}(\|F\|)\right) \sim \gamma_i.$$

Thus $\mathrm{ICoVaR}_{1-(\gamma_i|\gamma)}(F_i \mid \|F\|) \sim \mathrm{VaR}_{1-\gamma_i\gamma}(F_i)$. The asymptotics for the VaR follow from (1.8) and (1.9), yielding (3.16).

For (3.17), relations (3.2) and (3.13) imply that for $\gamma \to 0$ and $\kappa > \kappa_1 = \kappa_1(i)$,

$$\mathbb{P}\left(\|F\| > \mathrm{VaR}_{1-\kappa\gamma}(\|F\|) \mid F_i > \mathrm{VaR}_{1-\gamma}(F_i)\right) \quad \to \quad \sum_{j=1}^{d} K_j \mathbb{E} \min\left\{\kappa\frac{\|Ae_j\|^\alpha}{C_{\mathrm{ind}}^S}, \frac{A_{ij}^\alpha}{C_{\mathrm{ind}}^i}\right\} = \kappa\tau(i).$$

In particular, a simple rescaling gives for $\gamma_i \to 0$,

$$\mathbb{P}\left(\|F\| > \mathrm{VaR}_{1-\kappa\gamma_i}(\|F\|) \mid F_i > \mathrm{VaR}_{1-\gamma_i}(F_i)\right) \quad \to \quad \kappa\tau(i).$$

Letting $\gamma = \kappa\tau(i)$ gives for $\gamma \leq \kappa_1\tau(i)$, as $\gamma_i \to 0$,

$$\mathbb{P}\left(\|F\| > \mathrm{VaR}_{1-\frac{\gamma_i\gamma}{\tau(i)}}(\|F\|) \mid F_i > \mathrm{VaR}_{1-\gamma_i}(F_i)\right) \quad \to \quad \gamma.$$

Now (3.17) follows as before.

For (3.18), relations (3.3) and (3.14) give for $\gamma \to 0$ and $\kappa \leq \kappa_2(i,k)$,

$$\mathbb{P}\left(F_i > \mathrm{VaR}_{1-\gamma\kappa}(F_i) \mid F_k > \mathrm{VaR}_{1-\gamma}(F_k)\right) \quad \to \quad \sum_{j=1}^{d} K_j \mathbb{E} \min\left\{\kappa\frac{A_{ij}^\alpha}{C_{\mathrm{ind}}^i}, \frac{A_{kj}^\alpha}{C_{\mathrm{ind}}^k}\right\} = \kappa\tau(i,k).$$

Changing variables gives for $\gamma_k \to 0$,

$$\mathbb{P}\left(F_i > \mathrm{VaR}_{1-\gamma_k\kappa}(F_i) \mid F_k > \mathrm{VaR}_{1-\gamma_k}(F_k)\right) \quad \to \quad \kappa\tau(i,k).$$

Setting $\gamma = \kappa\tau(i,k)$ and requiring that $\gamma \leq \kappa_2(i,k)\tau(i,k)$ gives (3.18), provided that $\tau(i,k) \neq 0$.

The last assertion follows from the fact that F_i and F_k are independent, if agents i and k do not share an object. $\qquad\square$

Remark 3.5. The asymptotic behaviour of the risk measures is assessed in Theorem 3.4 through the exceedance probabilities conditioned on an extreme event. For example, in (3.18), agent k has already experienced a very large loss. This loss will have an effect on the loss of agent i if they share some objects in their portfolios. The more objects they share, the larger $\tau(i,k)$ will be. The unconditional VaR threshold $1 - \gamma_i$ at which $\mathbb{P}(F_i > t) = \gamma_i$ has to be adjusted to $1 - \gamma\frac{\gamma_k}{\tau(i,k)}$ if $\tau(i,k) \neq 0$. The larger $\tau(i,k)$, the larger $1 - \gamma\frac{\gamma_k}{\tau(i,k)}$ will be, hence, the higher the capital requirements for agent i.

The effect of the network on the agent in (3.17) indicates the dependence on $\tau(i)$, which increases with the number of connections of agent i. Again, the larger $\tau(i)$, the higher the capital requirements on agent i should be.

Even in (3.16) there is dependence of the network structure, which is reflected in $\kappa_0(i)$ as well as in C_{ind}^i. $\qquad\square$

4 Network effects

Throughout this section we restrict ourselves to the situation that the losses $V_1, \ldots, V_d$ are asymptotically independent. Furthermore, we exemplify our results based on a bipartite network model with independent edges. More precisely, we assume that the edge indicator variables $\{\mathbb{1}(i \sim j), 1 \leq i \leq q, 1 \leq j \leq d\}$ are independent Bernoulli random variables with $\mathbb{E}\mathbb{1}(i \sim j) = p_{ij}$. For the situation of a large claims insurance market as in (1.4), we have $A_{ij} = \frac{\mathbb{1}(i \sim j)}{\deg(j)}$, while for the capital investment problem as in (1.5), we have $A_{ij} = c_i \frac{\mathbb{1}(i \sim j)}{\deg(i)}$ (with $\frac{0}{0} := 0$).

This model allows us to compute Poisson approximations for the relevant constants in Proposition 3.1 and Theorem 3.4, which we present in Section 4.1 for the insurance network (1.4).

In Section 4.2 we present numerical simulations, now for the investor network (1.5) contrasting the homogeneous model with a Rasch-type inhomogeneous model.

4.1 Independent bipartite graph model: Poisson approximations

In this section we consider the insurance example, where agents are reinsurance companies, objects are possible catastrophic claims, and $A_{ij} = \frac{\mathbb{1}(i \sim j)}{\deg(j)}$. If d and q are large, we can provide Poisson approximations for the quantities C_{ind}^S, C_{ind}^i, $\mathbb{E}A_{ij}\|Ae_j\|^{\alpha-1}$, $\mathbb{E}A_{ij}^{\alpha-1}\|Ae_j\|$, and $\mathbb{E}A_{kj}^{\alpha-1}A_{ij}$ concerning the CoTE and CoVaR, which appear in Proposition 3.1 We define by $X \sim \text{Pois}(\lambda)$ a Poisson-distributed random variable X with mean $\lambda > 0$. We shall use the following Poisson variables;

$$X_j^{i,k} \sim \text{Pois}(\lambda_j^{i,k}) \quad \text{with} \quad \lambda_j^{i,k} = \sum_{l=1, l\neq i,k}^{q} p_{li},$$

$$X_j^{i} \sim \text{Pois}(\lambda_j^{i}) \quad \text{with} \quad \lambda_j^{i} = \sum_{l=1, l\neq i}^{q} p_{li}, \text{ and}$$

$$X_j \sim \text{Pois}(\lambda_j) \quad \text{with} \quad \lambda_j = \sum_{k=1}^{q} p_{kj}.$$

Proposition 4.1 from [20] gives that

$$\left|\mathbb{E}A_{ij}^{\alpha} - p_{ij}\mathbb{E}(1 + X_j^{i})^{-\alpha}\right| \leq p_{ij} \min\{1, (\lambda_j^{i})^{-1}\} \sum_{k=1,\ldots,q; k\neq i} p_{kj}^{2} =: B(i, j), \tag{4.1}$$

and, for the r-norm for some $r \geq 1$,

$$\left|\mathbb{E}\|Ae_j\|^{\alpha} - \mathbb{E}\left[\mathbb{1}\{X_j \geq 1\}(1 + X_j)^{\alpha(1/r-1)}\right]\right| \leq \min\{1, (\lambda_j)^{-1}\} \sum_{k=1}^{q} p_{kj}^{2} =: B(j). \tag{4.2}$$

We shall also employ

$$B(i, j, k) := \min\{1, (\lambda_j^{i,k})^{-1}\} \sum_{\ell=1,\ldots,q; \ell\neq i} p_{\ell j}^{2}. \tag{4.3}$$

The following is an immediate consequence of (1.6), (4.1) and (4.2).

Lemma 4.1. *With the notation as above*

$$\left| C_{\text{ind}}^i - \sum_{j=1}^d K_j p_{ij} \mathbb{E}(1 + X_j^i)^{-\alpha} \right| \leq \sum_{j=1}^d K_j B(i,j). \tag{4.4}$$

$$\left| C_{\text{ind}}^S - \sum_{j=1}^d K_j \mathbb{E}\big[\mathbb{1}\{X_j \geq 1\}(1 + X_j)^{-\alpha\frac{r-1}{r}}\big] \right| \leq \sum_{j=1}^d K_j B(j). \tag{4.5}$$

For $r \geq 1$, with $B(i,j)$ given in (4.1), $B(j)$ given in (4.2), and $B(i,j,k)$ given in (4.3) we obtain the following results for the quantities in Proposition 3.1.

Proposition 4.2. *For the quantities from Proposition 3.1, let*

$$M_1 = \min\left\{\frac{\kappa}{C_{\text{ind}}^i}, \frac{1}{C_{\text{ind}}^S}\right\}, \quad M_2 = \min\left\{\frac{1}{C_{\text{ind}}^i}, \frac{\kappa}{C_{\text{ind}}^S}\right\}, \quad M_3 = \min\left\{\frac{\kappa}{C_{\text{ind}}^i}, \frac{1}{C_{\text{ind}}^k}\right\}.$$

Then

$$\left| \mathbb{E}\min\left\{\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} - p_{i,j}\mathbb{E}\min\left\{\frac{(1 + X_j^i)^{-\alpha+\frac{\alpha}{r}}}{C_{\text{ind}}^S}, \kappa\frac{(1 + X_j^i)^{-\alpha}}{C_{\text{ind}}^i}\right\} \right| \leq M_1 B(i,j), \tag{4.6}$$

$$\left| \mathbb{E}\min\left\{\kappa\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} - p_{i,j}\mathbb{E}\min\left\{\kappa\frac{(1 + X_j^i)^{-\alpha+\frac{\alpha}{r}}}{C_{\text{ind}}^S}, \frac{(1 + X_j^i)^{-\alpha}}{C_{\text{ind}}^i}\right\} \right| \leq M_2 B(i,j), \tag{4.7}$$

and for $i \neq k$

$$\left| \mathbb{E}\min\left\{\kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}, \frac{A_{kj}^\alpha}{C_{\text{ind}}^k}\right\} - p_{ij}p_{kj}M_3\mathbb{E}\big[(2 + X_j^{i,k})^{-\alpha}\big] \right| \leq p_{ij}p_{kj}M_3 B(i,j,k). \tag{4.8}$$

Moreover, if $\alpha > 1$, then

$$\left| \mathbb{E}A_{ij}\|Ae_j\|^{\alpha-1} - p_{ij}\mathbb{E}\big[\big(1 + X_j^i\big)^{-\frac{\alpha(r-1)+1}{r}}\big] \right| \leq B(i,j), \tag{4.9}$$

$$\left| \mathbb{E}A_{ij}^{\alpha-1}\|Ae_j\| - p_{ij}\mathbb{E}\big[\big(1 + X_j^i\big)^{\frac{1}{r}-\alpha}\big] \right| \leq B(i,j), \tag{4.10}$$

and for $i \neq k$,

$$\left| \mathbb{E}A_{kj}^{\alpha-1}A_{ij} - p_{ij}p_{kj}\mathbb{E}\big[(2 + X_j^{i,k})^\alpha\big] \right| \leq p_{ij}p_{kj}\min\{1, (\lambda_j^{i,k})^{-1}\} \sum_{\ell=1,\ldots,q;\ell\neq i} p_{\ell j}^2. \tag{4.11}$$

Proof. We compute

$$\|Ae_j\|^\alpha = \Big(\sum_{k=1}^q \frac{\mathbb{1}(k \sim j)}{\deg(j)^r}\Big)^{\frac{\alpha}{r}} = \Big(\frac{1}{\deg(j)^{r-1}}\Big)^{\frac{\alpha}{r}}\mathbb{1}(\deg(j) > 0). \tag{4.12}$$

With (4.12), we obtain

$$\min\left\{\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} = \mathbb{1}(i \sim j)\min\left\{\frac{\deg(j)^{-\alpha+\frac{\alpha}{r}}}{C_{\text{ind}}^S}, \kappa\frac{\deg(j)^{-\alpha}}{C_{\text{ind}}^i}\right\}$$

$$= \mathbb{1}(i \sim j)\min\left\{\frac{(1 + \sum_{k\neq i}\mathbb{1}(k \sim j))^{-\alpha+\frac{\alpha}{r}}}{C_{\text{ind}}^S}, \kappa\frac{(1 + \sum_{k\neq i}\mathbb{1}(k \sim j))^{-\alpha}}{C_{\text{ind}}^i}\right\}$$

and, consequently,

$$\mathbb{E}\min\left\{\frac{\|Ae_j\|^\alpha}{C_{\text{ind}}^S}, \kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}\right\} = p_{ij}\mathbb{E}\min\left\{\frac{(1+\sum_{k\neq i}\mathbb{1}(k\sim j))^{-\alpha+\frac{\alpha}{r}}}{C_{\text{ind}}^S}, \kappa\frac{(1+\sum_{k\neq i}\mathbb{1}(k\sim j))^{-\alpha}}{C_{\text{ind}}^i}\right\}.$$
(4.13)

Now consider the function

$$k(x) = \min\left\{\frac{(1+x)^{-\alpha+\frac{\alpha}{r}}}{C_{\text{ind}}^S}, \kappa\frac{(1+x)^{-\alpha}}{C_{\text{ind}}^i}\right\}.$$

If $C_{\text{ind}}^S \geq 1$ or $\frac{\kappa}{C_{\text{ind}}^i} \geq 1$, then $k(x) \in [0,1]$. In general,

$$0 \leq k(x) \leq \min\left\{\frac{1}{C_{\text{ind}}^S}, \frac{\kappa}{C_{\text{ind}}^i}\right\} = M_1$$

with M_1 as in (4.6). Hence

$$t(x) = M_1^{-1}k(x) = \max\left\{C_{\text{ind}}^S, \frac{C_{\text{ind}}^i}{\kappa}\right\}k(x) \in [0,1].$$

Now we use a standard approximation result in total variation distance, Eq. (1.23), p. 8, from [3]. This result states that, if S is the sum of n independent Bernoulli random variables with success probabilities p_i, $\mathbb{E}S = \lambda = \sum_{i=1}^n p_i$, and $Z \sim \text{Pois}(\lambda)$, then

$$\sup_{h:\mathbb{Z}^+\to[0,1]} |\mathbb{E}k(S) - \mathbb{E}k(Z)| \leq \min\{1,\lambda^{-1}\}\sum_{i=1}^n p_i^2.$$
(4.14)

Applying (4.14) to the function $t(x)$ and keeping (4.13) in mind yields (4.6). Now (4.7) follows similarly. Finally,

$$\begin{aligned}
\min\left\{\kappa\frac{A_{ij}^\alpha}{C_{\text{ind}}^i}, \frac{A_{kj}^\alpha}{C_{\text{ind}}^k}\right\} &= \min\left\{\kappa\frac{\frac{\mathbb{1}(i\sim j)}{\deg(j)^\alpha}}{C_{\text{ind}}^i}, \frac{\frac{\mathbb{1}(k\sim j)}{\deg(j)^\alpha}}{C_{\text{ind}}^k}\right\}\\
&= \mathbb{1}(i\sim j)\mathbb{1}(k\sim j)\Big(2+\sum_{\ell\neq i,k}\mathbb{1}(\ell\sim j)\Big)^{-\alpha}\min\left\{\kappa\frac{1}{C_{\text{ind}}^i}, \frac{1}{C_{\text{ind}}^k}\right\}\\
&= M_3\mathbb{1}(i\sim j)\mathbb{1}(k\sim j)\Big(2+\sum_{\ell\neq i,k}\mathbb{1}(\ell\sim j)\Big)^{-\alpha}.
\end{aligned}$$

As the positive function $k(x) = (2+x)^{-\alpha}$ is bounded by 1 and as $\sum_{\ell=1,\dots,q;\ell\neq i,k}\mathbb{1}(\ell\sim j)$ is a sum of independent Bernoulli variables, (4.14) can be applied, and (4.8) follows.

For the Conditional Tail Expectations, with (4.12),

$$A_{ij}\|Ae_j\|^{\alpha-1} = \left(\frac{1}{\deg(j)^{r-1}}\right)^{\frac{\alpha-1}{r}}\frac{\mathbb{1}(i\sim j)}{\deg(j)} = \left(\frac{\mathbb{1}(i\sim j)}{\deg(j)}\right)^{\frac{\alpha(r-1)+1}{r}} = A_{ij}^{\frac{\alpha(r-1)+1}{r}}.$$

Hence, (4.1) applies and yields (4.9). Similarly, with (4.12),

$$A_{ij}^{\alpha-1}\|Ae_j\| = \frac{\mathbb{1}(i\sim j)}{\deg(j)^{\alpha-1}}\left(\frac{1}{\deg(j)^{r-1}}\right)^{\frac{1}{r}} = A_{ij}^{\alpha-\frac{1}{r}}.$$
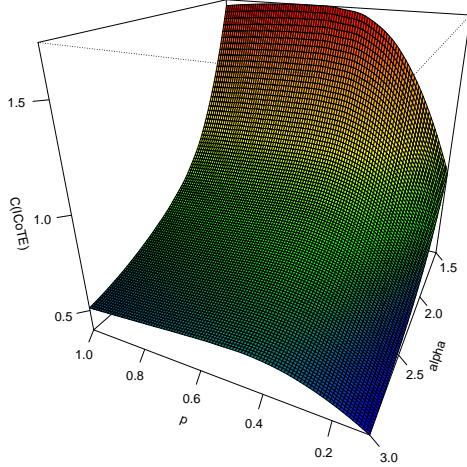
Figure 2: Homogeneous bipartite network (risk of an agent given the system is under stress): $C(\text{ICoTE})$ as a function of the tail index $\alpha \in [1.5, 3]$ and the market activity $p \in (0.01, 1.0]$.

Again (4.1) applies, and yields (4.10). For the last part we mimick the proof of Proposition 4.1 in [20]. By the independence of the edges,

$$\mathbb{E}\big[A_{kj}^{\alpha-1} A_{ij}\big] = \mathbb{E}\Big[\mathbb{1}(i \sim j)\mathbb{1}(k \sim j)\frac{1}{\deg(j)^{\alpha}}\Big] = p_{ij}p_{kj}\mathbb{E}\Big[\Big(2 + \sum_{\ell=1,\dots,q;\ell\neq i,k} \mathbb{1}(\ell \sim j)\Big)^{-\alpha}\Big].$$

Again (4.14) can be applied and the bound (4.11) follows. $\qquad\qquad\qquad\square$

Remark 4.3. Using (4.4) and (4.5) the constants M_1, M_2, and M_3, as well as the expressions on the left-hand side of Proposition 4.2, could be bounded further if desired. $\qquad\square$

Remark 4.4. Proposition 4.2 gives an exact bound on the distance to Poisson; no asymptotic regime is suggested. Hence, it can be interpreted in different asymptotic regimes.

If the number d of objects increases, while the number q of agents is such that $q = o(\sqrt{d})$, and the number of objects which an agent would connect to, stays constant in expectation, in a fashion so that $p_{ij} \sim \frac{c}{d}$ for a fixed c, then $B(j)$ and $B(i,j,k)$ are of order q/d^{-2}, $B(i,j)$ is of order qd^{-3}; as long as $q = o(\sqrt{d})$ the Poisson approximation will be suitable.

Similarly, if the number q of agents increases and the number d of objects only increases as $d = o(\sqrt{q})$, and if $p_{ij} \sim c/q$ for a fixed c, the Poisson approximation will be suitable. $\qquad\square$

4.2 Independent bipartite graph model: numerical results

In this section we consider the investment example, where agents are investors and objects are possible investments. All investors distribute their investment capital equally among all of their investments, which they have chosen as in (1.5). Furthermore, we set all $K_j = 1$.

We are interested in the asymptotic constants given in Proposition 3.1 as well as in Theorem 3.4 concerning the CoTE and CoVaR. We considered all conditioning situations: investor conditioned on system's stress, system conditioned on investor's stress, and investor conditioned

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on another investor's stress. For illustration purposes we present the first two situations in terms of CoTE and the last one in terms of CoVaR.

The conditional systemic risk measures ICoTE (risk of agent i given the system is under stress) in (3.4) and SCoTE (the system's risk given agent i is under stress) in (3.5), respectively, are determined by the constants

$$C(\text{ICoTE}) := C^i(\text{ICoTE}) := \frac{\alpha}{\alpha - 1}(C_{\text{ind}}^S)^{1/\alpha - 1} \sum_{j=1}^{d} K_j \mathbb{E}[A_{ij} \| Ae_j \|^{\alpha-1}], \qquad (4.15)$$

and

$$C(\text{SCoTE}) := C^i(\text{SCoTE}) := \frac{\alpha}{\alpha - 1}(C_{\text{ind}}^i)^{1/\alpha - 1} \sum_{j=1}^{d} K_j \mathbb{E}[A_{ij}^{\alpha-1} \| Ae_j \|], \qquad (4.16)$$

which capture all necessary market information. The third example, the MCoVaR (risk of agent i given agent k is under stress) from (3.18) is associated with the constant

$$C(\text{MCoVaR}) := C^{i|k}(\text{MCoVaR}) := (C_{\text{ind}}^i \tau(i,k))^{1/\alpha} = \sum_{j=1}^{d} K_j \mathbb{E}[\mathbb{1}(k \sim j) A_{ij}^{\alpha}] \qquad (4.17)$$

for $\tau(i,k) \neq 0$ from (3.11). For the independent bipartite graph model $\tau(i,k) \neq 0$ holds whenever $p_{ik} > 0$. All plots starts with $p = 0.01$, which corresponds to a market with (almost) no activity, ranging to $p = 1$. For α we choose the interval $\alpha \in [1.5, 3]$ to cover situations where the losses have finite mean, and the interval $\alpha \in [0.8, 3]$ if there is no restriction that the mean of the losses should exist, as the mean only exists for $\alpha > 1$.

4.2.1 Homogeneous bipartite graph model

For this model all edge probabilities $p_{ij} = p \in [0, 1]$ are equal, such that all agents behave exchangeably. Then the market ranges from no investment activity at all ($p = 0$) to a complete bipartite graph ($p = 1$) reflecting that each investor holds every investment. We consider the situation of $q = 5$ investors and $d = 5$ investments.

In Figure 2 we plot the constant $C(\text{ICoTE})$ from (4.15) as a proxy for the risk of an agent given the system is under stress as a function of the tail index α for $\alpha \in [1.5, 3]$ and the probability p, where the plot starts with $p = 0.01$ corresponding to a market with (almost) no activity, ranging to $p = 1$, where each investor holds every investment. Considering a curve in p for fixed α, the graph increases first—since the probability that investors have invested at all (and do not only hold cash) increases in p, the probability that they are exposed to risk at all increases. With further increasing connectivity in the market, we recognize that there is a positive effect of risk diversification: the curve decreases after having attained a local maximum. This effect can be observed for all α under consideration, though it is stronger for larger α corresponding to lighter tails.

In Figure 3 we plot the constant $C(\text{SCoTE})$ from (4.16) as a proxy for the risk of the system given an agent is under stress as a function of $\alpha \in [1.5, 3]$ and the probability p ranging from 0.01 to 1. If we fix α and consider a curve in p, we recognize for lighter tails the same behaviour
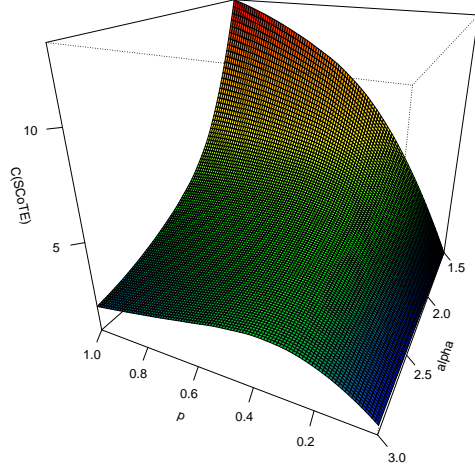
Figure 3: Homogeneous bipartite network (risk of the system given an investor is under stress): C(SCoTE) as a function of the tail index $\alpha \in [1.5, 3]$ and the market activity $p \in (0.01, 1.0]$.

as in Figure 2; i.e., after an increase due to increasing the investment probability, there is a positive effect of risk diversification in the way that the constant decreases again for p large enough. However, for smaller α corresponding to heavier tails the non-monotonous behaviour of the curve vanishes, so the risk increases throughout. Consequently, an increased connectivity in the market (often considered as diversification) does not always lower the corresponding risk.

In Figure 4 we plot the constant $C(\text{MCoVaR})$ from (4.17) as a proxy for the risk of one agent given another agent is under stress as a function of the tail index $\alpha \in [0.8, 3]$ and the probability p ranging from 0.01 to 1. Considering again curves in p for fixed α we recognize the effects of the previous plots: For larger α there is a non-monotonous behaviour of the curve indicating a positive effect of risk diversification shown by a final decrease of the curve for growing network connectivity. This effect is not observable for intermediate values of α, where the curves are monotonically increasing in p. In addition, if $\alpha < 1$, there are not only no positive effects of risk diversification, but the change in the curvature from concave to convex actually indicates that risk accelerates to rise as the network connectivity increases, so we discover negative effects of diversification. That diversification is not preferrable in infinite-mean models ($\alpha \leq 1$) has already been observed in non-conditional situations; see e.g. [16, 20, 23].

4.2.2 Rasch-type bipartite graph model

In the homogeneous example agents are exchangeable, so all investors are of the same type. Now we allow for different types of investors by considering a Rasch-type model as motivated in Section 4.2.3 of [20]. We assume that $p_{ij} = p\,\beta_i \delta_j$ for suitably chosen parameters $\beta_i, \delta_j > 0$ and a free parameter $p \in [0, 1]$. Here, the parameter β_i gives a measure for the risk proneness of investor i, while the parameter δ_j reflects the attractiveness of investment j; the parameter p again indicates the activity in the market.

In our market there are 5 investors and, in contrast to the homogeneous model, we have two
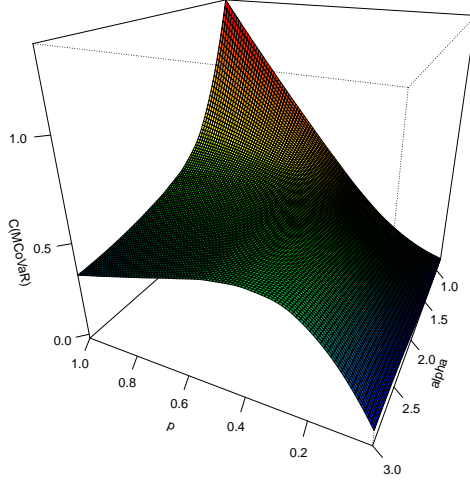
Figure 4: Homogeneous bipartite network (risk of one investor given another one is under stress): $C(\text{MCoVaR})$ as a function of the tail index $\alpha \in [1.5, 3]$ and the market activity $p \in (0.01, 1.0]$.

different types of investors corresponding to $\beta_1 = 0.2$ (a single risk averse investor 1) and $\beta_2 = 1$ (four risk prone investors with same risk affinity).

The situation for the investments is given by a vector δ, where in **Scenario 1** one investment is dominantly attractive in contrast to all the others, given as $\delta = (0.2, 0.2, 0.2, 0.2, 0.9)$. In **Scenario 2** we have a market, in which only one investment is unattractive opposed to the rest, which are attractive given as $\delta = (0.9, 0.9, 0.9, 0.9, 0.2)$.

Figure 5 depicts the constant $C(\text{ICoTE})$ from (4.15), which represents the conditional risk of an investor given the market is under stress. In Scenario 1 there is only one attractive investment. The risk obviously increases with the probability p; i.e. with the market activity. The conditional risk of the risk averse investor remains always smaller than that of a risk prone investor. Comparing with the homogeneous investor of Figure 2, the risk of the homogeneous investor lies between the risk of the risk averse and the risk prone investor. Moreover, the risk also increases when α becomes smaller, corresponding to heavier tails. A diversification effect cannot be observed for this scenario. In contrast to the homogeneous model there is no diversification effect visible.

Figure 6 shows the same constant $C(\text{ICoTE})$ from (4.15), but now for Scenario 2; i.e., there are 4 attractive investments. Here the risk averse investor's conditional risk increases with p, and decreases with α. For the risk prone investor the situation changes: the conditional risk for the risk prone investor increases first with p, but with further increasing connectivity in the market, there is a positive effect of risk diversification. This effect can be observed for all $\alpha \in [1.5, 3]$, though it is stronger for larger α; i.e. for lighter tails.

Figure 7 illustrates the conditional systemic risk given an investor is in distress by the constant C(SCoTE) from (4.16) for a portfolio with one attractive risk. The market risk conditional on the risk averse investor's distress is always smaller than that conditional on the risk prone investor being under stress. For small α and very large connectivity, this conditional risk is doubled compared to the risk averse investor. In particular, for small α there is a change in the
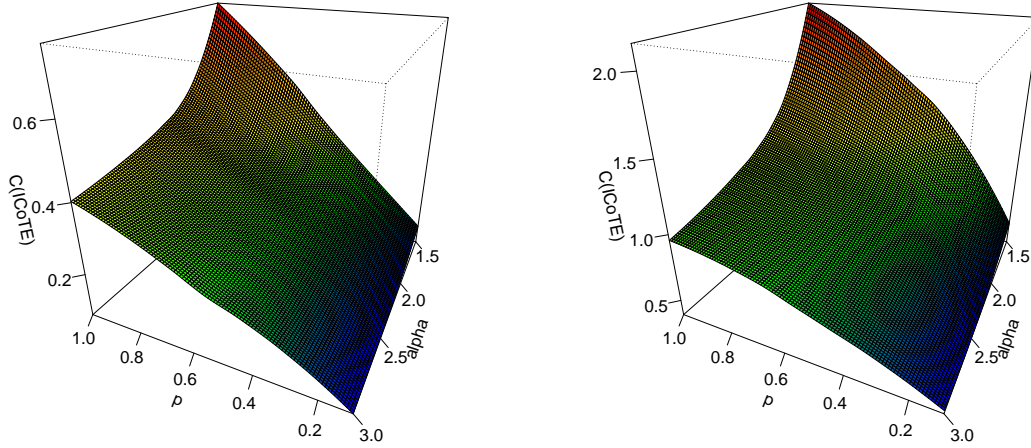
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Figure 5: Rasch model (risk of an investor given the system is under stress for Scenario 1): $C(\text{ICoTE})$ as a function of the tail index $\alpha \in [1.5, 3]$ and the market activity $p \in (0.01, 1.0]$. Left: risk averse investor. Right: risk prone investor.
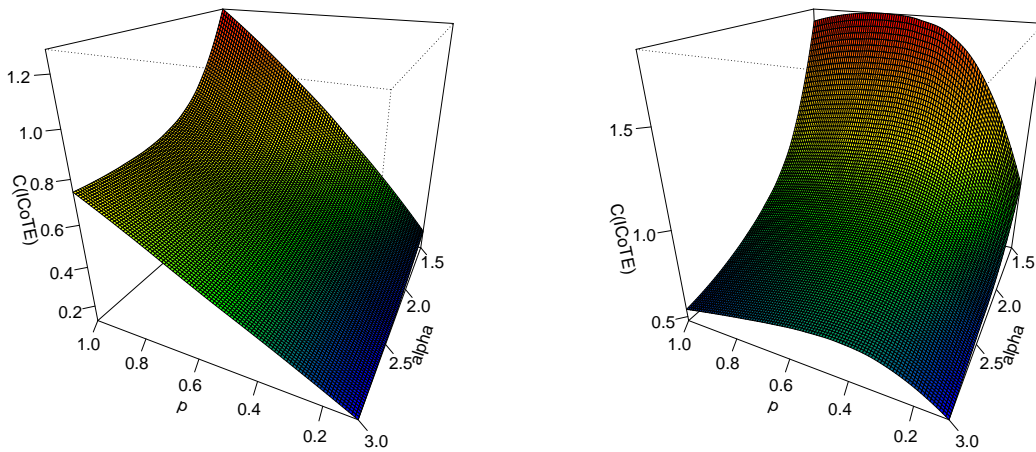

Figure 6: Rasch model (risk of an investor given the system is under stress for Scenario 2): $C(\text{ICoTE})$ as a function of the tail index $\alpha \in [1.5, 3]$ and the market activity $p \in (0.01, 1.0]$. Left: risk averse investor. Right: risk prone investor.

curvature indicating that risk accellerates to rise as the network connectivity increases. This confirms the obvious fact that a risk prone investor may be much more dangerous for the system than a risk averse one.

Figure 8 shows the same constant for Scenario 2, where 4 attractive risks are available for investment. If we fix α and consider a curve in p, we notice a profound difference when the risk averse investor is under stress compared to when the risk prone investor is under stress. In the left-hand plot we see a monotonous increase for increasing p and decreasing α. In the right-hand plot, however, there is a positive effect of risk diversification for not too small α, i.e. for not too heavy tailed investment risk, when the network connectivity increases. It is remarkable that this diversification effect can compensate the stress of a risk prone investor better than the stress of the risk averse investor: the risk for lighter tailed risk and large p is smaller.
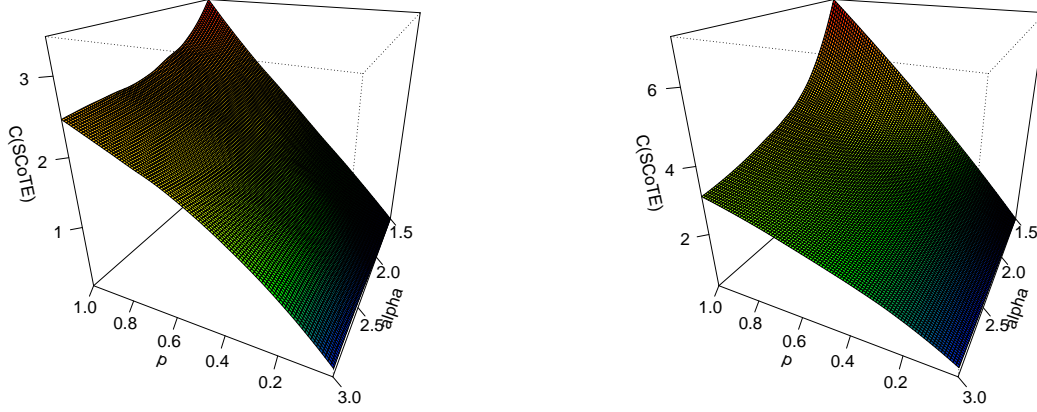
Figure 7: Rasch model (risk of the system given an investor is under stress for Scenario 1): C(SCoTE) as a function of the tail index α for $\alpha \in [1.5, 3]$ and with market activity $p \in (0.01, 1.0]$. Left: risk averse investor. Right: risk prone investor.

Figures 9-12 show the constant C(MCoVaR) from (4.17), which represents the conditional risk of one investor given another is under stress. Here we plot C(MCoVaR) for $\alpha \in [0.8, 3]$; i.e., including the infinite mean case. As these curves have different curvatures in different areas, we show the same figure twice, the left plot as before, and the right plot presenting it at a different angle.

Figures 9 and 10 consider Scenario 1; i.e., for one attractive risk in the market. Figure 9 shows the risk of the risk averse investor given that a risk prone investor is under stress. For fixed small market activity p the risk increases moderately (and almost linearly) when α decreases. When p becomes larger, then the risk increases further, but now it also increases with α. For an (almost) complete graph corresponding to p near 1, the risk increases with α in a concave fashion. For small α and small p the plot increases almost linearly, for large α and p the curvature becomes positive.

Figure 10 shows the risk of the risk prone investor given that a risk averse investor is under stress again for Scenario 1 with one attractive investment in the market. At its largest point, for $p = 1$ and $\alpha = 3$ the risk remains below the value of Figure 9. So, the risk prone investor takes still some benefit from diversifying his investments. The stress situation of the risk averse investor as the conditioning event stems from a loss in one investment on which the averse investor concentrates the capital or at least a huge fraction of it.

Figures 11 and 12 correspond to Figures 9 and 10 for Scenario 2, where 4 attractive investments are in the market. The conditional risk of the risk averse investor, shown in Figure 11 has similar curvature as for Scenario 1, but it is larger throughout. The conditional risk of the risk prone investor exhibits again some diversification effect when the market connectivity exceeds some p. Thus the threshold p varies with α and is non-trivial.
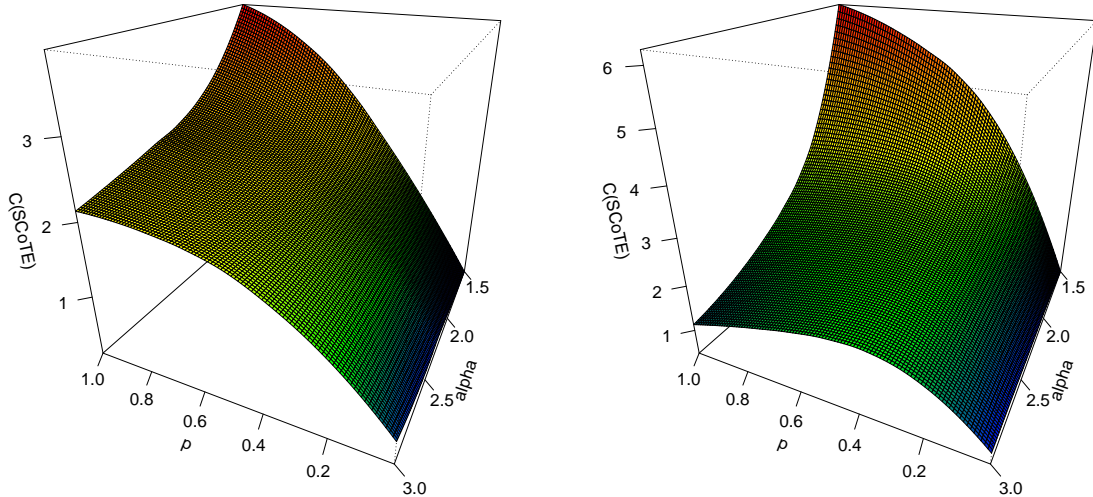
Figure 8: Rasch model (risk of the system given an agent is under stress for Scenario 2): C(SCoTE) as a function of the tail index α for $\alpha \in [1.5, 3]$ and with market activity $p \in (0.01, 1.0]$. Left: risk averse agent. Right: risk prone agent

5 Conclusion and Outlook

We have investigated conditional systemic risk measures for a agent-object market given by a bipartite graph structure. Within the appropriate framework of multivariate regular variation, a classic setting for heavy-tailed distributions, we have formulated and investigated the asymptotic behaviour of Individual, Systemic, and Mutual Conditional Value-at-Risk and Conditional Tail Expectation ; these notions are applicable to very high risk settings.

For the insurance example, where the agents (reinsurance companies) insure possible catastrophic claims, we provide Poisson approximations for the conditional risk measures represented by a number of constants. Based on our theoretical results, we have investigated the network effect for different scenarios. For an envisioned high-risk investment example we present numerical results for bipartite graph models. For the homogeneous model, where all agents behave exchangeably, we have plotted and interpreted the Individual and the Systemic Conditional Tail Expectation as well as the Mutual Conditional Value-at-Risk. For a Rasch-type bipartite graph model we have investigated different scenarios concerning the risk proneness of the agents as well as the attractiveness of the different investments.

Our work is applied in [19] to a real data set of operational risk data. In that paper we also develop a new statistical methodology for the estimation of the distribution of the joint exposures, taking the discreteness of the dependence structure into account.

This paper shows that understanding the behaviour of conditional risk under different market scenarios can be informative both for agents and for the regulator.
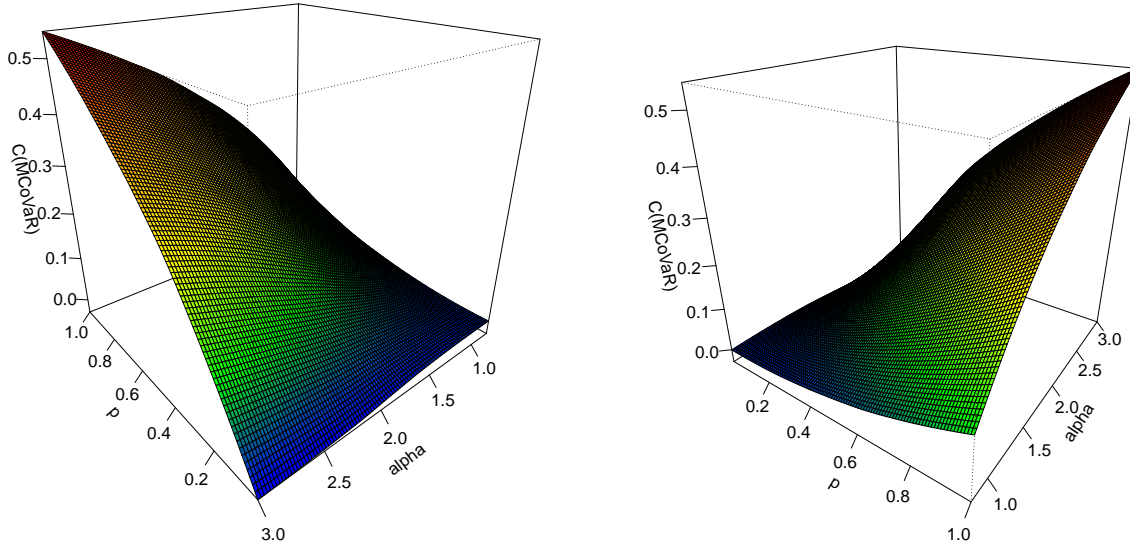
Figure 9: Rasch model (risk of the risk averse investor given the risk prone investor is under stress for Scenario 1): Left plot: $C(\text{MCoVaR})$ as a function of the tail index $\alpha \in [0.8, 3]$ and the market activity $p \in (0.01, 1.0]$. Right plot: left plot rotated by 180 degrees.
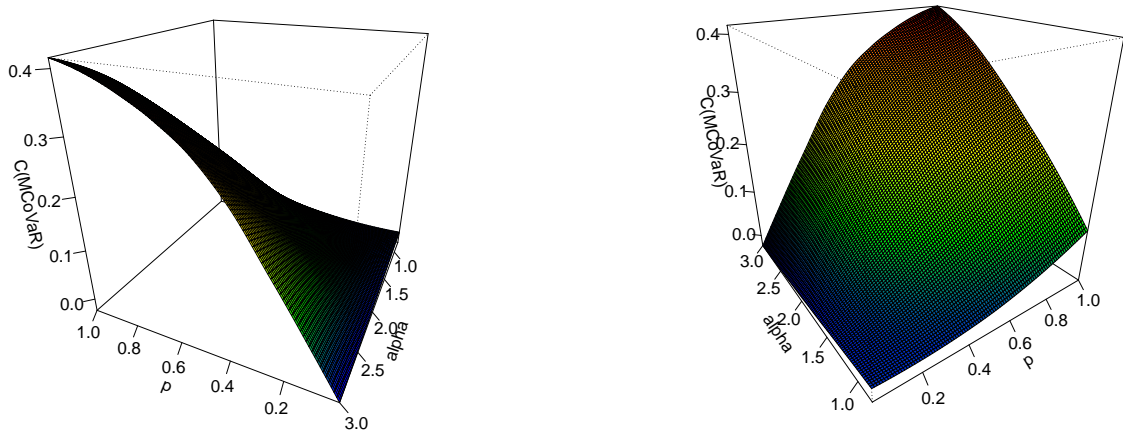

Figure 10: Rasch model (risk of the risk prone investor given the risk averse investor is under stress for Scenario 1): Left plot: $C(\text{MCoVaR})$ as a function of the tail index $\alpha \in [0.8, 3]$ and the market activity $p \in (0.01, 1.0]$. Right plot: left plot rotated by 90 degrees.
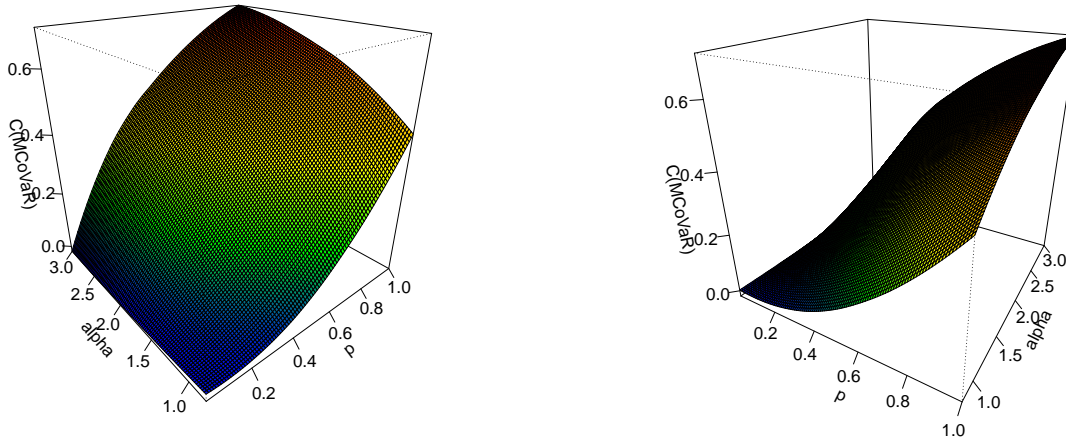
Acknowledgements

Figure 11: Rasch model (risk of the risk averse investor given the risk prone investor is under stress for Scenario 2): Left plot: $C(\text{MCoVaR})$ as a function of the tail index $\alpha \in [1.5, 3]$ and with market activity $p \in (0.01, 1.0]$. Right plot: left plot rotated by 180 degrees.
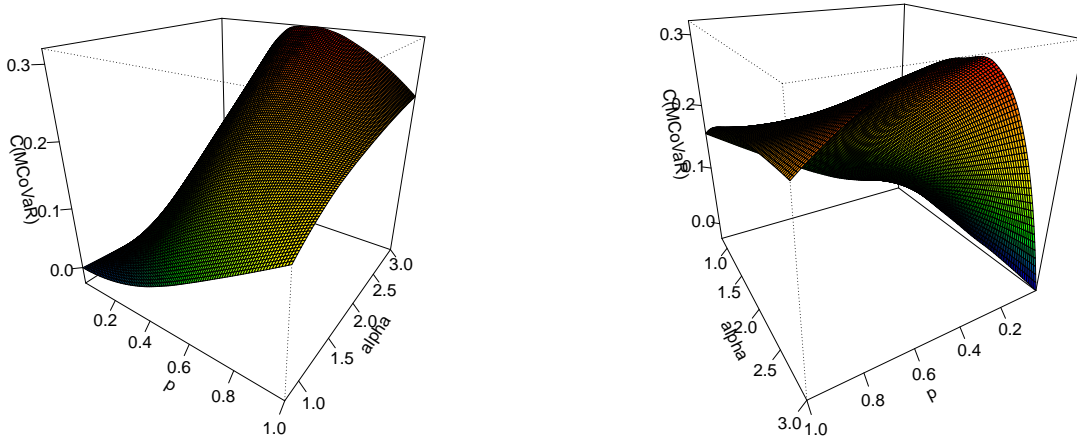

Figure 12: Rasch model (risk of the risk prone investor given the risk averse investor is under stress for Scenario 2): Left plot: $C(\text{MCoVaR})$ as a function of the tail index $\alpha \in [1.5, 3]$ and the market activity $p \in (0.01, 1.0]$. Right plot: left plot rotated by 180 degrees.

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